习题12-1 **中山大學 本科生考试草稿纸** 2011/6-1

P.325.1. 证明等式 (1) $\int_{-\pi}^{\pi}\sin mx\cdot\sin nx\,dx=0$; (2) $\int_{-\pi}^{\pi}\sin mx\cdot\cos nx\,dx=0.\ (m\neq n)$

证 (1) $\int_{-\pi}^{\pi}\sin mx\cdot\sin nx\,dx=\int_{-\pi}^{\pi}-\frac{1}{2}[\cos(m+n)x-\cos(m-n)x]dx=-\frac{1}{2}\left[\frac{\sin(m+n)x}{m+n}-\frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi}$
$=0$

(2) $\int_{-\pi}^{\pi}\sin mx\cdot\cos nx\,dx=\int_{-\pi}^{\pi}\frac{1}{2}[\sin(m+n)x+\sin(m-n)x]=\frac{1}{2}\left[\frac{-\cos(m+n)x}{m+n}+\frac{-\cos(m-n)x}{m-n}\right]_{-\pi}^{\pi}=0$

P.325.2 证明范数 (f,g) 有性质：$(f,\ c_1g_1+c_2g_2)=c_1(f,g_1)+c_2(f,g_2)$.
c_1,c_2 为常数。

证：由定义 $(f,g)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cdot g(x)dx$

从而 $(f,\ c_1g_1+c_2g_2)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cdot[c_1g_1+c_2g_2]dx=\frac{1}{\pi}\int_{-\pi}^{\pi}[c_1f\cdot g_1+c_2f\cdot g_2]$

$=c_1\cdot\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cdot g_1(x)dx+c_2\cdot\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cdot g_2(x)dx$

$=c_1(f,g_1)+c_2(f,g_2)$

P.325.3 证明 $|f(x),g(x)|\le\|f(x)\|\cdot\|g(x)\|$.

证：由于对任何 λ，$\|f(x)-\lambda g(x)\|^2\geq 0$，$\Longrightarrow\frac{1}{\pi}\int_{-\pi}^{\pi}[f-\lambda g]\cdot[f-\lambda g]dx$

$=\frac{1}{\pi}\int_{-\pi}^{\pi}[f^2-2\lambda f\cdot g+\lambda^2g^2]dx\geq 0$

也即 $\frac{1}{\pi}\int_{-\pi}^{\pi}f^2(x)dx-2\lambda\cdot\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cdot g(x)dx+\lambda^2\cdot\frac{1}{\pi}\int_{-\pi}^{\pi}g^2(x)dx\geq 0$

也即 $\frac{1}{\pi}\int_{-\pi}^{\pi}g^2(x)dx\cdot\lambda^2-\frac{2}{\pi}\int_{-\pi}^{\pi}f(x)\cdot g(x)dx\cdot\lambda+\frac{1}{\pi}\int_{-\pi}^{\pi}f^2(x)dx\geq 0$

必有 $\left[\frac{-2}{\pi}\int_{-\pi}^{\pi}f(x)\cdot g(x)dx\right]^2-4\cdot\frac{1}{\pi}\int_{-\pi}^{\pi}g^2(x)dx\cdot\frac{1}{\pi}\int_{-\pi}^{\pi}f^2(x)dx\leq 0$

从而 $\left[\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cdot g(x)dx\right]^2\leq\frac{1}{\pi}\int_{-\pi}^{\pi}f^2(x)dx\cdot\frac{1}{\pi}\int_{-\pi}^{\pi}g^2(x)dx$

开方得：$\left|\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cdot g(x)dx\right|\leq\sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi}f^2(x)dx}\cdot\sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi}g^2(x)dx}$，即：$|(f,g)|\leq\|f\|\cdot\|g(x)\|$

习题12-2 中山大學 本科生考试草稿纸 2011/2/6-2

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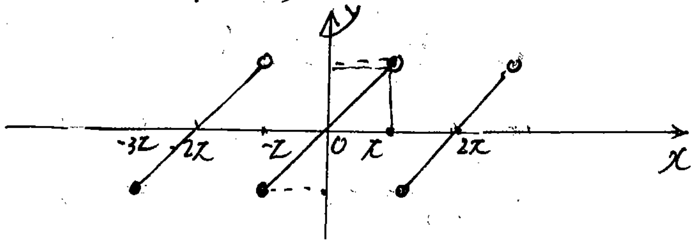
P.339.1. 设 $y=f(x)$ 是以 2π 为周期的函数，它在 $[-\pi,\pi)$ 中的表达式分别由下列各式给出，求出 $f(x)$ 的傅氏级数及其和函数。

(1) $f(x)=x,\quad -\pi\le x<\pi$；



解：$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}xdx=0$

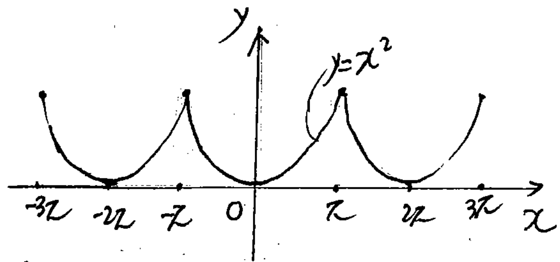
$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}x\cdot\cos nx\,dx=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cdot\sin nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin nx\,dx=\frac{2}{\pi}\int_0^{\pi}x\sin nx\,dx=\frac{-2}{n\pi}\int_0^{\pi}x\,d\cos nx$$

$$=-\frac{2}{n\pi}\left[x\cdot\cos nx\Big|_0^{\pi}-\int_0^{\pi}\cos nx\,dx\right]=-\frac{2}{n\pi}[\pi\cdot(-1)^n-0]=\frac{(-1)^{n+1}\cdot 2}{n}.$$

$$f(x)\sim\sum_{n=1}^{\infty}\frac{2\cdot(-1)^{n+1}}{n}\cdot\sin nx=2\left(\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x+\cdots\right)=\begin{cases}x, & -\pi<x<\pi\\ 0 & x=\pm\pi\end{cases}$$

(2) $f(x)=x^2,\quad -\pi\le x\le\pi$.



解：$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2dx=\frac{2}{\pi}\int_0^{\pi}x^2dx=\frac{2\pi^2}{3}.$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}x^2\cos nx\,dx$$

$$=\frac{2}{n\pi}\int_0^{\pi}x^2\,d\sin nx=\frac{2}{n\pi}\left[x^2\sin nx\Big|_0^{\pi}-\int_0^{\pi}\sin nx\cdot 2x\,dx\right]$$

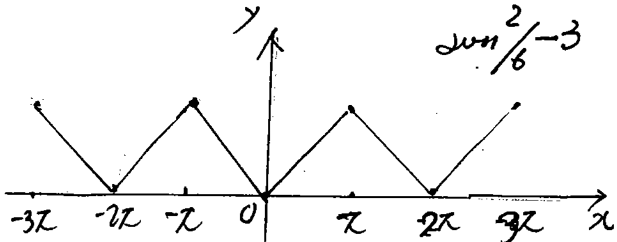
$$=\frac{2}{n\pi}\left(0+\frac{2}{n}\int_0^{\pi}x\,d\cos nx\right)=\frac{4}{n^2\pi}\left[x\cos nx\Big|_0^{\pi}-\int_0^{\pi}\cos nx\,dx\right]=\frac{4}{n^2\pi}[\pi\cdot(-1)^n]=(-1)^n\cdot\frac{4}{n^2}.$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cdot\sin nx\,dx=0$$

$$f(x)\sim\frac{\pi^2}{3}+\sum_{n=1}^{\infty}(-1)^n\cdot\frac{4}{n^2}\cos nx=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cdot\cos nx=x^2,\quad(-\pi\le x\le\pi)$$

Jun 2/6-3

(3) $f(x)=|x|,\quad \pi\le x\le\pi.$



解: $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}|x|dx=\frac{2}{\pi}\int_0^{\pi}xdx=\pi.$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cdot\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}|x|\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}x\cdot\cos nx\,dx=\frac{2}{n\pi}\int_0^{\pi}x\,d\sin nx$$

$$=\frac{2}{n\pi}[x\cdot\sin nx\Big|_0^{\pi}-\int_0^{\pi}\sin nx\,dx]=\frac{2}{n\pi}[0+\frac{1}{n}\cos nx\Big|_0^{\pi}]=\frac{2}{n^2\pi}[(-1)^n-1]$$

$$=\begin{cases}0, & n=2k\\ \frac{-4}{(2k-1)^2\cdot\pi}, & n=2k-1.\end{cases}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cdot\sin nx\,dx=0$$

$$f(x)\sim\frac{\pi}{2}-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos(2k-1)x=\frac{\pi}{2}-\frac{4}{\pi}(\cos x+\frac{1}{3^2}\cos 3x+\frac{1}{5^2}\cos 5x+\cdots\cdots)$$

$$=|x|,\qquad -\pi\le x\le\pi.$$

(4) $$f(x)=\begin{cases}-2 & -\pi\le x<0\\ 1 & 0\le x\le\pi\end{cases}$$

解: $$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}[\int_{-\pi}^{0}-2dx+\int_0^{\pi}1\cdot dx]=\frac{1}{\pi}[-2x\Big|_{-\pi}^{0}+x\Big|_0^{\pi}]=\frac{1}{\pi}(-2\pi+\pi)=-1$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cdot\cos nx\,dx=\frac{1}{\pi}[\int_{-\pi}^{0}-2\cos nx\,dx+\int_0^{\pi}1\cdot\cos nx\,dx=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cdot\sin nx\,dx=\frac{1}{\pi}[\int_{-\pi}^{0}(-2)\cdot\sin nx\,dx+\int_0^{\pi}1\cdot\sin nx\,dx]$$

$$=\frac{1}{\pi}[\frac{2}{n}\cos nx\Big|_{-\pi}^{0}-\frac{1}{n}\cos nx\Big|_0^{\pi}]=\frac{1}{\pi}\{\frac{2}{n}(1-(-1)^n)-\frac{1}{n}[(-1)^n-1]\}$$

$$=\frac{1}{n\pi}[2-2(-1)^n-(-1)^n+1]=\frac{3}{n\pi}[1-(-1)^n]$$

$$=\begin{cases}0, & n=2k\\ \frac{6}{n\pi} & n=2k-1\end{cases}$$

$$f(x)\sim-\frac{1}{2}+\sum_{k=1}^{\infty}\frac{6}{(2k-1)\pi}\sin(2k-1)x=-\frac{1}{2}+\frac{6}{\pi}(\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\cdots\cdots)$$

$$=\begin{cases}-2 & -\pi<x<0\\ -\frac{1}{2} & x=0,\ x=\pm\pi\\ 1 & 0<x<\pi\end{cases}$$

中山大學 本科生考试草稿纸

2011/6-4

$$\sin^4 x=(\sin^2 x)^2=\left(\frac{1-\cos 2x}{2}\right)^2=\frac{1-2\cos 2x+\cos^2 2x}{4}=\frac{1-2\cos 2x+\frac{\cos 4x+1}{2}}{4}$$

$$=\frac{3}{8}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$$

P.339. 1.(5) $f(x)=\sin^4 x$；$-\pi\le x\le\pi$.

解：$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\sin^4 x\cdot\sin nx\,dx=0$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{2}{\pi}\int_0^{\pi}\sin^4 x\,dx=\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\sin^4 x\,dx=\frac{4}{\pi}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{3}{4}.$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cdot\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{3}{8}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x\right)\cdot\cos nx\,dx$$

由三角函数系的正交性可知，$n\ne 2$, $n\ne 4$ 时 $a_n=0$.

$$a_2=\frac{2}{\pi}\int_0^{\pi}\sin^4 x\cdot\cos 2x\,dx=\frac{4}{\pi}\int_0^{\pi}\sin^4 x(1-2\sin^2 x)\,dx$$

$$=\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\sin^4 x\,dx-\frac{8}{\pi}\int_0^{\frac{\pi}{2}}\sin^6 x\,dx=\frac{4}{\pi}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}-\frac{8}{\pi}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=-\frac{1}{2}$$

$$a_4=\frac{1}{\pi}\int_{-\pi}^{\pi}\sin^4 x\cdot\cos 4x\,dx=\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\sin^4 x(1-8\sin^2 x+8\sin^4 x)\,dx$$

$$=\frac{4}{\pi}\left[\int_0^{\frac{\pi}{2}}\sin^4 x\,dx-8\int_0^{\frac{\pi}{2}}\sin^6 x\,dx+8\int_0^{\frac{\pi}{2}}\sin^8 x\,dx\right]$$

$$=\frac{4}{\pi}\left[\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}-8\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}+8\cdot\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]=\frac{3}{4}-5+5\times\frac{7}{8}=\frac{3}{4}-\frac{5}{8}=\frac{1}{8}$$

$$f(x)\sim\frac{3}{8}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x=\sin^4 x,\qquad -\pi\le x\le\pi.$$

(6) $f(x) = \begin{cases} e^x, & -\pi \le x < 0 \\ 1, & 0 \le x < \pi \end{cases}$

2011/2/6 − 5.

解：$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}[\int_{-\pi}^{0} e^x dx + \int_0^{\pi} 1\cdot dx]$

$= \frac{1}{\pi}[e^0 - e^{-\pi} + \pi] = \frac{\pi + 1 - e^{-\pi}}{\pi}$.

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos nx\,dx = \frac{1}{\pi}[\int_{-\pi}^{0} e^x \cos nx\,dx + \int_0^{\pi} 1\cdot\cos nx\,dx]$

$= \frac{1}{\pi}\int_{-\pi}^{0} \cos nx\,de^x + 0 = \frac{1}{\pi}[\cos nx\cdot e^x\Big|_{-\pi}^{0} - \int_{-\pi}^{0} e^x d\cos nx]$

$= \frac{1}{\pi}[1 - (-1)^n\cdot e^{-\pi} + n\int_{-\pi}^{0} e^x \sin nx\,dx]$

$= \frac{1}{\pi}[1 - (-1)^n\cdot e^{-\pi} + n\int_{-\pi}^{0} \sin nx\,de^x]$

$= \frac{1}{\pi}[1 - (-1)^n\cdot e^{-\pi} + n\cdot(e^x\sin nx\Big|_{-\pi}^{0} - \int_{-\pi}^{0} e^x\cdot n\cdot\cos nx\,dx)]$

$= \frac{1}{\pi}[1 - (-1)^n\cdot e^{-\pi} - n^2\int_{-\pi}^{0} e^x\cdot\cos nx\,dx]$

记有：$\int_{-\pi}^{0} e^x\cdot\cos nx\,dx = 1 - (-1)^n\cdot e^{-\pi} - n^2\int_{-\pi}^{0} e^x\cdot\cos nx\,dx$

$\int_{-\pi}^{0} e^x\cos nx\,dx = \frac{1-(-1)^n\cdot e^{-\pi}}{1+n^2}$; $a_n = \frac{1}{\pi}\cdot\frac{1-(-1)^n\cdot e^{-\pi}}{1+n^2}$.

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}[\int_{-\pi}^{0} e^x\cdot\sin nx\,dx + \int_0^{\pi} 1\cdot\sin nx\,dx]$

$= \frac{1}{\pi}[\int_{-\pi}^{0} \sin nx\,de^x - \frac{1}{n}\cos nx\Big|_{-\pi}^{0}] = \frac{1}{\pi}\{e^x\sin nx\Big|_{-\pi}^{0} - \int_{-\pi}^{0} e^x d\sin nx - \frac{(-1)^n - 1}{n}\}$

$= \frac{1}{\pi}[0 - n\int_{-\pi}^{0} e^x\cdot\cos nx\,dx - \frac{(-1)^n - 1}{n}]$

$= \frac{1}{\pi}\{(-n)\cdot\frac{1-(-1)^n\cdot e^{-\pi}}{1+n^2} + \frac{1-(-1)^n}{n}\} = \frac{1}{\pi}\cdot[\frac{-n+(-1)^n\cdot n e^{-\pi}}{1+n^2} + \frac{1-(-1)^n}{n}]$

$f(x) \sim \frac{\pi + 1 - e^{-\pi}}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\{[\frac{-n+(-1)^n\cdot n e^{-\pi}}{1+n^2} + \frac{1-(-1)^n}{n}]\cdot\sin nx + \frac{1+(-1)^n e^{-\pi}}{1+n^2}\cdot\cos nx\}$

$= \begin{cases} e^x, & -\pi < x < 0 \\ \frac{e^{-\pi}+1}{2} & x = \pm\pi \\ 1 & 0 \le x < \pi \end{cases}$

中山大學 本科生考试草稿纸 2011/6-6

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P.340.2 P.340.3. 将函数 $f(x)=\frac{x^2}{4}-\frac{\pi x}{2}$ $(0\le x\le\pi)$ 展开余弦级数。（正弦）

解:2. 对 $f(x)$ 进行偶延拓。 $b_n=0$

$$a_0=\frac{2}{\pi}\int_0^\pi f(x)dx=\frac{2}{\pi}\int_0^\pi(\frac{x^2}{4}-\frac{\pi x}{2})dx=\frac{2}{\pi}(\frac{\pi^3}{12}-\frac{\pi^3}{4})=-\frac{4\pi^2}{12}=-\frac{\pi^2}{3}$$

$$a_n=\frac{2}{\pi}\int_0^\pi f(x)\cdot\cos nx\,dx=\frac{2}{\pi}\int_0^\pi(\frac{x^2}{4}-\frac{\pi x}{2})\cdot\cos nx\,dx$$

$$=\frac{2}{n\pi}[\int_0^\pi\frac{x^2}{4}d\sin nx-\int_0^\pi\frac{\pi x}{2}\cdot d\sin nx]$$

$$=\frac{2}{n\pi}\{[\frac{x^2}{4}\sin nx\Big|_0^\pi-\int_0^\pi\sin nx\cdot d\frac{x^2}{4}]-(\frac{\pi x}{2}\cdot\sin nx\Big|_0^\pi-\int_0^\pi\sin nx\,d\frac{\pi x}{2})\}$$

$$=\frac{2}{n\pi}[-\frac{1}{2}\int_0^\pi x\cdot\sin nx\,dx+\frac{\pi}{2}\int_0^\pi\sin nx\,dx]$$

$$=\frac{1}{n^2\pi}[\int_0^\pi x\,d\cos nx-\pi\cos nx\Big|_0^\pi]=\frac{1}{n^2\pi}[x\cos nx\Big|_0^\pi-\int_0^\pi\cos nx\,dx-\pi(-1)^n+\pi]$$

$$=\frac{1}{n^2\pi}[\pi\cdot(-1)^n-\pi(-1)^n+\pi]=\frac{1}{n^2}$$

$$f(x)\sim-\frac{\pi}{6}+\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}=-\frac{\pi}{6}+\cos x+\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}+\cdots\cdots=\frac{x^2}{4}-\frac{\pi x}{2},\ 0\le x\le\pi.$$

解3. 对 $f(x)$ 进行奇延拓。 $a_n=0$

$$b_n=\frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx=\frac{2}{\pi}\int_0^\pi(\frac{x^2}{4}-\frac{\pi x}{2})\cdot\sin nx\,dx$$

$$=\frac{2}{n\pi}[-\int_0^\pi\frac{x^2}{4}d\cos nx+\frac{\pi}{2}\int_0^\pi x\,d\cos nx]$$

$$=\frac{2}{n\pi}\{-\frac{1}{4}[x^2\cos nx\Big|_0^\pi-\int_0^\pi 2x\cos nx\,dx]+\frac{\pi}{2}[x\cdot\cos nx\Big|_0^\pi-\int_0^\pi\cos nx\,dx]\}$$

$$=\frac{2}{n\pi}\{-\frac{\pi^2}{4}\cdot(-1)^n+\frac{2}{4n}\int_0^\pi x\,d\sin nx+\frac{\pi^2}{2}(-1)^n\}$$

$$=\frac{2}{n\pi}\{\frac{\pi^2}{4}\cdot(-1)^n+\frac{1}{2n}[x\sin nx\Big|_0^\pi-\int_0^\pi\sin nx\,dx]\}$$

$$=\frac{2}{n\pi}\{\frac{\pi^2}{4}\cdot(-1)^n+\frac{1}{2n^2}\cos nx\Big|_0^\pi\}=\frac{\pi}{2n}(-1)^n+\frac{(-1)^n-1}{n^3\pi}.$$

$$f(x)\sim\sum_{n=1}^{\infty}[\frac{\pi(-1)^n}{2n}+\frac{(-1)^n-1}{n^3\pi}]\cdot\sin nx=\frac{x^2}{4}-\frac{\pi x}{2}\quad,\quad 0\le x\le\pi.$$

P.340.4. 求函数 $f(x)=\begin{cases}\sin\frac{\pi x}{l}, & 0\le x<\frac{l}{2}\\ 0 & \frac{l}{2}\le x<l.\end{cases}$ 的付氏正弦级数，并写出其和函数。 2001 $\frac{2}{6}$—7

解：对 $f(x)$ 进行奇延拓，则 $a_n=0$

$$b_1=\frac{2}{l}\int_0^{\frac{l}{2}}\sin\frac{\pi x}{l}\cdot\sin\frac{\pi x}{l}\,dx=\frac{2}{l}\int_0^{\frac{l}{2}}\sin^2\frac{\pi x}{l}\,dx=\frac{1}{l}\int_0^{\frac{l}{2}}(1-\cos\frac{2\pi x}{l})\,dx=\frac{1}{2}$$

$$n\ne 1,\ b_n=\frac{2}{l}\int_0^{\frac{l}{2}}f(x)\cdot\sin\frac{n\pi x}{l}\,dx=\frac{2}{l}\int_0^{\frac{l}{2}}\sin\frac{\pi x}{l}\cdot\sin\frac{n\pi x}{l}\,dx$$

$$=-\frac{1}{l}\int_0^{\frac{l}{2}}[\cos\frac{(n+1)\pi}{l}x-\cos\frac{(n-1)\pi}{l}x]\,dx$$

$$=(-1)\cdot[\frac{1}{(n+1)\pi}\int_0^{\frac{l}{2}}\cos\frac{(n+1)\pi}{l}x\,d\frac{(n+1)\pi}{l}-\frac{1}{(n-1)\pi}\int_0^{\frac{l}{2}}\cos\frac{(n-1)\pi}{l}x\,d\frac{(n-1)\pi}{l}x]$$

$$=-\frac{1}{\pi}[\frac{1}{n+1}\sin\frac{(n+1)\pi}{2}-\frac{1}{n-1}\sin\frac{(n-1)\pi}{2}]$$

$$=-\frac{1}{\pi}(\frac{1}{n+1}\cos\frac{n\pi}{2}+\frac{1}{n-1}\cos\frac{n\pi}{2})=-\frac{1}{\pi}\cdot\frac{2n}{n^2-1}\cos\frac{n\pi}{2}=-\frac{2}{\pi}\cdot\frac{n}{n^2-1}\cos\frac{n\pi}{2}$$

$$=\begin{cases}0, & n=2k-1\\ -\frac{1}{\pi}\cdot\frac{4k}{4k^2-1}(-1)^k & n=2k\end{cases}$$

$$f(x)\sim\frac{1}{2}\sin\frac{\pi x}{l}-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k\cdot k}{4k^2-1}\sin\frac{2k\pi x}{l}=\begin{cases}\sin\frac{\pi x}{l} & 0<x<\frac{l}{2}\\ 0 & \frac{l}{2}<x<l\\ \frac{1}{2} & x=\frac{l}{2}\\ 0 & x=l\end{cases}$$

P.340.5 将 $f(x)=3,\ (0\le x\le\pi)$ 展开成正弦级数.

并推出：$\frac{\pi}{4}=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}$.

解：对 $f(x)$ 进行奇延拓。$a_n=0$, $b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx\,dx=\frac{2}{\pi}\int_0^{\pi}3\cdot\sin nx\,dx$

$$=\frac{6}{\pi}\int_0^{\pi}\sin nx\,dx=-\frac{6}{n\pi}(\cos nx)\Big|_0^{\pi}=\frac{6}{n\pi}[1-(-1)^n]$$

$$=\begin{cases}0, & n=2k\\ \frac{12}{n\pi}, & n=2k-1.\end{cases}$$

$$f(x)\sim\frac{12}{\pi}(\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\cdots\cdots)=3,\qquad 0<x<\pi$$

即 $3=\frac{12}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\cdot\sin(2k-1)x,\qquad 0<x<\pi,$

$$\frac{\pi}{4}=\sum_{k=1}^{\infty}\frac{1}{2k-1}\cdot\sin(2k-1)x,$$

取 $x=\frac{\pi}{2}$，则 $\frac{\pi}{4}=\sum^{\infty}\frac{1}{2k-1}\sin(2k-1)\cdot\frac{\pi}{2}=\sum_{k=1}^{\infty}\frac{1}{2k-1}[\sin k\pi\cos\frac{\pi}{2}-\cos k\pi\cdot\sin\frac{\pi}{2}]=\sum_{k=1}^{\infty}\frac{1}{2k-1}(-1)^{k-1}$,

中山大學 本科生考试草稿纸 2011/6-8

P.340.6 求函数 $f(x)=\frac{1}{2}-\frac{\pi}{4}\sin x$, $(0\le x\le\pi)$ 的付氏余弦级数.

解：对 $f(x)$ 进行偶延拓。$b_n=0$

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx=\frac{2}{\pi}\int_0^{\pi}(\frac{1}{2}-\frac{\pi}{4}\sin x)dx=\frac{2}{\pi}(\frac{\pi}{2}+\frac{\pi}{4}\cos x|_0^{\pi})=\frac{2}{\pi}(\frac{\pi}{2}-\frac{\pi}{2})=0$$

$$a_n=\frac{2}{\pi}\int_{-\pi}^{\pi}f(x)\cdot\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}(\frac{1}{2}-\frac{\pi}{4}\sin x)\cdot\cos nx\,dx$$

$$=\frac{2}{\pi}(0-\frac{\pi}{4}\int_0^{\pi}\sin x\cdot\cos nx\,dx)=-\frac{1}{2}\int_0^{\pi}\sin x\cdot\cos nx\,dx$$

$$=-\frac{1}{4}\int_0^{\pi}[\sin(n+1)x-\sin(n-1)x]dx=-\frac{1}{4}[-\frac{\cos(n+1)x}{(n+1)}\Big|_0^{\pi}+\frac{\cos(n-1)x}{n-1}\Big|_0^{\pi}]$$

$$=-\frac{1}{4}[-\frac{(-1)^{n+1}-1}{n+1}+\frac{(-1)^{n-1}-1}{n-1}]=-\frac{1}{4}(\frac{-(-1)^{n+1}+1}{n+1}+\frac{(-1)^{n}-1}{n-1})=-\frac{1}{4}\cdot\frac{(-2)\cdot[1-(-1)^{n+1}]}{n^2-1}$$

$$=\frac{1}{2}\cdot\frac{1-(-1)^{n+1}}{(n+1)(n-1)}=\begin{cases}0 & n=2k\\ \frac{1}{(2k-1)(2k+1)} & n=2k-1\end{cases}$$

$$f(x)\sim\frac{1}{1\cdot3}\cos2x+\frac{1}{3\cdot5}\cos4x+\frac{1}{5\cdot7}\cos6x+\cdots\cdots\frac{1}{(2k-1)(2k+1)}\cos(2kx)+\cdots\cdots$$

$$=\frac{1}{2}-\frac{\pi}{4}\sin x,\quad 0\le x\le\pi.$$

P.340.7 求函数 $f(x)=\frac{\pi-x}{2}$, $(0<x<2\pi)$ 的傅氏正弦级数。

解：对 $f(x)$ 进行奇延拓。$a_n=0$

$$b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cdot\sin nx\,dx=\frac{2}{\pi}\int_0^{\pi}\frac{\pi-x}{2}\cdot\sin nx\,dx=\frac{1}{\pi}[\int_0^{\pi}\pi\sin nx\,dx-\int_0^{\pi}x\cdot\sin nx\,dx]$$

$$=\frac{1}{\pi}\{\pi\cdot(-1)\cdot\frac{1}{n}\cos nx|_0^{\pi}+\frac{1}{n}\int_0^{\pi}x\,d\cos nx\}$$

$$=\frac{1}{\pi}\{-\frac{\pi}{n}[(-1)^n-1]+\frac{1}{n}[x\cdot\cos nx|_0^{\pi}-\int_0^{\pi}\cos nx\,dx]\}$$

$$=\frac{\pi}{n\pi}[1-(-1)^n+(-1)^n]=\frac{1}{n}.$$

$$f(x)\sim\sum_{n=1}^{\infty}\frac{1}{n}\sin nx=\frac{\pi-x}{2},\ (0<x<2\pi)$$

P.340.8 求下列级数的值。　　　　　　　　　　　　　　　　don $\frac{2}{6}$—9.

(1) $\sum_{n=1}^{\infty}\frac{1}{n}\sin n=\frac{\pi-1}{2}$.　(由7题结果)

(2) $\frac{1}{2^2}+\frac{1}{4^2}+\cdots\cdots+\frac{1}{(2n)^2}+\cdots\cdots$

由第2题结果: $\frac{x^2}{4}-\frac{\pi x}{2}=-\frac{\pi}{6}+\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$ ，$0\le x\le\pi$

从而, 当 $x=0$ 时. $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi}{6}$

$$\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}=\sum_{n=1}^{\infty}\frac{1}{(2n)^2}=\frac{1}{4}\cdot\frac{\pi}{6}=\frac{\pi}{24}.$$

P.340.9 设 $f(x)$ 以 T 为周期, 它在一个周期内的表达式为: $f(t)=\begin{cases}0, & -\frac{T}{2}\le t<0\\ A\sin\omega t, & 0\le t\le\frac{T}{2}\end{cases}$

其中 $\omega=\frac{2\pi}{T}$ ，$A>0$. 求 $f(t)$ 的傅氏展开式。

解: $2l=T$, $l=\frac{T}{2}$, $a_0=\frac{1}{l}\int_{-l}^{l}f(x)dx=\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(x)dx=\frac{2}{T}\int_0^{\frac{T}{2}}A\sin\omega t\,dt=-\frac{2A}{\omega T}[\cos\omega t]_0^{\frac{T}{2}}$

$$=-\frac{A}{\pi}[\cos\pi-1]=\frac{2A}{\pi}.$$

$$a_1=\frac{2A}{T}\int_0^{\frac{T}{2}}\sin\omega t\cdot\cos\omega t\,dt=0$$

$n\ne1$ 时, $a_n=\frac{2A}{T}\int_0^{\frac{T}{2}}\sin\omega t\cdot\cos n\omega t\,dt=\frac{A}{T}\int_0^{\frac{T}{2}}[\sin(n+1)x-\sin(n-1)x]\,dx$

$$=\frac{A}{T}\left[\frac{-\cos(n+1)\omega x}{(n+1)\omega}\Big|_0^{\frac{T}{2}}+\frac{\cos(n-1)\omega x}{(n-1)\omega}\Big|_0^{\frac{T}{2}}\right]=\frac{A}{T\cdot\omega}\left[\frac{-\cos(n+1)\pi+1}{(n+1)}+\frac{\cos(n-1)\pi-1}{(n-1)}\right]$$

$$=\frac{A}{2\pi}\left[\frac{-(-1)^{n+1}+1}{n+1}+\frac{(-1)^{n-1}-1}{n-1}\right]=-\frac{A}{\pi}\cdot\frac{(-1)^n+1}{n^2-1}=\begin{cases}0 & n=2k+1\\ \frac{-2A}{(4k^2-1)\cdot\pi} & n=2k\end{cases}$$

$$b_1=\frac{2A}{T}\int_0^{\frac{T}{2}}\sin^2\omega t\,dt=\frac{2A}{T\omega}\int_0^{\frac{T}{2}}\frac{1-\cos2\omega t}{2}d(\omega t)=\frac{2A}{T\omega}\cdot\frac{1}{2}\cdot\omega\cdot\frac{T}{2}=\frac{A}{2}$$

$n\ne1$, $b_n=\frac{2}{T}\int_0^{\frac{T}{2}}A\sin\omega t\cdot\sin n\omega t\,dt=\frac{1}{T}\int_0^{T}[\cos(n+1)\omega t-\cos(n-1)\omega t]dt=0$

从而 $f(t)\sim\frac{A}{\pi}+\frac{A}{2}\sin\omega t-\frac{2A}{\pi}\sum_{k=1}^{\infty}\frac{\cos2k\omega t}{4k^2-1}=f(t)$　$(-\infty<t<+\infty)$

中山大學 考试草稿纸 2011 2/6—10

P.340.10 设函数 $f(x)$ $(-\pi\le x\le\pi)$ 的傅氏系数为：a_0, a_n, b_n $(n=1,2,\cdots)$

求 $g(x)=f(-x)$，$(-\pi\le x\le\pi)$ 的傅氏系数 A_0, A_n, B_n $(n=1,2,3,\cdots\cdots)$.

解：$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$，$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx dx$，$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx dx$.

$$A_0=\frac{1}{\pi}\int_{-\pi}^{\pi}g(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}f(-x)dx=\frac{1}{\pi}\int_{\pi}^{-\pi}f(t)(-dt)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)dt=a_0$$

$$A_n=\frac{1}{\pi}\int_{-\pi}^{\pi}g(x)\cdot\cos nx dx=\frac{1}{\pi}\int_{-\pi}^{\pi}f(-x)\cdot\cos nx dx=\frac{1}{\pi}\int_{\pi}^{-\pi}f(t)\cos nt(-dt)=a_n$$

$$B_n=\frac{1}{\pi}\int_{-\pi}^{\pi}g(x)\sin nx dx=\frac{1}{\pi}\int_{-\pi}^{\pi}f(-x)\sin nx dx=\frac{1}{\pi}\int_{\pi}^{x}f(t)(-\sin nt)\cdot(-dt)=-b_n.$$

P.340.11. 设 $f(x)$ 是以 2π 为周期的连续函数，a_0, a_n, b_n 为其傅氏系数，

求 $F(x)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cdot f(t+x)dt$ 的傅氏系数 A_0, A_n, B_n $(n=1,2,\cdots\cdots)$

解：
$$A_0=\frac{1}{\pi}\int_{-\pi}^{\pi}F(x)dx=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\left[\int_{-\pi}^{\pi}f(t)\cdot f(t+x)dt\right]dx=\frac{1}{\pi^2}\int_{-\pi}^{\pi}f(t)dt\int_{-\pi}^{\pi}f(t+x)dx$$

$$=\frac{1}{\pi^2}\int_{-\pi}^{\pi}f(t)dt\int_{-\pi}^{\pi}f(t+x)d(t+x)=\frac{1}{\pi^2}\int_{-\pi}^{\pi}f(t)dt\int_{t-\pi}^{t+\pi}f(u)du$$

$$=\frac{1}{\pi^2}\int_{-\pi}^{\pi}f(t)dt\cdot\int_{-\pi}^{\pi}f(u)du=\left(\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)dt\right)\left(\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)dt\right)=a_0^2$$

$$A_n=\frac{1}{\pi}\int_{-\pi}^{\pi}F(x)\cdot\cos nx dx=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}f(t)\cdot f(t+x)dt\right)\cos nx dx$$

$$=\frac{1}{\pi^2}\int_{-\pi}^{\pi}f(t)dt\int_{-\pi}^{\pi}f(t+x)\cdot\cos nx dx \quad \text{，令 } t+x=u，\text{则 } x=u-t$$

$$=\frac{1}{\pi^2}\cdot\int_{-\pi}^{\pi}f(t)dt\int_{t-\pi}^{t+\pi}f(u)\cdot\cos n(u-t)du$$

$$=\frac{1}{\pi^2}\cdot\int_{-\pi}^{\pi}f(t)dt\int_{t-\pi}^{t+\pi}f(u)\cdot[\cos nu\cos nt+\sin nu\sin nt]du$$

$$=\left(\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos nt\right)\cdot\left(\frac{1}{\pi}\int_{-\pi}^{\pi}f(u)\cos nu du\right)+\left(\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin nt dt\right)\cdot\left(\frac{1}{\pi}\int_{-\pi}^{\pi}f(u)\sin nu du\right)$$

$$=a_n^2+b_n^2$$

同理可证：$B_n=0$