unit3_tutorials

R V COLLEGE OF ENGINEERING
(An autonomous institution affiliated to VTU, Belgaum)
DEPARTMENT OF MATHEMATICS

FUNDAMENTS OF LINEAR ALGEBRA, CALCULUS AND NUMERICAL METHODS (MAT211AT)
Multivariable Functions and Partial Differentiation

TUTORIAL SHEET-1

1. If $x = r\cos\theta, y = r\sin\theta$, then $\left(\frac{\partial x}{\partial\theta}\right)^2 + \left(\frac{\partial y}{\partial\theta}\right)^2 =$ r^2 ______ Ans: r ?
2. If $z = x\sin y + y\sin x$, then $\frac{\partial^2 z}{\partial x\partial y} =$ $\cos y + \cos x$ ______ Ans: $\cos y + \cos x$
3. If $z = e^{2x^2+xy}$, then $\frac{\partial z}{\partial y} =$ xe^{2x^2+xy} ______ Ans: xe^{2x^2+xy}
4. The steady state temperature of a metal sheet is $T(x,y) = x^2 - a^2y^2$. The values of 'a' for which $T(x,y)$ satisfies the Laplace equation $T_{xx} + T_{yy} = 0$ are ± 1 ______. Ans: $\pm a$
5. If $u = y\cos(xy)$ then $\frac{\partial u}{\partial y}$ at the point $(1,\pi)$ is -1 ______. Ans: -1
6. If $V = f(x-ct) + g(x+ct)$ where f and g are arbitrary functions of $x - ct$ and $x + ct$ respectively and c is a constant, then show that $\frac{\partial^2 V}{\partial t^2} = c^2\frac{\partial^2 V}{\partial x^2}$
7. If $u = \frac{x+y}{x-y}$, verify that $\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial^2 u}{\partial y\partial x}$
8. If $u = ae^{-gx}\sin(nt - gx)$, where a, g and n are positive constants and $\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial x^2}$, show that $g = \sqrt{\frac{n}{2\mu}}$.
9. If V is the volume and S is the total surface area of rectangular box of length x, breadth y and height z, find
   (i) the rate of change of V with respect to x if y = 4 and z = 12.
   (ii) the rate of change of S with respect to z if x = 3 and y = 4.
   Ans: (i) 48 (ii) 14
10. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, then show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$.

$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$.

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TUTORIAL SHEET-2

1. Given $z = xy^2 + x^3y$ where x and y are functions of t with $x(1) = 1$, $y(1) = 2$, $x'(1) = 3$ and $y'(1) = 4$. The value of $\frac{dz}{dt}$ at t = 1 is 50 ______. Ans: 16 ?
2. For the implicit function $(\cos x)^y = (\sin y)^x$, $\frac{dy}{dx} =$ ______.
   Ans: $-\left[\frac{y(\cos x)^{y-1}(-\sin x) - (\sin y)^x\log\sin y}{(\cos x)^y\log(\cos x) - x(\sin y)^{x-1}\cos y}\right]$
3. Given 't' represents time and $u = x^2 - y^2$, $x = \frac{1}{t}$, $y = e^t$ then the rate of change of u with respect to 't' is ______. Ans: $\frac{-2}{t^3} - 2e^{2t}$
4. For the implicit function $e^x - e^y = 2xy$, $\frac{dy}{dx} =$ ______. Ans: $\left[\frac{e^x - 2y}{e^y + 2x}\right]$
5. Given, $x^2 + y^2 + 3xz = 1$ and $x + y = 1$, then $\frac{dz}{dx} =$ ______ Ans: $\frac{-(2x+3)}{3} + \frac{2y}{3x}$
6. If $z = z(x,y)$, $x = e^u\sin v$, $y = e^v\cos v$, then $\frac{\partial z}{\partial u}$ ______. Ans: $\frac{\partial z}{\partial x}e^u\sin v$
7. If $u = xyz$ where $x = e^{-t}$, $y = e^{-t}\sin^2 t$, $z = \sin t$, then find $\frac{du}{dt}$.
   Ans: $e^{-2t}\sin^2 t(3\cos t - 2\sin t)$
8. If $z = x^2 + 2xy + 4y^2$ and $y = e^{3x}$, find $\frac{dz}{dx}$. Ans: $2(x + e^{3x}) + 2(x + 4e^{3x})3e^{3x}$
9. If z is a function of x and y and if $x = e^u\sin v$, $y = e^u\cos v$, prove that
   (i) $\frac{\partial z}{\partial u} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$
   (ii) $\frac{\partial z}{\partial x} = e^{-u}\left(\sin v\frac{\partial z}{\partial u} + \cos v\frac{\partial z}{\partial v}\right)$
10. If $u = f(x-y, y-z, z-x)$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.
11. If $u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$, determine $\frac{\partial^2 u}{\partial x\partial y}$.

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TUTORIAL SHEET-3

1. Match the following:

i)	If $x = e^u\sin v$, $y = e^v\cos v$ then $J\left(\frac{x,y}{u,v}\right) =$ ______.	a)	$\frac{1}{4v\sin 2u}$
ii)	If (a,b) is a stationary point of f(x,y) and $f_{xx} = 3$, $f_{xy} = 2$ and $f_{yy} = 2$ at this point then the nature of (a,b) is____.	b)	minimum
iii)	The nature of the point (1, -1) to the function $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 7$ is ______.	c)	$e^{2u}(\sin v\cos v - \sin^2 v)$
		d)	$\frac{v\sin 2u}{4}$
iv)	If $\frac{\partial(x,y)}{\partial(u,v)} = v\sin 2u$ and $\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{1}{4}$ then $\frac{\partial(x,y)}{\partial(r,\theta)} =$ ______.	e)	Neither maximum nor minimum
		f)	Saddle point
		g)	maximum
		h)	$e^{2u}(\sin v\cos v - \sin^2 v) - e^{u}\cos v$

Ans: (i) - (c) (ii) - (g) (iii) - (e) (iv) - (a)

2. Find the extreme values of $\sin x + \sin y + \sin(x+y)$. **Ans:** Maximum value at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, maximum value is $\frac{3\sqrt{3}}{2}$
3. Find the volume of largest rectangular parallelepiped that can be inscribed in an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Ans: Maximum volume $= \frac{8abc}{3\sqrt{3}}$
4. Find the maximum and minimum distances of the point (1, 2, 3) from the sphere $x^2 + y^2 + z^2 = 56$ using Lagrange's Method of undetermined multipliers. **Ans:** Minimum distance at (2, 4, 6) $= \sqrt{14}$, Maximum distance at (-2,-4,-6), maximum distance $= \sqrt{126}$
5. Show that $u = \frac{x^2 - y^2}{x^2 + y^2}$, $v = \frac{2xy}{x^2 + y^2}$ are functionally dependent and find the relation between them. Ans: $u^2 + v^2 = 1$.
6. For $u = xyz$, $v = yz + zx + xy$, $w = x + y + z$, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$. Ans: $(y-z)(z-x)(x-y)$.
7. If $x = e^v\sec u$, $y = e^v\tan u$, then verify that $J' = 1$.

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(1) $y = r\sin\theta$; $\left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2$

$\frac{dy}{d\theta} = r\cos\theta$; $= r^2\cos^2\theta + r^2\sin^2\theta$

$x = r\cos\theta$; $= r^2$

$\frac{dx}{d\theta} = -r\sin\theta$

(3) $z = e^{2x^2+xy}$

$\frac{dz}{dy} = xe^{2x^2+xy}$

(4) $T(x,y) = x^2 - a^2y^2 = y$

$T_{xx} + T_{yy} = 0$

$\frac{d^2y}{dx^2} + \frac{d^2y}{dy^2} = 0$

$\frac{d}{dx}(2x) + \frac{d}{dy}(-a^2(2y)) = 0$

$2 - 2a^2 = 0$

$2 = 2a^2$

$a^2 = 1 \Rightarrow a = \pm 1$

(2) $z = x\sin y + y\sin x$

$\frac{d}{dx}\left(\frac{dz}{dy}\right) = \frac{d}{dx}(x\cos y + \sin x)$

$= \cos y + \cos x$

(5) $u = y\cos(xy)$

$\frac{\partial u}{\partial y} = (1)\cos(xy) + y(-\sin(xy))\cdot x$

at $x = 1, y = \pi$

$= \cos\pi - \pi\cancel{\sin\pi}^{\,0}$

$= -1$

(7) $u = \frac{x+y}{x-y}$; $\frac{d^2u}{dydx} = \frac{d}{dy}\left(\frac{du}{dx}\right)$

$\frac{d^2u}{dxdy} = \frac{d}{dx}\left(\frac{du}{dy}\right)$

$= \frac{d}{dx}\left(\frac{(1)(x-y) - (x+y)(-1)}{(x-y)^2}\right)$

$= \frac{d}{dx}\left(\frac{x - \cancel{y} + x + \cancel{y}}{(x-y)^2}\right)$

$= \frac{d}{dx}\left(\frac{2x}{(x-y)^2}\right)$

$= 2\left(\frac{(1)(x-y)^2 - x(2(x-y))}{(x-y)^4}\right)$

$= 2\left(\frac{x^2 + y^2 - \cancel{2xy} - 2x^2 + \cancel{2xy}}{(x-y)^4}\right)$

$= 2\frac{(y^2 - x^2)}{(x-y)^4}$ (=)

$\frac{x+y}{x-y}$

$= \frac{d}{dy}\left(\frac{(1)(x-y) - (x+y)(1)}{(x-y)^2}\right)$

$= \frac{d}{dy}\left(\frac{\cancel{x} - y - \cancel{x} - y}{(x-y)^2}\right)$

$= \frac{d}{dy}\left(\frac{-2y}{(x-y)^2}\right)$

$= -2\left(\frac{(1)(x-y)^2 - y(2(x-y)(-1))}{(x-y)^4}\right)$

$= -2\frac{(x^2 + y^2 - \cancel{2xy} + \cancel{2xy} - 2y^2)}{(x-y)^4}$

$= -2\frac{(x^2 - y^2)}{(x-y)^4} = \frac{2(y^2 - x^2)}{(x-y)^4}$

(6) $V = f(x-ct) + g(x+ct)$

$\rightarrow \frac{dV}{dt} = f'(x-ct)\cdot(-c) + g'(x+ct)\cdot c$

$\frac{d^2V}{dt^2} = +c^2 f''(x-ct) + c^2 g''(x+ct)$

$\rightarrow \frac{dV}{dx} = f'(x-ct) + g'(x+ct)$

$\frac{d^2V}{dx^2} = f''(x-ct) + g''(x+ct)$

$\Rightarrow \frac{d^2V}{dt^2} = c^2\frac{d^2V}{dx^2}$

(8) $u = ae^{-gx}\sin(nt - gx)$

$\rightarrow \frac{\partial u}{\partial t} = ae^{-gx}\cos(nt - gx)\cdot n$

$\rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$

$= \frac{\partial}{\partial x}\left(a\left(e^{-gx}\cos(nt-gx)(-g) + e^{-gx}(-g)\sin(nt-gx)\right)\right)$

$= \frac{\partial}{\partial x}\left(-age^{-gx}\left(\cos(nt-gx) + \sin(nt-gx)\right)\right)$

$= -ag\left[e^{-gx}(-g)(\cos x + \sin x) + e^{-gx}(-\sin x(-g) + \cos x(-g))\right]$

$= ag^2e^{-gx}(\cos x + \cancel{\sin x} - \cancel{\sin x} + \cos x)$

$= 2ag^2e^{-gx}\cos(nt - gx)$

$\frac{du}{dt} = \mu\frac{d^2u}{dx^2}$

$\cancel{ae^{-gx}}\,\cancel{\cos(nt-gx)}\cdot n = \mu\cdot 2\cancel{a}g^2\cancel{e^{-gx}}\,\cancel{\cos(nt-gx)}$

$n = \mu(2g^2)$

$g = \sqrt{\frac{n}{2\mu}}$

(9) $V = xyz$ $S = 2(xy + yz + xz)$

(i) $\left.\frac{\partial V}{\partial x}\right|_{y=4,\, z=12} = yz = 4\times 12 = 48$

(ii) $\left.\frac{\partial S}{\partial z}\right|_{x=3,\, y=4} = 2(y + x) = 2(4+3) = 14$

(10) $u = \log(x^3 + y^3 + z^3 - 3xyz)$

$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

$\rightarrow \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$

$\rightarrow \frac{\partial u}{\partial x} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} \times (3x^2 - 3yz)$

$\rightarrow \frac{\partial u}{\partial y} = \frac{1}{''} \times (3y^2 - 3xz)$

$\rightarrow \frac{\partial u}{\partial z} = \frac{1}{''} \times (3z^2 - 3xy)$

$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{1}{''}\,(3x^2 + 3y^2 + 3z^2 - 3yz - 3xz - 3xy)\right)$

$\left(\frac{3(x^2 + y^2 + z^2 - xy - yz - xz)}{x^3 + y^3 + z^3 - 3xyz}\right)$

$\left(\frac{3(\;''\;)}{(x+y+z)(\;''\;)}\right)$

$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$

$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \ldots$

$= 3\left(\frac{-1}{(x+y+z)^2}\right) + \ldots$

$= -\frac{3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2} + \ldots$

$= \frac{-9}{(x+y+z)^2}$

[1] $z = xy^2 + x^3y$ $z \rightarrow x, y \rightarrow t$

$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$

$= [y^2 + 3x^2y](x'(t)) + [x(2y) + x^3](y'(t))$

at $t = 1$, $x = 1$, $y = 2$, $x'(1) = 3$, $y'(1) = 4$.

$\frac{dz}{dt} = [2^2 + 3(1)^2(2)](3) + [(1)(2\times 2) + (1)^3](4)$

$= 30 + 20$

$= 50$

[2] $(\cos x)^y = (\sin y)^x \rightarrow$ implicit

$f = (\cos x)^y - (\sin y)^x$

$\frac{dy}{dx} = -\frac{f_x}{f_y}$ $\langle a^x = a^x\log a\rangle$

$= -\frac{[y(\cos x)^{y-1}\times(-\sin x) - \sin y^x\cdot\log(\sin y)]}{[\cos x^y\cdot\log(\cos x) - x(\sin y)^{x-1}\cos y]}$

[3] $u = x^2 - y^2$; $x = \frac{1}{t}$, $y = e^t$

$\frac{du}{dt} = \frac{du}{dx}\cdot\frac{dx}{dt} + \frac{du}{dy}\cdot\frac{dy}{dt}$

$= 2x\left(\frac{-1}{t^2}\right) + -2y\cdot e^t$

[4] $e^x - e^y = 2xy$

$f: e^x - e^y - 2xy = 0$

$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{[e^x - 2y]}{-e^y - 2x}$

$= e^x - 2y$

[5] $x^2 + y^2 + 3xz = 1$ $x + y = 1$

$2x + 2y\frac{dy}{dx} + 3\left(x\frac{dz}{dx} + z\right) = 0$ $\frac{dx}{dx} + \frac{dy}{dx} = 0$

$\frac{dy}{dx} = -1$

[5] $x^2+y^2+3xz=1$

$2x+2y\frac{dy}{dx}+3\left(x\frac{dz}{dx}+z\right)=0$

$2x-2y+3x\frac{dz}{dx}+3z=0$

$3x\frac{dz}{dx}=2y-2x-3z$

$\frac{dz}{dx}=\frac{2y}{3x}-\frac{2}{3}-\frac{\cancel{3}z}{\cancel{3}x}$

$=\frac{2y}{3x}-\frac{2}{3}-\frac{z}{x}$

$x+y=1$

$\frac{dx}{dx}+\frac{dy}{dx}=0$

$\frac{dy}{dx}=-1$

$(x+y)^2=x^2+y^2+2xy$

$\frac{1-x^2-y^2}{3x}$

$1-(x^2+y^2)$

$\frac{1-[(x+y)^2+2xy)]}{3x}$

$\frac{x^2}{x^2}+\frac{y^2}{x^2}+\frac{3xz}{x^2}=\frac{1}{x^2}$

$1+\frac{y^2}{x^2}+\frac{3z}{x}=\frac{1}{x^2}$

$\frac{3z}{x}=\frac{1-y^2}{x^2}-1$

$-\frac{z}{x}=\frac{1}{3}-\frac{(1-y^2)}{3x^2}$

$x+y=1$

$y=1-x$

$y^2=1+x^2-2x$

$1-y^2=2x-x^2$

$\frac{2y}{3x}\left(-\frac{2}{3}+\frac{1}{3}\right)-\frac{(1-y^2)}{3x^2}$

$\frac{2y}{3x}\quad -\frac{1}{3}\quad -\frac{(1-y^2)}{3x^2}$

$-\frac{(2x-x^2)}{3x^{\cancel{2}}}$

$-\frac{1}{3}$

$\frac{2y}{3x}-\frac{(2-x)}{3x}$

$+\frac{\cancel{x}}{3x}\quad \cancel{-\frac{1}{3}}$

$\frac{2y-2}{3x}$

$\frac{2(y-1)}{3x}$

$\frac{du}{dt}=\frac{du}{dx}\cdot\frac{dx}{dt}+\frac{du}{dy}\cdot\frac{dy}{dt}$

$=2x\left(-\frac{1}{t^2}\right)+-2y\cdot e^t$

$=\frac{2}{t}\left(-\frac{1}{t^2}\right)-2e^{2t}$

$=\frac{-2}{t^3}-2e^{2t}$

$\frac{dy}{dx}=\frac{-f_x}{f_y}=\frac{-[e^x-2y]}{-e^y-2x}$

$=\frac{e^x-2y}{e^y+2x}$

[6] z — x (u, v), y (u, v)

$x=e^u\sin v \qquad y=e^v\cos v$

$\frac{dz}{du}=\frac{\partial z}{\partial x}\cdot\frac{dx}{du}+\frac{\partial z}{\partial y}\cdot\frac{dy}{du}$

$=\frac{\partial z}{\partial x}\cdot e^u\sin v+\cancel{\frac{\partial z}{\partial y}(0)}$

$=\frac{\partial z}{\partial x}(\sin v\, e^u)$

[7] $u=xyz$

$x=e^{-t},\ y=e^{-t}\sin^2 t,\ z=\sin t$

u — x, y, z — t

$\frac{du}{dt}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial t}$

$=yz(-e^{-t})+xz(-e^{-t}\sin^2t+e^{-t}(2\sin t\cos t))$

$+xy\cos t$

$=-e^{-2t}\sin^3t+e^{-2t}\sin t(2\cos t-\sin t)$

$+e^{-2t}\sin^2t\cos t$

$=e^{-2t}\sin t(-\sin t+2\cos t-\sin t+\cos t)$

$\frac{du}{dt}=e^{-2t}\sin^2 t(3\cos t-2\sin t)$

[8] $z=x^2+2xy+4y^2,\ y=e^{3x}$

$\frac{dz}{dx}=2x+2\left(x\frac{dy}{dx}+y\right)+4(2y)\frac{dy}{dx}$

$\frac{dy}{dx}=3e^{3x}$

$=2x+2(x(3e^{3x})+e^{3x})$

$+8(e^{3x})(3e^{3x})$

$=2x+2e^{3x}(3x+1)+24e^{6x}$

$=2x+6xe^{3x}+2e^{3x}+24e^{6x}$

$=2(x+e^{3x})+6e^{3x}(x+4e^{3x})$

[10] $u=f(x-y,\ y-z,\ z-x)$

$r=x-y,\quad s=y-z,\quad t=z-x$

u — r (x, y), s (y, z), t (x, z)

$\rightarrow \frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$

$=\frac{\partial u}{\partial r}(1)+\frac{\partial u}{\partial s}(0)+\frac{\partial u}{\partial t}(-1)$

$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t}$

$\rightarrow \frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$

$=\frac{\partial u}{\partial r}(-1)+\frac{\partial u}{\partial s}(1)+\frac{\partial u}{\partial t}(0)$

$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial s}-\frac{\partial u}{\partial r}$

$\rightarrow \frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$

$=\frac{\partial u}{\partial r}(0)+\frac{\partial u}{\partial s}(-1)+\frac{\partial u}{\partial t}(1)$

$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial t}-\frac{\partial u}{\partial s}$

$\rightarrow \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\cancel{\frac{\partial u}{\partial r}}-\cancel{\frac{\partial u}{\partial t}}+\cancel{\frac{\partial u}{\partial s}}-\cancel{\frac{\partial u}{\partial r}}+\cancel{\frac{\partial u}{\partial t}}-\cancel{\frac{\partial u}{\partial s}}=0$

[11] $u=x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\tan^{-1}\left(\frac{x}{y}\right)$

$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$

$=\frac{\partial}{\partial x}\left(x^{\cancel{2}}\cdot\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{\cancel{x}}\right)-2\cancel{y}\cdot\frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{-x}{y^{\cancel{2}}}\right)$

$=\frac{\partial}{\partial x}\left(\frac{x^3}{x^2+y^2}+\frac{2xy}{x^2+y^2}\right)$

$=\frac{\partial}{\partial x}\left(\frac{x^3+2xy}{x^2+y^2}\right)$

$=\frac{(3x^2+2y)(x^2+y^2)-(x^3+2xy)(2x+y^2)}{(x^2+y^2)^2}$

$=\frac{3x^4+3x^2y^2+2x^2y+2y^3-2x^4-x^3y^2-4x^2y-2xy^3}{(x^2+y^2)^2}$

$=\frac{x^4+3x^2y^2-2x^2y+2y^3-x^3y^2-2xy^3}{(x^2+y^2)^2}$

[9] z — x (u, v), y (u, v)

$x=e^u\sin v \qquad y=e^u\cos v$

PT:

(i) $\frac{\partial z}{\partial u}=x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$

$\rightarrow \frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$

$=\frac{\partial z}{\partial x}\cdot e^u\sin v+\frac{\partial z}{\partial y}\cdot e^u\cos v$

$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot x+\frac{\partial z}{\partial y}\cdot y$ proved

ii) $\frac{\partial z}{\partial x}=e^{-u}\left(\sin v\frac{\partial z}{\partial u}+\cos v\frac{\partial z}{\partial v}\right)$

$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$

$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot e^u\cos v+\frac{\partial z}{\partial y}\cdot e^u(-\sin v)$

$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot y+\frac{\partial z}{\partial y}\cdot(-x)$

$\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}(x+y)+\frac{\partial z}{\partial y}(y-x)$

RHS $=$

$e^{-u}\left\{\sin v\left(\frac{\partial z}{\partial x}e^u sv+\frac{\partial z}{\partial y}\cdot e^u cv\right)\right.$

$\left.+\cos v\left(\frac{\partial z}{\partial x}e^u cv+\frac{\partial z}{\partial y}e^u(-sv)\right)\right\}$

$=e^{-u}\left[\frac{\partial z}{\partial x}e^u s^2v+\cancel{\frac{\partial z}{\partial y}e^u cvsv}\right.$

$\left.+\frac{\partial z}{\partial x}e^u c^2v-\cancel{\frac{\partial z}{\partial y}e^u cvsv}\right]$

$=\cancel{e^{-u}}\frac{\partial z}{\partial x}\cancel{e^u}(s^2v+c^2v)$

$=\frac{\partial z}{\partial x}=LHS$