UNIT 3: MULTIVARIABLE CALCULUS:

Let $Z = f(x,y)$

kept constant

geometrically represent slope

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$ provided the limit exists

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$$ — 2° derivatives

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \cdot \partial y}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \cdot \partial z}$$

NOTE: $\dfrac{\partial^2 z}{\partial x \cdot \partial y} = \dfrac{\partial^2 z}{\partial x \cdot \partial y}$ holds true ← partial differentials are differentiable and continuous

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial x^2}\right) = \frac{\partial^3 z}{\partial y \cdot \partial x^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y^2}\right) = \frac{\partial^3 z}{\partial x \cdot \partial y^2}$$

(Q) Find $f_{(x)}$ & $f_{(y)}$ of the following

(A)(1) $f(x,y) = x^y$

$$\frac{\partial f}{\partial x} = yx^{y-1} \ ; \ \frac{\partial f}{\partial y} = x^y \cdot \log(x)$$

(2) $\sin^{-1}(2x-3y)$

$$\frac{\partial f}{\partial x} = \frac{+1}{\sqrt{1-(2x-3y)^2}} \cdot 2 \qquad \frac{\partial f}{\partial y} = \frac{1}{\sqrt{1-(2x-3y)^2}} (-3)$$

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(3) $\tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial f}{\partial x} = \frac{1}{1+(y/x)^2}\left(\frac{-y}{x^2}\right)$$

$$\Rightarrow \frac{-y}{x^2+y^2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

$$\Rightarrow \frac{\partial^2 f}{\partial x \cdot \partial y} = \frac{\partial^2 f}{\partial y \cdot \partial x}$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) \Rightarrow -\frac{\{x^2+y^2(1)-y(2y)\}}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial^2 f}{\partial y \cdot \partial x} \Rightarrow \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \Rightarrow \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$$

$$\Rightarrow \frac{x^2+y^2(1)-x(2x)}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{y^2-x^2}{(x^2+y^2)^2} \Rightarrow \therefore \frac{\partial^2 f}{\partial x \cdot \partial y} = \frac{\partial^2 f}{\partial y \cdot \partial x}$$

(1) $f = x\cos y + ye^x$

(2) $f = e^{xy} \cdot \ln y$

(3) $f = \frac{2y}{y+\cos x}$

(4) $f = \sqrt{x^2+y^2}$

(1) If $u = \log(x^3+y^3+z^3-3xyz)$;

then prove that (i) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$

(1) $\frac{\partial f}{\partial x} = \cos y + ye^{x}$

$\frac{\partial f}{\partial y} =$

(1) Let $u = \log(x^3 + y^3 + z^3 - 3xyz)$

differentiate 'u' partially w.r.t. x, y, z respectively

$$\frac{\partial y}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}(3x^2-3yz)$$

$$\frac{\partial y}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz}(3y^2-3xz)$$

$$\frac{\partial y}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz}(3z^2-3xy)$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3\cancel{(x^2+y^2+z^2-xy-yz-zx)}}{\cancel{x^3+y^3+z^3-3xyz}}$$

$$= \frac{3}{x+y+z}$$

LHS $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u.$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u.$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2}$$

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(Q) If $z = f(x+ct) + \phi(x-ct)$; prove that $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$

(A) Let $z = f(x+ct) + \phi(x-ct)$

$$\frac{\partial z}{\partial x} = f'(x+ct)(1) + \phi'(x-ct)\cdot(1)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ct)c^2 + \phi''(x-ct)(-c)^2$$

$$= c^2 \{f''(x+ct) + \phi''(x-ct))$$

From eq (1)

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial z^2}{\partial x^2}$$

(Q) If $\theta = t^n e^{-r^2/4t}$, what value of 'n' will make $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t}$

(A) $\theta = t^n e^{-r^2/4t}$

$$\frac{\partial\theta}{\partial r} = t^n e^{-r^2/4t}\left(\frac{-2r}{4t}\right)$$

$$= \frac{-1}{2} t^{n-1} r e^{-r^2/4t}$$

$$r^2 \frac{\partial\theta}{\partial r} = \frac{-1}{2} t^{n-1} r^3 e^{-r^2/4t}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial\theta}{\partial r}\right) = \frac{-1}{2} t^{n-1} \{r^3 e^{-r^2/4t}\cdot\left(\frac{-2r}{4t}\right) + e^{-r^2/4t}\cdot 3r^2\}$$

$$= \frac{-1}{2} t^{n-1} \{\frac{-1}{2} r^4 t^{-1} e^{-r^2/4t} + e^{-r^2/4t}\cdot 3r^2\}$$

$$= e^{-r^2/4t}\left[\frac{1}{4} t^{n-2} r^4 - \frac{3}{2} t^{n-1}\cdot r^2\right]$$

$$\frac{1}{\partial^2}\left\{\frac{\partial}{\partial r}\left(r^2 \frac{\partial\theta}{\partial r}\right)\right\}$$

$$= e^{-r^2/4t}\left[1/4\, t^{n-2} r^2 - 3/2\, t^{n-1}\right]$$

$$\frac{\partial \theta}{\partial t} = t^n e^{-r^2/4t}\left(\frac{+r^2}{4t^2}\right) + e^{-r^2/4t} \cdot n t^{n-1}$$

$$= e^{-r^2/4t}\left(\frac{t^{n-2}}{4} \cdot r^2 + n t^{n-1}\right) \longrightarrow \boxed{n = -3/2}$$

NOTE: The f^n which satisfies laplace eq^n

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \longrightarrow \text{laplace equation 2-D}$$

$u \rightarrow$ harmonic function.

(Q) Show that $\theta = \tan^{-1}(y/x)$ satisfies the laplace equation.

(A) $\dfrac{\partial^2 \theta}{\partial x^2} + \dfrac{\partial^2 \theta}{\partial y^2}$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+(y/x)^2}\left(\frac{-y}{x^2}\right) = \frac{x^2}{x^2+y^2} \cdot \frac{-y}{x^2} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right) = \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \theta}{\partial x}\right) = -y\left(\frac{-1 \times 2x}{(x^2+y^2)^2}\right) = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial \theta}{\partial y}\right) = \frac{-2y \cdot x}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

θ satisfies laplace equation

(Q) Show that $v = e^x \cos y$ satisfies the laplace equation.

(Q) show that $v = e^{x-at}\cos(x-at)$ is the solution of one dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \cdot \frac{\partial^2 y}{\partial x^2}$$

(A) $\frac{\partial y}{\partial t} = e^{x-at} \cdot \cos(x-at) \cdot (-a) \mp e^{x-at} \cdot \sin(x-at)(-a)$

$= -a[e^{x-at} \cdot \cos(x-at) - e^{x-at}\sin(x-at)]$

$= -a[y - e^{x-at}\sin(x-at)]$

$\frac{\partial^2 y}{\partial t^2} = -a\left[\frac{\partial y}{\partial t} - e^{x-at}\sin(x-at)\cdot(-a)\right] - e^{x-at}\cos(x-at)(-a)]$

$= -a^2[e^{x-at}\cancel{\cos(x-at)}(-a) + e^{x-at}\sin(x-at) + e^{x-at}$
$\sin(x-at) + \cancel{a e^{x-at}\cos(x-at)}]$

$= e^{\cancel{x-at}}\cancel{\cos(x-at)}(-a^2\cancel{a}) + e^{x-at}\sin(x-at)(a^2+\cancel{a})$

$= 2a^2 e^{x-at}\sin(x-at)$ —— ①

$\frac{\partial y}{\partial x} = e^{x-at}\cos(x-at) - e^{x-at}\sin(x-at)$

$\frac{\partial^2 y}{\partial x^2} = e^{x-at}\cancel{\cos(x-at)} - e^{x-at}\sin(x-at) - \cancel{e^{x-at}\cos(x-at)}$
$- e^{x-at}\sin(x-at)$

$= -2e^{x-at}\sin(x-at)$ —— ②

sub ② in ①

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$$

(Q) Show that D-Alembert solution $y = \phi(y+ax) + \psi(y-ax)$ satisfy the wave equation, $y_{xx} = a^2 y_{yy}$

(A) $\frac{\partial y}{\partial x} = \phi(y+ax)\cdot a + \psi(y-ax)\cdot(-a)$

(Q) Show that $y = \frac{1}{\sqrt{t}} e^{-x^2/4a^2t}$ is the solution of one dimensional heat equation.

(A) $\frac{\partial y}{\partial t} = a^2 \frac{\partial^2 y}{\partial x^2}$

$$\Rightarrow \frac{\partial y}{\partial t} = \frac{-1}{2(t)^{3/2}} e^{-x^2/4a^2t} + \frac{1}{\sqrt{t}} e^{-x^2/4a^2t} \cdot \frac{-x^2}{4a^2}$$

$$= \frac{-1}{2(t)^{3/2}} e^{-x^2/4a^2t} + \frac{1}{\sqrt{t}} e^{-x^2/4a^2t} \cdot \frac{x^2}{4a^2}$$

$$\frac{\partial y}{\partial t} = \frac{-1}{2(t)^{3/2}} e^{-x^2/4a^2t} + \frac{1}{\sqrt{t}} e^{-x^2/4a^2t} \cdot \frac{x^2}{4a^2}$$

$$\cancel{\frac{\partial^2 y}{\partial t^2}}, \frac{\partial y}{\partial x} = \frac{1}{\sqrt{t}} e^{-x^2/4a^2t} \cdot \frac{-2x}{4a^2t}$$

$$(1) = \frac{1}{\sqrt{t}} e^{-x^2/4a^2t} \cdot \frac{-2x}{4a^2t}$$

$$\frac{\partial^2 y}{\partial x^2} \Rightarrow \frac{1}{\sqrt{t}} e^{-x^2/4a^2t} \cdot \frac{-2x}{4a^2t} \cdot \frac{-2x}{}$$

$\partial(\sqrt{t})^{-1/2}$

$-1(\sqrt{t})^{-1-1}$

$-1(\sqrt{t})^{-2}$

t^{-1}

$\frac{-1}{t}$

$t^{1/2}$

$-\frac{1}{2}t^{-1-1/2}$

$-\frac{1}{2}t^{-3/2}$

t^{-1}

$-1t^{-2}$

$\frac{-1}{t^2}$

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(Q) Show that $v = e^{a\theta} \cos(a \log r)$ satisfies the laplace equation

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0 \longrightarrow \text{polar}$$

(A) $\frac{\partial v}{\partial r} = \left[-\sin(a \log r)\frac{a}{r}\right] e^{a\theta}$

$$\frac{d^2 v}{dr^2} = e^{a\theta}\left\{\left[-\cos(a\log r)\left(\frac{a}{r}\right)^2\right] - \left[\sin(a\log r)\left(\frac{-a}{r^2}\right)\right]\right\}$$

$$\frac{dv}{d\theta} = a e^{a\theta}(\cos)[a \log r]$$

$$\frac{d^2 v}{d\theta^2} = a^2 e^{a\theta}(\cos(a\log r))]$$

$$\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr} + \frac{1}{r^2}\frac{d^2 v}{d\theta^2} = 0$$

$\Rightarrow$ $$-\frac{a^2}{r^2} e^{a\theta}\cancel{\cos(a\log r)} + \frac{a}{r^2}\left[\cancel{e^{a\theta}\sin(a\log r)}\right] - \frac{a}{r^2} \cancel{e^{a\theta}\sin(a\log r)} + \cancel{\frac{a^2}{r} e^{a\theta}\cos(a\log r)}$$

$= 0$

(Q) Show that $u = e^{r\cos\theta}\cos(r\sin\theta)$, $v = e^{r\cos\theta}\sin(r\sin\theta)$ satisfy the CR eq^n (Cauchy-Reiman).

(A) $u_r = \frac{1}{r} V_\theta$; $v_r = -\frac{1}{r} u_\theta$

$=$ $$u_r = \frac{du}{dr} = e^{r\cos\theta}\cdot\cos\theta\cdot\cos(r\sin\theta) + e^{r\cos\theta}(-\sin(r\sin\theta)\cdot\sin\theta)$$

$$= e^{r\cos\theta}[\cos\theta\cos(r\sin\theta) - \sin\theta\sin(r\sin\theta)]$$

$$= e^{r\cos\theta}\cos(\theta + r\sin\theta)$$

$$v_r = \frac{du}{d\theta}$$

(*) (Q) given $x^x y^y z^z = c$, then show that at $x=y=z$,

$$\frac{\partial^2 z}{\partial x \cdot \partial y} = -(x \log ex)^{-1}$$

(A) $x\log x + y\log y + z\log z = \log c$

Diff wrt y partially

$$y \cdot \frac{1}{y} + \log y + z \cdot \frac{1}{z}\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = -\frac{(1+\log y)}{(1+\log z)} \to \text{diff wrt } x$$

$$\frac{\partial^2 z}{\partial y \cdot \partial x} = -\frac{\left[(1+\log z)\cdot 0 - (1+\log y)\cdot \frac{1}{z}\cdot\frac{\partial z}{\partial x}\right]}{(1+\log z)^2}$$

$$= \frac{1}{z}\left[\frac{1+\log y}{(1+\log z)^2}\right]\cdot\frac{\partial z}{\partial x} \to (1)$$

$$\text{If } \left[\frac{\partial z}{\partial x} = \frac{-(1+\log x)}{1+\log z}\right] \to (2)$$

sub (1) in (2)

$$\frac{\partial^2 z}{\partial x \cdot \partial y} = \frac{1}{z}\cdot\frac{(1+\log y)(1+\log x)}{(1+\log z)^3}$$

at $x=y=z$; $z=x$; $y=x$

$$\frac{\partial^2 z}{\partial x \cdot \partial y} = \frac{-1}{x}\frac{(1+\log x)^2}{(1+\log x)^3} = \frac{-1}{x(1+\log x)}$$

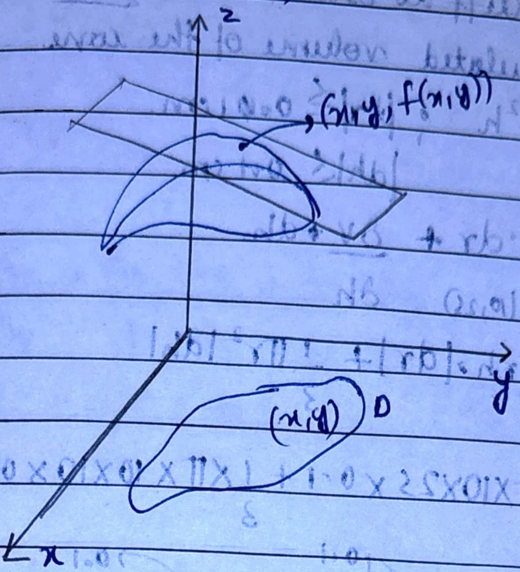
$$= \frac{-1}{x(\log_e + \log_x)} = -(x\log ex)^{-1}$$

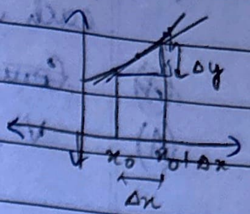
Total differential:

let $z = f(x,y)$ → equation of a surface.

$z = 4x - y + 3$ → equation of a plane

$\Delta y = f(x_0 + \Delta x) - f(x_0)$





$dy = f'(x) \cdot dx$

$z = f(x,y)$, $D \subseteq R^2$ ↓ set of

Approximation of z when x and y are changed

$$\boxed{dz = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} dy}$$

Ex1 Find total differentials of

(a) $z = e^{-2x} \cos(2\pi t)$

$dz = -e^{-2x} \cdot 2\cos(2\pi t) \cdot dx \cancel{+} - \sin(2\pi t) e^{-2x} \cdot dt \cdot 2\pi$

⇒

(b) $L = xze^{-y^2 - z^2}$

$dL = ze^{-y^2-z^2} \cdot dx + (xe^{-y^2-z^2} + xz \cancel{(-2z)} \cdot e^{-y^2-z^2} \cdot (-2z)) \cdot dz$
$+ (xze^{-y^2-z^2} \cdot (-2y)) \cdot dy$

(2) If $z = x^2 - xy + 3y^2$; (x,y) changes from $(3,-1)$ to $(2.96, -0.95)$

(A) $dz = (2x - y)^{(3,-1)} (-0.04) \cancel{+} (x + 6y)^{(3,-1)} (0.05)$

Compare the values of Δz and dz.

$\Delta z = (2.96)^2 + (2.96)(0.95) + 3(0.95)^2 - (3)^2 + (3)(-1) - 3(1)^2$

$= (2.96+3)(\cancel{2.96} - 0.04) + 3((0.95+1)(0.95-1)) +$

$(2.96)(0.95) + (3)(-1) = -0.7189$

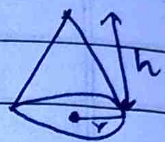
$= (6+1)(0.04) - 9(0.05)$

$= -0.73$

(3) The base radius and height of a right circular cone are measured as 10cm and 25cm respectively with possible error in measurement is as much as 0.1cm each. Use the diff to estimate the max error in the calculated volume of the cone.

(A) $V = \frac{1}{3}\pi r^2 h$; $|dr| \le 0.01$ cm.

$|dh| \le 0.1$ cm



$dV = \underset{(10,25)}{\frac{\partial V}{\partial r}} \cdot dr + \frac{\partial V}{\partial h} \cdot dh$

$= \frac{2}{3}\pi r h \cdot |dr| + \frac{1}{3}\pi r^2 |dh|$

$= \frac{2}{3}\pi \times 10 \times 25 \times 0.1 + \frac{1}{3} \times \pi \times 10 \times 10 \times 0.1$

$= 523.5 \times \overset{0.1}{|dr|} + 104.7 \overset{0.1}{|dh|}$

$= 62.82 \text{ cm}^3$

(Q) The dimension of a rectangle box is measured to be 75cm, 60cm and 40cm and each measurement correct to within 0.2cm. Use differentials to estimate the largest possible error when the volume of the box is calculated for these measurements.

(A) $V = lbh$

$dV = bh(dl) + lh(db) + lb(dh)$

$= 60 \times 40 \times 0.2 + 75 \times 40 \times 0.2 + 75 \times 60 \times 0.2$

$= \cancel{0.2}$

$= 1980 \text{ cm}^3$

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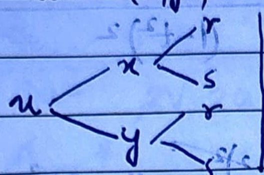
Total derivative $\to u \langle {x \atop y} \rangle t$ $\quad \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{}$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt}$$

Differentiation of composite f^n

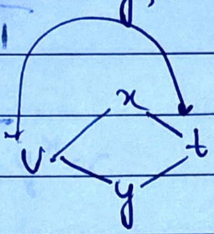
If u and v are f^n of x and y; x and y are f^n of r and ~~x~~ s, then u and v are composite f^n of r and s.

$[u = u(x,y);\ v = v(x,y),\ x = x(r,s);\ y = y(r,s)]$



$\frac{\partial u}{\partial r} =$

NOTE: If u and v are f^n of x and y, x and y are f^n of t, i.e. $u = u(x,y)$; $x = x(t)$; $y = y(t)$.



then total derivative of u wrt t is given by $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$

$u \to$ depends on (x and y) $\to$ two independent variables

differentiation of implicit f^n ( )

(Q) $f(x,y) = c \to$ An implicit f^n of x and y.

by total derivative $df =$ ~~$f'(x,y)$~~ $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0$ $\quad$ derivative of constant is 0.

$$\boxed{\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}} = \frac{-f_x}{f_y}$$

(Q) Homework $u = xy + yz + zx$

$x = t\cos t$; $y = t\sin t$; $z = t$

Find $\frac{du}{dt}$ (rate of change of u wrt t at $t = \pi/4$)

(Q) If $u = x^2 + y^2 + z^2$; $x = e^{2t}$, $y = e^{2t}\cos 3t$; $z = e^{2t}\sin 3t$ find du/dt as a total derivative of t.

(A) $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$

$= 2x \cdot e^{2t} \cdot 2 + 2y[e^{2t} \cdot 2\cos 3t - e^{2t}\sin 3t \cdot 3] + 2z[e^{2t} \cdot 2 \cdot \sin 3t + e^{2t} \cdot \cos 3t \cdot 3]$

$= 2e^{4t} \cdot 2 + 2e^{2t}\cos 3t[e^{2t} \cdot 2\cos 3t - e^{2t} \cdot \sin 3t \cdot 3] + 2e^{2t}\sin 3t[e^{2t} \cdot 2\sin 3t + e^{2t} \cdot 3\cos 3t]$

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(Q) $Z = \tan^{-1}\left(\frac{x}{y}\right)$; $x = 2t$; $y = 1 - t^2$

$\frac{dz}{dt} = \frac{2}{1+t^2}$ using total derivative

(A) $z = \tan^{-1}\left(\frac{2t}{1-t^2}\right)$, $\frac{dz}{dt} = \frac{1}{1+\left(\frac{2t}{1-t^2}\right)^2} \cdot \left[\frac{2(1-t^2) - 2t(-2t)}{(1-t^2)^2}\right]$

$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$

$= \frac{\cancel{(1-t^2)^2}}{(1-t^2)^2 + (2t)^2} \cdot \frac{2 - 2t^2 + 4t^2}{\cancel{(1-t^2)^2}}$

$\frac{2 + 2t^2}{1 + t^4 - 2t^2 + 4t^2}$

$\frac{2(1+t^2)}{1+t^4+2t^2} = \frac{2\cancel{(1+t^2)}}{(1+t^2)^{\cancel{2}}}$

$\frac{2}{(1+t^2)}$.

HW $Z = xy^2 + yx^2$, $x = at^2$, $y = 2at$ find $\frac{dz}{dt}$ using total derivatives at $t = 1$.

(Q) $v = \sin(x^2+y^2)$, $ax^2 + by^2 = c^2$, find $\frac{du}{dx}$.

(A) $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$

$\cos(x^2+y^2) \cdot 2x + \cos(x^2+y^2) \cdot 2y \cdot \frac{dy}{dx}$ —— ①

$f(x,y) = ax^2 + by^2 = c^2$

$\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-\cancel{2}ax}{\cancel{2}by} = \frac{-ax}{by}$

$\frac{du}{dx} = \cos(x^2+y^2) \cdot 2x + \cos(x^2+y^2) \cdot 2y \cdot \left(\frac{-ax}{by}\right)$.

(Q) $z = \sqrt{x^2+y^2}$ and $x^3 + y^3 + 3axy = 5a^2$, S.T. the rate of change of z wrt x at $x = y = a$ is 0 using total derivative.

(A) $\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x + \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y \cdot \frac{dy}{dx}$

$3x^2 + 3y^2 \cdot \frac{dy}{dx} + 3ay + 3ax \cdot \frac{dy}{dx} = 0$

$(3y^2 + 3ax)y' = -3ay - 3x^2$

$$y' = \frac{-3ay-3x^2}{3y^2+3ax}$$

$\sqrt{x}$, $(x)^{1/2-1}$, $x^{-1/2}$, $\frac{1}{2\sqrt{x}}$

$$\frac{2x}{2\sqrt{x^2+y^2}} + \frac{2y}{2\sqrt{x^2+y^2}} \cdot \frac{(-3ay-3x^2)}{3y^2+3ax} =$$

$$\frac{\cancel{2}a}{\cancel{2}\sqrt{2a^2}} + \frac{\cancel{2}a}{\cancel{2}\sqrt{2a^2}} \cdot \frac{(-3a^2-3a^2)}{3a^2+3a^2}$$

$$\frac{\cancel{a}}{\cancel{a}\sqrt{2}} + \frac{\cancel{a}}{\cancel{a}\sqrt{2}} \cdot \frac{-\cancel{6a^2}}{\cancel{6a^2}}$$

$$\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0 //$$

(Q) Find $\frac{dy}{dx}$ if $x^y + y^x = c$.

(A) $$\frac{dy}{dx} = \frac{-f_x}{f_y} = -\frac{yx^{y-1} + y^x \cdot \log(y)}{x^y \log(x) + xy^{x-1}}$$

(Q) $\frac{dy}{dx}$ if $(\cos y)^x = (\sin(x))^y$

$$f(x,y) = (\cos y)^x - (\sin x)^y$$

$$\frac{dy}{dx} = -\frac{(\cos y)^x \log(\cos y) - y(\sin x)^{y-1} \cdot \cos x}{x(\cos y)^{x-1} \cdot -\sin y - (\sin x)^y \cdot \log(\sin x)}$$

differentiation of composite f^n

If u and v are f^n of x and y → x and y are f^n of r and s, then u and v are composite f^n of r and s.

$$u = u(x,y);\quad x = x(r,s);\quad y = y(r,s)$$

$$\begin{tikzcd} & x \arrow[r] \arrow[dr] & r \\ u \arrow[ur] \arrow[dr] & & s \\ & y \arrow[ur] \arrow[r] & s \end{tikzcd}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}.$$

$$\begin{tikzcd} & x \arrow[r] \arrow[dr] & r \\ v \arrow[ur] \arrow[dr] & & r \\ & y \arrow[ur] \arrow[r] & s \end{tikzcd}$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial v}{\partial s} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial s}.$$

(2) $u = u(x,y)$; $x = x(r,s,t)$; $y = y(r,s,t)$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\begin{tikzcd}
 & & r \\
 & x \arrow[ur] \arrow[r] \arrow[dr] & s \\
 & & t \\
u \arrow[uur] \arrow[r] & y \arrow[r] \arrow[ur] \arrow[dr] & s \\
 & & t
\end{tikzcd}$$

$$\Rightarrow \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}$$

similarly do for v.

(Q) $z = f(x,y)$; $x = e^u \sin v$; $y = e^u \cos v$, show that $\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = e^{2u}\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]$

(A)
$$\begin{tikzcd} & x \arrow[r] \arrow[dr] & u \\ z \arrow[ur] \arrow[dr] & & v \\ & y \arrow[r] \arrow[ur] & v \end{tikzcd}$$

LHS: $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$

$= \frac{\partial z}{\partial x}\cdot e^u \sin v + \frac{\partial z}{\partial y}\cdot e^u \cos v$

$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$

$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot e^u \cos v + \frac{\partial z}{\partial y}\cdot(-e^u \sin v)$

LHS $\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 \cdot e^{2u}\cdot \sin^2 v + \left(\frac{\partial z}{\partial y}\right)^2 e^{2u}\cdot\cos^2 v + \cancel{2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\cdot e^u \sin v\cdot e^u \cos v} +$

$\left(\frac{\partial z}{\partial x}\right)^2 e^{2u}\cos^2 v + \left(\frac{\partial z}{\partial y}\right)^2 e^{2u}\cdot\sin^2 v - \cancel{2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\cdot e^u \cos v\cdot e^u \sin v}$

$= e^{2u}\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]$

(Q) $u = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$; S.T. $u_x^2 + u_y^2 = u_r^2 + \frac{1}{r^2}u_\theta^2$

(A) $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cdot\cos\theta + \frac{\partial u}{\partial y}\cdot\sin\theta$

$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}\cdot(-r\sin\theta) + \frac{\partial u}{\partial y}(+r\cos\theta)$

$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial u}{\partial y}\right)^2\sin^2\theta + \cancel{2\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}\sin\theta\cdot\cos\theta}$

$+ \cancel{\frac{1}{r^2}}\left[\left(\frac{\partial u}{\partial x}\right)^2 \cdot \cancel{r^2}\sin^2\theta + \left(\frac{\partial u}{\partial y}\right)^2 \cancel{r^2}\cos^2\theta - \cancel{2\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}\cdot r^2\sin\theta\cos\theta}\right]$

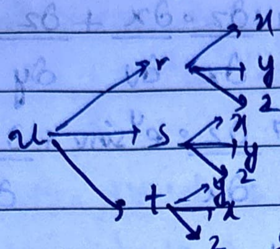
$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 =$ RHS

$\frac{\partial u}{\partial x}$ → means diff u with x keeping y and z constant $\qquad u = x+y+z$

(Q) $u = f(x-y, y-z, z-x)$, find $u_x + u_y + u_z$.

(A) $u_x = \frac{\partial u}{\partial x}$ ⇒

Let $r = x-y$, $s = y-z$; $t = z-x$



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r}\cdot(1) + \cancel{\frac{\partial u}{\partial s}\cdot(0)} + \frac{\partial u}{\partial t}\cdot(-1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r}\cdot(-1) + \frac{\partial u}{\partial s}(1) + \cancel{\frac{\partial u}{\partial t}}\cdot(0)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

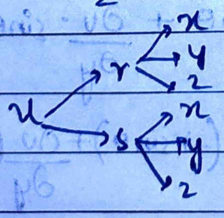
$$= \cancel{\frac{\partial u}{\partial r}(0)} + \frac{\partial u}{\partial s}\cdot(-1) + \frac{\partial u}{\partial t}\cdot(1)$$

$$u_x + u_y + u_z = \cancel{\frac{\partial u}{\partial r}} - \cancel{\frac{\partial u}{\partial t}} - \cancel{\frac{\partial u}{\partial r}} + \cancel{\frac{\partial u}{\partial s}} - \cancel{\frac{\partial u}{\partial s}} + \cancel{\frac{\partial u}{\partial t}}$$

$$= 0$$

(Q) $u = f\left(\frac{x}{z}, \frac{y}{z}\right)$, find the value $xu_x + yu_y + zu_z$

(A) Let $r = \frac{x}{z}$, $s = \frac{y}{z}$



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\left(\frac{1}{z}\right) + \cancel{\frac{\partial u}{\partial s}(0)} \qquad \frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}(x) + \frac{\partial u}{\partial s}(y)$$

$$\frac{\partial u}{\partial y} = \cancel{\frac{\partial u}{\partial r}(0)} + \frac{\partial u}{\partial s}\left(\frac{1}{z}\right) \qquad \cancel{\frac{x}{z}}\cdot\frac{\partial u}{\partial r} + \cancel{\frac{y}{z}}\frac{\partial u}{\partial s} - \frac{xz}{z}\frac{\partial u}{\partial r} + \ldots = 0$$

$- yz\,\partial u/\partial s$

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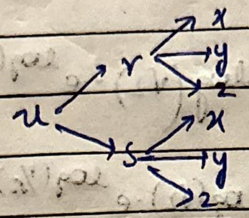
(Q) $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, $x^2u_x + y^2u_y + z^2u_z$

(A) $r = \frac{y-x}{xy}$; $s = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z}$.

$\hookrightarrow \frac{1}{x} - \frac{1}{y}$

$\Rightarrow$



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}$$

x^{-1}

$\frac{-1}{x^2}$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\left(\frac{-1}{x^2}\right) + \frac{\partial u}{\partial s}\left(\frac{-1}{x^2}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\left(\frac{1}{y^2}\right) + \frac{\partial u}{\partial s}(0)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}\left(\frac{1}{z^2}\right)$$

$$\cancel{x^2\cdot\frac{1}{x^2}\left(-\frac{\partial u}{\partial r}\right)} - \cancel{x^2\cdot\frac{1}{x^2}\left(\frac{\partial u}{\partial s}\right)} + \cancel{y^2\cdot\frac{1}{y^2}\cdot\frac{\partial u}{\partial r}} + \cancel{z^2\cdot\frac{1}{z^2}\frac{\partial u}{\partial s}}$$

$= 0$

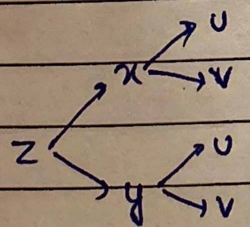
(Q) If $z = 4e^x \log y$, $x = \log(\cos v)$; $y = u\sin v$. $\to$ (u and v.)

$\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}$ at $(2, \pi/4)$

$4e^x\cdot\frac{1}{y}\cdot\sin v.$

$4e^2\cdot 1$

(A)



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot(0) + \frac{\partial z}{\partial y}\cdot\sin v \quad \to (2, \pi/4)$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial y}\cdot$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{-\sin v}{\cos v} + \frac{\partial z}{\partial y}\cdot u\cos v.$$

$$\frac{\partial z}{\partial u} = 4e^x\cdot\frac{1}{y}\cdot\frac{1}{\sqrt{2}}$$

$x = \log(1/\sqrt{2})$; $y = 2\times 1/\sqrt{2}$

$\partial z/\partial y = 4e^x \cdot 1/y$ $\quad \partial z/\partial x = 4\log y\, e^x$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot -\tan v + \frac{\partial z}{\partial y}\cdot u\cos v$$

$$= -\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\cdot 2\times\frac{1}{\sqrt{2}}$$

$$= +\frac{4e^x}{y} + 2\times\frac{1}{\sqrt{2}} - 4\log y\, e^x$$

$$= \frac{4e^{\log(1/\sqrt{2})}}{\cancel{2\times}}\cancel{\sqrt{2}}\cdot\cancel{2\times}\frac{1}{\cancel{\sqrt{2}}} - 4\log(\sqrt{2})\cdot e^{\log(1/\sqrt{2})}$$

$$4e^{\log(1/\sqrt{2})} - 4\log(\sqrt{2})\cdot e^{\log(1/\sqrt{2})}$$

$$\frac{\partial z}{\partial v} = 4e^{\log(1/\sqrt{2})}\left[1 - 4\log(\sqrt{2})\right]$$

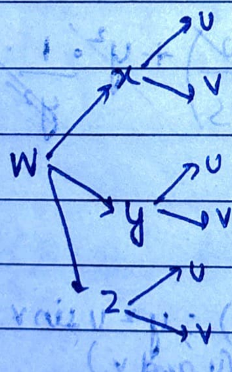
$$\frac{\partial z}{\partial u} = \frac{4e^x}{y}\cdot\frac{1}{\sqrt{2}} = \frac{\overset{2}{\cancel{4}}e^{\log(1/\sqrt{2})}}{\cancel{\sqrt{2}}}\cdot\frac{1}{\cancel{\sqrt{2}}} = 2e^{\log(1/\sqrt{2})}$$

$\frac{\partial w}{\partial u}, \frac{\partial w}{\partial v}$

Homework (Q) $w = xy + yz + zx$; $x = u+v$; $y = u-v$; $z = uv$ at $(1/2, 1)$

(Q) $v = f(2x-3y, 3x-4z, 4z-2x)$ find $v_x + v_y + v_z$

(A)



$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}$$

(Q) $x = u(1-v)$; $y = uv$; show $JJ' = 1$

(A) $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$

$= \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u - \cancel{uv} + \cancel{uv}$

$= u$

$x = u - uv$

$= u - y$

$x + y = u$

$v = y/u = \frac{y}{x+y}$

$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{(x+y)(1) - y(1)}{(x+y)^2} \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix}$

$= \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2} = \frac{x+y}{(x+y)^2} = \frac{1}{x+y} = \frac{1}{u}$

$\therefore J \cdot J' = u \cdot 1/u = 1$

(Q) If $x = e^u \cos v$; $y = e^u \sin v$, prove that $J \cdot J' = 1$

(A) $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \partial x/\partial v \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$x^2 + y^2 = e^{2u}$

$\ln(x^2+y^2) = 2u$

$u = \frac{\ln(x^2+y^2)}{2}$

$= \begin{vmatrix} \cos v\, e^u & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{vmatrix} = e^{2u}\cos^2 v + e^{2u}\sin^2 v$

$= e^{2u}$

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$u = \frac{\ln(x^2+y^2)}{2}$

$y/x = \tan v$

$\tan^{-1}(y/x) = v.$

$$= \begin{vmatrix} \frac{\cancel{2}x}{(x^2+y^2)\cdot\cancel{2}} & \frac{\cancel{2}y}{\cancel{2}(x^2+y^2)} \\ \frac{-y}{1+(y/x)^2\cdot x^2} & \frac{1}{1+(y/x)^2}\cdot\frac{1}{x} \end{vmatrix}$$

$$= \frac{x}{(x^2+y^2)}\cdot\frac{y^2}{x(x^2+y^2)} + \frac{y}{1+(y/x)^2\cdot x^2}\cdot\frac{y}{x^2+y^2}$$

$\frac{x^2}{x^2+y^2}\cdot\frac{1}{x}$

$$= \frac{y^2}{(x^2+y^2)^2} + \frac{y^2}{(x^2+y^2)^2}$$

$\frac{xy\cdot x^2}{x^2+y^2\cdot x}$

$$= \frac{2y^2}{(x^2+y^2)^2} = \frac{2(}{(e^{2u})^2} \longrightarrow \frac{x^2+y^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2 = e^{2u}}$$

$J\cdot J' = e^{2u}\cdot 1/e^{2u} = 1$

(Q) $x = r\cos\theta;\ y = r\sin\theta$, prove that $J\cdot J' = 1$

(Q) Show that $u = \frac{x+y}{1-xy}$; $v = \tan^{-1}x + \tan^{-1}y$ are functionally dependent also find the relation between them.

(A) $$\frac{\partial u}{\partial x} = \frac{(1-xy)(1) - (-y)(x+y)}{(1-xy)^2}$$

$$= \frac{1-\cancel{xy}+\cancel{xy}+y^2}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1+x^2}{(1-xy)^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0.$$ so u and v are functionally dependent

$v = \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}(u)$

$\therefore \boxed{\tan v = u}$

(Q) ST $u = x+y+z$; $v = x^2+y^2+z^2$; $w = xy+yz+zx$ are functionally dependent and find the relation b/w them.

(A) $\frac{\partial u}{\partial x} = 1$; $\frac{\partial v}{\partial x} = 2x$; $\frac{\partial w}{\partial x} = y+z$

$\frac{\partial u}{\partial y} = 1$; $\frac{\partial v}{\partial y} = 2y$; $\frac{\partial w}{\partial y} = x+z$

$\frac{\partial u}{\partial z} = 1$; $\frac{\partial v}{\partial z} = 2z$; $\frac{\partial w}{\partial z} = x+y$

$$\Rightarrow J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & x+y \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & x+z & x+y \end{vmatrix} (2) \qquad R_1 \longrightarrow R_1+R_2+R_3$$

$$= \begin{vmatrix} x+y+z+1 & x+y+z+1 & x+y+z+1 \\ x & y & z \\ y+z & x+z & x+y \end{vmatrix} = 0.$$

$u = (x+y+z)^2 \Rightarrow x^2+y^2+z^2+2(xy+yz+zx)$

$\boxed{u^2 = v+2w}$

(Q) $x = r\cos\theta;\ y = r\sin\theta;\ z = z$, find Jacobian of $J\left(\frac{x, y, z}{r, \theta, z}\right)$

$\underbrace{x = r\cos\theta;\ y = r\sin\theta}$ → cylindrical polar curves.

(A) $$J\left(\frac{x,y,z}{r,\theta,z}\right) = \begin{vmatrix} dx/dr & dy/dr & dz/dr \\ dx/d\theta & dy/d\theta & dz/d\theta \\ dx/dz & dy/dz & dz/dz \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$\Rightarrow 1(\cos\theta\, r\cos\theta + r\sin\theta \cdot \sin\theta)$

$\Rightarrow r\cos^2\theta + r\sin^2\theta$

$= r(1) = r$

(Q) $x = r\sin\theta\cos\phi;\ y = r\sin\theta\sin\phi;\ z = r\cos\theta$ → spherical polar curves.

(A) $dx/dr \Rightarrow \sin\theta\cos\phi$ $\qquad dy/dr = \sin\theta\sin\phi$ $\qquad \partial z/\partial r \Rightarrow \cos\theta$

$dx/d\theta \Rightarrow r\cos\theta\cos\phi$ $\qquad \partial y/\partial\theta = r\cos\theta\sin\phi$ $\qquad \partial z/\partial\theta = -r\sin\theta$

$dx/d\phi = -r\sin\theta\sin\phi$ $\qquad \partial y/\partial\phi \Rightarrow r\sin\theta\cos\phi$ $\qquad \partial z/\partial\phi = 0$

$$\Rightarrow J\left(\frac{x,y,z}{r,\theta,\phi}\right) = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{vmatrix}$$

$\Rightarrow \sin\theta\cos\phi(r^2\sin^2\theta\cos\phi) - \sin\theta\sin\phi(-r^2\sin^2\theta\sin\phi)$

$= r^2\sin^3\theta\cos^2\phi + r^2\sin^3\theta\sin^2\phi$

$= r^2\sin^3\theta(1) = r^2\sin^3\theta$

$\begin{vmatrix} \cos\theta & -r \\ \sin\theta & r\cos \end{vmatrix}$

$\begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix}$

(Q) If $u = x^2 - 2y^2;\ v = 2x^2 - y^2;\ x = r\cos\theta;\ y = r\sin\theta$; find $\frac{\partial(u,v)}{\partial(r,\theta)}$

(A) u and v are composite f^n of r and θ.

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$J = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} 2x & -4y \\ 4x & -2y \end{vmatrix}$$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -4y \\ 4x & -2y \end{vmatrix} \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = (-4xy + 16xy)(r)$$

$\Rightarrow (12xy \cdot r) = 6r^3\sin\cos$

(Q) If $u = x^2+3y^2-z^2$; $v = \cancel{2x^2-y^2}$ $4x^2yz$; $w = 2z^2 - xy$, find Jacobian of

$J\left(\frac{u,v,w}{x,y,z}\right)$ at $(1,-1,0)$

(A) $\frac{\partial u}{\partial x} = 2x\,(1) = 2$ $\quad \frac{\partial v}{\partial x} = 8xyz\,(8\times -1, 0) = 0$ $\quad \frac{\partial w}{\partial x} = -y\,(1)$

$\frac{\partial u}{\partial y} = 6y\,(1) = -6$ $\quad \frac{\partial v}{\partial y} = 4xz\,(4\times 1\times 0) = 0$ $\quad \frac{\partial w}{\partial y} = -x\,(-1)$

$\frac{\partial u}{\partial z} = -2z\,(1) = -2$ $\quad \frac{\partial v}{\partial z} = 4x^2y\,(4\times 1\times -1) = -4$ $\quad \frac{\partial w}{\partial z} = 4z\,(0)$

$$\begin{vmatrix} 2 & -6 & -2 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix}$$

$= 2(-4) + 6(4) - 2\cancel{(0)}$

$= -8 + 24 = \cancel{32}\ 16$

Homework

(Q) ① $x = r\cos\theta$; $y = r\sin\theta$; ST $JJ' = 1$

② $u + v = e^x\cos y$; $u - v = e^x\sin y$; $J\left(\frac{u,v}{x,y}\right)$

③ $u = \frac{x}{y-z}$; $v = \frac{y}{z-x}$; $w = \frac{z}{x-y}$, find $J\left(\frac{u,v,w}{x,y,z}\right)$

④ $u = \frac{x}{\sqrt{1-r^2}}$; $v = \frac{y}{\sqrt{1-r^2}}$; $r^2 = x^2+y^2$; ST $J\left(\frac{u,v}{x,y}\right) = \frac{1}{(1-r^2)^2}$

(5) $u = x+y$, $v = \frac{x}{x+y}$; $J\frac{(u,v)}{(x,y)} = ?$

(6) $u = \frac{yz}{x}$; $v = \frac{zx}{y}$; $w = \frac{xy}{z}$; $J\left(\frac{u,v,w}{x,y,z}\right) = ??$

(7) $x = a(u+v)$; $y = b(u-v)$; $u = r^2\cos 2\theta$; $v = r^2\sin 2\theta$. $\frac{\partial(x,y)}{\partial(r,\theta)} = ??$

(A) $\frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(r,\theta)}$

⇒ Maxima and Minima:

$y = f(x) \rightarrow f'(x) = 0$ critical/stationary points

$f''(x) < 0 \rightarrow$ maxima

$f''(x) > 0 \rightarrow$ minima

Maxima: $z = f(x,y)$ be a f^n of two independent variables x and y, then we use the following notation for the derivatives of z

$$p = \frac{\partial z}{\partial x};\ q = \frac{\partial z}{\partial y};\ r = \frac{\partial^2 z}{\partial x^2};\ s = \frac{\partial^2 z}{\partial x \cdot \partial y};\ t = \frac{\partial^2 z}{\partial y^2}$$

The necessary and sufficient condition for the f^n $z = f(x,y)$ to have extremum is $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

Working rule:

To find the extremum of $z = f(x,y)$.

① Find the stationary points $\left(\frac{\partial f}{\partial x} = 0;\ \frac{\partial f}{\partial y} = 0\right)$

② Find the derivatives $r = \frac{\partial^2 z}{\partial x^2}$; $s = \frac{\partial^2 z}{\partial x \cdot \partial y}$; $t = \frac{\partial^2 z}{\partial y^2}$ at the stationary points

③ If at the stationary points

(i) $rt - s^2 > 0$, $r < 0$, f^n is ~~minimum~~, maximum

(ii) $rt - s^2 > 0$; $r > 0$, f^n is minimum

(iii) $rt - s^2 = 0$, is the ~~point of inflection~~ saddle point.

(iv) $rt - s^2 < 0$; further study is required

(Q) Find the extreme values of $u = x^3y^2(1 - x - y)$

(A) $u = x^3y^2 - x^4y^2 - x^3y^3$

$\partial f/\partial x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$

$\Rightarrow y^2 x^2 [3 - 4x - 3y] = 0$

$\hookrightarrow 4x + 3y = 3$ — ①

$\partial u/\partial y = 2x^3y - 2x^4y - 3y^2x^3$

$= yx^3 [2 - 2x - 3y] = 0$

$2x + 3y = 2$ — ②

$2x + 3y = 2$

$(-)\ 4x + 3y = 3$

$-2x = -1$

$\boxed{x = 1/2}$; $\boxed{y = 1/3}$

$2 \times 1/2 + 3y = 2$

$3y = 1$

Stationary points

$(0,0), (0,1), (3/4, 0), (0, 2/3),$

$(1,0), (1/2, 1/3)$

$$r = \frac{\partial^2 z}{\partial x^2} \Rightarrow \frac{\partial}{\partial x}\left(3x^2y^2 - 4x^3y^2 - 3x^2y^3\right)$$

$$\Rightarrow 6xy^2 - 12x^2y^2 - 6xy^3 \quad (1/2, 1/3)$$

$$\Rightarrow \cancel{6} \times 1/2 \times 1/\cancel{9} - \cancel{12} \times 1/\cancel{4} \times 1/\cancel{9} - \cancel{6} \times 1/\cancel{2} \times 1/\cancel{27}$$

$$= \cancel{1/3} - \cancel{1/3} - 1/9 = -1/9$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(2x^3y - 2x^4y - 3y^2x^3\right) \quad (1/2, 1/3)$$

$$= 6x^2y - 8x^3y - \cancel{6}\,9x^2y^2$$

$$= \cancel{6} \times 1/\cancel{4} \times 1/\cancel{3} - \cancel{8} \times 1/\cancel{8} \times 1/3 - 9 \times 1/4 \times 1/\cancel{9}$$

$$= 1/2 - 1/3 - 1/4$$

$$= \frac{6-4-3}{12} = \frac{6-7}{12} = -1/12$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(2x^3y - 2x^4y - 3y^2x^3\right)$$

$$= 2x^3 - 2x^4 - 6yx^3 \quad (1/2, 1/3)$$

$$= \cancel{2} \times 1/\cancel{8} - \cancel{2} \times 1/\cancel{16} - \cancel{6} \times 1/\cancel{3} \times 1/8$$

$$= -1/8$$

$$rs - t^2 = -1/9 \cdot -1/12 - 1/64$$

$$= 1/108 - 1/64 = \underline{> 0,\ r < 0} \rightarrow \text{maximum}$$

$$= \cancel{\frac{16}{1728}} \Rightarrow$$

2 | 64, 108
2 | 32, 54
16 | 16, 27
~~38~~
27 × 32 × 54

32
54
0128
1600
1728
~~108~~

(Q) $u = xy(a-x-y)$; $f(x,y) = \sin x + \sin y + \sin(x+y)$

(A)

Lagrange's method ~~of~~ of undetermined multipliers:

~~By using~~ Let $f(x,y,z)$ be a function of three independent variables, x,y,z.
subject to the constraint $\phi(x,y,z)=c$.

Working rule:

$F(x,y,z) = f(x,y,z) + \lambda(\phi(x,y,z)-c)$ where λ is a parameter.

Eliminating λ from $\frac{\partial F}{\partial x}=0$; $\frac{\partial F}{\partial y}=0$, $\frac{\partial F}{\partial z}=0$ and $\phi(x,y,z)=c$, we get a stationary points.

(Q) Temperature at any point in space is given by $T = 400xyz^2$, find the highest temperature on the unisphere $x^2+y^2+z^2=1$.

(A) $F(x,y,z) = 400xyz^2 + \lambda(x^2+y^2+z^2-1)$

$\partial F/\partial x = 400yz^2 + \lambda(2x) = 0 \Rightarrow \boxed{\lambda = \frac{-200yz^2}{x}}$

$\partial F/\partial y = 400xz^2 + \lambda(2y) = 0$

$\boxed{\lambda = \frac{-200xz^2}{y}}$

$\partial F/\partial z = 0 \Rightarrow 800xyz + \lambda(2z) = 0$

$\boxed{\lambda = -400xy.}$

$\Rightarrow \frac{-200yz^2}{x} = \frac{-200xz^2}{y} = \boxed{y^2=x^2} \to \boxed{y=\pm x}$

$\Rightarrow \frac{-200xz^2}{y} = -\overset{2}{\cancel{400}}xy$

$\boxed{z^2=2y^2} \longrightarrow \boxed{z=\pm\sqrt{2}y}$

$x^2+y^2+z^2=1$

$x^2+x^2+2x^2=1$

$4x^2=1$

$\boxed{x=\pm 1/2} = y$; $\boxed{z = \sqrt{2}/2 = \pm 1/\sqrt{2}}$

$x = 1/2, y = 1/2; z = 1/\sqrt{2}$

highest temperature $= 400xyz^2$

$= \overset{50}{\cancel{400}} \times 1/2 \times 1/2 \times 1/2$

$= 50.$

(Q) Find the dimension of a rectangular parallelopiped, that can be enclosed in the elipse $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$

(A) Let $x, y, z \rightarrow$ length, breadth, height of the parallelopiped

Volume of parallelopiped $= 2x \cdot 2y \cdot 2z$
$= 8xyz$

$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$

$F(x,y,z) = 8xyz + \lambda\left(x^2/a^2 + y^2/b^2 + z^2/c^2 - 1\right)$

$\partial F/\partial x = 8yz + \lambda \cdot \dfrac{2x}{a^2} = 0$

$\lambda = -\dfrac{\overset{4}{\cancel{8}}yz \cdot a^2}{\cancel{2}x} = -\dfrac{4a^2yz}{x}$

$\partial F/\partial y = 8xz + \lambda\left(2y/b^2\right) = 0$

$\lambda = -\dfrac{\overset{4}{\cancel{8}}xz \cdot b^2}{\cancel{2}y} = -\dfrac{4xzb^2}{y}$

$\partial F/\partial z = 8xy + \lambda\left(\dfrac{2z}{c^2}\right) = 0$

$\lambda = -\dfrac{\overset{4}{\cancel{8}}xy \cdot c^2}{\cancel{2}z} = -\dfrac{4c^2xy}{z}$

$-\dfrac{4a^2y\cancel{z}}{x} = -\dfrac{4x\cancel{z}b^2}{y} \quad a^2y^2 = b^2x^2 \quad bx = ay \quad y = bx/a$

$-\dfrac{4\cancel{x}zb^2}{y} = -\dfrac{4c^2\cancel{x}y}{z} \quad z^2b^2 = y^2c^2 \quad bz = cy$

$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$

$= x^2/a^2 + b^2x^2/a^2 + c^2x^2/a^2 = 1$

$\dfrac{x^2(1+b^2+c^2)}{a^2} = 1 \quad x^2(1+b^2+c^2) = a^2$

$x = \dfrac{a}{\sqrt{1+b^2+c^2}}$; $y = \dfrac{ba}{\sqrt{1+b^2+c^2}}$; $z = \dfrac{c}{\sqrt{1+b^2+c^2}}$

$8xyz = \dfrac{8abc}{\left(\sqrt{1+b^2+c^2}\right)^3}$

$3x^2/a^2 = 1$

$3x^2 = a^2$

$x^2 = a^2/3$

$x = \pm a/\sqrt{3}$

$y = b/\sqrt{3}$

$z = c/\sqrt{3}$

$\dfrac{8 \times abc}{3\sqrt{3}}$

$x/y = a/b$

$z/y = c/b$, $y/z = b/c$

$x/y \cdot y/z = a/b \cdot b/c$

$x/z = a/c$

$cx = az$

(Q) Find the point on the plane $2x+3y-z=5$ which is the nearest to the origin. (plane)

(A) Let the point $P(x,y,z)$ on the plane, then the distance from the origin (OP)

$OP = \sqrt{x^2+y^2+z^2}$

$OP^2 = x^2+y^2+z^2$

$F(x,y,z) = x^2+y^2+z^2+\lambda(2x+3y-z-5)$

$\partial F/\partial x = 2x+\lambda(2) = 0$

$2\lambda = -2x$

$\lambda = -\cancel{2}x/\cancel{2} = -x$ —— ①

$\partial F/\partial y = 2y+3(\lambda) = 0$

$3\lambda = -2y$

$\lambda = -2y/3$ —— ②

$\partial F/\partial z = 2z+\lambda(-1) = 0$

$\lambda = 2z$ —— ③

$-x = 2z = -2y/3$ $\quad -x/2$

$2x+3\left(\frac{+3x}{+2}\right)-\left(\frac{+2y}{3(2)}\right) = 5$

$\cancel{2x+9x+y/3}$ $\quad 2x+\frac{9x}{2}+\frac{x}{2} = 5$

$2x+5x = 5$

$7x = 5$

$x = 5/7$

$\frac{49}{49}$

$OP = \sqrt{25/49+225/196+25/196}$

$\sqrt{\frac{100+25+225}{196}}$ $\quad \sqrt{\frac{350}{196}}$

225
250
100
350

$z = -5/14$

$y = \frac{3\times 5}{7\times 2} = 15/14$

$P = (5/7, 15/14, -5/14)$

$OP = \sqrt{\frac{350}{14}}$

$= \frac{5\sqrt{14}}{14}$

$= \frac{5}{\sqrt{14}}$ //

(Q) A rectangular box open at the top is to have the volume of $32 ft^3$, find the dimension of the box such that the total surface area is minimum.

(A) $V = 32 ft^3$; Let $P(x,y,z)$

$A = \cancel{2(xy+yz+zx)}$ $2xy+2yz+2x$

$V = xyz$

$F(x,y,z) = 2xy + 2yz + zx + \lambda(xyz - 32)$

$\partial F/\partial x = 2y + z + \cancel{x}\, yz\lambda = 0$

$\lambda(yz) = -2y - z$

$\lambda = \dfrac{-2y-z}{yz}$

$\partial F/\partial y = 2x + 2z + \lambda(xz) = 0$

$\lambda(xz) = -2x - 2z$

$\lambda = \dfrac{-2x-2z}{xz}$

$\partial F/\partial z = 2y + x + \lambda(xy) = 0$

$\lambda(xy) = -2y - x$

$\lambda = \dfrac{-2y-x}{xy}$

$\dfrac{-2y-z}{\cancel{yz}} = \dfrac{-2y-x}{\cancel{xy}} = \cancel{-2xy}-\cancel{xz} = \cancel{-2yz}-\cancel{xz}$

$\boxed{x = z}$

$\dfrac{-2x-2z}{\cancel{x}z} = \dfrac{-2y-x}{\cancel{x}y} = -2\cancel{xy}-\cancel{2yz} = \cancel{-2yz}-\cancel{xz}$

$+2y = z$

$\boxed{y = z/2}$

$xyz = 32$

$z \cdot \left(\dfrac{z}{2}\right) \cdot z = 32$ $\quad\boxed{x = 4\,ft}$

$z^3/2 = 32$ $\quad\boxed{y = 2\,ft}$

$z^3 = 64$

$\boxed{z = 4\,ft}$

$SA = \cancel{8}\;16 + 16 + 8$

$= 40$

(Q) Represent 24 as the sum of 3 parts such that the product of the first part, square of the second part and cube of the third part is max^m.

(A) $x \cdot y^2 \cdot z^3 = f(x,y,z)$

$F(x,y,z) = xy^2z^3 + \lambda(x+y+z-24)$

$\partial F/\partial x = y^2z^3 + \lambda \Rightarrow \lambda = -y^2z^3$

$\partial F/\partial y = x2y \cdot z^3 + \lambda \Rightarrow \lambda = -2xyz^3$

$\partial F/\partial z = 3z^2 \cdot xy^2 + \lambda \Rightarrow \lambda = -3xy^2z^2$

$\cancel{-}y^{\cancel{2}}\cancel{z^3} = \cancel{-}2x\cancel{y}\cancel{z^3}$

$y = 2x \rightarrow x = y/2$

$-2\cancel{xy}z^{\cancel{3}} = -3\cancel{xy}^{\cancel{2}}z^{\cancel{2}}$

$2z = 3y$

$z = 3y/2$

$y/2 + y + 3y/2 = 24$

$3y = 24$

$\boxed{y = 8}$

$\boxed{x = 4}$; $\boxed{z = \frac{3 \times \cancel{8}^{4}}{\cancel{2}} = 12}$

$xy^2z^3 = 4 \times 64 \times 1778$

2