Notations for the code in "Eval3.ipynb"

Now W is no longer a vector, it is a $32 \times 10\,000$-matrix.
Using the Einstein notation (exponents mean there is a sum in this variable, so it means there is a hidden multiplication of tensors):

$$\begin{aligned}
\mathtt{R}_{abxy\alpha}(W,\mathtt{P},\mathtt{Q}) &:= \mathtt{P} \boxtimes_W \mathtt{Q}(a,b\,|\,x,y) \\
&= \mathbf{1}^i \Big[\Big(\Big(A_{xy\alpha i}{}^k(W)\, \mathtt{P}_k{}^j \Big) \Big(B_{xy\alpha}{}^{j\ell}(W)\, \mathtt{Q}_{\ell i} \Big) \Big)_{ab}\, C_{abxy\alpha i}{}^j(W)\, D_{abxy\alpha i}{}^j(W) \Big] \mathbf{1}_j \\
i.e. \quad \mathtt{R}(W,\mathtt{P},\mathtt{Q}) &= \boxed{ \Big[\mathbf{1}_{2\times 2} \otimes \Big(\big(A(W) \cdot \mathtt{P} \big) * \big(B(W) \cdot \mathtt{Q} \big)^{\top(0,1,3,2)} \Big) * C(W) * D(W) \Big] \cdot \mathbf{1}_4 \cdot \mathbf{1}_4 },
\end{aligned}$$

where $a,b,x,y \in \{0,1\}$, and $i,j,k,\ell \in \{0,1,2,3\}$, and $m,n \in \{0,\dots,31\}$, and $\alpha \in \{0, 9\,999\}$, and where:

- $\boxed{*}$ is the component-wise multiplication ;
- By convention $\boxed{T \cdot U}$ is the tensor multiplication of T and U through the last entry of T and the first one of U ;
- $\boxed{T^{\top\sigma(0,1,\dots,n-1)}}$ (for some permutation $\sigma \in \mathfrak{S}(\{0,\dots,n-1\})$) is the tensor whose i-th entry is the σ_i-th entry of T (if $|\operatorname{\mathtt{supp}} \sigma| = 2$, then $\top\sigma$ is simply the transpose) ;
- $\boxed{\otimes}$ is the Kronecker product. When we do $\mathbb{1} \otimes T$, we duplicate the tensor T so that we obtain a new tensor with more entries ;
- $A_{xy\alpha ik}(W) = \Big(A^{(1)}_{xik}{}^m\, W_{m\alpha} + A^{(2)}_{xik\alpha} \Big)_y \Big(A^{(3)}_{yik}{}^n\, W_{n\alpha} + A^{(4)}_{yik\alpha} \Big)_x$,

  $$i.e. \quad \boxed{ A(W) = \Big(\Big[\mathbf{1}_2 \otimes \Big(A^{(1)} \cdot W + A^{(2)} \Big) \Big]^{\top(1,0,2,3,4)} * \Big[\mathbf{1}_2 \otimes \Big(A^{(3)} \cdot W + A^{(4)} \Big) \Big] \Big)^{\top(0,1,4,2,3)} } \;;$$
- $B_{xy\alpha j\ell}(W) = \Big(B^{(1)}_{xj\ell}{}^m\, W_{m\alpha} + B^{(2)}_{yj\ell\alpha} \Big)_y \Big(B^{(3)}_{yj\ell}{}^n\, W_{n\alpha} + B^{(4)}_{yj\ell\alpha} \Big)_x$,

  $$i.e. \quad \boxed{ B(W) = \Big(\Big[\mathbf{1}_2 \otimes \Big(B^{(1)} \cdot W + B^{(2)} \Big) \Big]^{\top(1,0,2,3,4)} * \Big[\mathbf{1}_2 \otimes \Big(B^{(3)} \cdot W + B^{(4)} \Big) \Big] \Big)^{\top(0,1,4,2,3)} } \;;$$
- $C_{abxy\alpha ij}(W) = \Big(C^{(1)}_{axij}{}^m\, W_{m\alpha} + C^{(2)}_{axij\alpha} \Big)_{by}$,

  $$i.e. \quad \boxed{ C(W) = \Big(\mathbf{1}_{2\times 2} \otimes \Big(C^{(1)} \cdot W + C^{(2)} \Big) \Big)^{\top(2,0,3,1,6,4,5)} } \;;$$
- $D_{abxy\alpha ij}(W) = \Big(D^{(1)}_{byij}{}^m\, W_{m\alpha} + D^{(2)}_{byij\alpha} \Big)_{ax}$,

  $$i.e. \quad \boxed{ D(W) = \Big(\mathbf{1}_{2\times 2} \otimes \Big(D^{(1)} \cdot W + D^{(2)} \Big) \Big)^{\top(0,2,1,3,6,4,5)} }.$$

Computation of $A^{(1)}$ and $A^{(2)}$. We have :

$$\begin{bmatrix}
(1-f_1(x,0)) & (1-f_1(x,0)) & f_1(x,0) & f_1(x,0) \\
(1-f_1(x,0)) & (1-f_1(x,0)) & f_1(x,0) & f_1(x,0) \\
(1-f_1(x,1)) & (1-f_1(x,1)) & f_1(x,1) & f_1(x,1) \\
(1-f_1(x,1)) & (1-f_1(x,1)) & f_1(x,1) & f_1(x,1)
\end{bmatrix}_{ik} = \Big(A^{(1)}_{xik}{}^m\, W_{m\alpha} + A^{(2)}_{xik\alpha} \Big)_{ik}$$

for $x \in \{0,1\}$, and $i,k \in \{0,\dots,3\}$, and $m \in \{0,\dots,31\}$, and $\alpha \in \{0, 9\,999\}$.
We have for all x and for $k = 0,1$ (1st and 2nd columns of the above matrix) and then $k = 2,3$ (3rd and 4th columns):

$$\left(A^{(1)\,m}_{xi0}\right)_{im} = \begin{bmatrix} \begin{smallmatrix} x-1 & 0 & -x & 0 \\ x-1 & 0 & -x & 0 \\ 0 & x-1 & 0 & -x \\ 0 & x-1 & 0 & -x \end{smallmatrix} & \mathbf{0}_{4\times 28} \end{bmatrix}_{im} = \left(A^{(1)\,m}_{xi1}\right)_{im},$$
$$= -\left(A^{(1)\,m}_{xi2}\right)_{im} = -\left(A^{(1)\,m}_{xi3}\right)_{im}.$$

And for all x, α:

$$\left(A^{(2)}_{xik\alpha}\right)_{ik} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}_{ik} = \begin{bmatrix} \mathbf{1}_{4\times 2} & \mathbf{0}_{4\times 2} \end{bmatrix}_{ik}.$$

Computation of $A^{(3)}$ and $A^{(4)}$. We have :

$$\begin{bmatrix} (1-g_1(y,0)) & g_1(y,0) & (1-g_1(y,0)) & g_1(y,0) \\ (1-g_1(y,1)) & g_1(y,1) & (1-g_1(y,1)) & g_1(y,1) \\ (1-g_1(y,0)) & g_1(y,0) & (1-g_1(y,0)) & g_1(y,0) \\ (1-g_1(y,1)) & g_1(y,1) & (1-g_1(y,1)) & g_1(y,1) \end{bmatrix}_{ik} = \left(A^{(3)\,n}_{yik} W_{n\alpha} + A^{(4)}_{yik\alpha}\right)_{ik}$$

for $y \in \{0,1\}$, and $i,k \in \{0,\dots,3\}$, and $n \in \{0,\dots,31\}$, and $\alpha \in \{0, 9\,999\}$.
We have for all y and $k = 0,2$ (1st and 3rd columns of the above matrix) and then for $k = 1,3$ (2nd and 4th columns):

$$\left(A^{(3)\,n}_{yi0}\right)_{i,n} = \begin{bmatrix} \mathbf{0}_{4\times 4} & \begin{smallmatrix} y-1 & 0 & -y & 0 \\ 0 & y-1 & 0 & -y \\ y-1 & 0 & -y & 0 \\ 0 & y-1 & 0 & -y \end{smallmatrix} & \mathbf{0}_{4\times 24} \end{bmatrix}_{i,n} = \left(A^{(3)\,n}_{yi2}\right)_{i,n},$$
$$= -\left(A^{(3)\,n}_{yi1}\right)_{i,n} = -\left(A^{(3)\,n}_{yi3}\right)_{i,n}.$$

And for all y, α:

$$\left(A^{(4)}_{yik\alpha}\right)_{ik} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}_{ik} = \begin{bmatrix} \mathbf{1}_{4\times 1} & \mathbf{0}_{4\times 1} & \mathbf{1}_{4\times 1} & \mathbf{0}_{4\times 1} \end{bmatrix}_{ik}.$$

Computation of $B^{(1)}$ to $B^{(4)}$. After doing as before, we obtain:

$$B^{(2)}_{yj\ell\alpha} = A^{(2)}_{yj\ell\alpha} \qquad \text{and} \qquad B^{(4)}_{yj\ell\alpha} = A^{(4)}_{yj\ell\alpha}.$$

As we just change f_1 into f_2 and g_1 into g_2, we have for all x:

$$\left(B^{(1)\,m}_{xj0}\right)_{j,m} = \begin{bmatrix} \mathbf{0}_{4\times 8} & \begin{smallmatrix} x-1 & 0 & -x & 0 \\ x-1 & 0 & -x & 0 \\ 0 & x-1 & 0 & -x \\ 0 & x-1 & 0 & -x \end{smallmatrix} & \mathbf{0}_{4\times 20} \end{bmatrix}_{j,m}$$
$$= \left(B^{(1)\,m}_{xj1}\right)_{j,m} = -\left(B^{(1)\,m}_{xj2}\right)_{j,m} = -\left(B^{(1)\,m}_{xj3}\right)_{j,m},$$

and for all y:

$$
\begin{aligned}
\left(B^{(3)\,n}_{yj0}\right)_{j,n} &= \left[\begin{array}{ccccccc} & y-1 & 0 & -y & 0 & \\ \mathbf{0}_{4\times 12} & 0 & y-1 & 0 & -y & \mathbf{0}_{4\times 16} \\ & y-1 & 0 & -y & 0 & \\ & 0 & y-1 & 0 & -y & \end{array}\right]_{j,n} \\
&= -\left(B^{(3)\,n}_{yj1}\right)_{j,n} = \left(B^{(3)\,n}_{yj2}\right)_{j,n} = -\left(B^{(3)\,n}_{yj3}\right)_{j,n}.
\end{aligned}
$$

Computation of $C^{(1)}$ and $C^{(2)}$. We have:

$$
-(-1)^a \left[\begin{array}{cc|cc} f_3(x,0,0) & f_3(x,0,0) & f_3(x,1,0) & f_3(x,1,0) \\ f_3(x,0,0) & f_3(x,0,0) & f_3(x,1,0) & f_3(x,1,0) \\ \hline f_3(x,0,1) & f_3(x,0,1) & f_3(x,1,1) & f_3(x,1,1) \\ f_3(x,0,1) & f_3(x,0,1) & f_3(x,1,1) & f_3(x,1,1) \end{array}\right]_{ij} = \left(C^{(1)\,m}_{axij} W_{m\alpha}\right)_{ij}
$$

For all a, x and for $j \in \{0,1\}$ (1st and 2nd columns):

$$
\left(C^{(1)\,m}_{axi0}\right)_{i,m} = -(-1)^a \left[\begin{array}{cccccccccc} & 1-x & 0 & 0 & 0 & x & 0 & 0 & 0 & \\ \mathbf{0}_{4\times 16} & 1-x & 0 & 0 & 0 & x & 0 & 0 & 0 & \mathbf{0}_{4\times 8} \\ & 0 & 1-x & 0 & 0 & 0 & x & 0 & 0 & \\ & 0 & 1-x & 0 & 0 & 0 & x & 0 & 0 & \end{array}\right]_{i,m} = \left(C^{(1)\,m}_{axi1}\right)_{i,m},
$$

and for $j \in \{2,3\}$ (3rd and 4th columns):

$$
\left(C^{(1)\,m}_{axi2}\right)_{i,m} = -(-1)^a \left[\begin{array}{cccccccccc} & 0 & 0 & 1-x & 0 & 0 & 0 & x & 0 & \\ \mathbf{0}_{4\times 16} & 0 & 0 & 1-x & 0 & 0 & 0 & x & 0 & \mathbf{0}_{4\times 8} \\ & 0 & 0 & 0 & 1-x & 0 & 0 & 0 & x & \\ & 0 & 0 & 0 & 1-x & 0 & 0 & 0 & x & \end{array}\right]_{i,m} = \left(C^{(1)\,m}_{axi3}\right)_{i,m}.
$$

And we have $C^{(2)}_{axij\alpha} = (1-a)\,\mathbf{1}_{xij\alpha}$.

Computation of $D^{(1)}$ and $D^{(2)}$. We have:

$$
-(-1)^b \left[\begin{array}{cc|cc} g_3(y,0,0) & g_3(y,1,0) & g_3(y,0,0) & g_3(y,1,0) \\ g_3(y,0,1) & g_3(y,1,1) & g_3(y,0,1) & g_3(y,1,1) \\ \hline g_3(y,0,0) & g_3(y,1,0) & g_3(y,0,0) & g_3(y,1,0) \\ g_3(y,0,1) & g_3(y,1,1) & g_3(y,0,1) & g_3(y,1,1) \end{array}\right]_{ij} = \left(D^{(1)\,m}_{byij} W_{m\alpha}\right)_{ij}.
$$

For all y and $j \in \{0,2\}$ (1st and 3rd columns):

$$
\left(D^{(1)\,m}_{byi0}\right)_{i,m} = -(-1)^b \left[\begin{array}{ccccccccc} & 1-y & 0 & 0 & 0 & y & 0 & 0 & 0 \\ \mathbf{0}_{4\times 24} & 0 & 1-y & 0 & 0 & 0 & y & 0 & 0 \\ & 1-y & 0 & 0 & 0 & y & 0 & 0 & 0 \\ & 0 & 1-y & 0 & 0 & 0 & y & 0 & 0 \end{array}\right]_{i,m} = \left(D^{(1)\,m}_{byi2}\right)_{i,m},
$$

and for $j \in \{1,3\}$ (2nd and 3rd columns):

$$
\left(D^{(1)\,m}_{byi1}\right)_{i,m} = -(-1)^b \left[\begin{array}{ccccccccc} & 0 & 0 & 1-y & 0 & 0 & 0 & y & 0 \\ \mathbf{0}_{4\times 24} & 0 & 0 & 0 & 1-y & 0 & 0 & 0 & y \\ & 0 & 0 & 1-y & 0 & 0 & 0 & y & 0 \\ & 0 & 0 & 0 & 1-y & 0 & 0 & 0 & y \end{array}\right]_{i,m} = \left(D^{(1)\,m}_{byi3}\right)_{i,m}.
$$

And $D^{(2)}_{byij\alpha} = (1-b)\,\mathbf{1}_{yij\alpha}$.