Reinforcement Learning China Summer School

RLChina 2021

Game Theory: Advanced Topics

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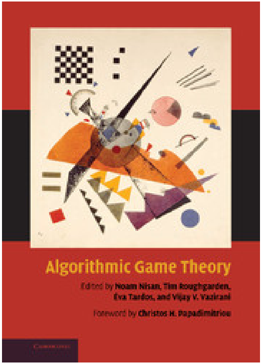
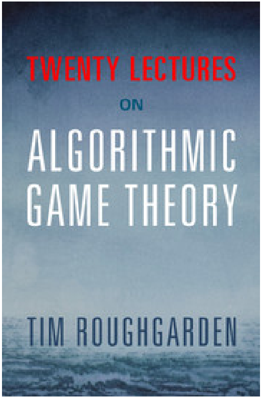
Main References





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Our Goal in MD

Goal

Understand how to design systems with strategic participants that have good performance guarantees.

Single-Item Auctions

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- A seller, n (strategic) bidders and one item
- Each bidder i has a private value $v_i \geq 0$.
- Quaslinear utility: $u_i = v_i - p$ if she wins at price p, $u_i = 0$ o.w.

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1. Each bidder i privately sends a bid b_i to the seller.
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3. The seller decides the selling price.

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3. The seller decides the selling price. Many different ways...

First-Price Auctions

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First-Price Auctions

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- Predict what will happen?

- Your value is sum of your birth month and the day of your birth, i.e. from 2 (Jan 1) to 43 (Dec 31).
- The other bidder whose value is constructed with the same way.
- How do you bid to max your expected utility?
- What if there are one more bidder in the auction?

Second-Price Auctions

- The winner pays the second-highest bid.
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- Fix a bidder i, and her value is v_i, other bids $\mathbf{b}_{-i}$. Let $B = \max_{j \neq i} b_j$, we consider two cases:
- If $v_i < B$, easy.
- If $v_i \geq B$, the maximum utility she can obtain is $\max\{0, v_i - B\} = v_i - B$.

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- If $v_i < B$, easy.
- If $v_i \geq B$, the maximum utility she can obtain is $\max\{0, v_i - B\} = v_i - B$.
- Another property, individual rationality, i.e. always get non-negative utility.

Ideal Auctions

Dominant-Strategy Incentive Compatible (DSIC)

An auction is *DSIC* if bidding truthfully is always a dominant strategy for every bidder and it can help them get non-negative utility.

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Theorem [Second-Price Auctions are Ideal]

1. It is DSIC.
2. Welfare maximization: $\sum_{i=1}^{n} v_i x_i$, where x_1 is 1 if bidder i wins, 0 o.w.
3. It is computationally efficient.

Multi-Item Auctions

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- Each advertiser is only interested in the click on her link, the private value is v_i. So the expected value from slot j is $v_i\alpha_j$.
- Two things: allocations and payments.
- Given truthful bidding, greedy allocation is optimal.
- Say bids are $b_1 \geq \cdots \geq b_n$. The price of bidder i is

$$p_i(\mathbf{b}) = \sum_{j=i}^{k} b_{j+1} \frac{\alpha_j - \alpha_{j+1}}{\alpha_i}.$$

(By Myerson's Lemma)

- A famous simplified version is the generalized second-price (GSP) auction.
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- If setting $b_1' = 5$, then $u_1' = \alpha_2(v_1 - p_1') = 0.4 \cdot (7 - 1) = 2.4$.

Combinatorial Auctions

- n bidders with a set M of m items.
- Each bidder i has a value function $v_i : 2^M \to \mathbb{R}_{\geq 0}$.
- The outcome is $\mathbf{S} = (S_1, \ldots, S_n)$, where $\cup_i S_i \subseteq M$ and S_i's are disjoint each other.

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- Charge an agent her externality,

$$p_i(\mathbf{b}) = \underbrace{\left(\max_{\mathbf{S}} \sum_{j \neq i} b_j(\mathbf{S}) \right)}_{\text{without } i} - \underbrace{\sum_{j \neq i} b_j(\mathbf{S}^*)}_{\text{with } i},$$

where $\mathbf{S}^* = \mathbf{x}(\mathbf{b})$.

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Preliminaries

- We focus on two-player games, a.k.a, bimatrix games ($A, B \in [0,1]^{n\times n}$).
- Mixed strategies: $x, y \in \Delta^n$, where $\Delta^n := \left\{z \in \mathbb{R}^n_{\geq 0} \mid \sum_{i=1}^n z_i = 1\right\}$.
- Nash equilibrium (x, y): if $x^T A y \geq x'^T A y$ and $x^T B y \geq x^T B y'$.

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- Nash equilibrium (x, y): if $x^T Ay \geq x'^T Ay$ and $x^T By \geq x^T By'$.
- Our goal: argue that finding an NE is intractable.

Complexity 101

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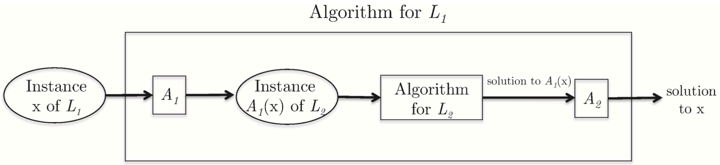

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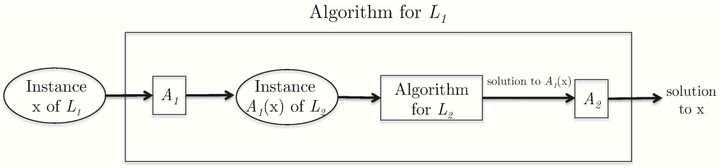

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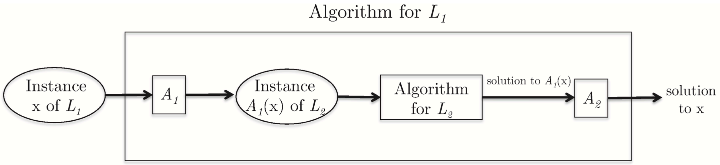

- T(otal) FNP: every instance has at least one solution.
- PPAD $\subseteq$ TFNP

PPAD

Problem $\textsc{End-of-A-Line}$

Input: two boolean circuits $S, P : \{0,1\}^n \to \{0,1\}^n$, such that $P(0^n) = 0^n \neq S(0^n)$.

Output: $x \in \{0,1\}^n$ such that $P(S(x)) \neq x$ or $S(P(x)) \neq x \neq 0^n$.

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- Problem X is in PPAD if there is a reduction from X to END-OF-A-LINE.

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- **P**olynomial **P**arity **A**rgument on a **D**irected graph
- Problem X is in PPAD if there is a reduction from X to END-OF-A-LINE.
- X is PPAD-hard if there is a reversed reduction.

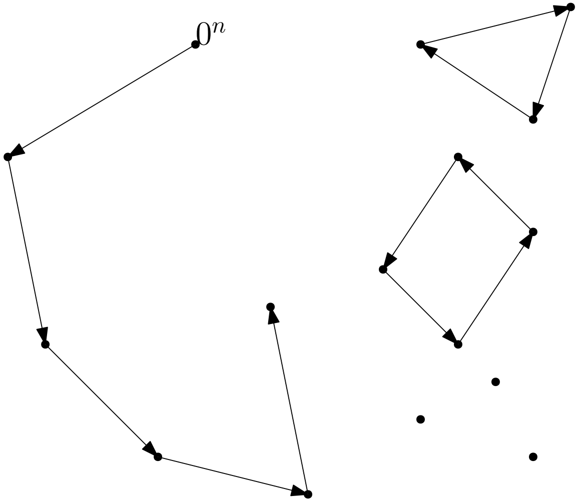

Figure: Sketch of End-of-A-Line

Proof Overview — Reductions![1]

1. End-of-A-Line to 2D-End-of-A-Line
2. to Discrete Fixed-Point
3. Finding a fixed point with Boolean Circuits
4. Simulating gates with game gadgets
5. Putting all gates together

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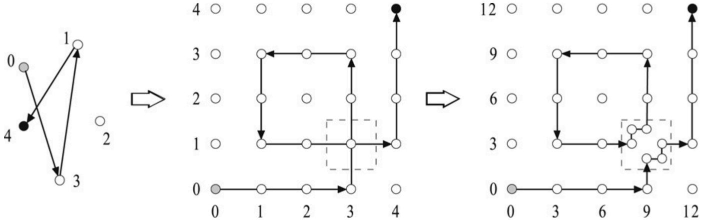
Embedding a Path on a 2D Space



Generalized Circuits

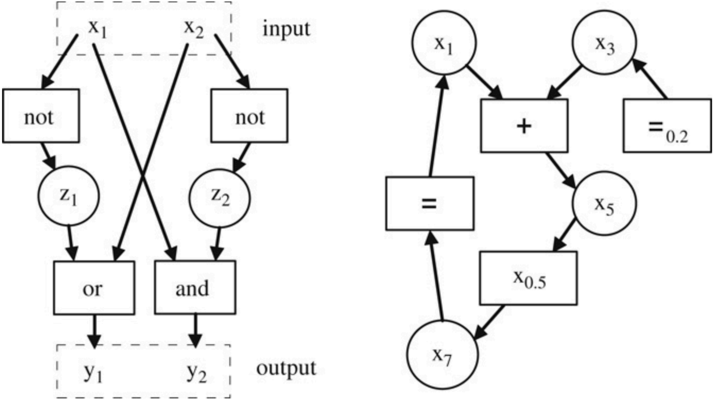


Figure: Two differences: 1. type of gates 2. has cycle

An Addition Gate Gadget

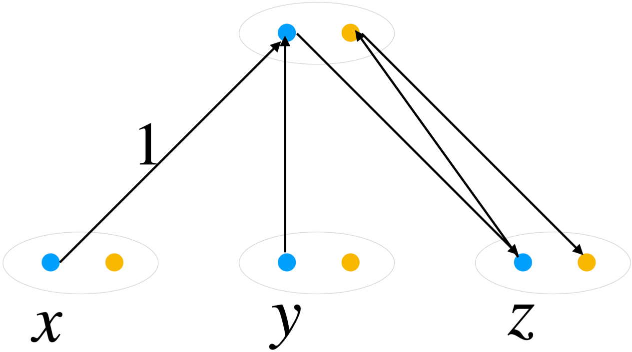


Figure: We make sure each player acts as an output at most once.

All Gates

$\mathbf{L}[T]$ **and** $\mathbf{R}[T]$**, where gate** $T = (G, v_1, v_2, v, \alpha)$
Set $\mathbf{L}[T] = (L_{i,j}) = \mathbf{R}[T] = (R_{i,j}) = 0$, $k = \mathcal{C}(v)$, $k_1 = \mathcal{C}(v_1)$ and $k_2 = \mathcal{C}(v_2)$
$G_\zeta :\quad L_{2k-1,2k} = L_{2k,2k-1} = R_{2k-1,2k-1} = 1,\ R_{i,2k} = \alpha, \forall\, i : 1 \leq i \leq 2K.$
$G_{\times\zeta} : L_{2k-1,2k-1} = L_{2k,2k} = R_{2k-1,2k} = 1,\ R_{2k_1-1,2k-1} = \alpha.$
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$G_\wedge :\quad L_{2k-1,2k-1} = L_{2k,2k} = R_{2k_1-1,2k-1} = R_{2k_2-1,2k-1} = 1,\ R_{i,2k} = 3/(2K),\ \forall\, i \in [2K].$
$G_\neg :\quad L_{2k-1,2k} = L_{2k,2k-1} = R_{2k_1-1,2k-1} = R_{2k_1,2k} = 1.$

The Final Step

Generalized Matching Pennies: $G^* = (A^*, B^*)$, where

$$A^* = \begin{pmatrix} M & M & 0 & 0 & \cdots & 0 & 0 \\ M & M & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & M & M & \cdots & 0 & 0 \\ 0 & 0 & M & M & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & M & M \\ 0 & 0 & 0 & 0 & \cdots & M & M \end{pmatrix},$$

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Property

For every game $G = (A, B)$ with all entries in $A - A^*, B - B^*$ are small, every approximate NE (x, y) of G are almost "pairwise" uniform.

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- Individual Rationality, Incentive Compatibility and Allocative Efficiency.
- VCG mechanism.
- General Games are (PPAD-)hard to solve. But we have some approximation algorithms.

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- Market Equilibria, PPA etc.

Thanks for listening!

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https://lozycs.github.io/