Supplementary: How does the Combined Risk Affect the Performance of Unsupervised Domain Adaptation Approaches?

Proofs of Theorems

Proof of Theorem 1

Theorem 1 is a corollary of Theorem 2. If we set the transformation space $\mathcal{G} = \{\mathbf{I}\}$, where $\mathbf{I}$ is the identity map from $\mathcal{X}$ to $\mathcal{X}$, then Theorem 1 can be concluded by Theorem 2 directly.

Proof of Theorem 2

Using triangle inequality of ℓ, we have

$$\begin{aligned} R_t^\ell(\mathbf{C} \circ \mathbf{G}) &\le R_t^\ell(\mathbf{C}' \circ \mathbf{G}, \mathbf{C} \circ \mathbf{G}) + R_t^\ell(\mathbf{C}' \circ \mathbf{G}), \\ R_s^\ell(\mathbf{C}' \circ \mathbf{G}, \mathbf{C} \circ \mathbf{G}) &\le R_s^\ell(\mathbf{C}' \circ \mathbf{G}) + R_s^\ell(\mathbf{C} \circ \mathbf{G}), \end{aligned} \tag{1}$$

where $\mathbf{C}'$ is any scoring function in $\mathcal{H}$.

Above inequalities imply that

$$\begin{aligned} &R_t^\ell(\mathbf{C} \circ \mathbf{G}) - R_s^\ell(\mathbf{C} \circ \mathbf{G}) \\ \le& R_t^\ell(\mathbf{C}' \circ \mathbf{G}, \mathbf{C} \circ \mathbf{G}) - R_s^\ell(\mathbf{C}' \circ \mathbf{G}, \mathbf{C} \circ \mathbf{G}) \\ &+ R_s^\ell(\mathbf{C}' \circ \mathbf{G}) + R_t^\ell(\mathbf{C}' \circ \mathbf{G}). \end{aligned} \tag{2}$$

According to inequality (2), it is easy to check that

$$\begin{aligned} &R_t^\ell(\mathbf{C} \circ \mathbf{G}) - R_s^\ell(\mathbf{C} \circ \mathbf{G}) \\ \le& \sup_{\mathbf{C}, \mathbf{C}' \in \mathcal{H}} \big| R_t^\ell(\mathbf{C}' \circ \mathbf{G}, \mathbf{C} \circ \mathbf{G}) - R_s^\ell(\mathbf{C}' \circ \mathbf{G}, \mathbf{C} \circ \mathbf{G}) \big| \\ &+ \min_{\mathbf{C}' \in \mathcal{H}} \big(R_s^\ell(\mathbf{C}' \circ \mathbf{G}) + R_t^\ell(\mathbf{C}' \circ \mathbf{G}) \big) \\ \le& d_{\mathcal{H}}^\ell(P_{\mathbf{G}(X_s)}, P_{\mathbf{G}(X_t)}) \\ &+ \min_{\mathbf{C}' \in \mathcal{H}} \big(R_s^\ell(\mathbf{C}' \circ \mathbf{G}) + R_t^\ell(\mathbf{C}' \circ \mathbf{G}) \big). \end{aligned}$$

The proof has been completed.

Proof of Theorem 3

Step 1.

$$\begin{aligned} &R_s^\ell(\mathbf{C}_t \circ \mathbf{G}) + R_t^\ell(\mathbf{C}_s \circ \mathbf{G}) + \delta \\ =& \big(R_s^\ell(\mathbf{C}_t \circ \mathbf{G}) + R_t^\ell(\mathbf{C}_t \circ \mathbf{G}) \big) + \big(R_s^\ell(\mathbf{C}_s \circ \mathbf{G}) + R_t^\ell(\mathbf{C}_s \circ \mathbf{G}) \big) \\ \ge& 2 \min_{\mathbf{C}' \in \mathcal{H}} \big(R_s^\ell(\mathbf{C}' \circ \mathbf{G} \circ \mathbf{G}) + R_t^\ell(\mathbf{C}' \circ \mathbf{G} \circ \mathbf{G}) \big) = 2\lambda^\ell(\mathbf{G}). \end{aligned}$$

Step 2. According to Theorem 2, if we set $\mathbf{C} = \mathbf{C}_s$, we have

$$R_t^\ell(\mathbf{C}_s \circ \mathbf{G}) \le R_s^\ell(\mathbf{C}_s \circ \mathbf{G}) + d_{\mathcal{H}}^\ell(P_{\mathbf{G}(X_s)}, P_{\mathbf{G}(X_t)}) + \lambda^\ell(\mathbf{G}). \tag{3}$$

If we exchange the source domain and the target domain, then we use Theorem 2 and set $\mathbf{C} = \mathbf{C}_t$. We have

$$R_s^\ell(\mathbf{C}_t \circ \mathbf{G}) \le R_t^\ell(\mathbf{C}_t \circ \mathbf{G}) + d_{\mathcal{H}}^\ell(P_{\mathbf{G}(X_s)}, P_{\mathbf{G}(X_t)}) + \lambda^\ell(\mathbf{G}). \tag{4}$$

Combining the inequality (3) and the inequality (4), we have

$$R_t^\ell(\mathbf{C}_s \circ \mathbf{G}) + R_s^\ell(\mathbf{C}_t \circ \mathbf{G}) \le \delta + 2d_{\mathcal{H}}^\ell(P_{\mathbf{G}(X_s)}, P_{\mathbf{G}(X_t)}) + 2\lambda^\ell(\mathbf{G}).$$

Combining the results of **Step 1** and **Step 2**, we have proved the result.

Proof of Theorem 4

Using triangle inequality of ℓ_s, ℓ_t, we have

$$\begin{aligned} R_t^{\ell_t}(\mathbf{C} \circ \mathbf{G}) &\le R_t^{\ell_t}(\mathbf{C}' \circ \mathbf{G}, \mathbf{C} \circ \mathbf{G}) + R_t^{\ell_t}(\mathbf{C}' \circ \mathbf{G}), \\ R_s^{\ell_s}(\mathbf{C}' \circ \mathbf{G}, \mathbf{C} \circ \mathbf{G}) &\le R_s^{\ell_s}(\mathbf{C}' \circ \mathbf{G}) + R_s^{\ell_s}(\mathbf{C} \circ \mathbf{G}), \end{aligned}$$

where $\mathbf{C}'$ is any scoring function in $\mathcal{H}$.

The above inequalities imply that

$$\begin{aligned} &R_t^{\ell_t}(\mathbf{C} \circ \mathbf{G}) - R_s^{\ell_s}(\mathbf{C} \circ \mathbf{G}) \\ \le& R_t^{\ell_t}(\mathbf{C}' \circ \mathbf{G}, \mathbf{C} \circ \mathbf{G}) - R_s^{\ell_s}(\mathbf{C}' \circ \mathbf{G}, \mathbf{C} \circ \mathbf{G}) \\ &+ R_s^{\ell_s}(\mathbf{C}' \circ \mathbf{G}) + R_t^{\ell_t}(\mathbf{C}' \circ \mathbf{G}). \end{aligned} \tag{5}$$

According to inequality (5), it is easy to check that

$$\begin{aligned} &R_t^{\ell_t}(\mathbf{C} \circ \mathbf{G}) - R_s^{\ell_s}(\mathbf{C} \circ \mathbf{G}) \\ \le& \sup_{\mathbf{C}' \in \mathcal{H}} \big(R_t^{\ell_t}(\mathbf{C}' \circ \mathbf{G}, \mathbf{C} \circ \mathbf{G}) - R_s^{\ell_s}(\mathbf{C}' \circ \mathbf{G}, \mathbf{C} \circ \mathbf{G}) \big) \\ &+ \min_{\mathbf{C}' \in \mathcal{H}} \big(R_s^{\ell_s}(\mathbf{C}' \circ \mathbf{G}) + R_t^{\ell_t}(\mathbf{C}' \circ \mathbf{G}) \big) \\ \le& d_{\mathbf{C}, \mathcal{H}}^{\ell_s \ell_t}(P_{\mathbf{G}(X_s)}, P_{\mathbf{G}(X_t)}) \\ &+ \min_{\mathbf{C}' \in \mathcal{H}} \big(R_s^{\ell_s}(\mathbf{C}' \circ \mathbf{G}) + R_t^{\ell_t}(\mathbf{C}' \circ \mathbf{G}) \big). \end{aligned}$$

The proof is completed.