UNIT-IV
INVERSE LAPLACE TRANSFORM

Topic Learning Objectives:

Upon Completion of this unit, students will be able to:

- Obtain inverse Laplace transform of rational functions.
- Solve ordinary linear differential equations

Definition

Let $L\{f\ (t)\} = F(s)$, then f(t) is defined as the inverse Laplace transform of $F(s)$ and is denoted by $L^{-1}\{F(s)\}$. Thus $L^{-1}\{F(s)\}\ =\ f\ (t)$.

Linearity Property

Let $L^{-1}\{F(s)\}\ =\ f\ (t)$ and $L^{-1}\{G(s)\}\ =\ g(t)$ and a and b be any two constants,
Then $L^{-1}\ [a\ F(s)\ +\ b\ G(s)]\ =\ a\ L^{-1}\{F(s)\}\ +\ b\ L^{-1}\{\ G(s)\}$

Table of Inverse Laplace Transforms

F(s)	$f(t) = L^{-1}\{F(s)\}$
$\frac{1}{s}, s > 0$	1
$\frac{1}{s-a}, s > a$	e^{at}
$\frac{s}{s^2+a^2}, s > 0$	$\cos at$
$\frac{1}{s^2+a^2}, s > 0$	$\frac{\sin at}{a}$
$\frac{1}{s^2-a^2}, s > \lvert a \rvert$	$\frac{\sinh at}{a}$
$\frac{s}{s^2-a^2}, s > \lvert a \rvert$	$\cosh at$
$\frac{1}{s^{n+1}}, s > 0$ $n = 0, 1, 2, 3, \ldots$	$\frac{t^n}{n!}$
$\frac{1}{s^{n+1}}, s > 0$ n $\neq$-1, -2, -3, …..	$\frac{t^n}{\Gamma(n+1)}$

Examples

1. Find the inverse Laplace transforms of the following:

$$(i)\frac{1}{2s-5} \quad (ii)\frac{s+b}{s^2+a^2} \quad (iii)\frac{2s-5}{4s^2+25}+\frac{4s-9}{9-s^2}$$

Solution:

$$(i)\ L^{-1}\left\{\frac{1}{2s-5}\right\}=\frac{1}{2}L^{-1}\left\{\frac{1}{s-5/2}\right\}=\frac{1}{2}e^{\frac{5t}{2}}$$

$$(ii)\ L^{-1}\left\{\frac{s+b}{s^2+a^2}\right\}=L^{-1}\left\{\frac{s}{s^2+a^2}\right\}+b\,L^{-1}\left\{\frac{1}{s^2+a^2}\right\}=\cos at+\frac{b}{a}\sin at$$

$$(iii)\ L^{-1}\left[\frac{2s-5}{4s^2+25}+\frac{4s-8}{9-s^2}\right]=\frac{2}{4}L^{-1}\left\{\frac{s-5/2}{s^2+\frac{25}{4}}\right\}-4L^{-1}\left\{\frac{s-9/2}{s^2-9}\right\}$$

$$=\frac{1}{2}\left[\cos\frac{5t}{2}-\sin\frac{5t}{2}\right]-4\left[\cos h3t-\frac{3}{2}\sin h3t\right]$$

Shifting property [Evaluation of $L^{-1}[F(s-a)]$]

If $L^{-1}[F(s)]=f(t)$, then $L^{-1}[F(s-a)]=e^{at}L^{-1}[F(s)]=e^{at}f(t)$

Examples:

1. Evaluate: $L^{-1}\left\{\frac{3s+1}{(s+1)^4}\right\}$.

Solution:

$$L^{-1}\left\{\frac{3(s+1-1)+1}{(s+1)^4}\right\}=3\,L^{-1}\left\{\frac{1}{(s+1)^3}\right\}-2\,L^{-1}\left\{\frac{1}{(s+1)^4}\right\}$$

$$=3e^{-t}\,L^{-1}\left\{\frac{1}{s^3}\right\}-2e^{-t}\,L^{-1}\left\{\frac{1}{s^4}\right\}$$

Using the formula

$L^{-1}\{\frac{1}{s^{n+1}}\}=\frac{t^n}{n!}$ and taking $n=2$ and 3, we get

$$=\frac{3e^{-t}t^2}{2}-\frac{e^{-t}t^3}{3}.$$

2. Evaluate: $L^{-1}\left\{\frac{s+2}{s^2-2s+5}\right\}$.

Solution:

$$\text{Given} = L^{-1}\left\{\frac{s+2}{(s-1)^2+4}\right\} = L^{-1}\left\{\frac{(s-1)+3}{(s-1)^2+4}\right\} = L^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\} + 3L^{-1}\left\{\frac{1}{(s-1)^2+4}\right\}$$

$$= e^t\, L^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\,e^t\; L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= e^t\;\cos 2t + \frac{3}{2}\,e^t \sin 2t$$

3. Evaluate: $L^{-1}\left\{\frac{2s+1}{s^2+3s+1}\right\}$.

Solution:

$$\text{Given} = 2L^{-1}\left\{\frac{\left(s+\frac{3}{2}\right)-1}{\left(s+\frac{3}{2}\right)^2-\frac{5}{4}}\right\} = 2\left[L^{-1}\left\{\frac{\left(s+\frac{3}{2}\right)}{\left(s+\frac{3}{2}\right)^2-\frac{5}{4}}\right\} - L^{-1}\left\{\frac{1}{\left(s+\frac{3}{2}\right)^2-\frac{5}{4}}\right\}\right]$$

$$= 2\left[e^{\frac{-3t}{2}}\, L^{-1}\left\{\frac{s}{s^2-\frac{5}{4}}\right\} - e^{\frac{-3t}{2}}\, L^{-1}\left\{\frac{1}{s^2-\frac{5}{4}}\right\}\right]$$

$$= 2e^{\frac{-3t}{2}}\left[\cosh\frac{\sqrt{5}}{2}t - \frac{2}{\sqrt{5}}\sinh\frac{\sqrt{5}}{2}t\right].$$

4. Evaluate: $L^{-1}\left\{\frac{2s^2+5s-4}{s^3+s^2-2s}\right\}$.

Solution:

We have $\frac{2s^2+5s-4}{s^3+s^2-2s} = \frac{2s^2+5s-4}{s(s^2+s-2)} = \frac{2s^2+5s-4}{s(s+2)(s-1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-1}$ -------- (1)

Then $2s^2+5s-4 = A(s+2)(s-1) + B\,s\,(s-1) + Cs\,(s+2)$

For $s = 0$, $A = 2$, for $s = 1$, $C = 1$ and for $s = -2$, $B = -1$.

Using these values in (1),

$$L^{-1}\left\{\frac{2s^2+5s-4}{s^3+s^2-2s}\right\}=L^{-1}\left\{\frac{2}{s}\right\}-L^{-1}\left\{\frac{1}{s+2}\right\}+L^{-1}\left\{\frac{1}{s-1}\right\}=2-e^{-2t}+e^{t}$$

5. Use the method of partial fractions to find the time signals corresponding to the Laplace transform function

$$F(s)=\frac{4s+5}{(s+1)^2(s+2)}$$

Solution:

Consider

$$\frac{4s+5}{(s+1)^2(s+2)}=\frac{A}{(s+1)^2}+\frac{B}{s+1}+\frac{C}{s+2}$$

Then $4s+5=A(s+2)+B(s+1)(s+2)+C(s+1)^2$

For $s=-1,\ A=1$, for $s=-2,\ C=-3$

Comparing the coefficients of s^2 to get $B+C=0$ so that $B=3$.

Using these values in (1) to get

$$\frac{4s+5}{(s+1)^2+(s+2)}=\frac{1}{(s+1)^2}+\frac{3}{(s+1)}-\frac{3}{s+2}$$

Hence

$$L^{-1}\left\{\frac{4s+5}{(s+1)^2+(s+2)}\right\}=e^{-t}L^{-1}\left\{\frac{1}{s^2}\right\}+3e^{-t}L^{-1}\left\{\frac{1}{s}\right\}-3e^{-2t}L^{-1}\left\{\frac{1}{s}\right\}$$
$$=te^{-t}+3e^{-t}-3e^{-2t}$$

5. Evaluate: $L^{-1}\left\{\dfrac{s^3}{s^4-a^4}\right\}$.

Solution:

$$Let\ \frac{s^3}{s^4-a^4}=\frac{A}{s-a}+\frac{B}{s+a}+\frac{Cs+D}{s^2+a^2} \qquad ------- \qquad (1)$$

Hence, $s^3=A(s+a)(s^2+a^2)+B(s-a)(s^2+a^2)+(Cs+D)(s^2-a^2)$

For $s=a, A=¼$, *for s = -a, B = ¼,* comparing the constant terms to get

D =a, (A-B) = 0,

Comparing the coefficients of s^3 to get $1=A+B+C$ and so $C=½$.

Using these values in (1),

$$\frac{s^3}{s^4-a^4}=\frac{1}{4}\left[\frac{1}{s-a}+\frac{1}{s+a}\right]+\frac{1}{2}\frac{s}{s^2+a^2}$$

Taking inverse transforms,

$$L^{-1}\left\{\frac{s^3}{s^4-a^4}\right\}=\frac{1}{4}\left[e^{at}+e^{-at}\right]+\frac{1}{2}\cos at=\tfrac{1}{2}[cosh\,at+\cos at]$$

6. Evaluate : $L^{-1}\left\{\frac{s}{s^4+s^2+1}\right\}$.

Solution:

Consider, $\frac{s}{s^4+s^2+1}=\frac{s}{(s^2+s+1)+(s^2-s+1)}=\frac{1}{2}\left[\frac{2s}{(s^2+s+1)(s^2-s+1)}\right]$

$$=\frac{1}{2}\left[\frac{(s^2+s+1)-(s^2-s+1)}{(s^2+s+1)(s^2-s+1)}\right]=\frac{1}{2}\left[\frac{1}{(s^2-s+1)}-\frac{1}{(s^2+s+1)}\right]$$

$$=\frac{1}{2}\left[\frac{1}{(s-\frac{1}{2})^2+\frac{3}{4}}-\frac{1}{(s+\frac{1}{2})^2+\frac{3}{4}}\right]$$

Therefore

$$L^{-1}\left\{\frac{s}{s^4+s^2+1}\right\}=\frac{1}{2}\left[e^{\frac{t}{2}}L^{-1}\left(\frac{1}{s^2+\frac{3}{4}}\right)-e^{-\frac{t}{2}}L^{-1}\left(\frac{1}{s^2+\frac{3}{4}}\right)\right]$$

$$=\frac{1}{2}\left[e^{\frac{t}{2}}\frac{\sin\frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}}-e^{-\frac{t}{2}}\frac{\sin\frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}}\right]$$

$$=\frac{2}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\sin h\left(\frac{t}{2}\right)$$

Evaluation of $L^{-1}[\,e^{-as}F(s)]$

If $L[f(t)]=F(s)$, then $L[f(t-a)H(t-a)]=\ e^{-as}F(s)$

$$L^{-1}[\,e^{-as}F(s)]=f(t-a)H(t-a)$$

Examples

1. $Evaluate: L^{-1}\left\{\frac{e^{-5s}}{(s-2)^4}\right\}$.

Solution:

$\mathrm{a}=5,\ \ F(s)=\frac{1}{(s-2)^4}$

$$f(t)=L^{-1}\{F(s)\}=L^{-1}\left\{\frac{1}{(s-2)^4}\right\}=e^{2t}L^{-1}\left\{\frac{1}{s^4}\right\}\quad=\frac{e^{2t}t^3}{6}$$

$$L^{-1}\left\{\frac{e^{-5s}}{(s-2)^4}\right\} = f(t-a)\,H(t-a)$$

$$= \frac{e^{2(t-5)}(t-5)^3}{6} H(t-5).$$

2. *Evaluate*: $L^{-1}\left[\frac{e^{-\pi s}}{s^2+1} + \frac{se^{-2\pi s}}{s^2+4}\right].$

Solution:

Given $= \{f_1(t-\pi)H(t-\pi) + f_2(t-2\pi)H(t-2\pi)\}$ (1)

$$f_1(t) = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

$$f_2(t) = L^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t$$

(1) leads to

Given $= \sin(t-\pi)\,H(t-\pi) + \cos 2(t-2\pi)\,H(t-2\pi)$

$= -\cos t\; H(t-\pi) + \cos(2t)\,H(t-2\pi).$

Inverse transform of logarithmic and trigonometric functions

If $L[tf(t)] = F(s)$ then $L[tf(t)] = L^{-1}\left[-\frac{d}{ds}F(s)\right] = tf(t)$

In general, $L^{-1}\left[(-1)^n \frac{d^n}{ds^n}F(s)\right] = t^n f(t)$

Examples

1. *Evaluate* $L^{-1}\left\{\log\left(\frac{s+a}{s+b}\right)\right\}.$

Let $F(s) = \log\left(\frac{s+a}{s+b}\right) = \log(s+a) - \log(s+b)$

Then $-\frac{d}{ds}F(s) = -\left[\frac{1}{s+a} - \frac{1}{s+b}\right]$

$$L^{-1}\left[-\frac{d}{ds}F(s)\right] = -\left[e^{-at} - e^{-bt}\right]$$

Or $t\;f(t) = e^{-bt} - e^{-at}$. Thus $f(t) = \frac{e^{-bt} - e^{-at}}{b}.$

2. Evaluate $L^{-1}\left\{\tan^{-1}\frac{a}{s}\right\}.$

Let $F(s) = \tan^{-1}\left(\dfrac{a}{s}\right)$

Then $-\dfrac{d}{ds}F(s) = \left[\dfrac{a}{s^2+a^2}\right]$ or $L^{-1}\left[-\dfrac{d}{ds}F(s)\right] = \sin at$ so that

or $t\,f(t) = \sin at,\quad f(t) = \dfrac{\sin at}{a}.$

Inverse transform of $\frac{F(s)}{s}$

$$\text{Since } L\int_0^t f(t)dt = \frac{F(s)}{s}\text{, we have } L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)dt$$

Examples

1. Evaluate $L^{-1}\left[\dfrac{1}{s(s^2+a^2)}\right]$.

Solution:

Let $F(s) = \dfrac{1}{s^2+a^2}$ so that $f(t) = L^{-1}F(s) = \dfrac{\sin at}{a}$

Then $L^{-1} = \dfrac{1}{s(s^2+a^2)} = L^{-1}\left[\dfrac{F(s)}{s}\right] = \displaystyle\int_0^t \frac{\sin at}{a}dt = \frac{(1-\cos at)}{a^2}$

2. $L^{-1}\left[\dfrac{1}{s^2(s+a)^2}\right]$.

$$L^{-1}\left\{\frac{1}{(s+a)^2}\right\} = te^{-at}$$

$$L^{-1}\left\{\frac{1}{s(s+a)^2}\right\} = \int_0^t e^{-at}t\,dt$$

$$= \frac{1}{a^2}\left[1-e^{-at}(1+at)\right]$$

Now $L^{-1}\left\{\dfrac{1}{s^2(s+a)^2}\right\} = \dfrac{1}{a^2}\displaystyle\int_0^t\left[1-e^{-at}(1+at)\right]dt$

$$= \frac{1}{a^3}\left[at\left(1+e^{-at}\right)+2\left(e^{-at}-1\right)\right].$$

Inverse Laplace transform of F(s) using Convolution theorem

If $L^{-1}[F(s)] = f(t)$ and $L^{-1}[G(s)] = g(t)$, then
$L^{-1}[F(s)G(s)] = \int_0^t f(u)g(t-u)du = f(t) * g(t)$

This expression is called the convolution theorem for inverse Laplace transform.

Examples:

Employ convolution theorem to evaluate the following:

1. $L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\}.$

Solution:

Let $F(s) = \frac{1}{s+a}, G(s)=\frac{1}{s+b}$

Taking inverse transform, we get $f(t) = e^{-at}, g(t) = e^{-bt}$

By convolution theorem,

$$L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\} = \int_0^t e^{-a(t-u)}e^{-bu}du \quad = e^{-at}\int_0^t e^{(a-b)u}du$$

$$= e^{-at}\left[\frac{e^{(a-b)t}-1}{a-b}\right] \quad = \frac{e^{-bt}-e^{-at}}{a-b}.$$

2. $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}.$

Solution:

Let $F(s)=\frac{1}{s^2+a^2}, G(s)=\frac{s}{s^2+a^2}$, then $f(t) = \frac{\sin at}{a}$, $g(t) = \cos at$

By convolution theorem,

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \int_0^t \frac{1}{a}\sin a(t-u)\cos au\, du$$

$$= \frac{1}{a}\int_0^t \frac{\sin at + \sin(at-2au)}{2}du,$$

$$= \frac{1}{2a}\left[u\sin at - \frac{\cos(at-2au)}{-2a}\right]_0^t = \frac{t\sin at}{2a}.$$

(3) $L^{-1}\left\{\frac{s}{(s-1)(s^2+1)}\right\}.$

Solution:

Let $F(s) = \frac{1}{s-1}, G(s)=\frac{s}{s^2+1}$

Then $f(t) = e^t, g(t) = sint$

$$L^{-1}\left\{\frac{1}{(s-1)(s^2+1)}\right\} = \int_0^t e^{t-u}\sin u\,du = e^t\left[\frac{e^{-u}}{2}(-\sin u - \cos u)\right]$$

$$= \frac{e^t}{2}\left[e^{-t}(-\sin t - \cos t) - (-1)\right] = \frac{1}{2}\left[e^t - \sin t - \cos t\right]$$

Exercise:

By employing convolution theorem evaluate the following:

1. $L^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\}$.

2. $L^{-1}\left\{\frac{s}{(s+2)(s^2+9)}\right\}$.

3. $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$.

4. $L^{-1}\left\{\frac{s}{(s^2+a^2)(s^2+b^2)}\right\}, a \neq b.$

5. $L^{-1}\left\{\frac{1}{s^2(s+2)^2}\right\}$.

6. $L^{-1}\left\{\frac{4s+5}{(s-1)^2(s+2)}\right\}$.

Answers:

(ii) $\frac{1}{13}(2\cos 3t + 3\sin 3t - 2e^{-2t})$ (iii) $\frac{1}{2a^3}(\sin at - at\cos at)$ (iv) $\frac{1}{a^2-b^2}(\cos bt - \cos at)$ (v) $\frac{1}{4}(e^{-2t}(t+1)+t-1)$

Laplace Transform Method for Differential Equations

As noted earlier Laplace transform technique is employed to solve initial-value problems. The solution of such a problem is obtained by using the Laplace Transform of the derivatives of function and then the inverse Laplace Transform.

The following are the expressions for the derivatives derived earlier.

$L[f'(t)] = s\,L\,f(t) - f(0)$

$L[f''(t) = s^2\,L\,f(t) - s\,f(0) - f'(0)$

$L\,[f'''(t) = s^3\,L\,f(t) - s^2\,f(0) - s\,f'(0) - f''(0)$

Examples

1. $y'' + 2y' - 3y = \sin t,\ \ y(0) = y'(0) = 0$

Solution:

Taking the Laplace transform of the given equation,

$$[s^2Ly(t) - sy(0) - y'(0)] + 2[s\ Ly(t) - y(0)] - 3\ L\ y(t) = \frac{1}{s^2+1}$$

Using the given conditions, we get

$$[s\,L\{y(t)\} - y(0)] + L\{y(t)\} = \frac{1}{(s+1)^2}$$

Using the given condition, this becomes

$$(s+1)L\{y(t)\} - 2 = \frac{1}{(s+1)^2} \text{ so that } L\{y(t)\} = \frac{2s^2+4s+3}{(s+1)^3}$$

Taking the inverse Laplace transform, we get

$$Y(s) = L^{-1}\left\{\frac{2s^2+4s+3}{(s+1)^3}\right\}$$

$$= L^{-1}\left[\frac{2(s+1-1)^2+4(s+1-1)+3}{(s+1)^3}\right]$$

$$= L^{-1}\left[\frac{2}{s+1} + \frac{1}{(s+1)^3}\right] = \frac{1}{2}e^{-t}(t^2+4)$$

$$L\,[y(t)][s^2+2s-3] = \frac{1}{s^2+1} \text{ or } L\,[y(t)] = \frac{1}{(s-1)(s+3)(s^2+1)} \text{ or}$$

$$y(t) = L^{-1}\left[\frac{1}{(s-1)(s+3)(s^2+1)}\right] = L^{-1}\left[\frac{A}{s-1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}\right]$$

$$= L^{-1}\left[\frac{1}{8}\,\frac{1}{s-1} - \frac{1}{40}\,\frac{1}{s+3} + \frac{-\frac{s}{10}-\frac{1}{5}}{s^2+1}\right] \text{ (method of partial fractions)}$$

$$= \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}(\cos t + 2\sin t).$$

2. Employ Laplace Transform method to solve the integral equation.

$$f(t) = 1 + \int_0^t f(u)\sin(t-u)\,du$$

Solution:

Taking Laplace transform of the given equation, we get

$$L\, f(t) = \frac{1}{s} + L\int_0^t f(u)\sin(t-u)du$$

By using convolution theorem here

$$L\, f(t) = \frac{1}{s} + Lf(t)\cdot L\sin t = \frac{1}{s} + \frac{L\, f(t)}{s^2+1} \text{ Thus}$$

$$L\, f(t) = \frac{s^2+1}{s^3} \quad \text{or} \quad f(t) = L^{-1}\left(\frac{s^2+1}{s^3}\right) = 1 + \frac{t^2}{2}.$$

3. A particle is moving along a path satisfying the equation $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 25x = 0$ where x denotes the displacement of the particle at time t. If the initial position of the particle is at $x = 20$ and the initial speed is 10, find the displacement of the particle at any time t using Laplace transforms.

Solution:

Given equation may be rewritten as

$$x''(t) + 6x'(t) + 25x(t) = 0$$

Here the initial conditions are $x(0) = 20$, $x'(0) = 10$.

Taking the Laplace transform of the equation, we get

$$L\{x(t)\}[s^2+6s+25] - 20s - 130 = 0 \quad or \quad L\{x(t)\} = \frac{20s+130}{s^2+6s+25}$$

so that

$$x(t) = L^{-1}\left[\frac{20s+130}{(s+3)^2+16}\right] \quad = L^{-1}\left[\frac{20(s+3)+70}{(s+3)^2+16}\right]$$

$$= 20\, L^{-1}\frac{s+3}{(s+3)^2+16} + 70\ L^{-1}\frac{1}{(s+3)^2+16}$$

$$= 20\, e^{-3t}\cos 4t + 35\,\frac{e^{-3t}\sin 4t}{2}.$$

4. A voltage Ee^{-at} is applied at t = 0 to a circuit of inductance L and resistance R. Show that the current at any time t is $\frac{E}{R-aL}\left[e^{-at} - e^{-\frac{Rt}{L}}\right]$.

Solution:

The circuit is an LR circuit. The differential equation with respect to the circuit is

$$L\frac{di}{dt} + Ri = E(t)$$

Here, L denotes the inductance, i denotes current at any time t and E(t) denotes the E.M.F.

It is given that $E(t) = E\, e^{-at}$. With this, we have

Thus, we have

$$L\frac{di}{dt}+Ri=Ee^{-at} \text{ or } Li'(t)+R\,i(t)=Ee^{-at}$$

Taking Laplace transform on both sides to get

$$L[L\{i'(t)\}]+R[L\,\{i(t)\}]=E\;L\,\{e^{-at}\}$$

$$L[s\,L\,\{i(t)\}-i(o)]+R[L\,\{i(t)\}]=E\,\frac{1}{s+a}$$

Since $i(0) = 0$, $\quad L\,\{i(t)\}[sL+R]=\dfrac{E}{s+a} \quad or$

$$L\,\{i(t)\}=\frac{E}{(s+a)(sL+R)}$$

Taking inverse transform $L^{-1}\{i(t)\}=L^{-1}\left\{\dfrac{E}{(s+a)(sL+R)}\right\}$

$$=\frac{E}{R-aL}\left[L^{-1}\left\{\frac{1}{s+a}\right\}-L\,L^{-1}\left\{\frac{1}{sL+R}\right\}\right]$$

Thus $i(t)=\dfrac{E}{R-aL}\left[e^{-at}-e^{-\frac{Rt}{L}}\right]$.

5. Mass spring damper system can be modeled using Newton's and Hooke's law. The differential equation representing this system is $\dfrac{d^2x}{dt^2}+6\dfrac{dx}{dt}+25x=4\sin\omega t$ with initial conditions $x(0)=x'(0)=0$.

Solution:

Given equation may be rewritten as

$$x''(t)+6x'(t)+25x(t)=4\sin\omega t$$

Here the initial conditions are $x(0)=20, x'(0)=10$.

Taking the Laplace transform of the equation, we get

$L\{x(t)\}[s^2+6s+25]=\dfrac{4\omega}{s^2+\omega^2}$ or $L\{x(t)\}=\dfrac{4\omega}{(s^2+\omega^2)(s^2+6s+25)}$

On resolving in to partial fractions (with $\omega=2$) leads to

$$x(t)=L^{-1}\left[\frac{8}{(s^2+4)(s^2+6s+25)}\right] \quad =L^{-1}\left[\frac{As+B}{s^2+4}+\frac{Cs+D}{s^2+6s+25}\right]$$

$=$

$$=\frac{4}{195}\;[7\sin 2\,t-4\cos 2\,t]+\frac{2}{195}\;[e^{-3t}(\;8\cos 4\,t-\sin 4\,t)].$$

Exercise:

Employ Laplace transform method to solve the following initial value problems.

(i) $\quad y''+5y'+6y=e^{-2t},\;\; y(0)=y'(0)=1$

(ii) $y''' + 2y'' - y' - 2y = 0, \quad y(0) = 1, \ y'(0) = 2 = y''(0)$

(iii) $y'' + 2y' + 5y = e^{-t}\sin t, \ \ y(0) = 0, \ y'(0) = 1.$

Answers:

(i) $3e^{-2t} - 2e^{-3t} + te^{-2t}$ (ii) $\frac{1}{3}e^{-2t} + \frac{5}{3}e^{-t} - e^{t}$ (iii) $\frac{1}{3}e^{-t}(\sin t + \sin 2t)$

Video links:

https://www.youtube.com/watch?v=6MXMDrs6ZmA
https://www.youtube.com/watch?v=wnnnv4wt-Lw
https://www.youtube.com/watch?v=N-zd-T17uiE
https://www.youtube.com/watch?v=l7nzLD3t4Uc