Wittgenstein's Philosophy of Mathematics

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WITTGENSTEIN'S PHILOSOPHY OF MATHEMATICS

FROM time to time Wittgenstein recorded in separate notebooks thoughts that occurred to him about the philosophy of mathematics. His recently published *Bemerkungen über die Grundlagen der Mathematik*[1] consists of extracts made by the editors from five of these. Neither it nor any of these notebooks was intended by its author as a book. That it cannot be considered, and ought not to be criticized, as such is therefore unsurprising, though disappointing. Many of the thoughts are expressed in a manner which the author recognized as inaccurate or obscure; some passages contradict others; some are quite inconclusive; some raise objections to ideas which Wittgenstein held or had held which are not themselves stated clearly in the volume; other passages again, particularly those on consistency and on Gödel's theorem, are of poor quality or contain definite errors. This being so, the book has to be treated as what it is—a selection from the jottings of a great philosopher. As Frege said of his unpublished writings, they are not all gold but there is gold in them. One of the tasks of the reader is therefore to extract the gold.

I encounter frequently in conversation the impression that this is typical of Wittgenstein's work in general; I have often heard the *Investigations* characterized as evasive and inconclusive. This seems to me a travesty of the truth; the book expresses with great clarity many forceful, profound, and quite definite ideas—though it is true that a hasty reader may sometimes be bewildered by the complexity of some of the thoughts. The contrast with the present volume is marked, and is due entirely to the different origins of the two books.

In the philosophy of mathematics, Platonism stands opposed to various degrees of constructivism. According to Platonism,

[1] *Bemerkungen über die Grundlagen der Mathematik*. By Ludwig Wittgenstein. Edited by G. H. von Wright, R. Rhees, and G. E. M. Anscombe. With English translation (*Remarks on the Foundations of Mathematics*) by G. E. M. Anscombe, on facing pages. Oxford, Basil Blackwell, 1956. Pp. xix, 196, each version.

mathematical objects are there and stand in certain relations to one another, independently of us, and what we do is to discover these objects and their relations to one another. The constructivist usually opposes to this the picture of our making, constructing, the mathematical entities as we go along. For the Platonist, the meaning of a mathematical statement is to be explained in terms of its truth-conditions; for each statement, there is something in mathematical reality in virtue of which it is either true or false. An example of the explanation of meaning in terms of truth and falsity is the truth-table explanation of the sentential connectives. For the constructivist, the general form of an explanation of meaning must be in terms of the conditions under which we regard ourselves as justified in asserting a statement, that is, the circumstances in which we are in possession of a proof. For instance, a statement made up of two statements joined by a connective is to be explained by explaining a claim to have proved the complex statement in terms of what a claim to have proved the constituent statements consists in; thus a claim to have proved ⌜A or B⌝ will be a claim to have a method leading either to a proof of A or to a proof of B. What in practice this will lead to will depend upon the degree of constructivism adopted; for example, if we confine ourselves to decidable statements, then the truth-tables will receive an acceptable interpretation and the whole classical logic will be applicable; if, on the other hand, we allow with the intuitionists a much wider range of mathematical statements to be considered as intelligible, then the law of excluded middle and many other classically valid laws will cease to hold generally. But in either case it is the notion of proof and not the notions of truth and falsity which is for the constructivist central to the account of the meaning of mathematical statements.

We may regard Platonism and the various varieties of constructivism not as rivals but merely as means of demarcating different areas of mathematics with respect not to subject matter but to methods of proof. In this case there are only the essentially mathematical problems of formulating clearly the different conceptions and investigating in detail the mathematical consequences of each. If, on the other hand, one regards the different schools as rivals, there remains the philosophical problem of deciding which of the various accounts is correct. Wittgenstein's book is intended as a contribution to the latter task only. It seems natural to suppose that the philosophical task and the mathematical go hand in hand, for the precise formulation of a conception is not irrelevant to deciding on its correctness, and unexpected consequences of adopting it may lead one to revise one's opinion as to its value. Wittgenstein will have none of this: for him philosophy and mathematics have nothing to say to one another; no mathematical discovery can have any bearing on the philosophy of mathematics.[2] It would seem that he is theoretically committed also to the converse, that no philosophical opinion could, or at least ought to, affect the procedure of the mathematician. This comes out to some extent in his discussion of the law of excluded middle in mathematics. Against one who insisted that either the sequence "77777" occurs in the development of π or it does not, he employs arguments similar to those of the intuitionists; and yet it appears that he is not wishing to question the validity in a mathematical proof of, for example, argument by cases, but only to reprove someone who in the course of philosophical reflection wishes to insist on the law of excluded middle.[3] Yet this is not to be taken too seriously, for Wittgenstein would always be able to claim that, while he had not shown that certain mathematical procedures were *wrong*, still he had shown them not to have the interest we were inclined to attach to them. Certainly in his discussion of Cantor he displays no timidity about "interfering with the mathematicians."[4] I think that there is no ground for Wittgenstein's segregation of philosophy from mathematics but that this springs only from a general tendency of his to regard discourse as split up into a number of distinct islands with no communication between them (statements of natural science, of philosophy, of mathematics, of religion).

As Frege showed, the nominalist objection to Platonism—that talk about "abstract entities" is unintelligible—is ill-taken; if we

[2] Cf. V, 13, 19; IV, 52; also *Investigations*, II, xiv; I, 124.
[3] IV, 10.
[4] I, App. II.

believe in the objectivity of mathematics, then there is no objection to our thinking in terms of mathematical objects, nor to the picture of them as already there waiting to be discovered that goes with it. Nor is formalism a real alternative. The formalist insists that the content of a mathematical theorem is simply that *if* there is any domain for which the axioms hold good, then the theorem will also hold good for that domain; and he will add that so long as we do not know the axioms to be categorical, a statement of the theory need not be either true or false. But he will not reject the classical logic, since he will agree that in any particular domain for which the axioms hold, the statement will be either true or false; and furthermore, he will allow that any given statement either does or does not follow from the axioms. Since the statement that there exists a proof of a given statement from given axioms is in exactly the same position as, say, an existence-statement in number theory for which we have neither proof nor disproof, the formalist has gained no advantage; he has merely switched from one kind of mathematical object—numbers—to another—formal proofs.

Wittgenstein adopts a version (as we shall see, an extreme version) of constructivism; for him it is of the essence of a mathematical statement that it is asserted as the conclusion of a *proof*, whereas I suppose that for a Platonist a being who had *direct* apprehension of mathematical truth, not mediated by inferences, would not be a complete absurdity. There are many different lines of thought converging upon Wittgenstein's constructivism; I shall deal first with his conception of logical necessity.

A great many philosophers nowadays subscribe to some form of conventionalist account of logical necessity, and it is perhaps difficult to realize what a liberation was effected by this theory. The philosophical problem of necessity is twofold: what is its source, and how do we recognize it? God can ordain that something shall hold good of the actual world; but how can even God ordain that something is to hold good in all possible worlds? We know what it is to set about finding out if something *is* true; but what account can we give of the process of discovering whether it *must* be true? According to conventionalism, all necessity is imposed by us not on reality, but upon our language; a statement is necessary by virtue of our having chosen not to count anything as falsifying it. Our recognition of logical necessity thus becomes a particular case of our knowledge of our own intentions.

The conventionalism that is so widespread is, however, a modified conventionalism. On this view, although all necessity derives from linguistic conventions that we have adopted, the derivation is not always direct. Some necessary statements are straightforwardly registers of conventions we have laid down; others are more or less remote *consequences* of conventions. Thus "Nothing can at the same time be green and blue all over" is a direct register of a convention, since there is nothing in the ostensive training we give in the use of color-words which shows that we are not to call something on the borderline between green and blue "both green and blue." "Nothing can be both green and red," on the other hand, is necessary in consequence of the meanings of "green" and "red" as shown in the ostensive training. We did not need to adopt a special convention excluding the expression "both green and red" from our language, since the use by someone of this expression would already show that he had not learned what he was supposed to have learned from the ostensive training.

When applied to mathematics, this modified conventionalism results in the sort of account of mathematical truth with which we are so familiar from logical positivist writings. The axioms of a mathematical theory are necessary in virtue of their being direct registers of certain conventions we have adopted about the use of the terms of the theory; it is the job of the mathematician to discover the more or less remote consequences of our having adopted these conventions, which consequences are epitomized in the theorems. If it is inquired what is the status of the logical principles in accordance with which we pass from axioms to theorems, the reply is that to subscribe to these principles is again the expression of the adoption of linguistic conventions, in this case conventions about the use of "if," "all," and so forth. This account is entirely superficial and throws away all the advantages of conventionalism, since it leaves unexplained the status of the assertion that certain conventions have certain consequences. It appears that if we adopt the conventions registered by the axioms, together with those registered by the principles of inference, then we *must* adhere to the way of talking embodied in the theorem; and *this* necessity must be one imposed upon us, one that we meet with. It cannot itself express the adoption of a convention; the account leaves no room for any further such convention.

Wittgenstein goes in for a full-blooded conventionalism; for him the logical necessity of any statement is always the *direct* expression of a linguistic convention. That a given statement is necessary consists always in our having expressly decided to treat that very statement as unassailable; it cannot rest on our having adopted certain other conventions which are found to involve our treating it so. This account is applied alike to deep theorems and to elementary computations. To give an example of the latter, the criterion which we adopt in the first place for saying that there are *n* things of a certain kind is to be explained by describing the procedure of counting. But when we find that there are five boys and seven girls in a room, we say that there are twelve children altogether, without counting them all together. The fact that we are justified in doing this is not, as it were, implicit in the procedure of counting itself; rather, we have chosen to adopt a *new* criterion for saying that there are twelve children, different from the criterion of counting up all the children together. It would seem that, if we have genuinely distinct criteria for the same statement, they may clash. But the necessity of "$5 + 7 = 12$" consists just in this, that we do not count anything as a clash; if we count the children all together and get eleven, we say, "We must have miscounted."

This account is very difficult to accept, since it appears that the mathematical proof drives us along willy-nilly until we arrive at the theorem. (Of course, we learned "$5 + 7 = 12$" by rote; but we could produce an argument to prove it if the need arose.) But here Wittgenstein brings in the considerations about rules presented in the *Investigations* and elsewhere. A proof proceeds according to certain logical principles or rules of inference. We are inclined to suppose that once we have accepted the axioms

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from which the proof starts, we have, as it were, no further active part to play; when the proof is shown us, we are mere passive spectators. But in order to follow the proof, we have to recognize various transitions as applications of the general rules of inference. Now even if these rules had been explicitly formulated at the start, and we had given our assent to them, our doing so would not in itself constitute recognition of each transition as a correct application of the rules. Once we have the proof, we shall indeed say that anyone who does not accept it either cannot really have understood or cannot really have accepted the rules of inference; but it does not have to be the case that there was anything in what he said or did before he rejected the proof which revealed such a misunderstanding or rejection of the rules of inference. Hence at each step we are free to choose to accept or reject the proof; there is nothing in our formulation of the axioms and of the rules of inference, and nothing in our minds when we accepted these before the proof was given, which of itself shows whether we shall accept the proof or not; and hence there is nothing which *forces* us to accept the proof. If we accept the proof, we confer necessity on the theorem proved; we "put it in the archives" and will count nothing as telling against it. In doing this we are making a new decision, and not merely making explicit a decision we had already made implicitly.

A natural reaction to this is to say that it is true enough when we have not formulated our principles of inference, or have formulated them only in an imprecise form, but that it does not apply at all when we have achieved a strict formalization. Wittgenstein's hostility to mathematical logic is great; he says that it has completely distorted the thinking of philosophers.[5] Because this remark as it stands is so plainly silly, it is difficult to get a clear view of the matter. Consider a favorite example of Wittgenstein's: you train someone to obey orders of the form "Add *n*" with examples taken from fairly small numbers, then give him the order "Add one" and find that he adds two for numbers from 100 to 199, three for numbers from 200 to 299, and so forth. Wittgenstein says that there need have been nothing

[5] IV, 48.

either in what you said to him during the training or in what "went on in your mind" then which of itself showed that this was not what you intended. This is certainly true, and shows something important about the concept of intention (it is a very striking case of what Wittgenstein means when he says in the *Investigations* that if God had looked into my mind, he would not have been able to see there whom I meant). But suppose the training was not given only by example, but made use also of an explicit formulation of the rule for forming from an Arabic numeral its successor. A machine can follow this rule; whence does a human being gain a freedom of choice in this matter which the machine does not possess?

It would of course be possible to argue that someone might appear to understand a rule of inference in a formal system —a substitution rule, say—and yet later reject a correct application of it; but it remains that we can see *in* the precise wording of the rule that that application was warranted. It might be replied that this is to take for granted the ordinary understanding of the words or symbols in terms of which the rule is framed; an explanation of these words or symbols would be something like Wittgenstein's idea of a rule for interpreting the rule. It is undoubtedly true and important that, while in using a word or symbol we are in some sense following a rule, this rule cannot in its turn be formulated in such a way as to leave no latitude in its interpretation, or if it can, the rules for using the words in terms of which this rule is formulated cannot in their turn be so formulated. But such considerations seem to belong to the theory of meaning in general, rather than having any particular relevance to the philosophy of mathematics. Rather, it seems that to someone who suggests that Wittgenstein's point about the scope left in deciding on the correctness of an application of a rule of inference is to be countered by concentrating on rules of inference in formal systems we ought to reply by referring to what Wittgenstein calls the "motley" of mathematics.[6] He wishes, like the intuitionists, to insist that we cannot draw a line in advance round the possible forms of argument that may

[6] II, 46, 48.

be used in mathematical proofs. Furthermore, it might be pointed out that a formal system does not *replace* the intuitive proofs as, frequently, a precise concept replaces a vague intuitive one; the formal system remains, as it were, answerable to the intuitive conception, and remains of interest only so long as it does not reveal undesirable features which the intuitive idea does not possess. An example would be Gödel's theorem, which shows that provability in a formal system cannot do duty as a substitute for the intuitive idea of arithmetical truth.

Suppose we are considering a statement of some mathematical theory. To avoid complications, assume that the theory is complete, that is, that it can be completely formalized, but that we are not thinking of any particular formal system. Then a Platonist will say that there exists either a proof or a disproof of the statement; the fact that the statement is true, if it is true, consists in the existence of such a proof even though we have not yet discovered it. Now if there exists a proof, let us suppose that there is somewhere an actual document, as yet unseen by human eyes, on which is written what purports to be a proof of the statement. Then Wittgenstein will reply that all the same there does not yet exist a proof, since when we discover the document it is still up to us to decide whether or not we wish to count it as a proof. It is evident that, if this is correct, then all motive for saying with the Platonist that there either *is* or *is not* a proof, that the statement must be either true or false, and so forth, has gone. What is not clear to me is that rejecting the Platonist's conception involves adopting this line about proofs; it seems to me that a man might hold that, once the proof was discovered, we had no choice but to follow it, without allowing the correctness of saying, before the proof was discovered, that either there is a proof or there is not. I will return to this later.

Wittgenstein's conception is extremely hard to swallow, even though it is not clear what one wishes to oppose to it. The proof is supposed to have the effect of persuading us, inducing us, to count such-and-such a form of words as unassailably true, or to exclude such-and-such a form of words from our language. It seems quite unclear how the proof accomplishes this remarkable feat. Another difficulty is the scarcity of examples. We naturally think that, face to face with a proof, we have no alternative but to accept the proof if we are to remain faithful to the understanding we already had of the expressions contained in it. For Wittgenstein, accepting the theorem is adopting a new rule of language, and hence our concepts cannot remain unchanged at the end of the proof. But we could have rejected the proof without doing any more violence to our concepts than is done by accepting it; in rejecting it we could have remained equally faithful to the concepts with which we started out. It seems extraordinarily difficult to take this idea seriously when we think of some particular actual proof. It may of course be said that this is because we have already accepted the proof and thereby subjected our concepts to the modification which acceptance of the proof involved; but the difficulty of believing Wittgenstein's account of the matter while reading the proof of some theorem with which one was not previously familiar is just as great. We want to say that we do not know what it would be like for someone who, by ordinary criteria, already understood the concepts employed, to reject this proof. Of course we are familiar with someone's simply not following a proof, but we are also familiar with the remedy, namely to interpolate simpler steps between each line of the proof. The examples given in Wittgenstein's book are—amazingly for him—thin and unconvincing. I think that this is a fairly sure sign that there is something wrong with Wittgenstein's account.

Consider the case of an elementary computation, for example "$5 + 7 = 12$." There might be people who counted as we do but did not have the concept of addition. If such a person had found out by counting that there were five boys and seven girls in a classroom, and were then asked how many children were present, he would proceed to count all the children together to discover the answer. Thus he would be quite prepared to say that on one occasion there were five boys, seven girls, and twelve children altogether, but on another occasion five boys, seven girls, and thirteen children altogether. Now if we came across such a person, we should know what kind of arguments to bring to show him that in such circumstances he must have miscounted on one occasion, and that whenever there are five boys and seven girls there are twelve children. If he accepts these arguments it will be quite true that he will have adopted a new criterion for saying that there are twelve children present, and again a new criterion for saying, "I must have miscounted." Before, he would say, "I miscounted," only when he noticed that he had, for example, counted one of the children twice over; now he will say, "I miscounted," when he has not observed anything of this kind, simply on the ground that he got the result that there were five boys, seven girls, and thirteen children. But we wish to say that even before we met this person and taught him the principles of addition, it would have been true that if he had counted five boys, seven girls, and thirteen children, he would have been wrong even according to the criteria he himself then acknowledged. That is, he must have made a mistake in counting; and if he made a mistake, then there must have been something that he did which, if he had noticed it, he himself would then have allowed as showing that he had miscounted.

If we say that if he counted five boys, seven girls, and thirteen children, then there must have been something which, if he had noticed it, he would have regarded as a criterion for having miscounted, then the effect of introducing him to the concept of addition is not to be simply described as persuading him to adopt a new criterion for having miscounted; rather, he has been induced to recognize getting additively discordant results as a *symptom* of the presence of something he already accepted as a criterion for having miscounted. That is, learning about addition leads him to say, "I miscounted," in circumstances where he would not before have said it; but if, before he had learned, he had said, "I miscounted," in those circumstances, he would have been right by the criteria he then possessed. Hence the necessity for his having miscounted when he gets additively discordant results does not, as it were, get its whole being from his now recognizing such results as a criterion for having miscounted.

If on the other hand we say that it is possible to count five boys, seven girls, and thirteen children without there being anything other than the fact of getting these results such that, if we had noticed it, we should have regarded it as a ground for saying that we had miscounted, then it appears to follow that one can make a mistake in counting (according to the criteria *we* recognize for having miscounted) without having made any particular mistake; that is, one cannot say that if one has miscounted, then either one counted this boy twice, or one counted that girl twice, or . . . . But this is absurd: one cannot make some mistake without there having been some particular mistake which one has made. It might be replied that we can choose to say that if one has miscounted, then either . . ., and that that is in fact what we do choose to say. But if a disjunction is true, then at least one of its limbs must be true; and if a statement is true, there must be something such that if we knew of it, we would regard it as a criterion for the truth of the statement. Yet the assumption from which we started is that someone counts five boys, seven girls, and thirteen children (and hence says that he must have miscounted) and that there is nevertheless nothing apart from his having got these results which (if he knew of it) he would regard as showing that he had miscounted; and hence there can be nothing which (if he knew of it) would show the truth of any one of the disjuncts of the form "He counted that boy twice," and so forth. One might put it by saying that if a disjunction is true, God must know which of the disjuncts is true; hence it cannot be right to count something as a criterion for the truth of the disjunction whose presence does not guarantee the existence of something which would show the truth of some one particular disjunct. For example, it would be wrong to regard ⌜Either if it had been the case that P, it would have been the case that Q, or if it had been the case that P, it would have been the case that not Q⌝ as a logical law, since it is perfectly possible to suppose that however much we knew about the kind of fact which we should regard as bearing on the truth of the disjunct counterfactuals, we should still know nothing which we should count as a reason for accepting either the one or the other.

It is certainly part of the meaning of the word "true" that if a statement is true, there must be something in virtue of which it is true. "There is something in virtue of which it is true" means: there is something such that if we knew of it we should regard it as a criterion (or at least as a ground) for asserting the statement. The essence of realism is this: for any statement which has a definite sense, there must be something in virtue of which either it or its negation is true. (Realism about the realm of mathematics is what we call Platonism.) Intuitionists do not at all deny the first thesis; for them one is justified in asserting a disjunction only when one has a method for arriving at something which would justify the assertion of some one particular limb of the disjunction. Rather, they deny the second thesis: there is no reason for supposing in general that, just because a statement has a quite definite use, there must be something in virtue of which either it is true or it is false. One must beware of saying that logical truths are an exception, that there is nothing in virtue of which they are true; on the contrary, for the realist we are justified in asserting ⌜P or not P⌝ because there must be something in virtue of which either P or ⌜Not P⌝ is true, and hence in any case there must be something in virtue of which ⌜P or not P⌝ is true.

Now there seems here to be one of the big differences between Wittgenstein and the intuitionists. He appears to hold that it is up to us to decide to regard any statement we happen to pick on as holding necessarily, if we choose to do so.[7] The idea behind this appears to be that, by laying down that something is to be regarded as holding necessarily, we thereby in part determine the sense of the words it contains; since we have the right to attach what sense we choose to the words we employ, we have the right to lay down as necessary any statement we choose to regard as such. Against this one would like to say that the senses of the words in the statement may have already been fully determined, so that there is no room for any further determination. Thus, if one takes a classical (realist) view, the general form of explanation of the sense of a statement consists in the stipulation of its truth-conditions (this is the view taken by Wittgenstein in the *Tractatus* and also the view of Frege). Thus the sense of the sentential operators is to be explained by means

[7] Cf. V, 23, last par. on p. 179.

of truth-tables; it is by reference to the truth-tables that one justifies taking certain forms as logically true.

Since the intuitionist rejects the conception according to which there must be for every statement something in virtue of which either it is true or it is false (and does not regard it as possible to remedy the situation by the introduction of further truth-values), for him the fundamental form of an explanation of a statement's meaning consists in stating the criteria we recognize as justifying the assertion of the statement (in mathematics, this is in general the possession of a proof). We thus specify the sense of the sentential operators, of "or," for example, by explaining the criteria for asserting the complex statement in terms of the criteria for asserting the constituents; hence, roughly speaking, we are justified in asserting ⌜P or Q⌝ only when we are justified either in asserting P or in asserting Q. A logical law holds in virtue of these explanations; by reference to them we see that we shall *always* be justified in asserting a statement of this form.

Wittgenstein's quite different idea, that one has the right simply to *lay down* that the assertion of a statement of a given form is to be regarded as always justified, without regard to the use that has already been given to the words contained in the statement, seems to me mistaken. If Wittgenstein were right, it appears to me that communication would be in constant danger of simply breaking down. The decision to count a particular form of statement as logically true does not affect only the sense of statements of that form; the senses of all sorts of other statements will be infected, and in a way that we shall be unable to give a direct account of, without reference to our taking the form of statement in question as logically true. Thus it will become impossible to give an account of the sense of any statement without giving an account of the sense of every statement, and since it is of the essence of language that we understand *new* statements, this means that it will be impossible to give an account of the use of our language at all. To give an example: suppose someone were to choose to regard as a logical law the counter-factual disjunction I mentioned above. We try to object to his claim that this is logically valid by observing that either he must admit that a disjunction may be true when neither limb is true, or that a counterfactual may be true when there is nothing in virtue of which it is true, that is, nothing such that if we knew of it we should regard it as a ground for asserting the counterfactual. But he may respond by denying that these consequences follow; rather, he adduces it as a consequence of the validity of the law that there must be something such that if we knew of it we should count it as a ground either for asserting ⌜If it had been the case that P, then it would have been the case that Q⌝ or for asserting ⌜If it had been the case that P, then it would have been the case that not Q.⌝ For example, he will say that there must be something in which either the bravery or the cowardice of a man consisted, even if that man had never encountered danger and hence had never had an opportunity to display either courage or cowardice. If we hold that he is entitled to regard anything as a logical law which he chooses so to regard, then we cannot deny him the right to draw this conclusion. The conclusion follows from the disjunction of counterfactuals which he elected to regard as logically true in the first place, together with statements we should all regard as logically true; and in any case, he must have the right to regard the conclusion itself as logically true if he so chooses. He will thus conclude that either a man must reveal in his behavior how he would behave in all possible circumstances, or else that there is inside him a sort of spiritual mechanism determining how he behaves in each situation.

Now we know from the rest of Wittgenstein's philosophy how repugnant such a conclusion would be to him; but what right would he have, on his own account of the matter, to object to this man's reaching this conclusion? It is all very well to say, "Say what you like once you know what the facts are"; but how are we to be sure that we can tell anyone what the facts are if it may be that the form of words we use to tell him the facts has for him a different sense as a result of his having adopted some logical law which we do not accept? It might be said that once we discover this difference in the understanding of a certain form of words, we must select another form of words which he does understand as we do and which expresses what we wanted to say; but how are we to know that there is a form of words which does the trick? If we ask him how he understands a certain statement, and he gives the same explanation of it that we should give, this is no guarantee that he in fact understands it as we do; for the mere fact that he recognizes certain forms as logically true which we do not recognize means that he may be able to construct arguments leading to the given statement as a conclusion and with premises that we accept, although we should not accept the argument; that is, he will regard himself as entitled to assert the statement in circumstances in which we should not regard ourselves as entitled to assert it. (An analogy, *not* strictly parallel, is this: we might imagine a classicist and an intuitionist giving explanations of the meaning of the existential quantifier which sounded exactly the same. Yet for all that the classicist will make existential assertions in cases in which the intuitionist will not, since he has been able to arrive at them by means of arguments which the intuitionist will not accept.) Now in the case we are imagining, it is essential to suppose that our man is not capable of giving any general kind of explanation of the words he uses such that we can, from this explanation, derive directly the meaning he attaches to any sentence composed of these words. For if he could give such an explanation, we could see from the explanation why the logical law which he accepts but we do not *is* necessary if the words in it are understood as he understands them. We should thus have a justification for taking statements of that form to be logical laws parallel to the justification of the laws of classical logic in terms of an explanation of meaning by reference to truth-conditions and the justification of intuitionist logic in terms of the explanation by reference to assertibility-conditions. But the whole point of the example was that this was a case of simply laying down a certain form of statement as logically true without the requirement of a justification of this kind.

This attitude of Wittgenstein's to logical necessity may in part explain his ambivalence about the law of excluded middle in mathematics. If a philosopher insists on the law of excluded middle, this is probably the expression of a realist (Platonist) conception of mathematics which Wittgenstein rejects: he insists that ⌜P or not P⌝ is true because he thinks that the general form of explanation of meaning is in terms of truth-conditions, and that for any mathematical statement possessing a definite sense there must be something in virtue of which either it is true or it is false. On the other hand, if a mathematician wishes to use a form of argument depending upon the law of excluded middle (for example, ⌜If P, then Q⌝; ⌜If not P, then Q⌝; therefore, Q), Wittgenstein will not object, since the mathematician has the right to regard the form of words ⌜P or not P⌝ as holding necessarily if he chooses to do so.

To return to the example of the people who counted but did not have addition, it seems likely that someone who accepted Wittgenstein's viewpoint would wish to reject the alternative: either when one of these people counted five boys, seven girls, and thirteen children there must have been something which, if he had noticed it, would have been for him evidence of his having miscounted, or else he could have done so when there was nothing which would have shown him he had miscounted. He would reject it on the ground that it is unclear whether the alternative is being posed in *our* language or in the language of the people in question. *We* say that he must have miscounted, and hence that he must either have counted this boy twice, or . . ., and hence that there was something which if he had noticed it would have shown him that he had miscounted, and we say this just on the ground that his figures do not add up. But he would have no reason for saying it, and would assert that he had probably counted correctly. Now we must not ask whether what we say or what he says is *true*, as if we could stand outside both languages; we just *say* this, that is, we count his having got discordant results as a criterion for saying it, and he does not. Against this I wish, for the reasons I have stated, to set the conventional view that in deciding to regard a form of words as necessary, or to count such-and-such as a criterion for making a statement of a certain kind, we have a responsibility to the sense we have already given to the words of which the statement is composed.

It is easy to see from this why Wittgenstein is so obsessed in this book with an empiricist philosophy of mathematics. He does not wish to accept the empiricist account, but it has a strong allure for him; again and again he comes back to the question, "What is the difference between a calculation and an experiment?". The fact is that even if we decide to *say* that we must have made a mistake in counting when we count five boys, seven girls, and thirteen children, our mere decision to treat this result as a criterion for having made a mistake cannot of itself make it probable that in such circumstances we shall be able to find a mistake; that is, if Wittgenstein's account of the matter is correct. Nevertheless, getting such a discrepancy in counting is a very sure sign in practice that we shall be able to find a mistake, or that if we count again we shall get results that agree. It is because it is such a sure sign in practice that it is possible—or useful—for us to put "$5 + 7 = 12$" in the archives. Thus for Wittgenstein an empirical regularity lies behind a mathematical law.[8] The mathematical law does not *assert* that the regularity obtains, because we do not treat it as we treat an assertion of empirical fact, but as a necessary statement; all the same, what leads us to treat it in this way is the empirical regularity, since it is only because the regularity obtains that the law has a useful application.[9] What the relation is between the regularity and the proof which induces us to put the law in the archives Wittgenstein does not succeed in explaining.

To avoid misunderstanding, I must emphasize that I am not proposing an alternative account of the necessity of mathematical theorems, and I do not know what account should be given. I have merely attempted to give reasons for the natural resistance one feels to Wittgenstein's account, reasons for thinking that it must be wrong. But I believe that whether one accepts Wittgenstein's account or rejects it, one could not after reflecting on it remain content with the standard view which I have called modified conventionalism.

Wittgenstein's constructivism is of a much more extreme kind than that of the intuitionists. For an intuitionist, we may say that every natural number is either prime or composite because we have a method for deciding, for each natural number, whether it is prime or not. Wittgenstein would deny that we have such

[8] III, 44.
[9] E.g., II, 73, 75.

a method. Normally one would say that the sieve of Eratosthenes was such a method; but with a large number one would not —*could* not—use the sieve, but would resort to some more powerful criterion. It will be said that this is a mere practical, not a theoretical, matter, due to the comparative shortness of our lives. But if some fanatic devoted his life to computing, by means of the sieve, the primality of some very large number proved to be prime by more powerful means, and arrived at the conclusion that it was composite, we should not abandon our proof but say that there must be some error in his computations. This shows that we are taking the "advanced" test, and not the sieve, as the *criterion* for primality here: we use the theorem as the standard whereby we judge the computation, and not conversely. The computation is of no use to us because it is not *surveyable*. A mathematical proof, of which computations are a special case, is a proof in virtue of our using it to serve a certain purpose; namely, we put the conclusion or result in the archives, that is, treat it as unassailable and use it as a standard whereby to judge other results. Now something cannot serve this purpose, and hence is not a mathematical proof, unless we are able to exclude the possibility of a mistake's having occurred in it. We must be able to "take in" a proof, and this means that we must be certain of being able to reproduce the *same* proof. We cannot in general *guarantee* that we shall be able to repeat an experiment and get the same result as before. Admittedly, if we get a different result, we shall look for a relevant difference in the conditions of the experiment; but we did not have in advance a clear conception of just what was to count as a relevant difference. (It is not quite clear whether in saying that we must be able to reproduce a proof Wittgenstein means that one must be able to copy from the written proof before one and be certain that one has copied without error, or that one must be able to read the proof and understand it so that one could write it down without referring to the original written proof, so that the possibility of a misprint becomes more or less irrelevant. It does not seem to affect the argument which interpretation is adopted.)

Thus the computation, for a very large number proved prime by other means, of its primality by means of Eratosthenes' sieve would not be a mathematical proof but an experiment to see whether one could do such enormous computations correctly; for the computation would be unsurveyable in the sense explained. Now what the word "prime" means as applied to large numbers is shown by what we accept as the *criterion* for primality, what we take as the standard whereby to assess claims that a number is prime or is composite. The sense of the word "prime" is not therefore given once for all by the sieve of Eratosthenes. Hence we should have no right to assert that every number is either prime or composite, since for any criterion we may adopt there will be a number so large that the application of the criterion to it will not be surveyable. This throws light on Wittgenstein's insistence that the sense of a mathematical statement is determined by its proof (or disproof),[10] that finding a proof alters the concept. One is inclined to think that such a statement as "There is an odd perfect number" is fixed quite definitely in advance, and that our finding a proof or a disproof cannot alter that already determinate sense. We think this on the ground that we are in possession of a method for determining, for *any* number, whether or not it is odd and whether or not it is perfect. But suppose that the statement were to be proved, say by exhibiting a particular odd perfect number. This number would have to be very large, and it is unthinkable that it should be proved to be perfect by the simple method of computing its factors by means of the sieve and adding them all up. The proof would probably proceed by giving a new method for determining perfection, and this method would then have been adopted as our *criterion* for saying of numbers within this range whether or not they are perfect. Thus the proof determines, for numbers of this size, what the *sense* of the predicate "perfect" is to be.

This constructivism, more severe than any version yet proposed, has been called "strict finitism" by Mr. G. Kreisel and "anthropologism" by Dr. Hao Wang. It was adumbrated by Professor Paul Bernays in his *Sur le platonisme dans les mathématiques*.[11] As presented by Bernays, it would consist in concentrating on practical rather than on theoretical possibility. I have tried to

[10] But cf., e.g., V, 7.
[11] *L'enseignement mathématique*, XXXIV (1935), 52-69.

explain how for Wittgenstein this is not the correct way in which to draw the contrast.

It is a matter of some difficulty to consider just what our mathematics would look like if we adopted this "anthropologistic" standpoint. Would the Peano axioms survive unaltered? "Every number has a successor" would mean, in this mathematics, that if a number is accessible (that is, if we have a notation in which it can be surveyably represented) then its successor is accessible, and this at first seems reasonable. On the other hand, it seems to lead to the conclusion that *every* number is accessible, and it is clear that, whatever notation we have, there will be numbers for which there will not be a surveyable symbol in that notation. The problem seems similar to the Greek problem of the heap: if I have something that is not a heap of sand, I cannot turn it into a heap by adding one grain of sand to it. One might solve the present difficulty by arguing as follows. Let us say that we "get to" a number if we actually write down a surveyable symbol for it. Then we may say: if I get to a number, I can get to its successor. From this it follows that if I *can* get to a number, then it is possible that I can get to its successor; that is, if a number is accessible, then its successor is possibly accessible. Unless we think that "possibly possibly p" implies "possibly p" it does not follow that if a number is accessible, its successor is accessible. We should thus have to adopt a modal logic like S2 or M which does not contain the law (in Polish notation) "CMMpMp." Another consideration pointing in the same direction is the following. "Surveyable," "accessible," and so forth, are *vague* concepts. It is often profitable to substitute for a vague concept a precise one, but that would be quite out of place here; we do not want to fix on some definite number as the last accessible number, all bigger numbers being definitely inaccessible. Now the vagueness of a vague predicate is ineradicable. Thus "hill" is a vague predicate, in that there is no definite line between hills and mountains. But we could not eliminate this vagueness by introducing a new predicate, say "eminence," to apply to those things which are neither definitely hills nor definitely mountains, since there would still remain things which were neither definitely hills nor definitely eminences, and so ad infinitum. Hence if we are looking for a logical theory suitable for sentences containing vague predicates, it would be natural to select a modal logic like S2 or M with infinitely many modalities (interpreting the necessity-operator as meaning "definitely"). Thus a suggestion for a propositional calculus appropriate to an anthropologistic mathematics would be one bearing to the modal system M the same relation as intuitionistic propositional calculus bears to S4. (This system would probably have to have axioms of a similar form to those originally given by Heyting, namely, they would frequently be implications whose antecedent was a conjunction, and would have a rule of adjunction as primitive; for, as has been pointed out to me by Mr. E. J. Lemmon, under Tarski's or Gödel's translation an implication whose consequent contains implication reiterated more often than does the antecedent does not usually go over into a valid formula of M, precisely because we do not have in M "CLpLLp.") Another suggestion, made by Dr. Wang, is that anthropologistic logic would coincide with intuitionist, but that the number theory would be weaker.

Wittgenstein uses these ideas to cast doubt upon the significance attached by some philosophers to the reductionist programs of Frege and Russell. We may think that the real meaning of and justification for such an equation as "$5 + 7 = 12$" has been attained if we interpret it as a statement in set theory or in a higher-order predicate calculus; but the fact is that not only the proof but the statement of the proposition in the primitive notation of these theories would be so enormously long as to be quite unsurveyable. It might be replied that we can shorten both the proof and the statement by using defined symbols; but then the definitions play an essential role, whereas for Russell definitions are *mere* abbreviations, so that the real formal statement and formal proof are those in primitive notation. For Wittgenstein notation is not a mere outward covering for a thought which is in itself indifferent to the notation adopted. The proof in primitive notation is not what "really" justifies us in asserting "$5 + 7 = 12$" since we never do write down this proof; if someone were to write it down and obtain the result "$5 + 7 = 11$," we should—appealing to schoolroom addition as a standard—say that he must have made a mistake; we do not even write down the proof with defined symbols; what, if anything, could be called the justification of "$5 + 7 = 12$" would be the proof that we actually do carry out that every addition sum "could" be formulated and proved within our formal logical system, and this proof uses methods far more powerful than the rules for ordinary schoolroom addition.

I now revert to the opposing *pictures* used by Platonists and constructivists—the picture of our making discoveries within an already existing mathematical reality and the picture of our constructing mathematics as we go along. Sometimes people—including intuitionists—argue as though it were a matter of first deciding which of these pictures is correct and then drawing conclusions from this decision. But it is clear that these are only pictures, that is, that the dispute as to which is correct must find its substance elsewhere—that such a dispute ought to be capable of being expressed without reference to these pictures. On the other hand, such pictures have an enormous influence over us, and the desire to be able to form an appropriate picture is almost irresistible. If one does not believe in the objectivity of mathematical truth, one cannot accept the Platonist picture. Wittgenstein's main reason for denying the objectivity of mathematical truth is his denial of the objectivity of *proof* in mathematics, his idea that a proof does not *compel* acceptance; and what fits this conception is obviously the picture of our constructing mathematics as we go along. Now suppose that someone disagrees with Wittgenstein over this and holds that a good proof is precisely one which imposes itself upon us, not only in the sense that once we have accepted the proof we use rejection of it as a criterion for not having understood the terms in which it is expressed, but in the sense that it can be put in such a form that no one could reject it without saying something which would have been recognized before the proof was given as going back on what he had previously agreed to. Is such a person bound to adopt the Platonist picture of mathematics? Clearly not; he can accept the objectivity of mathematical proof without having to believe also in the objectivity of mathematical truth. The intuitionists, for example, usually speak as though they believed in the former without believing in the latter. It is true that

A. Heyting, for instance, writes, "As the meaning of a word can never be fixed precisely enough to exclude every possibility of misunderstanding, we can never be mathematically sure that [a] formal system expresses correctly our mathematical thoughts."[12] But intuitionists incline to write as though, while we cannot delimit in advance the realm of all possible intuitionistically valid proofs, still we can be certain for particular proofs given, and particular principles of proof enunciated, that they are intuitionistically correct. That is to say, the point involved here concerns what Wittgenstein calls the motley of mathematics; the question whether a certain statement is provable cannot be given a mathematically definite formulation since we cannot foresee in advance all possible forms of argument that might be used in mathematics. Still, I suppose that someone might deny even this, in the sense that he claimed for some particular logical framework that every theorem that could be proved intuitionistically could be proved within this framework (though perhaps the proof given might not be reproducible within the framework), and yet remain essentially an intuitionist. For the strongest arguments for intuitionism seem to be quite independent of the question of the objectivity of mathematical proof—whether the proof once given compels acceptance, and whether the concept of valid proof can be made precise. The strongest arguments come from the insistence that the general form of explanation of meaning, and hence of the logical operators in particular, is a statement not of the truth-conditions but of the assertibility-conditions. We learn the meaning of the logical operators by being *trained* in their use, and this means being trained to assert complex statements in certain kinds of situation. We cannot, as it were, extract from this training more than was put into it, and unless we are concerned with a class of decidable statements the notions of truth and falsity cannot be used to give a description of the training we receive. Hence a general account of meaning which makes essential use of the notions of truth and falsity (or of any other number of truth-values) is not of the right form for an explanation of meaning.

[12] *Intuitionism, an Introduction* (Amsterdam, 1956), p. 4.

It is clear that considerations of this kind have nothing to do with mathematics in particular, but are of quite general application. They also have a close connection with Wittgenstein's doctrine that the meaning is the use; and I believe that the *Investigations* contains implicitly a rejection of the classical (realist) Frege-*Tractatus* view that the general form of explanation of meaning is a statement of the truth-conditions.[13] This provides a motive for the rejection by Wittgenstein and the intuitionists of the Platonist picture quite independent of any considerations about the non-objective character of mathematical proof and the motley of mathematics. On the other hand, it is not clear that someone such as I have described, who accepted the considerations about meaning but rejected the considerations about proof, would be happy with the usual constructivist picture of our making up our mathematics. After all, the considerations about meaning do not apply only to mathematics but to all discourse; and while they certainly show something mistaken in the realist conception of thought and reality, they surely do not imply outside mathematics the extreme of subjective idealism—that we *create* the world. But it seems that we ought to interpose between the Platonist and the constructivist picture an intermediate picture, say of objects springing into being in response to our probing. We do not *make* the objects but must accept them as we find them (this corresponds to the proof imposing itself on us); but they were not already there for our statements to be true or false of before we carried out the investigations which brought them into being. (This is of course intended only as a picture; but its point is to break what seems to me the false dichotomy between the Platonist and the constructivist pictures which surreptitiously dominates our thinking about the philosophy of mathematics.)

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[13] Cf. also *Remarks*, I, App. I, 6.