Ch1. Experiments, Models, and Probability

School of Electronics Engineering

[Review] Set Theory

Set Theory

Basic notations

Sets are represented by { }

- ✓ ex) $C = \{1, 4, 9, 16, 25\},\ D = \{x^2 \mid x = 1, 2, 3, 4, 5\}$

$x \in A$: x is an *element* of the set A (c.f., $x \notin A$)

$A \subset B$: set A is a *subset* of B

- ✓ $A \subset B$ if $x \in A$ and then $x \in B$

$A = B$: sets A and B are *equal*

- ✓ $A = B$ if and only if $A \subset B$ and $B \subset A$

S: *universal* set, ϕ: *null (empty)* set

Set Theory

Basic set operations

$A \cup B$: the *union* of sets A and B

$$x \in A \cup B \text{ if and only if } x \in A \text{ or } x \in B$$

$A \cap B$: the *intersection* of sets A and B

$$x \in A \cap B \text{ if and only if } x \in A \text{ and } x \in B$$

$A - B$: Set *minus* or set *subtraction*

$$\text{If } x \in A - B, \text{ then } x \in A \text{ but } x \notin B$$

A^c: the *complement* of a set A

$$x \in A^c \text{ if and only if } x \notin A$$

Set Theory

De Morgan's law

$$(A \cup B)^c = A^c \cap B^c$$

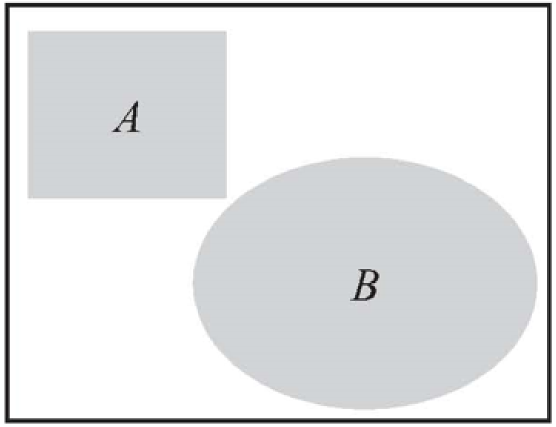
Set Theory

Mutually exclusive sets

A collection of sets $A_1, \dots, A_n$ is *mutually exclusive* if and only if

$$A_i \cap A_j = \phi, \qquad i \neq j$$

mutually exclusive $\approx$ *disjoint*

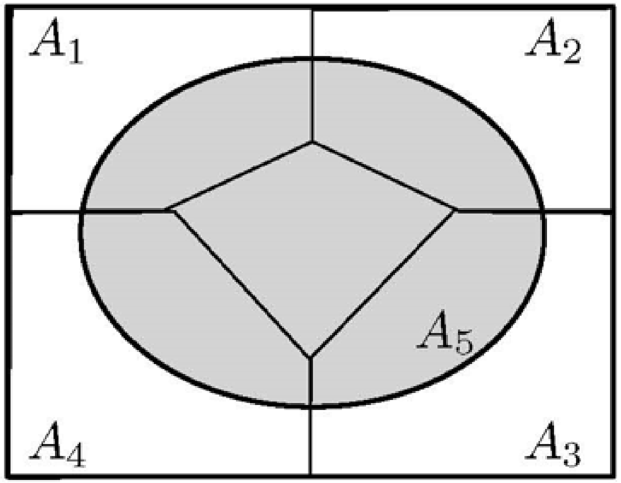


Set Theory

Collectively exhaustive sets

A collection of sets $A_1, \dots, A_n$ is *collectively exhaustive* if and only if

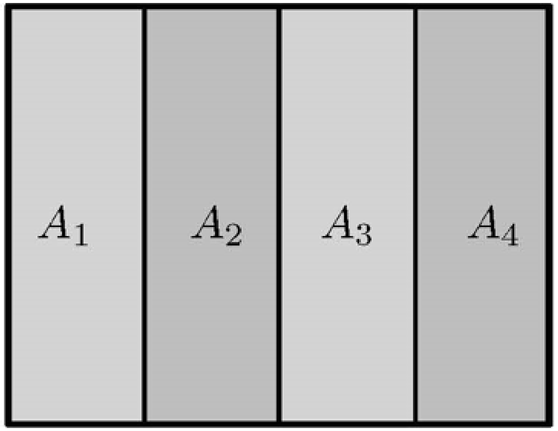
$$A_1 \cup A_2 \cup \cdots \cup A_n = S$$



Set Theory

Partitions

A collection of sets $A_1, \dots, A_n$ is a *partition* if it is both *mutually exclusive* and *collectively exhaustive*



Applying Set Theory to Probability

Set Theory to Probability

Experiments

An experiment consists of a procedure and observations. There is *uncertainty* in what will be observed (model)

Some examples:

- ✓ Flip a coin. Did it land with heads or tails facing up?
- ✓ Walk to a bus stop. How long do you wait for the arrival of a bus?
- ✓ Give a lecture. How many students are attended?
- ✓ Transmit a signal into the air. What signal arrives at the receiver?

Set Theory to Probability

Experiment (cont.)

Coin flip example

- ✓ Procedure: flip a coin
- ✓ Observation: head or tail
- ✓ Model: head and tail are equally likely

Set Theory to Probability

Outcome

An outcome of an experiment is *any possible observation* of that experiment

Set Theory to Probability

Sample space

The sample space of an experiment is the *finest-grain*, *mutually exclusive*, *collectively exhaustive* set of all possible outcomes

Set Theory to Probability

Event

An event is *a set of outcomes* of an experiment

PROBABILITY AXIOMS

Axioms of Probability

Probability measure: $P[\cdot]$

A function that maps events in the sample space to real numbers such that

Axiom 1 For any event A, $P[A] \geq 0$

Axiom 2 $P[S] = 1$

Axiom 3 For any *countable* collection $A_1, A_2, \ldots$ of *mutually exclusive events*

$$P[A_1 \cup A_2 \cup \cdots] = P[A_1] + P[A_2] + \cdots$$

Axioms of Probability

Theorem 1.2

For mutually exclusive events A_1 and A_2,

$$P[A_1 \cup A_2] = P[A_1] + P[A_2]$$

Axioms of Probability

Theorem 1.3

If $A = A_1 \cup A_2 \cup \cdots \cup A_m$ and $A_i \cap A_j = \phi$ for $i \neq j$, then

$$P[A] = \sum_{i=1}^{m} P[A_i]$$

Axioms of Probability

Theorem 1.4

The probability measure $P[\cdot]$ satisfies

(a) $P[\phi] = 0$

(b) $P[A^c] = 1 - P[A]$

(c) For any A and B (*not necessarily mutually exclusive*)

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

(d) If $A \subset B$, then $P[A] \leq P[B]$

Axioms of Probability

Theorem 1.5

The probability of an event $B = \{s_1, s_2, \ldots, s_m\}$ is the sum of probability of the outcomes contained in the event:

$$P[B] = \sum_{i=1}^{m} P[\{s_i\}]$$

Axioms of Probability

Theorem 1.6

For an experiment with sample space $S = \{s_1, \ldots, s_n\}$ in which each outcome s_i is *equally likely*,

$$P[s_i] = \frac{1}{n}, \qquad 1 \le i \le n$$

Axioms of Probability

Example

Roll a six-sided die in which all faces are equally likely. What is the probability of each outcome? Find the probabilities of the events: (a) roll 4 or higher, (b) roll an even number, and (c) roll the square of an integer.

Conditional Probability

Conditional Probability

Definition

The conditional probability of the event A given the occurrence of the event B is

$$P[A|B] = \frac{P[A \cap B]}{P[B]}$$

Note: $P[A|B]$ is defined when $P[B] > 0$

Conditional Probability

Theorem 1.7

$P[A|B]$ has the following properties corresponding to the axioms of probability

Axiom 1 $P[A|B] \geq 0$

Axiom 2 $P[B|B] = 1$

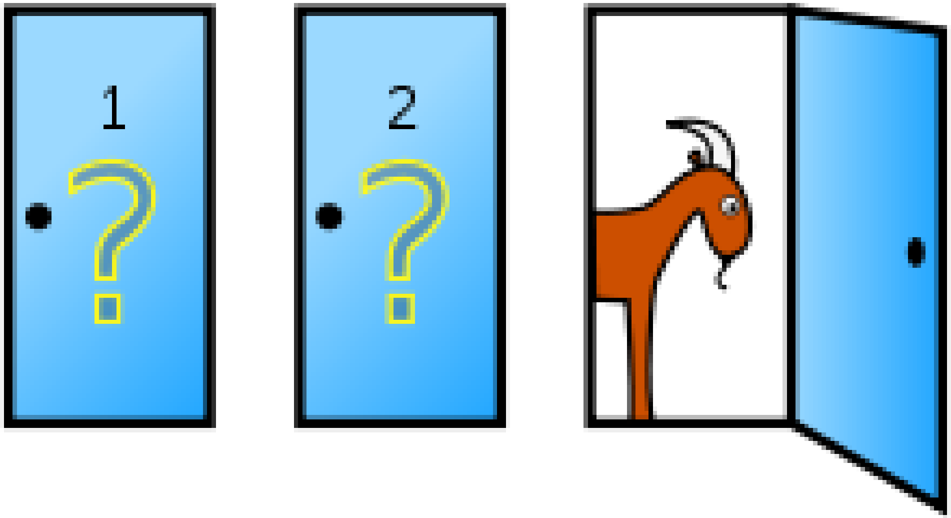
Axiom 3 If $A = A_1 \cup A_2 \cup \cdots$ with $A_i \cap A_j = \phi$ for $i \neq j$, then

$$P[A|B] = P[A_1|B] + P[A_2|B] + \cdots$$

Example of Probability

Monty Hall problem

You're given the choice of three doors: a car and two goats are behind the doors



Partitions and the Law of Total Probability

Theorem 1.8

For a partition $B = \{B_1, B_2, \dots\}$ and any event A in the sample space, let $C_i = A \cap B_i$. Then C_i and C_j are mutually exclusive for $i \neq j$ and

$$A = C_1 \cup C_2 \cup \cdots$$

Partitions and the Law of Total Probability

Theorem 1.9

For any event A, and partition $\{B_1, B_2, \ldots, B_m\}$,

$$P[A] = \sum_{i=1}^{m} P[A \cap B_i]$$

Law of Total Probability

Theorem 1.10 (law of total probability)

For a partition $\{B_1, B_2, \ldots, B_m\}$ with $P[B_i] > 0$ for all i,

$$P[A] = \sum_{i=1}^{m} P[A|B_i]P[B_i]$$

Bayes' Theorem

Theorem 1.11 (Bayes' theorem)

$$P[B|A] = \frac{P[A|B]P[B]}{P[A]}$$

INDEPENDENCE

Independence

Definition 1.6 (two independent events)

Events A and B are independent if and only if

$$P[A \cap B] = P[A]P[B]$$

Independence

Definition 1.7 (three independent events)

A_1, A_2 and A_3 are mutually independent if and only if

(a) A_1 and A_2 are independent

(b) A_2 and A_3 are independent

(c) A_3 and A_1 are independent

(d) $P[A_1 \cap A_2 \cap A_3] = P[A_1]P[A_2]P[A_3]$

Independence

Example 1.23

Consider a sample space $S = \{1, 2, 3, 4\}$, where all outcomes are equally likely. Are the events $A_1 = \{1, 3, 4\}, \mathrm{A}_2 = \{2, 3, 4\}$, and $A_3 = \phi$ mutually independent?

Independence

Definition 1.8 (more then two independent events)

If $n \geq 3$, the events $A_1, A_2, \ldots, A_n$ are mutually independent if and only if

(a) All collections of $n-1$ events chosen from $A_1, A_2, \ldots, A_n$ are mutually independent

(b) $P[A_1 \cap A_2 \cap \cdots \cap A_n] = P[A_1]P[A_2] \ldots P[A_n]$

Independence

Independent vs. mutually exclusive