

Automated Reasoning

Lecture 13: Rewriting II

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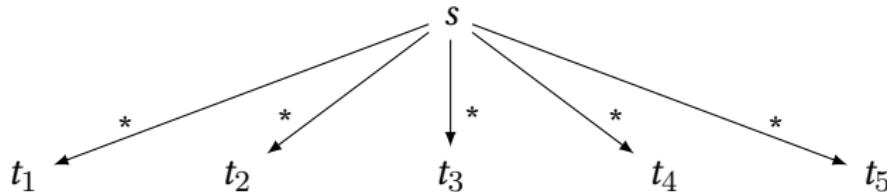
Recap

- ▶ Previously: Rewriting
 - ▶ Definition of Rewrite Rule of Inference
 - ▶ Termination
 - ▶ Rewriting in Isabelle
- ▶ This time: More of the same!
 - ▶ Canonical normal forms
 - ▶ Confluence
 - ▶ Critical Pairs
 - ▶ Knuth-Bendix Completion

Canonical Normal Form

For some rewrite rule sets, order of application might affect result.

We might have:



where all of t_1, t_2, t_3, t_4, t_5 are in normal form after multiple (zero or more) rewrite rule applications.

If all the normal forms are identical we can say we have a **canonical** normal form for s .

This is a very nice property!

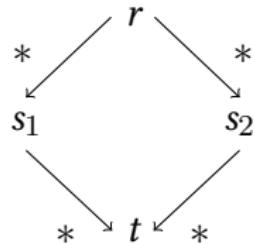
- ▶ Means that order of rewrite rule application doesn't matter
- ▶ In general, means our rewrites are simplifying the expression in a canonical (safe) way.

Confluence and Church-Rosser

How do we know when a set of rules yields canonical normal forms?

A set of rewrite rules is **confluent** if for all terms r , s_1 , s_2 such that $r \rightarrow^* s_1$ and $r \rightarrow^* s_2$ there exists a term t such that $s_1 \rightarrow^* t$ and $s_2 \rightarrow^* t$.

A set of rewrite rules is **Church-Rosser** if for all terms s_1 and s_2 such that $s_1 \leftrightarrow^* s_2$, there exists a term t such that $s_1 \rightarrow^* t$ and $s_2 \rightarrow^* t$.



Theorem

Church-Rosser is equivalent to confluence.

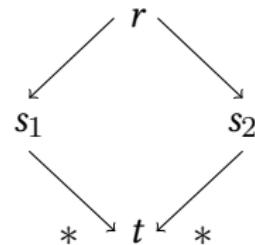
Theorem

For terminating rewrite sets, these properties mean that any expression will rewrite to a canonical normal form.

Local Confluence

The properties of Church-Rosser and confluence can be difficult to prove. A weaker definition is useful:

A set of rewrite rules is **locally confluent** if for all terms r, s_1, s_2 such that $r \rightarrow s_1$ and $r \rightarrow s_2$ there exists a term t such that $s_1 \rightarrow^* t$ and $s_2 \rightarrow^* t$.



Theorem (Newman's Lemma)

$$\text{local confluence} + \text{termination} = \text{confluence}$$

Also: local confluence is decidable (due to Knuth and Bendix)

Both theorem and the decision procedure use idea of **critical pairs**

Choices in Rewriting

How can choices arise in rewriting?

- ▶ Multiple rules apply to a single redex: **order might matter**
- ▶ Rules apply to multiple redexes:
 - ▶ if they are separate: **order does not matter**
 - ▶ if one contains the other: **order might matter**

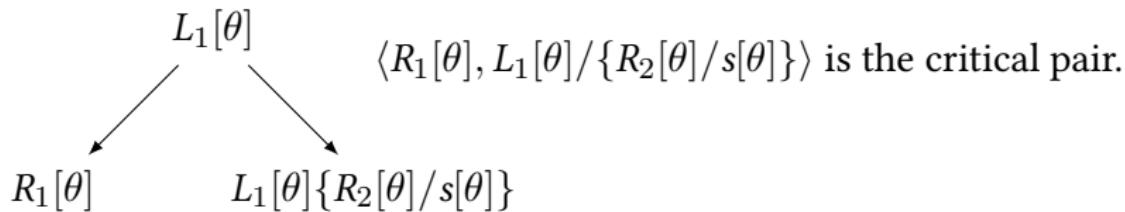
We are interested in cases where the order matters:

Rules	Rewrites	Critical Pair
$X^0 \Rightarrow 1$	0^0 rewrites to 0 and to 1	$\langle 0, 1 \rangle$
$0^Y \Rightarrow 0$		
$X \cdot e \Rightarrow X$	$(x \cdot e) \cdot z$ rewrites to	$\langle x \cdot z, x \cdot (e \cdot z) \rangle$
$(X \cdot Y) \cdot Z \Rightarrow X \cdot (Y \cdot Z)$	$x \cdot z$ and $x \cdot (e \cdot z)$	

Critical Pairs

Given two rules $L_1 \Rightarrow R_1$ and $L_2 \Rightarrow R_2$, we are concerned with the case when there exists a *non-variable* sub-term s of L_1 such that $s[\theta] = L_2[\theta]$, with most general unifier θ .

Applying these rules in different orders gives rise to a **critical pair**, where $L_1[\theta]\{R_2[\theta]/s[\theta]\}$ denotes replacing $s[\theta]$ by $R_2[\theta]$ in $L_1[\theta]$.



Note: the variables in the two rules should be *renamed* so they do **not** share any variable names.

Note: A rewrite rule may have critical pairs with itself e.g. consider the rule $f(f(x)) \Rightarrow g(x)$.

With $W \cdot e \Rightarrow W$ and $(X \cdot Y) \cdot Z \Rightarrow X \cdot (Y \cdot Z)$, where X , Y and Z are variables, we can have $\theta = [W/X, e/Y]$, **any other?**

Critical Pairs: Example

Consider the rewrite rules:

$$\begin{array}{ccc} \overbrace{f(f(x, y), z)}^{L_1} & \Rightarrow & \overbrace{f(x, f(y, z))}^{R_1} \\ s & & \\ \overbrace{f(i(x_1), x_1)}^{L_2} & \Rightarrow & \overbrace{e}^{R_2} \end{array}$$

The mgu θ , given our choice of non-variable subterm s of L_1 , is given by $\theta = \{i(x_1)/x, x_1/y\}$ and by considering:

$$\begin{array}{ccc} f(f(i(x_1), x_1), z) & & \\ \searrow & & \searrow \\ f(i(x_1), f(x_1, z)) & & f(e, z) \end{array}$$

We get the critical pair $\langle f(i(x_1), f(x_1, z)), f(e, z) \rangle$.

Testing for Local Confluence

If we can **conflate** (join) all the critical pairs, then have **local confluence**.

Conflation for a critical pair $\langle s_1, s_2 \rangle$ is when there is a t such that $s_1 \longrightarrow^* t$ and $s_2 \longrightarrow^* t$.

An algorithm to test for local confluence (assuming termination):

1. Find all the critical pairs in set of rewrite rules R
2. For each critical pair $\langle s_1, s_2 \rangle$:
 - 2.1 Find a normal form s'_1 of s_1 ;
 - 2.2 Find a normal form s'_2 of s_2 ;
 - 2.3 Check $s'_1 = s'_2$, if not then fail.

Establishing Local Confluence

Sometimes a set of rules is not locally confluent

$X \cdot e \Rightarrow X$
 $f \cdot X \Rightarrow X$ is not locally confluent: $\langle f, e \rangle$ does not conflate.

We can add the rule $f \Rightarrow e$ to make this critical pair joinable.

However, adding new rules requires **care**:

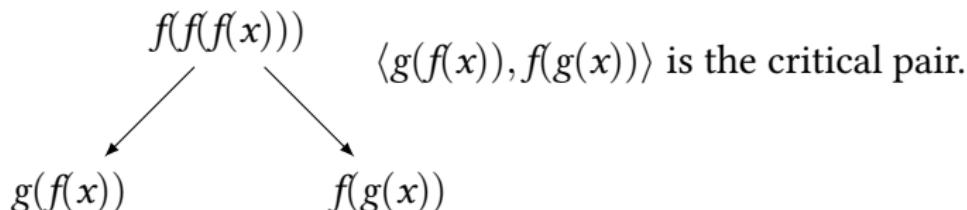
- ▶ Must preserve termination
- ▶ Might give rise to *new* critical pairs and so we may need to check local confluence again.

Establishing Local Confluence: Example

Consider the set R consisting of just one rewrite rule, with x a variable:

$$f(f(x)) \Rightarrow g(x)$$

which has exactly one critical pair (CP) when it is overlapped with a *renamed* copy of itself $f(f(y)) \Rightarrow g(y)$. The lhs $f(f(x))$ unifies with the subterm $f(y)$ of the renamed lhs to produce the mgu $\{f(x)/y\}$:



- ▶ This CP is not joinable, so R is not locally confluent.
- ▶ Adding the rule $f(g(x)) \Rightarrow g(f(x))$ to R makes the pair joinable.
- ▶ The enlarged R is terminating (how?), but
- ▶ (After renaming) new CP: $\langle g(g(z)), f(g(f(z))) \rangle$ arises (how?);
- ▶ LC test: it is joinable, $f(g(f(z))) \rightarrow g(f(f(z))) \rightarrow g(g(z))$.

Knuth-Bendix (KB) Completion Algorithm

Start with a set R of terminating rewrite rules

While there are non-conflatable critical pairs in R :

1. Take a critical pair $\langle s_1, s_2 \rangle$ in R
2. Normalise s_1 to s'_1 and s_2 to s'_2 (and we know $s'_1 \neq s'_2$)
3. if $R \cup \{s'_1 \Rightarrow s'_2\}$ is terminating then

$$R := R \cup \{s'_1 \Rightarrow s'_2\}$$

else if $R \cup \{s'_2 \Rightarrow s'_1\}$ is terminating then

$$R := R \cup \{s'_2 \Rightarrow s'_1\}$$

else Fail

- ▶ If KB succeeds then we have a locally confluent and terminating (and hence confluent) rewrite set (KB may run forever!)
- ▶ Depends on the termination check: define a measure and use that to test for termination.

Summary

- ▶ Rewriting (Bundy Ch. 9)
 - ▶ Local confluence
 - ▶ Local confluence + Termination = Confluence
 - ▶ Canonical Normal Forms
 - ▶ Critical Pairs and Knuth-Bendix Completion