Ch8. Random Vectors

School of Electronics Engineering

Random Vector

Definition 8.1

A random vector is a column vector $\mathbf{X} = [X_1, \ldots, X_n]^T$, where each X_i is a random variable

Random Vector

Definition 8.2

A sample value of a random vector is a column vector $\mathbf{x} = [x_1, \ldots, x_n]^T$, where the i-th component x_i is a sample value of a random variable X_i

Random Vector

Definition 8.3

(a) The CDF of a random vector $\mathbf{X}$ is

$$F_{\mathbf{X}}(\mathbf{x}) = F_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$$

(b) The PMF of a discrete random vector $\mathbf{X}$ is

$$P_{\mathbf{X}}(\mathbf{x}) = P_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$$

(c) The PDF of a continuous random vector $\mathbf{X}$ is

$$f_{\mathbf{X}}(\mathbf{x}) = f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$$

Random Vector

Definition 8.4

For random vectors $\mathbf{X}$ with n, $\mathbf{Y}$ with m components:

(a) The joint CDF of $\mathbf{X}$ and $\mathbf{Y}$ is

$$F_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y}) = F_{X_1,\dots,X_n,Y_1,\dots,Y_m}(x_1,\dots,x_n,y_1,\dots,y_m)$$

(b) The joint PMF of discrete random vectors $\mathbf{X}$ and $\mathbf{Y}$ is

$$P_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y}) = P_{X_1,\dots,X_n,Y_1,\dots,Y_m}(x_1,\dots,x_n,y_1,\dots,y_m)$$

(c) The joint PDF of continuous random vectors $\mathbf{X}$ and $\mathbf{Y}$ is

$$f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y}) = f_{X_1,\dots,X_n,Y_1,\dots,Y_m}(x_1,\dots,x_n,y_1,\dots,y_m)$$

Random Vector

Example 8.1

Random vector $\mathbf{X}$ has PDF

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} 6e^{-\mathbf{a}^T\mathbf{x}}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Where $\mathbf{a} = [1, 2, 3]^T$. What is the CDF of $\mathbf{X}$?

Independent Random Vectors

Definition 8.5

Random vectors $\mathbf{X}$ and $\mathbf{Y}$ are independent if

Discrete: $P_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y}) = P_{\mathbf{X}}(\mathbf{x})P_{\mathbf{Y}}(\mathbf{y})$

Continuous: $f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y}) = f_{\mathbf{X}}(\mathbf{x})f_{\mathbf{Y}}(\mathbf{y})$

Independent Random Vectors

Example 8.2

Random variables $Y_1, \ldots, Y_4$ have the joint PDF

$$f_{Y_1,\ldots,Y_4}(y_1, \ldots, y_4) = \begin{cases} 4, & 0 \le y_1 \le y_2 \le 1, \quad 0 \le y_3 \le y_4 \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Let $\mathbf{V} = [Y_1, Y_4]^T$ and $\mathbf{W} = [Y_2, Y_2]^T$. Are $\mathbf{V}$ and $\mathbf{W}$ independent?

Functions of Random Vectors

Theorem 8.1

For random variable $W = g(\mathbf{X})$,

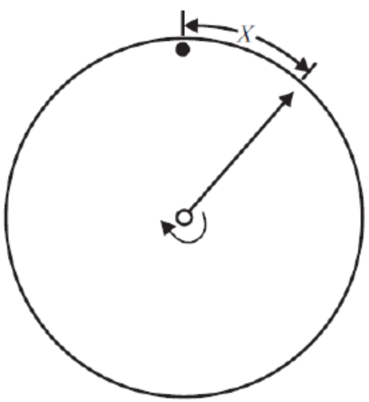
Discrete: $P_W(w) = P[W = w] = \sum_{\mathbf{x}:g(\mathbf{x})=w} P_{\mathbf{X}}(\mathbf{x})$

Continuous: $F_W(w) = P[W \leq w] = \int \dots \int_{g(\mathbf{x})\leq w} f_{\mathbf{X}}(\mathbf{x})\, dx_1 \dots dx_n$

Functions of Random Vectors

Example 8.3

Consider the spinning wheel experiment. Let Y be a random variable which is the maximum position of pointers in n experiment. Find the CDF and PDF of Y



Functions of Random Vectors

Theorem 8.2

Let $\mathbf{X}$ be a vector of n *i.i.d.* continuous random variables, each with CDF $F_X(x)$ and PDF $f_X(x)$

(a) For $Y = \max\{X_1, \dots, X_n\}$, $F_Y(y) = \left(F_X(y)\right)^n$, $f_Y(y) = n\left(F_X(y)\right)^{n-1} f_X(y)$

(b) For $W = \min\{X_1, \dots, X_n\}$,

$$F_W(w) = 1 - \left(1 - F_X(w)\right)^n, \; f_W(w) = n\left(1 - F_X(w)\right)^{n-1} F_X(w)$$

Functions of Random Vectors

Theorem 8.3

For a random vector $\mathbf{X}$, the random variable $g(\mathbf{X})$ has expected value

Discrete: $E[g(\mathbf{X})] = \sum_{x_1 \in S_{X_1}} \dots \sum_{x_n \in S_{X_n}} g(\mathbf{x}) P_{\mathbf{X}}(\mathbf{x})$

Continuous: $E[g(\mathbf{X})] = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} g(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) \, dx_1 \dots dx_n$

Functions of Random Vectors

Theorem 8.4

When the component of $\mathbf{X}$ are independent random variables,

$$E[g_1(X_1)g_2(X_2)\ldots g_n(X_n)] = E[g_1(X_1)]E[g_2(X_2)]\ldots E[g_n(X_n)]$$

Functions of Random Vectors

Theorem 8.5

Given the continuous random vectors $\mathbf{X}$, $\mathbf{Y}$ is derived such that $Y_k = aX_k + b$ for constant $a > 0$ and b. The CDF and PDF of $\mathbf{Y}$ are

$$F_{\mathbf{Y}}(\mathbf{y}) = F_{\mathbf{X}}\left(\frac{y_1 - b}{a}, \dots, \frac{y_n - b}{a}\right), \qquad f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{a^n} f_{\mathbf{X}}\left(\frac{y_1 - b}{a}, \dots, \frac{y_n - b}{a}\right)$$

Expected Value Vector and Correlation Matrix

Definition 8.6

Expected value of a random vector is a column vector

$$E[\mathbf{X}] = \boldsymbol{\mu}_{\mathbf{X}} = \left[E[X_1], \dots, E[X_n]\right]^T$$

Expected Value Vector and Correlation Matrix

Example 8.4

If $\mathbf{X} = [X_1, X_2, X_3]^T$. What are the components of $\mathbf{X}\mathbf{X}^T$?

Expected Value Vector and Correlation Matrix

Definition 8.7

For a random matrix $\mathbf{A}$ with random variables A_{ij} as its i,j-th component element, $E[\mathbf{A}]$ is a matrix with i,j-th element $E[A_{ij}]$

Expected Value Vector and Correlation Matrix

Definition 8.8 (correlation matrix)

Correlation of a random vector $\mathbf{X}$ is an $n \times n$ matrix $\mathbf{R_X}$ with i, j-th element $[\mathbf{R_X}]_{i,j} = E[X_i X_j]$.

Expected Value Vector and Correlation Matrix

Definition 8.9 (covariance matrix)

Covariance of a random vector $\mathbf{X}$ is an $n \times n$ matrix $\mathbf{C}_X$ with i, j-th element $[\mathbf{C}_X]_{i,j} = Cov[X_i, X_j]$.

Expected Value Vector and Correlation Matrix

Theorem 8.7

For a random vector $\mathbf{X}$ with correlation matrix $\mathbf{R}_{\mathbf{X}}$, covariance matrix $\mathbf{C}_{\mathbf{X}}$, and vector expected value $\boldsymbol{\mu}_{\mathbf{X}}$,

$$\mathbf{C}_{\mathbf{X}} = \mathbf{R}_{\mathbf{X}} - \boldsymbol{\mu}_{\mathbf{X}}(\boldsymbol{\mu}_{\mathbf{X}})^{T}$$

Expected Value Vector and Correlation Matrix

Example 8.7

Find the expected value $E[\mathbf{X}]$, correlation matrix $\mathbf{R_X}$, and covariance matrix $\mathbf{C_X}$ of the 2D random vector $\mathbf{X}$ with PDF

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} 2, & 0 \le x_1 \le x_2 \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Expected Value Vector and Correlation Matrix

Definition 8.10

Cross-correlation of random vectors, X with n components and Y with m components, is an $n \times m$ matrix R_{XY} with i, j-th element

$R_{XY}(i,j) = E\left[X_i Y_j\right]$

$$\mathbf{R_{XY}} = E[\mathbf{X}\mathbf{Y}^T]$$

Two Random Vectors

Definition 8.11

Cross-covariance of a pair of random vectors $\mathbf{X}$ with n, $\mathbf{Y}$ with m components, is an $n \times m$ matrix $\mathbf{C_{XY}}$ with $[\mathbf{C_{XY}}]_{ij} = Cov[X_i, Y_j]$

Two Random Vectors

Theorem 8.8

For a random vector $\mathbf{X}$ with n components, random vector $\mathbf{Y}$ with m components is derived as $\mathbf{Y} = \mathbf{AX} + \boldsymbol{b}$,

$$\boldsymbol{\mu}_{\mathbf{Y}} = \mathbf{A}\boldsymbol{\mu}_{\mathbf{X}} + \mathbf{b}$$

$$\mathbf{R}_{\mathbf{Y}} = \mathbf{A}\mathbf{R}_{\mathbf{X}}\mathbf{A}^{T} + (\mathbf{A}\boldsymbol{\mu}_{\mathbf{X}})\mathbf{b}^{T} + \mathbf{b}(\mathbf{A}\boldsymbol{\mu}_{\mathbf{X}})^{T} + \mathbf{b}\mathbf{b}^{T}$$

$$\mathbf{C}_{\mathbf{Y}} = \mathbf{A}\mathbf{C}_{\mathbf{X}}\mathbf{A}^{T}$$

Two Random Vectors

Theorem 8.9

For two random vectors $\mathbf{X}$ and $\mathbf{Y}$ such that $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$

$$\mathbf{R}_{\mathbf{XY}} = \mathbf{R}_{\mathbf{X}}\mathbf{A}^T + \boldsymbol{\mu}_{\mathbf{X}}\mathbf{b}^T, \qquad \mathbf{C}_{\mathbf{XY}} = \mathbf{C}_{\mathbf{X}}\mathbf{A}^T$$

Gaussian Random Vectors

Definition 8.12

$\mathbf{X}$ is the Gaussian $(\boldsymbol{\mu}_\mathbf{X}, \mathbf{C}_\mathbf{X})$ random vector with expected value $\boldsymbol{\mu}_\mathbf{X}$ and covariance $\mathbf{C}_\mathbf{X}$ if and only if

$$f_\mathbf{X}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}[\det(\mathbf{C}_\mathbf{X})]^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_\mathbf{X})^T \mathbf{C}_\mathbf{X}^{-1}(\mathbf{x}-\boldsymbol{\mu}_\mathbf{X})\right)$$

where $\det(\mathbf{C}_\mathbf{X}) > 0$

Gaussian Random Vectors

Theorem 8.10

A Gaussian random vector $\mathbf{X}$ has independent components if and only if $\mathbf{C}_{\mathbf{X}}$ is a diagonal matrix

Gaussian Random Vectors

Example 8.10

The outdoor temperature at 6:00, 12:00, 18:00 are Gaussian random variables, X_1, X_2, X_3 with variance 16 degrees^2. The expected values are 50, 62, and 58 degrees. The covariance matrix is given by

$$C_X = \begin{bmatrix} 16.0 & 12.8 & 11.2 \\ 12.8 & 16.0 & 12.8 \\ 11.2 & 12.8 & 16.0 \end{bmatrix}$$

Write the joint PDF of X_1 and X_2

Gaussian Random Vectors

Definition 8.13

The n-th standard normal random vector $\mathbf{Z}$ is the n-dimensional Gaussian random vector with $\boldsymbol{\mu}_{\mathbf{X}} = \mathbf{0}$ and $\mathbf{C}_{\mathbf{Z}} = \mathbf{I}$

Gaussian Random Vectors

Theorem 8.12

For a Gaussian $(\boldsymbol{\mu}_{\mathbf{X}}, \mathbf{C}_{\mathbf{X}})$ random vector, let $\mathbf{A}$ be an $n \times n$ matrix with $\mathbf{A}\mathbf{A}^T = \mathbf{C}_{\mathbf{X}}$. Then the random vector $\mathbf{Z} = \mathbf{A}^{-1}(\mathbf{X} - \boldsymbol{\mu}_{\mathbf{X}})$ is a standard normal random vector

Gaussian Random Vectors

Theorem 8.13

Given the n-dimensional standard normal random vector $\mathbf{Z}$, an invertible $n \times n$ matrix $\mathbf{A}$, and an n-dimensional vector $\mathbf{b}$,

$\mathbf{X} = \mathbf{AZ} + \mathbf{b}$ is an n-dimensional Gaussian random vector with $\boldsymbol{\mu}_{\mathbf{X}} = \mathbf{b}$ and $\mathbf{C}_{\mathbf{X}} = \mathbf{AA}^T$