# Lecture 4
# Portfolio Mathematics and Capital Allocation

# Recap from Last Lecture

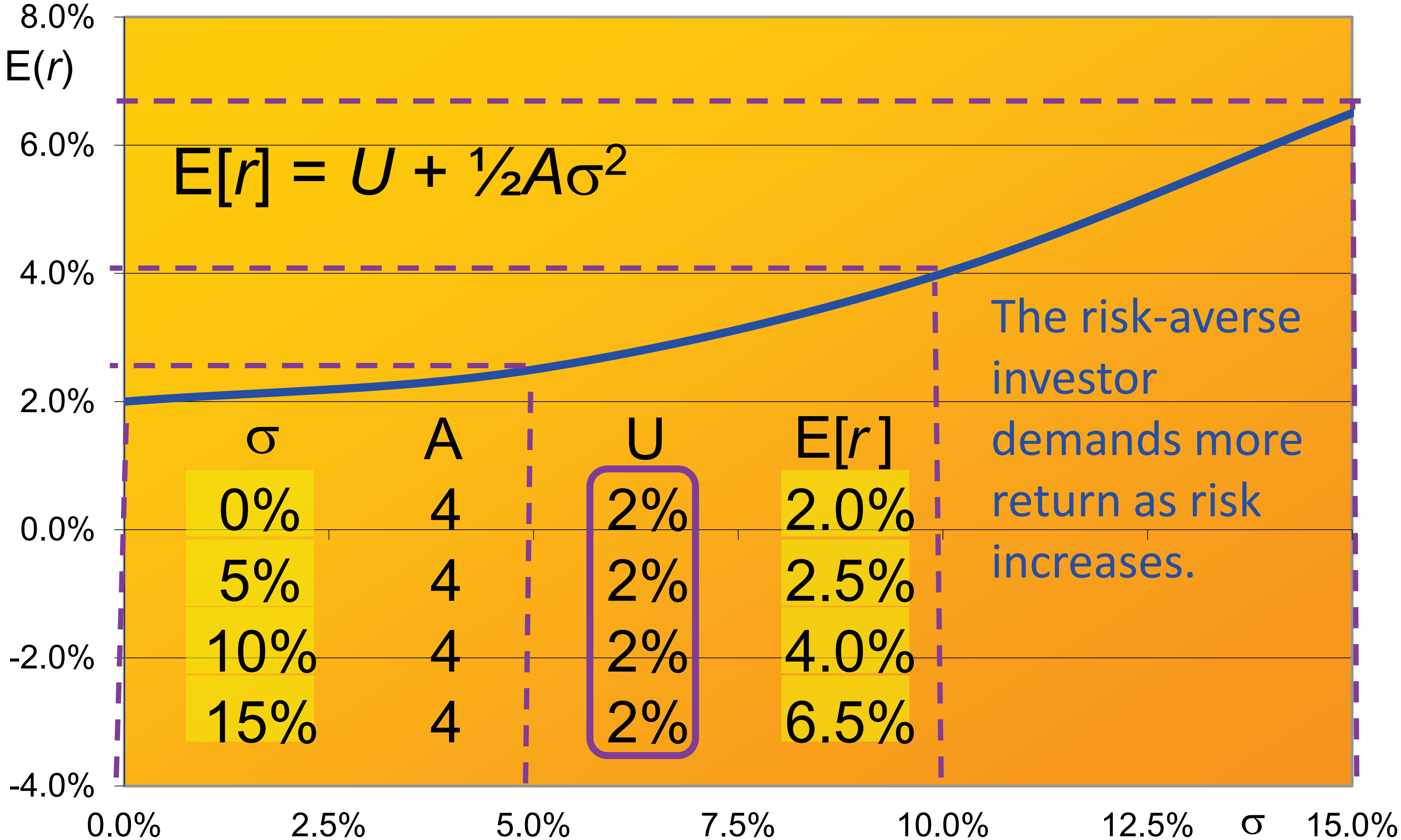
- Expected vs. Realized Return & Risk
  - Expected: $\mu$, $\sigma^2$, Realized: R(r), $s^2$
- Value at Risk (VaR)
  - Another measure of risk, works well for any distribution
- Riskfree Prospects & the Risk Premium
- Risk Preferences & Returns Utility

# Risk: Averse, Neutral, Loving

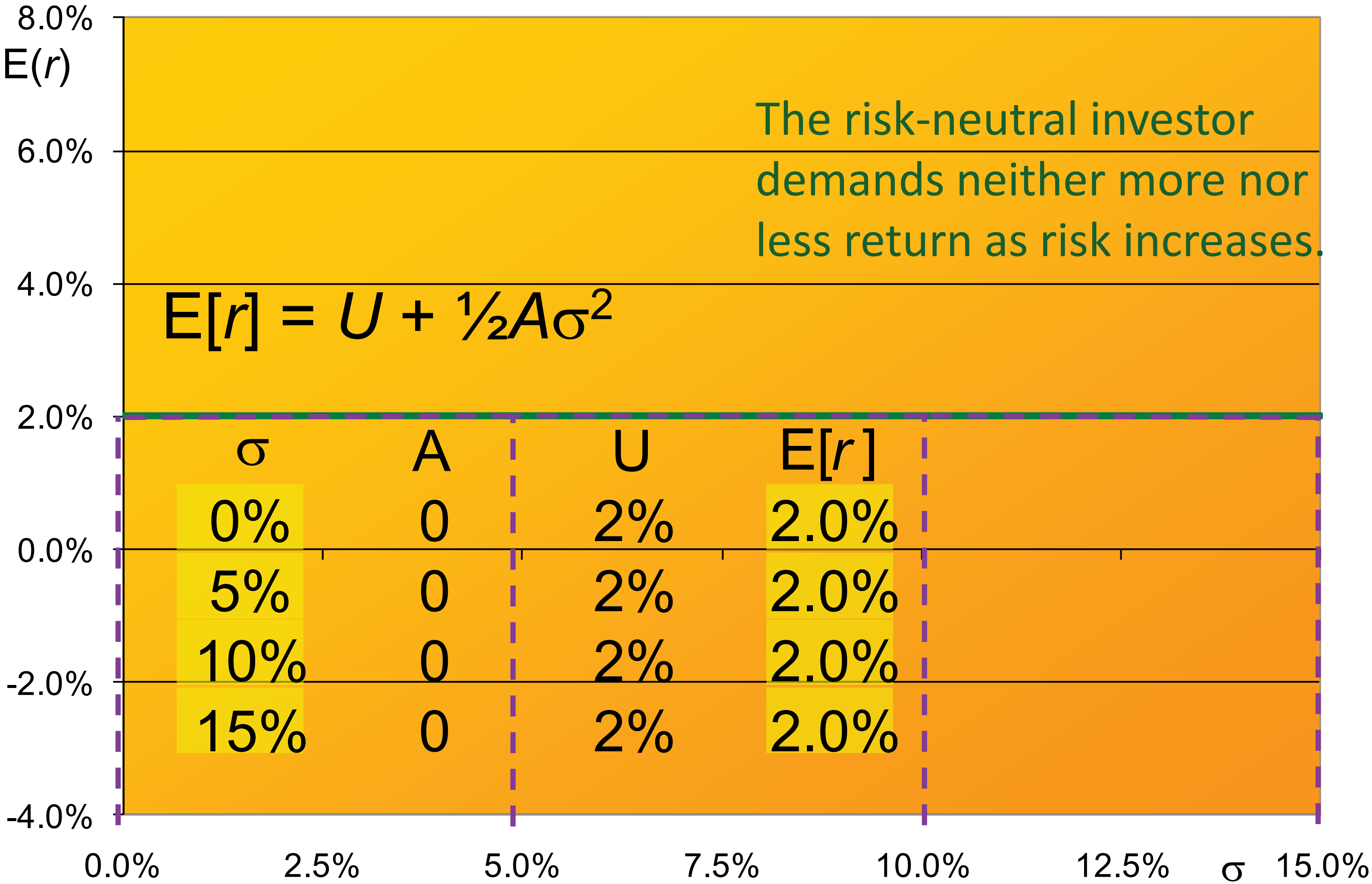
- Utility function: $U = E(r) – 1/2\ A\sigma^2$
  - Risk averse: $A > 0$ (usual case)
  - Risk neutral: $A = 0$
  - Risk loving: $A < 0$

# Utility Indifference: Risk Averse

Utility Indifference: The return-risk combinations that are equally agreeable (i.e., generate the same utility level) to a given investor.



# Utility Indifference: Risk Neutral

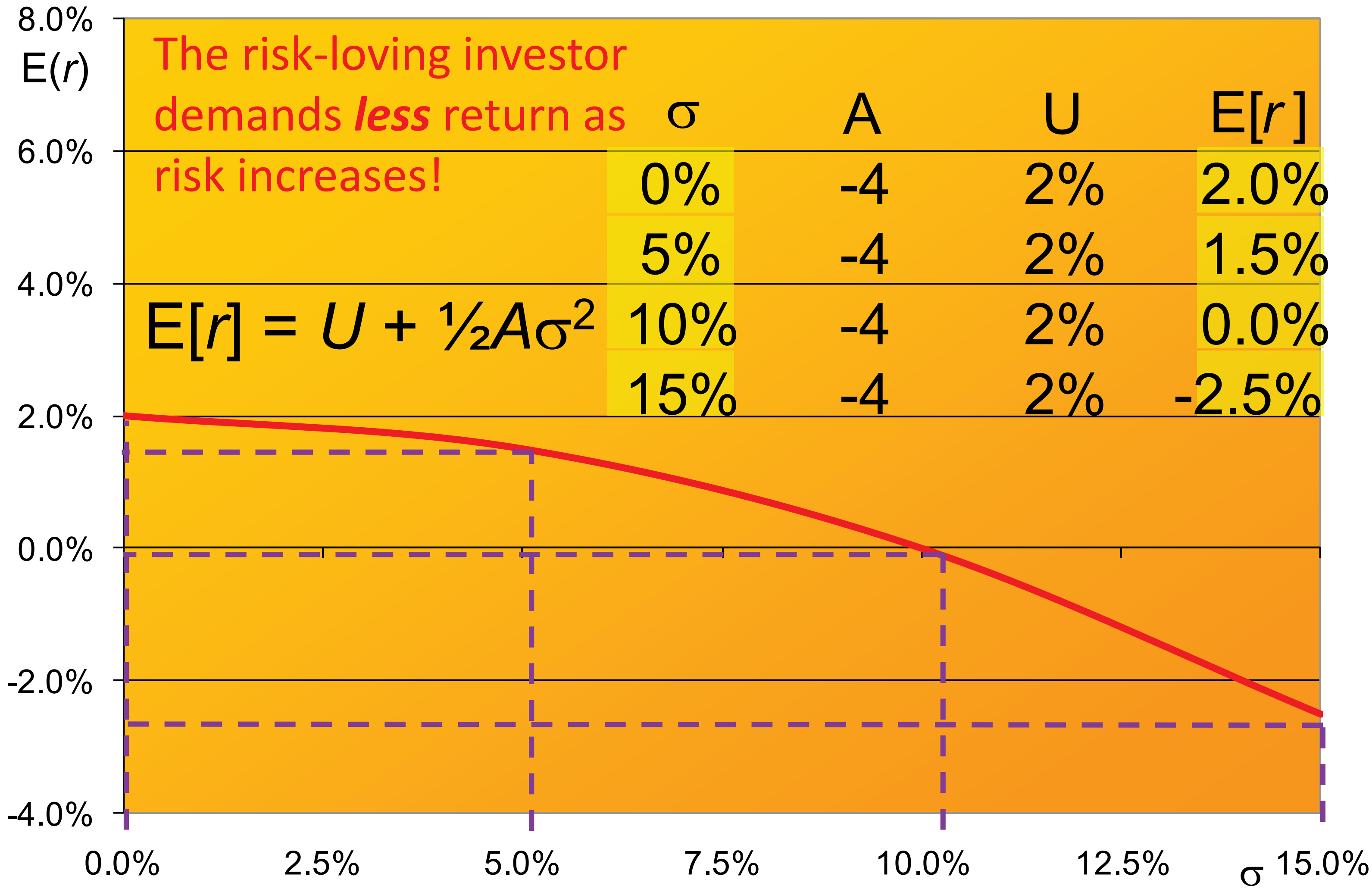



The risk-neutral investor demands neither more nor less return as risk increases.

$$E[r] = U + \frac{1}{2}A\sigma^2$$

| σ | A | U | E[r] |
|---|---|---|---|
| 0% | 0 | 2% | 2.0% |
| 5% | 0 | 2% | 2.0% |
| 10% | 0 | 2% | 2.0% |
| 15% | 0 | 2% | 2.0% |

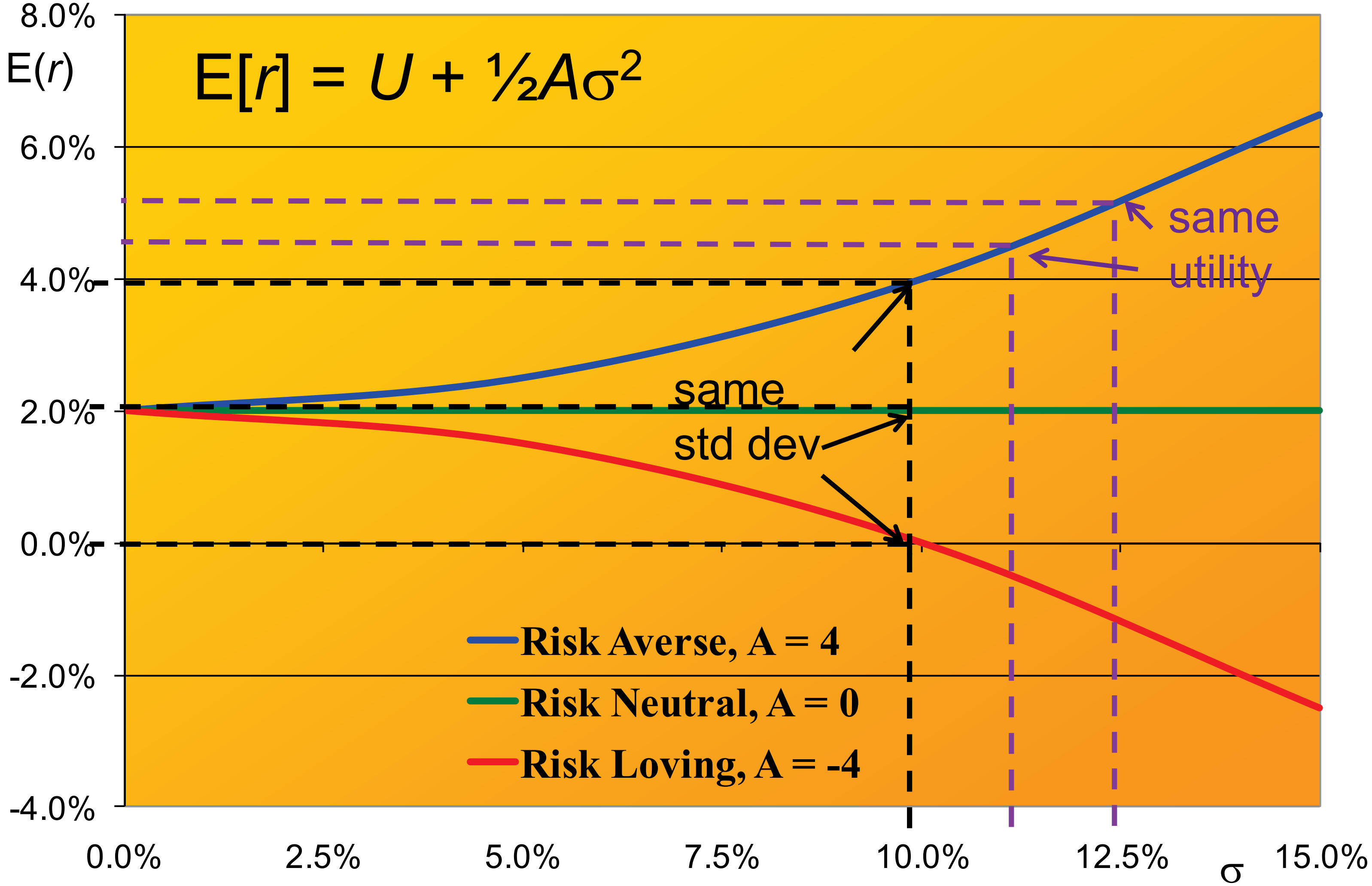
# Utility Indifference: Risk Loving



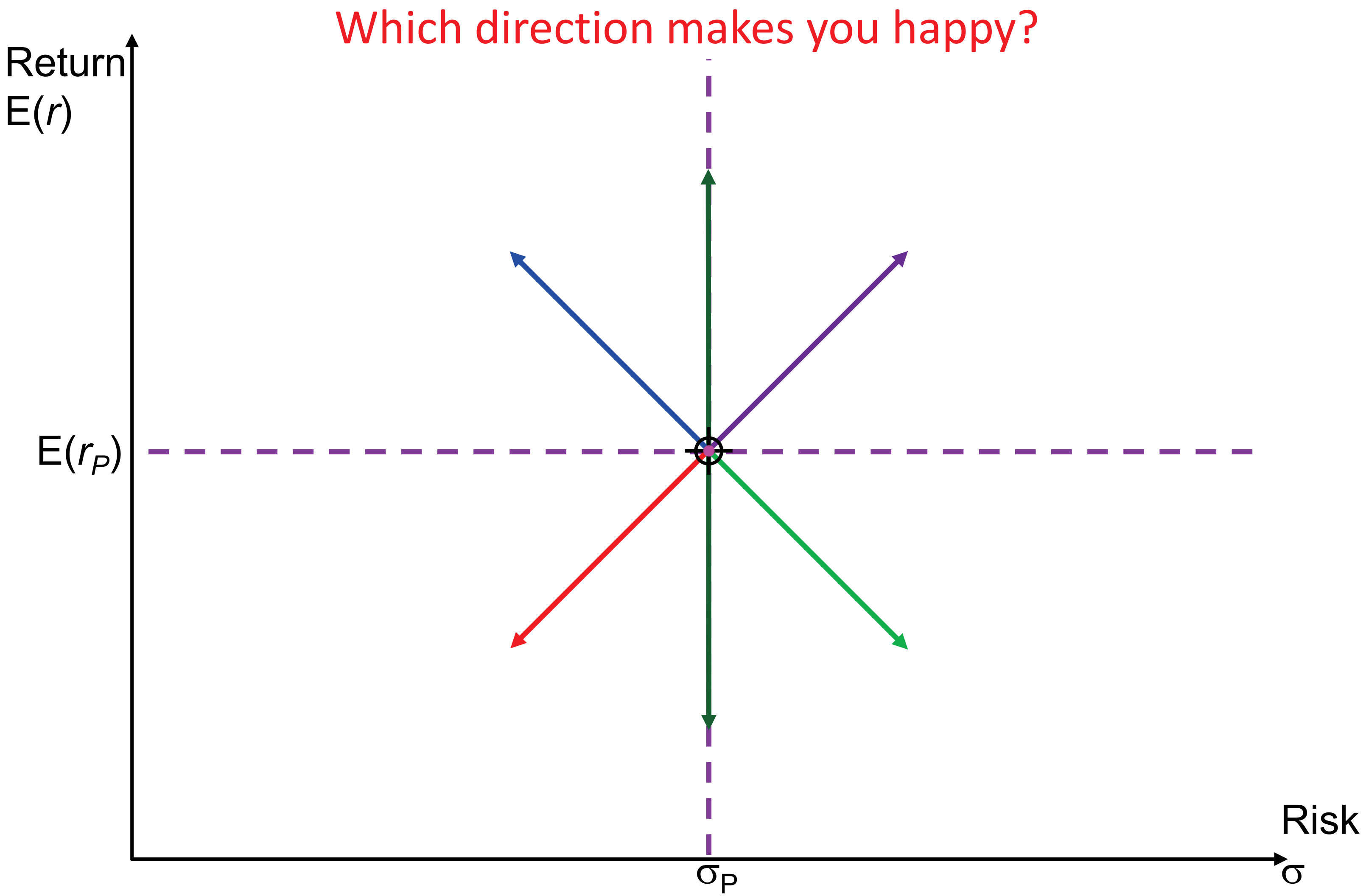

$E[r] = U + \frac{1}{2} A\sigma^2$

| σ | A | U | E[*r*] |
|---|---|---|---|
| 0% | -4 | 2% | 2.0% |
| 5% | -4 | 2% | 1.5% |
| 10% | -4 | 2% | 0.0% |
| 15% | -4 | 2% | -2.5% |

# Utility Indifference Curve



# Mean-Variance Criterion



# Outline of Today's Lecture

- **Risk & Return for Security Portfolios**
- Allocation Decision Levels
- Complete Portfolio Return & Risk
- The Capital Allocation Line

# Expected Portfolio Return

- Expected Portfolio Return, $E(r_P)$

$$E(r_P) = w_1 E(r_1) + w_2 E(r_2)$$

where $w_1$ and $w_2$ are portfolio weights, or the fraction of wealth the investor puts into each security.

- E.g., Invest \$25 in $r_1$ and \$75 in $r_2$:

$$w_1 = 25/100,\ w_2 = 75/100$$

# Example

- Suppose you have \$100 to invest, which you split equally between stocks 1 & 2:

$$w_1 = \$50/\$100 = 50\% = w_2$$

$$E(r_P) = w_1 E(r_1) + w_2 E(r_2)$$

$$E(r_P) = 0.50(4\%) + 0.50(8\%) = 6\%$$

This portfolio is expected to return 6%.

# Portfolio Variance

- Portfolio Variance, $\sigma_P^2 = E[r_P - E(r_P)]^2$

$$\sigma_P^2 = E[w_1 r_1 + w_2 r_2 - E(w_1 r_1 + w_2 r_2)]^2$$

$$= E[w_1 r_1 - E(w_1 r_1) + w_2 r_2 - E(w_2 r_2)]^2$$

$$= E[w_1 r_1 - E(w_1 r_1)]^2 + E[w_2 r_2 - E(w_2 r_2)]^2$$

$$+ 2E[(w_1 r_1 - E(w_1 r_1))(w_2 r_2 - E(w_2 r_2))]$$

# Portfolio Variance

$$\sigma_P^2 = \mathrm{Var}(w_1 r_1) + \mathrm{Var}(w_2 r_2) + 2\mathrm{Cov}(w_1 r_1, w_2 r_2)$$

$$= w_1^2 \mathrm{Var}(r_1) + w_2^2 \mathrm{Var}(r_2) + 2 w_1 w_2 \mathrm{Cov}(r_1, r_2)$$

$$\sigma_P^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \sigma_{12}$$

where $\sigma_{12} = \mathrm{Cov}(r_1, r_2)$, which we call

the **covariance** of the securities.

# Covariance

- Measures if two variables move together

$$\sigma_{12} = \mathrm{Cov}(r_1,r_2) = E[((r_1\text{-}E(r_1))((r_2\text{-}E(r_2))]$$

  If $\sigma_{12} > 0$, they are **positively** related

  If $\sigma_{12} < 0$, they are **negatively** related

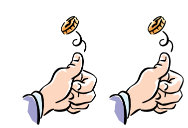
  If $\sigma_{12} = 0$, they are **unrelated**

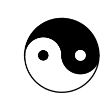
- Note that $\sigma_{22} = \sigma_2^{\,2} = E[(r_2\text{-}E(r_2))]^2$ means variance is a variable's own covariance

# Correlation: $\rho$

- A normalized measure of covariance
- $\rho_{12} = \dfrac{\sigma_{12}}{\sigma_1 \sigma_2}$

$0 < \rho_{12} \leq 1 \rightarrow$ Variables are **positively** correlated 

$\rho_{12} = 0 \rightarrow$ Variables **uncorrelated** 

$-1 \leq \rho_{12} < 0 \rightarrow$ Variables are **negatively** correlated 

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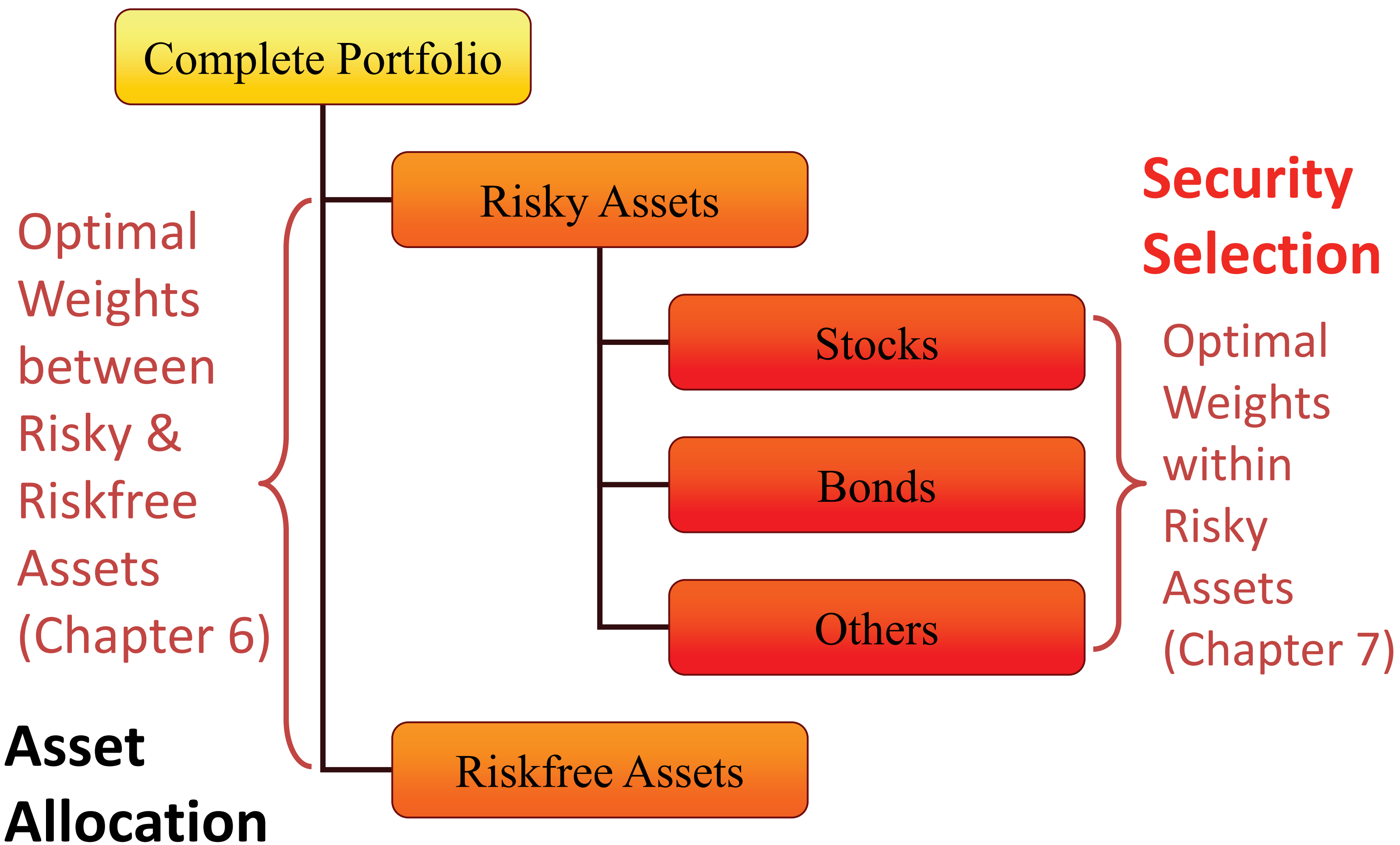
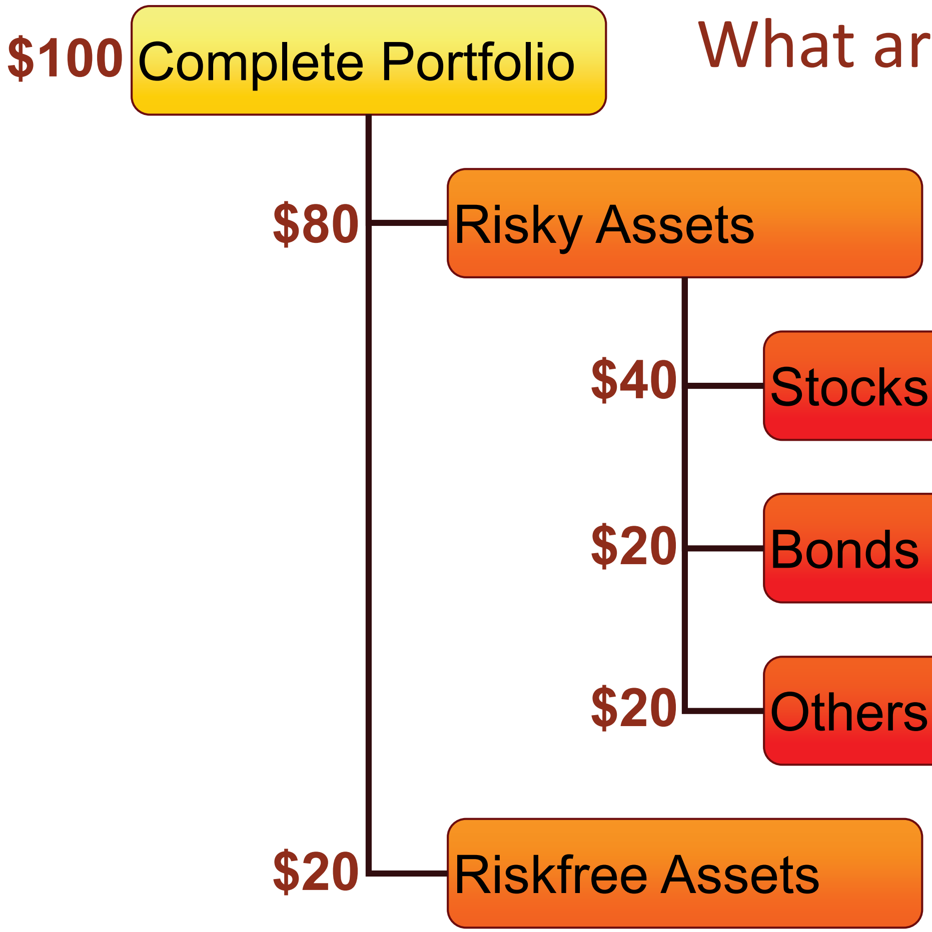
# Allocation Decision Levels



# Allocation Weights Example

What are the optimal *y* & *w*?




$y = 80/100 = 80\%$

$w_S = 40/80 = 50\%$

$w_B = 20/80 = 25\%$

$w_O = 20/80 = 25\%$

$1 - y = 20/100 = 20\%$

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- ✓ Allocation Decision Levels
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# Complete Portfolio Return

$$E(r_C) = yE(r_P) + (1 - y)r_f$$

$$\rightarrow \qquad E(r_C) = r_f + y\underbrace{[E(r_P) - r_f]}$$

Riskfree return:
reward for inflation and deferred consumption ***without*** risk.

Risk weight (quantity of risk taken)

Risk premium:
reward for taking risk (price of risk)

# Complete Portfolio Risk

$$\sigma_C^2 = y^2\sigma_P^2 + (1 - y)^2\sigma_f^2 + 2y(1 - y)\sigma_{Pf}$$

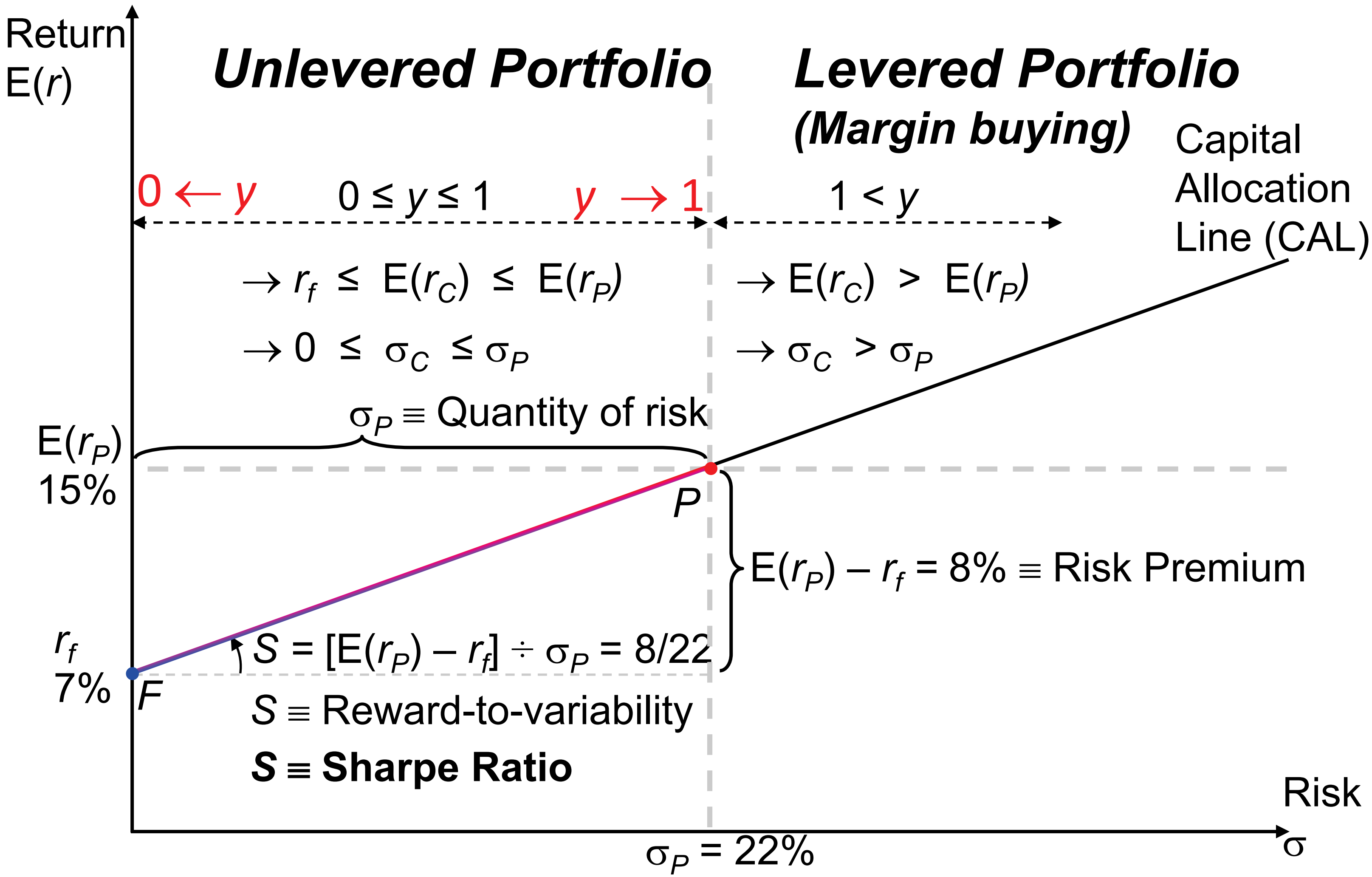
But $\sigma_{Pf} = \sigma_f^2 = 0$ by definition, so:

$\sigma_C^2 = y^2\sigma_P^2 \quad \rightarrow \sigma_C = y\sigma_P$

- Suppose $E(r_P) = 15\%$, $\sigma_P = 22\%$, $r_f = 7\%$:

1. $E(r_C) = r_f + y[E(r_P) - r_f] = 7\% + y8\%$, and
2. $\sigma_C = y22\%$

# Possible Complete Portfolios



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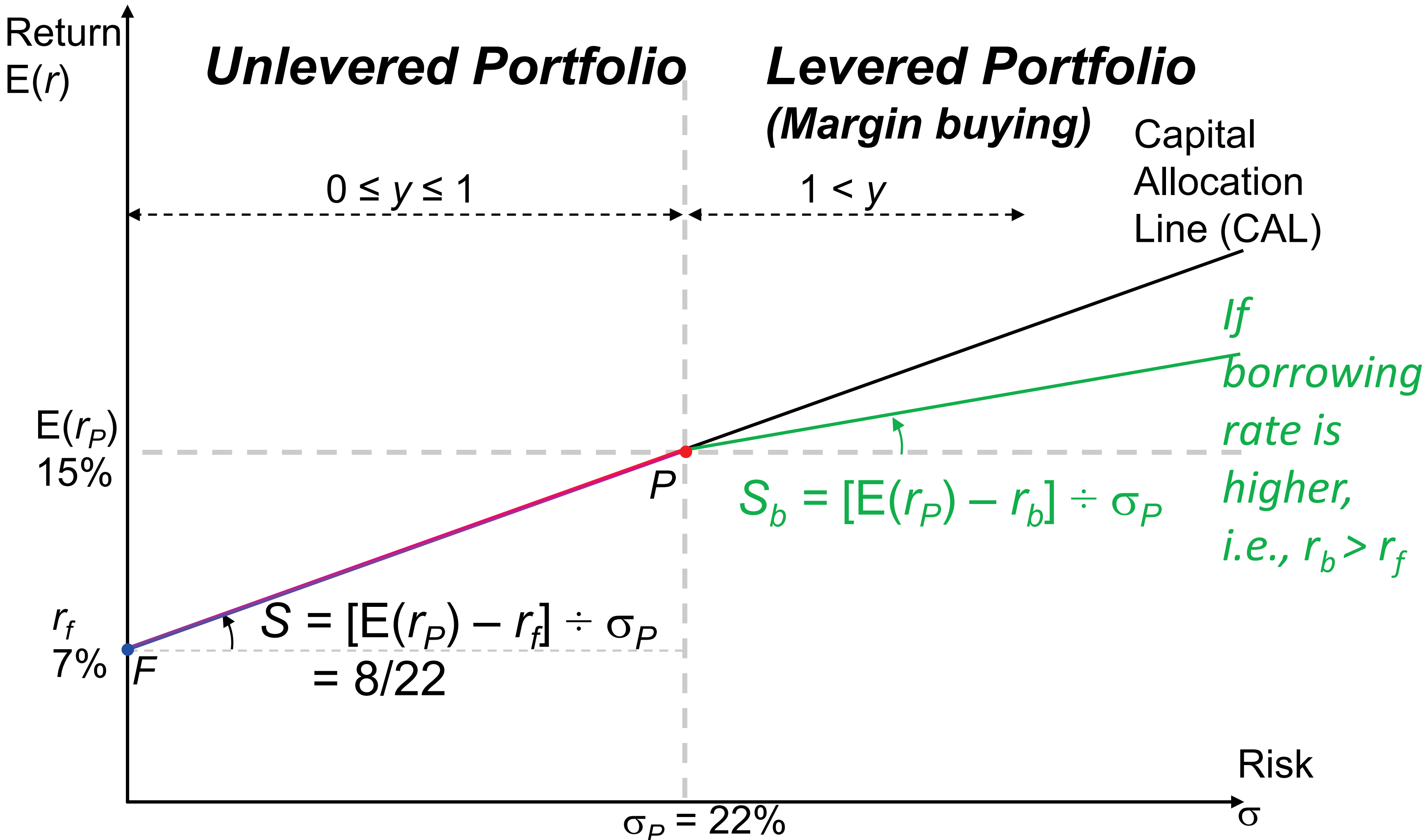
# The Capital Allocation Line



# Reference

- Investments book
  - Chapters 5-6