Ch13. Stochastic Process

School of Electronics Engineering

Stochastic Process

Definition 13.1

A stochastic process $X(t)$ consists of an experiment with a probability measure $P[\cdot]$ defined on a sample space S and a function that assigns a time function $x(t, s)$ to each outcome s in the sample space

Stochastic Process

Definition 13.2

A sample function $x(t, s)$ is the time function associated with outcome s of an experiment

Definition 13.3

The ensemble of a stochastic process is the set of all possible time functions

Stochastic Process

Averages

Ensemble average

- ✓ For given $t = t_0$, $X(t_0)$ is a random variable
- ✓ Expected value of $X(t_0)$ is called ensemble average

Time average

- ✓ For a sample s_0, sample function $x(t, s_0)$ is a time function
- ✓ Time average refers to the average of $x(t, s_0)$ for all possible t

Stochastic Process

Example 13.1

Starting at launch time $t = 0$, let $X(t)$ denote the temperature on the surface of a space shuttle. A temperature sequence $x(t, s)$ with each launch s can be modeled as a stochastic process.

Stochastic Process

Example 13.3

Starting on Jan. 1st, we measure the noontime temperature at Newark Airport every day for one year. This experiment generates a sequence, $C(1), C(2), \dots, C(365)$, which can be considered as a stochastic process.

Stochastic Process

Example 13.5

Suppose that at time instants $T = 0, 1, 2, \dots,$ we roll a die and record the outcome N_T where $1 \leq N_T \leq 6$. We then define the random process $X(t)$ such that for $T \leq t < T + 1$, $X(t) = N_T$

Stochastic Process

Definition 13.4

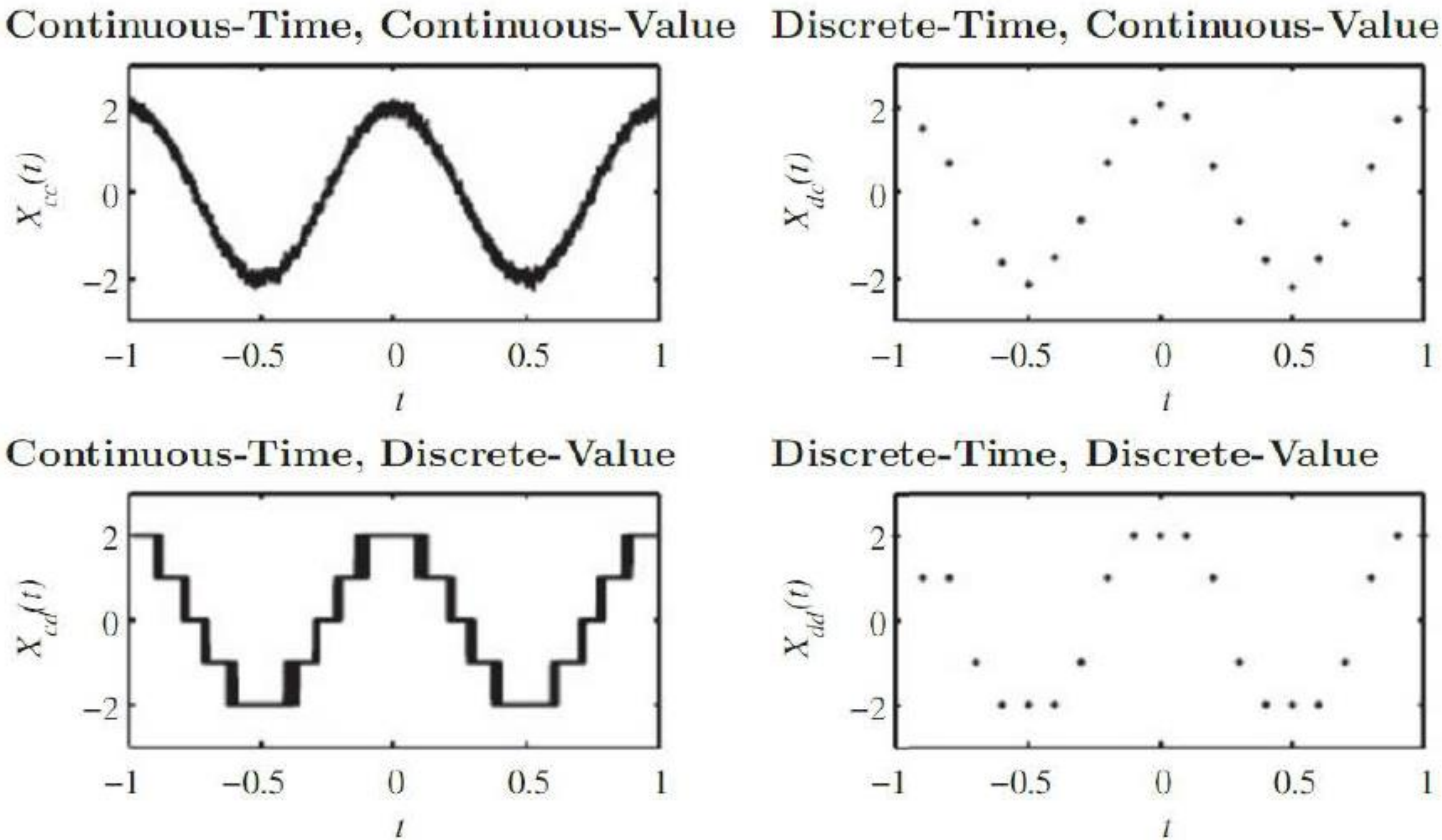
$X(t)$ is a discrete-value process if the set of all possible values of $X(t)$ at all times t is a countable set S_X; otherwise $X(t)$ is a continuous-value process

Stochastic Process

Definition 13.5

$X(t)$ is a discrete-time process if $X(t)$ is defined only for a set of time instants, $t_n = nT$, where T is a constant and n is an integer; otherwise $X(t)$ is a continuous-time process

Stochastic Process



Stochastic Process

Definition 13.6

A random sequence X_n is an ordered sequence of random variables $X_0, X_1, \ldots$

Random Variables from Random Processes

Example 13.7

In example 13.5 of repeatedly rolling a die, what is the PMF of $X(3.5)$?

Random Variables from Random Processes

Example 13.8

Let $X(t) = R|\cos 2\pi f t|$ be a rectified cosine signal having a random amplitude R with the exponential $(\lambda = 10)$ PDF. What is the PDF of $X(t)$?

Random Sequences

Theorem 13.1

Let X_n denote an *i.i.d.* random sequence. For a discrete-value process, the sample vector $\mathbf{X} = \left[X_{n_1}, \ldots, X_{n_k}\right]^T$ has joint PMF

$$P_{\mathbf{X}}(\mathbf{x}) = P_X(x_1)P_X(x_2)\ldots P_X(x_k) = \prod_{i=1}^{k} P_X(x_i)$$

Random Sequences

Definition 13.7 (Bernoulli process)

A Bernoulli (p) process X_n is an *i.i.d.* random sequence in which each X_n is a Bernoulli (p) random variable

Random Sequences

Example 13.10

In a common model for communications, the output $X_1, X_2, \ldots$ of a binary source is modeled as Bernoulli ($p = 1/2$) process. Find the joint PMF of $\mathbf{X} = [X_1, \ldots, X_n]^T$.

Expected Value and Correlation

Definition 13.11

The expected value of a stochastic process $X(t)$ is the deterministic function

$$\mu_X(t) = E[X(t)]$$

Expected Value and Correlation

Definition 13.12

The autocovariance function of the stochastic process $X(t)$ is

$$C_X(t,\tau) = Cov[X(t), X(t+\tau)]$$

Similarly, the autocovariance function of random sequence X_n is

$$C_X[m,k] = Cov[X_m, X_{m+k}]$$

Expected Value and Correlation

Definition 13.13

The autocorrelation function of the stochastic process $X(t)$ is

$$R_X(t,\tau) = E[X(t)X(t+\tau)]$$

Expected Value and Correlation

Theorem 13.9

The autocorrelation and autocovariance function of a process $X(t)$ satisfy

$$C_X(t,\tau) = R_X(t,\tau) - \mu_X(t)\mu_X(t+\tau)$$

Stationary Process

Definition 13.14

A stochastic process $X(t)$ is stationary if and only if for all sets of time instants $t_1, \dots, t_m$ and any time difference τ

$$f_{X(t_1),\dots,X(t_m)}(x_1, \dots, x_m) = f_{X(t_1+\tau),\dots,X(t_m+\tau)}(x_1, \dots, x_m)$$

Stationary Process

Theorem 13.10

For a stationary process $X(t)$ and constants $a > 0$ and b, $Y(t) = aX(t) + b$ is also a stationary process

Stationary Process

Theorem 13.11

A stationary process $X(t)$ have the following properties for all t:

(a) $\mu_X(t) = \mu_X$

(b) $R_X(t,\tau) = R_X(0,\tau) = R_X(\tau)$

(c) $C_X(t,\tau) = R_X(\tau) - \mu_X^2 = C_X(\tau)$

Stationary Process

Example 13.16

At the receiver of an AM radio, the received signal contains a cosine carrier signal at the carrier frequency f_c with a random phase Θ that is a sample value of the uniform $(0, 2\pi)$ random variable. The received carrier signal is

$$X(t) = A\cos(2\pi f_c t + \Theta)$$

What are the expected value and autocorrelation of the process $X(t)$?

Stationary Process

Example 13.16 (cont.)

Wide Sense Stationary Stochastic Process

Definition 13.15

$X(t)$ is a wide sense stationary stochastic process if and only if for all t,

$$E[X(t)] = \mu_X, \qquad R_X(t,\tau) = R_X(0,\tau) = R_X(\tau)$$

Wide Sense Stationary Stochastic Process

Example 13.17

From example 13.16, $X(t)$ is a wide sense stationary process.

Wide Sense Stationary Stochastic Process

Theorem 13.12

For a wide sense stationary process $X(t)$,

$$R_X(0) \geq 0, \qquad R_X(\tau) = R_X(-\tau), \qquad R_X(0) \geq |R_X(\tau)|$$

Wide Sense Stationary Stochastic Process

Definition 13.16

The average power of a wide sense stationary process $X(t)$ is $E[X^2(t)] = R_X(0)$

Cross-Correlation

Definition 13.17

The cross-correlation of continuous-time random processes $X(t)$ and $Y(t)$ is

$$R_{XY}(t,\tau) = E[X(t)Y(t+\tau)]$$

Cross-Correlation

Definition 13.18

Continuous-time random processes $X(t), Y(t)$ are jointly wide sense stationary if $X(t)$ and $Y(t)$ are both wide sense stationary, and the cross-correlation depends only on the time difference as

$$R_{XY}(t,\tau) = R_{XY}(\tau)$$

Cross-Correlation

Example 13.18

Suppose we are interested in $X(t)$ but we can observe

$$Y(t) = X(t) + N(t),$$

where $N(t)$ is a noise process. Assume $X(t)$ and $N(t)$ are independent wide sense stationary processes with $E[X(t)] = \mu_X$ and $E[N(t)] = 0$.

(a) Is $Y(t)$ wide sense stationary?

(b) Are $X(t)$ and $Y(t)$ jointly wide sense stationary?

(c) Are $Y(t)$ and $N(t)$ jointly wide sense stationary?

Cross-Correlation

Example 13.18 (cont.)

Cross-Correlation

Example 13.19

X_n is a wide sense stationary random sequence with autocorrelation function $R_X[k]$. Consider $Y_n = (-1)^n X_n$.

(a) Express autocorrelation function of Y_n in terms of $R_X[k]$

(b) Express cross-correlation function of X_n, Y_n in terms of $R_X[k]$

(c) Is Y_n wide sense stationary?

(d) Are X_n and Y_n jointly wide sense stationary?

Cross-Correlation

Example 13.19 (cont.)

Cross-Correlation

Theorem 13.14

If $X(t)$ and $Y(t)$ are jointly wide sense stationary continuous-time processes, then

$$R_{XY}(\tau) = R_{YX}(-\tau)$$

Gaussian Process

Definition 13.19

$X(t)$ is a Gaussian process if and only if $X = [X(t_1), \dots, X(t_k)]^T$ is a Gaussian random vector for any integer $k > 0$ and any set of time instants $t_1, t_2, \dots, t_k$

Gaussian Process

Theorem 13.15

If $X(t)$ is a wide sense stationary Gaussian process, then $X(t)$ is a stationary Gaussian process

Gaussian Process

Definition 13.20

$W(t)$ is a white Gaussian noise process if and only if $W(t)$ is a stationary Gaussian process with $\mu_W = 0$ and $R_W(\tau) = \eta_0 \delta(\tau)$