ODE proofs of sum formulas

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The most important fact about an ordinary homogeneous linear differential equation is that its solutions form a vector space. That is, if functions $f(z)$ and $g(z)$ are sent to 0 by the differential operator

$$D = \frac{d^n}{dz^n} + a\frac{d^{n-1}}{dz^{n-1}} + ... + p\frac{d}{dz} + q$$

then so is $\alpha f(z) + \beta g(z)$, aka

$$D\big[\, \alpha f(z) + \beta g(z) \,\big] = 0,$$

and that the dimension of this vector space is n.

We will use this fact to prove the argument sum laws of e^z, $cos(z)$, $sin(z)$, $cosh(z)$, and $sinh(z)$. Additionally, we show that $cos(z)$ and $cosh(z)$ are even while $sin(z)$ and $sinh(z)$ are odd. The conceptual novelty of this approach is that it allows us to establish these facts without using geometry or a series representation of the functions. As a conclusion we will establish Euler's famous formula $e^{i\pi} = -1$. We will take for granted that these 5 functions are entire on the complex plane.

Theorem 1 The function e^x has the sum formula $e^{x+y} = e^x e^y$.

Proof The ODE definition of the function e^x is the unique solution to the differential equation:

$$e^z = \begin{cases} \frac{d}{dz}f - f = 0 \\ f(0) = 1 \end{cases}.$$

Since it has order 1 this means the set e^x is a basis of the solutions of the differential operator $\frac{d}{dx} - 1$. Since we are taking y to be constant with respect to x we have that e^{x+y} is also a solution of $\frac{d}{dx} - 1$. This means we can write

$$e^{x+y} = \lambda e^x$$

for some number λ. By using our initial condition we have

$$e^{0+y} = \lambda e^0 \Longrightarrow \lambda = e^y.$$

Theorem 2 The euclidean trigonometric functions have angle sum formulas

$$cos(x+y) = cos(x)cos(y) - sin(x)sin(y),$$

$$sin(x+y) = cos(x)sin(y) + cos(y)sin(x).$$

Proof The ODEs which define $cos(z)$ and $sin(z)$ are

$$cos(z) = \begin{cases} \frac{d^2}{dz^2}f + f = 0 \\ f(0) = 1, f'(0) = 0 \end{cases} \qquad sin(z) = \begin{cases} \frac{d^2}{dz^2}f + f = 0 \\ f(0) = 0, f'(0) = 1 \end{cases}$$

which come from insisting that these functions parametrize the unit circle $x^2+y^2 = 1$, but that is an investigation for elsewhere. These two equations means that the set $\{\, cos(z), sin(z) \,\}$ is a basis for solutions of $f'' + f = 0$.

Since $cos(x+y)$ and $sin(x+y)$ also satisfy the condition $f''+f=0$, they can be written as a linear combination of $cos(x)$ and $sin(x)$, for instance

$$cos(x+y)=\alpha cos(x)+\beta sin(x)$$

but we need a second equation to solve for our constants, so we take the derivative to get

$$-sin(x+y)=-\alpha sin(x)+\beta cos(x).$$

By plugging our initial conditions into our equations we get

$$cos(0+y)=\alpha cos(0)+\beta sin(0)$$

$$-sin(0+y)=-\alpha sin(0)+\beta cos(0)$$

which tells us $\alpha=cos(y)$ and $\beta=-sin(y)$ as required.

Theorem 3 The hyperbolic trigonometric functions have angle sum formulas

$$cosh(x+y)=cosh(x)cosh(y)+sinh(x)sinh(y),$$

$$sinh(x+y)=cosh(x)sinh(y)+cosh(y)sinh(x).$$

Proof The ODEs which define $cosh(z)$ and $sinh(z)$ are

$$cosh(z)=\begin{cases}\frac{d^2}{dz^2}f-f=0\\ f(0)=1, f'(0)=0\end{cases}\qquad sinh(z)=\begin{cases}\frac{d^2}{dz^2}f-f=0\\ f(0)=0, f'(0)=1\end{cases}$$

which come from insisting that these functions parametrize the unit hyperbola $x^2-y^2=1$. These two equations means that the set $\{\,cosh(z),\, sinh(z)\,\}$ is a basis for solutions of $f''-f=0$.

Since $cosh(x+y)$ and $sinh(x+y)$ also satisfy the condition $f''-f=0$ we can write

$$cosh(x+y)=\alpha cosh(x)+\beta sinh(x)$$

but we need a second equation to solve for our constants, so we take the derivative to get

$$sinh(x+y)=\alpha sinh(x)+\beta cosh(x).$$

By plugging our initial conditions into our equations we get

$$cosh(0+y)=\alpha cosh(0)+\beta sinh(0)$$

$$sinh(0+y)=\alpha sinh(0)+\beta cosh(0)$$

which tells us $\alpha=cosh(y)$ and $\beta=sinh(y)$ as required.

Theorem 4 The function $cos(z)$ is even and $sin(z)$ is odd.

Proof the function $cos(-z)$ also satisfies $f'' + f = 0$ so we can write

$$cos(-z) = \alpha cos(z) + \beta sin(z)$$

and differentiating we also get

$$sin(-z) = -\alpha sin(z) + \beta cos(z).$$

As usual, plugging in the initial condition gives us $\alpha = 1$ and $\beta = 0$.

Theorem 5 The function $cosh(z)$ is even and $sinh(z)$ is odd.

Proof The proof is left as an exercise to the reader.

Theorem 6 Euler's most beautiful formula $e^{i\pi} = -1$ can be proved without using series.

Proof The function e^{iz} also satisfies the differential equation $f'' + f = 0$ so we can write the two equations

$$e^{iz} = \alpha cos(z) + \beta sin(z)$$
$$ie^{iz} = -\alpha sin(z) + \beta cos(z).$$

By plugging in our initial conditions we get $\alpha = 1$ and $\beta = i$ so that

$$e^{iz} = cos(z) + isin(z).$$

Then we just plug in π.

If the reader is familiar with hyperbolic complex numbers, they might be amused by using this technique to prove

$$e^{jz} = cosh(z) + jsinh(z).$$

Additionally, this technique can also be used to show $cos(iz) = cosh(z)$ and that $sin(iz) = sinh(z)$, which shows that Euclidean and Minkowski geometry are related by an imaginary dilation.