unit3_tutorials

R V COLLEGE OF ENGINEERING
(An autonomous institution affiliated to VTU, Belgaum)
DEPARTMENT OF MATHEMATICS

FUNDAMENTS OF LINEAR ALGEBRA, CALCULUS AND NUMERICAL METHODS (MAT211AT)
Multivariable Functions and Partial Differentiation

TUTORIAL SHEET-1

1. If $x = r\cos\theta$, $y = r\sin\theta$, then $\left(\frac{\partial y}{\partial\theta}\right)^2 + \left(\frac{\partial x}{\partial\theta}\right)^2 =$ r^2 Ans: r ?
2. If $z = x\sin y + y\sin x$, then $\frac{\partial^2 z}{\partial x\partial y} =$ $\cos y + \cos x$ Ans: $\cos y + \cos x$
3. If $z = e^{2x^2+xy}$, then $\frac{\partial z}{\partial y} =$ xe^{2x^2+xy} Ans: xe^{2x^2+xy}
4. The steady state temperature of a metal sheet is $T(x,y) = x^2 - a^2y^2$. The values of 'a' for which $T(x,y)$ satisfies the Laplace equation $T_{xx} + T_{yy} = 0$ are ±1. Ans: $\pm a$
5. If $u = y\cos(xy)$ then $\frac{\partial u}{\partial y}$ at the point $(1,\pi)$ is -1. Ans: -1
6. If $V = f(x-ct) + g(x+ct)$ where f and g are arbitrary functions of $x-ct$ and $x+ct$ respectively and c is a constant, then show that $\frac{\partial^2 V}{\partial t^2} = c^2\frac{\partial^2 V}{\partial x^2}$
7. If $u = \frac{x+y}{x-y}$, verify that $\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial^2 u}{\partial y\partial x}$
8. If $u = ae^{-gx}\sin(nt - gx)$, where a, g and n are positive constants and $\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial x^2}$, show that $g = \sqrt{\frac{n}{2\mu}}$.
9. If V is the volume and S is the total surface area of rectangular box of length x, breadth y and height z, find
   (i) the rate of change of V with respect to x if y =4 and z= 12.
   (ii) the rate of change of S with respect to z if x = 3 and y = 4.
   Ans: (i) 48 (ii)14
10. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, then show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$.

$a^3+b^3+c^3 = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$
$-3abc$

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TUTORIAL SHEET-2

1. Given $z = xy^2 + x^3y$ where x and y are functions of t with $x(1) = 1$, $y(1) = 2$, $x'(1) = 3$ and $y'(1) = 4$. The value of $\frac{dz}{dt}$ at t = 1 is 50. Ans: 16 ?
2. For the implicit function $(\cos x)^y = (\sin y)^x$, $\frac{dy}{dx} =$ ____.
   Ans: $-\left[\frac{y(\cos x)^{y-1}(-\sin x)-(\sin y)^x\log(\sin y)}{(\cos x)^y\log(\cos x)-x(\sin y)^{x-1}\cos y}\right]$
3. Given 't' represents time and $u = x^2 - y^2$, $x = \frac{1}{t}$, $y = e^t$ then the rate of change of u with respect to 't' is ____. Ans: $\frac{-2}{t^3} - 2e^{2t}$
4. For the implicit function $e^x - e^y = 2xy$, $\frac{dy}{dx} =$ ____. Ans: $\left[\frac{e^x-2y}{e^y+2x}\right]$
5. Given, $x^2 + y^2 + 3xz = 1$ and $x + y = 1$, then $\frac{dz}{dx} =$ ____ Ans: $\frac{-(2x+3)}{3} + \frac{2z}{3x}$
6. If $z = z(x,y)$, $x = e^u\sin v$, $y = e^v\cos v$, then $\frac{\partial z}{\partial u}$ ____. Ans: $\frac{\partial z}{\partial x}e^u\sin v$
7. If $u = xyz$ where $x = e^{-t}$, $y = e^{-t}\sin^2 t$, $z = \sin t$, then find $\frac{du}{dt}$.
   Ans: $e^{-2t}\sin^2 t(3\cos t - 2\sin t)$
8. If $z = x^2 + 2xy + 4y^2$ and $y = e^{3x}$, find $\frac{dz}{dx}$. Ans: $2(x + e^{3x}) + 2(x + 4e^{3x})3e^{3x}$
9. If z is a function of x and y and if $x = e^u\sin v$, $y = e^u\cos v$, prove that
   (i) $\frac{\partial z}{\partial u} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$
   (ii) $\frac{\partial z}{\partial x} = e^{-u}\left(\sin v\frac{\partial z}{\partial u} + \cos v\frac{\partial z}{\partial v}\right)$
10. If $u = f(x-y, y-z, z-x)$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.
11. If $u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$, determine $\frac{\partial^2 u}{\partial x\partial y}$

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TUTORIAL SHEET-3

1. Match the following:

i)	If $x = e^u\sin v$, $y = e^v\cos v$ then $J\left(\frac{x,y}{u,v}\right) =$ ____.	a)	$\frac{1}{4v\sin 2u}$
ii)	If (a,b) is a stationary point of f(x,y) and $f_{xx} = 3$, $f_{xy} = 2$ and $f_{yy} = 2$ at this point then the nature of (a,b) is____.	b)	minimum
iii)	The nature of the point (1, -1) to the function $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 7$ is ____.	c)	$e^{u+v}(\sin v\cos v - \sin^2 v)$
		d)	$\frac{v\sin 2u}{4}$
iv)	If $\frac{\partial(x,y)}{\partial(u,v)} = v\sin 2u$ and $\frac{\partial(x,y)}{\partial(r,\theta)} = \frac{1}{4}$ then $\frac{\partial(u,v)}{\partial(r,\theta)} =$ ____.	e)	Neither maximum nor minimum
		f)	Saddle point
		g)	maximum
		h)	$e^{u+v}(\sin v\cos v - \sin^2 v) - e^u\cos v$

Ans: (i) - (c) (ii) -(g) (iii) – (e) (iv) – (a)

2. Find the extreme values of $\sin x + \sin y + \sin(x+y)$. **Ans:** Maximum value at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, maximum value is $\frac{3\sqrt{3}}{2}$
3. Find the volume of largest rectangular parallelepiped that can be inscribed in an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Ans: Maximum volume $= \frac{8abc}{3\sqrt{3}}$
4. Find the maximum and minimum distances of the point (1, 2, 3) from the sphere $x^2 + y^2 + z^2 = 56$ using Lagrange's Method of undetermined multipliers. **Ans:** Minimum distance at (2, 4, 6) = $\sqrt{14}$, Maximum distance at (-2,-4,-6), maximum distance = $\sqrt{126}$
5. Show that $u = \frac{x^2-y^2}{x^2+y^2}$, $v = \frac{2xy}{x^2+y^2}$ are functionally dependent and find the relation between them. Ans: $u^2 + v^2 = 1$.
6. For $u = xyz$, $v = yz + zx + xy$, $w = x + y + z$, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$. Ans: $(y-z)(z-x)(x-y)$.
7. If $x = e^v\sec u$, $y = e^v\tan u$, then verify that $J' = 1$.

① $y = r\sin\theta$; $\left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2$
$\frac{dy}{d\theta} = r\cos\theta$ $= r^2\cos^2\theta + r^2\sin^2\theta$
$x = r\cos\theta$ $= r^2$
$\frac{dx}{d\theta} = -r\sin\theta$

② $z = x\sin y + y\sin x$
$\frac{d}{dx}\left(\frac{dz}{dy}\right) = \frac{d}{dx}(x\cos y + \sin x)$
$= \cos y + \cos x$

③ $z = e^{2x^2+xy}$
$\frac{dz}{dy} = xe^{2x^2+xy}$

④ $T(x,y) = x^2 - a^2y^2 = y$
$T_{xx} + T_{yy} = 0$
$\frac{d^2y}{dx^2} + \frac{d^2y}{dy^2} = 0$
$\frac{d}{dx}(2x) + \frac{d}{dy}(-a^2(2y)) = 0$
$2 - 2a^2 = 0$
$2 = 2a^2$
$a^2 = 1 \Rightarrow a = \pm1$

⑤ $u = y\cos(xy)$
$\frac{\partial u}{\partial y} = (1)\cos(xy) + y(-\sin(xy))\cdot x$
at $x = 1, y = \pi$
$= \cos\pi - \pi\cancel{\sin\pi}^{\,0}$
$= -1$

⑥ $V = f(x-ct) + g(x+ct)$
→ $\frac{dV}{dt} = f'(x-ct)\cdot(-c) + g'(x+ct)\cdot c$
$\frac{d^2V}{dt^2} = +c^2f''(x-ct) + c^2g''(x+ct)$
→ $\frac{dV}{dx} = f'(x-ct) + g'(x+ct)$
$\frac{d^2V}{dx^2} = f''(x-ct) + g''(x+ct)$
⇒ $\frac{d^2V}{dt^2} = c^2\frac{d^2V}{dx^2}$

⑦ $u = \frac{x+y}{x-y}$; $\frac{d^2u}{dydx} = \frac{d}{dy}\left(\frac{du}{dx}\right)$

$\frac{x+y}{x-y}$

$\frac{d^2u}{dxdy} = \frac{d}{dx}\left(\frac{du}{dy}\right)$
$= \frac{d}{dx}\left(\frac{(1)(x-y)-(x+y)(-1)}{(x-y)^2}\right)$
$= \frac{d}{dx}\left(\frac{x-\cancel{y}+x+\cancel{y}}{(x-y)^2}\right)$
$= \frac{d}{dx}\left(\frac{2x}{(x-y)^2}\right)$
$= 2\left(\frac{(1)(x-y)^2 - x(2(x-y))}{(x-y)^4}\right)$
$= 2\left(\frac{x^2+y^2-\cancel{2xy}-2x^2+\cancel{2xy}}{(x-y)^4}\right)$
$= 2\frac{(y^2-x^2)}{(x-y)^4}$ (=)

$= \frac{d}{dy}\left(\frac{(1)(x-y)-(x+y)(1)}{(x-y)^2}\right)$
$= \frac{d}{dy}\left(\frac{\cancel{x}-y-\cancel{x}-y}{(x-y)^2}\right)$
$= \frac{d}{dy}\left(\frac{-2y}{(x-y)^2}\right)$
$= -2\left(\frac{(1)(x-y)^2 - y(2(x-y)(-1))}{(x-y)^4}\right)$
$= -2\frac{(x^2+y^2-\cancel{2xy}+\cancel{2xy}-2y^2)}{(x-y)^4}$
$= \frac{-2(x^2-y^2)}{(x-y)^4} = \frac{2(y^2-x^2)}{(x-y)^4}$

⑧ $u = ae^{-gx}\sin(nt-gx)$
→ $\frac{\partial u}{\partial t} = ae^{-gx}\cos(nt-gx)\cdot n$
→ $\frac{\partial^2u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$
$= \frac{\partial}{\partial x}\left(a\left(e^{-gx}\cos(nt-gx)(-g) + e^{-gx}(-g)\sin(nt-gx)\right)\right)$
$= \frac{\partial}{\partial x}\left(-age^{-gx}\left(\cos(nt-gx) + \sin(nt-gx)\right)\right)$
$= -ag\left[e^{-gx}(-g)(\cos X + \sin X) + e^{-gx}(-\sin X(-g) + \cos X(-g))\right]$
$= ag^2e^{-gx}(\cos X + \cancel{\sin X} - \cancel{\sin X} + \cos X)$
$= 2ag^2e^{-gx}\cos(nt-gx)$

$\frac{du}{dt} = \mu\frac{d^2u}{dx^2}$
$\cancel{a}\cancel{e^{-gx}}\cancel{\cos(nt-gx)}\cdot n = \mu\cdot2\cancel{a}g^2\cancel{e^{-gx}}\cancel{\cos(nt-gx)}$
$n = \mu(2g^2)$
$g = \sqrt{\frac{n}{2\mu}}$

⑨ $V = xyz$ $S = 2(xy + yz + xz)$
(i) $\left.\frac{\partial V}{\partial x}\right|_{y=4,\,z=12} = yz = 4\times12 = 48$
(ii) $\left.\frac{\partial S}{\partial z}\right|_{x=3,\,y=4} = 2(y+x) = 2(4+3) = 14$

⑩ $u = \log(x^3+y^3+z^3-3xyz)$
$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$
→ $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$
→ $\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}\times(3x^2-3yz)$
→ $\frac{\partial u}{\partial y} = \frac{1}{''}\times(3y^2-3xz)$
→ $\frac{\partial u}{\partial z} = \frac{1}{''}\times(3z^2-3xy)$
$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{1}{''}(3x^2+3y^2+3z^2-3yz-3xz-3xy)\right)$
$\left(\frac{3(x^2+y^2+z^2-xy-yz-xz)}{x^3+y^3+z^3-3xyz}\right)$
$\left(\frac{3(\;'')}{(x+y+z)(\;'')}\right)$
$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$
$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \dots$
$= 3\left(\frac{-1}{(x+y+z)^2}\right) + \dots$
$= -\frac{3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2} + \dots$
$= \frac{-9}{(x+y+z)^2}$

$\boxed{1}$ $z = xy^2 + x^3y$ $z \to x, y \to t$
$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$
$= [y^2 + 3x^2y](x'(t)) + [x(2y) + x^3](y'(t))$
at $t = 1$, $x = 1$, $y = 2$, $x'(1) = 3$, $y'(1) = 4$
$\frac{dz}{dt} = [2^2 + 3(1)^2(2)](3) + [(1)(2\times2) + (1)^3](4)$
$= 30 + 20$
$= 50$

$\boxed{2}$ $(\cos x)^y = (\sin y)^x$ → implicit
$f = (\cos x)^y - (\sin y)^x$
$\frac{dy}{dx} = -\frac{f_x}{f_y}$ $\langle a^x = a^x\log a\rangle$
$= -\frac{[y(\cos x)^{y-1}\times(-\sin x) - \sin y^x\cdot\log(\sin y)]}{[\cos x^y\cdot\log(\cos x) - x(\sin y)^{x-1}\cos y]}$

$\boxed{3}$ $u = x^2 - y^2$; $x = \frac{1}{t}$, $y = e^t$
$\frac{du}{dt} = \frac{du}{dx}\cdot\frac{dx}{dt} + \frac{du}{dy}\cdot\frac{dy}{dt}$
$= 2x\left(\frac{-1}{t^2}\right) + -2y\cdot e^t$

$\boxed{4}$ $e^x - e^y = 2xy$
$f: e^x - e^y - 2xy = 0$
$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{[e^x - 2y]}{-e^y - 2x}$
$= e^x - 2y$

$\boxed{5}$ $x^2 + y^2 + 3xz = 1$ $x + y = 1$
$2x + 2y\frac{dy}{dx} + 3\left(x\frac{dz}{dx} + z\right) = 0$ $\frac{dx}{dx} + \frac{dy}{dx} = 0$
$du = -1$

5 $x^2+y^2+3xz=1$

$2x+2y\frac{dy}{dx}+3\left(x\frac{dz}{dx}+z\right)=0$

$2x-2y+3x\frac{dz}{dx}+3z=0$

$3x\frac{dz}{dx}=2y-2x-3z$

$\frac{dz}{dx}=\frac{2y}{3x}-\frac{2}{3}-\frac{\cancel{3}z}{\cancel{3}x}$

$=\frac{2y}{3x}-\frac{2}{3}-\frac{z}{x}$

$x+y=1$

$\frac{dx}{dx}+\frac{dy}{dx}=0$

$\frac{dy}{dx}=-1$

$(x+y)^2=x^2+y^2+2xy$

$\frac{1-x^2-y^2}{3x}$

$\frac{1-(x^2+y^2)}{}$

$\frac{1-[(x+y)^2+2xy]}{3x}$

$\frac{x^2}{x^2}+\frac{y^2}{x^2}+\frac{3xz}{x^2}=\frac{1}{x^2}$

$1+\frac{y^2}{x^2}+\frac{3z}{x}=\frac{1}{x^2}$

$x+y=1$

$y=1-x$

$y^2=1+x^2-2x$

$1-y^2=2x-x^2$

$\frac{3z}{x}=\frac{1-y^2}{x^2}-1$

$-\frac{z}{x}=\frac{1}{3}-\frac{(1-y^2)}{3x^2}$

$\frac{2y}{3x}\left(-\frac{2}{3}+\frac{1}{3}\right)-\frac{(1-y^2)}{3x^2}$

$-\frac{1}{3}-\frac{(1-y^2)}{3x^2}$

$\frac{2y}{3x}$ $-\frac{(2\cancel{x}-x^{\cancel{2}})}{3x^{\cancel{2}}}-\frac{1}{3}$

$\frac{2y}{3x}-\frac{(2-x)}{3x}$

$\frac{2y}{3x}$ $+\frac{\cancel{x}}{\cancel{3x}}-\cancel{\frac{1}{3}}$

$\frac{2y-2}{3x}$

$\frac{2(y-1)}{3x}$

$\frac{du}{dt}=\frac{du}{dx}\cdot\frac{dx}{dt}+\frac{du}{dy}\cdot\frac{dy}{dt}$

$=2x\left(-\frac{1}{t^2}\right)+-2y\cdot e^t$

$=\frac{2}{t}\left(-\frac{1}{t^2}\right)-2e^{2t}$

$=\frac{-2}{t^3}-2e^{2t}$

6 $z<^{x<^u_v}_{y<^u_v}$

$x=e^u\sin v$ $\quad y=e^v\cos v$

$\frac{dz}{du}=\frac{\partial z}{\partial x}\cdot\frac{dx}{du}+\frac{\partial z}{\partial y}\cdot\frac{dy}{du}$

$=\frac{\partial z}{\partial x}\cdot e^u\sin v+\cancel{\frac{\partial z}{\partial y}(0)}$

$=\frac{\partial z}{\partial x}(\sin v\, e^u)$

$\frac{dy}{dx}=\frac{-f_x}{f_y}=\frac{-[e^x-2y]}{-e^y-2x}$

$=\frac{e^x-2y}{e^y+2x}$

7 $u=xyz$

$x=e^{-t},\ y=e^{-t}\sin^2 t,\ z=\sin t$

$u<^{x}_{y}_{z}>t$

$\frac{du}{dt}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial t}$

$=yz(-e^{-t})+xz(-e^{-t}\sin^2 t+e^{-t}(2\sin t\cos t))$

$+xy\cos t$

$=-e^{-2t}\sin^3 t+e^{-2t}\sin t(2\cos t-\sin t)$

$+e^{-2t}\sin^2 t\cos t$

$=e^{-2t}\sin t(-\sin t+2\cos t-\sin t+\cos t)$

$\frac{du}{dt}=e^{-2t}\sin^2 t(3\cos t-2\sin t)$

8 $z=x^2+2xy+4y^2,\ y=e^{3x}$

$\frac{dy}{dx}=3e^{3x}$

$\frac{dz}{dx}=2x+2\left(x\frac{dy}{dx}+y\right)+4(2y)\frac{dy}{dx}$

$=2x+2(x(3e^{3x})+e^{3x})$

$+8(e^{3x})(3e^{3x})$

$=2x+2e^{3x}(3x+1)+24e^{6x}$

$=2x+6xe^{3x}+2e^{3x}+24e^{6x}$

$=2(x+e^{3x})+6e^{3x}(x+4e^{3x})$

10 $u=f(x-y,\ y-z,\ z-x)$

$r=x-y,\quad s=y-z,\quad t=z-x$

$\rightarrow\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$

$=\frac{\partial u}{\partial r}(1)+\frac{\partial u}{\partial s}(0)+\frac{\partial u}{\partial t}(-1)$

$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t}$

$\rightarrow\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$

$=\frac{\partial u}{\partial r}(-1)+\frac{\partial u}{\partial s}(1)+\frac{\partial u}{\partial t}(0)$

$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial s}-\frac{\partial u}{\partial r}$

$\rightarrow\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$

$=\frac{\partial u}{\partial r}(0)+\frac{\partial u}{\partial s}(-1)+\frac{\partial u}{\partial t}(1)$

$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial t}-\frac{\partial u}{\partial s}$

$\rightarrow\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\cancel{\frac{\partial u}{\partial r}}-\cancel{\frac{\partial u}{\partial t}}+\cancel{\frac{\partial u}{\partial s}}-\cancel{\frac{\partial u}{\partial r}}$

$+\cancel{\frac{\partial u}{\partial t}}-\cancel{\frac{\partial u}{\partial s}}=0$

11 $u=x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\tan^{-1}\left(\frac{x}{y}\right)$

$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$

$=\frac{\partial}{\partial x}\left(x^{\cancel{2}}\cdot\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{\cancel{x}}\right)-2y\cdot\frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{-x}{y^2}\right)$

$=\frac{\partial}{\partial x}\left(\frac{x^3}{x^2+y^2}+\frac{2xy}{x^2+y^2}\right)$

$=\frac{\partial}{\partial x}\left(\frac{x^3+2xy}{x^2+y^2}\right)$

$=\frac{(3x^2+2y)(x^2+y^2)-(x^3+2xy)(2x+y^2)}{(x^2+y^2)^2}$

$=\frac{3x^4+3x^2y^2+2x^2y+2y^3-2x^4-x^3y^2-4x^2y-2xy^3}{(x^2+y^2)^2}$

$=\frac{x^4+3x^2y^2-2x^2y+2y^3-x^3y^2-2xy^3}{(x^2+y^2)^2}$

9 $z<^{x<^u_v}_{y<^u_v}$

$x=e^u\sin v\quad y=e^u\cos v$

PT:

(i) $\frac{\partial z}{\partial u}=x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$

$\rightarrow\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$

$=\frac{\partial z}{\partial x}\cdot e^u\sin v+\frac{\partial z}{\partial y}\cdot e^u\cos v$

$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot x+\frac{\partial z}{\partial y}\cdot y$ proved

ii) $\frac{\partial z}{\partial x}=e^{-u}\left(\sin v\frac{\partial z}{\partial u}+\cos v\frac{\partial z}{\partial v}\right)$

RHS=

$e^{-u}\left(\sin v\left(\frac{\partial z}{\partial x}e^u sv+\frac{\partial z}{\partial y}e^u cv\right)\right.$

$\left.+\cos v\left(\frac{\partial z}{\partial x}e^u cv+\frac{\partial z}{\partial y}e^u(-sv)\right)\right)$

$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$

$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot e^u\cos v+\frac{\partial z}{\partial y}\cdot e^u(-\sin v)$

$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot y+\frac{\partial z}{\partial y}(-x)$

$\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}(x+y)+\frac{\partial z}{\partial y}(y-x)$

$=e^{-u}\left[\frac{\partial z}{\partial x}e^u s^2v+\cancel{\frac{\partial z}{\partial y}e^u cv\,su}\right.$

$\left.+\frac{\partial z}{\partial x}e^u c^2v-\cancel{\frac{\partial z}{\partial y}e^u cv\,sv}\right]$

$=\cancel{e^{-u}}\frac{\partial z}{\partial x}\cancel{e^u}(s^2v+c^2v)$

$=\frac{\partial z}{\partial x}=LHS$