UNIT-II

VECTOR INTEGRATION

Topic Learning Objectives:

Upon Completion of this unit, students will be able to:

- Understand the fundamentals of the integration of vector point function.
- Solve line, surface and volume integrals.
- Apply Green's Theorem, Stokes' Theorem and Gauss' Theorem in solving engineering problems.
- Estimate and apply the concepts of solenoidal and irrotational fields to calculate integrals of vector functions.

Line Integral: Any integral which is to be evaluated along a curve is called line integral.

If $\vec{F}(x, y, z)$ is a vector point function and C is any curve then $\int_C \vec{F}.d\vec{r}$ is called the vector line integral. (Tangential line integral or line integral)

NOTE:

1. C is a called path of integration.
2. If $\vec{F} = f_1\hat{\imath} + f_2\hat{\jmath} + f_3\hat{k}$ then $\int_C \vec{F}.d\vec{r} = \int_C f_1dx + f_2dy + f_3dz$.
3. When C is a simple closed curve, line integral is denoted by $\oint_C \vec{F}.d\vec{r}$ (means the line integral of $\vec{F}$ taken once around C in the anticlock wise direction).
4. If $\vec{F}$ represents force acting on a particle then the line integral $\int_C \vec{F}.d\vec{r}$ represents work done by a force $\overrightarrow{F}$.
5. If $\vec{F}$ represents the velocity of a fluid then $\int_C \vec{F}.d\vec{r}$ represents circulation of $\vec{F}$ around C.
6. Condition for $\vec{F}$ to be conservative is $\nabla \times \vec{\mathrm{F}} = 0$.
7. If $curl\,\vec{F} = 0$ then $\int_C \vec{F}.d\vec{r}$ is independent of path.

Problem 1. If $\vec{F} = (5xy - 6x^2)\hat{\imath} + (2y - 4x)\hat{\jmath}$. Evaluate $\int_C \vec{F}.d\vec{r}$ along $y = x^3$ in XY −plane from $(1, 1)$ to $(2, 8)$.

Solution: Given

$\vec{F} = (5xy - 6x^2)\hat{\imath} - (2y - 4x)\hat{\jmath}$

$\vec{r} = x\,\hat{\imath} + y\hat{\jmath} \Rightarrow \overrightarrow{dr} = dx\,\hat{\imath} + dy\hat{\jmath}$

$y = x^3 \Rightarrow dy = 3x^2dx$ and x: 1 to 2

Consider

$\vec{F}.d\vec{r} = (5x\,x^3 - 6x^2)dx + (2x^3 - 4x)3x^2dx$

$$\int_C \vec{F}.d\vec{r} = \int_1^2 (5x^4 - 6x^2 + 6x^5 - 12x^3)dx = [x^5 - 2x^3 + x^6 - 3x^4]_1^2 = 35.$$

Problem 2. Find the work done in moving a particle in the force field $\vec{F} = 3x^2\hat{\imath} + (2xz - y)\hat{\jmath} + z\hat{k}$ along

a) The straight line $(0,0,0)$ to $(2,1,3)$.
b) The curve $x = 2t^2, y = t, z = 4t^2 - t$ from $t = 0$ to $t = 1$.
c) The curve defined by $x^2 = 4y,\ 3x^3 = 8z$ from $x = 0$ to 2.

Solution: Work done $= \int_C \vec{F}.d\vec{r} = \int_C 3x^2dx + (2xz - y)dy + zdz.$ ---------- (i)

a) C is a straight line joining $(0,0,0)$ and $(2,1,3)$.

The equation of the line is given by $\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$

We have $x = 2t \Rightarrow dx = 2dt,\quad y = t \Rightarrow dy = dt,\quad z = 3t \Rightarrow dz = 3dt$

and $t = 0$ to 1 $[\because t = y,\ y = 0\ to\ 1]$

then equation (i)

$$\Rightarrow \int_C \vec{F}.d\vec{r} = \int_0^1 \{(3(2t)^2)(2dt) + (2(2t)(3t) - t)dt + (3t)3dt\}$$

$$= \int_0^1 (36t^2 + 8t)\,dt = \left[36\frac{t^3}{3} + 8\frac{t^2}{2}\right]\Bigg|_0^1 = 16.$$

b) Given curve $x = 2t^2 \Rightarrow dx = 4\,tdt,\quad y = t \Rightarrow dy = dt,\ \ z = 4t^2 - t \Rightarrow dz = (8t - 1)dt$

then (i) becomes

$$\int_C \vec{F}.d\vec{r} = \int_0^1 \{(3(2t^2)^2)(4tdt) + (2(2t^2)(4t^2 - t) - t)dt + (4t^2 - t)(8t - 1)dt\}$$

$$= \int_0^1 (48t^5 + 16t^4 + 28t^3 - 12t^2)\,dt = \frac{71}{5}$$

c) Given curve $x^2 = 4y \Rightarrow y = \frac{x^2}{4} \Rightarrow dy = \frac{x}{2}dx,\ 3x^3 = 8z \Rightarrow z = \frac{3x^3}{8} \Rightarrow dz = \frac{9}{8}x^2dx$ and x: 0 to 2 then (i) becomes

$$\int_C \vec{F}.d\vec{r} = \int_0^2 \left\{3x^2dx + \left(2x\left(\frac{3}{8}x^3\right) - \frac{x^2}{4}\right)\frac{x}{2}dx + \frac{3}{8}x^3\frac{9}{8}x^2\,dx\right\}$$

$$= \int_0^2 (3x^2 + \frac{3}{8}x^5 - \frac{x^3}{8} + \frac{27}{64}x^5)\,dx = 16.$$

Exercise:

1. If $\overrightarrow{F} = x^2\hat{\imath} + xy\hat{\jmath}$, evaluate $\int_C \vec{F}.d\vec{r}$ from $(0,0)$ to $(1,1)$

(a) along the line $y = x$ Ans: $\frac{2}{3}$

(b) along the parabola $y = \sqrt{x}$ Ans: $\frac{7}{12}$.

2. Find the total work done by the force represented by $\vec{F} = 3xy\hat{\imath} - y\hat{\jmath} + 2zx\hat{k}$ in moving a particle round the circle $x^2 + y^2 = 4, x = 2\cos\theta\,, y = 2\sin\theta\ \&\ z = 0, 0 \le \theta \le 2\pi$.

3. Find the circulation of $\vec{F}$ around the curve C, where C is the rectangle whose vertices are given by $(0,0), (1,0), \left(1,\frac{\pi}{2}\right)$ & $\left(0,\frac{\pi}{2}\right)$ and $\vec{F} = e^x \sin y\,\hat{\imath} + e^x \cos y\,\hat{\jmath}$.

4. If $\vec{F} = (2x + y^2)\hat{\imath} + (3y - 4x)\hat{\jmath}$ evaluate $\oint_C \vec{F}\cdot\overrightarrow{dr}$ around a triangle ABC in the xy −plane with $A(0,0)$ $B(2,0)$ and $C(2,1)$,

(a) In the counter clockwise direction. Ans: $-\frac{14}{3}$

(b) What is the value in the opposite direction? Ans: $\frac{14}{3}$

5. Evaluate the line integral $\int_C (x^2 + xy)dx + (x^2 + y^2)dy$, where $C: square: x = \pm 1,\ y = \pm 1$. Ans:0

NOTE: If circulation is "0" then $\int \vec{F}.d\vec{r}$ is irrotational.

GREEN'S THEOREM

Green's theorem in the plane transforms a line integral to a double integral in a plane.

Statement:

If R is a closed region in XY-plane, bounded by a simply closed curve C and if

$P(x,y)$ and $Q(x,y)$, $\frac{\partial}{\partial x}Q(x,y)$, $\frac{\partial}{\partial y}P(x,y)$ be continuous functions at every point in R, then

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

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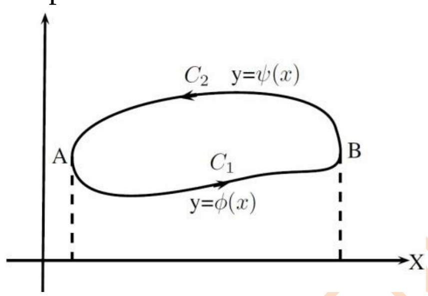
Proof: Suppose that C is a simply closed curve with the property that any line parallel to either axis meets the curve in at most two points.



Consider

$$\iint_R \left(-\frac{\partial P}{\partial y}\right) dx\,dy = \int_{x=a}^{b} \int_{y=\phi(x)}^{\psi(x)} \left(-\frac{\partial P}{\partial y}\right) dy\,dx = \int_{x=a}^{b} -P(x,y)\Big|_{\phi(x)}^{\psi(x)} dx$$

$$= \int_{x=a}^{b} \left[-P\big(x,\psi(x)\big) + P\big(x,\phi(x)\big)\right] dx$$

$$= \int_{b}^{a} P\big(x,\psi(x)\big)\,dx + \int_{a}^{b} P\big(x,\phi(x)\big)\,dx$$

$$\Rightarrow \int_{c_1} P(x,y)dx + \int_{c_2} P(x,y)dx = \int_{c} P(x,y)dx$$

Similarly,

$$\iint_R \left(\frac{\partial Q}{\partial x}\right) dx\,dy = \int_c Q(x,y)dx$$

$$\therefore \int_c P(x,y)dx + Q(x,y)dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

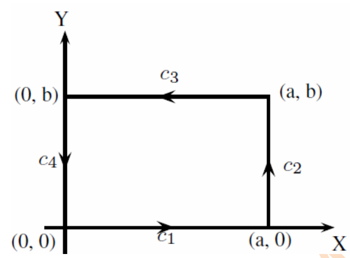
Problem 1. Verify Green's theorem in the plane for $\oint_C \{(x^2+y)dx - xy^2dy\}$ taken around the boundary of the rectangle whose vertices are $(0,\ 0), (a,0), (a,b)$ and $(0,b)$.

Solution: We have to verify

$$\oint_C P(x,y)dx + Q(x,y)dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

Consider

$$\oint_C Pdx + Qdy = \int_{c_1} Pdx + Qdy + \int_{c_2} Pdx + Qdy + \int_{c_3} Pdx + Qdy + \int_{c_4} Pdx + Qdy$$



$$\oint_C \{(x^2+y)dx - xy^2dy\} = \oint_C Pdx + Qdy$$

Along $C_1: y = 0 \Rightarrow dy = 0 \; and \; x: 0 \; to \; a$

$$\int_{c_1} \{(x^2+y)dx - xy^2dy\} = \int_0^a x^2dx = \left.\frac{x^3}{3}\right|_0^a = \frac{a^3}{3}.$$

Along $C_2: x = a \Rightarrow dx = 0 \; and \; y: 0 \; to \; b$

$$\int_{c_2} \{(x^2+y)dx - xy^2dy\} = \int_0^b -ay^2dy = -\left.\frac{ay^3}{3}\right|_0^b = -\frac{ab^3}{3}.$$

Along $C_3: y = b \Rightarrow dy = 0 \; and \; x: a \; to \; 0$

$$\int_{c_3} \{(x^2+y)dx - xy^2dy\} = \int_a^0 (x^2+b)\, dx = \left.\frac{x^3}{3} + bx\right|_a^0 = -\frac{a^3}{3} - ba.$$

Along $C_4: x = 0 \Rightarrow dx = 0 \; and \; y: b \; to \; 0$

$$\int_{c_3} \{(x^2+y)dx - xy^2dy\} = \int_b^0 0\, dy = 0$$

$$\therefore \oint_C \{(x^2+y)dx - xy^2dy\} = \oint_C Pdx + Qdy$$

$$= \frac{a^3}{3} - \frac{ab^3}{3} - \frac{a^3}{3} - ba + 0 = -ab\left(1+\frac{b^2}{3}\right) \qquad \ldots\ldots(1)$$

Next consider,

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \int_{x=0}^{a}\int_{y=0}^{b} (-y^2 - 1)\, dy\, dx = -\int_{x=0}^{a}\left[\frac{y^3}{3}+y\right]_0^b dx = -\int_0^a \left(\frac{b^2}{3}+b\right) dx$$

$$= -ab\left(1+\frac{b^2}{3}\right) \qquad \ldots\ldots(2)$$

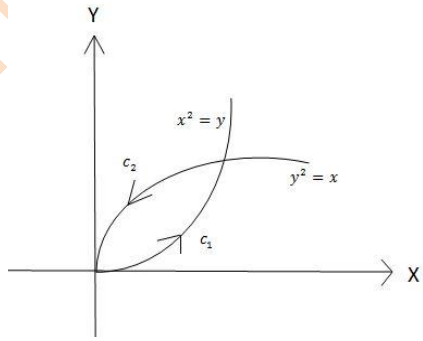
From (1) and (2), Green's theorem is verified.

Problem 2. Verify Green's theorem in the plane for $\int_c \{(x-y)dx + (x+y)dy\}$ taken around the boundary of the finite area in the positive quadrant included between $y = x^2$ & $x = y^2$.

Solution: We have to verify

$$\int_c P(x,y)dx + Q(x,y)dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

$\int_C P\,dx + Q\,dy = \int_{c_1} P\,dx + Q\,dy + \int_{c_2} P\,dx + Q\,dy = \int_{C_1}\{(x-y)dx + (x+y)dy\} + \int_{C_2}\{(x-y)dx + (x+y)dy\}$



Along C_1: $y = x^2 \Rightarrow dy = 2xdx$ and $x: 0\ to\ 1$

$$\int_{c_1} \{(x-y)dx + (x+y)dy\} = \int_0^1 \{(x-x^2)dx + (x+x^2)2xdx\}$$

$$= \int_0^1 (2x^3 + x^2 + x)dx = \left[2\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 = \frac{4}{3}.$$

Along C_2: x= $y^2 \Rightarrow dx = 2ydy$ and $y: 1\ to\ 0$

$$\int_{C_1} \{(x-y)dx + (x+y)dy\} = \int_1^0 \{(y^2 - y)2ydy + (y^2 + y)dy\}$$

$$= \int_1^0 (2y^3 - y^2 + y)dy = \left[2\frac{y^4}{4} - \frac{y^3}{3} + \frac{y^2}{2}\right]_1^0 = -\frac{2}{3}.$$

$$\therefore \int_C P\,dx + Q\,dy = \frac{4}{3} - \frac{2}{3} = \frac{2}{3} \qquad \ldots\ldots\ldots\ldots(1)$$

Now $P = x - y \Rightarrow \frac{\partial P}{\partial y} = -1$ and $Q = x + y \Rightarrow \frac{\partial Q}{\partial x} = 1$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{y=0}^{1} \int_{x=y^2}^{\sqrt{y}} (1+1)\,dx\,dy = \int_{y=0}^{1} \left(\int_{x=y^2}^{\sqrt{y}} 2\,dx\right) dy$$

$$= \int_{y=0}^{1} (2x)_{y^2}^{\sqrt{y}} dy = 2 \int_{y=0}^{1} \left(\sqrt{y} - y^2\right) dy$$

$$= 2\left(\frac{y^{\frac{3}{2}}}{\frac{3}{2}} - \frac{y^3}{3}\right)_0^1 = \frac{2}{3} \qquad \ldots\ldots\ldots\ldots(2)$$

From (1) and (2), Green's theorem is verified.

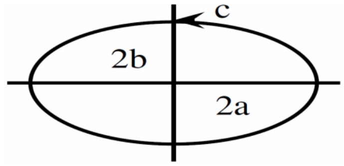
Problem 3. Show that area enclosed by a simple closed curve C is given by $\frac{1}{2}\oint\{xdy - ydx\}$. Using this, find the area bounded by the ellipse with axes $2a$ and $2b$.

Solution: we have

$$\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$P = -y \Rightarrow \frac{\partial P}{\partial y} = -1$ and $Q = x \Rightarrow \frac{\partial Q}{\partial x} = 1$

$$\frac{1}{2}\oint_C xdy - ydx = \frac{1}{2}\iint_R (1+1)dx\,dy$$



To find the area of the ellipse, the equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

In parametric form

$$x = a\cos\theta\,, \qquad y = b\sin\theta.$$

$$\Rightarrow dx = -a\sin\theta, \qquad dy = b\cos\theta$$

$$Area = \frac{1}{2}\oint_C xdy - ydx = \frac{1}{2}\oint_C \{a\cos\theta\, b\cos\theta - b\sin\theta\,(-a\sin\theta)\}\,d\theta$$

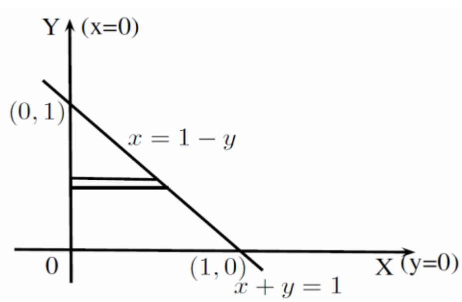
$$= \frac{1}{2}ab\int_0^{2\pi}(\cos^2\theta+\sin^2\theta)d\theta = \frac{1}{2}ab[2\pi] = \pi ab.$$

Problem 4. Using Green's theorem in the plane, evaluate $\int_C \{(2x^2-y^2)dx+(x^2+y^2)dy\}$, C is the boundary of the region bounded by $x=0, y=0, x+y=1$.

Solution: we have

$$\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$

$P = 2x^2 - y^2 \Rightarrow \frac{\partial P}{\partial y} = -2y$ and $Q = x^2+y^2 \Rightarrow \frac{\partial Q}{\partial x} = 2x$



$$\iint_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy = \int_{y=0}^{1}\int_{x=0}^{1-y}(2x+2y)dxdy = 2\int_0^1\left[\frac{x^2}{2}+xy\right]_0^{1-y}dy$$

$$= 2\int_0^1 \left(\frac{(1-y)^2}{2} + (1-y)y\right) dy$$

$$= 2\int_0^1 \left(\frac{1}{2}(1+y^2-2y\,) + (y-y^2)\right) dy$$

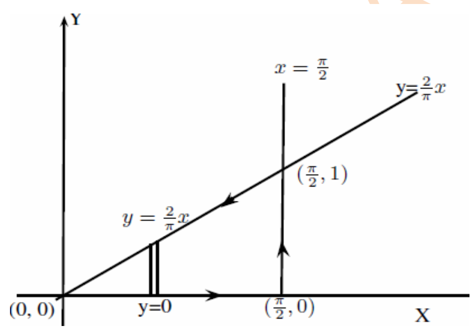
$$= 2\left[\frac{1}{2}\left(y+\frac{y^3}{3}-2\frac{y^2}{2}\right)+\left(\frac{y^2}{2}-\frac{y^3}{3}\right)\right]_0^1 = \frac{2}{3}.$$

Problem 5. Apply Green's theorem to evaluate $\int_c (y-\sin x)dx + \cos x\, dy$, where C is the triangle enclosed by the lines $y=0, x=\frac{\pi}{2}, and\ y=\frac{2}{\pi}x.$

Solution: we have

$$\oint_C \; Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

$P = y - \sin x \Rightarrow \frac{\partial P}{\partial y} = 1$ and $Q = \cos x \Rightarrow \frac{\partial Q}{\partial x} = -\sin x$



$$\int_C (y-\sin x)dx + \cos x\, dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

$$= \int_{x=0}^{\frac{\pi}{2}} \int_{y=0}^{\frac{2x}{\pi}} (-\sin x - 1) dy\, dx = -\int_0^{\frac{\pi}{2}} (\sin x + 1)[y]_0^{\frac{2x}{\pi}}\, dx$$

$$= -\frac{2}{\pi}\int_0^{\frac{\pi}{2}} (x\ \sin x + x)\, dx$$

$$= -\frac{2}{\pi}\left[-x\cos x + \sin x + \frac{x^2}{2}\right]_0^{\frac{\pi}{2}} = -\left(\frac{2}{\pi} + \frac{\pi}{4}\right).$$

Exercise:

1. Verify Green's theorem for $\int_C (e^{-x}\sin y)dx + (e^{-x}\cos y)dy$, where C is the rectangle, whose vertices are $(0,0), (\pi, 0), \left(\pi, \frac{\pi}{2}\right)$ *and* $\left(0, \frac{\pi}{2}\right)$. Ans: $[2(e^{-\pi} - 1)]$

2. Using Green's theorem, evaluate $\oint_C x^{-1}e^y dx + (e^y \ln x + 2x\,)dy$, where C is the bounded by $y = 2, y = x^4 + 1$. Ans: $\frac{16}{5}$

3. Using Green's theorem, evaluate $\oint_C (x^2 - \cosh y\, dx + (y + sinx)dy)$ where C is the boundary of the rectangle $0 \le x \le \pi,\ 0 \le y \le 1$. Ans: $\pi(\cosh 1 - 1)$

Surface Integral

Any integral which is to be evaluated over a surface is called surface integral.

<u>Physical interpretation:</u> The surface integral of a vector function $\vec{F}$ express the normal flux through a surface.

Note: If $\vec{F}$ represents velocity vector of a fluid, the surface integral represents the rate of flow of fluid through the surface.

1. The surface integral of a vector point function $\vec{F}$ over a surface S is defined as the integral of normal component of $\vec{F}$ taken over the surface S.
2. If $\vec{F}$ represents the velocity of a fluid $\oiint_S \vec{F}.\hat{n}\, ds$ gives the flux across the surface S.
3. If the flux of $\vec{F}$ across every closed surface S in a region R is zero. Then $\vec{F}$ is a solenoidal vector point function in the region R.
4. If $\vec{F}$ represents gravitational force, electric force or magnetic force in each case $\iint_S \vec{F}.\hat{n}\, ds$ gives corresponding flux.

Working Rule:

1. For the given surface $\emptyset$, find $\hat{n} = \frac{\nabla\emptyset}{|\nabla\emptyset|}$, $\hat{n}$ is outward unit normal vector to the surface.
2. Find $\vec{F}.\hat{n}$
3. If the projection of S is taken in YZ −plane, then $ds = \frac{dy\,dz}{|\hat{n}.\hat{\imath}|}$, where $\hat{\imath}$ is the unit vector along $x - axis$.
4. If the projection of S is taken in XY −plane, then $ds = \frac{dx\,dy}{|\hat{n}.\hat{k}|}$, where $\hat{k}$ is the unit vector along $z - axis$.
5. If the projection of S is taken in XZ −plane, then $ds = \frac{dx\,dz}{|\hat{n}.\hat{\jmath}|}$, where $\hat{\jmath}$ is the unit vector along $y - axis$.

NOTE: To evaluate any surface integral, it is convenient to evaluate the double integral of its projection on $xy, yz, or\ zx$ plane.

$$\iint_S \vec{F}.\hat{n}\,ds = \iint_S \vec{F}.\overrightarrow{ds} = \iint_R \vec{F}.\hat{n}\,\frac{dx\,dy}{|\hat{n}.\hat{k}|}$$

where R is the projection of S in $XY - plane$.

Problems 1. Evaluate $\iint_S \vec{F}.\hat{n}\,ds$ where $\vec{F} = 18z\hat{\imath} - 12\hat{\jmath} + 3y\hat{k}$ and S is the part of the $2x + 3y + 6z = 12$, located in first octant $(x = 0, y = 0, z = 0)$.

Solution: Given $2x + 3y + 6z = 12 \Rightarrow \frac{x}{6} + \frac{y}{4} + \frac{z}{2} = 1$

Let $\emptyset = 2x + 3y + 6z - 12$

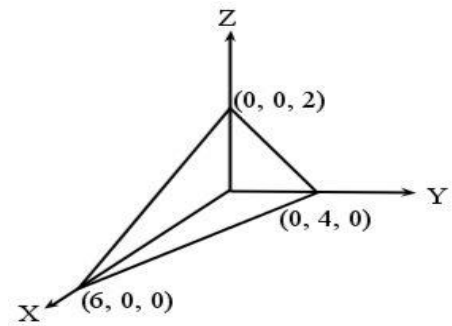
then $\nabla\emptyset = \frac{\partial\emptyset}{\partial x}\hat{\imath} + \frac{\partial\emptyset}{\partial y}\hat{\jmath} + \frac{\partial\emptyset}{\partial z}\hat{k} = 2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}$

$$\hat{n} = \frac{\nabla\emptyset}{|\nabla\emptyset|} = \frac{2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}}{\sqrt{2^2 + 3^2 + 6^2}} = \frac{2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}}{7}$$

$$\vec{F}.\hat{n} = \left(18z\hat{\imath} - 12\hat{\jmath} + 3y\hat{k}\right).\left(\frac{2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}}{7}\right) = \frac{36z - 36 + 18y}{7}$$

Projecting on to any plane (i.e, xy, yz or zx)

Projecting on to plane $xy - plane$



$2x + 3y = 12; \quad x: 0 \; to \; 6 \; ; \quad y: 0 \; to \; \frac{12-2x}{3}$ and $|\hat{n}.\hat{k}| = \frac{6}{7}$

$$\iint_S \vec{F}.\hat{n}\, ds = \iint_R \vec{F}.\hat{n} \frac{dx\, dy}{|\hat{n}.\hat{k}|} = \int_{x=0}^{6} \int_{y=0}^{\frac{12-2x}{3}} \frac{36\mathrm{z} - 36 + 18\mathrm{y}}{7} \frac{dy\, dx}{\frac{6}{7}}$$

$$= \frac{1}{6} \int_{x=0}^{6} \int_{y=0}^{\frac{12-2x}{3}} \left(36 \frac{12 - 2x - 3y}{6} - 36 + 18y\right) dy\, dx$$

$$= \frac{1}{6} \int_{x=0}^{6} \int_{y=0}^{\frac{12-2x}{3}} (36 - 12x)\, dy\, dx = \frac{1}{6} \int_{x=0}^{6} (36\, y - 12\, x\, y)\Big|_0^{\frac{12-2x}{3}} dx$$

$$= 2 \int_{x=0}^{6} (3 - x\,)y\Big|_0^{\frac{12-2x}{3}} dx = 2 \int_0^6 (3 - x\,) \left(\frac{12 - 2x}{3} - 0\right) dx$$

$$= 2 \int_0^6 (3 - x\,) \frac{12 - 2x}{3} dx = 24.$$

Problem 2. Evaluate $\iint_S \vec{F}.\hat{n}\, ds$ where $\vec{F} = y\hat{\imath} + 2x\hat{\jmath} - z\hat{k}$ and S is the surface of the plane $2x + y = 6$ included in the I octant cut by $z = 4$.

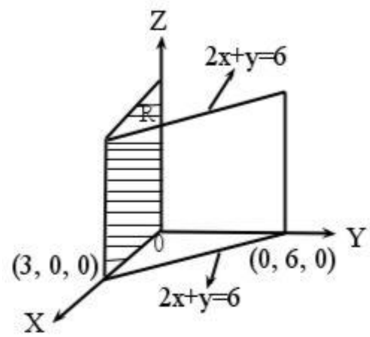
Solution: Let $\emptyset = 2x + y - 6$

then $\nabla\emptyset = \frac{\partial\emptyset}{\partial \mathrm{x}}\hat{\imath} + \frac{\partial\emptyset}{\partial \mathrm{y}}\hat{\jmath} + \frac{\partial\emptyset}{\partial \mathrm{z}}\hat{k} = 2\hat{\imath} + \hat{\jmath}$

$$\hat{n} = \frac{\nabla\emptyset}{|\nabla\emptyset|} = \frac{2\hat{\imath} + \hat{\jmath}}{\sqrt{4+1}} = \frac{2\hat{\imath} + \hat{\jmath}}{\sqrt{5}}$$

$$\vec{F}.\hat{n} = \left(y\hat{\imath} + 2x\hat{\jmath} - z\hat{k}\right).\left(\frac{2\hat{\imath} + \hat{\jmath}}{\sqrt{5}}\right) = \frac{2\mathrm{y} + 2\mathrm{x}}{\sqrt{5}}$$

Projecting on to $xz - plane$, we get
$|\hat{n}.\hat{j}| = \frac{1}{\sqrt{5}}$; $x: 0\ to\ 3$ and $z: 0\ to\ 4$



Now consider

$$\iint_S \vec{F}.\hat{n}\, ds = \iint_R \vec{F}.\hat{n}\, \frac{dx\, dz}{|\hat{n}.\hat{j}|}$$

$$= \int_{x=0}^{3}\int_{z=0}^{4} \frac{2\mathrm{y}+2\mathrm{x}}{\sqrt{5}}\, \frac{dz\, dx}{\frac{1}{\sqrt{5}}} = \int_{x=0}^{3}\int_{z=0}^{4} (2(6-2\mathrm{x})+2\mathrm{x})\, dz\, dx$$

$$= \int_{x=0}^{3}\int_{z=0}^{4} (12-2x)\, dz\, dx = \int_{x=0}^{3} (12-2x)\, dx \quad \int_{z=0}^{4} 1\, dz = 108.$$

Problem 3. Evaluate $\iint_S \vec{F}.\hat{n}\, ds$ where $\vec{F} = z\hat{\imath} + x\hat{\jmath} + 3y^2z\hat{k}$ where S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

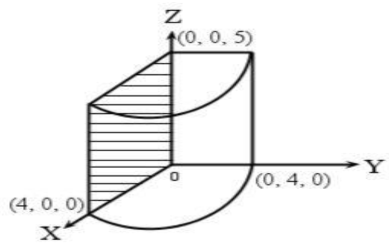
Solution: Given $x^2 + y^2 = 16$ is a right circular cylinder with base circle as $x^2 + y^2 = 16,\ \ z = 0$ and generates parallel to $z - axis$.

Let $\emptyset = x^2 + y^2 - 16$

then $\nabla\emptyset = \frac{\partial\emptyset}{\partial \mathrm{x}}\hat{\imath} + \frac{\partial\emptyset}{\partial \mathrm{y}}\hat{\jmath} + \frac{\partial\emptyset}{\partial \mathrm{z}}\hat{k} = 2x\hat{\imath} + 2y\hat{\jmath}$

$$\hat{n} = \frac{\nabla\emptyset}{|\nabla\emptyset|} = \frac{2x\hat{\imath} + 2y\hat{\jmath}}{\sqrt{4x^2 + 4y^2}} = \frac{2(x\hat{\imath} + y\hat{\jmath})}{2\sqrt{(x^2 + y^2)}} = \frac{x\hat{\imath} + y\hat{\jmath}}{\sqrt{16}} = \frac{x\hat{\imath} + y\hat{\jmath}}{4}$$

$$\vec{F}.\hat{n} = (z\hat{\imath} + x\hat{\jmath} + 3y^2z\hat{k}).\left(\frac{x\hat{\imath} + y\hat{\jmath}}{4}\right) = \frac{\mathrm{xz} + \mathrm{xy}}{4}$$



Projecting on to plane $xz - plane$

$$\iint_S \vec{F}.\hat{n}\, ds = \iint_R \vec{F}.\hat{n}\, \frac{dx\, dz}{|\hat{n}.\hat{j}|} = \int_{z=0}^{5}\int_{x=0}^{4} \frac{\mathrm{xz}+\mathrm{xy}}{4}\, \frac{dx\, dz}{\frac{y}{4}} \quad \because |\hat{n}.\hat{k}| = \frac{y}{4}$$

$$= \int_{z=0}^{5}\int_{x=0}^{4} \left(\frac{\mathrm{xz}}{y} + x\right) dx\, dz \;=\; \int_{z=0}^{5}\int_{x=0}^{4} \left(\frac{\mathrm{xz}}{\sqrt{16-x^2}} + x\right) dx\, dz$$

$$= \int_{z=0}^{5} z\, dz \int_{x=0}^{4} \frac{\mathrm{x}}{\sqrt{16-x^2}}\, dx + \int_{x=0}^{4} x\, dx \int_{z=0}^{5} 1\, dz \;= 90.$$

Exercise:

1. Find the surface integral over the parallelepiped $x = 0, y = 0, x = 1, y = 2, z = 3$ when $\vec{A} = 2xy\hat{\imath} + yz^2\hat{\jmath} + xz\hat{k}$ $\quad Ans: 33.$
2. If S is the surface of the sphere $x^2 + y^2 + z^2 = d^2$ and $\vec{A} = ax\hat{\imath} + by\hat{\jmath} + cz\hat{k}$, evaluate $\iint_S \vec{A} \cdot \hat{n}\, ds.$ $\;Ans: \frac{2\pi d^3}{3}(a + b + c)$
3. If $\vec{F} = 2y\hat{\imath} - 3\hat{\jmath} + x^2\hat{k}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6,$ show that $\iint_S \vec{F} \cdot \hat{n}\, ds = 132.$

Gauss divergence theorem

(Relation between surface and volume integrals)

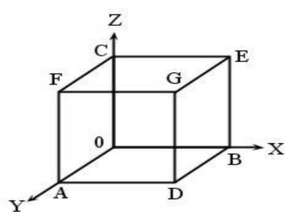
Statement: If V is the volume bounded by a closed surface S and $\vec{F}$ is a vector point function having continuous derivatives, then

$$\iint_S \vec{F}.\hat{n}\, ds = \iiint_V \nabla.\vec{F}\, dV,$$

where $\hat{n}$ is the unit normal drawn to S. ($\hat{n} \rightarrow$ outward unit normal i.e, normal vector away from the surface)

Problem 1. Verify Gauss divergence theorem for $\vec{F} = (x^3 - yz)\hat{\imath} - 2x^2y\hat{\jmath} + z\hat{k}$ taken over the suface of the cube bounded by the planes $x = y = z = 2$ and the coordinate planes.
Solution: We have to verfy that

$$\iint_S \vec{F}.\hat{n}\, ds = \iiint_V \nabla.\vec{F}\, dV$$



$$\iint_S \vec{F}.\hat{n}\, ds = \int_{S_1} \vec{F}.\hat{n}\, ds + \int_{S_2} \vec{F}.\hat{n}\, ds + \int_{S_3} \vec{F}.\hat{n}\, ds + \int_{S_4} \vec{F}.\hat{n}\, ds + \int_{S_5} \vec{F}.\hat{n}\, ds + \int_{S_6} \vec{F}.\hat{n}\, ds$$

where $S_1,\ S_2,\ S_3,\ S_4,\ S_5,\ S_6$ are the six faces of the cube.

$$\therefore \iint_S \vec{F}.\hat{n}\, ds = \iint_S \left((x^3 - yz)\hat{\imath} - 2x^2y\hat{\jmath} + z\hat{k}\right).\hat{n}\, ds$$

For S_1 $(DBEG)$ which is parallel to $yz - plane$ its equation is $x = 2, \hat{n} = i$ & $ds = dydz$.

Here $\hat{n} = i$, $\quad (x^3 - yz)i.i = x^3 - yz$ (remaining are zero).

$$\iint_{S_1} \left((x^3 - yz)\hat{\imath} - 2x^2y\hat{\jmath} + z\hat{k}\right).\hat{n}\, ds = \iint_{S_1} \left((x^3 - yz)\hat{\imath} - 2x^2y\hat{\jmath} + z\hat{k}\right)\cdot \hat{\imath}\, dy\, dz$$

$$= \int_{z=0}^{2}\int_{y=0}^{2} (8 - yz)\ dy\, dz = \int_{z=0}^{2} \left[8y - \frac{zy^2}{2}\right]_0^2\ dz = \int_{z=0}^{2} [16 - 2z]\ d$$

$$= \left[16z - \frac{2z^2}{2}\right]_0^2 = 32 - 4 = 28.$$

For S_2 $(OCEB)$ which is $xz - plane$, $\quad y = 0, \quad \hat{n} = -j \quad \& \ ds = dz\, dx$.

$$\iint_{S_2} \left((x^3 - yz)\hat{\imath} - 2x^2y\hat{\jmath} + z\hat{k}\right).\hat{n}\, ds = \int_{x=0}^{2}\int_{y=0}^{2} (2x^2y)\ dz\, dx = 0 \qquad \because y = 0$$

For S_3 $(OADB)$ which is $xy-plane,\quad z=0,\quad \hat{n}=-k\ \ \&\ ds=dx\,dy.$

$$\iint_{S_3}\left((x^3-yz)\hat{\imath}-2x^2y\hat{\jmath}+z\hat{k}\right).\hat{n}\,ds=\int_{y=0}^{2}\int_{x=0}^{2}(-z)\ dx\,dy=0\qquad\because z=0$$

For S_4 $(OCFA)$ which is $yz-plane,\quad x=0,\quad \hat{n}=-i\ \ \&\ ds=dy\,dz.$

$$\iint_{S_4}\left((x^3-yz)\hat{\imath}-2x^2y\hat{\jmath}+z\hat{k}\right).\hat{n}\,ds$$

$$=\int_{z=0}^{2}\int_{y=0}^{2}-(x^3-yz)\ dy\,dz=\int_{z=0}^{2}\int_{y=0}^{2}yz\ dy\,dz=\int_{z=0}^{2}z\,dz\int_{y=0}^{2}y\,dy$$

$$=\left[\frac{z^2}{2}\right]_0^2\left[\frac{y^2}{2}\right]_0^2=4.$$

For S_5 $(GFAD)$ which is parallel to $xz-plane,\ its\ equation\ is\ y=2,\quad \hat{n}=\hat{\jmath}\ \ \&\ ds=dx\,dz.$

$$\iint_{S_5}\left((x^3-yz)\hat{\imath}-2x^2y\hat{\jmath}+z\hat{k}\right).\hat{n}\,ds$$

$$=\int_{z=0}^{2}\int_{x=0}^{2}-2x^2y\ dx\,dz=\int_{z=0}^{2}\int_{x=0}^{2}-2x^2(2)\ dy\,dz=-4\int_{x=0}^{2}x^2\,dx\int_{z=0}^{2}1\,dz=$$

$$=-4\left[\frac{x^3}{3}\right]_0^2\ [z]_0^2=-4\times\frac{8}{3}\times 2=-\frac{64}{3}.$$

For S_6 $(GECF)$ which is parallel to $xy-plane,\ its\ equation\ is\ z=2,\quad \hat{n}=k\ \ \&\ ds=dx\,dy.$

$$\iint_{S_6}\left((x^3-yz)\hat{\imath}-2x^2y\hat{\jmath}+z\hat{k}\right).\hat{n}\,ds$$

$$=\int_{x=0}^{2}\int_{y=0}^{2}z\ dy\,dx=2\int_{x=0}^{2}1\,dx\int_{y=0}^{2}1\,dy=8.$$

$$\therefore \iint_S \left((x^3 - yz)\hat{\imath} - 2x^2y\hat{\jmath} + z\hat{k}\right).\hat{n}\, ds = 28 + 0 + 0 + 4 - \frac{64}{3} + 8 = \frac{56}{3}.$$

Now to evaluate $\iiint_V \nabla.\vec{F}\, dV$

Consider

$$\nabla.\vec{F} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left((x^3 - yz)\hat{\imath} - 2x^2y\hat{\jmath} + z\hat{k}\right) = 3x^2 - 2x^2 + 1 = x^2 + 1$$

$$\therefore \iiint_V \nabla.\vec{F}\, dV = \int_{z=0}^{2}\int_{y=0}^{2}\int_{x=0}^{2}(x^2+1)dx\, dy\, dz = \int_{z=0}^{2} 1\, dz \int_{y=0}^{2} 1\, dy \int_{x=0}^{2}(x^2+1)\, dx$$

$$= [z]_0^2\, [y]_0^2 \left[\frac{x^3}{3} + x\right]_{0_0}^{2^2} = (2)(2)\left(\frac{8}{3} + 2\right) = \frac{56}{3} \qquad \ldots\ldots\ldots\ldots(2)$$

From (1) and (2), we see that

$$\iint_S \vec{F}.\hat{n}\, ds = \iiint_V \nabla.\vec{F}\, dV = \frac{56}{3}$$

Hence the Gauss divergence theorem.

Problem 2. Verify divergence theorem $\vec{F} = xy\hat{\imath} - y\hat{\jmath} + 2z\hat{k}$ over the region bounded by the plane $x = 0, y = 0, z = 0\ \&\ 2x + 2y + z = 4$.

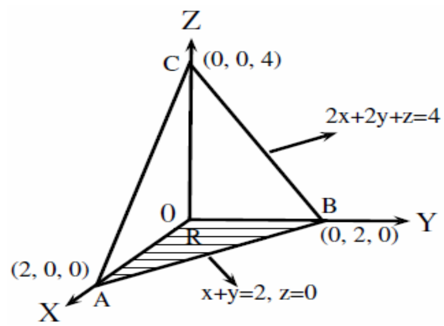
Solution: Let $\emptyset = 2x + 2y + z - 4$

$$i.e.\quad \frac{x}{2} + \frac{y}{2} + \frac{z}{4} = 1$$

$$\hat{n} = \frac{\nabla\emptyset}{|\nabla\emptyset|} = \frac{2\hat{\imath} + 2\hat{\jmath} + \hat{k}}{3}$$

$$\vec{F}.\hat{n} = \frac{2xy - 2y + 2z}{3}$$

Now project the surface on $xy - plane$



$$|\hat{n}.\hat{k}| = \left|\frac{2\hat{\imath}+2\hat{\jmath}+\hat{k}}{3}.\hat{k}\right| = \frac{1}{3}$$

$x: 0\ to\ 2$

$y: 0\ to\ 2-x$

plane $2x+2y+z=4 \Rightarrow z = 4-2x-2y$

$$\therefore \iint_S \vec{F}.\hat{n}\,ds = \iint_R \vec{F}.\hat{n}\,\frac{dx\,dy}{|\hat{n}.\hat{k}|} = \iint_R \left(\frac{2xy-2y+2z}{3}\right)\frac{dx\,dy}{\frac{1}{3}}$$

$$= 2\int_{x=0}^{2}\int_{y=0}^{2-x}(xy-y+4-2x-2y)\,dy\,dx$$

$$= 2\int_{x=0}^{2}\int_{y=0}^{2-x}(xy-3y-2x-2y+4)\,dy\,dx$$

$$= 2\int_{x=0}^{2}\left[\frac{xy^2}{2}-\frac{3y^2}{2}-2xy+4y\right]_0^{2-x}dx$$

$$= 2\int_{x=0}^{2}\left[\frac{x(2-x)^2}{2}-\frac{3(2-x)^2}{2}-2x(2-x)+4(2-x)\right]dx$$

$$= 2\int_{x=0}^{2}\left[\frac{1}{2}(4x+x^3-4x^2)-\frac{3}{2}(4+x^2-4x)-8x+2x^2+8\right]dx$$

$$= 2\left[\frac{1}{2}\left(\frac{4x^2}{2}+\frac{x^4}{4}-\frac{4x^3}{3}\right)-\frac{3}{2}\left(4x+\frac{x^3}{3}-\frac{4x^2}{2}\right)-\frac{8x^2}{2}+\frac{2x^3}{3}+8x\right]_0^2 = 4 \qquad (1)$$

Surface	Remarks	$\hat{n}$	ds	$\vec{F}.\hat{n}$
$S_1: AOB$	$xy - plane$ $z = 0$	$\hat{n} = -\hat{k}$	dx dy	-2z=0
$S_2: BOC$	$yz - plane$ $x = 0$	$\hat{n} = -\hat{\imath}$	dy dz	-xy=0
$S_3: AOC$	$xz - plane$ $y = 0$	$\hat{n} = -\hat{\jmath}$	dx dz	y=0
$S_4: ABC$	Projection on $xy - plane$	$\hat{n} = \frac{\nabla\emptyset}{\lvert\nabla\emptyset\rvert} = \frac{2\hat{\imath}+2\hat{\jmath}+\hat{k}}{3}$	$\frac{dx\,dy}{\lvert\hat{n}.\hat{k}\rvert}$	$\vec{F}.\hat{n} = \frac{2(xy-y+4-2x-2y)}{3}$

Now consider

$$\nabla.\vec{F} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).(xy\hat{\imath} - y\hat{\jmath} + 2z\hat{k}) = y + 1$$

$$\therefore \iiint_V \nabla.\vec{F}\,dV = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} (y+1)\,dz\,dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} [(y+1)z]_{z=0}^{4-2x-2y}\,dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} [(y+1)(4-2x-2y)]dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} [2y + 4 - 2xy - 2x - 2y^2]dy\,dx$$

$$= \int_{0}^{2}\left[\frac{2y^2}{2} + 4y - \frac{2xy^2}{2} - 2xy - \frac{2y^3}{3}\right]_0^{2-x} dy$$

$$= \int_0^2 \left[(2-x)^2 + 4(2-x) - x(2-x)^2 - 2x(2-x) - \frac{2}{3}(2-x)^3\right] dy = 4 \quad \ldots(2)$$

From (1) and (2), Gauss divergence theorem is verified.

Problem 3. Using divergence theorem, evaluate $\iint_S \left[(x^2 - yz)\hat{\imath} + (y^2 - zx)\hat{\jmath} + (z^2 - xy)\hat{k}\right].\hat{n}\,ds$, over the surface of the rectangular parallelepiped $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$.

Solution: We have

$$\iint_S \vec{F}.\hat{n}\, ds = \iiint_V \nabla.\vec{F}\, dV$$

$$\nabla.\vec{F} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left((x^2 - yz)\hat{\imath} + (y^2 - zx)\hat{\jmath} + (z^2 - xy)\hat{k}\right) = 2(x + y + z)$$

$$\therefore \iiint_V \nabla.\vec{F}\, dV = \int_{z=0}^{c}\int_{y=0}^{b}\int_{x=0}^{a} 2(x + y + z)\, dx\, dy\, dz$$

$$= 2\int_{z=0}^{c}\int_{y=0}^{b}\left(\frac{x^2}{2} + yx + zx\right)_0^a\, dy\, dz$$

$$= 2\int_0^c\int_0^b\left(\frac{a^2}{2} + ay + az\right) dy\, dz$$

$$= 2\int_0^c\left[\frac{a^2}{2}y + \frac{ay^2}{2} + azy\right]_0^b dz = 2\int_0^c\left[\frac{a^2b}{2} + \frac{ab^2}{2} + azb\right] dz$$

$$= \left[\frac{a^2b}{2}z + \frac{ab^2}{2}z + ab\frac{z^2}{2}\right]_0^c = abc(a + b + c).$$

Problem 4. Evaluate using divergence theorem $\iint_S \left[x^3\hat{\imath} + x^2y\hat{\jmath} + x^2z\hat{k}\right].\hat{n}\, ds$, where S is the surface consisting of the cylinder $x^2 + y^2 = a^2$ and the circular discs cut by the plane $z = 0$ & $z = b$.

Solution: Here

$\vec{F} = x^3\hat{\imath} + x^2y\hat{\jmath} + x^2z\hat{k}$

$$\nabla.\vec{F} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left(x^3\hat{\imath} + x^2y\hat{\jmath} + x^2z\hat{k}\right) = 5x^2$$

$$\iint_S \vec{F}.\hat{n}\, ds = \iiint_V \nabla.\vec{F}\, dV = \iiint_V 5x^2 dV$$

using cylindrical coordinates

$x = r\cos\theta,$

$y = r\sin\theta,$

$z = z,$

$dV = dx\, dy\, dz = r\, dr\, d\theta\, dz$

$r: 0\ to\ a, \quad \theta: 0\ to\ 2\pi, z: 0\ to\ b$

$$\therefore \iiint_V 5x^2 dV = \int_{z=0}^{b} \int_{\theta=0}^{2\pi} \int_{r=0}^{a} 5(r^2\cos\theta)\, r\, dr\, d\theta\, dz$$

$$= 5\int_{r=0}^{a} r^3 dr \int_{\theta=0}^{2\pi} \cos^2\theta\, d\theta \int_{z=0}^{b} 1\, dz = 5\frac{a^4}{4}\times\frac{1}{2}\times\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{2\pi}\times[z]_0^b$$

$$= \frac{5a^4 b\,\pi}{4}.$$

Problem 5. Using divergence theorem, $\iint_S \vec{F}.\hat{n}\, ds$, $\vec{F} = x^3\hat{\imath} + y^3\hat{\jmath} + z^3\hat{k}$ taken over the surface consisting of the hemisphere $x^2+y^2+z^2=a^2$ above the $xy-plane$ bounded by the $xy-plane$.

Solution: Here

$\vec{F} = x^3\hat{\imath} + y^3\hat{\jmath} + z^3\hat{k}$

$$\nabla.\vec{F} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).\left(x^3\hat{\imath} + y^3\hat{\jmath} + z^3\hat{k}\right) = 3(x^2+y^2+z^2) = 3a^2$$

Using Spherical coordinates

$\because x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$

$dx\, dy\, dz = r^2\sin\theta\; dr\, d\theta\, d\phi$

$r: 0\ to\ a$

$\theta: 0\ to \frac{\pi}{2} \qquad [verticle\ angle]$

$\phi: 0\ to\ 2\pi \qquad [Horizontal\ angle]$

$$\iint_S \vec{F}.\hat{n}\, ds = \iiint_V div(\vec{F})\; dV = 3\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{a} r^2(r^2\sin\theta)\, dr\, d\theta\, d\phi$$

$$= 3\int_{\phi=0}^{2\pi} 1\, d\phi \times \int_{\theta=0}^{\frac{\pi}{2}} \sin\theta\, d\theta \times \int_{r=0}^{a} r^4\, dr$$

$$= 3\times 2\pi\times\frac{a^5}{5}\times 1 = \frac{6\pi a^5}{5}.$$

Using Cartesian coordinates

$$\iint_S \vec{F}.\hat{n}\, ds = \iiint_V div(\vec{F})\, dV$$

$$= \int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}} 3(x^2+y^2+z^2)\, dz\, dy\, dx = \frac{6\pi a^5}{5}$$

Exercise:

1. Verify divergence theorem for $\vec{F} = 4xz\hat{\imath} - y^2\hat{\jmath} + yz\,\hat{k}$ taken over the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.
2. Using divergence theorem, evaluate $\iint_S \vec{r}.\hat{n}\, ds$ where S is the surface of the sphere$x^2 + y^2 + z^2 = 9$. $Ans: 108\pi$
3. Using divergence theorem, evaluate $\iint_S \vec{F}.\hat{n}\, ds$ over the entire surface S of the region above xy plane bounded by the cone $x^2 + y^2 = z^2$ the plane $z = 4$ where $\vec{F} = 4xz\hat{\imath} - xyz^2\hat{\jmath} + 3z\,\hat{k}$ $Ans: 704\pi$

STOKES THEOREM

(Relation between line and surface integral)

Statement: If S be an open surface bounded by a simple closed curve C and $\vec{F}$ be any vector point function having continuous first order partial derivatives, then

$$\oint_C \vec{F}.d\vec{r} = \iint_S curl\,\vec{F}.\hat{n}\, ds = \iint_S curl\,\vec{F}.d\vec{s}$$

where $\hat{n}$ is the outward drawn unit normal at any point to S.

Problem 1. Verify Stokes theorem for $\vec{F} = (x^2+y^2)\hat{\imath} - 2xy\hat{\jmath}$ taken round the rectangle bounded by the lines $x = \pm a, y = 0, y = b$.

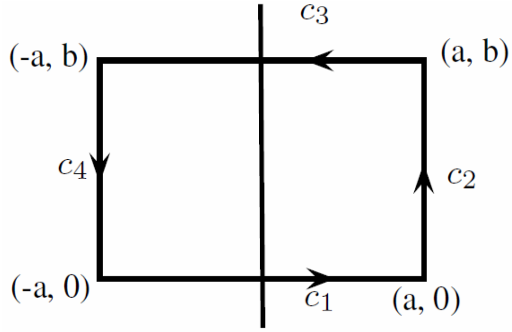
Solution: We have to prove that

$$\oint_C \vec{F}.d\vec{r} = \iint_S curl\,\vec{F}.\hat{n}\, ds$$

Now

$$\vec{F}.d\vec{r} = \left((x^2+y^2)\hat{\imath} - 2xy\hat{\jmath}\right).\left(dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}\right) = (x^2+y^2)dx - 2xy\, dy$$

$$\oint_C \vec{F}.d\vec{r} = \int_{C_1} \vec{F}.d\vec{r} + \int_{C_2} \vec{F}.d\vec{r} + \int_{C_3} \vec{F}.d\vec{r} + \int_{C_4} \vec{F}.d\vec{r} \qquad \ldots\ldots\ldots\ldots(1)$$



$$\oint_C \vec{F}.d\vec{r} = \oint_C (x^2+y^2)dx - 2xy\,dy$$

Along $C_1: y = 0 \Rightarrow dy = 0,\ \ x: -a\ to\ a$

$$\therefore \int_{C_1} \vec{F}.d\vec{r} = \int_{C_1} (x^2+y^2)dx - 2xy\,dy = \int_{-a}^{a} x^2 dx = \frac{2a^3}{3}$$

Along $C_2: x = a \Rightarrow dx = 0,\ \ y: 0\ to\ b$

$$\therefore \int_{C_2} \vec{F}.d\vec{r} = \int_{C_2} (x^2+y^2)dx - 2xy\,dy = \int 2ay\,dy = -ab^2$$

Along $C_3: y = b \Rightarrow dy = 0,\ \ x: a\ to -a$

$$\therefore \int_{C_3} \vec{F}.d\vec{r} = \int_{C_3} (x^2+y^2)dx - 2xy\,dy = \int_{a}^{-a} (x^2+b^2)\,dx = -\frac{2a^3}{3} - 2ab^2$$

Along $C_4: x = -a \Rightarrow dx = 0,\ \ y: b\ to\ 0$

$$\therefore \int_{C_4} \vec{F}.d\vec{r} = \int_{C_4} (x^2+y^2)dx - 2xy\,dy = \int_{b}^{0} -2(-a)y\,dy = -ab^2$$

$$\therefore (1) \Rightarrow \oint_C \vec{F}.d\vec{r} = \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 = -4ab^2 \qquad \ldots\ldots\ldots(2)$$

Next, consider

$$curl\,\vec{F} = \nabla\times\,\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = -4y\hat{k}$$

Rectangle in $xy - plane \Rightarrow\ \hat{n} = \hat{k}\ and\ ds = dx\,dy$

$$\iint_S curl\,\vec{F}.\hat{n}\,ds = \iint_R -4y\hat{k}.\hat{k}\,dx\,dy = -\int_{y=0}^{b}\int_{x=-a}^{a} 4y\,dx\,dy = -4\int_{-a}^{a} 1\,dx \times \int_{0}^{b} y\,dy$$

$$\therefore \iint_S curl\,\vec{F}.\hat{n}\,ds = -4ab^2 \qquad \ldots\ldots\ldots(3)$$

From (2) and (3), Stokes theorem is verified.

Problem 2. Verify Stokes theorem for $\vec{F} = (2x - y)\hat{\imath} - yz^2\hat{\jmath} - y^2z\hat{k}$ over the upper half surface of $x^2 + y^2 + z^2 = 1$ bounded by its projection on $xy - plane$.

Solution: The projection of upper half of the sphere $x^2 + y^2 + z^2 = 1$ in the $xy - plane$ $(z = 0)$ is the circle $x^2 + y^2 = 1$ and let C be its boundary.

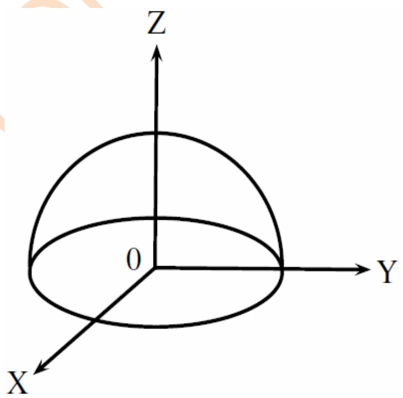
We have

$$\oint_C \vec{F}.d\vec{r} = \iint_S (\nabla \times \vec{F}).\hat{n}\,ds$$

Consider

$$\oint_C \vec{F}.d\vec{r} = \int_C \{(2x - y)dx - yz^2dy - y^2z\,dz\}$$

In $xy - plane, z = 0 \Rightarrow dz = 0$



$$\therefore \oint_C \vec{F}.d\vec{r} = \int_C \{(2x - y)dx - yz^2dy - y^2z\,dz\} = \int_C (2x - y)dx$$

Here C is the circle $x^2 + y^2 = 1$ whose parametric equation is given by

$x = \cos\theta, \qquad y = \sin\theta$

$\Rightarrow dx = -\sin\theta\,d\theta, \qquad dy = \cos\theta\,d\theta$

Here $\theta: 0\ to\ 2\pi$.

$$\therefore \oint_C \vec{F}.d\vec{r} = \int_0^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta)d\theta = \int_0^{2\pi} (\sin 2\theta + \sin^2\theta)\, d\theta = 0 + \int_0^{2\pi} \sin^2\theta\, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta = 4 \times \frac{1}{2} \times \frac{\pi}{2} = \pi.$$

Next, consider

$$curl\, \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = \hat{k}$$

On the xy −plane $\hat{n} = \hat{k}$ and $ds = dx\, dy$

$$\iint_S curl\, \vec{F}.\hat{n}\, ds = \iint_R \hat{k} \cdot \hat{k}\ dx\, dy = \iint_R 1\ dx\, dy$$

$= Area\ of\ circle\ (x^2 + y^2 = 1) = \pi \qquad \because r = 1$

Hence Stokes theorem is verified.

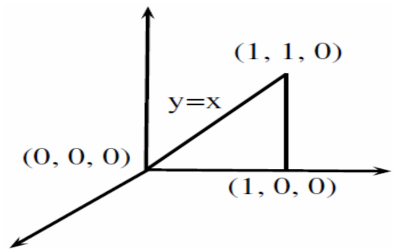
Problem 3. Evaluate by Stokes theorem $\oint_C (x + y)dx + (2x - z)dy + (y + z)dz$, C is the boundary of the triangular with vertices $(0, 0, 0), (1, 0, 0)$ and $(1, 1, 0)$.

Solution: By Stokes theorem we have

$$\oint_C \vec{F}.d\vec{r} = \iint_S curl\, \vec{F}.\hat{n}\, ds$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x + y & 2x - z & y + z \end{vmatrix} = 2\hat{\imath} + \hat{k}$$

In $xy - plane \Rightarrow\ \hat{n} = \hat{k}\ and\ ds = dx\, dy$



$$\iint_S curl\, \vec{F}.\hat{n}\, ds = \iint_R (2\hat{\imath} + \hat{k}).\hat{k}\, dx\, dy = \iint_R 1\ dx\, dy = Area\ of\ the\ triangle$$

$$= \frac{1}{2} \times 1 \times 1 = \frac{1}{2}.$$

Exercise:

1. Evaluate $\oint_C xy\,dx + xy^2\,dy$ by Stoke's theorem where C is the square in the xy plane with vertices $(1,0)$ $(-1,0)$ $(0,1)$ $(0,-1)$.
2. Verify Stokes's theorem where $\vec{A} = (2x - y)\hat{\imath} - yz^2\hat{\jmath} - y^2z\,\hat{k}$ and S: upper half of the surface of the sphere $x^2 + y^2 + z^2 = 1$ $Ans: \pi$
3. Evaluate $\oint_C 4z\,dx - 2x\,dy + 2x\,dz$ by Stoke's theorem where C is the ellipse $x^2 + y^2 = 1,\ z = y + 1.$ $Ans: -4\pi$

Video Links:

1. Line integral
   https://www.youtube.com/watch?v=7FUNdFN6ZKI
2. Surface integral
   https://www.youtube.com/watch?v=I1dfwKPV75A