

Notations for the code in "Eval2.ipyn"

Using the Einstein notation (exponents mean there is a sum in this variable, so it means there is a hidden multiplication of tensors):

$$\begin{aligned} R_{abxy}(W, P, Q) &:= P \boxtimes_W Q(a, b \mid x, y) \\ &= \mathbf{1}^i \left[\left(\left(A_{xyi}{}^k(W) P_k{}^j \right) \left(B_{xy}{}^{j\ell}(W) Q_{\ell i} \right) \right)_{ab} C_{abxy}{}^j(W) D_{abxy}{}^j(W) \right] \mathbf{1}_j \\ i.e. \quad R(W, P, Q) &= \left[\mathbf{1}_{2 \times 2} \otimes \left(\left(A(W) \cdot P \right) * \left(B(W) \cdot Q \right)^{\top(0,1,3,2)} \right) * C(W) * D(W) \right] \cdot \mathbf{1}_4 \cdot \mathbf{1}_4, \end{aligned}$$

where $a, b, x, y \in \{0, 1\}$, and $i, j, k, \ell \in \{0, 1, 2, 3\}$, and $m, n \in \{0, \dots, 31\}$, and where:

- $\boxed{*}$ is the component-wise multiplication ;
- By convention $\boxed{T \cdot U}$ is the tensor multiplication of T and U through the last entry of T and the first one of U ;
- $\boxed{T^{\top\sigma(0,1,\dots,n-1)}}$ (for some permutation $\sigma \in \mathfrak{S}(\{0, \dots, n-1\})$) is the tensor whose i -th entry is the σ_i -th entry of T (if $|\text{supp } \sigma| = 2$, then $\top\sigma$ is simply the transpose) ;
- $\boxed{\otimes}$ is the Kronecker product. When we do $\mathbf{1} \otimes T$, we duplicate the tensor T so that we obtain a new tensor with more entries ;
- $A_{xyik}(W) = \left(A_{xik}^{(1)m} W_m + A_{xik}^{(2)} \right)_y \left(A_{yik}^{(3)n} W_n + A_{yik}^{(4)} \right)_x$,
 $i.e. \quad A(W) = \left[\mathbf{1}_2 \otimes \left(A^{(1)} \cdot W + A^{(2)} \right) \right]^{\top(1,0,2,3)} * \left[\mathbf{1}_2 \otimes \left(A^{(3)} \cdot W + A^{(4)} \right) \right] ;$
- $B_{xyj\ell}(W) = \left(B_{xj\ell}^{(1)m} W_m + B_{xj\ell}^{(2)} \right)_y \left(B_{y j\ell}^{(3)n} W_n + B_{y j\ell}^{(4)} \right)_x$,
 $i.e. \quad B(W) = \left[\mathbf{1}_2 \otimes \left(B^{(1)} \cdot W + B^{(2)} \right) \right]^{\top(1,0,2,3)} * \left[\mathbf{1}_2 \otimes \left(B^{(3)} \cdot W + B^{(4)} \right) \right] ;$
- $C_{abxyij}(W) = \left(C_{axij}^{(1)m} W_m + C_{axij}^{(2)} \right)_{by}$,
 $i.e. \quad C(W) = \left(\mathbf{1}_{2 \times 2} \otimes \left(C^{(1)} \cdot W + C^{(2)} \right) \right)^{\top(2,0,3,1,4,5)} ;$
- $D_{abxyij}(W) = \left(D_{byij}^{(1)m} W_m + D_{byij}^{(2)} \right)_{ax}$,
 $i.e. \quad D(W) = \left(\mathbf{1}_{2 \times 2} \otimes \left(D^{(1)} \cdot W + D^{(2)} \right) \right)^{\top(0,2,1,3,4,5)} .$

Computation of $A^{(1)}$ and $A^{(2)}$. We have :

$$\begin{bmatrix} (1 - f_1(x, 0)) & (1 - f_1(x, 0)) & f_1(x, 0) & f_1(x, 0) \\ (1 - f_1(x, 0)) & (1 - f_1(x, 0)) & f_1(x, 0) & f_1(x, 0) \\ (1 - f_1(x, 1)) & (1 - f_1(x, 1)) & f_1(x, 1) & f_1(x, 1) \\ (1 - f_1(x, 1)) & (1 - f_1(x, 1)) & f_1(x, 1) & f_1(x, 1) \end{bmatrix}_{ik} = \left(A_{xik}^{(1)m} W_m + A_{xik}^{(2)} \right)_{ik}$$

for $x = 0, 1$, and $i = 0, \dots, 3$, and $k = 0, \dots, 3$, and $m = 0, \dots, 31$.

We have for all x and $k = 0, 1$ (1st and 2nd columns):

$$\left(A_{xi0}^{(1)m} \right)_{im} = \begin{bmatrix} x-1 & 0 & -x & 0 \\ x-1 & 0 & -x & 0 \\ 0 & x-1 & 0 & -x \\ 0 & x-1 & 0 & -x \end{bmatrix}_{im} \mathbf{0}_{4 \times 28} = \left(A_{xi1}^{(1)m} \right)_{im},$$

and for $k = 2, 3$ (3rd and 4th columns):

$$\left(A_{xi2}^{(1)m}\right)_{im} = -\left(A_{xi0}^{(1)m}\right)_{im} = \left(A_{xi3}^{(1)m}\right)_{im}.$$

And for all x :

$$\left(A_{xik}^{(2)}\right)_{ik} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}_{ik} = \begin{bmatrix} \mathbf{1}_{4 \times 2} & \mathbf{0}_{4 \times 2} \end{bmatrix}_{ik}.$$

Computation of $A^{(3)}$ and $A^{(4)}$. We have :

$$\begin{bmatrix} (1 - g_1(y, 0)) & g_1(y, 0) & (1 - g_1(y, 0)) & g_1(y, 0) \\ (1 - g_1(y, 1)) & g_1(y, 1) & (1 - g_1(y, 1)) & g_1(y, 1) \\ (1 - g_1(y, 0)) & g_1(y, 0) & (1 - g_1(y, 0)) & g_1(y, 0) \\ (1 - g_1(y, 1)) & g_1(y, 1) & (1 - g_1(y, 1)) & g_1(y, 1) \end{bmatrix}_{ik} = \left(A_{yik}^{(3)n} W_n + A_{yik}^{(4)}\right)_{ik}$$

for $y = 0, 1$, and $i = 0, \dots, 3$, and $k = 0, \dots, 3$, and $n = 0, \dots, 31$.

We have for all y and $k = 0, 2$ (1st and 3rd columns):

$$\left(A_{yi0}^{(3)n}\right)_{i,n} = \begin{bmatrix} & y-1 & 0 & -y & 0 & \\ \mathbf{0}_{4 \times 4} & 0 & y-1 & 0 & -y & \\ & y-1 & 0 & -y & 0 & \mathbf{0}_{4 \times 24} \\ & 0 & y-1 & 0 & -y & \end{bmatrix}_{i,n} = \left(A_{yi2}^{(3)n}\right)_{i,n},$$

and for $k = 1, 3$ (2nd and 4th columns):

$$\left(A_{yi1}^{(3)n}\right)_{i,n} = -\left(A_{yi0}^{(3)n}\right)_{i,n} = \left(A_{yi3}^{(3)n}\right)_{i,n}.$$

And for all y :

$$\left(A_{yik}^{(4)}\right)_{ik} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}_{ik} = \begin{bmatrix} \mathbf{1}_{4 \times 1} & \mathbf{0}_{4 \times 1} & \mathbf{1}_{4 \times 1} & \mathbf{0}_{4 \times 1} \end{bmatrix}_{ik}.$$

Computation of $B^{(1)}$ to $B^{(4)}$. After doing as before, we obtain:

$$B_{yjl}^{(2)} = A_{yjl}^{(2)} \quad \text{and} \quad B_{yjl}^{(4)} = A_{yjl}^{(4)}.$$

As we just change f_1 into f_2 and g_1 into g_2 , we have for all x :

$$\begin{aligned} \left(B_{xj0}^{(1)m}\right)_{j,m} &= \begin{bmatrix} & x-1 & 0 & -x & 0 & \\ \mathbf{0}_{4 \times 8} & x-1 & 0 & -x & 0 & \\ & 0 & x-1 & 0 & -x & \mathbf{0}_{4 \times 20} \\ & 0 & x-1 & 0 & -x & \end{bmatrix}_{j,m} \\ &= \left(B_{xj1}^{(1)m}\right)_{j,m} = -\left(B_{xj2}^{(1)m}\right)_{j,m} = -\left(B_{xj3}^{(1)m}\right)_{j,m}, \end{aligned}$$

and for all y :

$$\begin{aligned} \left(B_{yj0}^{(3)n}\right)_{j,n} &= \begin{bmatrix} & y-1 & 0 & -y & 0 & \\ \mathbf{0}_{4 \times 12} & y-1 & 0 & -y & 0 & \\ & y-1 & 0 & -y & 0 & \mathbf{0}_{4 \times 16} \\ & 0 & y-1 & 0 & -y & \end{bmatrix}_{j,n} \\ &= -\left(B_{yj1}^{(3)n}\right)_{j,n} = \left(B_{yj2}^{(3)n}\right)_{j,n} = -\left(B_{yj3}^{(3)n}\right)_{j,n}. \end{aligned}$$

Computation of $C^{(1)}$ and $C^{(2)}$. We have:

$$-(-1)^a \left[\begin{array}{cc|cc} f_3(x, 0, 0) & f_3(x, 0, 0) & f_3(x, 1, 0) & f_3(x, 1, 0) \\ f_3(x, 0, 0) & f_3(x, 0, 0) & f_3(x, 1, 0) & f_3(x, 1, 0) \\ \hline f_3(x, 0, 1) & f_3(x, 0, 1) & f_3(x, 1, 1) & f_3(x, 1, 1) \\ f_3(x, 0, 1) & f_3(x, 0, 1) & f_3(x, 1, 1) & f_3(x, 1, 1) \end{array} \right]_{ij} = \left(C_{axij}^{(1)m} W_m \right)_{ij}$$

For all a, x and for $j \in \{0, 1\}$ (1st and 2nd columns):

$$\left(C_{axi0}^{(1)m} \right)_{i,m} = -(-1)^a \left[\begin{array}{cccccccc} & 1-x & 0 & 0 & 0 & x & 0 & 0 & 0 \\ & 1-x & 0 & 0 & 0 & x & 0 & 0 & 0 \\ \mathbf{0}_{4 \times 16} & 0 & 1-x & 0 & 0 & 0 & x & 0 & 0 \\ & 0 & 1-x & 0 & 0 & 0 & x & 0 & 0 \end{array} \right]_{i,m} \mathbf{0}_{4 \times 8} = \left(C_{axi1}^{(1)m} \right)_{i,m},$$

and for $j \in \{2, 3\}$ (3rd and 4th columns):

$$\left(C_{axi2}^{(1)m} \right)_{i,m} = -(-1)^a \left[\begin{array}{cccccccc} & 0 & 0 & 1-x & 0 & 0 & 0 & x & 0 \\ & 0 & 0 & 1-x & 0 & 0 & 0 & x & 0 \\ \mathbf{0}_{4 \times 16} & 0 & 0 & 0 & 1-x & 0 & 0 & 0 & x \\ & 0 & 0 & 0 & 1-x & 0 & 0 & 0 & x \end{array} \right]_{i,m} \mathbf{0}_{4 \times 8} = \left(C_{axi3}^{(1)m} \right)_{i,m}.$$

And we have $C_{axij}^{(2)} = (1-a) \mathbf{1}_{xij}$.

Computation of $D^{(1)}$ and $D^{(2)}$. We have:

$$-(-1)^b \left[\begin{array}{cc|cc} g_3(y, 0, 0) & g_3(y, 1, 0) & g_3(y, 0, 0) & g_3(y, 1, 0) \\ g_3(y, 0, 1) & g_3(y, 1, 1) & g_3(y, 0, 1) & g_3(y, 1, 1) \\ \hline g_3(y, 0, 0) & g_3(y, 1, 0) & g_3(y, 0, 0) & g_3(y, 1, 0) \\ g_3(y, 0, 1) & g_3(y, 1, 1) & g_3(y, 0, 1) & g_3(y, 1, 1) \end{array} \right]_{ij} = \left(D_{byij}^{(1)m} W_m \right)_{ij}.$$

For all y and $j \in \{0, 2\}$ (1st and 3rd columns):

$$\left(D_{byi0}^{(1)m} \right)_{i,m} = -(-1)^b \left[\begin{array}{cccccccc} & 1-y & 0 & 0 & 0 & y & 0 & 0 & 0 \\ & 0 & 1-y & 0 & 0 & 0 & y & 0 & 0 \\ \mathbf{0}_{4 \times 24} & 1-y & 0 & 0 & 0 & y & 0 & 0 & 0 \\ & 0 & 1-y & 0 & 0 & 0 & y & 0 & 0 \end{array} \right]_{i,m} = \left(D_{byi2}^{(1)m} \right)_{i,m},$$

and for $j \in \{1, 3\}$ (2nd and 3rd columns):

$$\left(D_{byi1}^{(1)m} \right)_{i,m} = -(-1)^b \left[\begin{array}{cccccccc} & 0 & 0 & 1-y & 0 & 0 & 0 & y & 0 \\ & 0 & 0 & 0 & 1-y & 0 & 0 & 0 & y \\ \mathbf{0}_{4 \times 24} & 0 & 0 & 1-y & 0 & 0 & 0 & y & 0 \\ & 0 & 0 & 0 & 1-y & 0 & 0 & 0 & y \end{array} \right]_{i,m} = \left(D_{byi3}^{(1)m} \right)_{i,m}.$$

And $D_{byij}^{(2)} = (1-b) \mathbf{1}_{yij}$.