Archimedes, Gauss, and Cauchy

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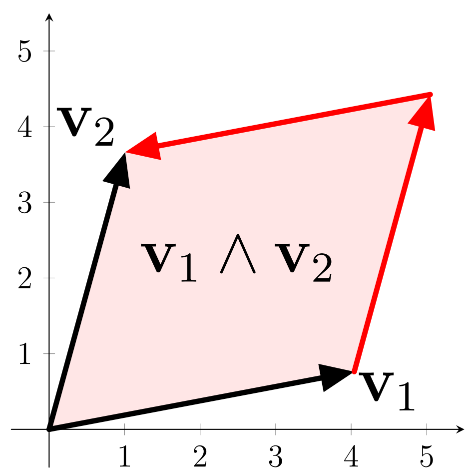
For the purposes of this paper we will say that a function f is *locally optimized* at a point $x = r$ if it's derivative is zero $f'(r) = 0$ at that point.

When Archimedes set about to compute the quadrature of a parabola, along the way he made the following two observations: (albeit without the benefit of our modern abstractions)

Optimizing the area of a triangle inscribed on a parabola Given the parabola $f(x) = ax^2 + bx + c$ and the inscribed segment between $P_1 = (t, f(t))$ and $P_2 = (s, f(s))$, for some constants a, b, c, t, s with $a \neq 0$, define the function $A^f_{t,s}(x)$ to be the (signed) area of the triangle with points P_1, P_2, and $(x, f(x))$. Then the unique point which locally optimizes this function occurs at $(t+s)/2$, and the tangent line to the parabola at that point is parallel to the segment between P_1 and P_2.

In this case Archimedes wanted to maximize the area of the triangle when $x \in (a, b)$, since the area can increase without bound otherwise. Later on he would prove that the area between the inscribed segment and the parabola was $\frac{4}{3}$ the area of this maximized inscribed triangle.

In order to prove this, let's start by finding the expression $A^f_{t,s}(x)$ for the area of a triangle with points P_1, P_2, and $(x, f(x))$. We will want to use a tool called *Gauss's shoelace formula* for computing the signed area of general polygons. If we have two vectors given by $\mathbf{v}_1 = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}}$ and $\mathbf{v}_2 = x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}}$, then the expression $\mathbf{v}_1 \wedge \mathbf{v}_2 = (x_1y_2 - x_2y_1)\hat{\mathbf{i}} \wedge \hat{\mathbf{j}} = -\mathbf{v}_2 \wedge \mathbf{v}_1$ measures the signed area of the counterclockwise parallelogram with sides v_1 and v_2 (relative to a chosen unit area of this plane $\hat{\mathbf{i}} \wedge \hat{\mathbf{j}}$).



We can obtain this scaling factor $x_1y_2 - x_2y_1$ (also known as a determinant) using the axioms of associativity, distributivity, and nilpotence $\mathbf{v} \wedge \mathbf{v} = 0$ (left as an exercise to the reader). These axioms justify calling $\wedge$ the *wedge product*. Since $\hat{\mathbf{i}} \wedge \hat{\mathbf{j}}$ is a nonzero constant we can exclude it going forward and focus on $x_1y_2 - x_2y_1$.

If we take this expression and divide it by two then it becomes the signed area of the triangle with one point at the origin as well as two points at the tips of $\mathbf{V}_1$ and $\mathbf{V}_2$.

Instead of making the argument to prove the shoelace formula in its full generality, we will simply state it in the case of a triangle. Proof of this special case is also left as an exercise.

Signed area of a triangle Given points P_1, P_2, and P_3 in a plane, and a chosen origin O in that plane, define the vectors $\mathbf{v}_i = P_i - O = (x_i, y_i)$ for $i = 1, 2, 3$ (according to your preferred pair of coordinate axis). Then the signed area A of the triangle formed by P_1, P_2, and P_3 is given by the expression

$$A = \frac{\mathbf{v}_1 \wedge \mathbf{v}_2 + \mathbf{v}_2 \wedge \mathbf{v}_3 + \mathbf{v}_3 \wedge \mathbf{v}_1}{2} = \frac{x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3}{2}\hat{\mathbf{i}} \wedge \hat{\mathbf{j}}.$$

Finishing the proof With this excellent tool at our disposal we can say that the signed area of our inscribed triangle is given by

$$2A^f_{t,s}(x) = tf(s) - sf(t) + (f(t) - f(s))x + (s - t)f(x),$$

which also give us that

$$2(A^f_{t,s})'(x) = f(t) - f(s) + (s - t)f'(x).$$

If we locally optimize $A^f_{t,s}(x)$ by writing $(A^f_{t,s})'(r) = 0$ then we can rearrange this to write

$$\begin{aligned} f'(r) &= \frac{f(s) - f(t)}{s - t} \\ \implies 2ar + b &= \frac{a(s^2 - t^2) + b(s - t)}{s - t} \\ \implies 2ar + b &= a(t + s) + b \\ \implies r &= \frac{t + s}{2} \end{aligned}$$

which verifies both of Archimedes's observations. $\square$

At this point the reader might realize that we have actually proven something much more interesting along the way if we consider this in light of Cauchy's *mean value theorem.*

MVT If a function $f(x)$ is smooth on the interval $[a, b]$, then there is at least one $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

What we have shown along the way to confirm Archimedes's observations is that whenever we try to locally optimize $A^f_{t,s}(x)$ of any smooth f the critical points we get all have the additional property that the tangent lines $y = f'(r)(x - r) + f(r)$ are parallel to the segment between $(t, f(t))$ and $(s, f(s))$. According to the mean value theorem, at least one of these critical points occurs between t and s, for any t and s.