

Algebraic Methods in Philosophical Logic

J. MICHAEL DUNN
and
GARY M. HARDEGREE

CLARENDON PRESS • OXFORD

2001

CONTENTS

1	Introduction	1
2	Universal Algebra	10
2.1	Introduction	10
2.2	Relational and Operational Structures (Algebras)	10
2.3	Subrelational Structures and Subalgebras	11
2.4	Intersection, Generators, and Induction from Generators	13
2.5	Homomorphisms and Isomorphisms	15
2.6	Congruence Relations and Quotient Algebras	19
2.7	Direct Products	25
2.8	Subdirect products and the Fundamental Theorem of Universal Algebra	28
2.9	Word Algebras and Interpretations	33
2.10	Varieties and Equational Definability	36
2.11	Equational Theories	37
2.12	Examples of Free Algebras	39
2.13	Freedom and Typicality	41
2.14	The Existence of Free Algebras; Freedom in Varieties and Subdirect classes	44
2.15	Birkhoff's Varieties Theorem	47
2.16	Quasi-varieties	49
2.17	Logic and Algebra: Algebraic Statements of Soundness and Completeness	51
3	Order, Lattices, and Boolean Algebras	55
3.1	Introduction	55
3.2	Partially Ordered Sets	55
3.3	Strict Orderings	58
3.4	Covering and Hasse Diagrams	60
3.5	Infima and Suprema	63
3.6	Lattices	67
3.7	The Lattice of Congruences	70
3.8	Lattices as Algebras	71
3.9	Ordered Algebras	74
3.10	Tonoids	77
3.11	Tonoid Varieties	82
3.12	Classical Complementation	85
3.13	Non-Classical Complementation	88
3.14	Classical Distribution	92
3.15	Non-Classical Distribution	98
3.16	Classical Implication	105
3.17	Non-Classical Implication	109
3.18	Filters and Ideals	115

4	Syntax	125
4.1	Introduction	125
4.2	The Algebra of Strings	125
4.3	The Algebra of Sentences	130
4.4	Languages as Abstract Structures: Categorical Grammar	133
4.5	Substitution Viewed Algebraically (Endomorphisms)	136
4.6	Effectivity	137
4.7	Enumerating Strings and Sentences	138
5	Semantics	141
5.1	Introduction	141
5.2	Categorical Semantics	142
5.3	Algebraic Semantics for Sentential Languages	144
5.4	Truth-Value Semantics	146
5.5	Possible Worlds Semantics	148
5.6	Logical Matrices and Logical Atlases	152
5.7	Interpretations and Valuations	155
5.8	Interpreted and Evaluationally Constrained Languages	158
5.9	Substitutions, Interpretations, and Valuations	162
5.10	Valuation Spaces	166
5.11	Valuations and Logic	169
5.12	Equivalence	172
5.13	Compactness	176
5.14	The Three-Fold Way	181
6	Logic	184
6.1	Motivational Background	184
6.2	The Varieties of Logical Experience	185
6.3	What Is (a) Logic?	187
6.4	Logics and Valuations	189
6.5	Binary Consequence in the Context of Pre-ordered Sets	191
6.6	Asymmetric Consequence and Valuations (Completeness)	194
6.7	Asymmetric Consequence in the Context of Pre-ordered Groupoids	196
6.8	Symmetric Consequence and Valuations (Completeness and Absoluteness)	199
6.9	Symmetric Consequence in the Context of Hemi-distributoids	202
6.10	Structural (Formal) Consequence	208
6.11	Lindenbaum Matrices and Compositional Semantics for Assertional Formal Logics	209
6.12	Lindenbaum Atlas and Compositional Semantics for Formal Asymmetric Consequence Logics	211
6.13	Scott Atlas and Compositional Semantics for Formal Symmetric Consequence Logics	213

6.14	Co-consequence as a Congruence	214
6.15	Formal Presentations of Logics (Axiomatizations)	216
6.16	Effectiveness and Logic	224
7	Matrices and Atlases	226
7.1	Matrices	226
7.1.1	Background	226
7.1.2	Łukasiewicz matrices/submatrices, isomorphisms	227
7.1.3	Gödel matrices/more submatrices	230
7.1.4	Sugihara matrices/homomorphisms	230
7.1.5	Direct products	232
7.1.6	Tautology preservation	232
7.1.7	Infinite matrices	233
7.1.8	Interpretation	234
7.2	Relations Among Matrices: Submatrices, Homomorphic Images, and Direct Products	237
7.3	Proto-preservation Theorems	239
7.4	Preservation Theorems	243
7.5	Varieties Theorem Analogs for Matrices	246
7.5.1	Unary assertional logics	246
7.5.2	Asymmetric consequence logics	247
7.5.3	Symmetric consequence logics	249
7.6	Congruences and Quotient Matrices	249
7.7	The Structure of Congruences	254
7.8	The Cancellation Property	257
7.9	Normal Matrices	262
7.10	Normal Atlases	266
7.11	Normal Characteristic Matrices for Consequence Logics	270
7.12	Matrices and Algebras	271
7.13	When is a Logic “Algebraizable”?	273
8	Representation Theorems	277
8.1	Partially Ordered Sets with Implication(s)	277
8.1.1	Partially ordered sets	277
8.1.2	Implication structures	278
8.2	Semi-lattices	287
8.3	Lattices	288
8.4	Finite Distributive Lattices	293
8.5	The Problem of a General Representation for Distributive Lattices	295
8.6	Stone’s Representation Theorem for Distributive Lattices	297
8.7	Boolean Algebras	300
8.8	Filters and Homomorphisms	302
8.9	Maximal Filters and Prime Filters	302

8.10	Stone's Representation Theorem for Boolean Algebras	303
8.11	Maximal Filters and Two-Valued Homomorphisms	305
8.12	Distributive Lattices with Operators	313
8.13	Lattices with Operators	317
9	Classical Propositional Logic	321
9.1	Preliminary Notions	321
9.2	The Equivalence of (Unital) Boolean Logic and Frege Logic	322
9.3	Symmetrical Entailment	324
9.4	Compactness Theorems for Classical Propositional Logic	326
9.5	A Third Logic	333
9.6	Axiomatic Calculi for Classical Propositional Logic	334
9.7	Primitive Vocabulary and Definitional Completeness	335
9.8	The Calculus BC	337
9.9	The Calculus $D(\mathbf{BC})$	341
9.10	Asymmetrical Sequent Calculus for Classical Propositional Logic	346
9.11	Fragments of Classical Propositional Logic	348
9.12	The Implicative Fragment of Classical Propositional Logic: Semi-Boolean Algebras	349
9.13	Axiomatizing the Implicative Fragment of Classical Propositional Logic	350
9.14	The Positive Fragment of Classical Propositional Logic	352
10	Modal Logic and Closure Algebras	356
10.1	Modal Logics	356
10.2	Boolean Algebras with a Normal Unitary Operator	358
10.3	Free Boolean Algebras with a Normal Unitary Operator and Modal Logic	361
10.4	The Kripke Semantics for Modal Logic	361
10.5	Completeness	363
10.6	Topological Representation of Closure Algebras	364
10.7	The Absolute Semantics for $\mathbf{S5}$	367
10.8	Henle Matrices	367
10.9	Alternation Property for $\mathbf{S4}$ and Compactness	369
10.10	Algebraic Decision Procedures for Modal Logic	370
10.11	$\mathbf{S5}$ and Pretabularity	375
11	Intuitionistic Logic and Heyting Algebras	380
11.1	Intuitionistic Logic	380
11.2	Implicative Lattices	381
11.3	Heyting Algebras	383
11.4	Representation of Heyting Algebras using Quasi-ordered Sets	383
11.5	Topological Representation of Heyting Algebras	384

11.6	Embedding Heyting Algebras into Closure Algebras	386
11.7	Translation of \mathbf{H} into $\mathbf{S4}$	386
11.8	Alternation Property for \mathbf{H}	387
11.9	Algebraic Decision Procedures for Intuitionistic Logic	388
11.10	\mathbf{LC} and Pretabularity	390
12	Gaggles: General Galois Logics	394
12.1	Introduction	394
12.2	Residuation and Galois Connections	395
12.3	Definitions of Distributoid and Tonoid	398
12.4	Representation of Distributoids	400
12.5	Partially Ordered Residuated Groupoids	406
12.6	Definition of a Gaggle	408
12.7	Representation of Gaggles	409
12.8	Modifications for Distributoids and Gaggles with Identities and Constants	412
12.9	Applications	414
12.10	Monadic Modal Operators	415
12.11	Dyadic Modal Operators	417
12.12	Identity Elements	420
12.13	Representation of Positive Binary Gaggles	421
12.14	Implication	422
	12.14.1 Implication in relevance logic	423
	12.14.2 Implication in intuitionistic logic	424
	12.14.3 Modal logic	424
12.15	Negation	425
	12.15.1 The gaggle treatment of negation	425
	12.15.2 Negation in intuitionistic logic	426
	12.15.3 Negation in relevance logic	427
	12.15.4 Negation in classical logic	429
12.16	Future Directions	430
13	Representations and Duality	431
13.1	Representations and Duality	431
13.2	Some Topology	433
13.3	Duality for Boolean Algebras	435
13.4	Duality for Distributive Lattices	438
13.5	Extensions of Stone's and Priestley's Results	441
	References	445
	Index	455