Numerical Methods - 1

① Interpolation

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad \text{---} \quad x_n \quad x_{n+1}$ $\quad y = f(x)$ is tabulated as:

$y_0 \quad y_1 \quad y_2 \quad y_3 \quad y_4 \quad \text{---} \quad y_n$

→ The process of finding the dependent variable y at some intermediate values of independent variable x is called interpolation.

→ The process of estimating y outside the range $x_0 \text{---} x_n$ is called extrapolation.

② Forward differences

Let $y = f(x)$ be tabulated for the equally spaced values of x as

x - x_0	$x_0 + h$ (x_1)	$x_0 + 2h$ (x_2)	$x_0 + 3h$ (x_3)	----
y - y_0	y_1	y_2	y_3	—

→ The differences $y_1 - y_0$, $y_2 - y_1$, $y_3 - y_2$, --- $y_n - y_{n-1}$

denoted by. $\Delta y_0 \quad \Delta y_1 \quad \Delta y_2 \quad \ldots \quad \Delta y_{n-1}$

where Δ is called forward difference operator or called forward differences

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0)$$
$$= \boxed{y_2 - 2y_1 + y_0}$$

$$\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$$

$$\boxed{\Delta^n y_0 = y_n - {}^nC_1 y_{n-1} + {}^nC_2 y_{n-2} - {}^nC_3 y_{n-3} \text{---} + (-1)^n y_n}$$

$\boxed{\text{Note}}$ → $\boxed{\Delta f(x) = f(x+h) - f(x)}$

③ Backward differences

→ $y_1 - y_0$, $y_2 - y_1$, $y_3 - y_2$, —

$\nabla y_1, \quad \nabla y_2, \quad \nabla y_3, \ldots$

∇ - nabla - Backward difference operator or backward differences

$$\nabla^2 y_4 = \nabla(\nabla y_4) = \nabla(y_4 - y_3) = \nabla y_4 - \nabla y_3 = y_4 - y_3 - (y_3 - y_2)$$
$$= \boxed{y_4 - 2y_3 + y_2}$$

$$\nabla^2 y_2 = y_2 - 2y_1 + y_0 = \Delta^2 y_0$$

$$\therefore \boxed{\nabla^2 y_2 = \Delta^2 y_0}$$

$$\nabla^2 y_3 = \Delta^2 y_1$$

Note – The n^{th} order differences of a polynomial of degree n, all constants; n+1 and above order → $\boxed{\text{differences are zero}}$

i.e., $f(x) = a_0 x^n + a_1 x^{n-1} + \cdots$

$$\boxed{\begin{array}{l}\Delta^n f(x) = a_0\, n!\, h^n \\ \Delta^{n+1} f(x) = 0\end{array}}$$ n – degree of polynomial

Q. $\Delta^{10}\left[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)\right]$

$= \Delta^{10}\left[abcd\, x^{10} + \cdots (\ )x^9 \cdots\right]$

$= abcd \times 10!\, x^0 = \boxed{abcd \times 10!}$

$\because \Delta^{10}(x^n) = 0$ for $\boxed{n < 10}$

④ **Forward difference table**

x	y	Ist diff	IInd diff	IIIrd diff	IVth diff
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

equal spacing

leading entries!

x – argument
y – entry

⑤ **Backward difference table**

x	y	Ist diff	IInd diff	IIIrd diff	IVth diff
x_0	y_0				
		∇y_1			
x_1	y_1		$\nabla^2 y_2$		
		∇y_2		$\nabla^3 y_3$	
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$
		∇y_3		$\nabla^3 y_4$	
x_3	y_3		$\nabla^2 y_4$		
		∇y_4			
x_4	y_4				

leading entries!

Q) Identify the differences → Δy_1, $\Delta^2 y_2$, $\Delta^3 y_1$ for following data.

x	y	I	II	III	IV
3	82				
		14			
5	96		-68		
		-54		136	
7	42		68		-203
		14		-67	
9	56		1		
		15			
11	71				

$\Delta y_1 \to -54$

$\Delta^2 y_2 \to 1$

$\Delta^3 y_1 \to -67$

Q) Consider backward diff. table → ∇y_3, $\nabla^2 y_2$, $\nabla^3 y_4$

x	y	I	II	III	IV	V
1	62					
		-4				
3	58		22			
		18		-62		
5	76		-40		166	
		-22		104		-370
7	54		64		-204	
		42		-100		
9	96		-36			
		6				
11	102					

$\nabla y_3 \to -22$

$\nabla^2 y_2 \to 22$

$\nabla^2 y_4 \to 104$

1) Find first order difference of following. forward!

① e^{ax+b}

$\Delta f(x) = f(x+h) - f(x)$

$\Delta(e^{ax+b}) = e^{a(x+h)+b} - e^{ax+b}$

$= \boxed{e^{ax+b}[e^{ah}-1]}$

② $\log(x+1)$

$\Delta f(x) = f(x+h) - f(x)$

$= \log(x+h+1) - \log(x+1)$

$= \boxed{\log\left(\frac{x+h+1}{x+1}\right)}$

③ $\sin(ax+b)$

$= \sin(a(x+h)+b) - \sin(ax+b)$

$= \sin(ax+ah+b) - \sin(ax+b)$

$= \sin((ax+b)+ah) - \sin(ax+b)$

$= \boxed{2\cos\left[ax+b+\frac{ah}{2}\right]\sin\left[\frac{ah}{2}\right]}$

④ $\tan^{-1}(ax+b)$

$\Delta \tan^{-1}(ax+b) = \tan^{-1}(ax+ah+b) - \tan^{-1}(ax+b)$

$= \tan^{-1}\left[\frac{(ah)}{1+(ax+ah+b)\tan(ax+b)}\right]$

$\tan^{-1}A - \tan^{-1}B$

$= \tan^{-1}\left[\frac{A-B}{1+AB}\right]$

⑤ $\Delta^2(ab^x)$

$\Delta(ab^x) = ab^{x+h} - ab^x$

$= ab^x[b^h-1]$

$\Delta^2(ab^x) = \Delta(ab^x)(b^h-1)$

$= a(b^h-1)[b^{x+h} - b^x]$

$= ab^x(b^h-1][b^h-1]$

$= \boxed{ab^x(b-1)^2}$

$\boxed{h=1}$ step length

- **Newton's Forward Interpolation Formula** [equally spaced x]
  (NFIF)

x	x_0	x_1	x_2	---	x_n
	y_0	y_1	y_2	---	y_n

To interpolate y at some $x \Rightarrow$ $\boxed{x = x_0 + ph}$
where p is a constant $\boxed{\dfrac{x - x_0}{h}}$

The formula is:

$$\boxed{y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots}$$

Note - Generally NFIF is used to interpolate y at some starting / beginning values of x.

- **Newton's Backward Interpolation Formula** [equally spaced x]
  (NBIF)

x	x_0	x_1	x_2	x_3	---
	y_0	y_1	y_2	y_3	---

To interpolate y at some $x \Rightarrow$ $\boxed{x = x_n + ph}$

$$\boxed{p = \frac{x - x_n}{h}}$$

The formula is:

$$\boxed{y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots}$$

Note - Generally NBIF is used to interpolate y at some end values of x.

Q

x	y	I	II	III	IV	V
1	1					
		7				
2	8		12			
		19		6		
3	27		18		0	
		37		6		0
4	64		24		0	
		61		6		
5	125		30			
		91				
6	216					

$f(2.5)$, → F

$f(4.5)$ → B

$f(5.5)$. → B

$f(3.5)$ → 2 ↳ F

Forward

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

$\boxed{x = 2.5}$ $\boxed{x_0 = 1}$, $\boxed{h = 1}$

$$\therefore p = \frac{x - x_0}{h} = \frac{2.5-1}{1} = \boxed{1.5}$$

$$\therefore y_{(2.5)} = 1 + 1.5(7) + \frac{(1.5)(0.5)(12)}{2!} + \frac{(1.5)(0.5)(-0.5) \times 6}{3!}$$

$$= \boxed{15.6250}$$

$y(3.5) =$ ~~$1 + (2.5)(7) + (2.5)$~~

$$p = \frac{3.5-1}{1} = 2.5$$

$$y(3.5) = 1 + 2.5(7) + \frac{(2.5)(1.5)(12)}{2} + \frac{(2.5)(1.5)(0.5)(6)}{6}$$

$$= \boxed{42.8750}$$

Backward

$$y = y_n + p\Delta y_n + \frac{p(p+1)}{2!}\Delta^2 y_n + \frac{p(p+1)(p+2)}{6}\Delta^3 y_n$$

$\rightarrow y(4.5) =$

$$p = \frac{x - x_n}{h} = \frac{4.5-6}{1} = (-1.5)$$

$$y = 216 + (-1.5)91 + \frac{(-1.5)(-0.5)}{2} \times 30$$
$$+ \frac{(-1.5)(-0.5)(0.5)}{6} \times 6$$

$$\boxed{91.1250}$$

$\rightarrow y(5.5)$. $p = \frac{5.5-6}{1} = (-0.5)$

$$y = 216 + (-0.5)(91) + \frac{(-0.5)(0.5)}{2} \times (30)$$
$$+ \frac{(-0.5)(0.5)(1.5)}{6} \times 6$$

$$= \boxed{166.3750}$$

Q) In an experiment values of output y, were recorded for input x from 1.0 to 3.5 at intervals of 0.5,

Estimate by using appropriate interpolation formulae, values of y for input ① $x = 1.2$ ② $x = 3.4$ ③ $x = 3.8$ — extrapolation

F (for ①), B (for ②), B (for ③)

x	y	I	II	III	IV	V
1.0	177					
		-11				
1.5	166		-9			
		-20		13		
2.0	146		4		-16	
		-16		-3		20
2.5	130		1		4	
		-15		1		
3.0	115		2			
		-13				
3.5	102					

$x = 1.2$. $x_0 = 1$

$p = \frac{1.2-1}{0.5} = \frac{0.2}{0.5} = \boxed{0.4}$

$\therefore y = y_0 + p(\Delta y_0) + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{24}\Delta^4 y_0 + \frac{p(p-1)(p-2)(p-3)(p-4)}{120}\Delta^5 y_0$

$= 177 + (0.4)(-11) + \frac{(0.4)(-0.6)}{2}(-9) + \frac{(0.4)(-0.6)(-1.6)}{6} \times 13 + \frac{(0.4)(-0.6)(-1.6)(-2.6)}{24} \times (-16) + \frac{(0.4)(-0.6)(-1.6)(-2.6)(-3.6)}{120} \times (20)$

$= \boxed{175.1366}$

$x = 3.8$. $x_n = 3.5$ $p = \frac{x - x_n}{h} = \frac{3.4-3.5}{0.5} = \frac{-0.1}{0.5} = \boxed{\cancel{-0.2}}$ 0.6

$y = y_n + p(\nabla y_n) + \dots$

$y = 102 + (0.6)(-13) + \frac{(0.6)(1.6)}{2}(2) + \frac{(0.6)(1.6)(2.6)}{6}(1) + \frac{(0.6)(1.6)(2.6)(3.6)}{24}(4) + \frac{(0.6)(1.6)(2.6)(3.6)(4.6)}{120}(20)$

$= \boxed{103.9626}$

Q. Details given below are regarding the monthly income of $\boxed{2000 \text{ adults}}$ in a colony. Using interpolation. Estimate the no. of persons who have monthly income b/w $\boxed{8000 - 10000.}$

Monthly income	Below
<5000	535
5000 - 10000	660
10000 - 15000	470
15000 - 20000	270
20000 - 25000	65

Monthly income below (x)	No. of persons (y)	I	II	III	IV
5000	535	660	-190	-10	5
10000	1195	470	-200	-5	
15000	1665	270	-205		
20000	1935	65			
25000	2000				

Forward $p = \frac{x - x_0}{h} = \frac{8000-5000}{5000} = \boxed{0.6}$

$$y(8000) = 535 + 0.6(660) + \frac{(0.6)(-0.4)(-190)}{2} + \frac{(0.6)(-0.4)(-1.4)(-10)}{6} + \frac{(0.6)(-0.4)(-1.4)(-2.4)(5)}{24}$$

No. of persons having income b/w 8K - 10K $= \boxed{953.0720}$

$y(10{,}000) - y(8000)$

$= 1195 - 953.0720$

$= \boxed{241.9280} \approx \boxed{242}$

- marks secured by 280 candidates in an examination given by following table. using interpolation estimate, no. of candidates who secured marks btw 45 and 50

Table

marks below	candidates	I	II	III	IV	V
30	35					
		49				
40	84		13			
		62		-1		
50	146		12		-45	
		74		-46		105
60	220		-34		60	
		40		14		
70	260		-20			
		20				
80	280					

$$p = \frac{45-30}{10} = \frac{15}{10} = 1.5$$

$$\therefore \Delta(45) = 35 + 1.5(49) + \frac{(1.5)(0.5)}{2}(13) + \frac{(1.5)(0.5)(-0.5)}{6}(-1)$$

$$+ \frac{(1.5)(0.5)(-0.5)(-1.5)}{24}(-45)$$

$$+ \frac{(1.5)(0.5)(-0.5)(-1.5)(-2.5)}{120}(105)$$

$$= \boxed{111.1523}$$

$$\therefore \Delta(50) - \Delta(45)$$

$$= 146 - 111.1523$$

$$= 34.8477$$

$$= \boxed{35}$$

Q Fit an interpolating polynomial for following data and hence find y (12)

x	y	I	II	III	IV
3	1	2	3	0	0
5	3	5	3	0	
7	8	8	3		
9	16	11			
11	27				

For fitting polynomial
→ forward differences

$$p = \frac{x - x_0}{h} = \frac{x-3}{2}$$

$$y(x) = 1 + \frac{(x-3)}{2}(2) + \frac{\frac{(x-3)}{2}\left(\frac{x-3}{2} - 1\right)}{2} \cdot (3)$$

$$= 1 + x - 3 + \frac{(x-3)(x-5) \times 3}{8}$$

$$= x - 2 + \frac{3}{8}[x^2 - 8x + 15]$$

$$= \frac{3}{8}x^2 - 2x + \frac{45}{8} - 2$$

$$= \boxed{\frac{3}{8}x^2 - 2x + \frac{29}{8}}$$

to get y(12) substitute 12

$$= \boxed{33.625}$$

Q Find polynomial also

x	y	I	II	III	IV
0	10	11	-26	78	-199
1	21	-15	52	-66	
2	6	37	-14		
3	43	23			
4	66				

$$p = \frac{x-0}{1}$$

$$10 + x(11) + \frac{x(x-1)(-26)}{2} + \frac{x(x-1)(x-2)(78)}{6} + \frac{x(x-1)(x-2)(x-3)(-144)}{24}$$

$$10 + 11x + (x^2 - x)(13) + \cancel{x(x-1)(x-2)}$$

$$(x^3 - 3x^2 + 2x) \times 13$$

$$- (x-3)(x^3 - 3x^2 + 2x) \times (6)$$

$$- (x^4 - 3x^3 + 2x^2 - 3x^3 + 9x^2 - 6x)\, 6$$

$$= \cancel{x^4(-6) + x^3(13 + 6) + x^2(-13 - 39 - 2 - 9)}$$

$$x^4(-6) + x^3(13 + 36) + x^2(-13 - 39 - 12 - 54)$$

$$+ x(11 + 13 + 26 + 36)$$

$$\boxed{-6x^4 + 49x^3 - 118x^2 + 86x + 10}$$

Q) The ~~tip~~ deflection d measured at various distances x, from one end of a cantilever are given below. Find d, when $x = 0.95$

x	y	I	II	III	IV	V
0.0	0					
		0.025				
0.2	0.025		0.0470			
		0.0820		−0.0295		
0.4	0.117		0.0175		−0.0045	
		0.0995		−0.034		$\boxed{0.0065}$
0.6	0.2165		−0.0165		0.0020	
		0.0830		−0.0320		
0.8	0.2995		−0.0485			
		0.0345				
1.0	0.334					
		(5)	(5)	(5)	(5)	(5)

$$p = \frac{0.95 - 1}{0.2} = \frac{-0.05}{0.2} = \cancel{-0.20} = \boxed{-0.25}$$

$$\therefore y(0.95) = 0.379 + (-0.25)(0.0395) + \frac{(-0.25)(\overset{0.75}{\cancel{-1.25}})}{2}(-0.0485)$$

$$+ \frac{(-0.25)(\overset{0.75}{\cancel{-1.25}})(\overset{1.75}{\cancel{-2.25}})}{6}(-0.0320)$$

$$+ \frac{(-0.25)(\overset{0.75}{\cancel{-1.25}})(\overset{1.75}{\cancel{-2.25}})(\overset{2.75}{\cancel{-3.25}})}{24}(0.0020)$$

$$+ \frac{(-0.25)(\overset{0.75}{\cancel{-1.25}})(\overset{1.75}{\cancel{-2.25}})(\overset{2.75}{\cancel{-3.25}})(\overset{3.75}{\cancel{-4.25}})}{120}(0.0065)$$

$$\boxed{0.3314}$$

---

* Lagrange Interpolation formula

Let $y = f(x)$ be tabulated as, where are not necessarily equal spaced.

x_0	x_1	...	x_n
y_0	y_1	...	y_n

Then the formula, to interpolate y at some x is

$$\boxed{\begin{aligned} y = {} & \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} \times y_0 \\ & + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} \times y_1 \\ & + \cdots \\ & + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} \times y_n \end{aligned}}$$

Imp!

→ Inverse Lagrange's Interpolation Formula

$y = f(x)$ be tabulated as

may be equally spaced or may not be.

x_0	---	x_n
y_0	---	y_n

To interpolate x at some y we use inverse Lagrange's interpolation formula

$$x = \frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)} \times x_0$$

$$+ \frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)} \times x_1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_2)\cdots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\cdots(y_n-y_{n-1})} \times x_n$$

Q From the table find, $y(9)$ and $y(15)$ using Lagrange's interpolation formula

	x	y
x_0	5	50
x_1	6	76
x_2	8	173
x_3	11	476
x_4	12	631
x_5	14	1040

$$y(9) = \frac{(9-76)(9-173)(9-476)(9-631)(9-1040)}{(5-76)(5-173)(5-476)(5-631)(5-1040)} \times 50$$

$$+ \frac{(9-50)(9-173)(9-476)(9-631)(9-1040)}{(6-50)(6-173)(6-476)(6-631)(6-1040)} \times 76$$

$$+ (9-50)(9-76)(9-476)(9-631)(9-1040) \times 173$$

$$y(x) = \frac{(x-6)(x-8)(x-11)(x-12)(x-14)}{(5-6)(5-8)(5-11)(5-12)(5-14)} \times 50$$

$$+ \frac{(x-5)(x-8)(x-11)(x-12)(x-14)}{(6-5)(6-8)(6-11)(6-12)(6-14)} \times 76$$

$$+\frac{(x-5)(x-6)(x-11)(x-12)(x-14)}{(8-5)(8-6)(8-11)(8-12)(8-14)} \times 173$$

$$+\frac{(x-5)(x-6)(x-8)(x-12)(x-14)}{(11-5)(11-6)(11-8)(11-12)(11-14)} \times 476$$

$$+\frac{(x-5)(x-6)(x-8)(x-11)(x-14)}{(12-5)(12-6)(12-8)(12-11)(12-14)} \times 631$$

$$+\frac{(x-5)(x-6)(x-8)(x-11)(x-12)}{(14-5)(14-6)(14-8)(14-11)(14-12)} \times 1040$$

$y(9) = 3.9683 - 19 + 144.1667 + 317.3333 - 225.3571 + 28.8889$

$\boxed{250.0001}$

$\boxed{1300}$

Q) Given that $y(1)=0$, $y(3)=12$, $y(4)=33$, $y(6)=135$. Find polynomial $p(x)$ and hence find using Lagrange interpolation, $y(5)$?

x_0	1	0
x_1	3	12
x_2	4	33
x_3	6	135

$$y(x) = \frac{(x-3)(x-4)(x-6)}{(1-3)(1-4)(1-6)} \times 0$$

$$+\frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)} \times 12$$

$$+\frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)} \times 33$$

$$+\frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)} \times 135$$

$(x-1)[(x^2-10x+24)2 + (x^2-9x+18)-\frac{11}{2} + (x^2-7x+12)\frac{9}{2}]$

$x^3+8x^2+3x-x^2-8x-3$

$\boxed{x^3+7x^2-5x-3}$

$\boxed{x^3-3x^2+5x-3}$

$(x-1)[2x^2-20x+48-\frac{11x^2}{2}+\frac{99x}{2}-\frac{198}{2} + \frac{9x^2}{2}-\frac{63x}{2}+54]$

$(x-1)[x^2-2x+3]$

$x^3-2x^2+3x-x^2+2x-3$

→ Using Lagrange interpolation method.

$y = f(x) = ?$ Find $y(2.5)$

x	y
3	8
2	26
1	32
-1	-40
0	14

$$y = \frac{(x-2)(x-1)(x-(-1))x}{(3-2)(3-1)(3+1)(3)} \times 8$$

$$+ \frac{(x-3)(x-1)(x-(-1))(x)}{(2-3)(2-1)(2+1)(2)} \times 26$$

$$+ \frac{(x-3)(x-2)(x-(-1))x}{(1-3)(1-2)(1+1)(1)} \times 32$$

$$+ \frac{(x-3)(x-2)(x-1)(x)}{(-1-3)(-1-2)(-1-1)(-1)} \times -40$$

$$+ \frac{(x-3)(x-2)(x-1)(x-(-1))}{(-3)(-2)(-1)(1)} \times 14$$

$x^3 - 2x^2 + (2-1-2)x + 2$

$(x^3 - 2x^2 - x + 2)x \times \frac{8}{9}$

$+ (x^3 - 3x^2 - x + 3)x \times \left(-\frac{13}{3}\right)$

$+ (x^3 - 4x^2 + x + 6)x \times 8$

$+ (x^3 - 6x^2 + 11x - 6)x \times -\frac{5}{3}$

$+ (x^3 - 6x^2 + 11x - 6)(x+1) \times -\frac{7}{3}$

$(x-a)(x-b)(x-c)$

$= x^3 - (a+b+c)x^2$

$+ (ab+bc+ca)x$

$- abc$

$x^3 - (3+1+-1)x^2$

$+ (3-1-3)x$

$+ 3$

$x^3 - (3+2-1)x^2$

$+ (6-2-3)x$

$+ 6$

$x^3 - (6)x^2 + (6+2+3)x$

$- (6)$

$x^3 - 6x^2 + 11x - 6$

$2x^3 - 18x^2 + 34x + 40$

Q

x	y
1	10
3	24
5	54
8	129.
x	100

$y(x) = 100$, $x = ?$

Reverse interpolation formula

$$x = \frac{(100-24)(100-54)(100-129)}{(10-24)(10-54)(10-129)} \times 1 + \frac{(100-10)(100-54)(100-129)}{(24-10)(24-54)(24-129)} \times 3 + \frac{(100-10)(100-24)(100-129)}{(54-10)(54-24)(54-129)} \times 5 + \frac{(100-10)(100-24)(100-54)}{(129-10)(129-24)(129-54)} \times 8$$

$$= 1.3831 + 8.1673 + 10.0182 + 2.6860$$

$$= \boxed{5.92}$$

Inverse interpolation $y = 12$, $x = ?$

x	y
12	13.6
20	5.8
25	3.4
30	2

$$x = \frac{(12-5.8)(12-3.4)(10)}{(13.6-5.8)(13.6-3.4)(11.6)} \times 12 + \frac{(12-13.6)(12-3.4)(12-2)}{(5.8-13.6)(5.8-3.4)(3.8)} \times 20 + \frac{(12-13.6)(12-5.8)(10)}{(3.4-13.6)(3.4-5.8)(1.4)} \times 25 + \frac{(12-13.6)(12-5.8)(12-3.4)}{(2-13.6)(2-5.8)(2-3.4)} \times 30$$

$$= 6.9330 + 38.6866 - 72.3623 + 41.4726$$

$$= \boxed{14.7298}$$

Numerical differentiation

let the fun $y = f(x)$ be tabulated as :

equally spaced

$x_0 \quad x_1 \quad -- \quad x_n$

$y_0 \quad y_1 \quad -- \quad y_n$

values given in table

To find $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, at some $x = x_0$ (specified values)

⇒ using Newtons forward interpolation formula.

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} - \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n\right]$$

Q $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ at $x = 1.05$,

1.25.

x	y
1	
1.05	
1.10	
1.15	
1.20	
1.25	
1.30	

x	y	I	II	III	IV	V	VI
1.	1.0247	0.0247	-0.0006	0.0002	-0.0010	0.0035	$\boxed{-0.0089}$
1.05	1.0488	0.0241	-0.0004	-0.0008	+0.0025	-0.0054	
1.10	1.0725	0.0237	-0.0012	0.0017	-0.0029		
1.15	1.095	0.0225	+0.0005	-0.0012			
1.20	1.118	0.0230	-0.0007				
1.25	1.1403	0.0223					
1.30							

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \frac{\Delta^5 y_0}{5} - \frac{\Delta^6 y_0}{6}\right]$$

$$\left(\frac{dy}{dx}\right)_{x=1.05} = \frac{1}{0.05}\left[0.0241 - \left(\frac{-0.0004}{2}\right) + \frac{(-0.0008)}{3} - \frac{0.0025}{4} + \frac{-0.0054}{5}\right]$$

$$= \boxed{0.4966}$$

Q Slider moves along with a fixed rod. Its distance x along the rod for various values of time being are given below. Find the velocity and accⁿ of the slider at t = 0.3.

t	x	I	II	III	IV	V
0.0	29.91	1.71	-0.46	-0.02	0.04	$\boxed{-0.05}$
0.1	31.62	1.25	-0.48	0.02	-0.01	
0.2	32.87	0.77	-0.46	0.01		
→ 0.3	33.64	0.31	-0.45			
0.4	33.95	-0.14				
0.5	33.81					

$$V = \frac{dx}{dt} = \frac{1}{0.1}\left[0.77 + \frac{-0.48}{2} + \frac{-0.02}{3}\right]$$

$$= \boxed{5.2333}$$

$$a = \frac{d^2x}{dt^2} = \frac{1}{(0.1)^2}\left[-0.48 - 0.02\right] = \boxed{-50}$$

Q: θ in radian. Turning of rod recorded for diff values of t (s)
Find angular velocity of rod when t = 0.4, 1.

t	θ	I	II	III	IV	V	VI
0	0						
		0.12					
0.2	0.12		0.25				
		0.37		0.01			
0.4	0.49		0.26		0		
		0.63		0.01		0	
0.6	1.12		0.27		0		0
		0.90		0.01		0	
0.8	2.02		0.28		0		
		1.18		0.01			
1	3.2		0.29				
		1.47					
1.2	4.67						

$$\omega = \frac{d\theta}{dt}\Big|_{0.4} = \frac{1}{h}\left[0.63 - \frac{0.27}{2} + \frac{0.01}{3}\right] \quad (h = 0.2)$$

$$= \boxed{2.4917} \text{ rad/s}$$

$$\omega = \frac{d\theta}{dt}\Big|_{1.0} = \frac{1}{h}\left[1.18 + \frac{0.28}{2} + \frac{0.01}{3}\right] \quad (h = 0.2)$$

$$= \boxed{6.6167} \text{ rad/s}^2$$

Q. A curve passes through points, find slope at $\boxed{x = 2}$

x	y	I	II	III
1	17			
		-8		
2	9		-20	
		-28		112
3	-19		92	
		64		
4	45			

$$\frac{dy}{dx}\Big|_{x=2} = \frac{1}{1}\left[-28 - \frac{92}{2}\right] = \boxed{-74}$$

Find out! $y'(1.1)$, $y''(1.1)$, $y'(1.6)$, $y''(1.6)$.
$y(1.25)$, $y(1.57)$.

x	y	I	II	III	IV	V	VI
1	23.967	1.2420	-0.1080	0.018	-0.006	0.003	$\boxed{0.006}$
1.1	25.209	1.1340	-0.09	0.012	-0.003	0.009	
1.2	26.393	1.0440	-0.078	0.009	0.006		
1.3	27.387	0.9660	-0.069	0.015			
1.4	28.353	0.8970	-0.054				
1.5	29.25	0.8430					
1.6	30.093						

$$\boxed{y'(1.1)} = \frac{1}{0.1}\left[1.1340 + \frac{0.09}{2} + \frac{0.012}{3} + \frac{0.003}{4} + \frac{0.009}{5}\right]$$

$$= \boxed{11.8555}$$

$$\boxed{y''(1.1)} = \frac{1}{(0.1)^2}\left[-0.09 - 0.012 + \frac{11}{12}(-0.003) - \frac{5}{6}(0.009)\right]$$

$$= \boxed{-11.2250}$$

$$\boxed{y'(1.6)} = \frac{1}{0.1}\left[0.8430 - \frac{0.054}{2} + \frac{0.015}{3} + \frac{0.006}{4} + \frac{0.009}{5} + \frac{0.006}{6}\right]$$

$$= \boxed{8.2530}$$

$$\boxed{y''(1.6)} = \frac{1}{(0.1)^2}\left[-0.054 + 0.015 + \frac{11}{12}(0.006) + \frac{5}{6}(0.009)\right]$$

$$= \boxed{-2.6}$$

$$\boxed{y(1.25)} \rightarrow p = \frac{1.25 - 1}{0.1} = \frac{0.25}{0.10} = \frac{25}{10} = \boxed{2.5}$$

$$y = 23.967 + 2.5(1.242) + \frac{(2.5)(1.5)}{2}[-0.108] + \frac{(2.5)(1.5)(0.5)}{6}[0.018]$$

$$+ \frac{(2.5)(1.5)(0.5)(-0.5)}{24}[-0.006] + \frac{(2.5)(1.5)(0.5)(-0.5)(-1.5)}{120}(0.003)$$

$+ \dfrac{(0.5)(1.5)(0.5)(-0.5)(-1.5)(-2.5)}{720}[0.006]$

$\rightarrow 27.0720 - 0.2025 + 0.0056 + 0.0002$

$= \boxed{26.8753} \rightarrow \boxed{y(1.25)}$

Numerical Integration

↳ The process of finding integrand for the numerical data is called numerical integration.

↳ Let the fun $y = f(x)$ be tabulated by means of (x_r, y_r)

$\boxed{x_r = x_0 + rh}$

$r = 0, 1, 2, \ldots$

- **Simpson's 1/3rd rule** — interval multiple of 2.

$y = f(x)$

$x_0 \; x_1 \; \text{---} \; x_n$

$y_0 \; y_1 \; \text{---} \; y_n$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 \ldots)\right]$$

$$= \frac{h}{3}\left[F + L + 4S_0 + 2S_E\right]$$

- **Simpson's 3/8th rule** — interval multiple of 3.

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\left[y_0 + y_n + 3(y_1 + y_2 + y_4 + y_5 + \ldots) + 2(y_3 + y_6 + y_9 + \ldots)\right]$$

- **Weddle's rule** — interval multiple of 6.

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{10}\Big[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6 +$$
$$y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}$$
$$+ y_{12} + 5y_{13} + y_{14} + 6y_{15} + y_{16} + 5y_{17} + y_{18}$$
$$+ \ldots\Big]$$

For 1/3rd rule - no. of intervals must be multiple of 2 or even.
For 3/8th rule - no. of intervals must be multiple of 3
For weddle rule - no. of intervals must be multiple of 6

Q By Newton 1/3rd rule ; 3/8th rule ; weddle rule.

$\int_0^{1.2} y\,dx = ?$

x	y	
0	0.146	(y_0)
0.2	0.161	(y_1)
0.4	0.176	(y_2)
0.6	0.19	(y_3)
0.8	0.204	(y_4)
1.0	0.217	(y_5)
1.2	0.23	(y_6)

ordinates - 7
Intervals - 6

Terms will not repeat ! → Golden rule !.

$$\int_0^{1.2} y\,dx = \frac{0.2}{3}\Big[\underbrace{0.146}_{y_0} + \underbrace{0.23}_{y_6} + 4\big[0.161 + \overset{y_3}{0.19} + \overset{y_5}{0.217}\big] + 2\big[\underbrace{0.176}_{y_2} + \underbrace{0.204}_{y_4}\big]\Big]$$

$\cancel{\boxed{0.2579}}$ $\boxed{0.2272}$ ✓

By 3/8th $= \frac{3}{8}(0.2)\Big[0.146 + 0.23 + 3\big[0.161 + 0.176 + 0.204 + 0.217\big] + 2\big[0.19 + 0.23\big]\Big]$

$= \cancel{\boxed{0.2618}}$

By weddle $= \frac{3}{10}[0.2]\Big[0.146 + 5\times 0.161 + 1\times 0.176 + 6\times 0.19 + 1\times 0.204 + 5\times 0.217 + 1\times 0.23\Big]$

$= \boxed{0.2272}$ ✓

Q) Using simpson's rule find area bounded b/w x = 0 & x = 4

x	0	0.5	1	1.5	2	2.5	3	3.5	4
y	4.6	3.8	2.8	2.2	2.5	3.2	3.8	4.0	4.0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

$$\int_0^4 y \cdot dx = \frac{0.5}{3}\left[4.6 + 4.0 + 4[3.8 + 2.2 + 3.2 + 4.0] + 2[2.8 + 2.5 + 3.8]\right]$$

$$= \boxed{13.2667}$$

Q) $\int_0^1 e^{-x^2} dx$ using simpsons $(3/8)^{th}$ rule?

$$h = \frac{b-a}{n} = \frac{1-0}{6} = \boxed{1/6}$$

	0	1/6	2/6	3/6	4/6	5/6	6/6
$\boxed{y = e^{-x^2}}$	1	0.9726	0.8948	0.7788	0.6412	0.4994	0.3679
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

↓
then CALC

$\int f(x) dx$ $\therefore \frac{3}{8} \times \frac{1}{6}\left[1 + 0.3679 + 3[0.9726 + 0.8948 + 0.6412 + 0.4994] + 2[0.7788]\right]$

$$= \boxed{0.7468}$$

$\int_0^{\pi/2} e^{\sin x} dx$ by dividing in 6 equal parts

$\boxed{\text{Keep in radian mode}}$

x	0	$\pi/12$	$2\pi/12$	$3\pi/12$	$4\pi/12$	$5\pi/12$	$6\pi/12$
$y = e^{\sin x}$	1	1.2954	1.6487	2.0281	2.3774	2.6272	2.7183
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

1/3rd rule

$$= \frac{\pi}{12(3)}\Big[1 + 2.7183 + 4(1.2954 + 2.0281 + 2.6272) + 2(1.6487 + 2.3774)\Big]$$

$$= \boxed{3.1044}$$

3/8 th rule

$$\frac{3}{8}\left[\frac{\pi}{12}\right]\Big[1 + 2.7183 + 3(1.2954 + 1.6487 + 2.3774 + 2.6272) + 2(2.0281)\Big]$$

$$\boxed{3.1043}$$

Weddle's rule

$$= \frac{3}{10} \times \frac{\pi}{12}\Big[1 + 5\times 1.2954 + 1\times 1.6487 + 6\times 2.0281 + 2.3774 + 5\times 2.6272 + 2.7183\Big]$$

$$= \boxed{3.1044}$$

Q The area bounded by the curve $y = f(x)$, the x-axis and the ordinates at $x = 1$, $x = 2.2$, revolves about the x-axis. Find the volume generated if $f(x)$ is given as

x	1	1.2	1.4	1.6	1.8	2.0	2.2
$y = f(x)$	1.5	1.94	2.46	3.06	3.74	4.3	4.52

Required volume = $\boxed{V = \int_1^{2.2} \pi y^2 \, dx}$

y^2	2.25	3.7636	6.0516	9.3636	13.9876	18.49	20.4304
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

In areas and volumes go for $1/3^{rd}$ rule.

$\therefore A = \frac{0.2}{3} [2.25 + 20.4304 + 4(3.7636 + 9.3636 + 18.49) + 2[6.0516 + 13.9876]]$

$\therefore \pi \times 12.6152 = \boxed{39.6317}$ $\boxed{V}$

Q The velocity of a train which starts from rest, ~~at~~ from a station and reaches next station in 12 minutes; is recorded at intervals of 2 minutes as follows.

t (hr)	0/60	2/60	4/60	6/60	8/60	10/60	12/60
t (min)	0	2	4	6	8	10	12
v (kmph)	0	20	36	60	64	42	0

~~v(kmph)~~

Find distance b/w stations.

Can commit a mistake! Take care!

$\therefore d = \frac{2}{60 \times 3} [20 + 42 + 4[36 + 64] + 2[60]]$

$= \boxed{6.4667}$

$3/8^{th}$ $12/60$

$d = \int_0 v \, dt = \frac{3 \times 2}{60 \times 8} [0 + 0 + 3[20 + 36 + 64 + 42] + 2[60]]$

$\boxed{7.575 \text{ km}}$

Q The table given below gives the areas of cross-sections measured in a road-project for earth work computation at intervals of 10 m.

x (in m)	0	10	20	30	40	50	60	70	80	90
A (m^2)	126	248	118	90	320	120	75	180		

Calculate volume of earth work required?

1/3 rd

$$\int_0^{80} A\,dx = \frac{10}{3}\left[126+90+4(248+90+120+180)+2(118+320+75)\right]$$

$$= \boxed{12646.6667}$$

Q A tank is discharging water through an orifice at a depth of x m below the surface of water whose area is A m^2. Values of A at diff values of x are as follows.

x	1.05	1.65	1.80	1.95	2.10	2.25	2.40
A	1.25	1.4	1.5	1.65	1.8	1.95	2.1
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Using formula $\boxed{0.018T = \int_{1.5}^{2.4} \frac{A}{\sqrt{x}}\,dx}$

1/3 of T, the time in seconds for water level to drop from 2.4 m to 1.5 m.

$\frac{A}{\sqrt{x}}$	1.0206	1.0899	1.1180	1.1816	1.2421	1.3	1.3555
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$= \frac{0.15}{3}\left[1.0206+1.3555+4[1.0899+1.1816+1.3]+2[1.1180+1.2421]\right]$$

$$= \frac{1.0691}{0.018} \qquad \boxed{T = 59.3953\ s}$$

Q Evaluate $\int_0^1 \frac{dx}{1+x^2}$, by taking $\overline{n=12}$ and using simpson's 1/3rd, 3/8th, weddle's rule, compare results with exact value

0	1/12	2/12	3/12	4/12	5/12	6/12
1	0.9931	0.9730	0.9412	0.9	0.8521	0.8
y_0	y_1	y_2	y_3	y_4	y_5	y_6

7/12	8/12	9/12	10/12	11/12	12/12
0.7461	0.6923	0.64	0.5902	0.5434	0.5
y_7	y_8	y_9	y_{10}	y_{11}	y_{12}

$$1/3^{rd} = \frac{1}{12\times 3}\Big[1+0.5+4[0.9931+0.9412+0.8521+0.7461+0.64+0.5434] + 2[0.9730+0.9+0.8+0.6923+0.5902]\Big]$$

$$= \boxed{0.7854}$$

$$3/8^{th} = \frac{3\times 1}{8\times 12}\Big[1+0.5+3[0.9931+0.9730+0.9+0.8521+0.7461+0.6923+0.5902+0.5434] + 2[0.9412+0.8+0.64]\Big]$$

$$= \boxed{0.7854}$$

$$\text{weddle} = \frac{3}{10}\times\frac{1}{12}\Big[1+5\times 0.9931+0.9730$$

$\frac{\pi}{4} = a$

$= \boxed{\text{same!}}$

Q Find the approximate mileage travelled by a train b/w 11:50 am to 12:30 pm from the following data.

T	11:50	12:00	12:10	12:20	12:30
v (miles/hr)	24.2	35	41.3	42.8	39.2

25.4 miles

Q The half ordinates in feet of the midship section of a vessel are

12.5 12.8 12.9 13 13 12.8 12.4

10 intervals

11.8 10.4 6.8 0.5

ordinates 2 feet apart. $\boxed{h = 2}$.

Find C.O.G of section $\boxed{(\bar{x}, 0)}$

$$\boxed{\bar{x} = \frac{\int_0^{20} x\,y\,dx}{\int_0^{20} y\,dx}}$$