Ch2. Sequential Experiments

School of Electronics Engineering

TREE DIAGRAMS

Tree Diagram

Tree diagram

Display the outcomes of the sub-experiments sequentially

The branches are (conditional) probabilities

Example: flip a coin and roll a six-sided die

Example 2.3

Suppose you have two coins, one biased, one fair, but you don't know which coin is which. Coin 1 is biased. It comes up heads with probability $3/4$, while coin 2 comes up heads with probability $1/2$. Suppose you pick a coin at random and flip it. Let C_i denote the event that coin i is picked. Let H and T denote the possible outcomes of the flip. Given that the outcome of the flip is a head, what is $\mathrm{P}[C_1|H]$, the probability that you picked the biased coin? Given that the outcome is a tail, what is the probability $\mathrm{P}[C_1|T]$ that you picked the biased coin?

COUNTING METHODS

Sampling without Replacement

Example 2.4

Choose 7 cards at random from a deck of 52 cards. Display the cards *in the order* in which you choose them. How many different sequences of cards are possible?

Sampling without Replacement

Theorem 2.1 (Fundamental principle of counting)

An experiment consists of two sub-experiments. If one has k outcomes and the other has n outcomes, then the experiment has kn outcomes

Sampling without Replacement

Example 2.5

Two sub-experiments: (1) flip a coin and observe either heads of tails and (2) roll a six-sided die and observe the number of spots. How many outcomes?

Sampling without Replacement

Theorem 2.2 (Permutation)

The number of *k-permutation* of n distinguishable object is

$$(n)_k = n(n-1)\ldots(n-k+1) = \frac{n!}{(n-k)!}$$

Sampling without Replacement

Theorem 2.3 (Combination)

The number of ways to choose k objects out of n distinguishable objects is

$$\binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{(n-k)!\,k!}$$

Sampling without Replacement

Definition 2.1 (n choose k)

For an integer $n \geq 0$, we define

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!\,(n-k)!} & k = 0, 1, \dots, n \\ 0 & \text{otherwise} \end{cases}$$

Note. $\binom{n}{k}$ is called a *binomial* coefficient

Sampling Types

Terminology

Sampling: choosing objects randomly from a collection

Sample: the chosen objects

Sampling *without* replacement

- ✓ Selected samples are removed from the collection
- ✓ Examples: permutation, combination

Sampling *with* replacement

- ✓ Selected samples can be selected again

Sampling with Replacement

Example

A laptop has two USB slots *A* and *B.* Each slot can be used for connecting a memory card (m), a camera (c) or a printer (p). How many ways can we use the USB slots? (there are plenty of m, c, p)

Sampling with Replacement

Theorem 2.4

Given m distinguishable objects, there are m^n ways to choose with replacement an ordered sample of n objects

Theorem 2.5

For n repetitions of a sub-experiment with sample space $S_{sub} = \{s_0, \ldots, s_{m-1}\}$, the sample space S of the sequential experiment has m^n outcomes

Sampling with Replacement

Example 2.11

There are 2^{10} binary sequences of length 10

Example

The letters A through Z can produce 26^4 four-letter words

Binomial and Multinomial Coefficient

Example 2.14

For five sub-experiments with sample space $S_{sub} = \{0, 1\}$, what is the number of observation sequences in which 0 appears 2 times and 1 appears 3 times?

Binomial and Multinomial Coefficient

Theorem 2.6

The number of observation sequence for n sub-experiments with sample space $S = \{0, 1\}$ with 0 appearing n_0 times and 1 appearing $n_1 = n - n_0$ times is $\binom{n}{n_0}$

Binomial and Multinomial Coefficient

Theorem 2.7

For n repetitions of a sub-experiment with sample space $S = \{s_0, \ldots, s_{m-1}\}$, the number of length $n = n_0 + \cdots + n_{m-1}$ observation sequence with s_i appearing n_i times is

$$\binom{n}{n_0, \ldots, n_{m-1}} = \frac{n!}{n_0!\, n_1! \ \ldots n_{m-1}!}$$

Binomial and Multinomial Coefficient

Binomial and Multinomial Coefficient

Definition 2.2 (Multinomial coefficient)

For an integer $n \geq 0$, we define

$$\binom{n}{n_0, \dots, n_{m-1}} = \begin{cases} \dfrac{n!}{n_0!\, n_1! \ \dots n_{m-1}!} & n = n_0 + \cdots + n_{m-1} \\ 0 & \text{otherwise} \end{cases}$$

Independent Trials

Independent Trials

Independent trials

A sequential experiments with identical sub-experiments

Sampling with replacement is an example of independent trials

Independent Trials

Example 2.16

What is the probability $P[E_{2,3}]$ of *two failures* and *three successes* in five independent trials with success probability p

Independent Trials

Theorem 2.8

The probability of n_0 failures and n_1 successes in $n = n_0 + n_1$ independent trials is

$$P\left[E_{n_0,n_1}\right] = \binom{n}{n_1}(1-p)^{n-n_1}p_1^{n} = \binom{n}{n_0}(1-p)^{n_0}p_1^{n-n_0}$$

Independent Trials

Theorem 2.9

A sub-experiment has sample space $S_{sub} = \{s_0, \ldots, s_{m-1}\}$ with $P[s_i] = p_i$. For $n = n_0 + \cdots + n_{m-1}$ independent trials, the probability of n_i occurrences of s_i is

$$P\left[E_{n_0,\ldots,n_{m-1}}\right] = \binom{n}{n_0, \ldots, n_{m-1}} p_0^{n_0} \ldots p_{m-1}^{n_{m-1}}$$

Independent Trials

Example 2.18

An internet packet carries audio, video, and text with probabilities 0.7, 0.2, and 0.1, respectively. Let $E_{a,v,t}$ denote the event that the router processes a audio, v video, and t text packets in a sequence of 100 packets.