Ch4. Continuous Random Variable

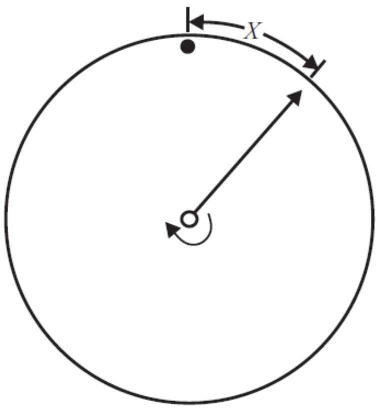
School of Electronics Engineering

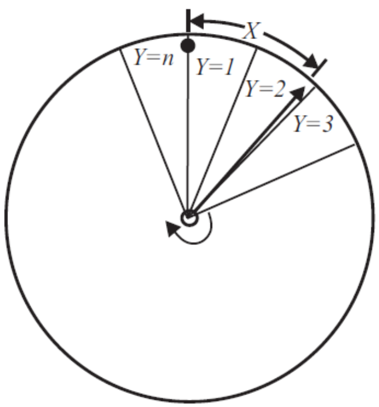
Basics of Continuous Random Variables

Continuous Random Variable

Example 4.1

Define X is the length of the arc from a certain reference point of the wheel of circumference 1 meter.





Continuous Random Variable

Definition

A random variable X is continuous if the range S_X consists of one or more intervals. For each $x \in S_X$, $P[X = x] = 0$

Cumulative Density Function (CDF)

Definition 4.1

The cumulative density function (CDF) of random variable X is

$$F_X(x) = P[X \leq x]$$

Cumulative Density Function (CDF)

Theorem 4.1

For any random variable X

(a) $F_X(-\infty) = 0$

(b) $F_X(\infty) = 1$

(c) $P[x_1 < X \leq x_2] = F_X(x_2) - F_X(x_1)$

Cumulative Density Function (CDF)

Definition 4.2

X is a continuous random variable if the CDF of $F_X(x)$ is a continuous function

Cumulative Density Function (CDF)

Quiz 4.2

The CDF of the random variable Y is

$$F_Y(y) = \begin{cases} \frac{y}{4}, & 0 \le y \le 4 \\ 0, & \text{otherwise} \end{cases}$$

Sketch the CDF of Y and find

(a) $P[Y \le -1]$

(b) $P[Y \le 1]$

(c) $P[2 < Y \le 3]$

(d) $P[Y > 1.5]$

Probability Density Function (PDF)

Definition 4.3

The probability density function (PDF) of a continuous random variable X is

$$f_X(x) = \frac{dF_X(x)}{dx}$$

Probability Density Function (PDF)

Theorem 4.2

For a continuous random variable X with PDF $f_X(x)$

(a) $f_X(x) \geq 0$ for all x

(b) $F_X(x) = \int_{-\infty}^{x} f_X(u)du$

(c) $\int_{-\infty}^{\infty} f_X(x)dx = 1$

Probability Density Function (PDF)

Theorem 4.3

$$P[x_1 < X \le x_2] = \int_{x_1}^{x_2} f_X(x)\,dx$$

Expected Value

Remind)

Expected value of a *discrete random variable* Y is defined as

$$E[Y] = \sum_{y \in S_Y} y_i P_Y(y_i)$$

Expected Value

Definition 4.4

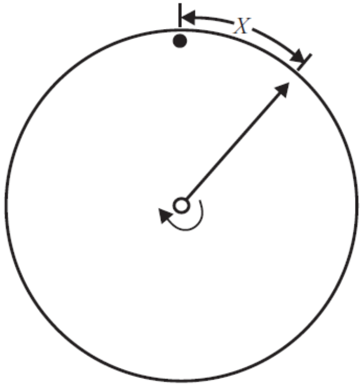
The expected value of a continuous random variable X is

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

Expected Value

Example 4.6

Find the PDF and expected value of the spinning wheel experiment (example 4.1)



Expected Value

Theorem 4.4

The expected value of a function, $g(X)$, of a random variable X is

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

Expected Value

Theorem 4.5

For any random variable X,

(a) $E[X - \mu_X] = 0$

(b) $E[aX + b] = aE[X] + b$

(c) $Var[X] = E[X^2] - \mu_X^2$

(d) $Var[aX + b] = a^2 Var[X]$

Expected Value

Quiz 4.4

The PDF of the random variable Y is

$$f_Y(y) = \begin{cases} \dfrac{3y^2}{2}, & -1 \le y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Sketch the PDF and find

(a) $E[Y]$

(b) $E[Y^2]$

(c) $Var[Y]$

Families of Continuous Random Variables

Uniform Random Variable

Definition 4.5

X is a uniform (a, b) random variable if the PDF of X is

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a \le x < b, \\ 0 & \text{otherwise} \end{cases}$$

where $a < b$

Uniform Random Variable

Theorem 4.6

If X is a uniform (a, b) random variable,

- The CDF of X: $F_X(x) = \begin{cases} 0, & x \leq a \\ \frac{x-a}{b-a}, & a < x \leq b \\ 1, & x > b \end{cases}$
- The expected value of X: $E[X] = \frac{b+a}{2}$
- The variance of X: $Var[X] = \frac{(b-a)^2}{12}$

Exponential Random Variable

Definition 4.6

X is an exponential (λ) random variable if the PDF of X is

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda > 0$

Exponential Random Variable

Example 4.10

The duration T of telephone call is often modeled as an exponential random variable. If $\lambda = \frac{1}{3}$, what is expected call duration?

Exponential Random Variable

Theorem 4.8

If X is an exponential (λ) random variable,

- The CDF of X: $F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$
- The expected value of X: $E[X] = \frac{1}{\lambda}$
- The variance of X: $Var[X] = \frac{1}{\lambda^2}$

Erlang Random Variable

Definition

X is an Erlang (n, λ) random variable if the PDF of X is

$$f_X(x) = \begin{cases} \dfrac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

where $\lambda > 0$ and $n \geq 1$ is an integer

Erlang Random Variable

Theorem 4.10

If X is an Erlang (n, λ) random variable, then

(a) $E[X] = \frac{n}{\lambda}$

(b) $Var[X] = \frac{n}{\lambda^2}$

Gaussian Random Variable

Gaussian Random Variable

Definition 4.8

X is a Gaussian (μ, σ) random variable if the PDF of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Gaussian Random Variable

Theorem 4.12

If X is a Gaussian (μ, σ) random variable

(a) $E[X] = \mu$

(b) $Var[X] = \sigma^2$

Gaussian Random Variable

Theorem 4.13

If X is a Gaussian (μ, σ), $Y = aX + b$ is Gaussian $(a\mu + b, a\sigma)$

Gaussian Random Variable

Definition 4.9

The standard normal random variable Z is the Gaussian (0, 1) random variable

Definition 4.10 (Standard normal CDF)

The CDF of the standard normal random variable Z is

$$\Phi(z) = \int_{-\infty}^{z} e^{-\frac{u^2}{2}}\, du$$

Table of Standard Normal CDF $\Phi(z)$

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
0.00	0.5000	0.50	0.6915	1.00	0.8413	1.50	0.9332	2.00	0.97725	2.50	0.99379
0.01	0.5040	0.51	0.6950	1.01	0.8438	1.51	0.9345	2.01	0.97778	2.51	0.99396
0.02	0.5080	0.52	0.6985	1.02	0.8461	1.52	0.9357	2.02	0.97831	2.52	0.99413
0.03	0.5120	0.53	0.7019	1.03	0.8485	1.53	0.9370	2.03	0.97882	2.53	0.99430
0.04	0.5160	0.54	0.7054	1.04	0.8508	1.54	0.9382	2.04	0.97932	2.54	0.99446
0.05	0.5199	0.55	0.7088	1.05	0.8531	1.55	0.9394	2.05	0.97982	2.55	0.99461
0.06	0.5239	0.56	0.7123	1.06	0.8554	1.56	0.9406	2.06	0.98030	2.56	0.99477
0.07	0.5279	0.57	0.7157	1.07	0.8577	1.57	0.9418	2.07	0.98077	2.57	0.99492
0.08	0.5319	0.58	0.7190	1.08	0.8599	1.58	0.9429	2.08	0.98124	2.58	0.99506
0.09	0.5359	0.59	0.7224	1.09	0.8621	1.59	0.9441	2.09	0.98169	2.59	0.99520
0.10	0.5398	0.60	0.7257	1.10	0.8643	1.60	0.9452	2.10	0.98214	2.60	0.99534
0.11	0.5438	0.61	0.7291	1.11	0.8665	1.61	0.9463	2.11	0.98257	2.61	0.99547
0.12	0.5478	0.62	0.7324	1.12	0.8686	1.62	0.9474	2.12	0.98300	2.62	0.99560
0.13	0.5517	0.63	0.7357	1.13	0.8708	1.63	0.9484	2.13	0.98341	2.63	0.99573
0.14	0.5557	0.64	0.7389	1.14	0.8729	1.64	0.9495	2.14	0.98382	2.64	0.99585
0.15	0.5596	0.65	0.7422	1.15	0.8749	1.65	0.9505	2.15	0.98422	2.65	0.99598
0.16	0.5636	0.66	0.7454	1.16	0.8770	1.66	0.9515	2.16	0.98461	2.66	0.99609
0.17	0.5675	0.67	0.7486	1.17	0.8790	1.67	0.9525	2.17	0.98500	2.67	0.99621
0.18	0.5714	0.68	0.7517	1.18	0.8810	1.68	0.9535	2.18	0.98537	2.68	0.99632
0.19	0.5753	0.69	0.7549	1.19	0.8830	1.69	0.9545	2.19	0.98574	2.69	0.99643
0.20	0.5793	0.70	0.7580	1.20	0.8849	1.70	0.9554	2.20	0.98610	2.70	0.99653
0.21	0.5832	0.71	0.7611	1.21	0.8869	1.71	0.9564	2.21	0.98645	2.71	0.99664
0.22	0.5871	0.72	0.7642	1.22	0.8888	1.72	0.9573	2.22	0.98679	2.72	0.99674
0.23	0.5910	0.73	0.7673	1.23	0.8907	1.73	0.9582	2.23	0.98713	2.73	0.99683
0.24	0.5948	0.74	0.7704	1.24	0.8925	1.74	0.9591	2.24	0.98745	2.74	0.99693
0.25	0.5987	0.75	0.7734	1.25	0.8944	1.75	0.9599	2.25	0.98778	2.75	0.99702
0.26	0.6026	0.76	0.7764	1.26	0.8962	1.76	0.9608	2.26	0.98809	2.76	0.99711
0.27	0.6064	0.77	0.7794	1.27	0.8980	1.77	0.9616	2.27	0.98840	2.77	0.99720
0.28	0.6103	0.78	0.7823	1.28	0.8997	1.78	0.9625	2.28	0.98870	2.78	0.99728
0.29	0.6141	0.79	0.7852	1.29	0.9015	1.79	0.9633	2.29	0.98899	2.79	0.99736
0.30	0.6179	0.80	0.7881	1.30	0.9032	1.80	0.9641	2.30	0.98928	2.80	0.99744
0.31	0.6217	0.81	0.7910	1.31	0.9049	1.81	0.9649	2.31	0.98956	2.81	0.99752
0.32	0.6255	0.82	0.7939	1.32	0.9066	1.82	0.9656	2.32	0.98983	2.82	0.99760
0.33	0.6293	0.83	0.7967	1.33	0.9082	1.83	0.9664	2.33	0.99010	2.83	0.99767
0.34	0.6331	0.84	0.7995	1.34	0.9099	1.84	0.9671	2.34	0.99036	2.84	0.99774
0.35	0.6368	0.85	0.8023	1.35	0.9115	1.85	0.9678	2.35	0.99061	2.85	0.99781
0.36	0.6406	0.86	0.8051	1.36	0.9131	1.86	0.9686	2.36	0.99086	2.86	0.99788
0.37	0.6443	0.87	0.8078	1.37	0.9147	1.87	0.9693	2.37	0.99111	2.87	0.99795
0.38	0.6480	0.88	0.8106	1.38	0.9162	1.88	0.9699	2.38	0.99134	2.88	0.99801
0.39	0.6517	0.89	0.8133	1.39	0.9177	1.89	0.9706	2.39	0.99158	2.89	0.99807
0.40	0.6554	0.90	0.8159	1.40	0.9192	1.90	0.9713	2.40	0.99180	2.90	0.99813
0.41	0.6591	0.91	0.8186	1.41	0.9207	1.91	0.9719	2.41	0.99202	2.91	0.99819
0.42	0.6628	0.92	0.8212	1.42	0.9222	1.92	0.9726	2.42	0.99224	2.92	0.99825
0.43	0.6664	0.93	0.8238	1.43	0.9236	1.93	0.9732	2.43	0.99245	2.93	0.99831
0.44	0.6700	0.94	0.8264	1.44	0.9251	1.94	0.9738	2.44	0.99266	2.94	0.99836
0.45	0.6736	0.95	0.8289	1.45	0.9265	1.95	0.9744	2.45	0.99286	2.95	0.99841
0.46	0.6772	0.96	0.8315	1.46	0.9279	1.96	0.9750	2.46	0.99305	2.96	0.99846
0.47	0.6808	0.97	0.8340	1.47	0.9292	1.97	0.9756	2.47	0.99324	2.97	0.99851
0.48	0.6844	0.98	0.8365	1.48	0.9306	1.98	0.9761	2.48	0.99343	2.98	0.99856
0.49	0.6879	0.99	0.8389	1.49	0.9319	1.99	0.9767	2.49	0.99361	2.99	0.99861

Gaussian Random Variable

Gaussian to standard normal random variable

For any Gaussian (μ, σ) random variable X

$$Z = \frac{X - \mu}{\sigma}$$

is the standard normal random variable

Gaussian Random Variable

Example 4.13

If X is a Gaussian $(61, 10)$ random variable, what is $P[51 < X < 71]$?

Gaussian Random Variable

Theorem 4.15

$$\Phi(-z) = 1 - \Phi(z)$$

Definition 4.11 (Standard normal *complementary* CDF)

$$Q(z) = P[Z > z] = \frac{1}{\sqrt{2\pi}} \int_{z}^{\infty} e^{-\frac{u^2}{2}} du = 1 - \Phi(z)$$

Delta Functions, Mixed Random Variables

Delta Function

Definition 4.12

Let

$$d_\epsilon(x) = \begin{cases} \frac{1}{\epsilon}, & -\frac{\epsilon}{2} \le x \le \frac{\epsilon}{2} \\ 0, & \text{otherwise} \end{cases}$$

The unit impulse (Delta) function is

$$\delta(x) = \lim_{\epsilon \to 0} d_\epsilon(\mathrm{x})$$

Delta Function

Theorem 4.16

For any continuous function $g(x)$

$$\int_{-\infty}^{\infty} g(x)\delta(x - x_0)dx = g(x_0)$$

Delta Function

Definition 4.13

The unit step function is

$$u(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Theorem 4.17

$$\int_{-\infty}^{x} \delta(v) dv = u(x)$$

Delta Function

Note)

The PDF of discrete random variable can be explained using Delta functions

Mixed Random Variable

Definition 4.14

X is mixed random variable if and only if $f_X(x)$ contains both impulses and non-zero finite values