Ch9. Sums of Random Variables

School of Electronics Engineering

Expected Values of Sums

Theorem 9.1

For any set of random variables $X_1, \ldots, X_n$, the sum $W_n = X_1 + \cdots + X_n$ has expected value

$$E[W_n] = E[X_1] + E[X_2] + \cdots + E[X_n]$$

Expected Values of Sums

Theorem 9.2

The variance of $W_n = X_1 + \cdots + X_n$ is

$$Var[W_n] = \sum_{i=1}^{n} Var[X_i] + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} Cov[X_i, X_j]$$

Expected Values of Sums

Theorem 9.3

When $X_1, \dots, X_n$ are uncorrelated,

$$Var[W_n] = Var[X_1] + \cdots + Var[X_n]$$

Expected Values of Sums

Example 9.1

$X_0, X_1, X_2, \ldots$ is a sequence of random variables with $E[X_i] = 0$ and $Cov[X_i, X_j] = 0.8^{|i-j|}$. Find the expected value and variance of a random variable $Y_i = X_i + X_{i-1} + X_{i-2}$

Expected Values of Sums

Example 9.2

At a party of $n \geq 2$ people, each throws a hat in a box and then blindly draws a hat from the box. Let V_n denote the number of people who draw their own hat. Find $E[V_n]$ and $Var[V_n]$.

Expected Values of Sums

Example 9.3

Continuing Example 9.2, suppose each person immediately returns to the box the hat that he/she draws. What is $E[V_n]$ and $Var[V_n]$?

Moment Generating Functions

Definition 9.1

For a random variable X, the moment generating function (MGF) of X is

$$\phi_X(s) = E[e^{sX}]$$

Moment Generating Functions

Theorem 9.4

A random variable X with MGF $\phi_X(s)$ has n-th moment

$$E[X^n] = \left.\frac{d^n \phi_X(s)}{ds^n}\right|_{s=0}$$

Random Variable	PMF or PDF	MGF $\phi_X(s)$
Bernoulli (p)	$P_X(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \\ 0 & \text{otherwise} \end{cases}$	$1-p+pe^s$
Binomial (n,p)	$P_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$	$(1-p+pe^s)^n$
Geometric (p)	$P_X(x) = \begin{cases} p(1-p)^{x-1} & x=1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{pe^s}{1-(1-p)e^s}$
Pascal (k,p)	$P_X(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}$	$\left(\dfrac{pe^s}{1-(1-p)e^s}\right)^k$
Poisson (α)	$P_X(x) = \begin{cases} \alpha^x e^{-\alpha}/x! & x=0,1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$	$e^{\alpha(e^s-1)}$
Disc. Uniform (k,l)	$P_X(x) = \begin{cases} \frac{1}{l-k+1} & x=k,\ldots,l \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{e^{sk}-e^{s(l+1)}}{1-e^s}$

Random Variable	PMF or PDF	MGF $\phi_X(s)$
Constant (a)	$f_X(x) = \delta(x-a)$	e^{sa}
Uniform (a,b)	$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{e^{bs}-e^{as}}{s(b-a)}$
Exponential (λ)	$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{\lambda}{\lambda - s}$
Erlang (n,λ)	$f_X(x) = \begin{cases} \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$	$\left(\dfrac{\lambda}{\lambda - s}\right)^n$
Gaussian (μ,σ)	$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$	$e^{s\mu + s^2\sigma^2/2}$

Moment Generating Functions

Example 9.4

X is an exponential random variable with MGF $\phi_X(s) = \lambda/(\lambda - s)$.

What are the first, second, and n-th moments of X?

Moment Generating Functions

Theorem 9.5

The MGF of $Y = aX + b$ is $\phi_Y(s) = e^{sb}\phi_X(as)$

Moment Generating Functions

Theorem 9.6

For a set of independent random variables $X_1, \ldots, X_n$, the MGF of $W = X_1 + \cdots + X_n$ is

$$\phi_W(s) = \phi_{X_1}(s)\phi_{X_2}(s) \ldots \phi_{X_n}(s)$$

Especially when $X_1, \ldots, X_n$ are *i.i.d.*, $\phi_W(s) = [\phi_X(s)]^n$

Moment Generating Functions

Theorem 9.7

If $K_1, \ldots, K_n$ are independent Poisson random variables, $W = K_1 + \cdots + K_n$ is a Poisson random variable

Moment Generating Functions

Theorem 9.8

The sum of n independent Gaussian random variables $W = X_1 + \cdots + X_n$ is a Gaussian random variable

Moment Generating Functions

Theorem 9.9

If $X_1, \dots, X_n$ are *i.i.d.* exponential (λ) random variables, then $W = X_1 + \cdots + X_n$ has Erlang PDF

$$f_W(w) = \begin{cases} \dfrac{\lambda^n w^{n-1} e^{-\lambda w}}{(n-1)!}, & w \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Central Limit Theorem

Theorem 9.10

Given $X_1, X_2, \ldots,$ a sequence of *i.i.d.* random variables with expected value μ_X and variance σ_X^2, the CDF of $Z_n = (\sum_{i=1}^{m} X_i - n\mu_X)/\sqrt{n\sigma_X^2}$ has the property

$$\lim_{n\to\infty} F_{Z_n}(z) = \Phi(z)$$

Central Limit Theorem

Definition 9.2

Let $W_n = X_1 + \cdots + X_n$ be the sum of n *i.i.d.* random variables, each with $E[X] = \mu_X$ and $Var[X] = \sigma_X^2$. The central limit theorem approximation to the CDF of W_n is

$$F_W(w) = \Phi\left(\frac{w - n\mu_X}{\sqrt{n\sigma_X^2}}\right)$$

Central Limit Theorem

Example 9.6

Consider a sequence of *i.i.d.* random variables X_i, where each random variable is uniform $(0, 1)$. Let $W_n = X_1 + \cdots + X_n$.

Central Limit Theorem

Example 9.7

Suppose $W_n = X_1 + \cdots + X_n$ is a sum of independent Bernoulli (p) random variables. We know that W_n has the binomial PMF

$$P_{W_n}(w) = \binom{n}{w} p^w (1-p)^{n-w}$$

Central Limit Theorem

Example 9.9

Transmit one million bits. Let A denote the event that there are at least 499,000 ones but no more than 501,000 ones. Find $P[A]$