Laplace_tutorials

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laplace transform...

RV College of Engineering

Department of Mathematics

VECTOR CALCULUS, LAPLACE TRANSFORM & NUMERICAL METHODS (MA221TA)

UNIT-III- LAPLACE TRANSFORM

TUTORIAL SHEET-1

Find the following:

1. $L\left[\sqrt{t}-\frac{1}{\sqrt{t}}\right]=$ ______. Ans: $\sqrt{\pi}\left(\frac{1}{2s^{3/2}}-\frac{1}{s^{1/2}}\right)$
2. $L[e^{-t}2^t]=$ ______. Ans: $\frac{1}{s+1-\log 2}$
3. $L[\sin 4t+\cos t\cos 2t]=$ ______. Ans: $\frac{4}{s^2+16}+\frac{1}{2}\left(\frac{s}{s^2+9}+\frac{s}{s^2+1}\right)$
4. $L[a^t]=$ ______. Ans: $\frac{1}{s-\log a}$

5. Evaluate the Laplace transform of the following signals:

(i) $f(t)=e^{-3t}(t+1)^2$ Ans: $\frac{2}{(s+3)^3}+\frac{2}{(s+3)^2}+\frac{1}{s+3}$

(ii) $f(t)=e^{2t}\sin 3t$ Ans: $\frac{3}{(s-2)^2+9}$

(iii) $f(t)=e^{at}\cosh bt$ Ans: $\frac{s-a}{(s-a)^2-b^2}$

(iv) $f(t)=e^{\frac{3}{2}t}\cos^3 t$ Ans: $\frac{1}{4}\left[\frac{s-\frac{3}{2}}{\left(s-\frac{3}{2}\right)^2+9}+\frac{3\left(s-\frac{3}{2}\right)}{\left(s-\frac{3}{2}\right)^2+1}\right]$

(v) $f(t)=t(1-\sqrt{t}e^t)^3$ Ans: $\frac{1}{s^2}-\frac{15\sqrt{\pi}}{8(s-3)^{7/2}}-\frac{9\sqrt{\pi}}{(s-1)^{5/2}}+\frac{6}{(s-2)^3}$

(vi) $f(t)=e^{at}\cos(bt+a)$ Ans: $\cos a\,\frac{s-a}{(s-a)^2+b^2}-\sin a\,\frac{b}{(s-a)^2+b^2}$

(vii) $f(t)=\sin t\sin 2t\sin 3t+e^{-3t}\cosh^2 2t$

Ans: $\frac{1}{4}\left[\frac{2}{s^2+4}+\frac{4}{s^2+16}-\frac{6}{s^2+36}+\frac{1}{s-1}+\frac{1}{s+7}+\frac{2}{s+3}\right]$

(viii) $f(t)=\begin{cases}t^2, & 0\le t<1\\ 2-t, & 1\le t<2\\ 7, & t\ge 2\end{cases}$. Also draw the graph of f(t).

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Ans: $\frac{e^{-2s}}{s^2}-\frac{3e^{-s}}{s^2}+\frac{7e^{-2s}}{s}-\frac{2e^{-s}}{s^3}+\frac{2}{s^3}$

(ix) $f(t)=\begin{cases}e^t, & 0<t<1\\ 0, & t>1\end{cases}$ Ans: $\frac{1}{1-s}[1+e^{1-s}]$

(x) $f(t)=\sin^5 t$ Ans: $\frac{5}{16(s^2+25)}-\frac{15}{16(s^2+9)}+\frac{5}{8(s^2+1)}$

(xi) $f(t)=\begin{cases}(t-a)^3, & t>a\\ 0, & t<a\end{cases}$ Ans: $\frac{6}{s^4}e^{-as}$

$L(t^n)=\frac{\Gamma(n+1)}{s^{n+1}}$

$\Gamma(1/2)=\sqrt{\pi}$

$\Gamma(n+1)=n\Gamma(n)$

① $L(t^{1/2})-L(t^{-1/2})$

$=\frac{\frac{1}{2}\Gamma(1/2)}{s^{\frac{1}{2}+1}}-\frac{\Gamma(1/2)}{s^{-1/2+1}}$

$=\frac{\sqrt{\pi}}{2s^{3/2}}-\frac{\sqrt{\pi}}{s^{1/2}}$

② $L(e^{-t}2^t)$ $2^t=e^{\log 2^t}=e^{t\log 2}$

$=\frac{1}{s+1-\log 2}$

③ $L(\sin 4t)+L(\cos t\cos 2t)$ $\left\{\frac{2\cos t\cos 2t}{2}=\cos t+\cos 3t\right.$

$=\frac{4}{s^2+16}+L\left(\frac{\cos t+\cos 3t}{2}\right)$

$=\frac{s}{s^2+16}+\frac{1}{2}\left[\frac{s}{s^2+1}+\frac{s}{s^2+9}\right]$

④ $L(a^t)=L\left[e^{\log a^t}\right]$

$=L\left[e^{t\log a}\right]$

$=\frac{1}{s-\log a}$

5(i) $L[f(t)]=L[e^{-3t}(t+1)^2]$

$\rightarrow L[t^2+2t+1]=\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}$

$L[e^{-3t}(t^2+2t+1)]=\frac{2}{(s+3)^3}+\frac{2}{(s+3)^2}+\frac{1}{(s+3)}$

5(ii) $L[\sin 3t]=\frac{3}{s^2+9}$

$L[e^{2t}\sin 3t]=\frac{3}{(s-2)^2+9}$

5(iii) $L[e^{at}\cosh(bt)]$

$\rightarrow L[\cosh(bt)]=\frac{s}{s^2-b^2}$

$L[e^{at}\cosh(bt)]=\frac{s-a}{(s-a)^2-b^2}$

5(iv) $L\left[e^{3/2t}\cos^3 t\right]$

$\rightarrow L[\cos^3 t]=L\left[\frac{3\cos t+\cos 3t}{4}\right]$

$=\frac{1}{4}\left[\frac{3s}{s^2+1}+\frac{s}{s^2+9}\right]$

$=\frac{1}{4}\left[\frac{3(s-3/2)}{(s-3/2)^2+1}+\frac{(s-3/2)}{(s-3/2)^2+9}\right]$

5(v) $L\left[t(1-\sqrt{t}e^t)^3\right]$ $(a-b)^3=a^3-b^3-3ab(a-b)$

$=L\left[t\left(1-t^{3/2}e^{3t}-3\sqrt{t}e^t+3te^{2t}\right)\right]$ $=a^3-b^3-3a^2b+3ab^2$

$=L\left[t-t^{5/2}e^{3t}-3t^{3/2}e^t+3t^2e^{2t}\right]$

$\rightarrow L(t)=\frac{1}{s^2}$, $L(t^{5/2})=\frac{\Gamma(5/2+1)}{s^{5/2+1}}=\frac{5/2\,\Gamma(5/2)}{s^{7/2}}$

$\left\{\Gamma(5/2)=\Gamma(3/2+1)=3/2\,\Gamma(3/2)=3/2\cdot\Gamma(1/2+1)=3/2\cdot 1/2\,\Gamma(1/2)=\frac{3}{4}\sqrt{\pi}\right\}$

$L(t^{5/2})=\frac{5}{2}\cdot\frac{3}{4}\frac{\sqrt{\pi}}{s^{7/2}}=\frac{15\sqrt{\pi}}{8s^{7/2}}$

$\rightarrow L(t^{5/2}e^{3t})=\frac{15\sqrt{\pi}}{8(s-3)^{7/2}}$

$\rightarrow L(t^{3/2})=\frac{3/2\,\Gamma(3/2)}{s^{3/2+1}}=\frac{\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right)}{s^{5/2}}=\frac{\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{5/2}}$

$L(t^{3/2})=\frac{3\sqrt{\pi}}{4s^{5/2}}\Rightarrow L(t^{3/2}e^t)=\frac{3\sqrt{\pi}}{4(s-1)^{5/2}}$

$\rightarrow L(t^2)=\frac{2!}{s^{2+1}}=\frac{2}{s^3}\rightarrow L(t^2e^{2t})=\frac{2}{(s-2)^3}$

$\rightarrow L\left[t-t^{5/2}e^{3t}-3t^{3/2}e^t+3t^2e^{2t}\right]$

$=\frac{1}{s^2}-\frac{15\sqrt{\pi}}{8(s-3)^{7/2}}-\frac{9\sqrt{\pi}}{4(s-1)^{5/2}}+\frac{6}{(s-2)^3}$

5(vi) $L[e^{at}\cos(bt+a)]$

$\rightarrow\cos(bt+a)=\cos(bt)\cos a-\sin(bt)\sin a$

$\{\cos(A+B)=\cos A\cos B-\sin A\sin B\}$

$L[\cos(bt+a)]=\frac{s}{s^2+b^2}\cos a-\frac{b}{s^2+b^2}\sin a$

$L[e^{at}(\cos(bt+a))]=\frac{(s-a)\cos a}{(s-a)^2+b^2}-\frac{b\sin a}{(s-a)^2+b^2}$

5(vii) $L\left[\sin t\sin 2t\sin 3t+e^{-3t}\cosh^2 2t\right]$

$\left\langle\sin A\sin B=\frac{1}{2}[\cos(A-B)-\cos(A+B)]\right\rangle$

$\rightarrow\sin t\sin 2t\sin 3t=\frac{(\cos t-\cos 3t)\sin 3t}{2}$

$=\frac{1}{2}(\cos t\sin 3t-\cos 3t\sin 3t)$

$\left\{\sin A\cos B=\frac{1}{2}[\sin(A+B)+\sin(A-B)],\ \cos A\sin A=\frac{\sin 2A}{2}\right\}$

$=\frac{1}{2}\cdot\frac{1}{2}(\sin 4t+\sin 2t-\sin 6t)$

$L(\sin t\sin 2t\sin 3t)=\frac{1}{4}\left(\frac{4}{s^2+16}+\frac{2}{s^2+4}-\frac{6}{s^2+36}\right)$

$\rightarrow\cosh^2 2t=\frac{1+\cosh 4t}{2}$ $\left\{\cos^2 x=\frac{1+\cos 2x}{2}\right\}$

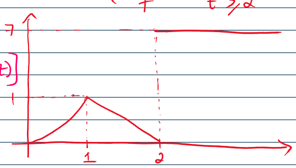
$\rightarrow L(\cosh^2 2t)=\frac{1}{2}\left(\frac{1}{s}+\frac{s}{s^2-16}\right)$

$L(e^{-3t}\cosh^2 2t)=\frac{1}{2}\left(\frac{1}{s+3}+\frac{s+3}{(s+3)^2-16}\right)$

$\rightarrow L\left[\sin t\sin 2t\sin 3t+e^{-3t}\cosh^2 2t\right]$

$=\frac{1}{4}\left(\frac{4}{s^2+16}+\frac{2}{s^2+4}-\frac{6}{s^2+36}\right)+\frac{1}{2}\left(\frac{1}{s+3}+\frac{s+3}{(s+3)^2-16}\right)$

5(viii) $f(t)=\begin{cases}t^2 & 0\le t<1\\ 2-t & 1\le t<2\\ 7 & t\ge 2\end{cases}$



$f(t)=t^2+(2-t-t^2)H(t-1)+(5+t)H(t-2)$

$\rightarrow L(t^2)=\frac{2!}{s^{2+1}}=\frac{2}{s^3}$ —①

$\rightarrow$ let $f(t-1)=2-t-t^2$

$f(t)=2-(t+1)-(t+1)^2$

$2-t-1-t^2-1-2t$

$f(t)=-t^2-3t$

$L(f(t))=\frac{-2!}{s^{2+1}}-3\cdot\frac{1!}{s^{1+1}}=\frac{-2}{s^3}-\frac{3}{s^2}=F(s)$

$\rightarrow L[(2-t-t^2)H(t-1)]=L[f(t-1)H(t-1)]$

$=e^{-s}F(s)$

$=\frac{-2e^{-s}}{s^3}-\frac{3e^{-s}}{s^2}$ —②

$\rightarrow f(t-2)=5+t$

$f(t)=7+t$

$L[f(t)]=7\cdot\frac{0!}{s^{0+1}}+\frac{1!}{s^{1+1}}=\frac{7}{s}+\frac{1}{s^2}=F(s)$

$\rightarrow L[(5+t)H(t-2)]=L[f(t-2)H(t-2)]$

$=e^{-2s}F(s)$

$=\frac{7e^{-2s}}{s}+\frac{e^{-2s}}{s^2}$ —③

5(ix) $f(t)=e^t+(0-e^t)H(t-1)$

$=e^t-(e^t)H(t-1)$

$\rightarrow f(t-1)=e^t$ $\rightarrow L(e^t)=\frac{1}{s-1}$ —①

$f(t)=e^{t+1}=e^t\cdot e^1$

$L(f(t))=\frac{e}{s-1}=F(s)$

$\rightarrow L[e^tH(t-1)]=L[f(t-1)H(t-1)]$

$=e^{-s}F(s)$

$=e^{-s}\cdot\frac{e}{s-1}=\frac{e^{1-s}}{s-1}$ —②

$\rightarrow L(e^t-e^tH(t-1))=$ ① $-$ ② $=\frac{1}{s-1}-\frac{e^{1-s}}{s-1}$

$=\frac{1}{s-1}(1-e^{1-s})$

5(x) $f(t)=\sin^5 t$

$=\sin t(\sin^2 t)^2$

$=\sin t\left(\frac{1-\cos 2t}{2}\right)^2$

$=\frac{1}{4}\sin t\left(1+\cos^2 2t-2\cos 2t\right)$

$=\frac{1}{4}\left[\sin t+\sin t\left(\frac{1+\cos 4t}{2}\right)-2\sin t\cos 2t\right]$

$=\frac{1}{4}\left[\sin t+\frac{1}{2}\sin t+\frac{1}{2}\sin t\cos 4t-(\sin 3t-\sin t)\right]$

$\langle 2\cos A\sin B=\sin(A+B)-\sin(A-B)\rangle$

$=\frac{1}{4}\left[\frac{3}{2}\sin t+\frac{1}{4}(\sin 5t-\sin 3t)-\sin 3t+\sin t\right]$

$=\frac{1}{4}\left[\frac{5}{2}\sin t+\frac{1}{4}\sin 5t-\frac{5}{4}\sin 3t\right]$

$=\frac{1}{4}\left[\frac{5}{2(s^2+1)}+\frac{5}{4(s^2+25)}-\frac{15}{4(s^2+9)}\right]$

$=\frac{5}{8(s^2+1)}+\frac{5}{16(s^2+25)}-\frac{15}{16(s^2+9)}$

5(xi) $f(t)=(t-a)^3+(0-(t-a)^3)H(t-a)$

$\rightarrow$ let $g(t-a)=(t-a)^3$

$g(t)=t^3$

$L(g(t))=\frac{3!}{s^{3+1}}=\frac{6}{s^4}=g(s)$

$\rightarrow L[(t-a)^3H(t-a)]$

$=L[g(t-a)H(t-a)]$

$=e^{-as}g(s)$

$=\frac{6e^{-as}}{s^4}$ —①

$\rightarrow L[(t-a)^3]=L[t^3-a^3-3t^2a+3ta^2]$

$= \frac{6e^{-as}}{s^4}$ — (1)

$\rightarrow L[(t-a)^3] = L[t^3 - a^3 - 3t^2a + 3ta^2]$

$= \frac{6}{s^4} - \frac{a^3}{s} - \frac{6a}{s^3} + \frac{3a^2}{s^2}$ — (2)

$\rightarrow L[f(t)] = (2) + (1)$

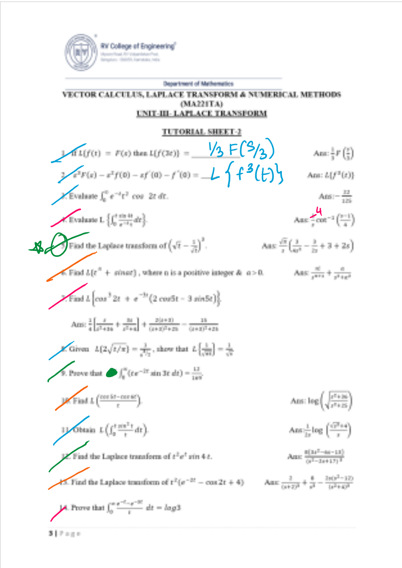
$\rightarrow L[(5+t)H(t-2)] = L[f(t-2)H(t-2)]$
$= e^{-2s}F(s)$

$= \frac{7e^{-2s}}{s} + \frac{e^{-2s}}{s^2}$ — (3)

$\rightarrow L[t^2 + (2-t-t^2)H(t-1) + (5+t)H(t-2)]$

$= (1) + (2) + (3)$

$= \frac{2}{s^3} - \frac{2e^{-s}}{s^3} - \frac{3e^{-s}}{s^2} + \frac{7e^{-2s}}{s} + \frac{e^{-2s}}{s^2}$



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UNIT-III- LAPLACE TRANSFORM

TUTORIAL SHEET-2

1. If $L(f(t)) = F(s)$ then $L(f(3t)) =$ ___ $\frac{1}{3}F(s/3)$ Ans: $\frac{1}{3}F\left(\frac{s}{3}\right)$
2. $s^3F(s) - s^2f(0) - sf'(0) - f''(0) =$ ___ $L\{f^3(t)\}$ Ans: $L\{f'''(t)\}$
3. Evaluate $\int_0^\infty e^{-t}t^2 \cos 2t\, dt$. Ans: $-\frac{22}{125}$
4. Evaluate $L\left\{\int_0^t \frac{\sin 4t}{e^{-t}t}dt\right\}$. Ans: $\frac{4}{s}\cot^{-1}\left(\frac{s-1}{4}\right)$
5. Find the Laplace transform of $\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3$. Ans: $\frac{\sqrt{\pi}}{s}\left(\frac{3}{4s^2} - \frac{3}{2s} + 3 + 2s\right)$
6. Find $L[t^n + \sin at]$, where n is a positive integer & $a > 0$. Ans: $\frac{n!}{s^{n+1}} + \frac{a}{s^2+a^2}$
7. Find $L\left[\cos^3 2t + e^{-3t}(2\cos 5t - 3\sin 5t)\right]$.

   Ans: $\frac{1}{4}\left[\frac{s}{s^2+36} + \frac{3s}{s^2+4}\right] + \frac{2(s+3)}{(s+3)^2+25} - \frac{15}{(s+3)^2+25}$
8. Given $L\{2\sqrt{t/\pi}\} = \frac{1}{s^{3/2}}$, show that $L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}}$
9. Prove that $\int_0^\infty (te^{-2t}\sin 3t\, dt) = \frac{12}{169}$.
10. Find $L\left(\frac{\cos 5t - \cos 6t}{t}\right)$. Ans: $\log\left(\sqrt{\frac{s^2+36}{s^2+25}}\right)$
11. Obtain $L\left(\int_0^t \frac{\sin^2 t}{t}dt\right)$. Ans: $\frac{1}{2s}\log\left(\frac{\sqrt{s^2+4}}{s}\right)$
12. Find the Laplace transform of $t^2e^t \sin 4t$. Ans: $\frac{8(3s^2-6s-13)}{(s^2-2s+17)^3}$
13. Find the Laplace transform of $t^2(e^{-2t} - \cos 2t + 4)$ Ans: $\frac{2}{(s+2)^3} + \frac{8}{s^3} - \frac{2s(s^2-12)}{(s^2+4)^3}$
14. Prove that $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}dt = \log 3$

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(3) $\int_0^\infty e^{-t}t^2 \cos 2t\, dt$

$\rightarrow L(\cos 2t) = \frac{s}{s^2+4}$

$L(t^2\cos 2t) = (-1)^2 \frac{d^2}{ds^2}[L(\cos 2t)]$

$= \frac{d}{ds}\left[\frac{1(s^2+4) - s(2s)}{(s^2+4)^2}\right]$

$= \frac{d}{ds}\left[\frac{4-s^2}{(s^2+4)^2}\right]$

$= \frac{-2s(s^2+4)^{\cancel{2}} - (4-s^2)\,2(\cancel{s^2+4})(2s)}{(s^2+4)^{\cancel{4}\,3}}$

$= \frac{-2s^3 - 8s - 16s + 4s^3}{(s^2+4)^3}$

$= \frac{2s^3 - 24s}{(s^2+4)^3}$

$L(e^{-t}t^2\cos 2t) = \frac{2(s+1)^3 - 24(s+1)}{((s+1)^2+4)^3}$

$\int_0^\infty e^{-t}t^2\cos 2t = \frac{2(1)^3 - 24(0+1)}{((0+1)^2+4)^3}$

$= \frac{2-24}{5^3} = \frac{-22}{125}$

(4) $L\left\{\int_0^t \frac{\sin 4t}{e^{-t}t}dt\right\}$

$\rightarrow L(\sin 4t) = \frac{4}{s^2+16}$

$L\left(\frac{\sin 4t}{t}\right) = \int_s^\infty \frac{4}{s^2+16}$

$= 4\tan^{-1}\left(\frac{s}{4}\right)\Big|_s^\infty$

$= 4[\pi/2 - \tan^{-1}(s/4)]$

$= 4\cot^{-1}(s/4)$

$\rightarrow L\left(\frac{e^t\sin 4t}{t}\right) = 4\cot^{-1}\left(\frac{s-1}{4}\right)$

$\rightarrow L\int_0^t \frac{e^t\sin 4t}{t}dt = \frac{4}{s}\cot^{-1}\left(\frac{s-1}{4}\right)$

$\Gamma(1/2) = \Gamma(-1/2+1)$

$\Gamma(1/2) = -1/2\,\Gamma(-1/2)$

$\rightarrow \Gamma(-1/2) = -2\Gamma(1/2)$
$= -2\sqrt{\pi}$

(5) $\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3 = t^{3/2} - t^{-3/2} - 3t^{1/2} + 3t^{-1/2}$

$\rightarrow L(t^{3/2}) = \frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{s^{3/2+1}} = \frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{5/2}} = \frac{3\sqrt{\pi}}{4s^{5/2}}$

$L(t^{-3/2}) = \frac{\Gamma(-3/2+1)}{s^{-3/2+1}} = \frac{\Gamma(-1/2)}{s^{-1/2}} = \frac{-2\sqrt{\pi}}{s^{-1/2}}$

$= \frac{1/2\,\Gamma(1/2)}{s^{-1/2}} = \frac{\sqrt{\pi}}{2s^{-1/2}}$

$L(t^{1/2}) = \frac{\Gamma(1/2+1)}{s^{1/2+1}} = \frac{1/2\,\Gamma(1/2)}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$

$L(t^{-1/2}) = \frac{\Gamma(-1/2+1)}{s^{-1/2+1}} = \frac{\Gamma(1/2)}{s^{1/2}} = \frac{\sqrt{\pi}}{s^{1/2}}$

(9) $e^{2t}\int_0^\infty te^{-2t}\sin 3t\, dt$

$\rightarrow L(\sin 3t) = \frac{3}{s^2+9}$

$L(t\sin 3t) = (-1)^1\frac{d}{ds}\frac{3}{s^2+9}$

$= -\frac{(-3(2s))}{(s^2+9)^2} = \frac{6s}{(s^2+9)^2}$

$L(e^{-2t}t\sin 3t) = \frac{6(s+2)}{[(s+2)^2+9]^2}$

$\int_0^\infty te^{-2t}\sin 3t = \frac{6(2)}{[2^2+9]^2} = \frac{12}{169}$

(6) $L(t^n + \sin at)$
$= \frac{n!}{s^{n+1}} + \frac{a}{s^2+a^2}$

(7) $L\{\cos^3 2t + e^{-3t}(2\cos 5t - 3\sin 5t)\}$

$\cos^3 2t = \frac{3\cos 2t + \cos 6t}{4}$

$L(\cos^3 2t) = \frac{1}{4}\left[3\frac{s}{s^2+4} + \frac{s}{s^2+36}\right] \rightarrow (1)$

$L(2\cos 5t - 3\sin 5t) = \frac{2s}{s^2+25} - \frac{15}{s^2+25}$

$L[e^{-3t}(2\cos 5t - 3\sin 5t)]$
$= \frac{2(s+3) - 15}{(s+3)^2+25}$ — (2)

$L[\sim\sim] = (1) + (2)$

(8) $L\{2\sqrt{t/\pi}\} = \frac{1}{s^{3/2}}$

$\rightarrow L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}}$ (S.T)

$\rightarrow$ let $f(t) = 2\sqrt{t/\pi}$ $\quad t^{1/2}$

$f'(t) = \frac{2}{\sqrt{\pi}}\cdot\frac{1}{2\sqrt{t}} = \frac{1}{\sqrt{\pi t}}$

$L\{f'(t)\} = s^1 L\{f(t)\} - s^{1-1}f(0)$

$L\left\{\frac{1}{\sqrt{\pi t}}\right\} = s\cdot\frac{1}{s^{3/2}} - 0$

$L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}}$ H.P

(10) $L\left(\frac{\cos 5t - \cos 6t}{t}\right)$

$\rightarrow L(\cos 5t - \cos 6t) = \frac{s}{s^2+25} - \frac{s}{s^2+36}$

$\rightarrow L\left\{\frac{\cos 5t - \cos 6t}{t}\right\} = \int_s^\infty \frac{s}{s^2+25} - \frac{s}{s^2+36}$

$= \frac{1}{2}\log|s^2+25| - \frac{1}{2}\log|s^2+36|\Big|_s^\infty$

$= \frac{1}{2}\log\left|\frac{s^2+25}{s^2+36}\right|\Big|_s^\infty$

$= -\frac{1}{2}\log\left(\frac{s^2+25}{s^2+36}\right) = \log\sqrt{\frac{s^2+36}{s^2+25}}$

(11) $L\left(\int_0^t \frac{\sin^2 t}{t}dt\right)$

$\rightarrow L(\sin^2 t) = L\left(\frac{1 - \cos 2t}{2}\right)$

$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$

$\rightarrow L\left(\frac{\sin^2 t}{t}\right) = \int_s^\infty \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)ds$

$= \frac{1}{2}\left[\log s - \frac{1}{2}\log|s^2+4|\right]_s^\infty$

$= \frac{1}{2}\left[\log\frac{s}{\sqrt{s^2+4}}\right]_s^\infty = \frac{1}{2}\log\frac{\sqrt{s^2+4}}{s}$

$\rightarrow L\left(\int_0^t \frac{\sin^2 t}{t}dt\right) = \frac{1}{2s}\log\frac{\sqrt{s^2+4}}{s}$

(12) $L(t^2e^t\sin 4t)$

$\rightarrow L(\sin 4t) = \frac{4}{s^2+16}$

$\rightarrow L(t^2\sin 4t) = (-1)^2\frac{d^2}{ds^2}\frac{4}{s^2+16}$

$= \frac{d}{ds}\left\{\frac{-4(2s)}{(s^2+16)^2}\right\}$

$= -\left[\frac{8(s^2+16)^{\cancel{2}} - 8s(2(\cancel{s^2+16}))(2s)}{(s^2+16)^{\cancel{4}\,3}}\right]$

$= -\left[\frac{8s^2 + 8\times 16 - 32s^2}{(s^2+16)^3}\right]$

$= \frac{24s^2 - 128}{(s^2+16)^3} = \frac{8(3s^2-16)}{(s^2+16)^3}$

$\rightarrow L(e^t t^2\sin 4t) = \frac{8(3(s-1)^2 - 16)}{[(s-1)^2+16]^3}$

$= \frac{8[3s^2 - 6s - 13]}{[s^2 - 2s + 17]^3}$

(13) $L\{t^2(e^{-2t} - \cos 2t + 4)\}$

$\rightarrow L(e^{-2t} - \cos 2t + 4)$
$= \frac{1}{s+2} - \frac{s}{s^2+4} + \frac{4}{s}$

$\rightarrow L\{t^2(e^{-2t} - \cos 2t + 4)\}$

$= (-1)^2\frac{d^2}{ds^2}\left[\frac{1}{s+2} - \frac{s}{s^2+4} + \frac{4}{s}\right] \Rightarrow -(4-s^2)$ s^2-4.

$= \frac{d}{ds}\left[\frac{-1}{(s+2)^2} - \left(\frac{1(s^2+4) - s(2s)}{(s^2+4)^2} + \frac{-4}{s^2}\right]\right.$

$= \frac{2}{(s+2)^3} + \frac{2s(s^2+4)^{\cancel{2}} - (s^2-4)(2(\cancel{s^2+4}))(2s)}{(s^2+4)^{\cancel{4}\,3}} + \frac{4(2)}{s^3}$

$= \frac{2}{(s+2)^3} + \frac{2s^3 + 8s - 4s^3 + 16s}{(s^2+4)^3} + \frac{8}{s^3}$

$= \frac{2}{(s+2)^3} + \frac{24s - 2s^3}{(s^2+4)^3} + \frac{8}{s^3}$

$= \frac{2}{(s+2)^3} - \frac{2s(s^2-12)}{(s^2+4)^3} + \frac{8}{s^3}$

(14) $\rightarrow L(e^{-t} - e^{-3t}) = \frac{1}{s+1} - \frac{1}{s+3}$

$\rightarrow L\left(\frac{e^{-t} - e^{-3t}}{t}\right) = \int^\infty \left(\frac{1}{\ldots} - \frac{1}{\ldots}\right)$

$$= \frac{8\,|3s^2-6s-13|}{[s^2-2s+17]^3}$$

$(s+2)^2 \qquad (s^2+4)^3 \qquad s^3$

(14) $\to L\{e^{-t}-e^{-3t}\} = \frac{1}{s+1} - \frac{1}{s+3}$

$\to L\left(\frac{e^{-t}-e^{-3t}}{t}\right) = \int_s^\infty \frac{1}{s+1} - \frac{1}{s+3}$

$= \log|s+1| - \log|s+3| \Big|_s^\infty$

$= \log\left|\frac{s+1}{s+3}\right| \Big|_s^\infty = \log\left(\frac{s+3}{s+1}\right)$

$\to \int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\,dt = \log\left|\frac{0+3}{0+1}\right| = \log 3$

RV College of Engineering

Department of Mathematics

VECTOR CALCULUS, LAPLACE TRANSFORM & NUMERICAL METHODS (MA221TA)

UNIT-III- LAPLACE TRANSFORM

TUTORIAL SHEET-3

1. A rectangular wave $f(t)$ of period $2a$, $a>0$ is defined by
$f(t) = \begin{cases} E, & 0 \le t \le a \\ -E, & a < t \le 2a \end{cases}$
Show that $L\{f(t)\} = \frac{E}{s}\tanh\left(\frac{as}{2}\right)$. Also draw the graph of the wave function.

2. A periodic function of period $2\pi/\omega$ is defined by
$f(t) = \begin{cases} E\sin\omega t, & 0 \le t < \pi/\omega \\ 0, & \pi/\omega \le t \le 2\pi/\omega \end{cases}$, where E & ω are positive constants.
Show that $L\{f(t)\} = \frac{E\omega}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$. Also draw the graph of the function.

3. Find the Laplace transform of function $f(t) = t\,u(t-4) - t^2\delta(t-2)$
Ans: $\frac{e^{-4s}}{s^2}(1+4s) - 4e^{-2s}$

4. Evaluate $\int_0^\infty t^m(\log t)^n\,\delta(t-3)\,dt$. Ans: $3^m \log 3^n$

5. Express $f(t) = \begin{cases} \sin t, & 0 < t \le \pi/2 \\ \cos t, & t > \pi/2 \end{cases}$ in terms of the unit step function and hence find its Laplace Transform. Ans: $\frac{1}{s^2+1}\left[1 - e^{-\frac{\pi s}{2}}(s+1)\right]$

6. Express $f(t) = \begin{cases} 2, & 0 < t \le \pi \\ 0, & \pi < t < 2\pi \\ \sin t, & t > 2\pi \end{cases}$ in terms of the unit step function and hence find its Laplace Transform.

7. Express $f(t) = \begin{cases} t^2, & 0 < t \le 2 \\ 4t, & t > 2 \end{cases}$ in terms of the unit step function and hence find its Laplace Transform. Ans: $\frac{2}{s^3} + e^{-2s}\left(-\frac{2}{s^3} + \frac{4}{s}\right)$

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8. Find Laplace of periodic function $f(t)$ of period $2a$, $a>0$ defined by
$f(t) = \begin{cases} t, & 0 \le t \le a \\ -t+2a, & a < t \le 2a \end{cases}$. Also draw the graph of the function. Ans: $\frac{1}{s^2}\tanh\frac{as}{2}$

7] $f(t) = t^2 + (4t - t^2)H(t-2)$

$\to L(t^2) = \frac{2}{s^3}$

$\to$ let $g(t-2) = 4t - t^2$

$g(t) = 4(t+2) - (t+2)^2$

$= \cancel{4t} + 8 - t^2 - 4 - \cancel{4t}$

$= 4 - t^2$

$L\{g(t)\} = \frac{4}{s} - \frac{2t}{s^3}$

$L\{g(t-2)H(t-2)\} = e^{-as}G(s)$

$= e^{-2s}\left(\frac{4}{s} - \frac{2t}{s^3}\right)$

$\to L\{f(t)\} = \frac{2}{s^3} + e^{-2s}\left(\frac{4}{s} - \frac{2t}{s^3}\right)$

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6) $f(t) = 2 + (0-2)H(t-\pi) + (\sin t - 0)H(t-2\pi)$

$\to L(2) = \frac{2}{s}$

$\to$ let $g(t-\pi) = -2$

$g(t) = -2$

$L\{g(t)\} = \frac{-2}{s} = G(s)$

$L\{g(t-\pi)H(t-\pi)\} = e^{-as}G(s)$

$= \frac{-2e^{-\pi s}}{s}$

$\to$ let $j(t-2\pi) = \sin t$

$j(t) = \sin(t+2\pi)$

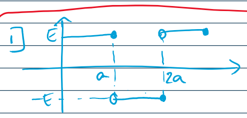
$= \sin t$

$L\{j(t)\} = \frac{1}{s^2+1} = J(s)$

$L\{g(t-2\pi)H(t-2\pi)\} = e^{-as}J(s)$

$\to L\{f(t)\} = \frac{2}{s} - \frac{2e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s^2+1}$

1]



$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$

$= \frac{1}{1-e^{-s(2a)}}\left[\int_0^a e^{-st}E\,dt + \int_a^{2a} e^{-st}(-E)\,dt\right]$

$= \frac{1}{1-e^{-2as}}\left[\frac{Ee^{-st}}{-s}\Big|_0^a - \frac{Ee^{-st}}{-s}\Big|_a^{2a}\right]$

$= \frac{1}{1-e^{-2as}}\left[\frac{-E}{s}e^{-sa} + \frac{E}{s} + \frac{E}{s}e^{-2as} - \frac{E}{s}e^{-sa}\right]$

$= \frac{1}{1-e^{-2as}}\left[\frac{E}{s}\left(1 - 2e^{-sa} + e^{-2as}\right)\right]$

$= \frac{E\,(1-e^{-as})^2}{s(1-e^{-2as})}$ let $x = e^{-as}$

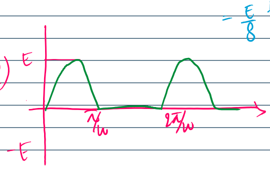
$= \frac{E}{s}\,\frac{(1-x)^2}{1-x^2}$

$= \frac{E}{s}\,\frac{(1-x)^{\cancel{2}}}{\cancel{(1-x)}(1+x)} = \frac{E}{s}\,\frac{1-e^{-as}}{1+e^{-as}}$

$= \frac{E}{s}\,\frac{e^{\frac{as}{2}} - e^{-\frac{as}{2}}}{e^{\frac{as}{2}} + e^{-\frac{as}{2}}}$

$= \frac{E}{s}\tanh\left(\frac{as}{2}\right)$

2)



LATE

$L\{f(t)\} = \frac{1}{1-e^{-s\left(\frac{2\pi}{\omega}\right)}}\left[\int_0^{\pi/\omega} E\sin\omega t\, e^{-st}\,dt + 0\right]$

$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\;\frac{E\omega\left(1+e^{-s\pi/\omega}\right)}{s^2+\omega^2}$

$= \frac{E\omega}{(s^2+\omega^2)\left(1-e^{-\frac{\pi s}{\omega}}\right)}$

3] $f(t) = t\,u(t-4) - t^2\delta(t-2)$

$\to$ let $f(t-4) = t$

$f(t) = t+4$

$L\{f(t)\} = \frac{1}{s^2} + \frac{4}{s} = F(s)$

$\to L\{f(t-4)u(t-4)\} = e^{-as}F(s)$

$= e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right) = \frac{e^{-4s}(1+4s)}{s^2}$

$\to f(t) = t^2 \quad a = 2$

$f(a) = f(2) = 2^2 = 4$

$\to L\{t^2\delta(t-2)\} = e^{-2s}(4)$

$\to L\{t\,u(t-4) - t^2\delta(t-2)\}(s)$

$= \frac{e^{-4s}(1+4s)}{s^2} - 4e^{-2s}$

4] $\int_0^\infty t^m(\log t)^n\,\delta(t-3)\,dt$

$\to$ let $f(t) = t^m(\log t)^n$

$a = 3$,

$f(a) = f(3) = 3^m(\log 3)^n$

$\to L\{t^m(\log t)^n\delta(t-3)\}$

$= e^{-3s}3^m(\log 3)^n$

$\to \int_0^\infty t^m(\log t)^n\delta(t-3)\,dt$

$= 3^m(\log 3)^n$

5] $f(t) = \sin t + (\cos t - \sin t)H(t-\pi/2)$

$\to L(\sin t) = \frac{1}{s^2+1}$

$\to$ let $g(t-\pi/2) = \cos t - \sin t$

$g(t) = \cos(t+\pi/2) - \sin(t+\pi/2)$

$= -\sin t - \cos t$

$L\{g(t)\} = \frac{-1}{s^2+1} - \frac{s}{s^2+1} = \frac{-(1+s)}{s^2+1} = F(s)$

$\to L\{g(t-\pi/2)H(t-\pi/2)\} = e^{-\frac{\pi s}{2}}F(s)$

$= \frac{-e^{-\pi s/2}(1+s)}{s^2+1}$

$\to L\{f(t)\} = \frac{1 - e^{-\pi s/2}(1+s)}{s^2+1}$

$I = \int_0^{\pi/\omega} \sin\omega t\, e^{-st}\,dt$

$I = \sin\omega t\,\frac{e^{-st}}{-s} - \int \omega\cos\omega t\,\frac{e^{-st}}{-s}\,dt$

$= -\frac{\sin\omega t\, e^{-st}}{s} + \frac{\omega}{s}\left(\cos\omega t\,\frac{e^{-st}}{-s} - \int \omega\sin\omega t\,\frac{e^{-st}}{-s}\right)$

$I = -\frac{\sin\omega t\, e^{-st}}{s} - \frac{\omega}{s^2}\left(\cos\omega t\, e^{-st} + \omega I\right)$

$I + \frac{\omega^2}{s^2}I = -\frac{\sin\omega t\, e^{-st}}{s} - \frac{\omega\cos\omega t\, e^{-st}}{s^2}$

$\frac{(s^2+\omega^2)}{s^{\cancel{2}}}\,I = -\frac{\sin\omega t\, e^{-st}}{\cancel{s}} - \frac{\omega\cos\omega t\, e^{-st}}{s^{\cancel{2}}}\Big|_0^{\pi/\omega}$

$= 0 + \omega\left(\frac{e^{-\frac{s\pi}{\omega}}}{s} + \frac{1}{s}\right)$

$I = \frac{\cancel{s}}{s^2+\omega^2} \times \frac{\omega}{\cancel{s}}\left(1 + e^{-s\pi/\omega}\right)$

8] $f(t) = \frac{1}{1-e^{-sT}}\Big(\underbrace{\int_0^a e^{-st}\,t\,dt}_{I_1} + \underbrace{\int_a^{2a} e^{-st}(-t+2a)\,dt}_{I_2}\Big)$

$I_1 = \int_0^a e^{-st}t\,dt$ ILATE

$= t\,\frac{e^{-st}}{-s} - \int 1\,\frac{e^{-st}}{-s}\,dt$

$= -\frac{t}{s}e^{-st} - \frac{1}{s}\frac{e^{-st}}{s}\Big|_0^a \to (1)$

$= -\frac{a}{s}e^{-as} - \frac{1}{s^2}e^{-as} - \left(0 - \frac{1}{s^2}\right)$

$I_1 = \frac{1}{s^2} - \frac{ae^{-as}}{s} - \frac{1}{s^2}e^{-as}$

$I_2 = \int_a^{2a} e^{-st}\,t\,dt + 2a\int_a^{2a} e^{-st}\,dt$

$= \frac{t}{s}e^{-st} + \frac{1}{s^2}e^{-st}\Big|_a^{2a} + 2a\frac{e^{-st}}{-s}\Big|_a^{2a}$

$= \frac{2a}{s}e^{-2as} + \frac{1}{s^2}e^{-2as} - \frac{a}{s}e^{-as} - \frac{1}{s^2}e^{-as} - \frac{2ae^{-2as}}{s} + \frac{2ae^{-as}}{s}$

$I_2 = \frac{1}{s^2}e^{-2as} - \frac{1}{s^2}e^{-as} + \frac{a}{s}e^{-as}$

$\to I_1 + I_2 = \frac{1}{s^2} - \frac{2}{s^2}e^{-as} + \frac{1}{s^2}e^{-2as}$

$= \frac{1}{s^2}\left(1 - 2e^{-as} + e^{-2as}\right)$

$= \frac{(1-e^{-as})^2}{s^2}$

$\to f(t) = \frac{1}{(1-e^{-2as})}\,\frac{(1-e^{-as})^2}{s^2}$

$= \frac{1}{s^2}\,\frac{(1-e^{-as})^{\cancel{2}}}{\cancel{(1-e^{-as})}(1+e^{-as})}$

$= \frac{1}{s^2}\,\frac{(1-e^{-as})}{(1+e^{-as})} = \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$