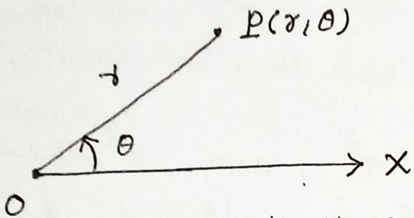
Differential Calculus - I

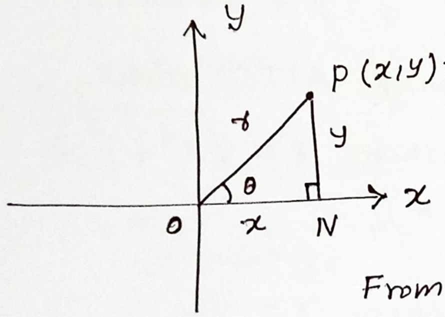
Polar Coordinates:

Let O be a fixed point called as pole and OX be a fixed line, known as initial line or polar axis. Any point P in plane can be described by pair (r, θ) known as polar coordinates of point. Here $r \geq 0$ and $0 \leq \theta \leq 2\pi$, r is known as radius vector and 'θ' is known as vectorial angle.



The equation of a curve in polar form is given by
$$r = f(\theta) \quad \text{or} \quad f(r,\theta) = 0$$
Polar coordinate system is used in situations where geometry is circular

Relation between cartesian coordinates and polar coordinates:



Let O be origin and $P(x,y)$ be any point in xy-plane.
Let $PN \perp x$-axis,
$OP = r$, $PN = y$

From the ΔOPN,
$$\cos(\theta) = \frac{ON}{OP} = \frac{x}{r} \quad \text{and}$$
$$\sin(\theta) = \frac{PN}{OP} = \frac{y}{r}$$

$$\therefore \left. \begin{aligned} x &= r.\cos(\theta) \\ y &= r.\sin(\theta) \end{aligned} \right\} \quad (1) \quad \text{and}$$

eliminating 'r' and 'θ' from these:
$$x^2 + y^2 = r^2, \quad \frac{y}{x} = \tan(\theta)$$
$$\therefore \left. \begin{aligned} r &= +\sqrt{x^2+y^2} \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned} \right\} \quad (2)$$

There is one to one correspondance between two sys. Ang polar

Eqns ① and ② gives the relation between two systems.

Now geometrically $r =$ constant $= a$ (say)

gives, $\sqrt{x^2+y^2} = a$

$\therefore x^2+y^2 = a^2$

Circles with center $(0,0)$ and radius 'a'

Similarly, $\theta =$ constant $= m$ (say) give

$\tan^{-1}\left(\frac{y}{x}\right) = m \quad \therefore \quad y = \tan(m)\, x$

Lines through origin and has slope $\tan(m)$.

The circles : $(x-a)^2+y^2 = a^2$, and $x^2+(y-b)^2 = b^2$

i.e. $x^2+a^2-2ax+y^2 = a^2$, $x^2+y^2+b^2-2by = b^2$

$\therefore x^2+y^2 = 2ax$ and $x^2+y^2 = 2by$

in polar form are :

$r = 2a\cos(\theta)$, $r = 2a\sin(\theta)$ respectively.

NOTE: The 'θ' is described using ASTC rules.

(i) If $(x,y) = (1,1)$, then $r = \sqrt{1^2+1^2} = \sqrt{2}$

$\theta = \tan^{-1}(1/1) = \pi/4$

(ii) If $(x,y) = (-1,1)$, then $r = \sqrt{2}$

$\theta = \tan^{-1}(-1) = -\pi/4$ or $3\pi/4$

(iii) If $(x,y) = (-1,-1)$, then $r = \sqrt{2}$

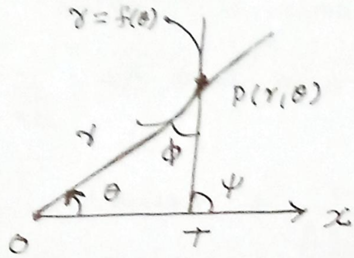
$\theta = \pi + \tan^{-1}(1) = 5\pi/4$

(iv) If $(x,y) = (1,-1)$, then $r = \sqrt{2}$

$\theta = 3\pi/2 + \pi/4 = 7\pi/4$

'r' measures the distance from origin and hence it is positive.

Angle between the radius vector and tangent to a polar curve:



Consider a differentiable polar curve $r = f(\theta)$, $P(r,\theta)$ be any point on the curve, draw PT tangent to curve, which makes an angle 'ψ' with x-axis (or polar axis)

Let $OP = r$ $\angle POT = \theta$ and ϕ is the angle between the radius vector and tangent.

Since exterior angle is equal to sum of interior opposite angles.

$$\psi = \theta + \phi \quad (1)$$

this can be used to find inclination of tangent and slope of tangent is $m = \tan(\psi)$.

Since $x = r.\cos(\theta) = f(\theta)\cos(\theta)$

$y = r\sin(\theta) = f(\theta)\sin(\theta)$

we have: $\frac{dx}{d\theta} = -f(\theta)\sin(\theta) + f'(\theta)\cos(\theta)$

$$\frac{dx}{d\theta} = -r\sin(\theta) + \frac{dr}{d\theta}\cos(\theta)$$

Similarly, $\frac{dy}{d\theta} = f(\theta)\cos(\theta) + f'(\theta)\sin(\theta)$

$$\frac{dy}{d\theta} = r\cos(\theta) + \frac{dr}{d\theta}\sin(\theta)$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos(\theta) + \frac{dr}{d\theta}\sin(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)} \quad (2)$$

but geometrically, $\frac{dy}{dx}$ = slope of tangent

$$\frac{dy}{dx} = \tan(\psi) = \tan(\theta + \phi) \quad \text{using (1)}$$

$$y' = \frac{\tan(\theta) + \tan(\phi)}{1 - \tan(\theta)\tan(\phi)} \quad (3)$$

from eqns. (2) and (3), we get:

$$\frac{\tan(\theta) + \tan(\phi)}{1 - \tan(\theta)\tan(\phi)} = \frac{r\cos(\theta) + \frac{dr}{d\theta}\sin(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)}$$

Multiply th Dividing RHS above equation both sides by $\left\{\frac{1}{\frac{dr}{d\theta}\cos(\theta)}\right\}$

$$\frac{\tan(\theta)+\tan(\phi)}{1-\tan(\theta)\cdot\tan(\phi)} = \frac{r\frac{1}{\left(\frac{dr}{d\theta}\right)} + \tan(\theta)}{1 - r\left(\frac{1}{\frac{dr}{d\theta}}\right)\tan(\theta)}$$

we obtain, $\tan(\phi) = r\cdot\frac{1}{\frac{dr}{d\theta}} = r\frac{d\theta}{dr}$

$$\therefore \boxed{\tan(\phi) = r\frac{d\theta}{dr}}$$

also, $\cot(\phi) = \frac{1}{r}\cdot\frac{dr}{d\theta}$

The above expression is useful to determine the angle between two intersecting polar curves.

It should be noted that since $r = f(\theta)$,

$$\log(r) = \log[f(\theta)]$$

and $\frac{1}{r}\frac{dr}{d\theta} = \left\{\frac{1}{f(\theta)}\cdot f'(\theta)\right\}$, LHS gives $\cot(\phi)$;

hence use of logarithmic differentiation is prefered in examples; however it should be noted that

$$\log_e(a\cdot b) = \log_e(a) + \log_e(b)$$

$$\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$$

$$\log(a^b) = b\log(a)$$

but $\log(a \pm b) \neq \log(a) \pm \log(b)$

$$\log(r) = \log(a) + \log[e^{\theta \cot(\alpha)}]$$

$$\log(r) = \log(a) + \theta \cdot \cot(\alpha)$$

$\therefore$ Diff. w.r.t 'θ' treating $\quad r = f(\theta)$

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + 1 \cdot \cot(\alpha)$$

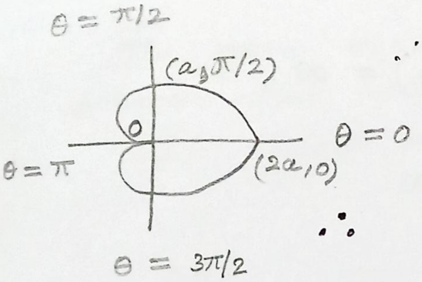
but, $\quad \cot(\phi) = \dfrac{1}{r}\dfrac{dr}{d\theta}$

$$\therefore \quad \cot(\phi) = \cot(\alpha)$$

$$\therefore \quad \phi = \alpha, \quad \text{a constant}$$

2. Find the angle between the tangent and radius vector for the cardioid $r = a(1+\cos\theta)$

Solution:



$$r = a(1+\cos\theta)$$

$$\therefore \ln(r) = \ln(a) + \ln(1+\cos\theta)$$

[here $\ln(r) = \log_e(r)$ notation]

$$\therefore \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1\,(-\sin\theta)}{[1+\cos(\theta)]}$$

$$\therefore \tan(\phi) = r\frac{d\theta}{dr} = -\frac{1+\cos(\theta)}{\sin(\theta)}$$

$$= -\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}$$

$$= -\frac{\cos(\theta/2)}{\sin(\theta/2)} = -\cot(\theta/2)$$

$$= \tan(\pi/2 + \theta/2)$$

$$\therefore \phi = (\pi/2 + \theta/2)$$

3. Determine the angle between the radius vector and tangent to the curve

$$\theta = \frac{1}{a}\sqrt{r^2 - a^2} - \cos^{-1}\left(\frac{a}{r}\right), \quad r \neq 0 \neq a$$

Solution: Here in the example $\theta = g(r)$, hence,

$$\frac{d\theta}{dr} = \frac{1}{a}\,\frac{2r}{2\sqrt{r^2-a^2}} + \frac{1}{\sqrt{1-\left(\frac{a^2}{r^2}\right)}}\left(\frac{-a}{r^2}\right)$$

$$\left[\because \frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}\right]$$

$$= \frac{r}{a}\cdot\frac{1}{\sqrt{r^2-a^2}} - \frac{r}{\sqrt{r^2-a^2}}\left(\frac{a}{r^2}\right)$$

$$= \frac{r}{a}\cdot\frac{1}{\sqrt{r^2-a^2}} - \frac{a}{r}\cdot\frac{1}{\sqrt{r^2-a^2}}$$

$$= \frac{1}{\sqrt{r^2-a^2}}\left(\frac{r}{a} - \frac{a}{r}\right) = \frac{(r^2-a^2)}{ar\sqrt{r^2-a^2}}$$

$$= \frac{\sqrt{r^2-a^2}}{ar}$$

$$\therefore \tan(\phi) = r\frac{d\theta}{dr} = r\cdot\frac{\sqrt{r^2-a^2}}{a\cdot r}$$

$$= \frac{\sqrt{r^2-a^2}}{a}$$

$$\therefore \phi = \tan^{-1}\left(\frac{\sqrt{r^2-a^2}}{a}\right)$$

4. Find the slope of the tangent for the curve

$$\frac{2a}{r} = 1 - \cos(\theta) \quad \text{at} \quad \theta = 2\pi/3$$

Solution: Given $\frac{2a}{r} = 1 - \cos(\theta)$

$$\therefore \log\left(\frac{2a}{r}\right) = \log[1 - \cos(\theta)]$$

$$\log(2a) - \log(r) = \log[1 - \cos(\theta)]$$

Diff. w.r.t 'θ' we get:

$$0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{1\{+\sin(\theta)\}}{[1 - \cos(\theta)]}$$

$$= \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)}$$

$$= \cot(\theta/2)$$

$$\therefore \tan(\phi) = r\frac{d\theta}{dr} = -\frac{1}{\cot(\theta/2)} = -\tan(\theta/2)$$

$$= \tan(-\theta/2)$$

$$\therefore \phi = -\theta/2$$

Now, $\psi = \theta + \phi = (\theta - \theta/2) = \frac{1}{2}\theta$

Slope of tangent $= \tan(\psi) = \tan(\theta/2)$

at $\theta = 2\pi/3$, Slope of tangent $= \tan(\pi/3)$

$$\therefore m = \sqrt{3}$$

Show that for the curve $\log(x^2+y^2) = k.\tan^{-1}\left(\frac{y}{x}\right)$ the angle between the radius vector and the tangent is the same at all points of the curve.

Solution: The curve is in cartesian form:

put $x = r\cos(\theta)$
$y = r\sin(\theta)$

$$\therefore \quad x^2+y^2 = r^2\left[\cos^2(\theta)+\sin^2(\theta)\right]$$

$$\therefore \quad x^2+y^2 = r^2$$

and $\tan(\theta) = \frac{y}{x}$ $\quad \therefore \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

Hence given curve reduces to

$$\log(r^2) = k.\theta$$

$$\therefore \quad 2\log(r) = k.\theta$$

or $\quad \log(r) = \left(\frac{1}{2}k\theta\right)$

$$\therefore \quad \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{2}k(1) = \frac{k}{2}$$

$$\therefore \quad \cot(\phi) = \frac{1}{r}\frac{dr}{d\theta} = \frac{k}{2}$$

$$\phi = \cot^{-1}\left(\frac{k}{2}\right) = \text{a constant}.$$

6. Determine the angle between the radius vector and tangent for the curve $r = a(1+\sin\theta)$ at any point θ; Hence find slope at $\theta = \pi/2$

Solution: Here $r = a(1+\sin\theta)$

$$\frac{dr}{d\theta} = a\cos(\theta)$$

Now $\tan(\phi) = r\frac{d\theta}{dr} = \frac{a(1+\sin\theta)}{a\cos\theta}$

Fin

$$\tan(\phi) = \frac{\{\cos^2(\theta/2) + \sin^2(\theta/2)\} + 2\sin(\tfrac{\theta}{2})\cos(\tfrac{\theta}{2})}{\left[\cos^2(\tfrac{\theta}{2}) - \sin^2(\tfrac{\theta}{2})\right]}$$

$$= \frac{\{\cos(\theta/2) + \sin(\theta/2)\}^2}{\left[\cos(\tfrac{\theta}{2}) - \sin(\tfrac{\theta}{2})\right]\left[\cos(\tfrac{\theta}{2}) + \sin(\tfrac{\theta}{2})\right]}$$

$$(\because a^2 - b^2 = (a-b)(a+b))$$

$$= \frac{[\cos(\theta/2) + \sin(\theta/2)]}{[\cos(\theta/2) - \sin(\theta/2)]} \times \frac{\left(\frac{1}{\cos\theta/2}\right)}{\left(\frac{1}{\cos\theta/2}\right)}$$

$$= \frac{1 + \tan(\theta/2)}{1 - \tan(\theta/2)}$$

$$= \frac{[\tan(\pi/4) + \tan(\theta/2)]}{\{\cancel{\tan}\ 1 - \tan(\pi/4)\tan(\theta/2)\}}$$

$$= \tan(\pi/4 + \theta/2)$$

$$\therefore \boxed{\phi = \left(\frac{\pi}{4} + \frac{\theta}{2}\right)}$$

Since $\psi = \theta + \phi = \frac{\pi}{4} + \frac{\theta}{2} + \theta = \frac{\pi}{4} + \frac{3\theta}{2}$

at $\theta = \pi/2$, $\psi = \frac{\pi}{4} + \frac{3\pi}{4} = \pi$

$\therefore$ slope of tangent is $\tan(\psi) = \tan(\pi) = 0$

7. Find the angle of intersection between the curves

$$r = \sin(\theta) + \cos(\theta) \quad , \quad r = 2\sin(\theta)$$

Sol: Here $r = \sin(\theta) + \cos(\theta)$, $r = 2\sin(\theta)$

$$\therefore \frac{dr}{d\theta} = \cos(\theta) - \sin(\theta) \, , \qquad \frac{dr}{d\theta} = 2\cos(\theta)$$

$$\therefore \tan(\phi_1) = r \cdot \frac{d\theta}{dr} \quad , \qquad \tan(\phi_2) = r \cdot \frac{d\theta}{dr}$$

$$= \frac{\sin(\theta) + \cos(\theta)}{\cos(\theta) - \sin(\theta)} \qquad = \frac{2\sin(\theta)}{2\cos(\theta)}$$

$$\tan(\phi_2) = \tan(\theta)$$

$$= \frac{\cos(\theta)[\tan(\theta) + 1]}{\cos(\theta)[1 - \tan(\theta)]} \qquad \therefore \phi_2 = \theta$$

$$= \frac{\tan(\pi/4) + \tan(\theta)}{1 - \tan(\frac{\pi}{4})\tan(\theta)}$$

$$\tan(\phi_1) = \tan(\pi/4 + \theta) \qquad \therefore \phi_1 = \left(\frac{\pi}{4} + \theta\right)$$

$\therefore$ Angle between the curves is $|\phi_1 - \phi_2|$

$$= \left| \left(\frac{\pi}{4} + \theta\right) - \theta \right| = \frac{\pi}{4}$$

8. Find the angle of intersection between the curves:

$$r = a\log(\theta) \quad \text{and} \quad r = \frac{a}{\log(\theta)}$$

Sol: Here $r = a\log(\theta)$, $r = \dfrac{a}{\log(\theta)}$

$$\therefore \frac{dr}{d\theta} = \frac{a}{\theta} \quad , \qquad \frac{dr}{d\theta} = -\frac{a}{[\log(\theta)]^2}\left(\frac{1}{\theta}\right)$$

$$\therefore \tan(\phi_1) = r\frac{d\theta}{dr} = \frac{a\log(\theta)}{(a/\theta)} = \theta \cdot \log(\theta)$$

$\therefore\ \phi_1 = \tan^{-1}\{\theta \log(\theta)\}$ and $\phi_2 = \tan^{-1}\{-\theta\log(\theta)\}$

$\phi_2 = -\tan^{-1}\{\theta\log(\theta)\}$

$\therefore$ Angle between the curves is :

$|\phi_1 - \phi_2| = 2\tan^{-1}\{\theta\log(\theta)\}$,

but θ is the angle of intersection (it is not explicit value here).

We have to solve for 'θ' by solving the pair of curves :

$$r = a\log(\theta), \quad r = \frac{a}{\log(\theta)}$$

Equating the R.H.S. we have $a\log(\theta) = \dfrac{a}{\log(\theta)}$

i.e. $[\log(\theta)]^2 = 1$

$\therefore\ \log(\theta) = 1 \Rightarrow \theta = e^1 = e$

$\therefore\ |\phi_1 - \phi_2| = 2\tan^{-1}(e) \quad [\because \log_e(e) = 1]$

Show that the curves $r = ae^\theta$ and $re^\theta = b$ intersect orthogonally.

Sol: Here $r = ae^\theta$, $\quad r = be^{-\theta}$

$\therefore\ \dfrac{dr}{d\theta} = ae^\theta, \quad \dfrac{dr}{d\theta} = -be^{-\theta}$

$\therefore\ \tan(\phi_1) = r\dfrac{d\theta}{dr} = \dfrac{ae^\theta}{ae^\theta} = 1$ and

$\tan(\phi_2) = r\dfrac{d\theta}{dr} = -\dfrac{be^{-\theta}}{be^{-\theta}} = -1$

$\therefore\ \tan(\phi_1)\cdot\tan(\phi_2) = (1)(-1) = -1$

$\therefore$ curves intersect orthogonally.

Show that the curves $r = a(1+\sin\theta)$ and $r = a(1-\sin\theta)$ intersect orthogonally.

Sol: Here $r = a(1+\sin(\theta))$ and $r = a(1-\sin\theta)$

$$\therefore \frac{dr}{d\theta} = a\cos(\theta) \quad , \quad \frac{dr}{d\theta} = -a\cos(\theta)$$

$$\tan(\phi_1) = r\frac{d\theta}{dr} = \frac{a(1+\sin\theta)}{a\cos(\theta)} = \frac{1+\sin(\theta)}{\cos(\theta)}$$

and $$\tan(\phi_2) = r.\frac{d\theta}{dr} = -\frac{a(1-\sin(\theta))}{a.\cos(\theta)} = -\frac{[1-\sin(\theta)]}{\cos(\theta)}$$

$$\therefore \tan(\phi_1).\tan(\phi_2) = -\frac{[1+\sin(\theta)]}{\cos\theta}.\frac{[1-\sin(\theta)]}{\cos\theta}$$

$$= -\frac{[1-\sin^2(\theta)]}{\cos^2(\theta)} = -\frac{\cos^2(\theta)}{\cos^2(\theta)} = -1$$

$\therefore$ Given pairs of curves intersect orthogonally.

11. Show that the curves $r = a(1+\cos\theta)$, $r^2 = a^2\cos(2\theta)$ intersect at an angle given by $3\sin^{-1}\left[\left(\frac{3}{4}\right)^{\frac{1}{4}}\right]$

Solution: Given $r = a(1+\cos\theta)$,

$$\frac{dr}{d\theta} = -a\sin(\theta)$$

$$\therefore \tan(\phi_1) = r\frac{d\theta}{dr} = -\frac{a(1+\cos\theta)}{a\sin(\theta)} = -\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}$$

$$= -\cot(\theta/2) = \tan(\pi/2+\theta/2)$$

$$\therefore \phi_1 = (\pi/2+\theta/2) \quad (1)$$

again, $r^2 = a^2\cos(2\theta)$, $\quad 2r\frac{dr}{d\theta} = -2a^2\sin(2\theta)$

$$\therefore \frac{dr}{d\theta} = -\frac{a^2\sin(2\theta)}{r}$$

$$\therefore \tan(\phi_2) = r\frac{d\theta}{dr} = -r.\frac{r}{a^2\sin(2\theta)} = -\frac{a^2\cos(2\theta)}{a^2\sin(2\theta)}$$

$$= -\cot(2\theta) = \tan(\pi/2+2\theta)$$

$$\therefore \phi_2 = (\pi/2+2\theta) \quad (2)$$

Angle between the curves is $|\phi_1 - \phi_2|$

$$= \left|\left(\frac{\pi}{2}+\frac{\theta}{2}\right)-\left(\frac{\pi}{2}+2\theta\right)\right| = \left|-3\theta/2\right| = 3\theta/2$$

We need to find 'θ', from given curves.

$$r = a(1+\cos\theta), \quad r^2 = a^2\cos(2\theta)$$

$$\therefore \quad a^2(1+\cos\theta)^2 = a^2\cos(2\theta)$$

$$1+2\cos(\theta)+\cos^2(\theta) = 2\cos^2(\theta)-1$$

$$\cos^2(\theta)-2\cos(\theta)-2=0$$, this is quadratic eqn.

$$\therefore \quad \cos(\theta) = \frac{2\pm\sqrt{4+8}}{2} = \frac{2\pm 2\sqrt{3}}{2}$$

$$\therefore \quad \cos(\theta) = (1\pm\sqrt{3})$$

but $(1+\sqrt{3})>1$, hence $\cos(\theta) \neq 1+\sqrt{3}$

$\cos(\theta) = 1-\sqrt{3}$ is the only choice

or $\quad 1-\cos(\theta) = \sqrt{3}$

$$2\sin^2\left(\frac{\theta}{2}\right) = \sqrt{3}$$

$$\therefore \quad \sin^2\left(\frac{\theta}{2}\right) = \frac{\sqrt{3}}{2} = \left(\frac{3}{4}\right)^{1/2}$$

$$\therefore \quad \sin\left(\frac{\theta}{2}\right) = \left(\frac{3}{4}\right)^{1/4}$$

$$\therefore \quad \theta = 2\cdot\sin^{-1}\left[\left(\frac{3}{4}\right)^{1/4}\right]$$

$$\therefore \quad |\phi_1-\phi_2| = \frac{3}{2}\theta = 3\cdot\sin^{-1}\left[\left(\frac{3}{4}\right)^{1/4}\right]$$