Ch5. Multiple Random Variables

School of Electronics Engineering

MULTIPLE RANDOM VARIABLES

Joint Cumulative Density Function (CDF)

Definition 5.1

The joint CDF of random variables X and Y is

$$F_{X,Y}(x, y) = P[X \leq x, Y \leq y]$$

Joint Cumulative Density Function (CDF)

Theorem 5.1

For any pair of random variables X, Y

(a) $0 \leq F_{X,Y}(x, y) \leq 1$

(b) $F_{X,Y}(\infty, \infty) = 1$

(c) $F_X(x) = F_{X,Y}(x, \infty)$

(d) $F_Y(x) = F_{X,Y}(\infty, y)$

(e) $F_{X,Y}(x, -\infty) = 0$

(f) $F_{X,Y}(-\infty, y) = 0$

(g) If $x \leq x_1$ and $y \leq y_1$, $F_{X,Y}(x, y) \leq F_{X,Y}(x_1, y_1)$

Joint Cumulative Density Function (CDF)

Theorem 5.2

$$P[x_1 < X \leq x_2,\ y_1 < Y \leq y_2] = F_{X,Y}(x_2, y_2) - F_{X,Y}(x_2, y_1)$$

$$-F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1)$$

Joint Cumulative Density Function (CDF)

Quiz 5.1

Express the following extreme values of the joint CDF $F_{X,Y}(x, y)$ as numbers or in terms of the CDFs $F_X(x)$ and $F_Y(y)$

(a) $F_{X,Y}(-\infty, 2)$

(b) $F_{X,Y}(\infty, \infty)$

(c) $F_{X,Y}(\infty, y)$

(d) $F_{X,Y}(\infty, -\infty)$

MULTIPLE DISCRETE RANDOM VARIABLES

Joint Probability Mass Function (PMF)

Definition 5.2

The joint PMF of discrete random variables X and Y is

$$P_{X,Y}(x, y) = P[X = x, Y = y]$$

Joint Probability Mass Function (PMF)

Example 5.3

Test two integrated circuits. The outcomes of each test are a (accepted) with probability 0.9 and r (rejected). Let X be the number of acceptable circuits and Y be the number of successful tests before you observe the first reject. Find the joint PMF.

Joint Probability Mass Function (PMF)

Theorem 5.3

For discrete random variables X and Y and any set B in the X, Y plane, the probability of the event $\{(X, Y) \in B\}$ is

$$P[B] = \sum_{(x,y)\in B} P_{X,Y}(x, y)$$

Joint Probability Mass Function (PMF)

Example 5.4

Continuing Example 5.3, find the probability of the event B, which is $X = Y$.

Joint Probability Mass Function (PMF)

Quiz 5.2

The joint PMF $P_{Q,G}(q,g)$ for random variables Q and G is given as

$P_{Q,G}(q,g)$	$g=0$	$g=1$	$g=2$	$g=3$
$q=0$	0.06	0.18	0.24	0.12
$q=1$	0.04	0.12	0.16	0.08

Calculate,

(a) $P[Q=0]$

(b) $P[Q=G]$

(c) $P[G>1]$

(d) $P[G>Q]$

Marginal Probability Mass Function (PMF)

Theorem 5.4

For discrete random variables X and Y with joint PMF $P_{X,Y}(x,y)$

$$P_X(x) = \sum_{y \in S_Y} P_{X,Y}(x,y), \qquad P_Y(y) = \sum_{x \in S_X} P_{X,Y}(x,y)$$

Marginal Probability Mass Function (PMF)

Example 5.5

Find the marginal PMFs for the random variables X and Y

$P_{X,Y}(x,y)$	$y=0$	$y=1$	$y=2$
$x=0$	0.01	0	0
$x=1$	0.09	0.09	0
$x=2$	0	0	0.81

Multiple Continuous Random Variables

Joint Probability Density Function (PDF)

Definition 5.3

The joint PDF of the continuous random variables X and Y is a function $f_{X,Y}(x, y)$ with the property

$$F_{X,Y}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u, v)\, dv\, du$$

Joint Probability Density Function (PDF)

Theorem 5.5

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \, \partial y}$$

Joint Probability Density Function (PDF)

Theorem 5.6

A joint PDF $f_{X,Y}(x, y)$ has the following properties

(a) $f_{X,Y}(x, y) \geq 0$ for all (x, y)

(b) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx\, dy = 1$

Joint Probability Density Function (PDF)

Theorem 5.7

The probability that the continuous random variables (X, Y) are in A is

$$P[A] = \iint_A f_{X,Y}(x, y)\, dx\, dy$$

Joint Probability Density Function (PDF)

Example 5.7

Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c, & 0 \le x \le 5, 0 \le y \le 3 \\ 0, & \text{otherwise} \end{cases}$$

Find the constant c and $P[A] = P[2 \le X < 3, 1 \le Y < 3]$

Joint Probability Density Function (PDF)

Example 5.8

Find the joint CDF $F_{X,Y}(x, y)$ when X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2, & 0 \le y \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Joint Probability Density Function (PDF)

Quiz 5.4

The joint PDF of random variables X and Y is

$$f_{X,Y}(x,y) = \begin{cases} cxy, & 0 \le x \le 1, 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the constant c and probability of the event $A = X^2 + Y^2 \le 1$

Marginal Probability Density Function (PDF)

Theorem 5.8

If X and Y are random variables with joint PDF $f_{X,Y}(x,y)$,

$$F_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,, \qquad F_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

Marginal Probability Density Function (PDF)

Quiz 5.5

The joint PDF of random variables X and Y is

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{6(x+y^2)}{5}, & 0 \le x < 1, 0 \le y \le 1 \\ 0 \quad , & \text{otherwise} \end{cases}$$

Find marginal PDFs

Independent Random Variables

Independent Random Variables

Definition 5.4

Random variables X and Y are independent if and only if

Discrete: $P_{X,Y}(x, y) = P_X(x)P_Y(y)$

Continuous: $f_{X,Y}(x, y) = f_X(x)f_Y(y)$

Independent Random Variables

Example 5.12

$$f_{X,Y}(x,y) = \begin{cases} 4xy, & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Are X and Y independent?

Independent Random Variables

Example 5.13

$$f_{U,V}(u,v) = \begin{cases} 24uv, & u \geq 0, v \geq 0, u + v \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Are U and V independent?

Expected Value, Covariance, and Correlation

Expected Value

Theorem 5.9

For random variables X and Y, the expected value of $W = g(X, Y)$ is

Discrete:

$$E[W] = \sum_{x \in S_X} \sum_{y \in S_Y} g(x, y) P_{X,Y}(x, y)$$

Continuous:

$$E[W] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dx\, dy$$

Expected Value

Theorem 5.10

$$E[a_1 g_1(X,Y) + \cdots + a_n g_n(X,Y)]$$
$$= a_1 E[g_1(X,Y)] + \cdots + a_n E[g_n(X,Y)]$$

Expected Value

Theorem 5.11

For any two random variables X and Y

$$E[X+Y] = E[X] + E[Y]$$

Variance

Theorem 5.12

The variance of the sum of two random variables is

$$Var[X+Y] = Var[X] + Var[Y] + 2E[(X-\mu_X)(Y-\mu_Y)]$$

Covariance

Definition 5.5

The covariance of two random variables X and Y is

$$Cov[X,Y] = E[(X-\mu_X)(Y-\mu_Y)]$$

Correlation Coefficient

Definition 5.6

The correlation coefficient of two random variables X and Y is

$$\rho_{X,Y} = \frac{Cov[X,Y]}{\sqrt{Var[X]\,Var[Y]}} = \frac{Cov[X,Y]}{\sigma_X\,\sigma_Y}$$

Correlation Coefficient

Theorem 5.13

If $\hat{X} = aX + b$ and $\hat{Y} = cY + d$,

(a) $\rho_{\hat{X},\hat{Y}} = \rho_{X,Y}$

(b) $Cov[\hat{X}, \hat{Y}] = acCov[X, Y]$

Correlation Coefficient

Theorem 5.14

$$-1 \le \rho_{X,Y} \le 1$$

Correlation Coefficient

Theorem 5.15

If X and Y are random variables such that $Y = aX + b$

$$\rho_{X,Y} = \begin{cases} -1, & a < 0 \\ 0, & a = 0 \\ 1, & a > 0 \end{cases}$$

Correlation

Definition 5.7

The correlation of X and Y is $r_{X,Y} = E[XY]$

Correlation

Theorem 5.16

(a) $Cov[X,Y] = r_{X,Y} - \mu_X \mu_Y$

(b) $Var[X+Y] = Var[X] + Var[Y] + 2Cov[X,Y]$

(c) If $X = Y$, $Cov[X,Y] = Var[X] = Var[Y]$ and $r_{X,Y} = E[X^2] = E[Y^2]$

Correlation

Example 5.17

Continuing Example 5.5, find $r_{X,Y}$ and $Cov[X,Y]$

$P_{X,Y}(x,y)$	$y=0$	$y=1$	$y=2$
$x=0$	0.01	0	0
$x=1$	0.09	0.09	0
$x=2$	0	0	0.81

Orthogonal and Uncorrelated

Definition 5.8

Random variables X and Y are orthogonal if $r_{X,Y} = 0$

Definition 5.9

Random variables X and Y are uncorrelated if $Cov[X, Y] = 0$

Multivariate Probability Models

Multivariate Probability Models

Definition 5.11

The joint CDF of $X_1, X_2, \dots, X_n$ is

$$F_{X_1,X_2,\dots,X_n}(x_1, x_2, \dots, x_n) = P[X_1 \le x_1, X_2 \le x_2, \dots, X_n \le x_n]$$

Definition 5.12

The joint PMF of discreate random variables $X_1, X_2, \dots, X_n$

$$P_{X_1,X_2,\dots,X_n}(x_1, x_2, \dots, x_n) = P[X_1 = x_1, \dots, X_n = x_n]$$

Multivariate Probability Models

Definition 5.13

The joint PDF of continuous random variables $X_1, X_2, \dots, X_n$

$$f_{X_1,X_2,\dots,X_n}(x_1, x_2, \dots, x_n) = \frac{\partial^n F_{X_1,X_2,\dots,X_n}(x_1, x_2, \dots, x_n)}{\partial x_1 \dots \partial x_n}$$

Multivariate Probability Models

Theorem 5.24

The probability of an event A expressed in terms of the random variables $X_1, \ldots, X_n$ is

Discrete: $$P[A] = \sum_{(x_1, x_2, \ldots, x_n) \in A} P_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$$

Continuous: $$P[A] = \int \ldots \int_A f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) dx_1\, dx_2\, \ldots, dx_n$$

Multivariate Probability Models

Theorem 5.25

For a joint PMF $P_{W,X,Y,Z}(w,x,y,z)$ of discrete random variables W,X,Y,Z, some marginal PMFs are

$$P_{X,Y,Z}(x,y,z) = \sum_{w \in S_W} P_{W,X,Y,Z}(w,x,y,z)$$

$$P_{W,Z}(w,z) = \sum_{x \in S_X} \sum_{y \in S_Y} P_{W,X,Y,Z}(w,x,y,z)$$

Multivariate Probability Models

Theorem 5.26

For a joint PDF $f_{W,X,Y,Z}(w,x,y,z)$ of continuous random variables W,X,Y,Z, some marginal PDFs are

$$f_{X,Y,Z}(x,y,z) = \int_{-\infty}^{\infty} f_{W,X,Y,Z}(w,x,y,z)\,dw$$

$$f_X(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{W,X,Y,Z}(w,x,y,z)dw\,dy\,dz$$

Multivariate Probability Models

Definition 5.14

Random variables $X_1, \ldots, X_n$ are independent if

Discrete: $$P_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = P_{X_1}(x_1)P_{X_2}(x_2) \ldots P_{X_n}(x_n)$$

Continuous: $$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = f_{X_1}(x_1)f_{X_2}(x_2) \ldots f_{X_n}(x_n)$$

Definition 5.15

Random variable $X_1, \ldots, X_n$ are independent and identically distributed (i.i.d) if

Discrete: $$P_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = P_X(x_1)P_X(x_2) \ldots P_X(x_n)$$

Continuous: $$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = f_X(x_1)f_X(x_2) \ldots f_X(x_n)$$