

Probabilistic graphical models: Homework 3

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Exercise 1.2 The complete log-likelihood is given by:

$$\begin{aligned}\ell_c(u_{1:T}, \theta) &= E_{p(q_{1:T}|u_{1:T})} [\log(p(q_1)) + \sum_{t=1}^T \log(p(q_t|u_t)) + \sum_{t=1}^{T-1} \log(p(q_{t+1}|q_t))] \\ &= \sum_{k=1}^K p(q_1 = k) \log(\pi_k) + \sum_{t=1}^T \sum_{k=1}^K p(q_t = k) \log(\mathcal{N}(\mu_k, \Sigma_k)[u_t]) + \sum_{t=1}^{T-1} \sum_{j,k=1}^K p(q_t = k, q_{t+1} = j) \log(A_{j,k})\end{aligned}$$

With $\mathcal{N}(\mu_k, \Sigma_k)[u_t] = p(q_t = k|u_t)$ the emission probabilities, $\sum_{k=1}^K \pi_k = 1$, $\forall j \in 1 \dots K : \sum_{k=1}^K A_{k,j} = 1$

Hence the Lagrangian is:

$$\mathcal{L}(\theta, \lambda, \mu) = \ell(u_{1:T}, \theta) + \lambda(\sum_{k=1}^K \pi_k - 1) + \sum_{j=1}^K \mu_j(\sum_{k=1}^K A_{k,j} - 1)$$

We compute the gradient and set it to zero to get:

$$\begin{aligned}\pi_k &= p(q_1 = k), \quad \mathcal{A}_{j,k} = \frac{\sum_{t=1}^{T-1} p(q_t = k, q_{t+1} = j)}{\sum_{i=1}^K \sum_{t=1}^{T-1} p(q_t = k, q_{t+1} = i)} \\ \mu_k &= \frac{\sum_{t=1}^T p(q_t = k) u_t}{\sum_{t=1}^T p(q_t = k)}, \quad \Sigma_k = \frac{\sum_{t=1}^T p(q_t = k) (u_t - \mu_k)(u_t - \mu_k)^T}{\sum_{t=1}^T p(q_t = k)}\end{aligned}$$

Exercise 1.4

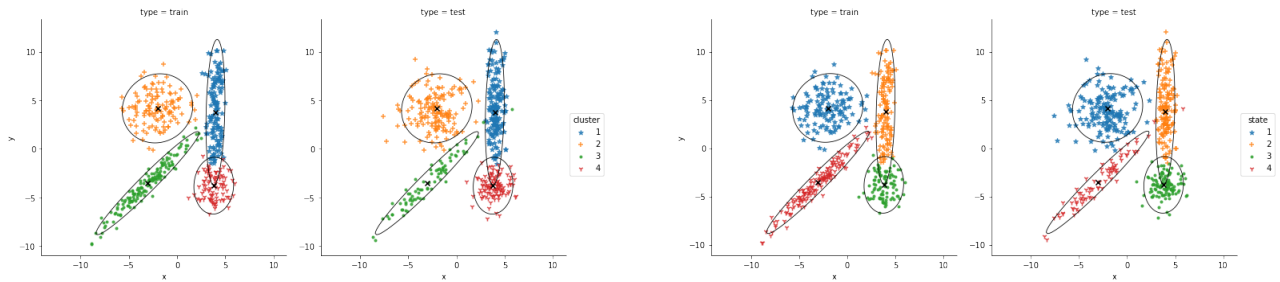


Figure 1: Plot of the training data and test data for Gaussian mixture (left) and HMM (right).

Exercise 1.5 The following table gives the log-likelihood divided by N_{train} and N_{test} respectively:

Method	Training	Test
GMM	-4.655	-4.818
HMM	-3.811	-3.918

We remark that HMM reach higher log-likelihoods, the clusters defined by each model are practically identical and we don't see any overfitting, for both models train log-likelihoods are higher on average than the test ones. While it is true that the latter doesn't generally mean that HMM is a better model, in our case it does since HMM is an extension of GMM in which we specify that $(q_t)_t$ is a Markov chain. Hence the higher log likelihood for HMM means that the hypothesis we added is more realistic than independence.