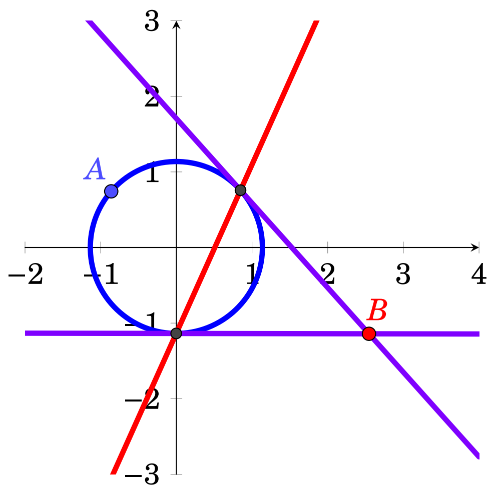
In 2021 I was investigating the idea of pole-polar duality in projective geometry. I had learned a few different methods for finding the lines passing through a point B tangent to a circle A, and I realized that it always came down to solving a quadratic equation when trying to find the lines individually. I had a hunch there might be a way to use techniques from linear algebra to find an equation for the product of those lines on the gamble that the coefficients of that product were rational. Im happy to report success, and that the formula holds in considerable generality. I exported the figures below from geogebra into tikz using the formula I obtained, which you can check yourself by clicking the link below and examining the definitions of the demonstration

https://www.geogebra.org/m/xkxnwkhs



Notation, Definitions, and Lemmas

Suppose that K is a field where $2 \neq 0$ and Φ is a symmetric $n \times n$ matrix with nonzero determinant. For vectors $\mathbf{a}, \mathbf{b} \in K^n$, define a Φ-*dot product* of vectors "$\cdot_\Phi$" by:

$$\mathbf{a} \cdot_\Phi \mathbf{b} = \mathbf{a}^T \Phi \mathbf{b}.$$

When $n = 2$ this looks like

$$\begin{bmatrix} x_1 & y_1 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = ax_1x_2 + bx_1y_2 + bx_2y_1 + cy_1y_2.$$

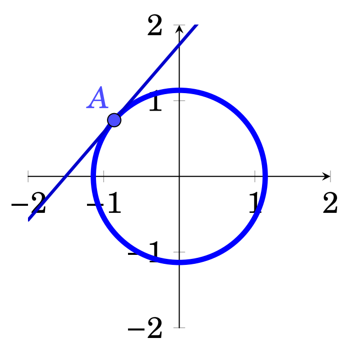
When $\Phi = I$ we are working in Euclidean geometry. For brevity usually we drop the subscript Φ as well as say $\mathbf{a} \cdot \mathbf{a} = \mathbf{a}^2$. Given a vector of variables $\mathbf{x}$, the condition that $det\Phi \neq 0$ is enough justification to call the vector equation $\mathbf{x}^2 = \mathbf{a}^2$ a *circle centered at the origin with radius* $\mathbf{a}$ (at least when $n = 2$). The quantity $\mathbf{a}^2$ is called the *quadrance* of $\mathbf{a}$, and we say that two vectors are *perpendicular* $\mathbf{a} \perp \mathbf{b}$ precisely when $\mathbf{a} \cdot \mathbf{b} = 0$. The quadrance of the radius a circle is also referred to as the quadrance of the circle. A nonzero vector with quadrance 0 is called a *null vector*, and a circle with quadrance 0 is called a null circle. Null vectors are perpendicular to themselves, and in the plane a null circle is a product of lines.

In this context we still have the Pythagorean theorem: if $\mathbf{a} + \mathbf{b} = \mathbf{c}$ then

$$\mathbf{a} \perp \mathbf{b} \iff \mathbf{a}^2 + \mathbf{b}^2 = \mathbf{c}^2.$$

which follows from linearity and symmetry. This is important because it allows us to define power of a point with respect to any conic, but first we need to do a little calculus and define *pole-polar duality*.

Lemma 1 *Given the circle* $\mathbf{x}^2 = \mathbf{a}^2$, *the equation of the line tangent to the circle at* $\mathbf{a}$ *is given by the equation* $\mathbf{x} \cdot \mathbf{a} = \mathbf{a}^2$.



Proof There are a lot of boring but straightforward proofs, but we are going to take a coordinate free approach for this one. Starting with the equation $\mathbf{x}^2 = \mathbf{a}^2$, set $\mathbf{x} = \mathbf{y} + \mathbf{a}$ so that our equation reads

$$\mathbf{y}^2 + 2\mathbf{y} \cdot \mathbf{a} = 0.$$

Since we are looking for the tangent line, toss out the $\mathbf{y}^2$ term, then undo the substitution so that

$$(\mathbf{x} - \mathbf{a}) \cdot \mathbf{a} = 0$$

as required. The reader is invited to check that this formula is correct (at least for the case of n=2) by some other method. An important consequence of this lemma is that *tangent lines are perpendicular to diameters*.

Now that we have tangent lines for points on the circle, we are closer to our goal. Suppose that $\mathbf{b}^2 \neq \mathbf{a}^2$ so that $\mathbf{b}$ does not lie on the same circle centered at the origin as $\mathbf{a}$. In projective geometry it is customary to call such points a "pole". Since there are two lines tangent to $\mathbf{x}^2 = \mathbf{a}^2$ passing through $\mathbf{b}$, call the points of tangency the lines make with the circle $\mathbf{a}_1$ and $\mathbf{a}_2$. Finding these points directly can be done by solving a quadratic equation, but we want to avoid that for our purposes.

Lemma 2 *The line* $\mathbf{x} \cdot \mathbf{b} = \mathbf{a}^2$ *contains the points of tangency* $\mathbf{a}_1$ *and* $\mathbf{a}_2$. *This line is called the "Polar" of* $\mathbf{b}$ *with respect to* $\mathbf{x}^2 = \mathbf{a}^2$.

Proof The tangent lines at $\mathbf{a}_1$ and $\mathbf{a}_2$ are given by the equations

$$\mathbf{x} \cdot \mathbf{a}_1 = \mathbf{a}^2$$

$$\mathbf{x} \cdot \mathbf{a}_2 = \mathbf{a}^2$$

insisting that $\mathbf{b}$ lie on both these lines are the conditions

$$\mathbf{b} \cdot \mathbf{a}_1 = \mathbf{a}^2$$

$$\mathbf{b} \cdot \mathbf{a}_2 = \mathbf{a}^2$$

therefore $\mathbf{a}_1$ and $\mathbf{a}_2$ lie on $\mathbf{b} \cdot \mathbf{x} = \mathbf{a}^2$ as required. Similar reasoning proves that if $\mathbf{b}$ lies on the polar of $\mathbf{c}$, then we also have that $\mathbf{c}$ lies on the polar of $\mathbf{b}$. This remarkable symmetry is called "Pole-Polar duality" and is another example of projective geometry allowing us to interchange points and lines. This proof only works when $n = 2$, but for $n = k$ the proof is the same with k linear constraints to consider before you conclude that the polar contains every point of tangency.

Using the Pythagorean theorem and the fact that tangent lines are perpendicular to diameters, we know that the quadrances of the segments from $\mathbf{b}$ to $\mathbf{a}_1$ and $\mathbf{a}_2$ are given by $\mathbf{b}^2 - \mathbf{a}^2$. This quantity is called *The power of the point* $\mathbf{b}$ with respect to the circle $\mathbf{x}^2 = \mathbf{a}^2$.

Finally we need to know that given polynomials in n variables $p_1(x_1, ..., x_n)$ and $p_2(x_1, ..., x_n)$ or $p_i(\mathbf{x})$ for short, then any common solution $\mathbf{a}$ of $p_i(\mathbf{x}) = 0$ is also a solution of $\alpha p_1(\mathbf{x}) + \beta p_2(\mathbf{x}) = 0$, for $\alpha, \beta \in K$.

Derivation of the formula

With all that housekeeping taken care of we can state and prove the main result

Theorem *The product of the lines tangent to* $\mathbf{x}^2 = \mathbf{a}^2$ *passing through a point* $\mathbf{b}$ *not on the circle is given by the equation*

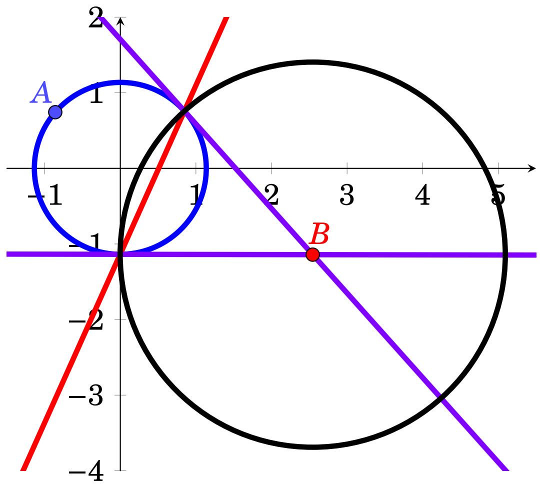
$$(\mathbf{x} \cdot \mathbf{b} - \mathbf{b}^2)^2 - (\mathbf{b}^2 - \mathbf{a}^2)(\mathbf{x} - \mathbf{b})^2 = 0$$

Proof The strategy is to find two quadratic equations which intersect on the lines we are looking for, and find the linear combination of those two quadratics which has

$\mathbf{b}$ as a solution. The way we do this will make the proof valid in every dimension as well. Since nontrivial linear combinations the circle $\mathbf{x}^2 = \mathbf{a}^2$ and the polar $\mathbf{x} \cdot \mathbf{b} = \mathbf{a}^2$ can only create circle passing through their intersection, they will not do. Instead we will consider the circle $(\mathbf{x} - \mathbf{b})^2 = \mathbf{b}^2 - \mathbf{a}^2$, or equivalently, the zeros of the polynomial

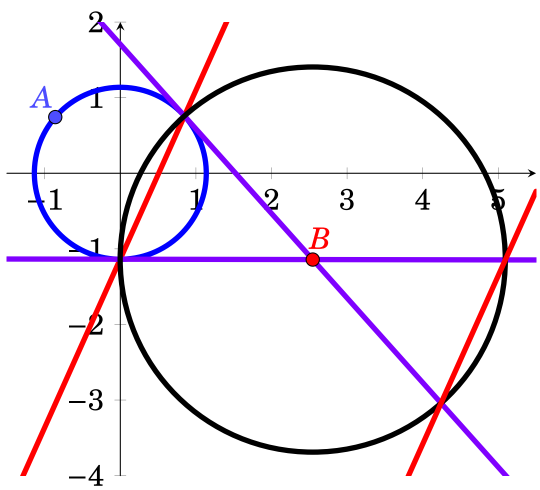
$$p_1(\mathbf{x}) = \mathbf{x}^2 - 2\mathbf{x} \cdot \mathbf{b} + \mathbf{a}^2.$$

This circle has the power of $\mathbf{b}$ as its quadrance so it also contains the points of tangency. The advantage this circle has is that it intersects the lines of tangency in two more points, namely the reflections of $\mathbf{a}_1$ and $\mathbf{a}_2$ in $\mathbf{b}$, as shown in the figure below.



Again we dont need direct access to these points, just the linear equation passing through them. We compute this by reflecting the polar of $\mathbf{b}$ in $\mathbf{b}$:

$$(2\mathbf{b} - \mathbf{x}) \cdot \mathbf{b} = \mathbf{a}^2$$



So the equation of the polar multiplied by its reflection is given by

$$(\mathbf{x}\cdot\mathbf{b}-\mathbf{a}^2)(2\mathbf{b}^2-\mathbf{x}\cdot\mathbf{b}-\mathbf{a}^2)=0$$

which we associate to the polynomial

$$p_2(\mathbf{x})=(\mathbf{x}\cdot\mathbf{b}-\mathbf{a}^2)(2\mathbf{b}^2-\mathbf{x}\cdot\mathbf{b}-\mathbf{a}^2).$$

Now we are looking for values of α and β which solve the equation

$$\alpha p_1(\mathbf{b})+\beta p_2(\mathbf{b})=0.$$

A straightforward solution is given by $\alpha = p_2(\mathbf{b})$ and $\beta = -p_1(\mathbf{b})$. The rest is just algebra:

$$\begin{aligned}
&p_2(\mathbf{b})p_1(\mathbf{x})-p_1(\mathbf{b})p_2(\mathbf{x})\\
&=(\mathbf{b}\cdot\mathbf{b}-\mathbf{a}^2)(2\mathbf{b}^2-\mathbf{b}\cdot\mathbf{b}-\mathbf{a}^2)(\mathbf{x}^2-2\mathbf{x}\cdot\mathbf{b}+\mathbf{a}^2)-(\mathbf{b}^2-2\mathbf{b}\cdot\mathbf{b}+\mathbf{a}^2)(\mathbf{x}\cdot\mathbf{b}-\mathbf{a}^2)(2\mathbf{b}^2-\mathbf{x}\cdot\mathbf{b}-\mathbf{a}^2)\\
&=(\mathbf{b}^2-\mathbf{a}^2)\Big[(\mathbf{b}^2-\mathbf{a}^2)(\mathbf{x}^2-2\mathbf{x}\cdot\mathbf{b}+\mathbf{a}^2)-(\mathbf{x}\cdot\mathbf{b})^2+2\mathbf{b}^2(\mathbf{x}\cdot\mathbf{b})-\mathbf{a}^2(2\mathbf{b}^2-\mathbf{a}^2)\Big]\\
&=(\mathbf{b}^2-\mathbf{a}^2)\Big[(\mathbf{b}^2-\mathbf{a}^2)(\mathbf{x}^2-2\mathbf{x}\cdot\mathbf{b})-(\mathbf{x}\cdot\mathbf{b})^2+2\mathbf{b}^2(\mathbf{x}\cdot\mathbf{b})-\mathbf{a}^2\mathbf{b}^2\Big]
\end{aligned}$$

If we add and subtract $\mathbf{b}^4$ from the inside of the square brackets it factors into

$$p_2(\mathbf{b})p_1(\mathbf{x})-p_1(\mathbf{b})p_2(\mathbf{x})=-(\mathbf{b}^2-\mathbf{a}^2)\Big[(\mathbf{x}\cdot\mathbf{b}-\mathbf{b}^2)^2-(\mathbf{b}^2-\mathbf{a}^2)(\mathbf{x}-\mathbf{b})^2\Big]$$

Since $-(\mathbf{b}^2-\mathbf{a}^2)$ is constant, we can ignore it. Thus the square brackets are the expression we desire.

A beautiful consequence of this fact is that if you have two circles with the same center, one null, and one with nonzero quadrance, then they are tangent at the complex projective line at infinity. An n=3 demonstration for the tangent cone to a sphere (using the same formula) can be found here

https://www.geogebra.org/m/yzmyjnqm.

An attentive reader might point out that we never used the assumption that $det\Phi \neq 0$. The formula does still work in that case, but admittedly I am biased towards a certain narative for the proof of this one. The proof presented is the first Ive found, but since then it has occurred to me that using

$$p_1(\mathbf{x})=\mathbf{x}^2-a^2, \qquad p_2(\mathbf{x})=(\mathbf{x}\cdot\mathbf{b}-\mathbf{a}^2)^2$$

may make the algebra more palatable.