Moment generating function

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In combinatorics an **exponential generating function** for the sequence $F = \{f_n\}_{n=0}^{\infty}$ is the power series

$$G_F(x) = \sum_{n=0}^{\infty} \frac{f_n x^n}{n!}.$$

There are many reasons why writing down this power series helps us deduce properties of the sequence f_n, but a very nice feature of $G_F(x)$ is

$$\frac{d^k}{dx^k} G_F(x)|_{x=0} = f_n$$

which tells us we can recover the terms of our sequence by taking derivatives and setting $x = 0$. The reason we get this convenience is that

$$\frac{d^k}{dx^k} G_F(x) = \sum_{n=k}^{\infty} \frac{n(n-1)...(n-k+1)}{n!} f_n x^{n-k} = \sum_{n=k} \frac{f_n x^{n-k}}{k!}$$

So if we set $x = 0$ then all we are left with is f_k the constant term of the derivative.

We can find a lot of use of exponential generating functions in statistics by combining them with the concept of the k^{th} moment of a random variable X with density f_X

$$M_X^k = \mathbb{E}(X^k) = \int_{\mathbb{R}} x^k f_X(x) dx.$$

We can define the **moment generating function** for the random variable X to be

$$M_X(t) = \sum_{n=0}^{\infty} \frac{M_X^n t^n}{n!}.$$

If we tried to compute each of these coefficients individually then we could possibly have a lot of work cut out for us to do so. Thankfully this definition allows us to find another way of computing this function all at once, which lets us go back and find the coefficients using derivatives, and that practically is much easier than solving integrals. We have the useful expression

$$M_X(t) = \mathbb{E}[e^{tX}].$$

<u>Proof</u> From our definitions we have

$$M_X(t) = \sum_{n=0}^{\infty} \frac{M_X^n t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{\mathbb{R}} x^n f_X(x) dx = \int_{\mathbb{R}} \sum_{n=0}^{\infty} \frac{t^n}{n!} x^n f_X(x) dx = \int_{\mathbb{R}} e^{tx} f(x) dx = \mathbb{E}[e^{tX}].$$

We are assuming that in our context it makes sense to switch integrals and infinite series.