Ch6. Probability Models of Derived Random Variables

School of Electronics Engineering

Introduction

Examples of derived random variables

- When voltage across an r_0 ohm resister is a random variable X, the power $Y = X^2/r_0$ is a random variable
- If X is the amplitude of signal and Y is the amplitude of noise, $W = X/Y$ is a random variable
- For two random variables X and Y, $W = X + Y$ is a random variable
- For two random variables X and Y, $W = \max(X, Y)$ is a random variable

Function of Two Discrete Random Variables

Theorem 6.1

For discrete random variables X and Y, the derived random variable $W = g(X, Y)$ has PMF

$$P_W(w) = \sum_{(x,y):g(x,y)=w} P_{X,Y}(x, y)$$

Function of Two Discrete Random Variables

Example 6.1

Two types of newsletters; (a) 40 cents per page, (b) 60 cents per page. R.V. L is the length of news letter and R.V. X is the cost per page. Find range and PMF of R.V. $W = g(L, X) = LX$.

$P_{L,X}(l,x)$	$x = 40$	$x = 60$
$l = 1$	0.15	0.1
$l = 2$	0.3	0.2
$l = 3$	0.15	0.1

R.V. = random variable

Function of Continuous Random Variables

Theorem 6.2

If $W = aX$, where $a > 0$, then W has CDF and PDF

$$F_W(w) = F_X\left(\frac{w}{a}\right), \qquad f_W(w) = \frac{1}{a} f_X\left(\frac{w}{a}\right)$$

Function of Continuous Random Variables

Example 6.3

The triangular PDF of X is

$$f_X(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the PDF of $W = aX$ and sketch the PDF for $a = 0.5, 1, 2$

Function of Continuous Random Variables

Theorem 6.3

$W = aX$, where $a > 0$

(a) If X is uniform (b, c), then W is uniform (ab, ac)

(b) If X is exponential (λ), then W is exponential (λ/a)

(c) If X is Erlang (n, λ), then W is Erlang $(n, \lambda/a)$

(d) If X is Gaussian (μ, σ), then W is Gaussian $(a\mu, a\sigma)$

Function of Continuous Random Variables

Theorem 6.4

If $W = \mathrm{X} + \mathrm{b}$,

$$F_W(w) = F_X(w-b), \qquad f_W(w) = f_X(w-b)$$

Function of Continuous Random Variables

Example 6.4

Suppose X is the continuous uniform (-1, 1) random variable.

Find the CDF and PDF of $W = X^2$

Function of Continuous Random Variables

Theorem 6.5

Let U be a uniform (0, 1) random variable, $F(x)$ be a cumulative distribution function with an inverse $F^{-1}(u)$ defined for $0 < u < 1$. The random variable $X = F^{-1}(U)$ has CDF $F_X(x) = F(x)$

Function of Continuous Random Variables

Example 6.5

U is the uniform (0, 1) random variable and $X = g(U)$. Derive $g(U)$ such that X is the exponential (1) random variable

Function of Continuous Random Variables

Example 6.6

For a uniform (0, 1) random variable U, find a function $g(\cdot)$ such that $X = g(U)$ has a uniform (a, b) distribution

Function of Discrete/Mixed Random Variables

Example 6.7

Let X be a random variable with CDF $F_X(x)$. Let Y be the output of a clipping circuit with the characteristic $Y = g(X)$ where

$$g(x) = \begin{cases} 1, & x \leq 0 \\ 3, & x > 0 \end{cases}$$

Express $F_Y(y)$ and $f_Y(y)$ in terms of $F_X(x)$ and $f_X(x)$

Function of Discrete/Mixed Random Variables

Example 6.8

V is Gaussian (0, 5) random variable. Find the CDF and PDF of random variable W, which is derived as

$$W = g(V) = \begin{cases} -10, & V \leq -10 \\ 10, & V \geq 10 \\ V, & \text{otherwise} \end{cases}$$

Function of Two Continuous Random Variables

Theorem 6.6

For continuous random variables X, Y, the CDF of $W = g(X, Y)$ is

$$F_W(w) = P[W \leq w] = \iint_{g(x,y)\leq w} f_{X,Y}(x, y)dx\,dy$$

Function of Two Continuous Random Variables

Theorem 6.7

For continuous R.V. X and Y, the CDF of $W = \max(X, Y)$ is

$$F_W(w) = F_{X,Y}(w, w) = \int_{-\infty}^{w} \int_{-\infty}^{w} f_{X,Y}(x, y)\, dx\, dy$$

Function of Two Continuous Random Variables

Example 6.10

X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \lambda\mu e^{-(\lambda x+\mu y)}, & x \geq 0, y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the PDF of $W = Y/X$

Sum of Two Random Variables

Theorem 6.8

The PDF of $W = X + Y$ is

$$f_W(w) = \int_{-\infty}^{\infty} f_{X,Y}(x, w - x)\, dx = \int_{-\infty}^{\infty} f_{X,Y}(w - y, y)\, dy$$

Sum of Two Random Variables

Example 6.11

Find the PDF of $W = X + Y$ when X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2, & 0 \le y \le 1, 0 \le x \le 1, x + y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Sum of Two Random Variables

Theorem 6.9

When X and Y are independent random variables, the PDF of $W = X + Y$ is

$$f_W(w) = \int_{-\infty}^{\infty} f_X(w-y) f_Y(y)\, dy = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\, dx$$