02435 - Decision-making under Uncertainty

Assignment 3

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Task1 - L-shaped method

1) The optimization problem proposed is decomposed using the L-shaped method. To do so, an artificial full recourse is first added to the stochastic program.
This is done by introducing the following elements in the program:

- Variables $v^+_{t,s}$ and $v^-_{t,s}$ (defined greater or equal to 0) measure the infeasibility in constraint (4).
- Parameters ϕ^+ and ϕ^- penalize these infeasibilities in the objective function.

Thus, the objective function is changed as follows:

$$\min \sum_{t\in\mathscr{T}} \left[\sum_{j\in\mathscr{J}} \left(c_j^{CHP} q_{j,t}^{CHP} - \sum_{s\in\mathscr{S}} \pi_s e_{t,s} \frac{1}{\phi_j} q_{j,t}^{CHP} \right) + \sum_{i\in\mathscr{I}} \sum_{s\in\mathscr{S}} \pi_s c_i^H q_{i,t,s}^H + \sum_{s\in\mathscr{S}} (\phi^+ v^+_{t,s} + \phi^- v^-_{t,s}) \right]$$

Constraint (4) is changed as follows:

$$\sum_{j\in\mathscr{J}} q_{j,t}^{CHP} + \sum_{i\in\mathscr{I}} q_{i,t,s}^H + v^+_{t,s} - v^-_{t,s} = d_{t,s} \qquad \forall t\in\mathscr{T}, \forall s\in\mathscr{S}$$

The introduction of this artificial full recourse is needed, because solutions to the master problem could lead to infeasible solutions in the subproblems. This is due to constraint (4). If the solutions obtained in the master problems are so that $\sum_{i\in\mathscr{I}} q^H_{i,t,s,fix} > d_{t,s}$, then the problem would be infeasible without this artificial full recourse added ($q_{j,t}^{CHP}$ cannot take negative values).

2) The problem can be decomposed into master and subproblems as follows:

Master problem

$$\min \sum_{t\in\mathscr{T}} \sum_{j\in\mathscr{J}} \left(c_j^{CHP} q_{j,t}^{CHP} - \sum_{s\in\mathscr{S}} \pi_s e_{t,s} \frac{1}{\phi_j} q_{j,t}^{CHP} \right) + \theta$$

s.t. $q_{j,t}^{CHP} \leq Q_j^{CHP} \qquad \forall j\in\mathscr{J}, \forall t\in\mathscr{T}$

$$\sum_{t\in\mathscr{T}} \left(E_{l,t} \sum_{j\in\mathscr{J}} q_{j,t}^{CHP} \right) + \theta \geq \sum_{t\in\mathscr{T}} e_{l,t} \qquad \forall l\in\mathscr{L}$$

$q_{j,t}^{CHP} \geq 0 \qquad \forall j\in\mathscr{J}, \forall t\in\mathscr{T}$

Subproblems

$$\min \sum_{t\in\mathscr{T}} \left[\sum_{i\in\mathscr{I}} \pi_s c_i^H q_{i,t,s}^H + (\phi^+ v_{t,s}^+ + \phi^- v_{t,s}^-) \right]$$

$$\begin{aligned}
\textbf{s.t. } & q_{i,t,s}^H \leq Q_i^H && \forall i \in \mathscr{I}, \forall t \in \mathscr{T} \quad \text{(2a)}\\
& \sum_{i\in\mathscr{I}} q_{i,t,s}^H + v_{t,s}^+ - v_{t,s}^- = d_{t,s} - \sum_{j\in\mathscr{J}} q_{j,t,fix}^{CHP} && \forall t \in \mathscr{T} \quad \text{(2b)}\\
& q_{i,t,s}^H \geq 0 && \forall i \in \mathscr{I}, \forall t \in \mathscr{T} \quad \text{(2c)}\\
& v_{t,s}^+, v_{t,s}^- \geq 0 && \forall t \in \mathscr{T} \quad \text{(2d)}
\end{aligned}$$

3) For each subproblem s, the dual objective function is expressed as follows:

$$\max \sum_{t\in\mathscr{T}} \left[\sum_{i\in\mathscr{I}} \lambda_{i,t,s} Q_i^H + \mu_{t,s} \left(d_{t,s} - \sum_{j\in\mathscr{J}} q_{j,t,fix}^{CHP} \right) \right]$$

, where $\lambda_{i,t,s}$ represent the dual variables associated to constraint 2a in subproblem s, and $\mu_{t,s}$ represent the dual variables associated to constraint 2b in subproblem s.

Considerering the constraints they are associated to, it holds that:

$\lambda_{i,t,s} \geq 0 \quad \forall (i,t,s) \in \mathscr{I} * \mathscr{T} * \mathscr{S}$

and

$\mu_{t,s} \in \mathbb{R} \quad \forall (t,s) \in \mathscr{T} * \mathscr{S}$

4) Applying the week duality theorem, it holds that the recourse function θ verifies the following inequality at cut level l:

$$\sum_{t\in\mathscr{T}} \left(E_{l,t} \sum_{j\in\mathscr{J}} q_{j,t} \right) + \theta \geq \sum_{t\in\mathscr{T}} e_{l,t}$$

, where cut coefficient $E_{l,t}$ and cut right-hand-side $e_{l,t}$ are expressed as:

$$E_{l,t} = \sum_{s\in\mathscr{S}} \pi_s \mu_{t,s}$$

$$e_{l,t} = \sum_{s\in\mathscr{S}} \pi_s \left(\sum_{i\in\mathscr{I}} \lambda_{i,t,s} Q_i^H + \mu_{t,s} d_{t,s} \right)$$

5) Using the formulas established above, the cut constraint for the master problem can be formulated as follows:

$$\sum_{t\in\mathscr{T}}\left(\sum_{s\in\mathscr{S}}\pi_s\mu_{t,s}\sum_{j\in\mathscr{J}}q_{j,t}^{CHP}\right)+\theta\geq\sum_{t\in\mathscr{T}}\sum_{s\in\mathscr{S}}\pi_s\left(\sum_{i\in\mathscr{I}}\lambda_{i,t,s}Q_i^H+\mu_{t,s}d_{t,s}\right)$$

Task 2 - (Meta)heuristics - General questions

A heuristic approach comes in very handy when trying to solve optimization problems which involve integer variables. In these problems, computational time grows exponentially with the size of the problem. The goal of a heuristics is to find a strategy that is specific to the problem at hand to get a good enough solution - not necessary the optimal one. This is the main advantage of such approach, as it allows to get good solutions rather than waiting for years to get the optimal one.

A metaheuristics is an improvement heuristics which relies on two main concepts: **intensification** and **diversification** of the solution. The latter provides a great advantage to this technique compared to other improvement heuristics which rely only on local search. It allows to escape local optima and to investigate other parts of the solution space.

The main disadvantages about heuristics are linked to their specificity. All the parameters have to be adapted to the problem at hand - e.g. the neighboring operator, or the weights given to intensification and diversification processes. Plus, it is not always obvious to know if a solution is better than another, which means that a proper fitness function needs to be defined. It becomes even more complex when dealing with stochastic programs, since the fitness function involves expected values, which must be either calculated - which is possibly very time-consuming - or approximated - which induces errors.

Task 3 - Robust optimization - General questions

1) The two main goals wanted to be reached when implementing a robust optimization process are:

- to take into account the uncertainty of the parameters in the model, and to make sure that the model is feasible for all values taken by the uncertain parameters
- to make sure that a small deviation in the data parameters would not lead to an infeasible problem

2) In stochastic programming, the constraints have to be fulfilled only for the values of the parameters considered. In robust optimization, they have to be fulfilled for every possible values taken by the uncertain parameters. Plus, unlike stochastic programming, there are only here-and-now decision variables in robust optimization.

3) Adjustable robust optimization is less conservative than normal robust optimization, because it allows some variables to react to the outcome of the uncertain parameters. In normal robust optimization, all the variables have to respect their constraints for the whole range of possible outcomes, while in adjustable robust optimization, some of them can take an optimal value depending on the outcome of the uncertainty.

Task4 - Reformulation to robust linear models

The aim of this exercise is to reformulate models which include uncertain parameters as robust counterparts given the corresponding uncertainty sets.

1) The model is formulated as follows:

$$\max 10x_1 + 20x_2 + 15x_3 \tag{3a}$$

$$
\begin{aligned}
\text{s.t. } & 5x_1 + 3x_2 + \tilde{a}x_3 \leq 10 && \text{(3b)}\\
& 7x_1 - 2x_2 - \tilde{b}x_3 \geq 5 && \text{(3c)}\\
& -2 \leq x_1 \leq 10 && \text{(3d)}\\
& 0 \leq x_2 \leq 15 && \text{(3e)}\\
& -10 \leq x_3 \leq 10 && \text{(3f)}
\end{aligned}
$$

Where $\tilde{a} \sim U(1,5)$ and $\tilde{b} \sim U(1,3)$. These 2 uncertainty sets can be reformulated as follows:

$\tilde{a} = \bar{a} + P^a\zeta^a$, with $\bar{a} = 3$, $\mid \zeta^a \mid \leq 1$ and $P^a = 2$

$\tilde{b} = \bar{b} + P^b\zeta^b$, with $\bar{b} = 2$, $\mid \zeta^b \mid \leq 1$ and $P^b = 1$

Constraints 3b and 3c are thus reformulated as follows:

$5x_1 + 3x_2 + \bar{a}x_3 + P^a\zeta^a x_3 \leq 10 \quad \forall -1 \leq \zeta^a \leq 1$

$7x_1 - 2x_2 - \bar{b}x_3 - P^b\zeta^b x_3 \geq 5 \quad \forall -1 \leq \zeta^b \leq 1$

- For constraint 3b, the worst case is represented by: $5x_1 + 3x_2 + \bar{a}x_3 + \mid P^a x_3 \mid \leq 10$ This formulation with absolute value can be linearized by introducing the variable u^a such as:

  $5x_1 + 3x_2 + \bar{a}x_3 + u^a \leq 10$
  $-u^a \leq P^a x_3 \leq u^a$
  $u^a \geq 0$

- For constraint 3c, the worst case is represented by: $7x_1 - 2x_2 - \bar{b}x_3 - \mid P^b x_3 \mid \geq 5$ This formulation with absolute value can be linearized by introducing the variable u^b such as:

  $7x_1 - 2x_2 - \bar{b}x_3 - u^b \geq 5$
  $-u^b \leq P^b x_3 \leq u^b$
  $u^b \geq 0$

Eventually, the robust formulation of model 3 is:

$$\max 10x_1 + 20x_2 + 15x_3 \tag{4a}$$

$$
\begin{aligned}
\text{s.t. } & 5x_1 + 3x_2 + 3x_3 + u^a \leq 10 && \text{(4b)}\\
& 7x_1 - 2x_2 - 2x_3 - u^b \geq 5 && \text{(4c)}\\
& -u^a \leq 2x_3 \leq u^a && \text{(4d)}\\
& -u^b \leq x_3 \leq u^b && \text{(4e)}\\
& -2 \leq x_1 \leq 10 && \text{(4f)}\\
& 0 \leq x_2 \leq 15 && \text{(4g)}\\
& -10 \leq x_3 \leq 10 && \text{(4h)}\\
& u^a \geq 0 && \text{(4i)}\\
& u^b \geq 0 && \text{(4j)}
\end{aligned}
$$

Solving this model with Julia JuMP using Gurobi yields the following results:

Objective value	40.32
x_1	1.129
x_2	1.452
x_3	0.000

Table 1: First robust model results

2) The model is formulated as follows:

$$\max 10x_1 + \tilde{c}x_2 + 15x_3 \qquad \text{(5a)}$$

$$
\begin{aligned}
\text{s.t. } & 5x_1 + 3x_2 + 3x_3 \leq 10 && \text{(5b)}\\
& 7x_1 - 2x_2 - 2x_3 \geq 5 && \text{(5c)}\\
& -2 \leq x_1 \leq 10 && \text{(5d)}\\
& 0 \leq x_2 \leq 15 && \text{(5e)}\\
& -10 \leq x_3 \leq 10 && \text{(5f)}
\end{aligned}
$$

, where $\tilde{c} \sim U(5, 35)$. The uncertainty in the objective function 5a can be modeled by introducing a variable $t \in \mathbb{R}$:

$\max 10x_1 + t + 15x_3$

, and by adding a constraint stating that:

$t \leq \tilde{c}x_2$

This uncertainty set can be reformulated as follows:

$\tilde{c} = \bar{c} + P^c\zeta^c$, with $\bar{c} = 20$, $\mid \zeta^c \mid \leq 1$ and $P^c = 15$

The newly introduced constraint can thus reformulated as follows:

$t \leq \bar{c}x_2 + P^c\zeta^c x_2 \quad \forall -1 \leq \zeta^c \leq 1$

For this constraint, since x_2 is positive, the worst case is represented by:

$t \leq \bar{c}x_2 - P^c x_2$

Eventually, the robust formulation of model 5 is:

$$\max 10x_1 + t + 15x_3 \tag{6a}$$

$$\begin{aligned}
\text{s.t. } & 5x_1 + 3x_2 + 3x_3 \leq 10 && \text{(6b)}\\
& t \leq 5x_2 && \text{(6c)}\\
& 7x_1 - 2x_2 - 2x_3 \geq 5 && \text{(6d)}\\
& -2 \leq x_1 \leq 10 && \text{(6e)}\\
& 0 \leq x_2 \leq 15 && \text{(6f)}\\
& -10 \leq x_3 \leq 10 && \text{(6g)}\\
& t \in \mathbb{R} && \text{(6h)}\\
& && \text{(6i)}
\end{aligned}$$

Solving this model with Julia JuMP using Gurobi yields the following results:

Objective value	33.06
x_1	1.129
x_2	0.000
x_3	1.452

Table 2: Second robust model results

3) The model is formulated as follows:

$$\max 10x_1 + 20x_2 + 15x_3 \tag{7a}$$

$$\begin{aligned}
\text{s.t. } & 5x_1 + \tilde{a}_2 x_2 + \tilde{a}_3 3x_3 \leq 10 && \text{(7b)}\\
& 7x_1 - 2x_2 - 2x_3 \geq 5 && \text{(7c)}\\
& -2 \leq x_1 \leq 10 && \text{(7d)}\\
& 0 \leq x_2 \leq 15 && \text{(7e)}\\
& -10 \leq x_3 \leq 10 && \text{(7f)}
\end{aligned}$$

, where $\tilde{a}_2$ and $\tilde{a}_3$ can take any value in $U = \{\tilde{a}_2 + \tilde{a}_3 \leq 8, \tilde{a}_2 + \tilde{a}_3 \geq 2, \tilde{a}_2, \tilde{a}_3 \geq 0\}$.

Since it is a "lower-or-equal" constraint, a robust counterpart of constraint 7b can be formulated as follows:

$5x_1 + \max_{(\tilde{a}_2, \tilde{a}_3) \in U} (\tilde{a}_2 x_2 + 3\tilde{a}_3 x_3) \leq 10$

To get a linear formulation of this constraint, the following subproblem needs to be solved:

$$\max \tilde{a}_2 x_2 + 3\tilde{a}_3 x_3 \tag{8a}$$

$$\text{s.t. } \tilde{a}_2 + \tilde{a}_3 \leq 8 \tag{8b}$$
$$\tilde{a}_2 + \tilde{a}_3 \geq 2 \tag{8c}$$
$$\tilde{a}_2 \geq 0 \tag{8d}$$
$$\tilde{a}_3 \geq 0 \tag{8e}$$

The dual problem of this problem is:

$$\min 8\lambda + 2\mu \tag{9a}$$

$$\text{s.t. } \lambda + \mu \geq x_2 \tag{9b}$$
$$\lambda + \mu \geq x_3 \tag{9c}$$
$$\lambda \geq 0 \tag{9d}$$
$$\mu \leq 0 \tag{9e}$$

Thus, a robust counterpart of constraint 7b using this dual formulation is:

$5x_1 + \min(8\lambda + 2\mu) \leq 10$
$\lambda + \mu \geq x_2$
$\lambda + \mu \geq x_3$
$\lambda \geq 0$
$\mu \leq 0$

Since the first constraint is a "lower-or-equal" constraint, the minimum operator can be dropped (if one value of $8\lambda + 2\mu$ fulfills the constraint, then the minimum value of $8\lambda + 2\mu$ also does).

Eventually, the robust formulation of model 7 is:

$$\max 10x_1 + 20x_2 + 15x_3 \tag{10a}$$

$$\text{s.t. } 5x_1 + 8\lambda + 2\mu \leq 10 \tag{10b}$$
$$7x_1 - 2x_2 - 2x_3 \geq 5 \tag{10c}$$
$$\lambda + \mu \geq x_2 \tag{10d}$$
$$\lambda + \mu \geq x_3 \tag{10e}$$
$$-2 \leq x_1 \leq 10 \tag{10f}$$
$$0 \leq x_2 \leq 15 \tag{10g}$$
$$-10 \leq x_3 \leq 10 \tag{10h}$$
$$\lambda \geq 0 \tag{10i}$$
$$\mu \leq 0 \tag{10j}$$
$$\tag{10k}$$

Solving this model with Julia JuMP using Gurobi yields the following results:

Objective value	31.25
x_1	1.053
x_2	0.592
x_3	0.592

Table 3: Third robust model results

4) The model is formulated as follows:

$$\max 10x_1 + 20x_2 + 15x_3 \tag{11a}$$

$$\begin{aligned}
\text{s.t. } & \tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \tilde{a}_3 x_3 \leq 10 && (11b)\\
& 7x_1 - 2x_2 - 2x_3 \geq 5 && (11c)\\
& -2 \leq x_1 \leq 10 && (11d)\\
& 0 \leq x_2 \leq 15 && (11e)\\
& -10 \leq x_3 \leq 10 && (11f)
\end{aligned}$$

, where $\tilde{a}_1 \sim U(1,8)$, $\tilde{a}_2 \sim U(2,9)$, $\tilde{a}_3 \sim U(1,5)$. These uncertainty sets can be represented as intervals:

- $[\bar{a}_1 - \hat{a}_1,\ \bar{a}_1 + \hat{a}_1]$, with $\bar{a}_1 = 4.5$ and $\hat{a}_1 = 3.5$
- $[\bar{a}_2 - \hat{a}_2,\ \bar{a}_2 + \hat{a}_2]$, with $\bar{a}_2 = 5.5$ and $\hat{a}_2 = 3.5$
- $[\bar{a}_3 - \hat{a}_3,\ \bar{a}_3 + \hat{a}_3]$, with $\bar{a}_3 = 3$ and $\hat{a}_3 = 2$

In this problem, at most two of the three parameters are allowed to deviate from their mean value. In order to formulate a robust counterpart of constraint 11b, the following subproblem needs to be solved:

$$\max \hat{a}_1 \mid x_1 \mid z_1 + \hat{a}_2 x_2 z_2 + \hat{a}_3 \mid x_3 \mid z_3 \tag{12a}$$

$$\begin{aligned}
\text{s.t. } & z_1 + z_2 + z_3 \leq 2 && (12b)\\
& 0 \leq z_1 \leq 1 && (12c)\\
& 0 \leq z_2 \leq 1 && (12d)\\
& 0 \leq z_3 \leq 1 && (12e)
\end{aligned}$$

No absolute sign was needed for variable x_2, as it is always positive. Constraint 12b represents the "budget of uncertainty".

The dual problem of this problem is:

$$\min 2\lambda + \mu_1 + \mu_2 + \mu_3 \tag{13a}$$

$$
\begin{aligned}
\text{s.t. } & \lambda + \mu_1 \geq \hat{a}_1 \mid x_1 \mid && \text{(13b)}\\
& \lambda + \mu_2 \geq \hat{a}_2 x_2 && \text{(13c)}\\
& \lambda + \mu_3 \geq \hat{a}_3 \mid x_3 \mid && \text{(13d)}\\
& \lambda \geq 0 && \text{(13e)}\\
& \mu_1 \geq 0 && \text{(13f)}\\
& \mu_2 \geq 0 && \text{(13g)}\\
& \mu_3 \geq 0 && \text{(13h)}
\end{aligned}
$$

Thus, a robust counterpart of constraint 11b using this dual formulation is:

$\bar{a}_1 x_1 + \bar{a}_2 x_2 + \bar{a}_3 x_3 + \min(2\lambda + \mu_1 + \mu_2 + \mu_3) \leq 10$
$\lambda + \mu_1 \geq \hat{a}_1 \mid x_1 \mid$
$\lambda + \mu_2 \geq \hat{a}_2 x_2$
$\lambda + \mu_3 \geq \hat{a}_3 \mid x_3 \mid$
$\lambda \geq 0 \; \mu_1 \geq 0 \; \mu_2 \geq 0 \; \mu_3 \geq 0$

Since the first constraint is a "lower-or-equal" constraint, the minimum operator can be dropped. Plus, the absolute values can be linearized similarly to question 3).

Eventually, the robust formulation of model 11 is:

$$
\max 10x_1 + 20x_2 + 15x_3 \tag{14a}
$$

$$
\begin{aligned}
\text{s.t. } & 4.5x_1 + 5.5x_2 + 3x_3 + 2\lambda + \mu_1 + \mu_2 + \mu_3 \leq 10 && \text{(14b)}\\
& 7x_1 - 2x_2 - 2x_3 \geq 5 && \text{(14c)}\\
& \lambda + \mu_1 \geq 3.5u_1 && \text{(14d)}\\
& \lambda + \mu_2 \geq 3.5x_2 && \text{(14e)}\\
& \lambda + \mu_3 \geq 2u_3 && \text{(14f)}\\
& -u_1 \leq x_1 \leq u_1 && \text{(14g)}\\
& -u_3 \leq x_3 \leq u_3 && \text{(14h)}\\
& -2 \leq x_1 \leq 10 && \text{(14i)}\\
& 0 \leq x_2 \leq 15 && \text{(14j)}\\
& -10 \leq x_3 \leq 10 && \text{(14k)}\\
& \lambda \geq 0 && \text{(14l)}\\
& \mu_1 \geq 0 && \text{(14m)}\\
& \mu_2 \geq 0 && \text{(14n)}\\
& \mu_3 \geq 0 && \text{(14o)}\\
& && \text{(14p)}
\end{aligned}
$$

Solving this model with Julia JuMP using Gurobi yields the following results:

Objective value	18.43
x_1	0.878
x_2	0.209
x_3	0.365

Table 4: Fourth robust model results

Task5 - Adjustable robust optimization

The aim of this exercise is to formulate a robust optimization model accounting for the uncertain efficiency of a machine in a goods production process. An adjustable robust optimization model will be then formulated to account for the possibility to extend one of the machine to produce more goods.

1) The efficiency of machine 2 is uncertain: $\eta_2 \in [0.6, 0.8]$. This uncertainty set can be formulated as follows: $\eta_2 = \bar{\eta_2} + \zeta P \quad \forall \zeta \in [-1, 1]$, with $\bar{\eta_2} = 0.7$ and $P = 0.1$

A robust linear optimization problem can be formulated as follows:

Sets

$\mathscr{I}$	$\{1, 2\}$	Machines
$\mathscr{T}$	$[\![1, 12]\!]$	Time units (months)

Parameters

c_i	Cost of producing a good with machine i [EUR/unit]
D_t	Demand for goods at time y [unit]
K_i	Maximal monthly production capacity of machine i [unit]
η_1	Efficiency of machine 1
$\bar{\eta_2}$	Mean efficiency of machine 2
P	Deviation from mean of machine 2 efficiency

Variables

$p_{i,t}$	$\mathbb{R}^+$	Production of machine i at time t [unit]

$$\min C = \sum_{i \in \mathscr{I}, t \in \mathscr{T}} c_i p_{i,t} \tag{15a}$$

$$\text{s.t. } \eta_1 p_{1,t} + (\bar{\eta_2} - P) p_{2,t} \geq D_t \quad \forall t \in \mathscr{T} \tag{15b}$$

$$p_{i,t} \leq K_i \quad \forall (i, t) \in \mathscr{I} * \mathscr{T} \tag{15c}$$

$$p_{i,t} \geq 0 \quad \forall (i, t) \in \mathscr{I} * \mathscr{T} \tag{15d}$$

$$\tag{15e}$$

- Equation 15a is the objective function. The aim is to minimize the cost of producing the goods with the machines.

- Constraint 15b states that the amount of **fault-free** goods produced must be greater than the demand at each time unit. Here, the efficiency of machine 2 has been chosen as the lower bound of its uncertainty set (worst case).
- Constraint 15c limits the monthly production of the machines to their maximal capacity.
- Constraint 15d states that the amount of goods produced must be positive.

The objective value is $C = 29772000$ EUR. The results of this optimization problem are given on table 5:

	1	2	3	4	5	6	7	8	9	10	11	12
Machine 1 (units$*10^3$)	317	363	334	394	309	153	174	132	303	353	370	341
Machine 2 (units$*10^3$)	650	650	650	650	650	650	650	650	650	650	650	650

Table 5: Monthly production of machines given by the RO problem

2) It is now possible to extend machine 2 to adjust the model to the outcome of the uncertain efficiency. This extension is modeled through a virtual machine (machine 3), with a production cost $c_3 = 3$ EUR/unit, a maximal capacity $K_3 = 200000$ units and an efficiency $\eta_3 = 0.8$.

The recourse decision is the production of virtual machine 3 $p_{3,t}$. It is modeled as a linear decision rule: $p_{3,t} = y_t + Q_t\zeta$.

An adjustable robust linear optimization problem can be formulated as follows:

Sets

$\mathscr{I}$	$\{1, 2\}$	Real machines
$\mathscr{T}$	$[\![1, 12]\!]$	Time units (months)

Parameters

c_i	Cost of producing a good with machine i [EUR/unit]
c_3	Cost of producing a good with machine 3 [EUR/unit]
D_t	Demand for goods at time y [unit]
K_i	Maximal monthly production capacity of machine i [unit]
K_3	Maximal monthly production capacity of machine i [unit]
η_1	Efficiency of machine 1
$\bar{\eta}_2$	Mean efficiency of machine 2
P	Deviation from mean of machine 2 efficiency
η_3	Efficiency of machine 3

Variables

$p_{i,t}$	$\mathbb{R}^+$	Production of machine i at time t [unit]
β_t	$\mathbb{R}$	Recourse function (cost of goods produced by virtual machine 3)
y_t	$\mathbb{R}$	Certain part of linear decision rule
Q_t	$\mathbb{R}$	Uncertain part of linear decision rule
α_t	$\mathbb{R}^+$	Absolute value of $Pp_{2,t} + \eta_3 Q_t$
γ_t	$\mathbb{R}^+$	Absolute value of Q_t

$$\min C = \sum_{i \in \mathscr{I}, t \in \mathscr{T}} c_i p_{i,t} + \sum_{t \in \mathscr{T}} \beta_t \tag{16a}$$

$$\text{s.t. } \eta_1 p_{1,t} + \bar{\eta}_2 p_{2,t} + \eta_3 y_t - \alpha_t \geq D_t \quad \forall t \in \mathscr{T} \tag{16b}$$

$$p_{i,t} \leq K_i \quad \forall (i,t) \in \mathscr{I} * \mathscr{T} \tag{16c}$$

$$y_t + \gamma_t \leq K_3 \quad \forall t \in \mathscr{T} \tag{16d}$$

$$c_3(y_t + \gamma_t) \leq \beta_t \quad \forall t \in \mathscr{T} \tag{16e}$$

$$-\alpha_t \leq Pp_{2,t} + \eta_3 Q_t \leq \alpha_t \quad \forall t \in \mathscr{T} \tag{16f}$$

$$-\gamma_t \leq Q_t \leq \gamma_t \quad \forall t \in \mathscr{T} \tag{16g}$$

$$p_{i,t} \geq 0 \quad \forall (i,t) \in \mathscr{I} * \mathscr{T} \tag{16h}$$

$$y_t - \gamma_t \geq 0 \quad \forall t \in \mathscr{T} \tag{16i}$$

$$\alpha_t, \gamma_t \geq 0 \quad \forall t \in \mathscr{T} \tag{16j}$$

$$\beta_t, y_t, Q_t \in \mathbb{R} \quad \forall t \in \mathscr{T} \tag{16k}$$

$$\tag{16l}$$

- Equation 16a is the objective function. The aim is to minimize the cost of producing the goods with the machines.
- Constraint 16b states that the amount of **fault-free** goods produced must be greater than the demand at each time unit. The productions of machines 2 and 3 both depend on the outcome of the same uncertain phenomenon represented by ζ. Thus, the worst case has been modeled through a single variable α_t, which ties together the response to the uncertainty of machines 2 and 3.
- Constraint 16c limits the monthly production of machines 1 and 2 to their maximal capacity.
- Constraint 16d limits the monthly production of machine 3 to its maximal capacity. Here, the production of machine 3 has been chosen as its maximal possible value (worst case).
- Constraint 16e represents the cost of goods produced by machine 3 at time t. Here, the production of machine 3 has been chosen as its maximal possible value (worst case).

- Constraint 16f calculates the absolute value of $Pp_{2,t} + \eta_3 Q_t$ for each time t.
- Constraint 16g calculates the absolute value of Q_t for each time t.
- Constraint 16h states that the amount of goods produced by machines 1 and 2 must be positive.
- Constraint 16i states that the amount of goods produced by machine 3 must be positive. Here, the production of machine 3 has been chosen as its minimal possible value (worst case).
- Constraint 16j states that variables representing absolute values must be positive.
- Constraint 16k states that the variables involved in the recourse function and in the linear decision rule can take any real value.

The objective value is $C = 29300750$ EUR. The results of this optimization problem are given on table 6:

	1	2	3	4	5	6	7	8	9	10	11	12
Machine 1 (units$*10^3$)	157	203	174	234	149	0	14	0	143	193	210	181
Machine 2 (units$*10^3$)	650	650	650	650	650	650	650	650	650	650	650	650
Machine 3_low (units$*10^3$)	200	200	200	200	200	191,25	200	165	200	200	200	200
Machine 3_high (units$*10^3$)	200	200	200	200	200	191,25	200	165	200	200	200	200

Table 6: Monthly production of machines given by the ARO problem

In table 6, two values of production were given for machine 3.

- The row *Machine3_low* corresponds to the production in case of a minimal fault-free production from machine 2. This corresponds to an outcome of -1 for the uncertainty ζ, which means an efficiency $\eta_2 = 0.6$. In this case, the monthly production of machine 3 at time t is given by $p_{3,t} = y_t - Q_t$.
- The row *Machine3_high* corresponds to the production in case of a maximal fault-free production from machine 2. This corresponds to an outcome of 1 for the uncertainty ζ, which means an efficiency $\eta_2 = 0.8$. In this case, the monthly production of machine 3 at time t is given by $p_{3,t} = y_t + Q_t$.

A clear strategy can be observed in table 6, with priority in terms of production given first to machine 2, then to machine 3 and finally to machine 1.
Indeed, even in the worst case ($\eta_2 = 0.6$), machine 2 is more interesting in terms of cost than machine 3 (its efficiency is 25% lower, but its unitary production cost is 33% lower). This explains why the production of machine 3 does not depend on the uncertainty (i.e $Q_t = 0 \quad \forall t \in \mathscr{T}$): no matter the efficiency of machine 2, machine 3 is always less interesting in terms of cost.
Similarly, machine 3 is more interesting in terms of cost than machine 1 (its efficiency is 20% lower, but its unitary production cost is 25% lower).