$$\langle \hat{G}^z(t) \rangle = \langle \psi | U^\dagger \hat{G}^z U | \psi \rangle \quad \text{with } U = e^{-iHt}$$

$$= \langle \psi | e^{iHt} \hat{G}^z e^{-iHt} | \psi \rangle$$

eigenbasis of G^z : $\left\{ \bigotimes_{i=1}^{N} \{ |\uparrow\rangle, |\downarrow\rangle \} \right\}$

$$H = J \sum_{i=1}^{N} \vec{I}_i \cdot \vec{I}_{i+1} + \sum_{i=1}^{N} B_i I_i^z$$

$$= J \sum_{i=1}^{N} \left(I_i^x I_{i+1}^x + I_i^y I_{i+1}^y + I_i^z I_{i+1}^z \right) + \sum_{i=1}^{N} B_i I_i^z$$

For $N=3$: $\{ |\uparrow\uparrow\uparrow\rangle, |\uparrow\uparrow\downarrow\rangle, |\uparrow\downarrow\uparrow\rangle, |\uparrow\downarrow\downarrow\rangle, |\downarrow\uparrow\uparrow\rangle, |\downarrow\uparrow\downarrow\rangle, |\downarrow\downarrow\uparrow\rangle, |\downarrow\downarrow\downarrow\rangle \}$ ‖ In eigenbasis $|\varphi_i\rangle$ of H: $H |\varphi_i\rangle = E_i |\varphi_i\rangle$

1. Simplifications for starters

   I. Only coupling in z-direction

   II. $B_i = 0 \quad \forall i$

   $\Rightarrow H^0 = J \sum_{i=1}^{N} I_i^z I_{i+1}^z$

$$\Rightarrow H^0_{11} = \langle \uparrow\uparrow\uparrow | J \sum_{i=1}^{3} I_i^z I_{i+1}^z | \uparrow\uparrow\uparrow \rangle = J \cdot \left(\tfrac{1}{4} + \tfrac{1}{4} \right) = J/2$$

$$H^0_{22} = \langle \uparrow\uparrow\downarrow | H^0 | \uparrow\uparrow\downarrow \rangle = J \cdot \left(\tfrac{1}{4} - \tfrac{1}{4} \right) = 0$$

$$H^0_{33} = \langle \uparrow\downarrow\uparrow | H^0 | \uparrow\downarrow\uparrow \rangle = J \cdot \left(-\tfrac{1}{4} - \tfrac{1}{4} \right) = -J/2$$

$\vdots$

$$H^0_{ii} = \langle s_1 s_2 s_3 | H^0 | s_1 s_2 s_3 \rangle = J \cdot \left(s_1 \cdot s_2 + s_2 \cdot s_3 \right)$$

$$= \frac{J}{4} \left(\mathrm{sgn}(s_1 \cdot s_2) + \mathrm{sgn}(s_2 \cdot s_3) \right)$$

$$H^0_{ii'} = \langle s_1 s_2 s_3 | H^0 | s_1' s_2' s_3' \rangle \qquad , \; i \neq i'$$

$$= \frac{J}{4}\left(\text{sgn}(s_1 \cdot s_2) + \text{sgn}(s_2 \cdot s_3)\right) \underbrace{\langle s_1 s_2 s_3 | s_1' s_2' s_3' \rangle}_{=0} = 0$$

$$\begin{array}{cccccccc} |\uparrow\uparrow\uparrow\rangle & |\uparrow\uparrow\downarrow\rangle & |\uparrow\downarrow\uparrow\rangle & |\uparrow\downarrow\downarrow\rangle & |\downarrow\uparrow\uparrow\rangle & |\downarrow\uparrow\downarrow\rangle & |\downarrow\downarrow\uparrow\rangle & |\downarrow\downarrow\downarrow\rangle \end{array}$$

$$\Rightarrow H^0 = \frac{J}{2} \begin{pmatrix} 1 & & & & & & & \\ & 0 & & & & & \text{\Large 0} & \\ & & -1 & & & & & \\ & & & 0 & & & & \\ & & & & 0 & & & \\ & \text{\Large 0} & & & & -1 & & \\ & & & & & & 0 & \\ & & & & & & & 1 \end{pmatrix}$$

2. Now with all spin directions:

$$H^1 = J \sum_{i=1}^{N} \vec{I}_i \, \vec{I}_{i+1} = J \sum_{i=1}^{N} I^z_i I^z_{i+1} + I^x_i I^x_{i+1} + I^y_i I^y_{i+1}$$

$$= \underbrace{J \sum_{i=1}^{N-1} I^z_i I^z_{i+1}}_{H^0} + \frac{1}{2}\left(S^+_i S^-_{i+1} + S^-_i S^+_{i+1}\right)$$

$$H^1_{ii} = H^0_{ii}$$

$$S_\pm |s, m_s\rangle \overset{\hbar=1}{=} \sqrt{s(s+1) - m_s(m_s \pm 1)} \, |s, m_s \pm 1\rangle$$

$$\Rightarrow S_+ |\tfrac{1}{2}, \tfrac{1}{2}\rangle = 0 \, , \; S_+ |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = |\tfrac{1}{2}, \tfrac{1}{2}\rangle$$

$$S_- |\tfrac{1}{2}, \tfrac{1}{2}\rangle = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \qquad S_- |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = 0$$

$$H^1_{ii'} = \langle s_1 s_2 s_3 | H^1 | s_1' s_2' s_3' \rangle$$

$$= \frac{1}{2}\left(\delta_{s_1\, s_1'+1}\, \delta_{s_2\, s_2'-1} + \delta_{s_1\, s_1'-1}\, \delta_{s_2\, s_2'+1} + \delta_{s_2\, s_2'+1}\, \delta_{s_3\, s_3'-1} + \delta_{s_2 s_2'-1}\, \delta_{s_2 s_2'+1}\right)$$

Integer values:
$(\uparrow = 1, \downarrow = 0)$

$$H^1 = \frac{J}{2} \cdot \begin{array}{cccccccc} 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ |\uparrow\uparrow\uparrow\rangle & |\uparrow\uparrow\downarrow\rangle & |\uparrow\downarrow\uparrow\rangle & |\uparrow\downarrow\downarrow\rangle & |\downarrow\uparrow\uparrow\rangle & |\downarrow\uparrow\downarrow\rangle & |\downarrow\downarrow\uparrow\rangle & |\downarrow\downarrow\downarrow\rangle \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

3) Now with magnetization terms:

$$H^2 = J \sum_{i=1}^{N} \vec{I}_i\, \vec{I}_{i+1} + \sum_{i=1}^{N} B_i\, I_i^z = H^1 + \underbrace{\sum_{i=1}^{N} B_i\, I_i^z}_{H^M}$$

$$H^M = \mathrm{diag}(B_1 \cdot s_1 + B_2 \cdot s_2 + B_3 \cdot s_3)$$

Diagonalization:

With eigenstates of H: $|\varphi_i\rangle$

$$H|\varphi_i\rangle = E_i|\varphi_i\rangle$$

$$\Rightarrow \langle\psi| e^{iHt} \hat{G}^z e^{-iHt} |\psi\rangle$$

$$= \sum_i \langle\psi|\varphi_i\rangle\langle\varphi_i| e^{iHt} \hat{G}_i^z e^{-iHt} |\varphi_i\rangle\langle\varphi_i|\psi\rangle$$

$$= \sum_i \langle\psi|\varphi_i\rangle\langle\varphi_i| e^{iE_it} \hat{G}_i^z e^{-iE_it} |\varphi_i\rangle\langle\varphi_i|\psi\rangle$$