Ch7. Conditional Probability Models

School of Electronics Engineering

Conditional Probability Models

Example 7.1

Let N equals the number of bytes in email. A conditioning event might be the event I that the email contains an image.

Conditional Probability Models

Definition 7.1

Given the event B with $P[B] > 0$, the conditional CDF of X is

$$F_{X|B}(x) = P[X \leq x \mid B]$$

Conditional Probability Models

Definition 7.2

Given the event B with $P[B] > 0$, the conditional PMF of X is

$$P_{X|B}(x) = P[X = x \mid B]$$

Conditional Probability Models

Definition 7.3

For a random variable X and an event B with $P[B] > 0$, the conditional PDF of X given B is

$$f_{X|B} = \frac{dF_{X|B}(x)}{dx}$$

Conditional Probability Models

Theorem 7.1

For a random variable X and an event $B \subset S_X$ with $P[B] > 0$, the conditional PDF of X given B is

$$\text{Discrete: } P_{X|B}(x) = \begin{cases} \frac{P_X(x)}{P[B]}, & x \in B \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Continuous: } f_{X|B}(x) = \begin{cases} \frac{f_X(x)}{P[B]}, & x \in B \\ 0, & \text{otherwise} \end{cases}$$

Conditional Probability Models

Example 7.2

A website distributes videos, where the length of each video is modeled as a random variable X with the PMF of

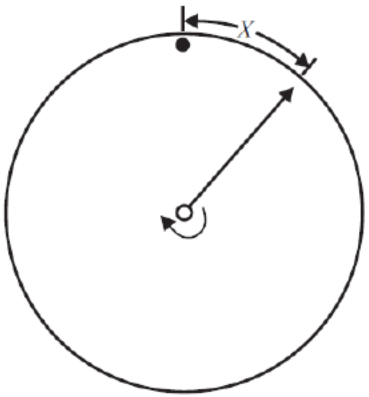
$$P_X(x) = \begin{cases} 0.15, & x = 1, 2, 3, 4 \\ 0.1, & x = 5, 6, 7, 8 \\ 0, & \text{otherwise} \end{cases}$$

Suppose the website has two servers, one for $X < 5$, the other one for $X \geq 5$. Find the PMF of video length in the 2nd server.

Conditional Probability Models

Example 7.3

For the pointer-spinning experiment of Example 4.1, find the conditional PDF of the pointer position for spins in which the pointer stops on the left side of the circle.



Conditional Probability Models

Example 7.4

Suppose X, the time in integer mins you wait for a bus, is modeled as a discrete (0, 20) uniform random variable. Suppose the bus has not arrived by the eight minute; what is the conditional PMF of your waiting time X?

Conditional Probability Models

Theorem 7.2

For random variable X resulting from an experiment with partition $B_1, \ldots, B_m$,

Discrete: $P_X(x) = \sum_{i=1}^{m} P_{X|B_i}(x) P[B_i]$

Continuous: $f_X(x) = \sum_{i=1}^{m} f_{X|B_i}(x)\, P[B_i]$

Conditional Probability Models

Example 7.7

Random variable X is a voltage at the receiver of a modem. When symbol “0” is transmitted, X is the Gaussian (-5, 2) random variable. When symbol “1” is transmitted, X is the Gaussian (5, 2) random variable. What is the PDF of X?

Conditional Expected Value

Theorem 7.3

Discrete X:

(a) For any $x \in B$, $P_{X|B}(x) \geq 0$

(b) $\sum_{x \in B} P_{X|B}(x) = 1$

(c) $P[C|B] = \sum_{x \in C} P_{X|B}(x)$

Continuous X:

(a) For any $x \in B$, $f_{X|B}(x) \geq 0$

(b) $\int_B f_{X|B}(x) dx = 1$

(c) $P[C|B] = \int_C f_{X|B}(x)\, dx$

Conditional Expected Value

Definition 7.4

The conditional expected value of random variable X given condition B is

Discrete: $E[X|B] = \sum_{x \in B} x P_{X|B}(x)$

Continuous: $E[X|B] = \int_{-\infty}^{\infty} x f_{X|B}(x)\, dx$

Conditional Expected Value

Theorem 7.4

For a random variable X resulting from an experiment with partitions $B_1, \ldots, B_m$

$$E[X] = \sum_{i=1}^{m} E[X|B_i]P[B_i]$$

Conditional Expected Value

Theorem 7.5

The conditional expected value of $Y = g(X)$ given condition B is

Discrete: $E[Y|B] = E[g(X)|B] = \sum_{x \in B} g(x) P_{X|B}(x)$

Continuous: $E[Y|B] = E[g(X)|B] = \int_{-\infty}^{\infty} g(x) f_{X|B}(x)\, dx$

Conditional Expected Value

Definition 7.5

The conditional variance of X given event B is

$$Var[X|B] = E\left[\left(X - \mu_{X|B}\right)^2 \middle| B\right] = E[X^2|B] - \mu_{X|B}^2$$

Conditioning Two Random Variables

Definition 7.6

For discrete random variables X and Y and an event B with $P[B] > 0$, the conditional joint PMF of X and Y given B is

$$P_{X,Y|B}(x, y) = P[X = x, Y = y \mid B]$$

Conditioning Two Random Variables

Theorem 7.6

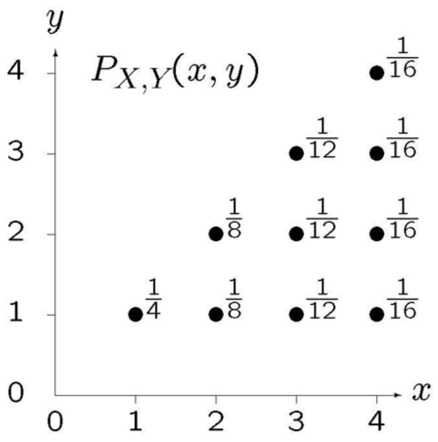
For any event B, a region of X, Y plane with $P[B] > 0$,

$$P_{X,Y|B}(x,y) = \begin{cases} \dfrac{P_{X,Y}(x,y)}{P[B]}, & (x,y) \in B \\ 0, & \text{otherwise} \end{cases}$$

Conditioning Two Random Variables

Example 7.9

If $B = \{X + Y \leq 4\}$, find the conditional PMF $P_{X,Y|B}(x, y)$



Conditioning Two Random Variables

Definition 7.7

Given an event B with $P[B] > 0$, the conditional joint PDF of X, Y is

$$f_{X,Y|B}(x,y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{P[B]}, & (x,y) \in B \\ 0, & \text{otherwise} \end{cases}$$

Conditioning Two Random Variables

Example 7.10

X and Y are random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{15}, & 0 \le x \le 5, 0 \le y \le 3 \\ 0, & \text{otherwise} \end{cases}$$

Find the conditional PDF of X, Y given the event $B = \{X + Y \ge 4\}$

Conditioning Two Random Variables

Theorem 7.7

For random variables X and Y and an event B with $P[B] > 0$, the conditional expected value of $W = g(X, Y)$ given B is

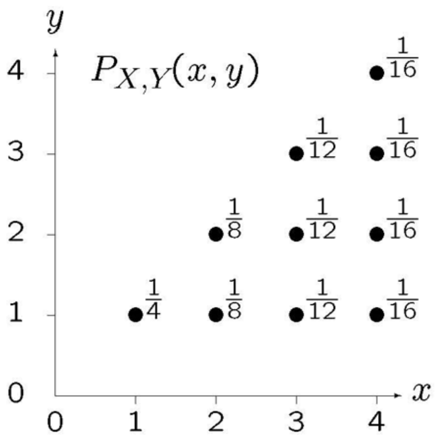
Discrete: $E[W|B] = \sum_{x \in S_X} \sum_{y \in S_Y} g(x, y) P_{X,Y|B}(x, y)$

Continuous: $E[W|B] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y|B}(x, y)\ dxdy$

Conditioning Two Random Variables

Example 7.11

Find the conditional expected value and variance of $W = X + Y$ given the event $B = \{X + Y \leq 4\}$.



Conditioning by a Random Variable

Definition 7.8

For any event $Y = y$ such that $P_Y(y) > 0$, the conditional PMF of X given $Y = y$ is

$$P_{X|Y}(x|y) = P[X = x \mid Y = y]$$

Conditioning by a Random Variable

Theorem 7.8

For discrete random variables X and Y with joint PMF $P_{X,Y}(x, y)$, and x and y such that $P_X(x) > 0$ and $P_Y(y) > 0$,

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x, y)}{P_Y(y)}, \qquad P_{Y|X}(y|x) = \frac{P_{X,Y}(x, y)}{P_X(x)}$$

Conditioning by a Random Variable

Definition 7.9

For y such that $f_Y(y) > 0$, the conditional PDF of X given $\{Y = y\}$ is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Conditioning by a Random Variable

Example 7.13

Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2, & 0 \le y \le x < 1 \\ 0, & \text{otherwise} \end{cases}$$

For $0 \le x \le 1$, find the conditional PDF $f_{Y|X}(y|x)$

Conditioning by a Random Variable

Theorem 7.9

For discrete random variables X and Y with joint PDF $P_{X,Y}(x,y)$, and x and y such that $P_X(x) > 0$ and $P_Y(y) > 0$,

$$P_{X,Y}(x,y) = P_{Y|X}(y|x)P_X(x) = P_{X|Y}(x|y)P_Y(y)$$

Conditioning by a Random Variable

Theorem 7.10

For continuous random variables X and Y with joint PDF $f_{X,Y}(x,y)$, and x and y such that $f_X(x) > 0$ and $f_Y(y) > 0$,

$$f_{X,Y}(x,y) = f_{Y|X}(y|x)f_X(x) = f_{X|Y}(x|y)f_Y(y)$$

Conditioning by a Random Variable

Example 7.14

Let R be the uniform $(0, 1)$ random variable. Given $R = r$, X is the uniform $(0, r)$ random variable. Find the conditional PDF of R given X

Conditioning by a Random Variable

Theorem 7.11

If X and Y are independent,

Discrete: $P_{X|Y}(x|y) = P_X(x), \quad P_{Y|X}(y|x) = P_Y(y)$

Continuous: $f_{X|Y}(x|y) = P_X(x), \quad f_{Y|X}(y|x) = f_Y(y)$

Expected Value Given a Random Variable

Definition 7.10

For any $y \in S_Y$, the conditional expected value of $g(X, Y)$ given $Y = y$ is

Discrete: $E[g(X,Y)|Y = y] = \sum_{x \in S_X} g(x,y) P_{X|Y}(x|y)$

Continuous: $E[g(X,Y)|Y = y] = \int_{-\infty}^{\infty} g(x,y) f_{X|Y}(x|y)\, dx$

Expected Value Given a Random Variable

Theorem 7.12

For independent random variables X and Y,

(a) $E[X|Y = y] = E[X]$ for all $y \in S_Y$

(b) $E[Y|X = x] = E[Y]$ for all $x \in S_X$

Expected Value Given a Random Variable

Definition 7.11

The conditional expected value $E[X|Y]$ is a function of random variable Y such that if $Y = y$, then $E[X|Y] = E[X|Y = y]$

Expected Value Given a Random Variable

Theorem 7.13 (Iterated expectation)

$$E\big[E[X|Y]\big] = E[X]$$