Banking and Financial Intermediation

FINA 4503

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Lecture notes 2

Variables and Utility Maximization

Random variables

- Variable is a symbol that represents a number. We will deal with two different variables: deterministic and random.
- Deterministic variables are the ones we know its value for sure (*e.g.*, the number of sessions in this course, the number of current customers of a bank).
- Random variables are the ones we do not know its exact value ex ante because there is element of chance (*e.g.*, temperature tomorrow, the number of customers of a bank from one year from now).
- Although we do not know the exact value of a random variable (*i.e.*, its realization), we do know the possible outcomes.
- The set of possible outcomes (*i.e.*, Ω) is called support.

Random variables

- Probability distribution function $f(\cdot)$ assigns probabilities to each possible outcome $x \in \Omega$.
  - distribution can be discrete *e.g.*, the number of students that will stay in the course after add/drop period.
  - or continuous *e.g.*, the temperature tomorrow.

Random variables

- The expected value of a random variable is the sum of the possible outcomes weighted by their probabilities.
- The expected value is represented by $E[x]$ and calculated as

Random variables

- We are rolling a dice. Define the outcome as X.
- What is the support of X?
- What is the distribution of X?
- What is the expected value of X?

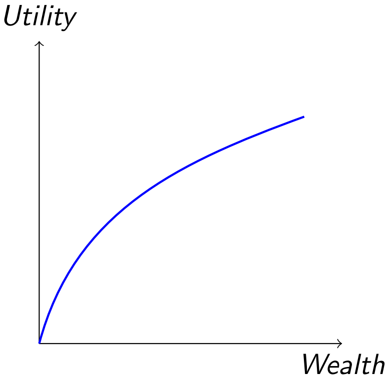
Random variables

- Law of iterated expectations says that your expectation on the conditional expected value of X is the same as your expectation of X.
- It can be written as

Utility and risk attitude

- We are always interested in maximizing the utility of individuals.
- Utility is a measure of preferences over a set of consumption goods. We represent utility as a function of consumption. In this course we will always assume that the consumption is a linear function of the income. Therefore, our utility function will always be an increasing function of income (or wealth depending on the context).
- Utility function $u(\cdot)$ can be of three different forms.

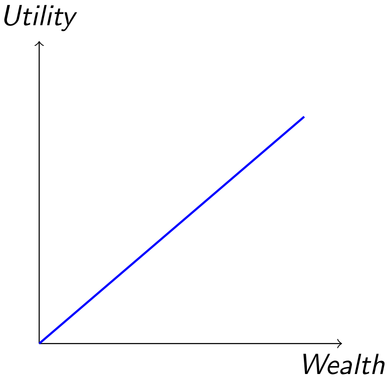
- Concave utility represents risk aversion.



- $u'' < 0$

Utility and risk attitude

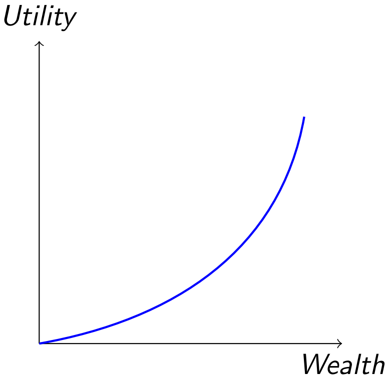
- Linear utility represents risk neutrality.



- $u'' = 0$

Utility and risk attitude

- Convex utility represents risk seeking.



- $u'' > 0$

- Consider an individual with initial wealth 100. There is a fair lottery which costs 50 and which yields either 0 or 100 with equal probability. Would an individual with utility function $\sqrt{W}$ purchase this lottery?

A lottery is fair if its price is equal to its expected value.

Utility and risk appetite

Optional exercise

- Consider an individual with initial wealth W_0. There is a fair lottery which costs $c < W_0$ and which yields any number between 0 and $\bar{x}$. Would a risk averse individual purchase this lottery?

Optimization

- Consider a maximization problem

$$\max_x u(x)$$

$$x \in [a, b]$$

- x is the decision variable.
- $u(x)$ is the objective function.
- $[a, b]$ is the set of feasible choices.
- We denote the solution by x^*.
- If there is an interior solution (*i.e.*, if $x^* \in (a, b)$), then $u'(x^*) = 0$. Otherwise (*i.e.*, if $x^* = a$ or $x^* = b$), we say that there is a corner solution. In that case the first derivative does not need to be zero at the optimum.
- Optimality requires $u''(x^*) \leq 0$.

Optimization

I expect you to be able to calculate simple derivatives such as

- $\dfrac{f(x)}{g(x)}$
- $f(x)^a$
- $ln\big(f(x)\big)$

Optimization example

- Suppose there are two investment opportunities. The first project yields 1 dollar next year for each dollar invested. The second project yields either nothing or 4 dollars with equal probability for each dollar invested. You have 100 dollars. You will invest all your wealth in these projects and then consume everything you will get (x) next year. How will you allocate your wealth over these two investment opportunities if your utility function is
    - $u = \sqrt{x}$
    - $u = x$
    - $u = x^2$.

Optimization example

Game theory

Game theory

- Many economic environments include interactions of individuals.
- Game theory provides us methods to understand how the optimal decisions are made when the action of one individual effects the payoff of others.

A problem is called an individual decision making problem when you do not need to worry about the responses of others.

Game theory

- An optimal portfolio problem (*e.g.*, how to allocate your money between a stock and a risk-free bond) is an individual decision making problem.
- Chess is a game theory problem because you need to take into account the next move of your opponent before choosing your move.
- Suppose you decide on how much time to invest in this course to get a good grade. Because your grade will be set according to your position on the score distribution, your effort (the number of hours to study) will be a function of other students' effort: if they work more, you will also work more.

Game theory

- We will call the decision making individuals involved in the problem as players.
- A feasible action of a player is called a (pure) strategy and the set of actions each player has is called strategy space or strategy set. We will denote strategy space by Θ.
- Strategy space can be discrete or continuous.

- Consider the previous example. One day before the exam, each student decides how many hours to spend on preparing for the exam.
- If a student studies x hours and if the average hours the rest of the class spends on studying is $\bar{x}$, then the probability that the student will receive a good grade is $\frac{x}{x+\bar{x}}$.
- The cost of studying is $\frac{x^2}{800}$ and if the student receives a good grade his utility increases by 1.
- Who are players?
- What is Θ (*i.e.*, their strategy space)?

- Sometimes it is more convenient to describe the payoffs of a game on a table. This representation is called normal form.
- *Meeting game*: Ying and Kai decide to meet at the library or in cafeteria to study together. Because they do not have phones they cannot communicate. Therefore, they should simultaneously decide whether to go to library or to cafeteria.

		Kai: *Library*	Kai: *Cafeteria*
Ying	*Library*	5, 5	3, 0
	Cafeteria	0, 3	2, 2

- Who are players? What is Θ?

- Consider a strategy $x \in \Theta$. If for each chosen action of other players there is always another strategy that pays off higher than (or equal to) x, then x is called a dominated strategy.
- Players never choose a dominated strategy. Therefore, one way to solve a game is to eliminate the dominated strategies.
- Is there a dominated strategy in *Meeting game*?

		Ying	
		Library	*Cafeteria*
Kai	*Library*	5, 5	3, 0
	Cafeteria	0, 3	2, 2

- *Prisoner's dilemma*: Two people are arrested for a crime. The police lack sufficient evidence to convict either suspect and need them to give testimony against each other. Each suspect is in a different cell.They have to choose whether to cooperate (do not testify) or defect (testify against his friend).

	Cooperate	*Defect*
Cooperate	$-1,-1$	$-3,0$
Defect	$0,-3$	$-2,-2$

- What is the solution of this game?

- How about the solution of this game (*Battle of sexes*):

	Football	*Opera*
Football	$3, 2$	$1, 1$
Opera	$0, 0$	$2, 3$

- There is no dominated strategy. We need to introduce the concept of equilibrium to solve this game.

Game theory

- (Nash) equilibrium describes the set of strategies chosen by all players such that given others' chosen actions, each player is happy to keep her own choice of action.
- In an equilibrium,
  - each player's strategy is best response to the strategies that he forecasts that his opponents will use,
  - and further that each player's forecast is correct.

In this course we will never be interested in how we reach an equilibrium. We are only interested in equilibrium strategies of players and their payoffs.

- What is the equilibrium of *Battle of sexes*?

	Football	*Opera*
Football	$3, 2$	$1, 1$
Opera	$0, 0$	$2, 3$

Game theory

- Some games have more than one equilibrium. When there are multiple equilibrium, it is difficult to know which equilibrium will be played.
- There are advanced techniques to eliminate some equilibria. These techniques are called refinements. We will not cover refinements in this course.

- What is the equilibrium of this game?

	L	R
U	$10, 10$	$0, 11$
D	$11, 0$	$1, 1$

- What is the equilibrium of this game?

	A	B	C
α	$1,3$	$0,1$	$3,0$
β	$2,2$	$1,0$	$4,3$
γ	$1,0$	$2,2$	$5,1$

- Another example: three players act simultaneously. The first player chooses U or D; the second player chooses L or R and the third player chooses the matrix. Find the equilibrium.

I

	L	R
U	$0,1,0$	$0,0,0$
D	$1,1,0$	$1,0,3$

II

	L	R
U	$2,2,2$	$0,0,0$
D	$2,2,0$	$2,2,2$

III

	L	R
U	$0,1,3$	$0,0,0$
D	$1,1,1$	$1,0,0$

- What is the equilibrium of this game (*Matching pennies*)?

	Head	*Tail*
Head	$1, -1$	$-1, 1$
Tail	$-1, 1$	$1, -1$

- In addition to pure strategies, there are also mixed strategies.
- Mixed strategies are probability distributions on pure strategies.
- For example, suppose the strategy space Θ is $\{Left, Right\}$. Then one mixed strategy would be: flip a coin and if tail comes play "*Left*", otherwise play "*Right*".
- A player uses a mixed strategy in equilibrium if she is indifferent between her pure strategies.
- Find the equilibrium of *Matching pennies*.

	Head	*Tail*
Head	$1, -1$	$-1, 1$
Tail	$-1, 1$	$1, -1$

- Is there a mixed strategy equilibrium in *Battle of sexes*?

	Football	*Opera*
Football	$3, 2$	$1, 1$
Opera	$0, 0$	$2, 3$

- What is the equilibrium of *Audit game*?

	Crime	*Honest*
Patrol	$2, -3$	$-1, 0$
Not	$-3, 3$	$0, 0$

Game theory - Optional material

- Every (finite) game has at least one equilibrium. Either in pure strategies or in mixed strategies.

Note that a pure strategy is a degenerate mixed strategy.

You will not be asked any question in the exams regarding mixed strategies.

- So far we studied normal form games and learned how to find the equilibrium. Normal form is useful when players move simultaneously. Sometimes players move in sequence. Such sequential games are better represented in extensive form.
- Consider *Matching pennies*, played in turns. Player 1 first chooses between *Head* and *Tail* and then player 2 upon observing player 1's action makes her choice.

- Each node represent the information set of a player. Sometimes a player does not know at which node he is at (*e.g.*, when the move of the previous player(s) is not observable). In that case we use a dotted line to define an information set.
- Represent the simultaneous-move *Matching pennies*.

Game theory

- In an extensive form, the part of the game that starts with a single node is called a subgame. In a subgame, the starting node should not share an information set with another node. Moreover, if a node in a particular information set is in the subgame, then all members of that information set belong to the subgame. The entire game itself is also a subgame.

- In a sequential game, it is reasonable to expect players to make their best moves when it is their turn. Such equilibrium is called a subgame perfect equilibrium.
- Strategies in a subgame perfect equilibrium constitute a Nash equilibrium in every subgame.

- Consider the *Battle of sexes* where row player moves first. What is the subgame perfect equilibrium?

	Football	*Opera*
Football	3, 2	1, 1
Opera	0, 0	2, 3

Game theory

- We use backward induction to find subgame perfect equilibrium. Consider *Centipede game* below.

- Find subgame perfect equilibrium of the *Entry* game below.

Game theory

- In all the games we analyzed so far players know the payoffs for any given set of strategy profiles.
- When some players do not know the payoffs, the game is said to have incomplete information.
- Many banking games we will cover during the course will have incomplete information. For example, credit customer knows whether her project will yield 1 or 10 while the banker thinks that there is 50% chance that the project will yield 10.

- There are two kinds of incomplete information games: adverse selection and moral hazard. Almost all the problems in information economics can be classified as either adverse selection or moral hazard or both.
- We will start with adverse selection.

- One way to model an incomplete information game is to introduce a prior move by nature that determines player 1's type.
- Possible set of types is called type space.
- We will call an incomplete information game with a type space adverse selection problem. For example, in the health insurance market, insurance company usually does not know the health condition (*i.e.*, the type) of the customer: the customer can be inclined to have a diabetic or not.
- Probability distribution of types is called beliefs.

Adverse selection problem is also called hidden information problem.

- Player 1 decides whether to build a new plant and simultaneously player 2 decides whether to enter. Imagine that player 2 is uncertain whether player 1's cost of building is high (*i.e.*, 4) or low (*i.e.*, 0), while player 1 knows her own cost. The payoffs (before player 1's cost) are given below.

High cost

	Enter	*Don't*
Build	$0, -1$	$2, 0$
Don't	$2, 1$	$3, 0$

Low cost

	Enter	*Don't*
Build	$3, -1$	$5, 0$
Don't	$2, 1$	$3, 0$

- Suppose first that player 2 believes that the probability of player 1 having high cost is 10%. What is the equilibrium?

- Suppose next that player 2 believes that the probability of player 1 having high cost is 90%. What is the equilibrium?

- Every adverse selection problem starts with an initial belief on types. Then, after observing actions of players, beliefs are revised using Bayesian updating rule. This process is called learning.

- $Pr(A|B) = \dfrac{Pr(B|A)\ x\ Pr(A)}{Pr(B)}$

- Suppose that 10% of students in the school are lazy. The rest of the students are hard workers. It is impossible to observe from outside whether a student is lazy or not. A hard worker always passes a course while a lazy has a 20% chance to pass the course.
- If you pick a random student what is the probability that he is lazy?
- You observe that a student has just passed a course. What is the probability that this student is lazy?

- Suppose you want to buy a car. There are two types of car in the market. Good ones and lemons. You are willing to pay 1000 dollars to a good one and 100 dollars to a lemon. It is impossible to know whether the car is good or not, but the seller knows whether his car is good or a lemon. The value of a good car to its seller is 500 dollars and a lemon is 0 dollars (*i.e.*, the owner of a good car is willing to sell if any only if the price is more than 500 dollars). You think that half of the cars in the market are lemons. What is the equilibrium? That is, what type of sellers will sell their cars and at which price?

- The equilibrium is called pooling if all types take the same action in an equilibrium of the adverse selection problem. If all different types take different actions, then the equilibrium is called separating.
- In a separating equilibrium, one can differentiate types by looking at their actions. Therefore, a separating equilibrium is sometimes called a revealing equilibrium.

There is also semi-separating equilibrium where only some types pool their actions.

- Is the equilibrium of the *Lemons market* pooling or separating?

- Suppose you believe that only 10% of cars in the market are good. What is the equilibrium? Is it pooling or separating?

- If the source of the asymmetric information is not nature (*i.e.*, is not types) but the actions of other player(s), then the problem is called moral hazard.
- For example, banker wants her credit customer to choose safe projects, but usually she cannot observe which projects are chosen by the customer. In that case, before extending the loan, she anticipates the optimal choice of the credit customer and design initial credit contract accordingly (*e.g.*, she might increase the credit interest rate to be compensated by the possible risk the customer will take).

Moral hazard problem is also called hidden action problem.

- Your car is worth 1000 dollars. At night, you can either park it on the street or can park in a private garage. If you park on the street, there is 10% probability that it will be stolen. The cost of parking in a garage is 2 dollars. Garage protects your car from thieves. Would you park in the garage?
- Suppose next that there is an insurance company which is ready to insure your car. If insurance premium is 1 dollar, would you insure your car? If yes, would you park on the street.
- Is 1 dollar a fair premium?

An important concept: limited liability

Limited liability

- One of the biggest achievements in the history is the introduction of limited liability in investments. It allows accumulation of capital which spurs economic growth.
- In modern corporations, an investor's financial liability is limited to the value of her investment in the company. In particular owners' liability for the firms' obligations are limited to the amount they put as equity. If a firm has more obligation than the value of its assets, the owner is not responsible.

Limited liability

- Throughout the course we will assume that owners of firms have limited liability.
- *Example:* Suppose a firm has 1.5 dollars obligation to a bank. Firm's assets will generate either 2 or 0.5 dollars with equal probability. Firm does not have any other asset or obligation. What are the expected payoffs of the bank and the owner?

Limited liability

- One consequence of limited liability is "risk shifting", also known as asset substitution. Firms might have incentive to take risky (and sometimes negative NPV) projects when the owners have limited liability.
- *Example:* Suppose a firm has 1 dollar obligation to a bank and 1 dollar cash to invest. There are two investment opportunities. First project generates 1.2 dollars for each dollar invested. Second project generates either nothing or 1.8 dollars with equal probability. Which project will be chosen by the firm?