

Eigenvector Geometry as a Universal Amplifier of Heavy-Tailed Fluctuations in Random Multiplicative Systems

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Heavy-tailed fluctuations and power-law distributions pervade physics, biology, and the social sciences, with numerous mechanisms proposed for their emergence. Kesten processes, which are multiplicative stochastic recursions with additive noise or reinjection, provide a canonical explanation, where power-law tails arise from transient supercritical excursions as eigenvalues intermittently cross the stability boundary. Here we uncover a distinct and more general mechanism in multidimensional systems: *non-normal eigenvector amplification*. In random non-normal matrices, the non-orthogonality of eigenvectors, quantified by the condition number κ , induces transient growth that increases the effective Lyapunov exponent $\gamma \simeq \gamma_0 + \langle \ln \kappa \rangle$ and lowers the tail exponent $\alpha \simeq -2\gamma/\sigma_\kappa^2$, where σ_κ^2 is the variance of $\ln \kappa$. As the system dimension N grows, κ typically increases proportionally, making non-normal amplification the dominant source of scale-free behavior. We illustrate this mechanism in two representative systems: (i) polymer stretching in turbulent flows, where intermittent extensions arise from eigenvector amplification of velocity gradients and (ii) financial return distributions, where extending one-dimensional GARCH/Kesten processes to a multidimensional setting yields a collective origin for heavy-tailed market fluctuations and explains their near-universal exponents across assets.

Heavy-tailed distributions are a hallmark of complex systems, arising in physics, finance, economics, biology, and many other domains. A canonical mechanism for their emergence is provided by the class of stochastic recursions studied by Kesten [1], where the interplay between multiplicative growth and additive noise naturally produces stationary distributions with power-law tails. Such *Kesten processes* have since become a cornerstone in the theoretical understanding of scale-free phenomena, with broad applications ranging from physics [2–4], biology [5], to financial markets [6–9], and to models of wealth and income distribution [10, 11].

The classical understanding attributes the emergence of power laws in Kesten-type processes to *spectral supercriticality*, i.e., episodes in which eigenvalues of the random multiplicative operator transiently cross the unit circle. In this view, power law tails reflect rare bursts of exponential growth balanced by global stability on average.

In this paper, we uncover a complementary and more general mechanism in multidimensional processes: *non-normal eigenvector amplification*. When random matrices are non-normal (not unitarily diagonalizable), transient growth can occur even when all eigenvalues lie strictly within the unit circle. This effect, arising from the non-orthogonality of eigenvectors, provides a distinct and generic route to heavy-tailed stationary distributions. Our analysis shows that non-normality not only lowers the tail exponent, thereby producing fatter tails, but can also shift the effective Lyapunov exponent, destabilizing the system in a way fundamentally different from spectral criticality.

The power law mechanism rooted in the generic geom-

etry of eigenvectors in high-dimensional random systems provides a unified explanation for heavy-tailed distributions in systems governed by multiplicative interactions, from turbulent flows to wealth and income dynamics. By identifying non-normal amplification as a fundamental source of heavy tails, our work extends the theoretical scope of Kesten processes and clarifies their relevance to the statistics of complex systems.

Our work builds on a broad literature that first recognized the dynamical significance of non-normal operators. Pioneering studies in hydrodynamic stability and atmospheric dynamics by Farrell and Ioannou [12–17] and in fluid mechanics by Trefethen and collaborators [18, 19] showed that non-normality can induce large transient amplifications even when all eigenvalues indicate linear stability. Related ideas were later extended to other fields, such as subcritical magnetic dynamos [20]. Here we generalize this perspective beyond hydrodynamics and linear response, showing that non-normal eigenvector amplification constitutes a universal route to heavy-tailed fluctuations in high-dimensional stochastic systems.

We consider the N -dimensional Kesten process

$$\mathbf{x}_{t+1} = \mathbf{A}_t \mathbf{x}_t + \boldsymbol{\eta}_t, \quad (1)$$

where (\mathbf{A}_t) is an i.i.d. sequence of random matrices and $\boldsymbol{\eta}_t$ is an additive noise term. Their long-term behavior is controlled by

$$\pi_t = \|\boldsymbol{\Pi}_t\| \quad \text{with} \quad \boldsymbol{\Pi}_t = \prod_{s=1}^t \mathbf{A}_s, \quad (2)$$

where $\boldsymbol{\Pi}_t$ denotes the product of matrices up to time t and π_t is its L_2 -norm. The stability of the process is

characterized by the *Lyapunov exponent* γ , defined as

$$\gamma := \lim_{t \rightarrow \infty} \frac{1}{t} \mathbb{E}[\ln \pi_t]. \quad (3)$$

A negative exponent ($\gamma < 0$) implies asymptotic stability, with convergence to a stationary distribution, whereas a positive exponent ($\gamma > 0$) indicates instability, with the system diverging and growing exponentially in time.

The *tail exponent* α quantifies the heaviness of the tail of the stationary distribution, namely

$$\mathbb{P}[\mathbf{n} \cdot \mathbf{x}_t > x_n] \sim x_n^{-\alpha}, \quad x_n \rightarrow \infty, \quad (4)$$

where $\mathbf{n} \cdot \mathbf{x}_t$ denotes any projection (the same asymptotics hold for the L_2 -norm). Consider the function

$$\phi(\alpha) := \lim_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{E}[\pi_t^\alpha], \quad (5)$$

which is convex, satisfies $\phi(0) = 0$, and has derivative $\phi'(0) = \gamma$. For $\gamma < 0$, the convexity of $\phi(\alpha)$ ensures that there exists a unique $\alpha > 0$ solving $\phi(\alpha) = 0$, which determines the tail exponent of the stationary distribution [1, 21]. For $\gamma \geq 0$, no positive solution exists and the process fails to admit a stationary heavy-tailed regime.

Normal Kesten processes: spectral criticality. When the random matrices \mathbf{A}_t are *normal* (i.e. unitarily diagonalizable), they can be written as $\mathbf{A}_t = \mathbf{U}_t \mathbf{\Lambda}_t \mathbf{U}_t^\dagger$ with \mathbf{U}_t unitary and $\mathbf{\Lambda}_t$ diagonal. Then the product norm has an upper bound given by $\pi_t \leq \prod_{s=1}^t \|\mathbf{A}_s\| = \prod_{s=1}^t \rho_s$, since, for normal matrices, the L_2 -norm $\|\mathbf{A}_t\|$ of \mathbf{A}_t is given by its spectral radius ρ_t . Using the monotonicity of the logarithm and the convexity of $\phi(\alpha)$ (5), we obtain the following bounds for the Lyapunov and tail exponents

$$\gamma \leq \ln \rho, \quad \alpha \geq -\frac{2 \ln \rho}{\sigma_\rho^2} \quad \text{for } \ln \rho < 0, \quad (6)$$

where $\ln \rho$ and σ_ρ^2 are respectively the expected value and variance of the logarithm of the spectral radius.

The upper bound for γ and the lower bound for α are reached for the one-dimension case for which the Kesten process (1) reduces to $x_{t+1} = \rho_t x_t + \eta_t$. When the multiplicative noise is lognormally distributed, $\ln \rho_t \sim \mathcal{N}(\ln \rho, \sigma_\rho^2)$, expression (3) gives $\gamma = \ln \rho$ and the solution of the equation $\phi(\alpha) = 0$ is exactly

$$\alpha = -\frac{2 \ln \rho}{\sigma_\rho^2}, \quad \text{for } \ln \rho < 0. \quad (7)$$

Hence, the one-dimensional Kesten process represents the *worst-case scenario* of the multidimensional Kesten process with normal matrices since the bounds (6) are exactly attained. Intuitively, the power law tail originates from the fact that sequences of successive $\rho_t > 1$, which generate transient exponential growth, have an exponentially small probability to occur as a function of their

durations, so that the stationary distribution reflects a balance between rare but strong amplification bursts and global stability enforced by $\ln \rho < 0$ [22]. In other words, the heavy tails arise from eigenvalues stochastically crossing the unit circle, placing the system for brief periods in a spectrally supercritical state. Additional details and general proofs are presented in the companion paper [23].

Non-normal Kesten processes: eigenvector amplification. When the random matrices \mathbf{A}_t are *non-normal*, i.e. not unitarily diagonalizable, an additional amplification mechanism appears. Writing $\mathbf{A}_t = \mathbf{P}_t \mathbf{\Lambda}_t \mathbf{P}_t^{-1}$, where $\mathbf{\Lambda}_t$ is a diagonal matrix, the matrix L_2 -norm of \mathbf{A}_t is no more given by its spectral radius ρ_t , but depends on the condition number $\kappa_t = \|\mathbf{P}_t\| \|\mathbf{P}_t^{-1}\|$ of the eigenbasis transformation, which quantifies the non-orthogonality of eigenvectors and the degree of non-normality ($\kappa > 1$ for non-normal matrices and $\kappa = 1$ for normal matrices).

By the classical bound for diagonalizable matrices (see, e.g., [24]), with equality (and $\kappa = 1$) when the eigenbasis is orthogonal, one has $\|\mathbf{A}_t\| \leq \kappa_t \rho_t$. The L_2 -norm π_t (2) of the product $\mathbf{\Pi}_t$ of matrices up to time t is thus bounded as $\pi_t \leq \left(\prod_{s=1}^t \rho_s\right) \left(\prod_{s=1}^t \kappa_s\right)$.

Results from random matrix theory show that, for broad classes of ensembles, eigenvectors and eigenvalues become asymptotically independent (see, e.g., [25, 26]), while perturbation arguments indicate that correlations between spectra and eigenbases are typically weak in high dimensions. In the Ginibre ensemble [27] (matrices with i.i.d. centered Gaussian entries, almost surely non-normal), this asymptotic independence is rigorously established provided the system remains away from spectral criticality (i.e., the spectral radius stays strictly inside the unit circle) and from eigenvalue degeneracies (i.e., eigenvalues remain well separated) [28]. These two conditions ensure spectral stability, making the assumption of eigenvalue-eigenvector independence both natural and robust in the generic Ginibre case. On this basis, we adopt the independence hypothesis, which allows us not only to bound but also to characterize the leading behavior of the Lyapunov and tail exponents (see companion paper [23]):

$$\gamma \leq \ln \rho + \mathbb{E}[\ln \kappa], \quad \alpha \geq -2 \frac{\ln \rho + \ln \kappa}{\sigma_\rho^2 + \sigma_\kappa^2}, \quad (8)$$

where $\ln \kappa := \mathbb{E}[\ln \kappa_t]$ and $\sigma_\kappa^2 := \text{Var}[\ln \kappa_t]$. Thus, non-normality can (i) *increase* the effective Lyapunov exponent, acting as a destabilizing force, and (ii) *decrease* the tail exponent, thereby producing heavier power law tails.

Crucially, this mechanism operates *even when the spectrum is strictly stable*, providing a new and generic route to apparent criticality $\gamma \rightarrow 0$ and to power law distributions. To see this, consider the case $\ln \rho < 0$ with $\sigma_\rho = 0$ for normal matrices ($\kappa = 1$): in this situation, the bound (6) and the exact expression (7) for α diverge, indicating the absence of power law tails. This

is expected, since the mechanism of intermittent supercriticality is absent. When non-normality is present, stochastic fluctuations of the condition number, quantified by the mean $\mathbb{E}[\ln \kappa]$ and variance σ_κ , increases γ by $\mathbb{E}[\ln \kappa]$ and give rise to power law tails with effective exponent $\alpha \simeq -2 \frac{\ln \rho + \mathbb{E}[\ln \kappa]}{\sigma_\kappa^2}$. The key intuition is that non-orthogonal eigenvectors allow transient amplifications: a vector multiplied by \mathbf{A}_t can be stretched due to constructive interference between nearly aligned eigen-directions [29]. In the product of random matrices, successive multiplications then act as random rotations, which can repeatedly project the state back into the most expanding direction. This recursive reinjection mechanism increases the effective Lyapunov exponent while simultaneously reducing the tail exponent, thereby amplifying large fluctuations, as illustrated in Figure 1(a). This simultaneous shift of the Lyapunov and tail exponents shows that non-normality can reshape both aspects of the dynamics at once, even when the spectrum itself remains strictly subcritical.

In high-dimensional random ensembles such as the Ginibre ensemble [27], the expected log-condition number scales as $\mathbb{E}[\ln \kappa] \sim \ln N$ [30], so that non-normal amplification inevitably grows with system size. This makes non-normality the dominant route to instability and the emergence of power law statistics in large systems, providing a generic mechanism for scale-free behavior.

Two-dimensional illustration. To build intuition, we illustrate the general results on a simple 2×2 example with purely off-diagonal structure,

$$\mathbf{A}_t = \rho_t \begin{pmatrix} 0 & z_t \\ z_t^{-1} & 0 \end{pmatrix}, \quad (9)$$

where ρ_t sets the spectral radius of \mathbf{A}_t and z_t encodes its degree of non-normality. The condition number is in fact equal to $\kappa_t = \max\{z_t, 1/z_t\}$. In this case, the Lyapunov exponent simplifies to $\gamma = \ln \rho_t$, and is independent of the non-normality parameter z_t .

The form (9) yields a convenient simplification for two-step products:

$$\mathbf{A}_t^{(2)} = \mathbf{A}_{2t} \mathbf{A}_{2t-1} = \rho_{2t} \rho_{2t-1} \begin{pmatrix} z_t^{(2)} & 0 \\ 0 & (z_t^{(2)})^{-1} \end{pmatrix}, \quad (10)$$

where $z_t^{(2)} = z_{2t}/z_{2t-1}$. Considering the case where $\text{Var}[\ln z_t^{(2)}] < \infty$, by the Central Limit Theorem, the variable $Z_t = \prod_{s=1}^t z_s^{(2)}$ converges to a log-normal distribution: $\ln Z_t \sim \mathcal{N}(0, 2t\sigma_z^2)$ and $\sigma_z^2 := \text{Var}[\ln z_1]$. Similarly, $\sum_{s=1}^{2t} \ln |\rho_s| \sim \mathcal{N}(2t \ln \rho, 2t\sigma_\rho^2)$ and $\sigma_\rho^2 := \text{Var}[\ln |\rho_1|]$. Hence, the logarithm of the product norm is approximately Gaussian for large t : $\ln \pi_{2t} \sim \mathcal{N}(2t \ln \rho, 2t(\sigma_\rho^2 + \sigma_z^2))$. Using definition (5) and the equation $\phi(\alpha) = 0$, we obtain the tail exponent $\alpha = -\frac{2 \ln \rho}{\sigma_\rho^2 + \sigma_z^2}$. Even well within the stable regime ($\ln \rho < 0$), non-normality reduces the

tail exponent, producing heavier tails while leaving the global stability criterion unchanged. This demonstrates that eigenvector geometry alone can generate and amplify fat-tailed stationary distributions. In particular, even in the absence of transient supercritical excursions ($\sigma_\rho = 0$), the non-normal amplification mechanism alone yields power law tails with $\alpha = -\frac{2 \ln \rho}{\sigma_z^2}$.

This case provides an intuition for how stochastic non-normality can destabilize an otherwise spectrally stable system. Suppose the non-normal parameter z_t remains constant over a time interval δt . If the eigenvalues λ have their norm smaller than 1, the system undergoes only a transient deviation before decaying exponentially. However, if after a duration δt the parameter switches as $z_t \rightarrow 1/z_t$, the dominant mode, i.e. the axis along which the transient deviation occurs, also switches. If the system has not had time to fully relax during δt , the new transient amplifies the residual from the previous one. Repeated switching between non-normal modes thus produces successive reinjections of growth, causing the global trajectory to inflate and eventually diverge. This mechanism is illustrated in Fig. 1(a) for the case of deterministic periodic switching $z_t \rightarrow 1/z_t$.

A more general setting than (9) incorporates random rotations, in which case the matrix takes the form

$$\mathbf{A}_t = \rho_t \mathbf{U}(\theta_t) \begin{pmatrix} 0 & z_t \\ z_t^{-1} & 0 \end{pmatrix} \mathbf{U}(\theta_t)^\dagger, \quad (11)$$

where $\mathbf{U}(\theta_t)$ is a random planar rotation. The impact of the rotations is quantified through the statistical behavior of the angular terms $\sum_{s=1}^t \ln |\sin(\theta_{2s} - \theta_{2s-1})|$ which, by the central limit theorem, converges to a random variable distributed according to $\mathcal{N}(-2t\mu_\theta, t\sigma_\theta^2)$ in the large t limit. This defines the average logarithmic penalty μ_θ from angular misalignment and the standard deviation σ_θ of this penalty. In other words, μ_θ quantifies the systematic reduction of growth due to imperfect reinjection into the most expanding direction after each random rotation. This leads to an additional contribution to the Lyapunov exponent as $\gamma \simeq \ln \rho + \mathbb{E}[\ln \kappa] - \mu_\theta$ [23], so that angular randomness reduces γ compared to the purely spectral and non-normal contributions. In this way, μ_θ encapsulates the average damping effect of the angular geometry on the Lyapunov exponents. The presence of rotations also modifies the power law exponent into $\alpha \approx -\frac{2\gamma}{\sigma_\rho^2 + \sigma_\kappa^2 + \sigma_\theta^2}$, reflecting the competing effects of angular randomness: on the one hand, μ_θ makes γ more negative, thereby increasing α , while on the other hand σ_θ^2 enlarges the denominator, which reduces α .

Figure 1(b) illustrates these predictions by measuring Lyapunov and tail exponents of the dynamics (1) with matrices (11). We fix $\ln \rho_t = -1$ (ensuring no spectral supercriticality), and sample $\ln z_t \sim \mathcal{N}(0, \sigma_z^2)$. We compare two cases: (i) no rotation, i.e. $\mathbf{U}(\theta_t) = \mathbf{I}$, and (ii) uniform random rotations with $\theta_t \sim \mathcal{U}[-\pi, \pi]$. Without

rotations, the system remains stable at all times ($\gamma < 0$) while the tail exponent α decreases as σ_z increases, in quantitative agreement with our theoretical analysis. By contrast, when rotations are included, the dynamics are stochastically reinjected along non-normal directions. In this case, not only does α decrease as predicted, but the Lyapunov exponent increases approximately linearly with $\mathbb{E}[\ln \kappa] \sim \sigma_z$, again in full agreement with our theoretical analysis. The general 2-dimensional case for real matrices is fully developed in the companion paper [23] but the results stays qualitatively the same.

General N -dimensional case. We revisit the general N -dimensional case with a more detailed analysis (see [23] for the full derivations). We begin with the decomposition

$$\mathbf{A}_t = \mathbf{P}_t \mathbf{\Lambda}_t \mathbf{P}_t^\dagger, \quad (12)$$

where $\mathbf{\Lambda}_t = \text{Diag}(\lambda_{i,t} | i = 1, \dots, N)$ comprises the eigenvalues, $\mathbf{P}_t = \mathbf{U}_t \mathbf{\Sigma}_t \mathbf{V}_t^\dagger$ is the eigenbasis transformation matrix, $\mathbf{\Sigma}_t = \text{Diag}(s_{1,t}, \dots, s_{N,t})$ contains the singular values of \mathbf{P}_t , and \mathbf{U}_t and \mathbf{V}_t are unitary matrices. We assume that the eigenvalues $\{\lambda_{i,t}\}$, singular values $\{s_{i,t}\}$, and the rotations $\mathbf{U}_t, \mathbf{V}_t$ are independent, and that all degrees of freedom are i.i.d. in time. The condition number quantifying the degree of non-normality is given by $\kappa_t = s_{\max,t}/s_{\min,t}$, which is ratio of the largest to smallest singular values. The norm of A_t obeys the inequality $\|\mathbf{A}_t\| \leq \|\mathbf{\Lambda}_t\| \kappa_t$, which yields the decomposition for $\ln \pi_t$ as $\ln \pi_t \approx \sum_{s=1}^t \ln |\lambda_s| + \sum_{s=1}^t \ln \kappa_s + \sum_{s=1}^{t-1} \ln |u_{\min,s} \cdot u_{\max,s+1}|$, where u_{\max} and u_{\min} are the singular vectors associated to the largest/smallest singular values of P_t . Taking expectations,

$$\gamma = \mathbb{E}[\ln |\lambda|] + \mathbb{E}[\ln \kappa] + \mathbb{E}[\ln |u_{\min} \cdot u_{\max}|], \quad (13)$$

and fluctuations combine additively in the denominator of the expression for

$$\alpha \approx -\frac{2\gamma}{\sigma_\lambda^2 + \sigma_\kappa^2 + \sigma_\theta^2}, \quad (14)$$

with σ_λ^2 and σ_κ^2 are the variances of $\ln |\lambda|$ and $\ln \kappa$, respectively, and $\sigma_\theta^2 = \text{Var}(\ln |u_{\min} \cdot u_{\max}|)$. Let us examine each of the three term in the right-hand-side of (13).

(i) *Eigenvalue mixing.* Writing $\lambda_{i,t} = e^{\ln \rho} \lambda_{i,t}^0$ with $\mathbb{E}[\ln |\lambda^0|] = 0$ and treating the mixed entry as a sum of $O(1/\sqrt{N})$ coefficients, the Central Limit Theorem gives (SM)

$$\mathbb{E}[\ln |\lambda|] = \ln \rho + \ln \sigma_0 - \frac{1}{2} \ln N - \frac{g - \ln 2}{2}, \quad \text{Var}[\ln |\lambda|] = \frac{\pi^2}{8}, \quad (15)$$

with g Euler's constant.

(ii) *Condition number by Extreme Value Theory (EVT).* Given that $\kappa_t = s_{\max,t}/s_{\min,t}$ is the ratio of the

largest to the smaller singular values of \mathbf{P}_t , assuming i.i.d. $\ln s_{i,t} \sim \mathcal{N}(0, \sigma^2)$, EVT yields

$$\mathbb{E}[\ln \kappa] \simeq 2\sigma\sqrt{2\ln N}, \quad \text{Var}[\ln \kappa] \simeq \frac{\pi^2 \sigma^2}{6\ln N}. \quad (16)$$

(iii) *Orientation reinjection.* For random unit vectors in \mathbb{R}^N ,

$$\mathbb{E}[\ln |u_{\min} u_{\max}|] = -\frac{1}{2} \ln N - \frac{g - \ln 2}{2}, \quad \text{Var}(\ln |u_{\min} u_{\max}|) = \frac{\pi^2}{8}. \quad (17)$$

Combining (15)-(17) in (13) gives the large- N Lyapunov estimate

$$\gamma \approx \ln \rho + \ln \sigma_0 - \ln N - \ln 2 - g + 2\sigma\sqrt{2\ln N}, \quad (18)$$

and the variance entering (14) is $\sigma_\lambda^2 + \sigma_\kappa^2 + \sigma_\theta^2 \simeq \frac{\pi^2}{8} + \frac{\pi^2 \sigma^2}{6\ln N} + \frac{\pi^2}{8} = \frac{\pi^2}{4} + \frac{\pi^2 \sigma^2}{6\ln N}$.

Near-critical tail exponent. Let us define the critical non-normal dispersion of the singular values $\sigma_c(N)$ as the value that solves $\gamma(\sigma) = 0$. Linearizing (18) around $\sigma = \sigma_c$ and inserting in (14) yields the power law tail exponent

$$\alpha \approx \frac{\sqrt{2\ln N}}{48} \frac{\pi^2}{3 + \frac{2\sigma^2}{\ln N}} (\sigma_c(N) - \sigma) + o\left(\frac{1}{\sqrt{\ln N}}\right). \quad (19)$$

The tail exponent decreases *linearly* with the distance to criticality, with slope scaling as $\sqrt{2\ln N}$ and with a denominator collecting the variance contributions from spectrum, condition number, and reinjection. In high dimensions, the interplay of these three sources of randomness creates a fine balance between stability and criticality. In particular, the critical non-normal variance $\sigma_c(N)$ scales as $\sqrt{\ln N}$, implying that larger systems can sustain greater heterogeneity before destabilizing when $\gamma \approx 0$. Moreover, the tail exponent α decreases as $\sigma \rightarrow \sigma_c(N)$, showing that high-dimensional non-normality produces heavy-tailed fluctuations even under spectral stability. Thus, dimensionality and non-normal amplification together drive the system closer to criticality, in the form of diverging moments and vanishing tail exponents.

To illustrate our theoretical predictions, we conducted numerical experiments for systems of dimension $N = 6, 10$, and 20 . The setup mirrors the generic scenario outlined above while remaining analytically tractable. At each step t , the random update matrix is drawn as (12) where $\mathbf{\Lambda}$ is diagonal with eigenvalues λ_i with $\ln \lambda_i \sim \mathcal{N}(-1, 0.1)$ to avoid the matrix $\mathbf{\Lambda}$ to be a multiple of the identity, and to have $\ln \rho \approx -1$ ensuring spectral stability. \mathbf{P}_t is sampled from its singular value decomposition (SVD), $\mathbf{P}_t = \mathbf{U}_t \mathbf{\Sigma}_t \mathbf{V}_t$, where \mathbf{U}_t and \mathbf{V}_t are independent Haar-distributed unitary matrices, and $\mathbf{\Sigma}_t$ is diagonal with i.i.d. singular values $s_{i,t}$ satisfying $\ln s_{i,t} \sim \mathcal{N}(0, \sigma)$. The role of σ in shaping the dynamics will be examined. The matrix $\mathbf{V}^\dagger \mathbf{A} \mathbf{V}$ is drawn at

the beginning of the simulation, and define the “normal” case of the system dynamic, and the $\mathbf{U}_t \mathbf{\Sigma}_t$ parts account for the non-normal stretching and rotation, which do not affect the matrix spectrum at all time.

The choice of sampling the matrices \mathbf{P}_t in this way, rather than from the Ginibre ensemble, is motivated by the fact that it allows us to control explicitly the statistics of the rotations \mathbf{U}_t , \mathbf{V} , as well as the condition number κ_t , while avoiding the need to invert \mathbf{P}_t numerically, which is an operation that would be unstable given the likelihood of ill-conditioning. One consequence of this construction is that the expected value of the log condition number grows only as $\ln \kappa \sim \sqrt{\ln N}$ with matrix dimension, implying a slower scaling of κ with N than in the Ginibre ensemble. In the latter case, the typical condition number grows linearly with dimension, $\kappa \sim N$, leading to $\mathbb{E}[\ln \kappa] \sim \ln N$. As a result, the non-normal amplifications observed in our construction should be viewed as conservative estimates, since the generic Ginibre case would exhibit even stronger effects. A further advantage is that this ensemble introduces two natural control parameters: (i) the *volatility of the log singular values* (σ); (ii) the *dimension of the system* (N). These two parameters can be combined into $\ln \kappa$, which, by Extreme Value Theory (EVT), scales as $\ln \kappa \sim \sigma \sqrt{\ln N}$. This setup ensures spectral stability, while fluctuations in κ_t probe non-normal amplification.

Figures 1(c) illustrate how the Lyapunov exponent depends on the singular value dispersion σ and the system size N . The main findings are: (i) for normal systems ($\sigma = 0$), γ is independent of N , as expected; (ii) introducing non-normality ($\sigma > 0$) leads to an approximately linear growth of γ with σ , with a larger slope when random rotations are present; (iii) the dominant finite- N effect is well captured by $\gamma \approx \gamma_0 + \mathbb{E}[\ln \kappa]$, where $\mathbb{E}[\ln \kappa] \sim \sqrt{\ln N}$, consistent with extreme-value arguments. At small σ , the growth of γ is initially quadratic, before crossing over to the linear regime (see SM for further details). Overall, the simulations confirm that non-normality raises the Lyapunov exponent, and the scaling of the log-condition number follows the theoretical $\sqrt{\ln N}$ behavior.

Figures 1(c) also report the behavior of the power-law tail exponent α . The main finding is the confirmation of the prediction (19) that α decreases approximately linearly with $\sigma_c(N) - \sigma$ when σ is close to but smaller than the critical value $\sigma_c(N)$, where the Lyapunov exponent γ crosses zero. In particular, the top panel shows that $\gamma \approx 0$ when $\sigma \sqrt{\ln N} \approx 2.7$, which is consistent with the linear extrapolation of α for $\alpha < 5$ observed in the bottom panel of Fig. 1(c). Moreover, α exhibits an approximate scaling with $\sigma \sqrt{\ln N}$, vanishing as $\sigma \rightarrow \sigma_c(N)$, in agreement with the simultaneous vanishing of γ . The bottom panel of Fig. 1(c) further reveals a crossover to an approximately constant α for $\sigma \sqrt{\ln N} < 1.5$, corresponding to the disappearance of non-normal eigenvector amplification. A particularly noteworthy outcome is

that the theoretical predictions, which are derived under asymptotic assumptions such as the law of large numbers, the central limit theorem, and large- N arguments, are already well supported by simulations with moderate system sizes $N = 6$ to 20. This robustness illustrates how asymptotic results can hold quantitatively even for relatively small N .

We demonstrate the relevance of the non-normal amplification mechanism in two applications.

Polymer stretching in turbulent flows as an instance of eigenvector amplification. The coil-stretch transition of single molecules in turbulent or random flows is described by repeated action of random velocity-gradient matrices, with entropic elasticity providing the reinjection mechanism that prevents collapse and giving rise to a power law distribution of the end-to-end polymer lengths [31, 32]. This behavior can be formalized by starting from the stochastic evolution equation for the end-to-end extension vector \mathbf{R} of the polymer molecule, $\dot{\mathbf{R}} = (\nabla \mathbf{v}) \mathbf{R} - \frac{1}{\tau} \mathbf{R}$, where $\nabla \mathbf{v}$ is the local velocity-gradient tensor and τ the polymer relaxation time. Introducing the normalized orientation $\mathbf{n} = \mathbf{R}/|\mathbf{R}|$, the instantaneous stretching rate is reduced to the scalar process $z(t) = n_a n_b \partial_a v_b = \mathbf{n}^\top (\nabla \mathbf{v}) \mathbf{n}$ and the log-extension obeys $r(t) = \ln \left(\frac{R(t)}{R_0} \right) = \int_0^t (z(t') - 1/\tau) dt'$. Using large-deviation theory for the time-integrated process $z(t)$, Refs.[31, 32] show that r acquires an exponential tail, which translates into a power-law tail for R : $P(R) \sim R^{2\alpha-1}$.

Viewed through our framework, the origin of these heavy tails becomes transparent. The key variable $z(t)$ depends not only on the eigenvalues of $\nabla \mathbf{v}$ but also on its projection onto the instantaneous orientation \mathbf{n} . Because velocity-gradient tensors in turbulence are inherently non-normal, their eigenvectors are non-orthogonal; transient alignments between \mathbf{n} and nearly co-linear eigendirections of $\nabla \mathbf{v}$ thus generate amplification events far larger than eigenvalues alone predict. These rare constructive alignments constitute the large-deviation events governing the tail of $P(R)$, providing a concrete physical basis for the observed power law distribution of polymer extensions.

Collective Origin of Heavy-Tailed Market Fluctuations. Since the 1980s, the standard framework to model volatility clustering and heavy tails in financial return distributions has been the family of GARCH processes. These models, and their stochastic volatility counterparts, are essentially one-dimensional Kesten recursions applied to single asset’s squared volatility: the conditional variance evolves multiplicatively with random coefficients plus an additive noise term, ensuring reinjection. As a consequence, GARCH dynamics predict stationary return distributions with power law tails. However, the exponents obtained when this one-dimensional Kesten model is calibrated to time series of returns are typically too large

($\alpha > 4$), yielding distributions that are much thinner than those observed empirically. Financial data instead show that the tail exponent α is typically between 2 and 4 across stocks, indices, currencies, and commodities [8, 33–36]. This long-standing discrepancy finds a natural resolution when one extends the analysis from scalar to multidimensional Kesten dynamics. In realistic markets, returns are not driven by isolated variance processes but by the interactions of many correlated stocks, indices, and portfolios, which together form a high-dimensional multiplicative system. As the system dimension N grows, non-normal eigenvector amplification lowers the effective tail exponent via (i) an increase of the Lyapunov exponent γ by $\mathbb{E}[\ln \kappa] \propto \ln N$, and (ii) the variance of $\ln \kappa$, which decreases α as $1/\sqrt{\ln N}$. Together, these effects provide a collective mechanism across financial assets for the emergence of heavier tails. Even when all eigenvalues remain subcritical, increasing N enhances the condition number of the eigenbasis, amplifying transient growth and thereby reducing α from the unrealistically large values predicted by scalar GARCH models to smaller values compatible with the empirically observed values. Crucially, because the amplification strength depends primarily on the non-normal interactions between the N assets encoded in the N -dimensional matrix and not on asset-specific parameters, the same effective exponent naturally arises across diverse markets and instruments as seen from expression (4).

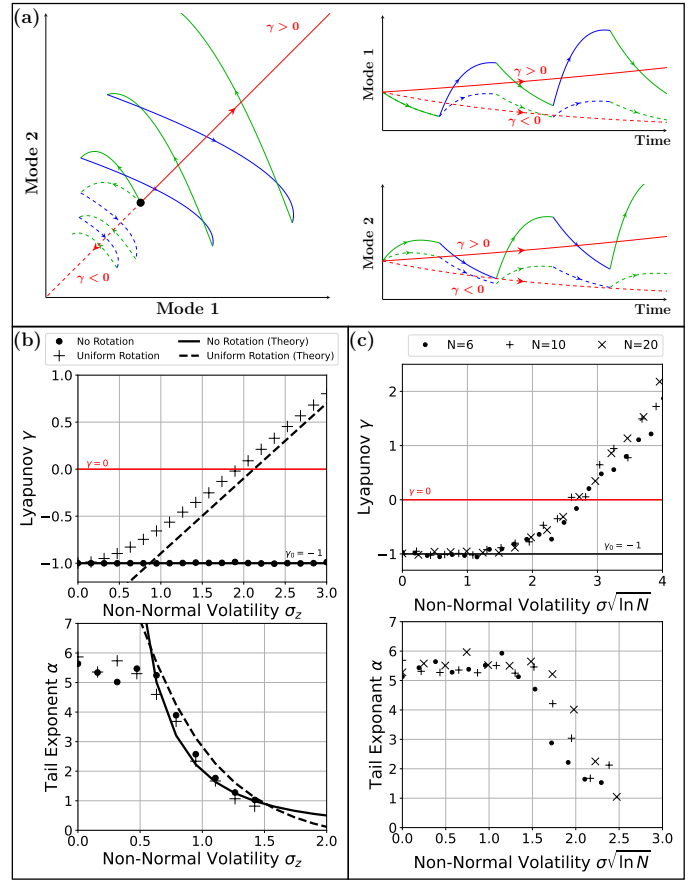


FIG. 1. Effect of stochastic non-normality on system critical behavior.

(a) Schematic illustration of non-normal instability in a two-dimensional system of the form (9), with constant spectral radius ρ . When the non-normal parameter switches periodically as $z_t \rightarrow 1/z_t$, the dominant mode alternates between the two axes, leading to cumulative reinjections and instability.

(b) Empirical Lyapunov and tail exponents measured for a two-dimensional Kesten process (11), with constant spectral radius $\ln \rho = -1$ and non-normal asymmetry $\ln z_t \sim \mathcal{N}(0, \sigma_z^2)$, so that $\mathbb{E}[\ln \kappa] = \sqrt{2/\pi} \sigma_z$. Symbols (\cdot) correspond to the case without rotations ($\mathbf{U}(\theta_t) = \mathbf{I}$), and (+) to uniform random rotations in the plane. Black solid and dashed lines show the theoretical predictions without and with rotations, respectively. The red horizontal line marks the critical threshold $\gamma = 0$.

(c) Same as (b) but generalized to N dimensions (12) for $N = 6, 10, 20$. The x -axis corresponds to the expected log-condition number $\mathbb{E}[\ln \kappa] \sim \sigma\sqrt{\ln N}$ in the extreme-value limit, with i.i.d. log-normal singular values $\ln s_{i,t} \sim \mathcal{N}(0, \sigma^2)$. The numerical procedures used to estimate Lyapunov and tail exponents are detailed in the SM.

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