Three Significant Quotients of $\mathbb{R}$

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After constructing $\mathbb{R}$ there are 3 interesting quotients by quadratic polynomials which tell us significant things about geometry and calculus. In order of relative familiarity to most, they are:

$$\mathbb{C} = \mathbb{R}[i]/(i^2+1) \qquad \mathbb{D} = \mathbb{R}[u]/(u^2) \qquad \mathbb{H} = \mathbb{R}[j]/(j^2-1).$$

As expected $\mathbb{C}$ is isomorphic to the well-known complex numbers which are essential for the fundamental theorem of algebra and euclidean geometry. The second construction $\mathbb{D}$ creates the *Dual Numbers* which are used in nonstandard analysis and automatic differentiation. Finally, I call $\mathbb{H}$ the *hyberbolic complex numbers* because this construction naturally models rotations of the Minkowski plane like $\mathbb{C}$ does for the Euclidean plane.

Theorem 1 The three quotients $\mathbb{C}$, $\mathbb{D}$, $\mathbb{H}$ are algebras of dimension 2 over $\mathbb{R}$.

Proof We can prove the stronger claim that $\mathbb{R}[x]/(q(x))$ is an algebra of dimension $deg(q)$ over R using the division algorithm as is standard. For polynomials $p(x)$ and $q(x)$ the division algorithm in $\mathbb{R}[x]$ tells us we can write

$$p(x) = d(x)q(x) + r_0 + r_1 x + ... + r_{deg(q)-1}x^{deg(q)-1}.$$

Passing to the quotient we have

$$p(x) \equiv r_0 + r_1 x + ... + r_{deg(q)-1}x^{deg(q)-1} \ mod(q(x)).$$

Since polynomials of degree at most $n-1$ form a vector space of dimension n, and these polynomials are closed under multiplication in our quotient, we are done.

First, let us state and prove the most centrally interesting property of $u \in \mathbb{D}$

Theorem 2 If $f(z)$ is analytic in a domain $D \subseteq \mathbb{C}$ then for $z \in D$ we have

$$f(z+u) = f(z) + f'(z)u$$

Proof By combining the property $u^2 = 0$ with the taylor expansion of $f(z)$ in D we have that

$$f(z+u) = \sum_{n\geq 0} a_n(z+u)^n = \sum_{n\geq 0} \frac{f^{(n)}(z)}{n!}(u)^n = f(z) + f'(z)u.$$

Much more could be said about arithmetic in $\mathbb{D}$ but I will pass over that for brevity, in favor of a substantial discussion of the similarities between $\mathbb{C}$ and $\mathbb{H}$.

The multiplication of two generic elements of $\mathbb{H}$ looks like

$$(a+bj)(c+dj) = ac + bd + (ad+bc)j$$

which tells us that whenever $a^2 - b^2 \neq 0$ we can write $(a+bj)^{-1} = \frac{a-bj}{a^2-b^2}$.

We call the quantity $|a+bj|_h = \sqrt{|a^2-b^2|}$ the hyperbolic magnitude of $a+bj$. it is straightforward to check that

$$|(a+bj)(c+dj)|_h = |(a+bj)|_h|(c+dj)|_h$$

so hyperbolic complex multiplication multiplies hyperbolic magnitudes.

In this context, we should consider 3 ”circles”

$$x^2 - y^2 = 1,$$

$$x^2 - y^2 = 0,$$

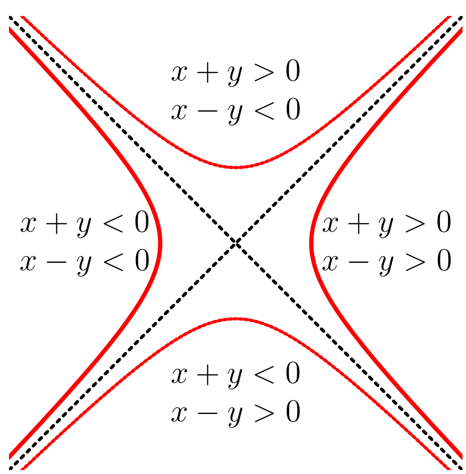
$$x^2 - y^2 = -1,$$

known respectively the hyperbolic unit, null, and anti-null circles.

Cosh and sinh are cooked up to be parts of the parametrization of $x^2 - y^2 = 1$ which has a velocity with constant hyperbolic magnitude. As in regular complex analysis we have the formula

$$e^{jt} = \sum_{n\geq 0} \frac{(jt)^n}{n!} = \sum_{n\geq 0} \frac{(t)^{2n}}{2n!} + j \sum_{n\geq 0} \frac{(t)^{2n+1}}{(2n+1)!} = cosh(t) + jsinh(t)$$

There is a defect present here that is not observed in cos and sin, making angles and polar coordinates in the hyperbolic setting a little more delicate to discuss. The problem is that cosh is an even function with positive coefficients and $cosh(0) = 1$, so $cosh(x) \geq 1$. This forces us to split the Minkowski plane into 4 quadrants relative to the null circle



and we need different formulas for our polar coordinates in each quadrant. Define the functions

$$h_1(t) = cosh(t) + jsinh(t) \quad h_2(t) = jh_1(t)) \quad h_3(t) = -h_1(t) \quad h_4(t) = -jh_1(t)$$

then we can use these to prove that hyperbolic complex multiplication adds angles.

Theorem 3 For hyperbolic angles θ and ϕ we have that $h_1(\theta)h_1(\phi) = h_1(\theta + \phi)$.

Proof We start by writing down

$$h_1(\theta)h_1(\phi) = (cosh(\theta) + jsinh(\theta))(cosh(\phi) + jsinh(\phi))$$

then expanding to get

$$cosh(\theta)cosh(\phi) + sinh(\theta)sinh(\phi) + j(cosh(\theta)sinh(\phi) + cosh(\phi)sinh(\theta))$$

from which we recognize that the real part is $cosh(\theta+\phi)$ and the imaginary part is $sinh(\theta+\phi)$. We could also have proved this much more simply by recognizing that $h_1(t) = e^{jt}$.

At this point there is more that we should say about factoring polynomials in $\mathbb{H}$, but that may go too far away from our goal of investigating PDE's, so the reader will have to consult "two ways to solve a cubic" for more details.

For now we will talk about differentiability of functions of hyperbolic complex variables.

Theorem 4 Suppose we have An appropriate condition function of a hyperbolic complex variable that can be written as a series around the origin

$$f(z = x + jy) = \sum_{n\geq 0} a_n(x + jy)^n = u(x, y) + jv(x, y)$$

then we have $\frac{\partial}{\partial x}u(x, y) = \frac{\partial}{\partial y}v(x, y) \qquad \frac{\partial}{\partial y}u(x, y) = \frac{\partial}{\partial x}v(x, y)$.

Proof If we take the partial derivatives of f with respect to x and y we get

$$\frac{\partial}{\partial x}f(x + jy) = \sum_{n\geq 0} a_n n(x + jy)^{n-1}, \qquad \frac{\partial}{\partial y}f(x + jy) = j\sum_{n\geq 0} a_n n(x + jy)^{n-1}.$$

From these two equations, we deduce the relation

$$\frac{\partial}{\partial x}f(x + jy) = j\frac{\partial}{\partial y}f(x + jy).$$

This imposes additional constraints on the real and imaginary parts

$$\frac{\partial}{\partial x}u(x, y) + j\frac{\partial}{\partial x}v(x, y) = j\frac{\partial}{\partial y}u(x, y) + \frac{\partial}{\partial y}v(x, y)$$

$$\implies \frac{\partial}{\partial x}u(x, y) = \frac{\partial}{\partial y}v(x, y) \qquad \frac{\partial}{\partial y}u(x, y) = \frac{\partial}{\partial x}v(x, y).$$

If we introduce a hyperbolic dot product

$$\vec{v}_1 \cdot_h \vec{v}_2 = x_1x_2 - y_1y_2$$

then we can say two vectors are minkowski perpendicular whenever $\vec{v}_1 \cdot_h \vec{v}_n = 0$.
We can also define a hyperbolic gradient operator

$$\nabla_h = \vec{e}_1\frac{\partial}{\partial x} - \vec{e}_2\frac{\partial}{\partial y}$$

by insisting that it satisfies $\nabla_h \cdot_h v_1 = x_1\frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y}$. Finally, we can define the hyperbolic laplace operator

$$\triangle_h = \nabla_h \cdot_h \nabla_h = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}.$$

Theorem 5 The real and imaginary parts of An appropriate condition $f(x+jy) = u(x,y) + jv(x,y)$ are Minkowski perpendicular

$$\nabla_h u(x,y) \cdot_h \nabla_h v(x,y) = 0.$$

Proof By expanding the left side we get

$$\nabla_h u(x,y) \cdot_h \nabla_h v(x,y) = (\vec{e}_1\frac{\partial}{\partial x}u(x,y) - \vec{e}_2\frac{\partial}{\partial y}u(x,y)) \cdot_h (\vec{e}_1\frac{\partial}{\partial x}v(x,y) - \vec{e}_2\frac{\partial}{\partial y}v(x,y))$$

$$= \frac{\partial}{\partial x}u(x,y)\frac{\partial}{\partial x}v(x,y) - \frac{\partial}{\partial y}u(x,y)\frac{\partial}{\partial y}v(x,y) = \frac{\partial}{\partial x}u(x,y)\frac{\partial}{\partial y}u(x,y) - \frac{\partial}{\partial y}u(x,y)\frac{\partial}{\partial x}u(x,y) = 0.$$

Theorem 6 An appropriate condition function $f(x+jy)$ satisfies the hyperbolic laplace equation

$$\triangle_h f(x+jy) = 0.$$

Proof Writing out the left side we have

$$\triangle_h f(x+jy) = (\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2})(u(x,y) + jv(x,y))$$

$$= \frac{\partial^2}{\partial x^2}u(x,y) - \frac{\partial^2}{\partial y^2}u(x,y) + j\big[\frac{\partial^2}{\partial x^2}v(x,y) - \frac{\partial^2}{\partial y^2}v(x,y)\big]$$

$$= \frac{\partial^2}{\partial x\partial y}u(x,y) - \frac{\partial^2}{\partial y\partial x}u(x,y) + j\big[\frac{\partial^2}{\partial x\partial y}v(x,y) - \frac{\partial^2}{\partial y\partial x}v(x,y)\big] = 0.$$