Ch3. Discrete Random Variables

School of Electronics Engineering

Random Variables

Random Variables

Definition 3.1

A random variable consists of an experiment with a probability measure $P[\cdot]$ defined on a sample space S and a function that *assigns a real number* to each outcome in S

Discrete Random Variable

Definition 3.2

X is a *discrete* random variable if the range of X is a countable set

$$S_X = \{x_1, x_2, \ldots\}$$

Discrete Random Variable

Example 3.1

The experiment is to attach a photo detector to an optical fiber and count the number of photons arriving in a one-microsecond time interval. Each observation is a random variable X

Probability Mass Function

Definition 3.3

The probability mass function (PMF) of the discrete random variable X is

$$P_X(x) = P[X = x]$$

Probability Mass Function

Example 3.5

When a basketball player shot *two free throws*, each shot is equally likely either to be good (g) or bad (b). Each shot that was good was worth 1 point. What is the PMF of X, the number of points that he scored?

Probability Mass Function

Theorem 3.1

For a discrete random variable X with PMF $P_X(x)$ and range S_X:

(a) For any x, $P_X(x) \geq 0$

(b) $\sum_{x \in S_X} P_X(x) = 1$

(c) For any event $B \subset S_X$, the probability that X is in the set B is

Since X is a function working on each outcome of the sample space

$$P[B] = \sum_{x \in B} P_X(x)$$

Probability Mass Function

Quiz 3.2

The random variable N has PMF

$$P_N(n) = \begin{cases} \dfrac{c}{n}, & n = 1, 2, 3 \\ 0, & \text{otherwise} \end{cases}$$

Find (a) The value of c

(b) $P[N = 1]$

(c) $P[N \geq 2]$

(d) $P[N > 3]$

FAMILIES OF DISCRETE RANDOM VARIABLES

Bernoulli (p) Random Variable

Example 3.6

- Flip a coin and observe whether the side facing up is heads or tails.

- Observe one bit transmitted by a modem that is downloading a file from the Internet. Let X be the value of the bit (0 or 1)

Bernoulli (p) Random Variable

Definition 3.4

X is a Bernoulli (p) random variable if the PMF of X has the form

$$P_X(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \\ 0 & \text{otherwise} \end{cases}$$

Geometric (p) Random Variable

Example 3.8

In a sequence of independent tests of integrated circuit, each circuit is rejected with probability p. Let X equal the number of tests *up to the first rejection*. Find the PMF.

Geometric (p) Random Variable

Definition 3.5

X is a geometric (p) random variable if the PMF of X has the form

$$P_X(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Binomial (n, p) Random Variable

Example 3.9

In a sequence of n *independent tests of integrated circuits*, each circuit is rejected with probability p. Let X equal to the number of rejects in the n tests. Find the PMF.

Binomial (n, p) Random Variable

Definition 3.6

X is a binomial (n, p) random variable if the PMF of X has the form

$$P_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

where $0 < p < 1$ and n is an integer $(n \geq 1)$

Pascal (k, p) Random Variable

Example 3.10

Perform independent tests of integrated circuits in which each circuit is rejected with probability p. Observe L, the number of tests performed until there are k rejects. Find the PMF.

Pascal (k, p) Random Variable

Definition 3.7

X is a Pascal (k, p) random variable if the PMF of X has the form

$$P_X(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}$$

where $0 < p < 1$ and k is an integer $(k \geq 1)$

Discrete Uniform (k, l) Random Variable

Definition 3.8

X is a discrete uniform (k, l) random variable if the PMF of X has the form

$$P_X(x) = \begin{cases} 1/(l-k+1) & x = k, k+1, \ldots, l \\ 0 & \text{otherwise} \end{cases}$$

where the parameters k and l are integers $(k < l)$

X is uniformly distributed between k and l

Poisson (α) Random Variable

Definition 3.9

X is a Poisson (α) random variable if the PMF of X has the form

$$P_X(x) = \begin{cases} \dfrac{\alpha^x e^{-\alpha}}{x!} & x = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

where the parameters $\alpha > 0$

Poisson (α) Random Variable

Example 3.13

The number of hits at a website in any time interval is a Poisson random variable. A particular site has on average $\lambda = 2$ hits per s. What is the probability that there are no hits in an interval of 0.25 s?

Poisson (α) Random Variable

Example

The number of database queries processed by a computer in any 10-second interval is a Poisson random variable with $\alpha = 5$ queries. What is the probability that there will be no queries processed in a 10-second interval? What is the probability that at least two queries will be processed in a 2-second interval.

Poisson (α) Random Variable

Theorem 3.8 (in later chapter)

PMF of Poisson random variable can be derived using binomial random variable

Cumulative Distribution Function (CDF)

Cumulative Distribution Function

Definition 3.10

The *cumulative distribution function* (CDF) of random variable X is

$$F_X(x) = P[X \leq x]$$

Cumulative Distribution Function

Example 3.15

Random variable X has PMF $P_X(x)$. Find CDF of X

$$P_X(x) = \begin{cases} 0.25, & x = 0 \\ 0.5, & x = 1 \\ 0.25, & x = 2 \\ 0, & \text{otherwise} \end{cases}$$

Cumulative Distribution Function

Theorem 3.2

For any discrete random variable X with range $S_X = \{x_1, x_2, \dots\}$ satisfying $x_1 \le x_2 \le \cdots$

(a) $F_X(-\infty) = 0$ and $F_X(\infty) = 1$

(b) For all $x' \ge x$, $F_X(x') \ge F_X(x)$

(c) For $x_i \in S_X$ and ϵ, an arbitrarily small positive number,

$$F_X(x_i) - F_X(x_i - \epsilon) = P_X(x_i)$$

(d) $F_X(x) = F_X(x_i)$ for all x such that $x_i \le x < x_{i+1}$

Cumulative Distribution Function

Theorem 3.2

For any discrete random variable X with range $S_X = \{x_1, x_2, \ldots\}$ satisfying $x_1 \leq x_2 \leq \cdots$

(a) Starts at zero and ends at one

(b) The CDF never decreases as it goes from left to right

(c) For a discrete random X, there is a jump (discontinuity) at each value of $x_i \in S_X$ and the height of the jump at x_i is $P_X(x_i)$

(d) Between jumps, the graph of the CDF of the discrete random variable is a horizontal line.

Cumulative Distribution Function

Theorem 3.3

For all $b > a$,

$$F_X(b) - F_X(a) = P[a < X \leq b]$$

Average and Expected Value

Averages and Expected Value

Definition 3.11 (Expected value)

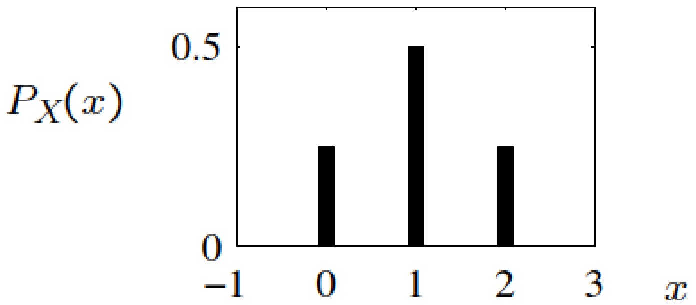
The expected value of X is

$$E[X] = \mu_X = \sum_{x \in S_X} x P_X(x)$$

Averages and Expected Value

Example 3.18

Find the expected value of X



Averages and Expected Value

Theorem 3.4

The Bernoulli (p) random variable X has expected value $E[X] = p$

Theorem 3.5

The geometric (p) random variable X has expected value $E[X] = \frac{1}{p}$

Averages and Expected Value

Theorem 3.6

The Poisson (α) random variable X has expected value $E[X] = \alpha$

Averages and Expected Value

Theorem 3.7

(a) For the binomial (n, p) random variable X, $E[X] = np$

(b) For the Pascal (k, p) random variable X, $E[X] = \frac{k}{p}$

(c) For the discrete uniform (k, l) random variable X, $E[X] = \frac{k+l}{2}$

Averages and Expected Value

Table of expected value

Random variable	Expected value
Bernoulli (p)	$E[X] = p$
Geometric (p)	$E[X] = \frac{1}{p}$
Binomial (n, p)	$E[X] = np$
Pascal (k, p)	$E[X] = \frac{k}{p}$
Poisson (α)	$E[X] = \alpha$
Discrete uniform (k, l)	$E[X] = \frac{(k+l)}{2}$

Functions of a Random Variable

Derived Random Variable

Definition 3.12

A function $Y = g(X)$ of random variable X is another random variable. Y is called *derived random variable*

Derived Random Variable

Theorem 3.9

For a discrete random variable X, the PMF of $Y = g(X)$ is

$$P_Y(y) = \sum_{x:g(x)=y} P_X(y)$$

Derived Random Variable

Example 3.20

Suppose all packages weigh 1, 2, 3, 4 pounds with equal probability, and a parcel shipping company offers 100, 190, 270, 340 cents for packages with weights 1, 2, 3, 4 pounds, respectively. What is the PMF of expected cost Y?

Further consider the case when the company offers 100 cents for 1,2 pounds and 200 cents for 3,4 pounds.

Expected Value of a Derived Random Variables

Theorem 3.10

Given a random variable X with PMF $P_X(x)$ and the derived random variable $Y = g(X)$, the expected value of Y is

$$E[Y] = \mu_Y = \sum_{x \in S_X} g(x) P_X(x)$$

Expected Value of a Derived Random Variables

Theorem 3.11

For any random variable X, $E[X - \mu_X] = 0$

Theorem 3.12

For any random variable X, $E[aX + b] = aE[X] + b$

VARIANCE AND STANDARD DEVIATION

Variance and Standard Deviation

Definition 3.13

The variance of random variable X is

$$Var[X] = E[(X - \mu_X)^2]$$

Definition 3.14

The standard deviation of random variable X is

$$\sigma_X = \sqrt{Var[X]}$$

Variance and Standard Deviation

Theorem 3.14

$$Var[X] = E[X^2] - (E[X])^2$$

Simply derived using the linearity.

Variance and Standard Deviation

Definition 3.15

For random variable X:

(a) The n-th *moment* is $E[X^n]$

(b) The n-th *central moment* is $E[(X-\mu_X)^n]$

Variance and Standard Deviation

Theorem 3.15

$$Var[aX + b] = a^2 Var[X]$$

Variance and Standard Deviation

Table of variance

Random variable	Variance
Bernoulli (p)	$Var[X] = p(1-p)$
Geometric (p)	$Var[X] = \frac{1-p}{p^2}$
Binomial (n, p)	$Var[X] = np(1-p)$
Pascal (k, p)	$Var[X] = \frac{k(1-p)}{p^2}$
Poisson (α)	$Var[X] = \alpha$
Discrete uniform (k, l)	$Var[X] = \frac{(l-k)(l-k+2)}{12}$