Reinforcement Learning China Summer School

RLChina 2021

Bayesian Brain

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Life and the universe

Life and the universe

Table of Contents

Table of Contents

What is “learning”?

- in biology, **learning** means
  a change of behaviour as a result of experience
  - in classical conditioning[1], animals can learn to identify a useful pattern in the environment by associating one stimulus with another:
    repeated given $\{$*ring-a-bell*, *food*$\}$, a dog will start to salivate (anticipate the upcoming of the food) when bell ringing again
- learned behaviours are **adaptive**, and thus are essential for animals to survive in the changing environment
  - e.g., they may learn not to eat certain foods if they have ever become ill after eating them
- more learned behaviours $\Longrightarrow$ more intelligent

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[1] Ivan Petrovitch Pavlov and William Gantt. “Lectures on conditioned reflexes: Twenty-five years of objective study of the higher nervous activity (behaviour) of animals.”. In: (1928).

Learning is not limited to biological systems

- *machine learning* is to answer the question of how computer programmes can improve their performance through experience
  - past experience exists in various forms, typically including (1) human-labelled or unlabelled data sets, (2) past interactions with the environments, or (3) data collected from realistic simulators
- with experience, computers can be trained to identify the associations, spot the underlying patterns, and make accurate predictions and forecasts the futures
- as a field, machine learning studies fundamental theory and algorithms and provides a principled solution for the computational methods of learning

Three levels in any information processing system[2]

- **computational theory**
  - what is the problem in a generic manner?
  - what is the computing goal?
  - what is the logic of strategy behind?
- **representation and algorithm**
  - how can the identified computational problems be solved?
  - what are the inputs/outputs and the algorithm for the transformation?
- **hardware implementation**
  - how can the representation and algorithm be realised physically?
    - human: biological
    - AI: silicon using transistors

[2] David Marr. "Vision: A computational investigation into the human representation and processing of visual information". In: *Inc., New York, NY* 2.4.2 (1982).

Human intelligent v.s. current AI[3]

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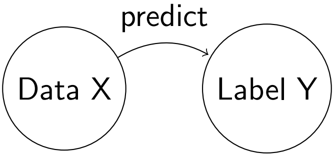
研表究明,汉字序顺并不定一影阅响读，事证实明了当你看这完句话之后才发字现都乱是的

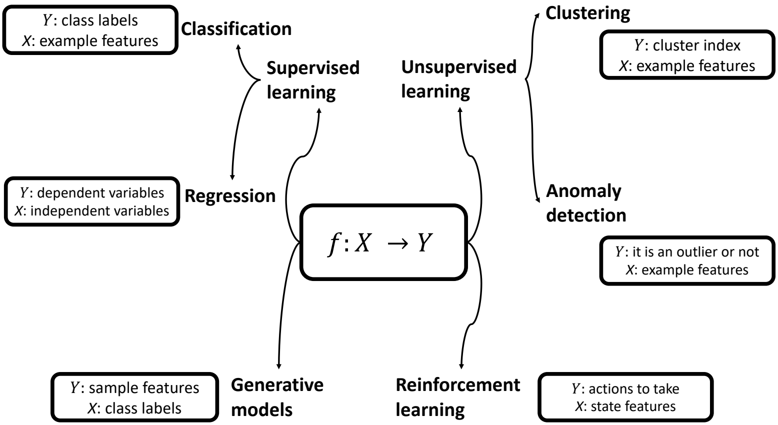
[3] Eliezer Sternberg. *NeuroLogic: The Brain's Hidden Rationale Behind Our Irrational Behavior*. Vintage, 2016.

Typical narrow machine learning



- mathematically, ml can loosely boil down to the question of finding a unknown function mapping $f : X \rightarrow Y$,
    - X is input feature space representing data points,
    - Y is label space representing the knowledge outputs
    - thus the mapping f represents a *knowledge discovery* process from a given data point $x \in X$ to the specific label $y \in Y$ associated with the data point:$y = f(x)$
- however, as f is not known a priori, the goal of the learning is to identify a hypothesis $h \in H$ from a predefined set H to approximate the unknown f, so that
    - the learned function h can *predict* the output variable $\hat{y}_{\text{new}} = h(x_{\text{new}})$ for a given new data point

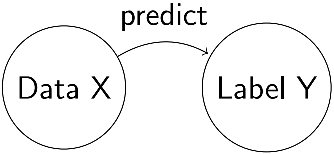
View narrow machine learning as function mapping



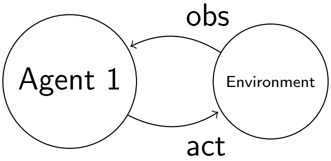
machine learning tasks are different in the experience available, the objective function, and the specific learning algorithms

Machine learning: reasoning under uncertainty

pattern recognition:



decision making (reinforcement learning):



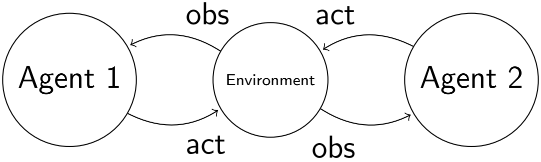
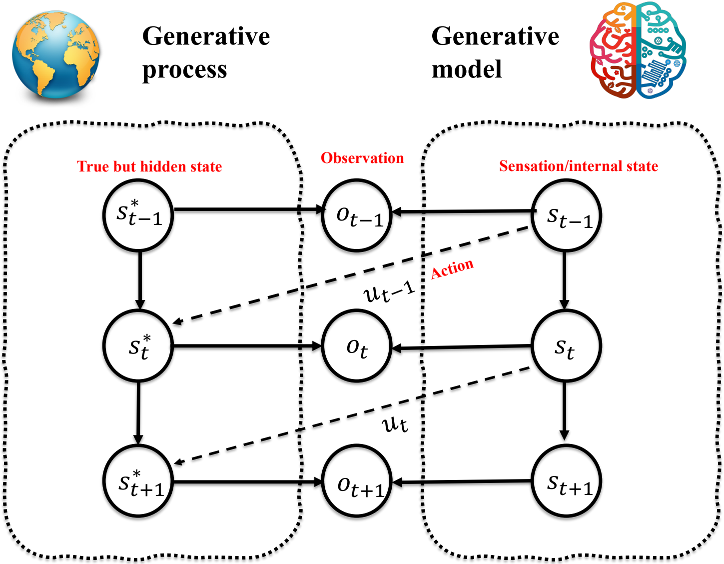


Table of Contents

A living system and its environment



Entropy

- *Entropy*: a measure of the average surprise (or uncertainty or disorder) of a random event sampled from a probability distribution or density
- definition: entropy of a random event X with possible outcomes $x_1, ..., x_n$:

$$H(p) = -\sum_i p(X = x_i) logp(X = x_i)$$

- low entropy means that, on average, the outcome is relatively predictable

Table of Contents

Variational Bayes[4]/Free energy principle[5]

- an agent builds a world model by having an internal representation of the world s from observation o as:

$$p(s|o) = \frac{p(o|s)p(s)}{\int_s p(o|s)p(s)\, dx}$$

- typically the denominator is intractable and one can approximate the posterior using a tractable function family $q(s) \in \mathbb{Q}$ by minimising a dis-similarity measure, e.g., KL Divergence:

$$KL(q(s)||p(s|o)) = \mathbb{E}_{q(s)}[\log \frac{q(s)}{p(s|o)}]$$

[4] Michael I Jordan et al. "An introduction to variational methods for graphical models". In: *Machine learning* 37.2 (1999), pp. 183–233.

[5] Karl Friston. "The free-energy principle: a unified brain theory?" In: *Nature reviews neuroscience* 11.2 (2010), pp. 127–138.

Variational Bayes[6]/Variational free energy[7]

- we then derive:

$$
\begin{aligned}
KL(q(s)||p(s|o)) &= \mathbb{E}_{q(s)}[\log \frac{q(s)}{p(s|o)}] \\
=&\mathbb{E}_{q(s)}[\log q(s) - \log p(s,o) + \log p(o)] \\
=&\mathbb{E}_{q(s)}[\log q(s) - \log p(s,o)] + \log p(o)
\end{aligned}
$$

- reorganising gives the measure of surprise (or negative model evidence or log marginal distribution):

$$
\begin{aligned}
-\log p(o) =&\mathbb{E}_{q(s)}[\log q(s) - \log p(s,o)] - D_{KL}[]q(s)||p(s|o)] \\
\leq&\mathbb{E}_{q(s)}[\log q(s) - \log p(s,o)] \equiv \mathbb{F}(q)
\end{aligned}
$$

where RHS is variational free energy (negative ELBO, Evidence Lower BOund)

[6] Michael I Jordan et al. “An introduction to variational methods for graphical models”. In: *Machine learning* 37.2 (1999), pp. 183–233.

[7] Karl Friston. “The free-energy principle: a unified brain theory?” In: *Nature reviews neuroscience* 11.2 (2010), pp. 127–138.

Free energy principle[9]

- the VFE can be decomposed in three principal ways:

$$\begin{aligned}
\mathbb{F}(q) &= \mathbb{E}_{q(s)}[\log q(s) - \log p(s,o)] \\
&= \underbrace{\mathbb{E}_{q(s)}[\log q(s)]}_{\text{Negative Entropy}} - \underbrace{\mathbb{E}_{q(s)}[\log p(s,o)]}_{\text{Energy}} \\
&= \underbrace{\mathbb{E}_{q(s)}[\log p(o|s)]}_{\text{Accuracy}} + \underbrace{D_{KL}[q(s)|p(s)]}_{\text{Complexity}} \\
&= \underbrace{D_{KL}[q(s)\|p(s|o)]}_{\text{Posterior Divergence}} \quad \underbrace{-\log p(o)}_{\text{Negative Log Model Evidence}}
\end{aligned}$$

- free energy principle: the goal of a living system is to minimise the free energy $\mathbb{F}$ in order to avoid surprising observations (states) $\rightarrow$ maintain homeostasis[8] (thus remain alive)

[8] The process whereby an open or closed system regulates its internal environment to maintain its states within bounds.

[9] Karl Friston. "The free-energy principle: a unified brain theory?" In: *Nature reviews neuroscience* 11.2 (2010), pp. 127–138.

Table of Contents

Extension to handle dynamics and control

- the real world is dynamic and thus we extend the model to have an observation $o = \{o_1, ..., o_T\}$ and a hidden state at each point in time $s = \{s_1, ..., s_T\}$
- also add action policy $\pi = \{u_1, ...u_T\}$ which interacts with the environment by altering the next hidden state
- the generative model: 1) $p\left(s_{t+1} \mid s_t, u_t\right), t > 1$; $p(s_1)$ 2) $p(o_t|s_t)$
- the Expected Free Energy (EFE), from time τ until the time horizon T:
  $\mathcal{G} = \mathbb{E}_{q(o_{\tau:T}, s_{\tau:T}, \pi)}\left[\ln q\left(s_{\tau:T}, \pi\right) - \ln \tilde{p}\left(o_{\tau:T}, s_{\tau:T}\right)\right]$
- where an agent's goals are encoded as (subjective) desired distribution over observations $\tilde{p}(o_{\tau:T})$[10]; thus we have $\tilde{p}\left(o_\tau, s_\tau\right) \approx \tilde{p}\left(o_\tau\right) q\left(s_\tau \mid o_\tau\right)$

[10] in ML, an optimality variable is added to encode the desires

Expected free energy[12]

- a temporal mean-field factorisation[11] is assumed:
  - the variational function: $q\left(s_{\tau:T},\pi\right) \approx q(\pi)\prod_{\tau}^{T} q(s \mid \pi)$
  - the gen. model: $\tilde{p}\left(o_{\tau:T},s_{\tau:T}\right) \approx \prod_{t}^{T} \tilde{p}\left(o_{\tau}\right) q\left(s_{\tau} \mid o_{\tau}\right)$
- as a result, they are independent between time steps:

$$\mathbb{G} = \mathbb{E}_{q(o_{\tau:T},s_{\tau:T},\pi)}\left[\ln q\left(s_{\tau:T},\pi\right) - \ln \tilde{p}\left(o_{\tau:T},s_{\tau:T}\right)\right]$$

$$=\mathbb{E}_{q(o_{\tau:T},s_{\tau:T}\mid\pi)q(\pi)}\left[\ln q\left(s_{\tau:T} \mid \pi\right) + \ln q(\pi) - \ln \tilde{p}\left(o_{\tau:T},s_{\tau:T}\right)\right]$$

$$=\mathbb{E}_{q(\pi)}\left[\ln q(\pi) + \mathbb{E}_{q(o_{\tau:T},s_{\tau:T}\mid\pi)}\left[\sum_{\tau}^{T}\left[\ln q(s \mid \pi) - \ln \tilde{p}\left(o_{\tau},s_{\tau}\right)\right]\right]\right]$$

$$=\mathbb{E}_{q(\pi)}\left[\ln q(\pi) - \left(-\sum_{t}^{T}\mathbb{G}_{\tau}(\pi)\right)\right] = \mathrm{D}_{KL}\left[q(\pi) \| e^{-\sum_{t}^{T}\mathbb{G}_{\tau}(\pi)}\right]$$

<u>where $\mathbb{G}_{\tau}(\pi) = \mathbb{E}_{q(o_{\tau},s_{\tau}\mid\pi)}\left[\ln q\left(s_{\tau} \mid \pi\right) - \ln \tilde{p}\left(o_{\tau},s_{\tau}\right)\right]$</u>

[11] David M Blei, Alp Kucukelbir, and Jon D McAuliffe. "Variational inference: A review for statisticians". In: *Journal of the American statistical Association* 112.518 (2017), pp. 859–877.

[12] Beren Millidge, Alexander Tschantz, and Christopher L Buckley. "Whence the expected free energy?" In: *Neural Computation* 33.2 (2021), pp. 447–482.

Expected free energy

- thus optimising $\mathbb{G}$ results in $q^*(\pi) = \mathrm{SoftMax}\Big(-\sum_t^T \mathbb{G}_\tau(\pi)\Big)$
- EFE at time τ, $\mathbb{G}_\tau$, can be further decomposed as:

$$
\begin{aligned}
&\mathbb{G}_\tau(\pi)\\
=&\mathbb{E}_{q(o_\tau, s_\tau \mid \pi)}\left[\ln q\left(s_\tau \mid \pi\right) - \ln \tilde{p}\left(o_\tau, s_\tau\right)\right]\\
\approx&\mathbb{E}_{q(o_\tau, s_\tau \mid \pi)}\left[\ln q\left(s_\tau \mid \pi\right) - \ln \tilde{p}\left(o_\tau\right) - \ln q\left(s_\tau \mid o_\tau\right)\right]\\
\approx&\underbrace{-\mathbb{E}_{q(o_\tau, \mid \pi)}\left[\ln \tilde{p}\left(o_\tau\right)\right]}_{\text{Extrinsic Value}} - \underbrace{\mathbb{E}_{q(o_\tau)} \mathrm{D}_{KL}\left[q\left(s_\tau \mid o_\tau\right) \| q\left(s_\tau \mid \pi\right)\right]}_{\text{Epistemic Value}}
\end{aligned}
$$

- thus, optimal policies are obtained by minimising the sum of the expected free energies
- EFE is estimated using the generative model to roll out predicted futures, and compute the EFE of those futures

Expected free energy

- similar to variational free energy, EFE can be also decomposed as:

$$
\begin{aligned}
&\mathbb{G}_\tau(\pi) \\
=&\mathbb{E}_{q(o_\tau, s_\tau \mid \pi)}\left[\ln q\left(s_\tau \mid \pi\right)-\ln \tilde{p}\left(o_\tau, s_\tau\right)\right] \\
\approx&\mathbb{E}_{q(o_\tau, s_\tau \mid \pi)}\left[\ln q\left(s_\tau \mid \pi\right)-\ln \tilde{p}\left(o_\tau\right)-\ln q\left(s_\tau \mid o_\tau\right)\right] \\
\approx&\underbrace{-\mathbb{E}_{q(o_\tau, s_\tau \mid \pi)}\left[\ln \tilde{p}\left(o_\tau\right)\right]}_{\text{Extrinsic Value}}-\underbrace{\mathbb{E}_{q(o_\tau)} \mathrm{D}_{KL}\left[q\left(s_\tau \mid o_\tau\right) \| q\left(s_\tau \mid \pi\right)\right]}_{\text{Epistemic Value}}
\end{aligned}
$$

- note that an epistemic uncertainty refers to the deficiencies by a lack of knowledge or information; reducible with more data or a better model
- the second term measures the reduced entropy of s_τ when observed o_τ $\rightarrow$ max its value $\rightarrow$ we intend to choose the policy such that $H[q(s|\pi]$ is high and strong dependency between states and observation (aka $H[q(s|o]$ is low)

Expected free energy

- one can also decompose it into the following:

$$\mathbb{G}_\tau(\pi) = \mathbb{E}_{q(o_\tau, s_\tau \mid \pi)}\left[\ln q\left(s_\tau \mid \pi\right) - \ln \tilde{p}\left(o_\tau, s_\tau\right)\right]$$
$$= \mathbb{E}_{q(o_\tau, s_\tau \mid \pi)}\left[\ln q\left(s_\tau \mid \pi\right) - \ln \tilde{p}\left(o_\tau \mid s_\tau\right) - \ln p\left(s_\tau\right)\right]$$
$$= \underbrace{-\mathbb{E}_{q(o_\tau, s_\tau \mid \pi)}\left[\ln \tilde{p}\left(o_\tau \mid s_\tau\right)\right]}_{\text{Accuracy}} + \underbrace{\mathbb{E}_{q(o_\tau \mid s_\tau)}\left[\mathrm{D}_{KL}\left[q\left(s_\tau \mid \pi\right) \| p\left(s_\tau\right)\right]\right]}_{\text{Complexity}}$$

- or this (typically used for computation):

$$\mathbb{G}_\tau(\pi) = \mathbb{E}_{q(o_\tau, s_\tau \mid \pi)}\left[\ln q\left(s_\tau \mid \pi\right) - \ln \tilde{p}\left(o_\tau\right) - \ln q\left(s_\tau \mid o_\tau\right)\right]$$
$$= \mathbb{E}_{q(o_\tau, s_\tau \mid \pi)}\Big[\cancel{\ln q\left(s_\tau \mid \pi\right)} - \ln \tilde{p}\left(o_\tau\right) - \ln p\left(o_\tau \mid s_\tau\right)$$
$$- \cancel{\ln q\left(s_\tau \mid \pi\right)} + \ln q\left(o_\tau\right)\Big]$$
$$= \underbrace{D_{KL}\left[q\left(o_t \mid \pi\right) \| \tilde{p}\left(o_t\right)\right]}_{\text{Expected Cost}} + \underbrace{\mathbb{E}_{q(s_t \mid \pi)}\Big[H\left[p\left(o_t \mid s_t\right)\right]\Big]}_{\text{Expected Ambiguity}}$$

Table of Contents

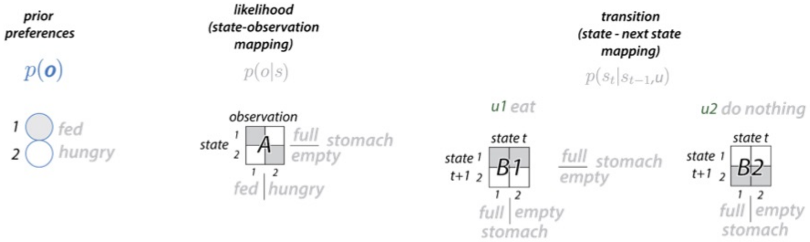
A simple example[13]

- suppose there is an agent interacts with an environment
- perception:
  - the environment has two hidden states $s \in \{1, 2\}$, e.g., *there is food in your stomach (1) or not (2)*
  - while s is NOT measurable, there is an observation $o = \{1, 2\}$, e.g., you feeling fed (1) or hungry (2)
  - assume $p(o|s)$ is given as a 2x2 likelihood matrix (called "A" matrix) which maps states to observations, e.g., if you fed, you have food and vice versa
- control:
  - transition prob. $p(s_t|s_{t-1}, u)$ maps the previous state to the next one, depending on action $u = \{u_1, u_2\}$
  - parameterised by a separate 2x2 transition matrix ("B") for each action, e.g., *either go get food ($u1$) or do nothing ($u2$); if $u1$, will have food in the next state, regardless of whether we have it now, and vice versa*

[13] https://medium.com/@solopchuk/tutorial-on-active-inference-30edcf50f5dc

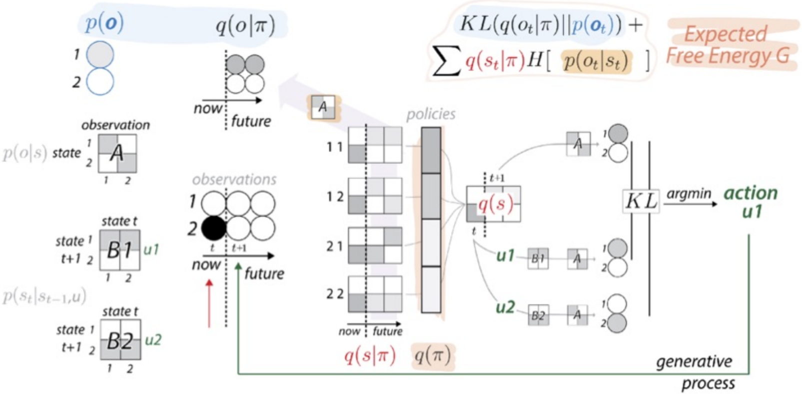
A simple example[14]

- preference (as a form of objective or reward)
  - we also have prior preferences $\tilde{p}(o)$, e.g., we like to be fed and not hungry, so we assign a higher probability to observation $o = 1$ fed. In other words, we express preferences over observations as probability $\tilde{p}(o)$



[14] https://medium.com/@solopchuk/tutorial-on-active-inference-30edcf50f5dc

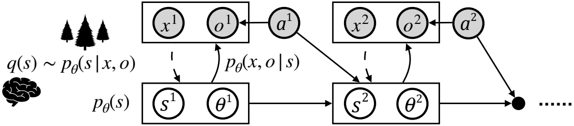
A simple example[15]



action selection: option 1) $q(s|\pi) \to q(o|\pi) \to \mathbb{G}(\pi) \to q(\pi)$
option 2 $u^*_{t+1} = \arg\max_u D_{KL}\left[AS_{t+1}||A(B(u))S_t\right]$, Bayesian model average method

[15] `https://medium.com/@solopchuk/tutorial-on-active-inference-30edcf50f5dc`

Unifying perception and control (Bayesian brain)[16]



- the latent state $\boldsymbol{s}$: true world configuration such as pixel assignment, the optimality o is a binary variable
- the perception model includes a bottom-up recognition model $q(\boldsymbol{s})$ and a top-down generative model $p(\boldsymbol{x}, o, \boldsymbol{s})$ (decomposed into the likelihood $p(\boldsymbol{x}, o|\boldsymbol{s})$ and the prior belief $p(\boldsymbol{s})$)
- the prior knowledge $\boldsymbol{\theta}$ represents the physical law of the environment (the property of each object)
- control is performed by taking an action a to change the environment state.

[16] Minne Li et al. "Joint Perception and Control as Inference with an Object-based Implementation". In: (2020).

Table of Contents

Bayesian decision principle

making decision under uncertainty can be interpreted as maximising expected utility in the face of uncertainty[17]

- the uncertainty is captured by the (hidden) *state of nature*: $z \in \mathbb{Z}$
- decisions (aka *actions*): $a \in \mathbb{A}$
- reward: $R(z, a)$
- the posterior distribution $p(z|o)$, where we can perform a statistical investigation to obtain information (denoted as $o = (o_1, o_2, ..., o_n) \in O$) about the nature state θ

(**Conditional Bayes Decision Principle**)

$$a^*|o = \arg\max_a E_{\theta \sim p(z|o)}[R(o, a)], \tag{1}$$

where the principle is the only fundamentally correct analysis[18].

Yet, the posterior can be difficult to obtain practically

[17] John Von Neumann and Oskar Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, 1953.

[18] James O Berger. *Statistical decision theory and Bayesian analysis*. Springer Science & Business Media, 2013.

The duality between inference and control

- the filtering problem is shown to be the dual of the noise-free regulator (control) problem[19]

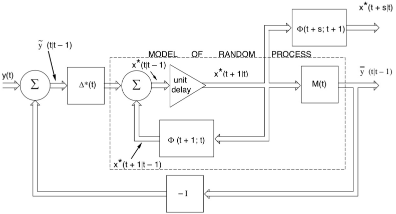


Figure: Optimal filter

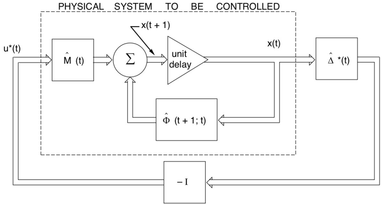
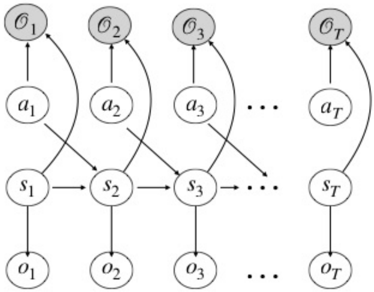


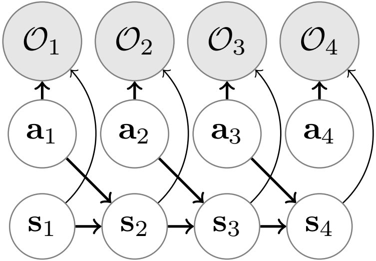
Figure: Optimal controller

[19] Rudolph Emil Kalman. "A new approach to linear filtering and prediction problems". In: (1960).

Control as inference:[20]



(a) states s are hidden



(b) states s are observable

- the idea: add an optimality variable $\mathcal{O}_t = 1$ and assume a biased probability with reward r:
  $p(\mathcal{O}_t = 1 \mid \mathrm{s}_t, \mathrm{a}_t) = \exp(r(\mathrm{s}_t, \mathrm{a}_t))$

[20] Sergey Levine. "Reinforcement learning and control as probabilistic inference: Tutorial and review". In: *arXiv preprint arXiv:1805.00909* (2018); Beren Millidge et al. "On the relationship between active inference and control as inference". In: *International Workshop on Active Inference*. Springer. 2020, pp. 3–11.

Control as inference:[21]

- let us look at an MDP model (states are observable)
- the evidence: $\mathcal{O}_t = 1$ for all $t \in \{1, ..., T\}$; thus ELBO is

$$\log p\left(\mathcal{O}_{1:T}\right) = \log \iint p\left(\mathcal{O}_{1:T}, \mathrm{s}_{1:T}, \mathrm{a}_{1:T}\right) d\mathrm{s}_{1:T} d\mathrm{a}_{1:T}$$

$$= \log \iint p\left(\mathcal{O}_{1:T}, \mathrm{s}_{1:T}, \mathrm{a}_{1:T}\right) \frac{q\left(\mathrm{s}_{1:T}, \mathrm{a}_{1:T}\right)}{q\left(\mathrm{s}_{1:T}, \mathrm{a}_{1:T}\right)} d\mathrm{s}_{1:T} d\mathrm{a}_{1:T}$$

$$= \log E_{(\mathrm{s}_{1:T}, \mathrm{a}_{1:T}) \sim q(\mathrm{s}_{1:T}, \mathrm{a}_{1:T})} \left[\frac{p\left(\mathcal{O}_{1:T}, \mathrm{s}_{1:T}, \mathrm{a}_{1:T}\right)}{q\left(\mathrm{s}_{1:T}, \mathrm{a}_{1:T}\right)}\right]$$

$$\geq E_{(\mathrm{s}_{1:T}, \mathrm{a}_{1:T}) \sim q(\mathrm{s}_{1:T}, \mathrm{a}_{1:T})} \left[\log p\left(\mathcal{O}_{1:T}, \mathrm{s}_{1:T}, \mathrm{a}_{1:T}\right) - \log q\left(\mathrm{s}_{1:T}, \mathrm{a}_{1:T}\right)\right]$$

[21] Sergey Levine. "Reinforcement learning and control as probabilistic inference: Tutorial and review". In: *arXiv preprint arXiv:1805.00909* (2018).

Control as inference:[22]

- we make use of the evidence $p\left(\mathcal{O}_{1:T}, \mathrm{s}_{1:T}, \mathrm{a}_{1:T}\right) = \left[p\left(\mathrm{s}_1\right)\prod_{t=1}^{T} p\left(\mathrm{s}_{t+1} \mid \mathrm{s}_t, \mathrm{a}_t\right)\right] \exp\left(\sum_{t=1}^{T} r\left(\mathrm{s}_t, \mathrm{a}_t\right)\right)$ and
- factorise the variational distribution as $q(\tau) \equiv q\left(\mathrm{s}_{1:T}, \mathrm{a}_{1:T}\right) = q\left(\mathrm{s}_1\right)\prod_{t=1}^{1} q\left(\mathrm{s}_{t+1} \mid \mathrm{s}_t, \mathrm{a}_t\right) q\left(\mathrm{a}_t \mid \mathrm{s}_t\right)$
- this leads to the final ELBO: $\log p\left(\mathcal{O}_{1:T}\right) \geq E_{q\left(\mathrm{s}_{1:T}, \mathrm{a}_{1:T}\right)}\left[\sum_{t=1}^{T} r\left(\mathrm{s}_t, \mathrm{a}_t\right) - \log q\left(\mathrm{a}_t \mid \mathrm{s}_t\right)\right]$
- where we define $q\left(\mathrm{s}_{t+1} \mid \mathrm{s}_t, \mathrm{a}_t\right) = p\left(\mathrm{s}_{t+1} \mid \mathrm{s}_t, \mathrm{a}_t\right)$

[22] Sergey Levine. "Reinforcement learning and control as probabilistic inference: Tutorial and review". In: *arXiv preprint arXiv:1805.00909* (2018).

Control as inference:[23]

- without using temporal mean-field factorisation as Active Inference, one can derive Value Iteration similar to a standard RL solution
- the ELBO can be decomposed recursively as:
$$E_{q(\mathrm{s}_t,\mathrm{a}_t)}\left[r\left(\mathrm{s}_t,\mathrm{a}_t\right)-\log\pi\left(\mathrm{a}_t\mid\mathrm{s}_t\right)\right]+$$
$$E_{q(\mathrm{s}_t,\mathrm{a}_t)}\left[E_{\mathrm{s}_{t+1}\sim p(\mathrm{s}_{t+1}\mid\mathrm{s}_t,\mathrm{a}_t)}\left[V\left(\mathrm{s}_{t+1}\right)\right]\right]=$$
$$E_{q(\mathrm{s}_t)}\left[-D_{\mathrm{KL}}\left(\pi\left(\mathrm{a}_t\mid\mathrm{s}_t\right)\|\frac{1}{\exp(V(\mathrm{s}_t))}\exp\left(Q\left(\mathrm{s}_t,\mathrm{a}_t\right)\right)\right)+V\left(\mathrm{s}_t\right)\right]$$
- where we define:
$$Q\left(\mathrm{s}_t,\mathrm{a}_t\right)\equiv r\left(\mathrm{s}_t,\mathrm{a}_t\right)+E_{\mathrm{s}_{t+1}\sim p(\mathrm{s}_{t+1}\mid\mathrm{s}_t,\mathrm{a}_t)}\left[V\left(\mathrm{s}_{t+1}\right)\right]$$
$$V\left(\mathrm{s}_t\right)\equiv\log\int_{\mathcal{A}}\exp\left(Q\left(\mathrm{s}_t,\mathrm{a}_t\right)\right)d\mathrm{a}_t$$

$$\rightarrow\pi\left(\mathrm{a}_t\mid\mathrm{s}_t\right)=\exp\left(Q\left(\mathrm{s}_t,\mathrm{a}_t\right)-V\left(\mathrm{s}_t\right)\right)$$

[23] Sergey Levine. "Reinforcement learning and control as probabilistic inference: Tutorial and review". In: *arXiv preprint arXiv:1805.00909* (2018).

Table of Contents

Agent modelling with maximum entropy objective[24]

- each agent pursues the maximal cumulative reward

$$\max \quad \eta^i(\pi_\theta) = \mathbb{E}\left[\sum_{t=1}^{\infty} \gamma^t R^i(s_t, a_t^i, a_t^{-i})\right], \qquad (2)$$

  with actions (a_t^i, a_t^{-i}) sampled from policy $(\pi^i_{\theta^i}, \pi^{-i}_{\theta^{-i}})$
- a strategy profile $(\pi^{1*}, \ldots, \pi^{n*})$ reaches optimum when:

$$\begin{aligned} &\mathbb{E}_{s\sim p_s, a_t^{i*}\sim\pi^{i*}, a_t^{-i*}\sim\pi^{-i*}}\left[\sum_{t=1}^{\infty} \gamma^t R^i(s_t, a_t^{i*}, a_t^{-i*})\right] \\ &\geq \mathbb{E}_{s\sim p_s, a_t^{i}\sim\pi^{i}, a_t^{-i}\sim\pi^{-i}}\left[\sum_{t=1}^{\infty} \gamma^t R^i(s_t, a_t^{i}, a_t^{-i})\right] \qquad (3) \\ &\forall \pi \in \Pi, i \in (1 \ldots n), \end{aligned}$$

  where $\pi = (\pi^i, \pi^{-i})$ and agent i's optimal policy is π^{i*}

[24] Zheng Tian et al. "A regularized opponent model with maximum entropy objective". In: *IJCAI* (2019).

Agent modelling with maximum entropy objective

- a lower bound on the likelihood of optimality of agent i:

$$\log P(\mathcal{O}^i_{1:T} = 1 | \mathcal{O}^{-i}_{1:T} = 1) \geq \sum_t \mathbb{E}_{(s_t, a^i_t, a^{-i}_t) \sim q} [R^i(s_t, a^i_t, a^{-i}_t)$$

$$+ H(\pi(a^i_t | s_t, a^{-i}_t)) - D_{\mathrm{KL}}(\rho(a^{-i}_t | s_t) || P(a^{-i}_t | s_t))] \tag{4}$$

$$= \sum_t \mathbb{E}_{s_t} [\underbrace{\mathbb{E}_{a^i_t \sim \pi, a^{-i}_t \sim \rho} [R^i(s_t, a^i_t, a^{-i}_t) + H(\pi(a^i_t | s_t, a^{-i}_t))]}_{\text{MEO}}$$

$$- \underbrace{\mathbb{E}_{a^{-i}_t \sim \rho} [D_{\mathrm{KL}}(\rho(a^{-i}_t | s_t) || P(a^{-i}_t | s_t))]]}_{\text{Regulariser of } \rho}. \tag{5}$$

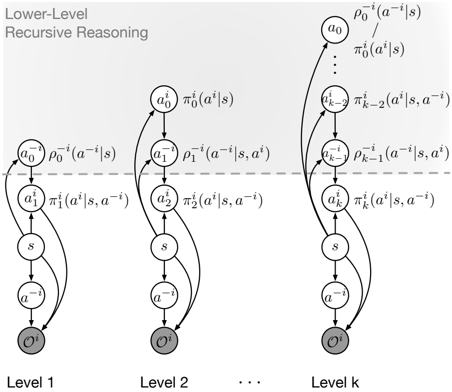
  - $\rho(a^{-i}_t | s_t, o^{-i}_t = 1)$ is agent i's opponent model
  - $\pi(a^i_t | s_t, a^{-i}_t, o^i_t = 1, o^{-i}_t = 1)$ is the agent i's conditional policy at optimum $(o^i_t = o^{-i}_t = 1)$ and
  - $P(a^{-i}_t | s_t, o^{-i}_t = 1)$ is the prior of opponent model
  - the prior $P(a^{-i}_t | s_t, o^{-i}_t = 1)$ is estimated empirically
  - we drop $(o^i_t = 1, o^{-i}_t = 1)$ in π, ρ and $P(a^{-i}_t | s_t)$

Recursive reasoning[25]

- Example: in the "beauty contest" game, players are asked to pick numbers from 0 to 100, and the player whose number is closest to 2/3 of the average wins a prize

[25] Colin F Camerer, Teck-Hua Ho, and Juin-Kuan Chong. "A cognitive hierarchy model of games". In: *The Quarterly Journal of Economics* 119.3 (2004), pp. 861–898.

Multi-agent generalized recursive reasoning[26]



- unobserved opponent policies are approximated by ρ^{-i}
- agent i recursively reasons about opponents (grey area)
- in the recursion, agents with higher-level beliefs take the best response to the lower-level thinkers' actions.

[26]Ying Wen et al. "Modelling Bounded Rationality in Multi-Agent Interactions by Generalized Recursive Reasoning". In: *IJCAI* (2020).

Table of Contents

Remarks

- AI: reasoning under uncertainty
- a learning system is an information system
- pattern recognition and decision making problems (include multiagent learning) can be unified and modelled by probabilistic inference such as Variational Bayes (e.g., active- inference or inference-as-control)
- as such information theory can be directly utilised
- however, we call for new research on information theory that deals with "intrinsic information exchange"

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