Multivariable Functions and Partial Differentiation

Topic Learning Objectives:

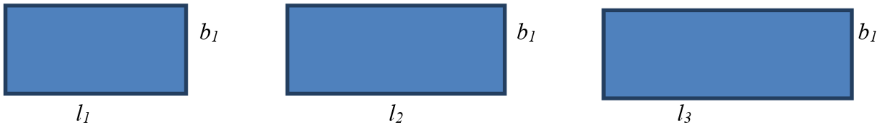
To recognize the geometrical and physical quantities which depend on more than one independent variable, evaluate the possible partial changes due to change in any of these variables or total change due to changes in all the variables. Study of applications of these concepts in finding extreme points on a surface and Jacobian as a transformation factor between different coordinate systems.

Introduction:

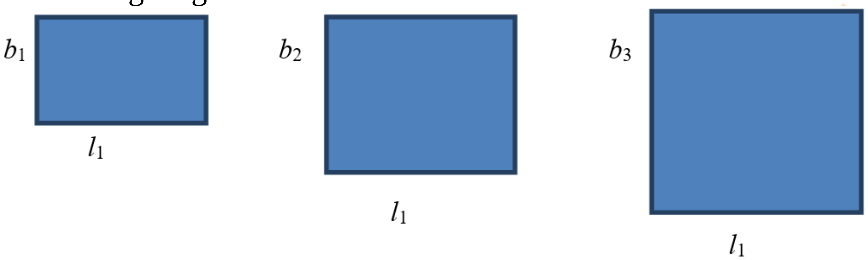
The physical quantities like displacement, density, temperature of metal plates etc. depend on more than one variable. These quantities vary with space and time, hence rate of change of these lead to concepts of partial derivatives.

For example, the area of a rectangle of length 'l' and breadth 'b' is given by $A = lb$. This can be represented as a general function $u = f(l, b)$. There are two types of changes possible with such a function resulting in two first order partial changes and one total differential.

The partial change in area of rectangle is due to changes in any one of the variables either length or breadth. When there are changes in both the variables at a time, it results in total change in area.
Partial change in area of rectangle due to change in only length maintaining breadth at the same value:



Partial change in area of rectangle due to change in only breadth maintaining length at same value:



Total change in area of rectangle due to changes in both length and breadth:

These concepts can be formally expressed as definitions as follows.

Definition: Let $z = f(x, y)$ be a function of two variables x and y. The first order partial derivative of z with respect to x, denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or z_x or f_x or $\boldsymbol{p}$ is defined as $\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$ provided the limit exists.

$\Rightarrow \left(\frac{\partial z}{\partial x}\right)$ is the partial derivative of z with respect to x, treating y as constant.

The first order partial derivative of z with respect to y, denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or z_y or f_y or $\mathbf{q}$ is defined as $\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x,y+\delta y)-f(x,y)}{\delta y}$ provided the limit exists.

$\Rightarrow \left(\frac{\partial z}{\partial y}\right)$ is the partial derivative of z with respect to y, treating x as constant.

Each of the first order partial derivatives being functions of (x, y), they can be further differentiated partially with respect to both x and y resulting in second order partial derivatives.

The partial derivatives $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$ or $\frac{\partial^2 f}{\partial x^2}$ or z_{xx} or f_{xx} or $\mathbf{r}$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \text{ or } \frac{\partial^2 f}{\partial y^2} \text{ or } z_{yy} \text{ or } f_{yy} \text{ or } \mathbf{t}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} \text{ or } z_{yx} \text{ or } f_{yx} \text{ or } \mathbf{s}$$

and $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}$ or z_{xy} or f_{xy} or $\mathbf{s}$

are second order Partial derivatives.

In most of the cases, it can be verified that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

(This is true at every point of continuity in any region of interest)

All the rules of differentiation applicable to functions of a single independent variable are applicable in partial differentiation also, the only difference is while differentiating partially with respect to one independent variable all other independent variables are treated as constants.

Examples:

1. Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, if

(a) $z = x^2 y - x \sin x y$

(b) $x + y + z = \log z$

Solution:

(a) Consider $z = x^2y - x\,sin\,x\,y$

Differentiating z with respect to x, keeping y as a constant, $\frac{\partial z}{\partial x} = (2x)y - \{x\,cos(xy)\,(y) + sin(xy)\,(1)\}$

i.e., $z_x = 2xy - xy\,cos\,x\,y - sin\,x\,y = xy(2 - cos\,x\,y) - sin\,x\,y$

Similarly, differentiating z with respect to y keeping x as a constant,

$\frac{\partial z}{\partial y} = x^2(1) - \{x\,cos\,x\,y(x) + sin(xy)\,(0)\}$

i.e., $z_y = x^2 - x^2\,cos\,x\,y = x^2(1 - cos\,x\,y)$

(b) Consider $x + y + z = log\,z$

i.e., $log\,z - z = x + y$ (1)

Differentiating (1) partially with respect to x, treating y as constant,

$$\frac{1}{z}\frac{\partial z}{\partial x} - \frac{\partial z}{\partial x} = 1 + 0 \Rightarrow \left(\frac{\partial z}{\partial x}\right)\left(\frac{1}{z} - 1\right) = 1$$

i.e., $\frac{\partial z}{\partial x}\left(\frac{1-z}{z}\right) = 1$ or $\frac{\partial z}{\partial x} = \left(\frac{z}{1-z}\right)$

Similarly, differentiating (1) partially with respect to y, treating x as

constant, $\frac{1}{z}\frac{\partial z}{\partial y} - \frac{\partial z}{\partial y} = 1 + 0\ or\ \left(\frac{\partial z}{\partial y}\right)\left(\frac{1}{z} - 1\right) = 1 \Rightarrow \frac{\partial z}{\partial y} = \left(\frac{z}{1-z}\right)$

$\therefore \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = \left(\frac{z}{1-z}\right)$.

2. If $\boldsymbol{\theta = t^n e^{-\frac{r^2}{4t}}}$ find the value of $\boldsymbol{n}$ such that $\boldsymbol{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}}$.

Solution: Consider $\theta = t^n e^{-\frac{r^2}{4t}}$ (1)

Differentiating (1) with respect to r, treating t as constant,

$$\frac{\partial \theta}{\partial r} = t^n e^{-\frac{r^2}{4t}}\left(\frac{-2r}{4t}\right) = \frac{-r}{2}t^{n-1}e^{\frac{-r^2}{4t}}$$

$$\therefore\ r^2\frac{\partial\theta}{\partial r} = -\frac{r^3}{2}t^{n-1}e^{\frac{-r^2}{4t}}$$

Now $\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{-3r^2}{2}t^{n-1}e^{-r^2/4t} + \left(-\frac{r^3}{2}\right)t^{n-1}e^{-r^2/4t}\left(-\frac{2r}{4t}\right)$

$\therefore \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \left(-\frac{3}{2}t^{n-1} + \frac{r^2}{4}t^{n-2}\right)e^{-r^2/4t}$ (2)

Again, differentiating (1) with respect to *t*, treating *r* as constant,

$\frac{\partial\theta}{\partial t} = nt^{n-1}e^{-\frac{r^2}{4t}} + t^n e^{-r^2/4t}\left(\frac{r^2}{4t^2}\right) = \left(nt^{n-1} + \frac{1}{4}r^2t^{n-2}\right)e^{-r^2/4t}$

(3)

Given $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t}$, equation (1) and (2) yield,

$$\left(-\frac{3}{2}t^{n-1} + \frac{r^2}{4}t^{n-2}\right)e^{\frac{-r^2}{4t}} = \left(nt^{n-1} + \frac{r^2}{4}t^{n-2}\right)e^{-r^2/4t}$$

i.e. $\left(n+\frac{3}{2}\right)\ t^{n-1}e^{-r^2/4t} = 0 \Rightarrow \left(n+\frac{3}{2}\right) = 0\ or\ n = -\frac{3}{2}$

3. If $\boldsymbol{z = e^{ax+by}f(ax-by)}$, prove that $\boldsymbol{b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz}$

Solution: Consider $z = e^{ax+by}f(ax-by)$ (1)

Differentiating (1) with respect to *x* using product and chain rules,

$$\frac{\partial z}{\partial x} = \{e^{ax+by}f^1(ax-by)(a)\} + \{f(ax-by)e^{ax+by}(a)\}$$

$\frac{\partial z}{\partial x} = ae^{ax+by}\{f(ax-by) + f'(ax-by)\}$ (2)

Differentiating (1) with respect to *y* using product and chain rules,

$$\frac{\partial z}{\partial y} = \{e^{ax+by}f'(ax-by)(-b)\} + \{f(ax-by)e^{ax+by}(b)\}$$

$\frac{\partial z}{\partial y} = be^{ax+by}\{f(ax-by) - f'(ax-by)\}$ (3)

Multiplying equation (2) by *b* and equation (3) by *a*, and adding

$b\frac{\partial z}{\partial x}+a\frac{\partial z}{\partial y}=abe^{ax+by}[f'(ax+by)+f(ax-by)+f(ax-by)-f'(ax-by)]=abe^{ax+by}[2f(ax-by)]=2ab[e^{ax+by}f(ax-by)]=2abz.$

4. If $\boldsymbol{u=e^{r\cos\theta}\cos(r\sin\theta)}$, $\boldsymbol{v=e^{r\cos\theta}\sin(r\sin\theta)}$, prove that $\boldsymbol{\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial\theta}}$ & $\boldsymbol{\frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial\theta}}$

Solution: Consider $u=e^{r\cos\theta}\cos(r\sin\theta)$, differentiating u with respect to r and θ partially, $\frac{\partial u}{\partial r}=e^{r\cos\theta}\{-\sin(r\sin\theta)(\sin\theta)\}+\cos(r\sin\theta)\{e^{r\cos\theta}(\cos\theta)\}$

i.e., $\frac{\partial u}{\partial r}=e^{r\cos\theta}\{\cos(r\sin\theta)\cos\theta-\sin(r\sin\theta)\sin\theta\}$

$\frac{\partial u}{\partial r}=e^{r\cos\theta}\{\cos(r\sin\theta+\theta)\}$ (1)

and $\frac{\partial u}{\partial\theta}=e^{r\cos\theta}\{-\sin(r\sin\theta)(r\cos\theta)\}+\cos(r\sin\theta)\{e^{r\cos\theta}(-r\sin\theta)\}$

$=-re^{r\cos\theta}\{\sin(r\sin\theta)\cos\theta+\cos(r\sin\theta)\sin\theta\}$

i.e., $-\frac{1}{r}\frac{\partial u}{\partial\theta}=e^{r\cos\theta}\{\sin(r\sin\theta+\theta)\}$ (2)

Again consider $v=e^{r\cos\theta}\sin(r\sin\theta)$

Differentiating v with respect to r & θ partially,

$$\frac{\partial v}{\partial r}=e^{r\cos\theta}\{\cos(r\sin\theta)\sin\theta\}+\sin(r\sin\theta)\{e^{r\cos\theta}(\cos\theta)\}$$

$$=e^{r\cos\theta}\{\sin(r\sin\theta)\cos\theta+\cos(r\sin\theta)\sin\theta\}$$

i.e., $\frac{\partial v}{\partial r}=e^{r\cos\theta}\{\sin(r\sin\theta+\theta)\}$

(3) $\frac{\partial v}{\partial\theta}=e^{r\cos\theta}\{\cos(r\sin\theta)(r\cos\theta)\}+\sin(r\sin\theta)\{e^{r\cos\theta}(-r\sin\theta)\}$

$=re^{r\cos\theta}\{\cos(r\sin\theta)\cos\theta-\sin(r\sin\theta)\sin\theta\}$

i.e., $\frac{1}{r}\frac{\partial v}{\partial\theta}=e^{r\cos\theta}\{\cos(r\sin\theta+\theta)\}$ (4)

Thus, from equations (1) and (4), $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, and from equations (2) and (3) $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

Examples:

For the following functions, verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$, where $\boldsymbol{u} = \boldsymbol{f(x, y)}$.

1. $u = x^y$

2. $u = sin^{-1}\left({}^{y}/_{x}\right)$

3. $u = e^x(x \sin y - y \sin y)$

Solution: 1. Differentiating $u = x^y$ partially with respect to y,

$\frac{\partial u}{\partial y} = x^y \log x \left(sin\,c\,e\ \frac{d}{dx}(a^x) = a^x \log a\right)$

Differentiating the above equation partially with respect to x,

$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(x^y \log x)$

i.e., $\frac{\partial^2 u}{\partial x \partial y} = x^y\left(\frac{1}{x}\right) + \log x\,(yx^{y-1})$ or $\frac{\partial^2 u}{\partial x \partial y} = x^{y-1}(1 + y \log x)$ (1)

Again, differentiating $u = x^y$ partially with respect to x, $\frac{\partial u}{\partial x} = yx^{y-1}$ so that

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(yx^{y-1})$$

i.e., $\frac{\partial^2 u}{\partial y \partial x} = y(x^{y-1} \log x) + x^{y-1}(1)$ or (2)

From (1) and (2), $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Solution: 2. Differentiating $u = sin^{-1}\left({}^{y}/_{x}\right)$ partially with respect to x,

$$\frac{\partial u}{\partial x}=\frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}}\left({}^{-y}/_{x^2}\right)=\frac{-y}{x\sqrt{x^2-y^2}}$$

$$\therefore \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=-\frac{x\sqrt{x^2-y^2}(1)-y\left\{x\left({}^{1}/_{2\sqrt{x^2-y^2}}\right)(-2y)\right\}}{x^2(x^2-y^2)}$$

i.e., $\frac{\partial^2 u}{\partial y\partial x}=-\left\{\frac{x(x^2-y^2)+xy^2}{x^2(x^2-y^2)^{3/2}}\right\}=-\frac{x}{(x^2-y^2)^{3/2}}$

(1)

Again, differentiating $u=sin^{-1}({}^{y}/_{x})$ partially with respect to y,

$\frac{\partial u}{\partial y}=\frac{1}{\sqrt{1-({}^{y}/_{x})^2}}\left({}^{1}/_{x}\right)=\frac{1}{\sqrt{x^2-y^2}}$

$\therefore \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial^2 u}{\partial x\partial y}=-\frac{\sqrt{x^2-y^2}(0)-(1)\frac{1}{2\sqrt{x^2-y^2}}(2x)}{x^2-y^2}=-\frac{x}{(x^2-y^2)^{\frac{3}{2}}}$ (2)

From (1) and (2), $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$

Solution: 3. Differentiating $u=e^x(x\sin y-y\sin y)$with respect to y, $\frac{\partial u}{\partial y}=e^x(x\cos y-y\cos y-\sin y)$

Differentiating with respect to x,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=e^x(x\cos y-y\cos y-\sin y)+e^x\cos y$$

or $\frac{\partial^2 u}{\partial x\partial y}=e^x\{(1+x)\cos y-y\cos y-\sin y\}$ (1)

differentiating u with respect to x,

$$\frac{\partial u}{\partial x}=e^x\{x\sin y-y\sin y\}+e^x\sin y$$

$$\frac{\partial u}{\partial x}=e^x\{(1+x)\sin y-y\sin y\}$$

Differentiating above expression with respect to y,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x} = e^x\{(1+x)\cos y - y\cos y - \sin y\} \tag{2}$$

From (1) and (2), $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Examples:

1. If $u = tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Solution: Differentiating $u = tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)$ partially with respect to x,

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{2xy}{x^2-y^2}\right)^2}\left\{\frac{(x^2-y^2)(2y)-2xy(2x)}{(x^2-y^2)^2}\right\}$$

i.e. $\frac{\partial u}{\partial x} = \frac{-2y}{(x^2+y^2)}$ Differentiating partially this with respect to x,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{-2y}{x^2+y^2}\right) = \frac{(x^2+y^2)(0)-(-2y)(2x)}{(x^2+y^2)^2}$$

i.e., $\frac{\partial^2 u}{\partial x^2} = \frac{4xy}{(x^2+y^2)^2}$ (1)

Again differentiating $u = tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)$ partially with respect to y,

$$\frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{2xy}{x^2-y^2}\right)^2}\left\{\frac{(x^2-y^2)(2x)-2xy(-2y)}{(x^2-y^2)^2}\right\}$$

$\frac{\partial u}{\partial y} = \frac{2x}{(x^2+y^2)}$ Differentiating partially this with respect to y,

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2}\right) = \frac{(x^2-y^2)(0)-(2x)(2y)}{(x^2+y^2)^2}$

i.e., $\frac{\partial^2 u}{\partial y^2} = \frac{-4xy}{(x^2+y^2)^2}$ (2)

Adding equations (1) and (2) we get $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, as desired.

Note:

(a) The equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is known as **Laplace's equation** in two dimensions which has variety of applications in field theory.

(b) The equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ is known as **Laplace's equation** in three dimensions where $u = u(x, y, z)$.

(c) Any function $u(x, y)$ satisfying Laplace equation is called **harmonic function.**

2. If $u = f(x + ay) + g(x - ay)$, show that $\frac{\partial^2 u}{\partial y^2} = a^2 \frac{\partial^2 u}{\partial x^2}$.

Solution: Differentiating $u = f(x + ay) + g(x - ay)$, partially with respect to x,

$$\frac{\partial u}{\partial x} = f'(x + ay)(1) + g'(x - ay)(1)$$

Again differentiating partially with respect to x,

$$\frac{\partial^2 u}{\partial x^2} = f''(x + ay)(1) + g''(x - ay)(1)$$

$$a^2 \frac{\partial^2 u}{\partial x^2} = a^2[f''(x + ay) + g''(x - ay)] \qquad (1)$$

Next, differentiating $u = f(x + ay) + g(x - ay)$, partially with respect to y

$$\frac{\partial u}{\partial y} = f'(x + ay)a + g.'(x - ay)(-a)$$

Again, differentiating partially with respect to y,

$$\frac{\partial^2 u}{\partial x^2} = f''(x + ay)(a^2) + g''(x - ay)(a^2)$$

i.e., $\frac{\partial^2 u}{\partial y^2} = a^2[f''(x + ay) + g''(x - ay)]$ (2)

From equations (1) and (2), $\frac{\partial^2 u}{\partial y^2} = a^2 \frac{\partial^2 u}{\partial x^2}$

Note:

(a) The equation $\frac{\partial^2 u}{\partial y^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ or the equation $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ is known as **one-dimensional wave equation.**

(b) The equation, $\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$ is known as **three-dimensional wave equation**

(c) The solution of the type $u = f(x + ay) + g(x - ay)$ is called D'Alembert's solution of wave equation.

3. If $u = e^{-2t} \cos 3x$, find the value of 'c' such that $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

Solution: Differentiating $u = e^{-2t} \cos 3x$, partially with respect to t

$$\frac{\partial u}{\partial t} = -2e^{-2t} \cos 3x \qquad (1)$$

Next, differentiating $u = e^{-2t} \cos 3x$ partially with respect to x,

$$\frac{\partial u}{\partial x} = -3e^{-2t} \sin 3x$$

Again, differentiating partially with respect to x,

$$\frac{\partial^2 u}{\partial x^2} = -9e^{-2t} \cos 3x$$

$\therefore \quad \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow -2e^{-2t} \cos 3x = c^2 9e^{-2t} \cos 3x \Rightarrow c^2 = \left(\frac{2}{9}\right)$ or $c = \sqrt{\frac{2}{9}}$

Note:

The equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ is called **one-dimensional heat equation** and the equation $\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$ is known as **three-dimensional heat equation.** The constant c in the equation is called dissipation coefficient.

4. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, prove that

(a) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \left(\frac{3}{x+y+z}\right)$

(b) $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=-\frac{9}{(x+y+z)^2}$

Solution:

(a) Differentiating u partially with respect to x, y, z

$\frac{\partial u}{\partial x}=\frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$,

similarly, $\frac{\partial u}{\partial y}=\frac{3y^2-3zx}{x^3+y^3+z^3-3xyz}$ (u is symmetric function)

and $\frac{\partial u}{\partial z}=\frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$

adding these partial derivatives

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$

$$=\frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$=\left(\frac{3}{x+y+z}\right)$$

(b) By definition, we have

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$

$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$

$=\frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$

$=\left(\frac{-3}{(x+y+z)^2}\right)+\left(\frac{-3}{(x+y+z)^2}\right)+\left(\frac{-3}{(x+y+z)^2}\right)=-\frac{9}{(x+y+z)^2}$

5. If $u=f(r)$, where $r=\sqrt{x^2+y^2+z^2}$, show that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=f^{11}(r)+\frac{2}{r}f^{1}(r)$

Solution: By data $r^2 = x^2 + y^2 + z^2$ differentiating partially with respect to x, $2r\frac{\partial r}{\partial x} = 2x$ or $\frac{\partial r}{\partial x} = \left(\frac{x}{r}\right)$.

Similarly, we get $\frac{\partial r}{\partial y} = \left(\frac{y}{r}\right)$ and $\frac{\partial r}{\partial z} = \left(\frac{z}{r}\right)$

Now, differentiating$u = f(r)$, partially with respect to x,

$\frac{\partial u}{\partial x} = f^1(r)\frac{\partial r}{\partial x}as\frac{\partial r}{\partial x} = \left(\frac{x}{r}\right)$,

$$\frac{\partial u}{\partial x} = f^1(r)\left(\frac{x}{r}\right)$$

Differentiating $\frac{\partial u}{\partial x} = x\left\{\frac{f^1(r)}{r}\right\}$partially with respect to x,

$$\frac{\partial^2 u}{\partial x^2} = x\left\{\frac{1}{r}f^{11}(r)\frac{\partial r}{\partial x} + f^1(r)\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x}\right\} + \frac{f^1(r)}{r}$$

i.e. $\frac{\partial^2 u}{\partial x^2} = x\left\{\frac{1}{r}f^{11}(r)\left(\frac{x}{r}\right) + f^1(r)\left(-\frac{1}{r^2}\right)\left(\frac{x}{r}\right)\right\} + \frac{f^1(r)}{r}$

or $\frac{\partial^2 u}{\partial x^2} = x\left(\frac{x}{r}\right)\left\{\frac{f^{11}(r)}{r} - \frac{f^1(r)}{r^2}\right\} + \frac{f^1(r)}{r}$

$\frac{\partial^2 u}{\partial x^2} = \frac{x^2}{r^2}f^{11}(r) + \frac{f^1(r)}{r}\left\{1 - \frac{x^2}{r^2}\right\}$ (1)

$$\frac{\partial^2 u}{\partial y^2} = \frac{y^2}{r^2}f^{11}(r) + \frac{f^1(r)}{r}\left\{1 - \frac{y^2}{r^2}\right\} \quad (2)$$

Similarly,

$\frac{\partial^2 u}{\partial z^2} = \frac{z^2}{r^2}f^{11}(r) + \frac{f^1(r)}{r}\left\{1 - \frac{z^2}{r^2}\right\}$ (3)

Adding equations. (1), (2) and (3)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{f^{11}(r)}{r^2}(x^2 + y^2 + z^2) + \frac{f^1(r)}{r}\left\{3 - \frac{x^2 + y^2 + z^2}{r^2}\right\}$$

$$= f^{11}(r) + \frac{f^1(r)}{r}(3-1) = f^{11}(r) + 2\frac{f^1(r)}{r}$$

Exercise:

1. Verify $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ where

(i) $z = x^3 + y^3 - 3axy$ (ii)$z = tan^{-1}(x^2 + y^2)$

(iii)$z = log(x \sin y + y \sin x)$

2. If$u = x^2 \, tan^{-1}\left({}^{y}/_{x}\right) - y^2 \, tan^{-1} \, {}^{x}/_{y}$, prove that$\frac{\partial^2 u}{\partial y \partial x} = \frac{x^2 - y^2}{x^2 + y^2}$

3. If$z = \frac{x^2 + y^2}{x \ + y}$, show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$

4. If $u = (x-y)(y-z)(z-x)$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

5. If $z = e^{xy}$, prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{z}\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y^2}\right)^2\right]$

6. If$u = \frac{y}{z} + \frac{z}{x}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

7. If $x = r\cos\theta\,, y = r\sin\theta$, show that$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$

Total derivatives, Differentiation of Composite and Implicit functions

Total differential and Total derivative:

Let $z = f(x, y)$ be a differentiable function of two variables, *x* and *y* then **total differential (or exact differential) '*dz*'** is defined by $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$ (1)

Further, if $z = f(x, y)$ where $x = x(t)$, $y = y(t)$ i.e. x and y are themselves functions of an independent variable t, then total **derivative of z** is given by

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \qquad (2)$$

Similarly, the total differential of a function $u = f(x, y, z)$ is defined by

$$du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \qquad (3)$$

Further, if $u = f(x, y, z)$ and if $x = x(t)$, $y = y(t)$, $z = z(t)$, then the total derivative of u is given by $\frac{du}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$. (4)

Differentiation of implicit functions:

An implicit function with x as an independent variable and y as the dependent variable is generally of the form $z = f(x, y) = 0$. This gives $\left(\frac{dz}{dx}\right) = \left(\frac{df}{dx}\right) = 0$. Then, by virtue of expression (2) above, $\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$ or $\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$

and hence $0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$, so that $\frac{dy}{dx} = -\frac{\left(\partial f/\partial x\right)}{\left(\partial f/\partial y\right)}$. (5)

Differentiation of composite functions:

Let z be function of x and y and that $x = \varphi(u, v)$ and $y = \phi(u, v)$ are functions of u and v then,

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \quad \& \frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

Similarly, if $z = f(u, v)$ are functions of u and v and if$u = \varphi(x, y)$ and $y = \phi(x, y)$ are functions of x and y then,

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \quad \& \frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}.$$

Note:

1) The above formulae can be extended to functions of three or more variables and expressions are called Chain rule for partial differentiation.

2) The second and higher order partial derivatives of $z = f(x, y)$ can be obtained by repeated applications of the above expressions.

Examples:

1. Find the total differential of

(i) $e^x[x \sin y + y \cos y]$ (ii)e^{xyz}

Solution:

(i) Let $z = f(x, y) = e^x[x \sin y + y \cos y]$ then

$$\frac{\partial z}{\partial x} = e^x[(1 + x) \sin y + y \cos y]$$

and$\frac{\partial z}{\partial y} = e^x[(1 + x) \cos y - y \sin y]$

Hence, using formula (1), $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = e^x[(1 + x) \sin y + y \cos y]dx + e^x[(1 + x) \cos y - y \sin y]dy$

(ii) Let $u = f(x,y,z) = e^{xyz}$ Then

$\frac{\partial u}{\partial x} = (yz)e^{xyz}; \frac{\partial u}{\partial y} = (xz)e^{xyz}; \frac{\partial u}{\partial z} = (xy)e^{xyz}$

$\therefore$Total differential of $z = f(x,y,z)$ is (see formula (3) above)

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = e^{xyz}(yzdx + zxdy + xydz)$$

2. Find $\frac{dz}{dt}$ if

(i) $z = xy^2 + x^2y$,where $x = at^2$,$y = 2at$

(ii) $z = tan^{-1}\left({}^{y}/_{x}\right)$,where$x = e^t - e^{-t}$,$y = e^t + e^{-t}$

Solution:

(i) Consider $z = xy^2 + x^2y$

$\frac{\partial z}{\partial x} = y^2 + 2xy \,\&\, \frac{\partial z}{\partial y} = 2xy + x^2$

Since $x = at^2 \,\&\, y = 2at$, $\frac{dx}{dt} = 2at \,\&\, \frac{dy}{dt} = 2a$

Hence, using formula (2),

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (y^2 + 2xy)(2at) + (2xy + x^2)(2a)$$

To get $\left(\frac{dz}{dt}\right)$ explicitly in terms of t, substitute

$x = at^2 \,\&\, y = 2at$, to get $\left(\frac{dz}{dt}\right) = (4a^2t^2 + 4a^2t^3)2at +$

$(2at^2 \times 2at + a^2t^4)2a = 2a^3(8t^3 + 5t^4)$

(ii) Consider$z = tan^{-1}\left({}^{y}/_{x}\right)$

$\frac{\partial z}{\partial x} = \frac{-y}{x^2+y^2}, \frac{\partial z}{\partial y} = \frac{x}{x^2+y^2}$

Since $x = e^t - e^{-t} \,\&\, y = e^t + e^{-t}$,

$\frac{dx}{dt} = e^t + e^{-t} = y \frac{dy}{dt} = e^t - e^{-t} = x$

Hence $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$ (see equation (2))

$$= \left(\frac{-y}{x^2+y^2}\right)y + \left(\frac{x}{x^2+y^2}\right)x = \left(\frac{x^2-y^2}{x^2+y^2}\right)$$

Substituting $x = e^t - e^{-t}$&$y = e^t + e^{-t}$,

$$\frac{dz}{dt} = \frac{-2}{e^{2t}+e^{-2t}}$$

3. Find $\left(\frac{dy}{dx}\right)$ if

(i) $\boldsymbol{x^y + y^x}$=c

(ii) $x + e^y = 2xy$

Solution:

(i) Let $z = f(x,y) = x^y + y^x$= Constant.

using formula (5)

$$\frac{dy}{dx} = -\frac{\partial f/_{\partial x}}{\partial f/_{\partial y}}$$

But $\frac{\partial f}{\partial x} = yx^{y-1} + y^x \log y$ and $\frac{\partial f}{\partial y} = x^y \log x + xy^{x-1}$

putting these in (1),

$$\frac{dy}{dx} = -\left\{\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}\right\}$$

(ii) Let $z = f(x,y) = e^x + e^y - 2xy$=Constant

Now, $\frac{\partial f}{\partial x} = e^x - 2yand\frac{\partial f}{\partial y} = e^y - 2x$ Using this in (1),

$$\frac{dy}{dx} = -\left\{\frac{\partial f/_{\partial x}}{\partial f/_{\partial y}}\right\} = -\left\{\frac{e^x - 2y}{e^y - 2x}\right\}$$

4. (i) If $z = f(x,y)$,where $x = r\cos\theta$,$y = r\sin\theta$ show that

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

(ii) If $z=f(x,y)$,where$x=e^u+e^{-v}$&$y=e^{-u}-e^v$, show that

$$\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}$$

Solution:

(i) As $x=r\cos\theta$ and$y=r\sin\theta$,

$\frac{\partial x}{\partial r}=\cos\theta,\frac{\partial x}{\partial \theta}=-r\sin\theta;\frac{\partial y}{\partial r}=\sin\theta\&\frac{\partial y}{\partial \theta}=r\cos\theta.$

Using chain rule,

$$\left(\frac{\partial z}{\partial r}\right)=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial z}{\partial x}(\cos\theta)+\frac{\partial z}{\partial y}(\sin\theta)$$

$$\left(\frac{\partial z}{\partial \theta}\right)=\frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}=\frac{\partial z}{\partial x}(-r\sin\theta)+\frac{\partial z}{\partial y}(r\cos\theta)$$

Squaring on both sides and adding the above equations,

$$\left(\frac{\partial z}{\partial r}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta+\left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta+2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\sin\theta\cos\theta$$

$$\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta+\left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta-2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\sin\theta\cos\theta$$

Adding the above equations,

$$\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2=\left\{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right\}(\cos^2\theta+\sin^2\theta)$$

$=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2.$

(ii) Given $x=e^u+e^{-v}$&$y=e^{-u}-e^v$,

$\frac{\partial x}{\partial u}=e^u,\frac{\partial x}{\partial v}=-e^{-v},\frac{\partial y}{\partial u}=-e^{-u}\&\frac{\partial y}{\partial v}=-e^v$

Using Chain rule (6)

$$\left(\frac{\partial z}{\partial u}\right)=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}=\frac{\partial z}{\partial x}(e^u)+\frac{\partial z}{\partial y}(-e^{-u})$$

$$\left(\frac{\partial z}{\partial v}\right)=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}=\frac{\partial z}{\partial x}(-e^{-v})+\frac{\partial z}{\partial y}(-e^{v})$$

$$\therefore\left(\frac{\partial z}{\partial u}\right)-\left(\frac{\partial z}{\partial v}\right)=\frac{\partial z}{\partial x}(e^u+e^{-v})-\frac{\partial z}{\partial y}(e^{-u}-e^{v})$$

$$=\frac{\partial z}{\partial x}x-\frac{\partial z}{\partial y}y$$

5. (i) If $\boldsymbol{u=f(xz,y/z)}$ then show that $\boldsymbol{x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}-z\frac{\partial u}{\partial z}=0}$

(ii) If $H=f(x-y,y-z,z-x)$, show that $\frac{\partial H}{\partial x}+\frac{\partial H}{\partial y}+\frac{\partial H}{\partial z}=0$

Solution:

(i) Let $u=f(v,w)$, where $v=xz$ and $w={}^{y}/_{z}$

$\frac{\partial v}{\partial x}=z,\frac{\partial v}{\partial y}=0,\frac{\partial v}{\partial z}=x\ \&\frac{\partial w}{\partial x}=0,\frac{\partial w}{\partial y}={}^{1}/_{z},\frac{\partial w}{\partial z}={}^{-y}/_{z^2}$

Using chain rule,

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial w}\frac{\partial w}{\partial x}=\frac{\partial u}{\partial v}(z)+\frac{\partial u}{\partial w}(0)=z\frac{\partial u}{\partial v}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial w}\frac{\partial w}{\partial y}=\frac{\partial u}{\partial v}(0)+\frac{\partial u}{\partial w}\left({}^{1}/_{z}\right)=\frac{1}{z}\frac{\partial u}{\partial w}$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial v}\frac{\partial v}{\partial z}+\frac{\partial u}{\partial w}\frac{\partial w}{\partial z}=\frac{\partial u}{\partial v}(x)+\frac{\partial u}{\partial w}\left({}^{-y}/_{z^2}\right)=x\frac{\partial u}{\partial v}-\frac{y}{z^2}\frac{\partial u}{\partial w}$$

From these,

$$x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}-z\frac{\partial u}{\partial z}=xz\frac{\partial u}{\partial v}-\frac{y}{z}\frac{\partial u}{\partial w}-z\left(x\frac{\partial u}{\partial v}-\frac{y}{z^2}\frac{\partial u}{\partial w}\right)=0$$

(ii) Let $H=f(u,v,w)$ Where $u=x-y,v=y-z,w=z-x$

Now $\frac{\partial u}{\partial x}=1,\frac{\partial u}{\partial y}=-1,\frac{\partial u}{\partial z}=0$

$$\frac{\partial v}{\partial x}=0,\frac{\partial v}{\partial y}=1,\frac{\partial v}{\partial z}=-1$$

$$\frac{\partial w}{\partial x} = -1, \frac{\partial w}{\partial y} = 0, \frac{\partial u}{\partial z} = 1$$

Using Chain rule,

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial H}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial H}{\partial w}\frac{\partial w}{\partial x} = \frac{\partial H}{\partial u}(1) + \frac{\partial H}{\partial v}(0) + \frac{\partial H}{\partial w}(-1)$$

$$\frac{\partial H}{\partial y} = \frac{\partial H}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial H}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial H}{\partial w}\frac{\partial w}{\partial y} = \frac{\partial H}{\partial u}(-1) + \frac{\partial H}{\partial v}(1) + \frac{\partial v}{\partial w}(0)$$

$$\frac{\partial H}{\partial z} = \frac{\partial H}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial H}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial H}{\partial w}\frac{\partial w}{\partial z} = \frac{\partial H}{\partial u}(0) + \frac{\partial H}{\partial v}(-1) + \frac{\partial H}{\partial w}(1)$$

Adding the above equations,

$\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial H}{\partial z} = 0.$

6. The length and breadth of a rectangle are increasing at the rate of 1.5cm/sec and 0.5 cm/sec respectively. Find the rate at which the area is increasing at the instant when length is 40 cm and breadth is 30 cm respectively.

Solution: Let x be the length, y be the breadth, s be the area at any time t

So $s = xy$

It is required to find $\frac{ds}{dt}$ when $x = 40cm$and $y = 30cm$, given that

$\frac{dx}{dt} = 1.5\ cm/\text{sec} \frac{dy}{dt} = 0.5cm/sec$

$$\frac{ds}{dt} = \frac{\partial s}{\partial x}\frac{dx}{dt} + \frac{\partial s}{\partial y}\frac{dy}{dt}$$

$$= y\frac{dx}{dt} + x\frac{dy}{dt} = 30 \times 1.5 + 40 \times 0.5 = 65$$

Therefore increasing at the rate $65cm^2/sec$.

7. If $z = f(u,v)$ and $u = ax + by, v = ay - bx$, show that

$$\boldsymbol{\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (a^2 + b^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)}$$

Proof: $\frac{\partial u}{\partial x} = a, \frac{\partial u}{\partial y} = b, \frac{\partial v}{\partial x} = -b, \frac{\partial v}{\partial y} = a$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}a - \frac{\partial z}{\partial v}b$$

Apply chain rule $z \to {}^{\partial z}/_{\partial x}$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial}{\partial u}\left(a\frac{\partial z}{\partial u} - b\frac{\partial z}{\partial v}\right)\cdot a + \frac{\partial}{\partial v}\left(a\frac{\partial z}{\partial u} - b\frac{\partial z}{\partial v}\right)\cdot(-b)$$

$$\frac{\partial^2 z}{\partial x^2} = a^2\frac{\partial^2 z}{\partial u^2} - ab\frac{\partial^2}{\partial u\partial v} - ab\frac{\partial^2}{\partial v\partial u} + b^2\frac{\partial^2 z}{\partial v^2}$$

Similarly, $\frac{\partial^2 z}{\partial y^2} = b^2\frac{\partial^2 z}{\partial u^2} + ab\frac{\partial^2}{\partial u\partial v} + ab\frac{\partial^2}{\partial v\partial u} + a^2\frac{\partial^2 z}{\partial v^2}$

$$LHS = (a^2 + b^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$$

Exercise:

1. Find the total differentials of

(i) $xyz + (xyz)^{-1}$ (ii) $x^2y + y^2z + z^2x$

ANS: (i) $\left[1 - \frac{1}{(xyz)^2}\right](yzdx + zxdy + xydz)$, (ii) $(2xy + z^2)dx + (2yz + x^2)dy + (y^2 + 2xz)$

2. Find $\left(\frac{du}{dt}\right)$ If

(i) $u = x^2 - y^2, x = e^t \cos t, y = e^t \sin t$

(ii) $u = \sin x\, y^2, x = \log t, y = e^t$

(iii) $u = xy + yz + zx, x = {}^1/_t, y = e^t, z = e^{-t}$

(iv) $u = \log(x + y + z), x = e^{-t}, y = \sin t, z = \cos t$

ANS: (i) $2e^{2t}$, (ii) $e^{2t}\left[\frac{1}{t} + 2\log t\right]\cos(e^{2t}\log t)$, (iii) $\frac{-(e^t + e^{-t})}{t^2} + \frac{2}{t}$, (iv) $\frac{(\cos t - \sin t - e^{-t})}{(\cos t + \sin t + e^{-t})}$

3. Find $\left(\frac{dy}{dx}\right)$ each of the following cases:-

(i) $x\sin(x - y) = (x + y)$ (ii) $(\cos x)^y = (\sin y)^x$

ANS: (i) $\frac{(x\cos(x-y)+\sin(x-y)-1)}{x\cos(x-y)+1}$, (ii) $\frac{y\tan x+\log\sin y}{\log\cos x-x\cot y}$

4. If $z=f(x,y)$ and $x=u-v, y=uv$ Prove that
(i) $(u+v)\frac{\partial z}{\partial x}=u\frac{\partial z}{\partial x}-v\frac{\partial z}{\partial v}$ (ii) $(u+v)\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}$

5. If $u=x^2-y^2, z=2r-3s+4, y=-r+8s-5$, Prove that $\frac{\partial u}{\partial r}=4x+2y$

6. If $z=f(r,s)$, where $r=x+at, y=y+bt$ Prove that $\frac{\partial u}{\partial t}=a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}$.

7. If $z=f(u,v)$, $u=x^2-y^2, v=2xy$ Show that $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=4(x^2+y^2)\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}\right)$

If $u=f(x,y), x+y=2e^{\theta}\cos\varphi, x-y=2ie^{\theta}\sin\varphi$ prove that $\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial \varphi^2}=4xy\frac{\partial^2 z}{\partial x\partial y}$.

Jacobians:

Carl Gustav Jacob Jacobi; 10 December 1804 – 18 February 1851) was a German mathematician, who made fundamental contributions to elliptic functions, dynamics, differential equations, and number theory.

If ‘u’ and ‘v’ are functions of variables ‘x’ and ‘y’, then the determinant

$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$ is called the Jacobian of u, v with respect to x, y and denoted by $\frac{\partial(u,v)}{\partial(x,y)}$. Likewise, it is possible to define Jacobian of more variables in similar manner. These have an important application in coordinate transformation from one system to another.

Note:

1. If J $=\frac{\partial(u,v)}{\partial(x,y)} \neq 0$, then $\frac{\partial(x,y)}{\partial(u,v)}$exists and is known as inverse of 'J' with JJ '=1.
2. If u,v are functions of r, s and r, s are functions of x, y then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} . \frac{\partial(r,s)}{\partial(x,y)}$
3. If J $=\frac{\partial(u,v)}{\partial(x,y)} = 0$, then u,v are said to be functionally dependent. In this case u can be expressed as a function of v and vice versa or functional dependence can be expressed as f(u,v) = 0.

Examples:

1. If x = r cosθ, y = r sinθ find $\frac{\partial(x,y)}{\partial(r,\theta)}$.

Solution: By definition we have

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

The above are the transformations between ~~Cartesian and~~ *scdsdaaaavvv* polar coordinates. The inverse transforms are $r = \sqrt{x^2+y^2}$and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

It can be verified that the Jacobian $J^1 = \frac{\partial(r,\theta)}{\partial(x,y)}$of these functions is 1/r and therefore $JJ^1 = 1$.

2. If $x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta$ find $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$

Solution: $J=\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=\begin{vmatrix} x_r & x_\theta & x_\phi \\ y_r & y_\theta & y_\phi \\ z_r & z_\theta & z_\phi \end{vmatrix}=$

$$\begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

On expanding by the last row we get, J = $r^2\sin\theta\,(\cos^2\theta + \sin^2\theta)$ = $r^2\sin\theta$

The above are the transformations between Cartesian and spherical coordinates. The inverse transforms are $r = \sqrt{x^2+y^2+z^2}$, $\theta = \tan^{-1}\left(\sqrt{\frac{x^2+y^2}{z^2}}\right)$ and $\varphi = \tan^{-1}\left(\frac{y}{x}\right)$.

It can be easily verified that the Jacobian $J^1=\frac{\partial(r,\theta,\varphi)}{\partial(x,y,z)}$ of these functions is $1/r^2\sin\theta$ and therefore $JJ^1 = 1$.

3.If $u = x + 3y^2 - z^3,\ v = 4x^2yz,\ w = 2z^2 - xy$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ at (1, -1, 0).

Solution: $J=\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix}$ at (1,-1, 0)

$J=\begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix}=20$

4. If x = u(1-v), y = uv then compute J and J^1 and verify that J J^1=1.

Solution: J = $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$ = u(1-v) + uv = u.

To find J^1, express u and v in terms of x and y.

Given x = u-uv → x= u-y → u = x + y, $v = \frac{y}{x+y}$

$J^1 = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix} = \frac{1}{x+y} = \frac{1}{u}$.

J J^1 =1.

5. If $u = \frac{x+y}{1-xy}$ and $v = tan^{-1}x + tan^{-1}y$ find $\frac{\partial(u,v)}{\partial(x,y)}$.

Solution: $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

$$= \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = 0$$

Since Jacobian is zero, u and v are functionally dependent.

Relation between u and v is tan $v = u$.

Exercise:

1. .If $u + v = e^x \cos y\,, u - v = e^x \sin y$, find $J\left(\frac{u,v}{x,y}\right)$.

[Hint: Start from $u = \frac{e^x}{2}(\cos y + \sin y)$ and$v = \frac{e^x}{2}(\cos y - \sin y)$

2. If $u = xyz, v = xy + yz + zx, w = x + y + z$, show that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (x - y)(y - z)(z - x).$$

3. If $x = e^u \sec v\,, y = e^u \tan v$, show that

$$\frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = 1.$$

4 .If $x + y + z = u, y + z = v, z = uvw$, find the value of $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

5. If $u = 2xy, v = x^2 - y^2, x = r\cos\theta\,, y = r\sin\theta\,, find\, \frac{\partial(u,v)}{\partial(r,\theta)}$.

Answers: 1. $-\frac{e^{2x}}{2}$, 4. u^2v, 5. - $4r^3$.

Maxima and Minima of *f(x, y)*:

In mathematics, the maximum and minimum (plural: maxima and minima) of a function, known collectively as extrema (singular: extremum), are the largest and smallest value that the function takes at a point within a given neighborhood.

A function *f* (*x*, *y*) is said to have a Maximum value at (*a*,*b*) if there exists a neighborhood point of (*a*,*b*) (say (*a*+*h*, *b*+*k*)) such that *f* (*a*, *b*) >*f* (*a*+*h*, *b*+*k*).

Similarly, a function *f* (*x*, *y*) is said to have a Minimum value at (*a*,*b*) if there exists a neighborhood point of (*a*,*b*) (say (*a*+*h*,*b*+*k*)) such that *f* (*a*, *b*) <*f* (*a*+*h*, *b*+*k*).

Necessary condition:

Consider a function *f* = *f*(*x*, *y*) where *x* and *y* are independent real variables. Suppose this function is continuous and has continuous partial derivatives of second order in a region *R* in the *x y* – plane. Let

(a, b) be a point in the region. Then (a, b) is called a extremum point of $f(x, y)$, if $f(x, b)$ is extremum at $x=a$ and $f(a, y)$ is extremum at $y=b$. Since $f(x, b)$ is a function of a single variable x, a necessary condition for the zero at $x = a$ that is $\frac{\partial f}{\partial x}(a, b) = 0$Similarly a necessary condition for $f(a, y)$ to be extremum at $y = b$ is that $\frac{\partial f}{\partial y}(a, b) = 0$. Thus, we have the following result.

A necessary condition for $f(x, y)$ to have an extremum at a point (a, b) is that $\frac{\partial f}{\partial x}(a, b) = 0$and $\frac{\partial f}{\partial y}(a, b) = 0$.

An extremum point of $f(x, y)$ is also called a stationary point or a critical point.

Working procedure for finding Maxima and Minima of f(x,y):

Given f (x, y)

1. Find p, q, r, s, t,
2. Solve p = 0, q = 0 simultaneously
3. Let the points be (x_1, y_1), (x_2, y_2)….. .At each such point find $rt - s^2$.
4. At (x_1, y_1) **$rt - s^2 > 0$, $r > 0 \rightarrow$ point of minima and $f(x_1, y_1)$ is minimum value of the function** **$rt - s^2 > 0$, $r < 0 \rightarrow$ point of maxima and $f(x_1, y_1)$ is maximum value of the function** **$rt - s^2 < 0 \rightarrow f(x_1, y_1)$ is neither maximum nor minimum and (x_1, y_1) is called saddle point** **$rt - s^2 = 0 \rightarrow$ further investigation needed**

Example

1. Show that z(x, y) = $xy(a$-x-$y)$, $a>0$, is maximum at the point (a/3,a/3).

Solution: For the given function,

we have $p = \frac{\partial f}{\partial x} = y(a - 2x - y) and\ q = \frac{\partial f}{\partial y} = x(a - x - 2y)$

Solving $p = 0$, $q = 0$, we get (a/3,a/3). Hence (a/3,a/3) is a critical point.

$$r = \frac{\partial^2 f}{\partial x^2} = -2y,\ s = \frac{\partial^2 f}{\partial x \partial y} = a-2x-2y,\ t = \frac{\partial^2 f}{\partial y^2} = -2x\ and\ rt - s^2$$
$$= 4xy - (a - 2x - 2y)^2$$

At the point (a/3,a/3), $r < 0$, and $rt - s^2 > 0$.

It follows that $f(x, y)$ is maximum at the point (a/3,a/3). The maximum value is (1/27) a^3.

2. Examine the following functions for extreme values $f = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

Solution: Given $f = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$f_x = 4x^3 - 4x + 4y$, $f_y = 4y^3 - 4x - 4y$

$f_{xy} = 4$, $f_{xx} = 12x^2 - 4$, $f_{yy} = 12y^2 - 4$

Solving $f_x = 0$, $f_y = 0$,

$4x^3 - 4x + 4y = 0 \rightarrow (1)$ and $4y^3 + 4x - 4y = 0 \rightarrow (2)$

Adding (1) & (2), we get $4(x^3 + y^3) = 0 \Rightarrow x^3 + y^3 = 0$ so, $y = -x$

Substitute $y = -x$ in (1), we get

$4x^3 - 4x - 4x = 0 \Rightarrow 4x^3 - 8x = 0 \Rightarrow x^3 - 2x = 0 \Rightarrow x(x^2 - 2) = 0$

$\Rightarrow x = 0\ \&\ x^2 - 2 = 0$

$\Rightarrow x = 0\ \&\ x = \pm\sqrt{2}$ and $y = 0\ \&\ y = \mp\sqrt{2}$

$\therefore$ The critical points are (0,0), $(+\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, +\sqrt{2})$

At (0,0):

$r = f_{xx}(0,0) = -4$, $s = f_{xy}(0,0) = 4$, $t = f_{yy}(0,0) = -4$ and $rt - s^2 = (-4)(-4) - 16 = 0$

$\Rightarrow$ further investigation needed.

At $(\pm\sqrt{2}, \mp\sqrt{2})$:

$r = f_{xx}(\pm\sqrt{2}, \mp\sqrt{2}) = 24 - 4 = 20$, $s = f_{xy}(\pm\sqrt{2}, \mp\sqrt{2}) = 4 > 0$ and $t = f_{yy}(\pm\sqrt{2}, \mp\sqrt{2}) = 20$

$\therefore s^2 - rt = 16 - (20)(20) = 16 - 400 = -384 < 0$

$\Rightarrow$The function is minimum at $(\pm\sqrt{2}, \mp\sqrt{2})$ and $f_{min} = f(\pm\sqrt{2}, \mp\sqrt{2}) = -8$

3. Find Maximum and Minimum of $2(x^2 - y^2) - x^4 + y^4$.

Solution: Let $u = 2(x^2 - y^2) - x^4 + y^4$

For the stationary points $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$.

$$u_x = 4x - 4x^3 = 0 \Rightarrow 4x(1 - x^2) = 0 \Rightarrow x = 0, 1, -1$$

$$u_y - 4y + 4y^3 = 0 \Rightarrow -4y(1 - y^2) = 0 \Rightarrow y = 0, 1, -1$$

Stationary points are $(0,0), (0,1), (0,-1), (1,0), (1,1), (1,-1), (-1,0), (-1,1)$ and $(-1,-1)$

$r = u_{xx} = 4 - 12x^2$, $s = u_{xy} = 0$, $t = u_{yy} = -4 + 12y^2$

Point	r	s	t	$rt - s^2$	r	Nature of the point
$(0,0)$	4	0	-4	< 0	> 0	*Saddle pt*
$(0,1)$	4	0	8	> 0	> 0	*Min pt*
$(0,-1)$	4	0	8	> 0	> 0	*Min pt*
$(1,0)$	-8	0	-4	> 0	< 0	*Max pt*
$(1,1)$	-8	0	8	< 0	< 0	*Saddle pt*
$(1,-1)$	-8	0	8	< 0	< 0	*Saddle pt*
$(-1,0)$	-8	0	-4	> 0	< 0	*Max pt*
$(-1,1)$	-8	0	8	< 0	< 0	*Saddle pt*
$(-1,-1)$	-8	0	8	< 0	< 0	*Saddle pt*

u is maximum at (1,0),(-1,0), Maximum value is 1. U is minimum at (0,1),(0,-1), minimum value is -1

4. A rectangle box, open at the top to have a volume of 32 cubic units, what must be the dimension so that total surface area of the box is a minimum.

Solution: Let x, y and z be the dimension of the box. (Length, breadth and height of the box) Then Volume $V = xyz$

Surface area is $s = xy + 2yz + 2zx = 192$

Given $V = 32$ so that $xyz = 32 \Rightarrow z = \frac{32}{xy}$

$$s = xy + 2y\frac{32}{xy} + 2x\frac{32}{xy} = xy + \frac{64}{x} + \frac{64}{y} = f(x, y)$$

$f_x = y - \frac{64}{x^2},\ f_y = x - \frac{64}{y^2}$, $f_x = 0,\ f_y = 0 \Rightarrow y - \frac{64}{x^2} = 0,\quad x - \frac{64}{y^2} = 0$

$$x^2y = 64, \quad y^2x = 64$$

$$\Rightarrow x^2y = y^2x \Rightarrow x = y \Rightarrow x^2y = x^3 = 64 \Rightarrow x = y = 4$$

Therefore $(4, 4)$ is critical point.

$$r = f_{xx} = \frac{128}{x^3} s = f_{xy} = 1 \quad t = f_{yy} = \frac{128}{y^3}$$

At $(4, 4) r = 2,\ s = 1,\ t = 2$

$rt - s^2 = 4 - 1 = 3 > 0$, r > 0.Therefore (4, 4) is minimum point.

Minimum volume $f(4, 4) = 48$.

Then dimensions of box which make surface area minimum are $x = 4,\ y = 4$ and $z = 2$

Exercise:

1) Find the extreme values of $f = x^3y^2\ (1 - x - y)$

2) Examine the function $f(x,y) = 1 + \sin(x^2 + y^2)$ for extremum.

3) Find the maxima and minima of the functions $f(x,y) = x^3 + y^3 - 3axy$, $a > 0$ is a constant.

4) The temp T at any point (x, y, z) in space is $T = 400xyz^2$. Find the highest temp on the surface of the unit sphere $x^2 + y^2 + z^2 = 1$.

ANS: (1) Maximum at (1/2,1/3), maximum value =1/432.

(2) Minimum at (0,0), minimum value =1

(3) Minimum at (a,a), minimum value = - a^3

(4) 50

Lagrange Multiplier method or Constrained Extrema

In practical situations one often comes across problems of determining extrema of functions of many variables with the variables constrained by certain conditions. In other words one requires maxima or minima of functions where the variables satisfy certain equality or inequality conditions, called constraints.

This is a special method of obtaining stationary values of a function of three independent variables when the variables are subject to some conditions. Suppose it is required to find extrema of $f(x, y, z)$ subject to the condition that $\varphi(x, y, z) = 0$

To find the values of x, y, z for which f(x, y, z) is stationary (maximum or minimum):

1. Find an auxiliary function F(x, y, z) defined by $F(x, y, z) = f(x, y, z) + \lambda\varphi(x, y, z)$, where λ is a constant.
2. Determine the partial derivatives $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ and equate each of these to zero. Thus we obtain three equations of the form
$F_1(x, y, z, \lambda) = 0, F_2(x, y, z, \lambda) = 0, F_3(x, y, z, \lambda) = 0.$
These equations together with the constrain gives a value for λ. For this value of λ, find x, y, z. Then (x, y, z) is a stationary point and the corresponding value is the stationary value. This method of obtaining

stationary values of f(x, y, z) is called the **Lagrange's method of multipliers**.

Examples:

1. The temperature at any point (x, y, z) in space $= 400xy^2z$. Find the highest temperature at the surface of unit sphere $x^2 + y^2 + z^2 = 1$ using Lagrange's method of undetermined multipliers.

Solution: Let $f(x, y, z) = 400xy^2z$

$$\emptyset(x, y, z) = x^2 + y^2 + z^2 - 1$$

$$F(x, y, z) = f(x, y, z) + \lambda\varphi(x, y, z)$$

$$F(x, y, z) = 400xy^2z + \lambda(x^2 + y^2 + z^2 - 1) \quad (1)$$

$$F_x = 400y^2z + 2\lambda x = 0 \Rightarrow \frac{200y^2z}{x} = -\lambda \quad (2)$$

$$F_y = 800xyz + 2\lambda y = 0 \Rightarrow \frac{400xyz}{y} = -\lambda \quad (3)$$

$$F_z = 400xy^2 + 2\lambda z = 0 \Rightarrow \frac{200xy^2}{z} = -\lambda \quad (4)$$

Solving (2),(3),(4), we get $x = \pm\frac{1}{2}, y = \pm\frac{1}{\sqrt{2}}, z = \pm\frac{1}{2}$

The highest temperature is $T = 400xy^2z = 400 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = 50.$

2. A tank is in the form of a rectangular box open at the top. Find its dimensions if its inner surface area is 192 m^2 and maximum capacity.

Solution: Let the length, breadth and height of the box be x, y, z meters respectively. Then volume $V = xyz$. It is given that $xy + 2yz + 2zx = 192$.

To determine x, y, z so that V, subject to the condition, is maximum.

$$V = xyz + \lambda(xy + 2yz + 2zx - 192)$$

At the stationary points, $\frac{\partial V}{\partial x} = 0, \frac{\partial V}{\partial y} = 0, \frac{\partial V}{\partial z} = 0.$

$$yz + \lambda(y + z) = 0 \quad (1)$$

$$zx + \lambda(y + 2z) = 0 \qquad (2)$$

$$xy + \lambda(2y + 2x) \qquad (3)$$

$$xy + 2yz + 2zx = 192) \qquad (4)$$

Solving (1), (2), (3) we get $x = -4\lambda, y = -4\lambda, z = -2\lambda$,
from (4), $\lambda = -2, x = 8, y = 8, z = 4; (\lambda =$
2 is not admissible because this give $x, y, z < 0$. Thus, for V to be maximum, length = 8m, breadth = 8, and height = 4m with volume = 256m^3.

Exercise:

1. A tent on a square base of side x has its four sides vertical, of height y and top in the form of a regular pyramid of height h. If the capacity of the tent is given, find x and y in terms of h in order that the canvas required for its construction is to be a minimum using Lagrange's multiplier method.
2. Find the dimensions of the largest rectangular parallelepiped that can be enclosed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ using Lagrange's multiplier method.

ANS: 1. $x = \sqrt{5}h, y = \frac{h}{2}$ 2. $x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$.

Video Links:

1. Partial derivatives
https://www.youtube.com/watch?v=P43qNhCj1Gg
https://www.youtube.com/watch?v=Q8mbXy0oJj8

2. Jacobian
https://www.khanacademy.org/math/multivariable-calculus/multivariable-derivatives/jacobian/v/the-jacobian-matrix

3. Multivariable Maxima and Minima
https://www.khanacademy.org/math/multivariable-calculus/applications-of-multivariable-derivatives/optimizing-multivariable-functions-videos/v/multivariable-maxima-and-minima

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