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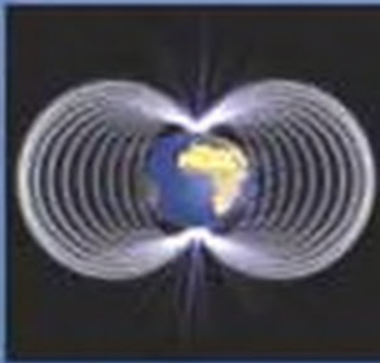
PHYSICS FOR DEGREE STUDENTS



B. Sc. First Year

AS PER UGC MODEL CURRICULUM
(For All Indian Universities)

C.L. ARORA
P.S. HEMNE

S. CHAND

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SECTION I

1. MECHANICS
2. OSCILLATIONS
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1 MECHANICS

LAWS OF MOTION

INTRODUCTION

Every motion, either linear or rotational is governed by certain laws. Mechanics deals with these laws and formulates them in a scientific manner. Linear motion is a translational motion in a straight line. The concept of 'force' is fully explained by Newton through his laws of motion. Force is responsible for linear acceleration while torque for angular acceleration. The basic ideas of state of rest and uniform motion were studied by various philosophers. According to Aristotle (384 – 322 BC), a constant force has to be applied on a body so as to keep it in motion with constant velocity. Later on, Galileo (1564 – 1642 AD) stated that no force is required for a body to move with uniform velocity. Newton (1642 – 1727) was the first person who formulated the laws differentiating 'state of rest' and 'state of uniform motion'. Newton's notion of space as 'Absolute space', isotropic nature for rotational invariance and homogeneity for translational invariance led to understand the laws of motion in inertial frame of reference. Any accelerated frame of reference is a non-inertial frame. Concept of Coriolis force and its applications are studied at the end of this chapter.

1.1 SOME IMPORTANT TERMS

To understand the motion of a body in different frame of references, the terms like particle, event, observer etc. frequently come across.

(i) Particle. A particle is idealy just a quantity of matter, having practically no linear dimensions but possesses only a position; the measure of this quantity of matter being the *mass* of the particle.

(ii) Event. An event stands for anything that occurs suddenly or instantaneously at a point in space. It, thus, involves both, a *position* and *time of occurance*.

(iii) Observer. A person or an equipment that can locate, record, measure and interpret an event is called an observer.

1.2 NEWTON'S NOTIONS OF SPACE

According to Newton '*Absolute space in its own nature without relation to anything external remains always similar and immovable'*. According to Newton, *main classical properties* of space are:

1. Space is three dimensional. The concept of space and time is fundamental to the study of mechanics. All objects occupy space and have a length, a breadth and a height. Space is, therefore, *three dimensional.* This is why the position of a point can be specified completely by the three co-ordinates (x, y, z) or (r, θ, ϕ).

2. Space is flat. Space is flat *i.e.*, it possesses *Eucledian flatness.* This means that the shortest distance between any ***two*** points in space is a straight line. If we take ***three*** points in space to form a triangle, the sum of angles is equal to π. If it is a right angled triangle, then the three sides are related by Pythagoras Theorem *i.e.*, Hypotenuse^2 = Base^2 + Altitude^2

However, according to latest theory, space is *not exactly flat* but somewhat curved. The departure from flatness is very small and can be ignored in the study of classical mechanics.

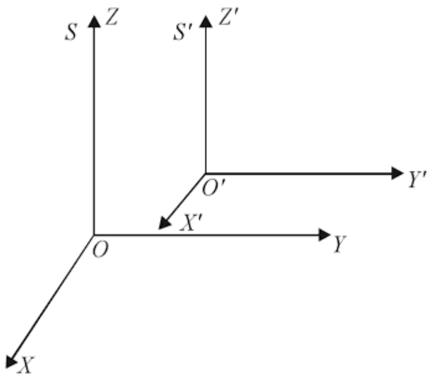

Fig. 1.1

3. Space is homogeneous. In other words, *space is everywhere alike. Homogeneity of free space* (a space in which fields and forces are absent) means *translational invariance* of its properties *i.e.*, the result of an experiment is not altered due to linear displacement of the co-ordinate systems.

4. Space is isotropic. It means that if we consider a point O in space and we move from this point in any direction, the properties are the same *i.e.*, there is nothing to distinguish one direction from the other. In other words, there is no preferred direction in space or one direction is as good as any other direction.

Distinction between homogeneity and isotropy. The *homogeneity* of space means *translational invariance* of the properties of space.

If we consider two co-ordinate systems S and S' displaced with respect to one another as shown in Fig. 1.1 then an experiment performed in system S will give exactly the same results in the system S'.

On the other hand, *isotropy* means *rotational invariance* of free space.

Thus, if we have two co-ordinate system S and S' rotating with respect to one another as shown in Fig. 1.2 then again an experiment performed in system S will give exactly the same result in system S'.

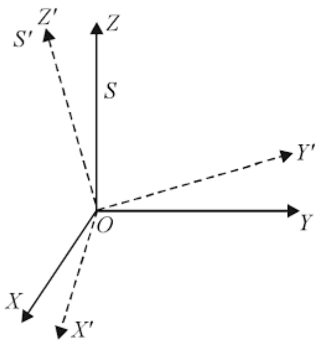

Fig. 1.2

1.3 NEWTON'S LAWS OF MOTION

Its our common experience that a book or a duster lying on a table will continue to remain there (state of rest) untill we remove it or displace it. Thus, all bodies, which are initially at rest will continue to remain at rest. Similarly, a body moving with uniform velocity (state of uniform motion), will continue its uniform motion. These, state of rest or uniform motion will be continued for an infinite time, unless acted upon by an external force. This is nothing but Newton's first law of motion. Newton's three laws of motion are as under.

First Law. *Every body continues to be in its state of rest or of uniform motion in a straight line unless acted upon by an external unbalanced force.*

Second Law. *The rate of change of linear momentum is proportional to the impressed force and takes place in the direction of the applied force.*

Third Law. *To every action, there is equal and opposite reaction. Thus, if* $\vec{F}_{12}$ and $\vec{F}_{21}$ are the forces exerted on each other by two interacting bodies respectively, then we have $\vec{F}_{12} = -\vec{F}_{21}$.

1.4 NEWTON'S FIRST LAW OF MOTION IS SIMPLY A SPECIAL CASE OF SECOND LAW

Newton's second law of motion gives a measure of a force as *the rate of change of linear momentum*. If $\vec{p}$ is the momentum of the particle, given by $\vec{p} = m\vec{v}$, where m is mass and $\vec{v}$ its velocity, then

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v})$$

$$= m\frac{d\vec{v}}{dt} \qquad (m \text{ being constant})$$

$$= m\vec{a}$$

so that, if $\vec{F} = 0$, then $\vec{a} = 0$ which is first law of motion, *i.e.* if no force acts, there will be no change in velocity and hence no acceleration. Thus, the Newton's first law of motion is simply a special case of the second law.

1.5 LIMITATIONS OF NEWTON'S LAWS OF MOTION

The followings are the limitations:

First Limitation. The relation $\vec{F} = m\vec{a}$ (second law) would not hold good in case m does not remain constant (refer article 1.4). The examples are:

(i) *A falling raindrop*, which gathers mass as it falls, due to more water vapour condensing around it. The mass of the drop continuously increases and therefore does not obey the condition for $\vec{F} = m\vec{a}$.

(ii) *A rocket*, which looses part of its mass as it moves forward, in the form of ejected burnt fuel, and

(iii) *Particles with relativistic velocities*; The mass of the particle is given by $m = m_0 / \sqrt{1 - \frac{v^2}{c^2}}$, where m_0 is the rest mass of the particle. The mass of the particle increases as its velocity becomes appreciable fraction of the velocity of light (c).

Second Limitation. Newton's third law of motion implies that the forces exerted by the two interacting bodies over each other are equal and opposite, provided both the forces are measured simultaneously. A simultaneous measurement of the two forces is possible for ordinary practical purposes, if the time taken by the interaction is sufficiently large compared with the time taken by light signal to travel one to the other. This means that the law *ceases to hold good for particles of atomic dimensions,* for which simultaneous measurement of two forces is almost impossible.

1.6 CONSERVATION OF LINEAR MOMENTUM

(From Newton's Laws of motion)

According to Newton's first law of motion, 'a body remains at rest or continues to move with a uniform velocity, so long as no external force is acting on it.' According to Newton's second law of motion, 'the rate of change of linear momentum of a body is proportional to the force acting on it.'

Proof : Using Newton's first law of motion, if a particle of mass m is moving with velocity $\vec{v}$, its linear momentum is $\vec{p} = m\vec{v}$. When a force $\vec{F}$ is applied to it, its momentum changes at the rate $\frac{d\vec{p}}{dt}$.

$$\therefore \qquad \vec{F} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt}$$

Clearly when $\qquad \vec{F} = 0, \; \frac{d\vec{p}}{dt} = 0$ or $\vec{p} = m\vec{v}$ = a constant

i.e., thus, if no force acts, the linear momentum is conserved. Thus Newton's first law of motion is only a special case of second law.

Proof : Using Newton's second law of motion, now consider a system of two particles which are acted only by forces of interaction like Coulomb's forces or gravitational forces and *no external forces act on the system.* Let $\vec{F}_{12}$ be the force exerted by the first particle on the second, known as **action** and $\vec{F}_{21}$ the force exerted by the second on the first, known as **reaction**, then according to Newton's third law of motion, *action and reaction being equal and opposite.*

$$\vec{F}_{12} = -\vec{F}_{21}$$

According to Newton's second law of motion

$$\vec{F}_{12} = \frac{d\vec{p}_2}{dt} = \frac{d(m_2\vec{v}_2)}{dt} = m_2\frac{d\vec{v}_2}{dt}$$

where m_2, $\vec{v}_2$ and $\vec{p}_2$ are the mass, velocity and momentum of the second particle.

Similarly $\qquad \vec{F}_{21} = \frac{d\vec{p}_1}{dt} = \frac{d}{dt}(m_1\vec{v}_1) = m_1\frac{d\vec{v}_1}{dt}$

where m_1, $\vec{v}_1$ and $\vec{p}_1$ are the mass, velocity and momentum of the first particle.

$$\because \qquad \vec{F}_{12} = -\vec{F}_{21} \qquad \vec{F}_{12} + \vec{F}_{21} = 0$$

or $\qquad \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = 0$ or $\frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = 0$

Integrating, we have $\vec{p}_1 + \vec{p}_2$ = a constant or $m_1\vec{v}_1 + m_2\vec{v}_2$ = a constant

The quantity $\vec{p}_1 + \vec{p}_2 = m_1\vec{v}_1 + m_2\vec{v}_2$ represents the total linear momentum of the system. Hence we conclude that

'If Newton's second and third law of motion hold good, the total linear momentum of the system of two particles remains constant'. The above law can be extended to a system of three or more interacting particles. The law of conservation of linear momentum is, therefore, a basic law and is stated as under.

"*The total linear momentum of a system of particles free from the action of external forces and subjected only to their mutual interaction remains constant, no matter how complicated the forces are.*"

1.7 COMPONENTS OF VELOCITY AND ACCELERATION

Motion of a particle in a straight line is known as linear motion. However, it is possible to treat motion in a curved path by considering the *tangential velocity* or by resolving the motion into two linear components. For example, as the earth rotates around the sun in its nearly circular orbit, its speed is approximately constant but its direction is constantly changing. Yet at any particular instant, its velocity has a definite direction. If in fig. 1.3, E represents the earth, it traces a curved path along the arc EA but the instantaneous value of its velocity is along the straight line EB, tangent to the path. Thus at any particular instant, its instantaneous velocity is along tangent to the path, and we refer it as tangential velocity.

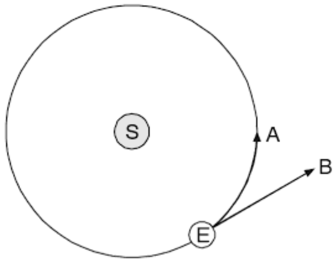

Fig. 1.3

Displacement. Consider a particle moving along a curve in a plane. The position of a point P in a co-ordinate system can be specified by a single vector *i.e.,* the displacement of the particle relative to the origin O of the co-ordinate system. This vector is called the *position vector* of the point and denoted by $\overrightarrow{OP} = \vec{r}$. It gives the magnitude as well as the direction of the *displacement.* If $\hat{r}$ is a *unit vector* along OP *i.e.*, along the direction of $\vec{r}$ then

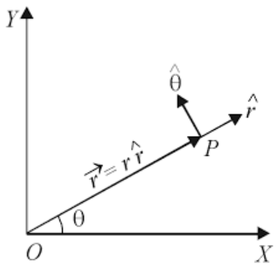

Fig. 1.4

$$\vec{r} = r\hat{r}$$

When a particle is moving along a curve in a plane it has a velocity and an acceleration.

Velocity. *The velocity is the derivative of displacment* $\vec{r}$ *with respect to time t.*

$$\therefore \qquad \text{Velocity } \vec{v} = \frac{d}{dt}(\vec{r}) = \frac{d}{dt}(r\hat{r}) = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{d\theta}\frac{d\theta}{dt}$$

But
$$\frac{d\hat{r}}{d\theta} = \hat{\theta}$$

$$\therefore \qquad \text{Velocity } \vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \qquad ...(i)$$

where
$$\dot{r} = \frac{dr}{dt} \text{ and } \dot{\theta} = \frac{d\theta}{dt}$$

or
$$\vec{v} = \vec{v}_r + \vec{v}_\theta \qquad ...(ii)$$

Components of velocity. The quantity $\vec{v_r} = \dot{r}\hat{r}$ is known as the *radial velocity* and is due to the change in magnitude of r, θ remaining constant and $\vec{v_\theta} = r\dot{\theta}\hat{\theta}$ is known as *transverse velocity* and is due to the change in θ, r remaining constant.

The magnitude of radial velocity $|\vec{v}_r| = \dot{r}$

and the magnitude of transverse velocity $|\vec{v_\theta}| = r\dot{\theta}$

If $\dot{\theta}$ = a constant = ω the angular velocity, then $|\vec{v_\theta}| = r\omega$

The magnitude of velocity $\vec{v}$ is

$$|\vec{v}| = [|\vec{v_r}|^2 + |\vec{v_\theta}|^2]^{\frac{1}{2}} = [\dot{r}^2 + r^2\dot{\theta}^2]^{½}$$

Acceleration. *Acceleration is the derivative of velocity $\vec{v}$ with respect to time t.*

$$\therefore \quad \text{Acceleration } \vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$$

$$= \ddot{r}\hat{r} + \dot{r}\left(\frac{d\hat{r}}{dt}\right) + \dot{r}\,\dot{\theta}\hat{\theta} + r\ddot{\theta}\,\hat{\theta} + r\,\dot{\theta}\left(\frac{d\hat{\theta}}{dt}\right)$$

$$= \ddot{r}\hat{r} + \dot{r}\left(\frac{d\hat{r}}{d\theta}\cdot\frac{d\theta}{dt}\right) + \dot{r}\,\dot{\theta}\hat{\theta} + r\,\ddot{\theta}\hat{\theta} + r\,\dot{\theta}\left(\frac{d\hat{\theta}}{d\theta}\cdot\frac{d\theta}{dt}\right)$$

But $$\frac{d\hat{r}}{d\theta} = \hat{\theta} \text{ and } \frac{d\hat{\theta}}{d\theta} = -\hat{r}$$

$$\therefore \quad \vec{a} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\,\hat{r}$$

$$= (\ddot{r} - r\dot{\theta}^2)\,\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\theta}$$

or $$\vec{a} = \vec{a_r} + \vec{a_\theta} = a_r\hat{r} + a_\theta\hat{\theta}$$

Components of acceleration. The quantity $\vec{a_r}$ is known as *radial acceleration.* Its magnitude $|\vec{a_r}| = \ddot{r} - r\dot{\theta}^2$ and its direction is along $\hat{r}$. It consists of two parts.

(*i*) The quantity $\ddot{r}$ gives the acceleration due to change in magnitude of $\dot{r}$. It has a positive sign as it is directed *away from* the centre.

(*ii*) The quantity $r\dot{\theta}^2$ gives the *centripetal acceleration* (if $\dot{\theta}$ = a constant = ω, $r\dot{\theta}^2 = r\omega^2$) due to change in θ. It has a negative sign as it is directed *towards* the centre.

The quantity $\vec{a_\theta}$ is known as *transverse acceleration*. Its magnitude $|\vec{a_\theta}| = r\ddot{\theta} + 2\dot{r}\dot{\theta}$ and its direction is along θ. This also consists of two parts.

(*i*) The quantity $r\ddot{\theta}$ gives the angular acceleration due to change in $\dot{\theta}$.

(*ii*) The quantity $2\dot{r}\dot{\theta}$ arises due to the interaction of linear and angular velocities due to changes in r and θ respectively. This is similar to *coriolis acceleration.*

$\vec{a_r}$ and $\vec{a_\theta}$ are perpendicular to each other. The magnitude of $|\vec{a}|$ is given by

$$|\vec{a}| = \left[|\vec{a_r}|^2 + |\vec{a_\theta}|^2\right]^{1/2}$$

1.8 COMPONENTS OF VELOCITY IN CARTESIAN COORDINATE SYSTEM (Two dimensional)

If x and y are cartesian co-ordinates of the point P having polar co-ordinates (r, θ) in a two dimensional system, then

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

$$\therefore \quad \cos\theta = \frac{x}{r};\ \sin\theta = \frac{y}{r} \quad \text{and} \quad r = (x^2 + y^2)^{1/2}$$

Differentiating $x = r\cos\theta$ and $y = r\sin\theta$ with respect to t we have

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\dot{\theta} \qquad ...(i) \quad \text{where } \dot{x} = \frac{dx}{dt}$$

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and $\dot{y} = \dot{r}\sin\theta + r\cos\theta\dot{\theta}$ ...(*ii*) where $\dot{y} = \frac{dy}{dt}$

Multiplying (*i*) by $\cos\theta$, (*ii*) by $\sin\theta$ and adding, we get

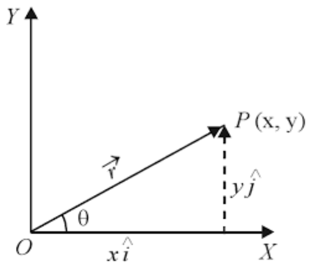

Fig. 1.5

$$\dot{x}\cos\theta + \dot{y}\sin\theta = \dot{r}\,(\cos^2\theta + \sin^2\theta) = \dot{r}$$

$$\therefore \quad \dot{r} = \dot{x}\cos\theta + \dot{y}\sin\theta = \frac{\dot{x}x + \dot{y}y}{r} = \frac{\dot{x}x + \dot{y}y}{(x^2 + y^2)^{1/2}}$$

$$\text{or} \quad |\vec{v}_r| = \frac{\dot{x}x + \dot{y}y}{(x^2 + y^2)^{1/2}}$$

Multiplying (*i*) by $\sin\theta$, (*ii*) by $\cos\theta$ and substracting (*i*) from (*ii*) we have

$$\dot{y}\cos\theta - \dot{x}\sin\theta = r\,(\cos^2\theta + \sin^2\theta)\,\dot{\theta}$$

$$\therefore \quad \dot{\theta} = \frac{\dot{y}\cos\theta - \dot{x}\sin\theta}{r} = \frac{\dot{y}x - \dot{x}y}{r^2} = \frac{\dot{y}x - \dot{x}y}{x^2 + y^2}$$

$$\therefore \quad |\vec{v}_\theta| = r\dot{\theta} = \frac{\dot{y}x - \dot{x}y}{(x^2 + y^2)^{\frac{1}{2}}} = \frac{\dot{y}x - \dot{x}y}{r} \quad \text{... }(iii)$$

1.9 COMPONENTS OF VELOCITY IN SPHERICAL POLAR COORDINATE SYSTEM

Displacement. The position of a point P in a co-ordinate system can be specified by a single vector *i.e.*, the *displacement* of the particle relative to the origin O of the co-ordinate system. This vector is called the *position vector* of the point and denoted by $\overrightarrow{OP} = \vec{r}$. It gives the *magnitude* as well as the *direction* of *displacement.* If $\hat{r}$, is a unit vector along $\overrightarrow{OP}$ *i.e.*, along the direction or $\vec{r}$, then

$$\vec{r} = r\hat{r}$$

If (r, θ, ϕ) are the polar co-ordinates of the particle P at any instant and (x, y, z) the corresponding cartesian co-ordinates, then

$$x = r\sin\theta\cos\phi;\ y = r\sin\theta\sin\phi;\ z = r\cos\theta$$

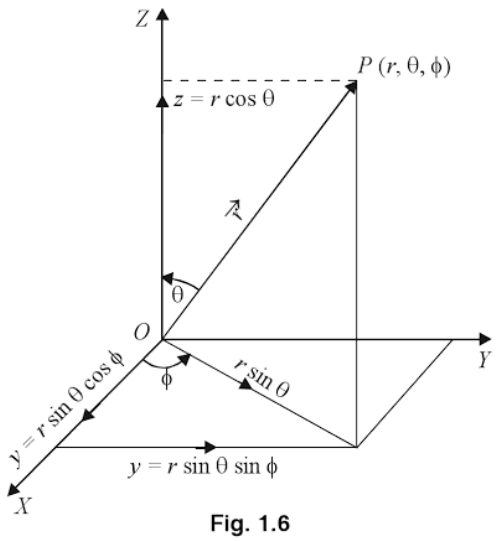

Fig. 1.6

As shown in Fig. 1.6 $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\therefore$ Displacement $\quad \vec{r} = r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$

Hence $\quad \hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$

Since the unit vector $\hat{\theta}$ in the direction of increasing θ is at right angles to $\hat{r}$.

$$\therefore \quad \hat{\theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$

As unit vector $\hat{\phi}$ is at right angles to the component $r\sin\theta$ and lies in the x-y plane

PA10

$\therefore \qquad \hat{\phi} = -\sin\phi\hat{i} + \cos\phi\hat{j}$

Velocity *is the time rate of change of displacement.*

$\because$ Displacement $\qquad \vec{r} = r\hat{r}$

Velocity $\qquad \vec{v} = \dfrac{d\vec{r}}{dt} = \dfrac{d}{dt}(r\hat{r}) = \dfrac{dr}{dt}\hat{r} + r\dfrac{d\hat{r}}{dt}$

Now $\qquad \dfrac{d\hat{r}}{dt} = \dfrac{d}{dt}[\sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}]$

$$= (\cos\theta\cos\phi\dot{\theta} - \sin\theta\sin\phi\dot{\phi})\hat{i} + (\cos\theta\sin\phi\dot{\theta} + \sin\theta\cos\phi\dot{\phi})\hat{j} - (\sin\theta\dot{\theta})\hat{k}$$

$$= \dot{\theta}(\cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k}) + \sin\theta\dot{\phi}(-\sin\phi\hat{i} + \cos\phi\hat{j})$$

$$\therefore \qquad \frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta} + \sin\theta\dot{\phi}\hat{\phi} \qquad ...(i)$$

$$\therefore \qquad \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi} \qquad ..(ii)$$

The above relation can be put in the form $\vec{v} = \vec{v}_r + \vec{v}_\theta + \vec{v}_\phi$.

Physical significance. Now $\vec{v}_r$ is the velocity along $\vec{r}$, $\vec{v}_\theta$ the velocity along θ and $\vec{v}_\phi$ the velocity along ϕ. The three velocity vectors $\vec{v}_r$, $\vec{v}_\theta$ and $\vec{v}_\phi$, therefore act as three orthogonal components in the polar co-ordinate system in the same way as $\vec{v}_x$, $\vec{v}_y$ and $\vec{v}_z$, are rectangular components in the cartesian co-ordinate system.

Further $\qquad |\vec{v}^2| = \vec{v}\cdot\vec{v} = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 = v_r^2 + v_\theta^2 + v_\phi^2$

where $\qquad v_r = |\vec{v}_r| = \dot{r} \quad v_\theta = |\vec{v}_\theta| = r\dot{\theta} \quad v_\phi = |\vec{v}_\phi| = r\sin\theta\dot{\phi}$

If a particle travels only in a straight line it has only the linear velocity $\vec{v} = \dot{r}\hat{r}$, if it moves along a circular path with a uniform velocity, it has only an angular velocity $\vec{v}_\theta = r\dot{\theta}\hat{\theta}$ where r is the radius of the circle, $\dot{\theta}$ the rate of change of angle. If the particle moves along a curved path in a plane it has both the components $\vec{v}_r$ along $\hat{r}$ and $\vec{v}_\theta$ along $\hat{\theta}$ at right angles to $\hat{r}$. When the particle moves in a curved path in space it has in addition the third components $r\sin\theta\dot{\phi}$ in the XY plane perpendicular to the YZ plane in which $\hat{r}$ and $\hat{\theta}$ lie. $r\sin\theta$ is the component of $\vec{r}$ in the XY plane and $r\sin\theta\dot{\phi}$ gives the angular velocity due to rate of change of ϕ in the direction $\hat{\phi}$ perpendicular to the vector $r\sin\theta$. A unit vector in the direction $r\sin\theta$ is denoted by $\hat{\rho}$.

1.10 COMPONENTS OF ACCELERATION IN CARTESIAN COORDINATE SYSTEM (TWO DIMENSIONAL)

If x and y are the cartesian co-ordinates of a point having polar co-ordinates r, θ, then

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

$$\therefore \qquad \dot{x} = \dot{r}\cos\theta - r\sin\theta\dot{\theta} \qquad ...(i)$$

$$\text{and} \qquad \dot{y} = \dot{r}\sin\theta + r\cos\theta\dot{\theta} \qquad ...(ii)$$

Differentiating equation (i) with respect to t, we have

$$\ddot{x} = \ddot{r}\cos\theta - \dot{r}\sin\theta\dot{\theta} - \dot{r}\sin\theta\dot{\theta} - r\cos\theta\dot{\theta}^2 - r\sin\theta\ddot{\theta}$$

$$= (\ddot{r} - r\dot{\theta}^2)\cos\theta - (r\ddot{\theta} + 2\dot{r}\dot{\theta})\sin\theta \qquad ...(iii)$$

$$\therefore \qquad \frac{x\dot{x} + y\dot{y}}{\rho} = \dot{r}\sin\theta + r\cos\theta\dot{\theta}$$

Multiplying both sides by $z = r\cos\theta$, we get

$$(x\dot{x} + y\dot{y})\frac{z}{\rho} = r\dot{r}\sin\theta\cos\theta + r^2\cos^2\theta\dot{\theta}$$

Now multiplying (*iii*) and (*iv*) (*a*) $\dot{z}\rho = r\dot{r}\sin\theta\cos\theta - r^2\sin^2\theta\dot{\theta}$

$$\therefore \qquad (x\dot{x} + y\dot{y})\frac{z}{\rho} - \dot{z}\rho = r^2\dot{\theta}$$

or
$$\dot{\theta} = \frac{(x\dot{x} + y\dot{y})\frac{z}{\rho} - \dot{z}\rho}{x^2 + y^2 + z^2}$$

or
$$r\dot{\theta} = |\vec{v}_\theta| = \frac{(x\dot{x} + y\dot{y})\frac{z}{\rho} - \dot{z}\rho}{(x^2 + y^2 + z^2)^{1/2}} \qquad ...(v)$$

Multiplying (*ii*) by $x = r\sin\theta\cos\phi$ and (*i*) by $y = r\sin\theta\sin\phi$ and substracting, we have

$$\begin{aligned} x\dot{y} - y\dot{x} &= r\dot{r}\sin^2\theta\sin\phi\cos\phi + r^2\sin\theta\cos\theta\dot{\theta}\sin\phi\cos\phi \\ &\quad + r^2\sin^2\theta\cos^2\phi\dot{\phi} - r\dot{r}\sin^2\theta\sin\phi\cos\phi \\ &\quad - r^2\sin\theta\cos\theta\dot{\theta}\sin\phi\cos\phi + r^2\sin^2\theta\sin^2\phi\dot{\phi} \\ &= r^2\sin^2\theta\dot{\phi} \end{aligned}$$

$$\therefore \qquad \dot{\phi} = \frac{x\dot{y} - y\dot{x}}{x^2 + y^2} \quad \text{or} \quad |\vec{v}_\phi| = r\sin\theta\dot{\phi} = \frac{x\dot{y} - y\dot{x}}{(x^2 + y^2)^{1/2}} \qquad ...(vi)$$

1.12 COMPONENTS OF ACCELERATION IN SPHERICAL POLAR COORDINATE SYSTEM

Acceleration. The velocity of a particle in spherical polar co-ordinates is given by

Velocity
$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\,\dot{\phi}\hat{\phi}$$

Acceleration is the derivative of $\vec{v}$ *with respect to time t.*

$\therefore$ Acceleration
$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt} + \dot{r}\sin\theta\dot{\phi}\hat{\phi}$$
$$+ r\cos\theta\dot{\theta}\dot{\phi}\hat{\phi} + r\sin\theta\ddot{\phi}\hat{\phi} + r\sin\theta\dot{\phi}\frac{d\hat{\phi}}{dt} \qquad ...(i)$$

Now unit vectors $\hat{r}, \hat{\theta}, \hat{\phi}$ in spherical polar co-ordinates in terms of unit vectors $\hat{i}, \hat{j}, \hat{k}$ in cartesian co-ordinates are given by

$$\hat{r} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\hat{k}$$

and
$$\hat{\phi} = -\sin\phi\hat{i} + \cos\phi\,\hat{j}$$

$$\therefore \qquad \frac{d\hat{r}}{dt} = \frac{d}{dt}[\sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\hat{k}]$$
$$= (\cos\theta\cos\phi\dot{\theta} - \sin\theta\sin\phi\dot{\phi})\hat{i}$$

$$+(\cos\theta\sin\phi\dot{\theta} + \sin\theta\cos\phi\dot{\phi})\hat{j} - (\sin\theta\dot{\theta})\hat{k}$$

$$= \dot{\theta}(\cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k})$$

$$+ \sin\theta\dot{\phi}(-\sin\phi\hat{i} + \cos\phi\hat{j})$$

$$= \dot{\theta}\hat{\theta} + \sin\theta\dot{\phi}\hat{\phi}$$

$$\frac{d\hat{\theta}}{dt} = \frac{d}{dt}[\cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k}]$$

$$= (-\sin\theta\dot{\theta}\cos\phi - \cos\theta\sin\phi\dot{\phi})\hat{i}$$

$$+ (-\sin\theta\dot{\theta}\sin\phi + \cos\theta\cos\phi\dot{\phi})\hat{j} - (\cos\theta\dot{\theta})\hat{k}$$

$$= -\dot{\theta}(\sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k})$$

$$+ \cos\theta\dot{\phi}(-\sin\phi\hat{i} + \cos\phi\hat{j})$$

$$= -\dot{\theta}\hat{r} + \cos\theta\dot{\phi}\hat{\phi}$$

and
$$\frac{d\hat{\phi}}{dt} = \frac{d}{dt}[-\sin\phi\hat{i} + \cos\phi\hat{j}]$$

$$= (-\cos\phi\dot{\phi}\hat{i} - \sin\phi\dot{\phi}\hat{j})$$

$$= -\dot{\phi}(\cos\phi\hat{i} + \sin\phi\hat{j})$$

Now $\cos\phi\hat{i} + \sin\phi\hat{j} = \sin\theta\hat{r} + \cos\theta\hat{\theta}$

$\therefore$
$$\frac{d\hat{\phi}}{dt} = -\sin\theta\dot{\phi}\,\hat{r} - \cos\theta\dot{\phi}\,\hat{\theta}$$

Substituting the values of $\frac{d\hat{r}}{dt}, \frac{d\hat{\theta}}{dt}$ and $\frac{d\hat{\phi}}{dt}$ in (*i*) we have

$$\vec{a} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\sin\theta\dot{\phi}\hat{\phi} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta}$$

$$- r\dot{\theta}^2\hat{r} + r\cos\theta\dot{\theta}\dot{\phi}\hat{\phi} + \dot{r}\sin\theta\dot{\phi}\hat{\phi} + r\cos\theta\dot{\theta}\dot{\phi}\hat{\phi}$$

$$+ r\sin\theta\ddot{\phi}\hat{\phi} - r\sin^2\theta\dot{\phi}^2\hat{r} - r\sin\theta\cos\theta\dot{\phi}^2\hat{\theta}$$

$$= (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2)\hat{\theta}$$

$$+ (2\dot{r}\dot{\phi}\sin\theta + 2r\cos\theta\dot{\theta}\dot{\phi} + r\sin\theta\ddot{\phi})\hat{\phi}$$

$$= \vec{a}_r + \vec{a}_\theta + \vec{a}_\phi$$

where
$$\vec{a}_r = (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2)\hat{r}$$

$$\vec{a}_\theta = (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2)\hat{\theta}$$

$$\vec{a}_\phi = (r\sin\theta\ddot{\phi} + 2\dot{r}\dot{\phi}\sin\theta + 2r\cos\theta\,\dot{\theta}\dot{\phi})\hat{\phi}$$

As $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$ are orthogonal to one another, the magnitude of acceleration

$$|\vec{a}| = [|\vec{a}_r|^2 + |\vec{a}_\theta|^2 + |\vec{a}_\phi|^2]^{1/2} \qquad \text{... }(ii)$$

1.13 FRAMES OF REFERENCE: INERTIAL AND NON-INERTIAL

In article 1.3, we have discussed the Newton's laws of motion and their limitations. An important point about the first two laws is that they do not hold good in each and every frame of reference. For a body at rest in one reference frame may not necessarily appear to be so in another. It may, for instance appear to be moving in a circle in a frame of reference rotating with respect to the first. It is only in a very special frame of reference that the two laws of motion hold good and it is the inertial frame. Inertial frame of reference is also known as Newtonian or Galelian frame of reference (because it was Galileo who first enunciated the law of inertia which we refer to as Newton's first law of motion). Motions in inertial frame and non-inertial frame of reference are discussed here:

Inertial frame. Newton's first law of motion is also known as the *law of inertia*. A reference frame *e.g.*, a co-ordinate system in which Newton's first law of motion holds good, is known as an *inertial frame of reference*. In an inertial frame a body continues in its state of rest or of uniform motion in a straight line as long as no external force acts on it. All the Newton's law of motion hold good in an inertial frame. All frames of reference moving with a *constant velocity* with respect to an inertial frame are also inertial frames of reference. So inertial frame of reference can be called an unaccelerated frame of reference.

Example. The best approximation to an inertial frame is the frame of reference in the intergalactic space.

Non-inertial frame. The basic laws of Physics are not changed in form in inertial frames of reference. But when a frame of reference is accelerated relative to an inertial frame the form of basic physical laws such as Newton's second law of motion becomes completely different. Such frames of reference having an accelerated motion relative to an inertial frame are called *non-inertial frames of reference*. Since a uniformly rotating frame has a centripetal acceleration it is also a non-inertial frame.

Example. A non-inertial frame is either a frame having *uniform linear acceleration* or a frame which is *uniformly rotating*.

Earth a non-inertial frame. Earth is not an inertial frame. It is a non-inertial frame. The earth is a rotating sphere as it is rotating about its own axis. It has a centripetal acceleration due to rotation.

In addition, earth also revolves round the sun in a period of one year in an orbit of radius 1.49×10^8 km.

$\therefore$ Its orbital velocity
$$v = \frac{2\pi r}{T} = \frac{2\pi \times 1.49 \times 10^8 \times 10^3}{365 \times 24 \times 60 \times 60} = 3 \times 10^4 \text{ ms}^{-1}$$

and centripetal acceleration of the earth due to revolution

$$= \frac{v^2}{r} = \frac{3 \times 10^4 \times 3 \times 10^4}{1.49 \times 10^8 \times 10^3} = 6 \times 10^{-3} \text{ ms}^{-2}$$

Thus, the reference frame attached to the earth is an *accelerated* frame and is, therefore, *non-inertial*.

For our every day purposes, the value of centripetal acceleration being small, is often neglected and the earth is taken to be an *inertial frame* of reference.

The laws of physics are the same in all inertial frames of reference. Moreover, it is impossible to detect the state of rest or of uniform motion, as the experimental results are the same. This is known as Newtonian principle of relativity.

1.14 NEWTONIAN PRINCIPLE OF RELATIVITY

It states '*By performing physical experiments entirely is one inertial frame, we cannot determine its motion with respect to some other inertial frame and the phenomenon occurring in a closed system are independent of any unaccelerated motion of the system as a whole*.'

In other words, the laws of physics are the same in all inertial frames of reference.

As an example, consider an observer A in a train at rest and another observer B standing outside on the platform. The observer A throws a ball vertically upward and finds that the ball returns vertically downward to him. The same thing is observed by the observer B standing on the platform.

Now, suppose the train starts moving with a uniform velocity along a straight track. If the observer A repeats his experiment *i.e.*, throws the ball vertically upward he again finds that the ball falls vertically downward to him. But the observer B standing on the platform finds that the ball thrown upward by the observer A in the moving train follows a parabolic path. Thus we find that :

(*i*) The observer in the train cannot distinguish whether the train is moving or not because he observes exactly the same thing whether the train is at rest or in uniform motion in a straight line. Therefore, we conclude that

'*It is not possible to detect the state of rest or of uniform rectilinear motion of a system by performing mechanical experiments within the system itself.*'

(*ii*) The vertically upward and vertically downward motion of the ball for observer A and the motion of the ball along the parabolic path for the observer B can be explained by the same laws of Mechanics *i.e.*, the laws of Mechanics remains unchanged due to uniform rectilinear motion of the frames of reference. Thus

'*All inertial frames are equivalent and the laws of Physics (nature) are the same in all inertial frames of reference.*'

1.15 GALELIAN TRANSFORMATIONS

When a physical phenomenon is observed in two inertial frames moving with uniform velocity relative to each other and the time interval registered in both the frames is the same, then the datas of results in one frame of reference can be transformed to those in the second frame. This process is known as Galelian transformation.

The equations which connect the position vectors of a particle (or event) it two-inertial frames are known as Galelian transformations.

Galelian transformation equations. We can derive Galelian transformation equation as under. We shall discuss two cases.

Case (*i*): When one inertial frame is moving relative to the other along positive direction of *x*-axis. Let S and S' be the two inertial frames, whose origins O and O' coincide at $t = t' = 0$. Their X-axis OX and $O'X'$ are along the same line and Y and Z-axis are parallel to each other as shown in Fig. 1.7. The frame S' is moving with a uniform velocity v with respect to the frame S along the positive direction of X-axis. Now consider an event P in the frame S specified by space co-ordinate x, y, z and time co-ordinate t. An event is completely known from the space co-ordinates of the point of its occurrence and from the time co-ordinate of its happening. An event can be a collision between two particles.

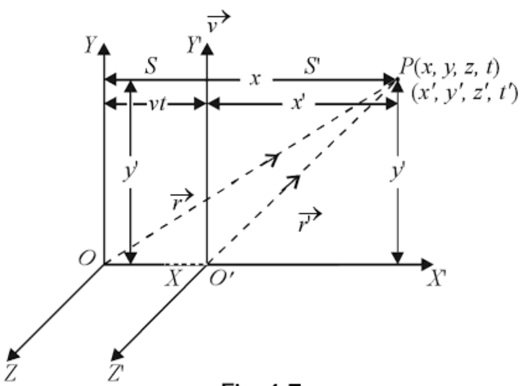

Fig. 1.7

Let the corresponding space and time co-ordinates of the same event in the inertial frame S' be x', y', z' and t'.

Then after a time t, $O'O = vt$ and $x' = x - vt$

$y' = y \quad z' = z \quad t' = t$

The last equation is based upon the universal nature of time as assumed in classical physics. It means that a clock in S measures the same time as a clock in S' provided these were synchronised at time $t = t' = 0$. These equations are known as Galelian transformation equations.

The inverse Galelian transformation equations can be written by changing v to $-v$, x to x', t to t' and so on. These are $x = x' + vt \quad y = y' \quad z = z' \quad t = t'$

If $\vec{r} = \overrightarrow{OP}$ is the position vector of the point P in the inertial frames S and $\vec{r'} = \overrightarrow{O'P}$ the position vector of P in the frame S', then $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{r'} = x'\hat{i} + y'\hat{j} + z'\hat{k}$

$\therefore \qquad \vec{r'} = (x - vt)\,\hat{i} + y\hat{j} + z\hat{k} = (x\hat{i} + y\hat{j} + z\hat{k}) - vt\,\hat{i} = \vec{r} - \vec{v}t$ where $\vec{v} = v\hat{i}$

Thus Galelian transformations can also be put as $\vec{r'} = \vec{r} - \vec{v}t \qquad t' = t$

The corresponding inverse Galelian transformations are $\vec{r} = \vec{r'} + \vec{v}t' \qquad t = t'$

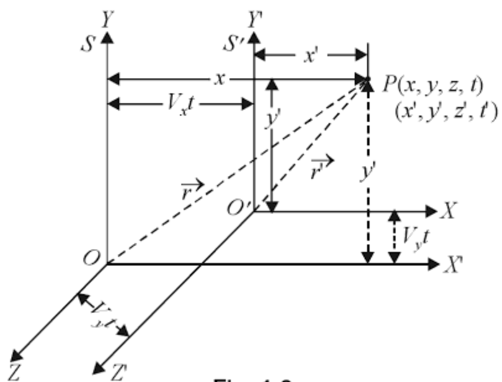

Fig. 1.8

(*ii*) **When one inertial frame is moving relative to the other in any direction.** Now let the frame S' move with respect to the frame S with uniform velocity $\vec{v}$ given by $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$ in any direction where v_x, v_y and v_z are the components of $\vec{v}$ along x, y and z directions respectively. The origins and the axis of the two frame are so chosen that these coincide at $t = t' = 0$. After a time t the frame S' is separated from the frame S by a distance $v_x t$, $v_y t$ and $v_z t$ along the x, y and z axis respectively (fig. 1.8). If x, y, z, t are the co-ordinates of an event P in the frame S and x', y', z', t' of the same event in the frame S', then $x' = x - v_x t \quad y' = y - v_y t \quad z' = z - v_z t \quad t' = t$

In this case
$$\vec{r_1} = (x - v_x t)\,\hat{i} + (y - v_y t)\,\hat{j} + (z - v_z t)\,\hat{k}$$
$$= (x\hat{i} + y\hat{j} + z\hat{k}) - (v_x\hat{i} + v_y\hat{j} + v_z\hat{k})\,t = \vec{r} - \vec{v}t$$
$$(\text{where } \vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k})$$

Thus Galelian transformations can also be put as $\vec{r'} = \vec{r} - \vec{v}t \qquad t' = t$

We shall, however, restrict our discussion to the simple case discussed in (*i*) where S' is moving along the x-axis relative to the frame S.

Galelian invariance. It states that '*The basic laws of mechanics do not change under Galelian transformations*'. The basic laws of Newtonian mechanics are the Newton's laws of motion, laws of conservation of momentum and energy etc. We shall prove below that a space interval is invariant under Galelian transformation.

Consider a rod at rest in the inertial frame S having its two ends at the points x_2 and x_1, then for the observer S

The length of the rod $= x_2 - x_1$

The observer S' will consider the rod to be moving with velocity $-v$ from O' to O and will observe the two ends to be at the points x_2' and x_1' at the **same time.**

For the observer S' the length of the rod $= x_2' - x_1'$

$$= (x_2 - vt_2) - (x_1 - vt_1) = (x_2 - x_1) - v\,(t_2 - t_1)$$

but the ends of the rod have been observed at one and the same time

$$\therefore \qquad t_1 = t_2 \text{ or } t_2 - t_1 = 0 \qquad \therefore\ x_2' - x_1' = x_2 - x_1 \qquad ...(i)$$

Relation (*i*) shows that the space interval or the distance between two points (or length) is in variant under Galelian transformations.

Thus we find that the space-interval measurement and time-interval measurement are independent of the relative motion of the inertial frames. Classical physics already assumes that mass of a body is constant and independent of its motion with respect to an inertial frame. *Thus all the three fundamental quantities length, mass and time are invariant of the relative motion of the observer.*

This property is known as **Galelian invariance.** Hence,

(*i*) The mass of a body is the same for all observers and it is independent of the motion of the observer *i.e.*, mass is an absolute quantity.

(*ii*) The motion has no effect on time. If the clocks in two inertial frames which are in uniform motion agree at one instance, they will agree at all later times *i.e.*, $t = t'$. Time is an absolute quantity.

(*iii*) The length of a rod does not change due to relative motion of an inertial frame with respect to another. Hence space interval is an invariant quantity.

1.16 NEWTON'S LAWS OF MOTION INVARIANT TO GALELIAN TRANSFORMATIONS

To prove that Newton's laws of motion are invariant to Galelian transformations we shall first prove that acceleration is invariant to these transformations. We shall consider the case when the coordinates axis of the two frames are parallel, their origins coincide at $t' = t = 0$ and the frame S' moves with a uniform velocity $\vec{v}$ with respect to the frame S in the $+X$ direction. In such a case Galelian transformation equations are $x' = x - vt \qquad y' = y \qquad z' = z \qquad t' = t$

Differentiating we get, $dx' = dx - vdt, \qquad dy' = dy$

$$dz' = dz \qquad dt' = dt$$

Dividing left hand side of each equation by dt' and right hand side by dt we have

$$\frac{dx'}{dt'} = \frac{dx}{dt} - v, \quad \frac{dy'}{dt'} = \frac{dy}{dt}, \quad \frac{dz'}{dt} = \frac{dz}{dt}$$

Now $\dfrac{dx'}{dt'} = u'_x = x$ – component of the velocity $\vec{u'}$ as measured in the frame S'

and $\dfrac{dx}{dt} = u_x = x$ – component of the velocity $\vec{u}$ as measured in the frame S

$$\therefore \qquad u'_x = u_x - v$$

Similarly $\qquad u'_y = u_y, \qquad u'_z = u_z$

Combining the equations for x, y and z components we have

$$u'_x\,\hat{i} + u'_y\,\hat{j} + u'_z\,\hat{k} = (u_x - v)\,\hat{i} + u_y\,\hat{j} + u_z\,\hat{k} = u_x\,\hat{i} + u_y\,\hat{j} + u_z\,\hat{k} - v\,\hat{i}$$

or $$\vec{u'} = \vec{u} - \vec{v} \qquad ...(i)$$

Differentiating equation (*i*) again we have $du'_x = d\,(u_x - v) = du_x$

Since $v = a$ constant

Similarly $du'_y = du_y$ and $du'_z = du_z$

Dividing left hand side of each equation by dt' and right hand side by dt, we have

$$\frac{du'_x}{dt'} = \frac{du_x}{dt}\ ;\ \frac{du'_y}{dt'} = \frac{du_y}{dt}\ ;\ \frac{du'_z}{dt'} = \frac{du_z}{dt}$$

or $$a'_x = a_x,\ a'_y = a_y,\ a'_z = a_z$$

where a'_x, a'_y, a'_z are the x, y and z components of acceleration in the frame S' and a_x, a_y and a_z in the frame S.

Combining the equations for x, y and z components we have

$$a'_x \hat{i} + a'_y \hat{j} + a'_z \hat{k} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k} \quad \text{or} \quad \vec{a'} = \vec{a} \qquad ...(ii)$$

Hence the components of acceleration of a particle are independent of the uniform velocity of one inertial frame of reference relative to another. In other words **acceleration is an invariant** for a transformation from one inertial frame to another which is having a uniform relative motion of translation.

Newton's laws of motion. Newton's second law of motion is the real law of motion and includes the first and the third law. In mathematical form the second law is stated as $\vec{F} = m\vec{a}$ where $\vec{F}$ is the force acting on a mass m in the frame S. If $\vec{F'}$ is the force acting on the same mass in the frame S' and $\vec{a'}$ is the acceleration produced in the mass as observed in the frame S', then

$$\vec{F'} = m\vec{a'}$$

As proved above in equation (ii) $\vec{a'} = \vec{a}$

$$\therefore \qquad \vec{F'} = \vec{F}$$

showing thereby the law is invariant to Galelian transformations. Hence **Newton's laws of motion** are invariant to Galelian transformations.

Mass, length, time, acceleration and force are the quantities which are invariant under Galelian transformation.

The two important laws in physics which are invariant to Galelian transformations are:

(i) Law of conservation of linear momentum, and

(ii) Law of conservation of energy.

1.17 NON-INERTIAL FRAMES:FICTITIOUS FORCES

A frame of reference in accelerated translational motion with respect to an inertial frame is a *non-inertial frame*. Moreover, a reference frame in uniform rotation with respect to an inertial frame is also non-inertial frame. In a non-inertial frame, an apparent or fictitious force appears to be acting. Let us consider both the cases and the origin of fictitious force.

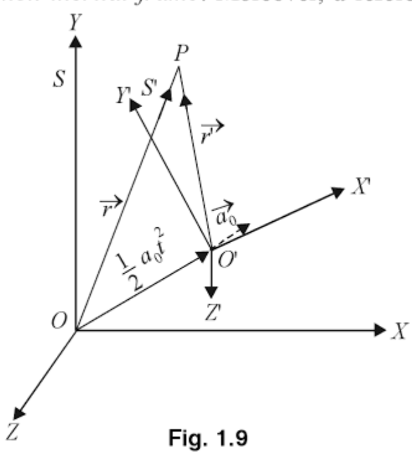

Fig. 1.9

Consider an inertial frame S with its origin O. Let S' be another frame with origin O' moving relative to O with uniform translational acceleration $\vec{a_0}$. When the origins O' and O coincide let $t' = t = 0$.

Suppose the frame S' starts from rest, then displacement $\overrightarrow{OO'}$ of the origin O' in time $t = \frac{1}{2}\vec{a_0}t^2$. If P is a particle moving in space and $\vec{r}$ and $\vec{r'}$ are its position vectors at time t in the frames S and S' respectively, then

$$\overrightarrow{OP} = \vec{r} \text{ and } \overrightarrow{O'P} = \vec{r'}$$

Now $\overrightarrow{OP} = \overrightarrow{OO'} + \overrightarrow{OP}$ or $\vec{r} = \vec{r}' + \frac{1}{2}\vec{a}_0 t^2$ ...(*i*)

Differentiating equation (*i*), twice we have

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}'}{dt} + \vec{a}_0 t$$

and $$\frac{d^2\vec{r}}{dt^2} = \frac{d^2\vec{r}'}{dt^2} + \vec{a}_0 \quad \text{or} \quad \vec{a}_i = \vec{a}' + \vec{a}_0 \qquad ...(ii)$$

where $\vec{a}_i$ and $\vec{a}'$ are the accelerations of the particle P in the frame S (inertial) and S' respectively.

If m is the mass of the particle, then multiplying both sides of relation (*ii*) by m, we have

$$m\vec{a}_i = m\vec{a}' + m\vec{a}_0 \quad \text{or} \quad \vec{F}_i = \vec{F}' + \vec{F}_0 \quad \text{or} \quad \vec{F}' = \vec{F}_i - \vec{F}_0$$

When no real force acts on the particle P in the inertial frame S, $\vec{F}_i = 0$. In such a case $\vec{F}' = -\vec{F}_0$.

Thus we find that in the frame S' the particle P appears to experience a force $-\vec{F}_0$ even when no force is acting on it in the inertial frame S. This force arises due to the uniform translational acceleration of the frame S' and is, therefore, a fictitious force. This fictitious force has not been applied from outside but arises because of the frame of reference selected by us. It is, therefore, a non-inertial force and the *frame of reference which is moving with a uniform translational acceleration with respect to an inertial frame is a non-inertial frame*.

Fictitious force. Consider a reference frame S' moving with an acceleration $\vec{a}_0$ relative to a frame S. If the frame S is an inertial frame, then the frame S' will be non-inertial frame and vice-versa.

Now consider a particle of mass m on which *no external force is acting* in the reference frame S. Then the acceleration of the particle in the frame S is **zero** but the particle will appear to have an acceleration $-\vec{a}_0$ relative to the frame S'. In other words, to an observer in the frame S' a force $\vec{F}_0 = -m\vec{a}_0$ will appear to be acting on the particle whereas for an observer in the frame S there is no force acting on the particle. *Such a force which appears only due to the acceleration of the frame of reference is called* **fictitious force, apparent** or **pesudo force.**

Total force. Suppose the particle has an acceleration $\vec{a}_i$ as observed in the inertial frame S, then according to Newton's second law of motion the force acting on the particle is given by $\vec{F}_i = m\vec{a}_i$

When this force is observed from the reference frame S', then the frame S may be assumed to be moving with an acceleration $-\vec{a}_0$ relative to S'. Hence the particle will appear to be acted upon by an additional force $-m\vec{a}_0$ in the system S' due to its acceleration.

∴ The total force acting on the particle as observed by an observer in the reference frame S'

$$= \vec{F}' = m\vec{a}_i - m\vec{a}_0 = m(\vec{a}_i - \vec{a}_0) \text{ or } \vec{F}' = \vec{F}_i - \vec{F}_0$$

which means that to an observer in the frame S' the total force acting on the particle appears to be $\vec{F}'$ and the net acceleration $(\vec{a}_i - \vec{a}_0)$. The force $\vec{F}_0 = -m\vec{a}_0$ is clearly the fictitious force. This force does not actually exist but appears to come into being only as a result of the acceleration of the frame S' with respect to the frame S. Hence

Fictitious force $\vec{F}_0$ = (mass) (Acceleration of non-inertial frame with sign reversed)

$$= m\,(-\vec{a}_0) = -\,m\,\vec{a}_0$$

Newton's second law of motion will also hold good in a non-inertial frame S' *i.e.*, S' will behave as an inertial frame if we add to the **true force** $\vec{F}_i$ a fictitious force $\vec{F}_0 = -\,m\,\vec{a}_0$ and take the total force $\vec{F}' = \vec{F}_i - \vec{F}_0$ and total acceleration as $\vec{a}' = \vec{a}_i - \vec{a}_0$.

The examples of fictitious force are given below:

Example 1: Centrifugal force. An interesting example of fictitious force is that of "*centrifugal force*'. Consider a point mass m at rest in a non-inertial frame *i.e.*, $\vec{a}_i = 0$. Suppose the non-inertial frame rotates uniformly about an axis fixed with respect to the inertial frame. The acceleration of the point mass with respect to the inertial frame can be written as $\vec{a}_0 = -\,\omega^2\,\vec{r}$

where $\vec{r}$ is the position vector of the particle and is directed outward to the particle from the axis.

Now $\qquad \vec{F}' = \vec{F}_i - \vec{F}_0$

But $\qquad \vec{F}_i = 0 \qquad \therefore \quad \vec{F}' = -\,\vec{F}_0 = -\,m\,\vec{a}_0 = -\,m\,\omega^2\,\vec{r}$

Example 2. Coriolis force. Another example of fictitious force is the Coriolis force, which acts on a particle in motion with respect to a rotatory frame of reference. This is explained in article 1.19.

1.18 EQUILIBRIUM STATE IN NON-INERTIAL FRAME

A particle attains an equilibrium state in a non-inertial frame. A particle is said to be in equilibrium when no net force is acting on it. This condition is satisfied when the external force acting on the particle is equal and opposite to the fictitious force due to the non-inertial frame.

The total force $\vec{F}'$ acting on a particle in a non-inertial frame is given by

$$\vec{F}' = \vec{F}_i - \vec{F}_0$$

where $\vec{F}_i$ is the external force acting on the particle as observed in an inertial frame and $\vec{F}_0$ is the fictitious force due to the non-inertial frame. Clearly $\vec{F}' = 0$ when $\vec{F}_i = \vec{F}_0$ and the particle is in equilibrium.

1.19 UNIFORMLY ROTATING FRAME:

(Coriolis force)

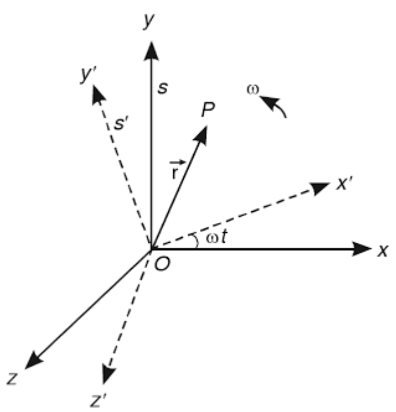

Fig. 1.10

Let us consider a reference frame S' (shown dotted) rotating with a uniform angular velocity ω with respect to an inertial frame S (Fig. 1.10).

Let us assume, both frames S and S' have a common origin O. Suppose now we have a moving particle P (like a fly or an ant walking on a rotating piece of card board). The value of its position vector $\vec{r}$ at any instant t will, obviously, be the same in either frame of reference (although its components will be different along the three axes).

If the particle be at rest with respect to the inertial frame S, it would appear to be moving with relative linear velocity $-\vec{\omega} \times \vec{r}$ in the non-inertial *i.e.,* rotating frame S'.

$$\vec{a} = \vec{a'} + 2\vec{\omega} \times \vec{V'} + \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \quad \text{... (ii)}$$

$$\left[\because \left(\frac{d\vec{r}}{dt}\right)_{S'} = \vec{V'}\right]$$

Since the angular velocity $\vec{\omega}$ is uniform, $\frac{d\vec{\omega}}{dt} = 0$

$$\vec{a} = \vec{a'} + 2\vec{\omega} \times \vec{V'} + \omega \times (\vec{\omega} \times \vec{r}) \quad \text{... (iii)}$$

Here $2\vec{\omega} \times \vec{V'}$ is called Coriolis acceleration after the name of its discoverer. The term appears only when the particle moves in the rotating frame, *i.e.*, $\vec{V'} \neq 0$. The term $\omega \times (\vec{\omega} \times \vec{r})$ is the centripetal acceleration.

$$\begin{matrix}\textit{Acceleration in}\\ \textit{the inertial}\\ \textit{Frame}\end{matrix} = \begin{matrix}\textit{Acceleration}\\ \textit{observed in rotating}\\ \textit{Frame}\end{matrix} + \begin{matrix}\textit{Coriolis}\\ \textit{acceleration}\end{matrix} + \begin{matrix}\textit{Centripetal}\\ \textit{acceleration}\end{matrix}$$

1.20 PHYSICAL SIGNIFICANCE OF FICTITIOUS FORCE.

The acceleration a and a′ are related by the equation

$$\vec{\boldsymbol{a}} = \vec{\boldsymbol{a}'} + 2\vec{\omega} \times \vec{\boldsymbol{v}'} + \vec{\omega} \times (\vec{\omega} \times \vec{\boldsymbol{r}})$$

Multiplying both sides by m, we have

$$\boldsymbol{m}\vec{\boldsymbol{a}} = \boldsymbol{m}\vec{\boldsymbol{a}'} + 2\boldsymbol{m}(\vec{\omega} \times \vec{\boldsymbol{v}'}) + \boldsymbol{m}\vec{\omega} \times (\vec{\omega} \times \vec{\boldsymbol{r}}) \quad \text{... (i)}$$

According to Newton’s second law of motion, the effective force on the praticle of mass m in the uniformly rotating frame S' is given by $m\vec{a'} = \vec{F'}$ and the true force in the inertial frame S is given by $m\vec{a} = \vec{F}$

Substituting in Eq. (i), we get

$$\therefore \quad \vec{\boldsymbol{F}} = \vec{\boldsymbol{F}'} + 2\boldsymbol{m}(\vec{\omega} \times \vec{\boldsymbol{v}'}) + \boldsymbol{m}\vec{\omega} \times (\vec{\omega} \times \vec{\boldsymbol{r}}) \quad \text{... (ii)}$$

$$\therefore \quad \vec{\boldsymbol{F}'} = \vec{\boldsymbol{F}} - 2\boldsymbol{m}(\vec{\omega} \times \vec{\boldsymbol{v}'}) - \boldsymbol{m}\vec{\omega} \times (\vec{\omega} \times \vec{\boldsymbol{r}}) \quad \text{... (iii)}$$

$$\vec{\boldsymbol{F}'} = \vec{\boldsymbol{F}} + \vec{\boldsymbol{F}_0}$$

where $\vec{\boldsymbol{F}} = -2\boldsymbol{m}(\vec{\omega} \times \vec{\boldsymbol{v}'}) - \boldsymbol{m}\vec{\omega} \times (\vec{\omega} \times \vec{\boldsymbol{r}})$ is the ‘Fictitious Force’ acting in the non-inertial frame.

∴ Fictitious force = *Coriolis force + Centrifugal force*

The Coriolis and Centrifugal forces are consequences of rotation of frame of reference.

Coriolis force. The Coriolis force is given by $-2m(\vec{\omega} \times \vec{v})$. *It is defined as the fictitious force which acts on a particle when it is in motion relative to a rotating frame of reference.*

It is proportional to the angular velocity $\vec{\omega}$ of the rotating frame and to the velocity $\vec{v}$ of the

1.21.1 Effect of Centrifugal Force on g.

The acceleration of a particle in a uniformly rotating frame is given by

$$\vec{a} = \vec{a}' + 2\vec{\omega} \times \vec{v}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \quad \text{... } (i)$$

when a particle is at rest in the rotating frame, the coriolis acceleration $2\vec{\omega} \times \vec{v}' = 0$. In such a case, the body is under the action of the centrifugal force only given by $\vec{\omega} \times (\vec{\omega} \times \vec{r})$. Therefore, acceleration of the particle with respect to the inertial frame will be

$$\vec{a} = \vec{a}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \quad \text{... } (ii)$$

or
$$\vec{a}' = \vec{a} - \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

The earth is rotating form west to east and a reference frame fixed in it is rotating frame of reference. To consider the effect of centrifugal force on the acceleration due to gravity we take angular velocity vector $\vec{\omega}$ along the y-axis and x-axis is taken perpendicular to it as shown in Fig. 1.12, then the actual value of $\vec{g}$ at a point P in the latitude λ will act along the direction PO where O is the centre of the earth. The components of $\vec{g}$ are

$-g \cos \lambda\, \hat{i}$ along x-axis and $-g \sin \lambda\, \hat{j}$ along y-axis

$\therefore \quad \vec{g} = -g\,(\cos \lambda\, \hat{i} + \sin \lambda\, \hat{j})$

Also $\vec{\omega} = \omega\, \hat{j}$

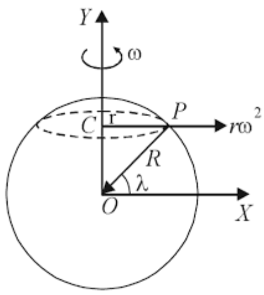

Fig. 1.12

From P draw PC perpendicular to OY, then $CP = \vec{r}$

where $\vec{r}$ is the radius of the circle in which a particle at P rotates.

Hence $\quad CP = \vec{r} = \vec{R} \cos \lambda = R \cos \lambda\, \hat{i}$

where R = magnitude of the radius of the earth.

The observed value of acceleration due to gravity at P taking into consideration the effect of centrifugal acceleration and neglecting the effect of Coriolis acceleration due to the rotation of the earth is given by

$$\vec{g}' = \vec{g} - \vec{\omega} \times (\vec{\omega} \times \vec{r})$$
$$= -g\,(\cos \lambda\, \hat{i} + \sin \lambda\, \hat{j}) - \omega\, \hat{j} \times (\omega\, \hat{j} \times R \cos \lambda\, \hat{i})$$
$$= -[(g \cos \lambda - \omega^2 R \cos \lambda)\, \hat{i} + g \sin \lambda\, \hat{j}]$$

$\therefore$ Magnitude of $g' = [(g \cos \lambda - \omega^2 R \cos \lambda)^2 + g^2 \sin^2 \lambda]^{1/2}$

$$= \left[g^2 \cos^2 \lambda \left(1 - \frac{\omega^2 R}{g}\right)^2 + g^2 \sin^2 \lambda \right]^{1/2}$$

$$= \left[g^2 \cos^2 \lambda \left(1 - \frac{2\omega^2 R}{g}\right) + g^2 \sin^2 \lambda \right]^{1/2}$$

since $\dfrac{\omega^2 R}{g} << 1$, its square is a negligible quantity.

$$\therefore \qquad g' = g\left[\cos^2\lambda - \frac{2\omega^2 R\cos^2\lambda}{g} + \sin^2\lambda\right]^{1/2}$$

$$= g\left[1 - \frac{2\omega^2 R}{g}\cos^2\lambda\right]^{1/2} = g\left[1 - \frac{\omega^2 R}{g}\cos^2\lambda\right] = g - \omega^2 R\cos^2\lambda$$

Taking $R = 6.38 \times 10^6$ m, the value of

$$R\omega^2 = 6.38 \times 10^6 \times \left(\frac{2\pi}{24 \times 60 \times 60}\right)^2 = 0.03372 \text{ ms}^{-2} = 3.372 \text{ cm/sec}^2$$

At the equator. At the equator $\lambda = 0$, $\cos\lambda = 1$

$$\therefore \qquad g' = g - \omega^2 R\cos^2\lambda = g - \omega^2 R = g - 3.372 \text{ cm/sec}^2$$

$\therefore$ Due to rotation of the earth the value of g at the equator decreases by 3.372 cm/sec^2.

At the poles. At the poles $\lambda = 90°$, $\cos\lambda = 0$. Therefore decrease in the value of g due to rotation is zero. The observed difference between the value of g at the poles and at the equator is about 5.2 cm/sec^2. This is due to the fact that earth is flattened at the poles and even if it were not rotating the value of g at the poles would have been greater than that at the equator.

At latitude 45°. At latitude 45°, the decrease in the value of $g = R\omega^2\cos^2\lambda = 3.372 \times \frac{1}{2} = 1.686$ cm/sec^2.

Hence if the earth were to stop rotating, the value of g at the equator will increase by 3.372 cm/sec^2 and at a point in latitude 45° by 1.686 cm/sec^2.

In other words, the effect of centrifugal force due to rotation of the earth on the acceleration due to gravity is *maximum* at the *equator* and *minimum* at the *poles*.

1.21.2 Deflection of plumb line

The direction of g' will not be exactly along the vertical OP but will be deviated from the vertical by a small angle say α.

The value of α can be found out by applying the law of sines to the triangle $OO'P$ (Fig. 1.13) in which various sides represent the acceleration vectors in magnitude and direction

$$\therefore \qquad \frac{g'}{\sin\lambda} = \frac{R\omega^2\cos\lambda}{\sin\alpha} \text{ or } \sin\alpha = \frac{R\omega^2\cos\lambda\sin\lambda}{g'}$$

As α is small $\qquad \sin\alpha = \alpha$

$$\therefore \qquad \alpha = \frac{R\omega^2\cos\lambda\sin\lambda}{g'} = \frac{R\,\omega^2}{2g'}\sin 2\lambda$$

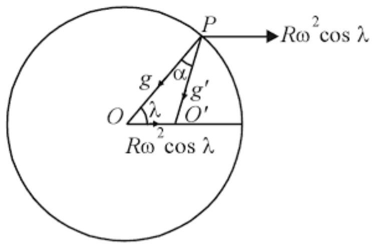

Fig. 1.13

Now $R\omega^2 = 3.372$ cm/sec^2 and taking $g' = 981$ cm/sec^2

$$\therefore \quad \frac{R\omega^2}{g'} = \frac{3.372}{981} = \frac{1}{291}$$

Deflection at latitude 45°. At a place $\lambda = 45°$, $2\lambda = 90°$, $\sin 2\lambda = 1$

$$= -2m\omega\,[-\dot{y}\sin\lambda\,\hat{i} + \dot{x}\sin\lambda\,\hat{j} - \dot{x}\cos\lambda\,\hat{k}] \qquad ...(i)$$

The magnitude of Coriolis force is given by

$$|\vec{F}_{\text{cor}}| = 2m\omega\,[\dot{y}^2\sin^2\lambda + \dot{x}^2\sin^2\lambda + \dot{x}^2\cos^2\lambda]^{1/2}$$

$$= 2m\omega\,[\dot{y}^2\sin^2\lambda + \dot{x}^2]^{1/2} = 2m\omega\,[v_y^2\sin^2\lambda + v_x^2]^{1/2}$$

In terms of the co-latitude ϕ, $|\vec{F}_{\text{cor}}| = 2m\omega\,[v_y^2\cos^2\phi + v_x^2]^{1/2}$

The **horizontial component** of Coriolis force is given by

$$(\vec{F}_{\text{cor}})_{\text{horizontal}} = -2m\omega\,[-\dot{y}\sin\lambda\,\hat{i} + \dot{x}\sin\lambda\,\hat{j}] = -2m\omega\sin\lambda\,[-\dot{y}\,\hat{i} + \dot{x}\,\hat{j}]$$

The **magnitude of horizontal component** is given by

$$|(\vec{F}_{\text{cor}})_{\text{H}}| = 2m\omega\sin\lambda\,[\dot{y}^2 + \dot{x}^2]^{1/2}$$

$$= 2m\omega\sin\lambda\,[v_x^2 + v_y^2]^{1/2} = 2m\omega\,v\sin\lambda$$

In terms of co-latitude ϕ, the magnitude of horizontal component is

$$= 2m\omega\cos\phi\,[v_x^2 + v_y^2]^{1/2} = 2m\,v\cos\phi$$

The horizontal component of Coriolis force has maximum value at the poles ($\lambda = 90°$) and zero at the equator ($\lambda = 0$).

The **vertical component of Coriolis force** is given by

$$(\vec{F}_{\text{cor}})_{\text{vertical}} = -2m\omega\,[-\dot{x}\cos\lambda\,\hat{k}] = +2m\omega\,\dot{x}\cos\lambda\,\hat{k}$$

The **magnitude of the vertical component** is given by

$$|\vec{F}_{\text{cor}}|_v = 2m\omega\,\dot{x}\cos\lambda = 2m\omega\,v_x\cos\lambda$$

and in terms of co-latitude $\qquad = 2m\omega\,v_x\sin\phi$

The vertical component $(\vec{F}_{\text{cor}})_{\text{vertical}} = +2\,m\omega\,\dot{x}\cos\lambda\,\hat{k}$ *i.e.*, it acts in the direction of $+Z$-axis which means vertically upward. At the equator if the body has an initial velocity x it will appear to be lifted upwards.

1.23 DIRECTION OF CORIOLIS FORCE IN NORTHERN AND SOUTHERN HEMISPHERES

The Coriolis force is given by $\vec{F}_{\text{cor}} = -2m\,(\vec{\omega} \times \vec{v})$ *i.e.*, by the negative of the cross product of $\vec{\omega}$ the angular velocity of rotation of the earth and $\vec{v}$ the linear velocity of the body. The direction of $\vec{\omega}$ the angular velocity vector is always along the axis of the earth from S to N.

In the northern hemisphere if a body is moving towards North (*i.e.*, along $+Y$-axis) an application of the rule for the direction of vector product of two vectors indicates that this force will act towards the East (*i.e.*, along $+X$-axis). On the other hand if the body is moving towards South (*i.e.*, along $-Y$-axis) the Coriolis force will act towards the West (*i.e.*, along $-X$-axis). In other words, in the northern hemisphere a moving body turns towards the **right.**

Similarly in the Southern hemisphere a moving body will turn towards the **left** due to the effect of Coriolis force.

1.24 APPLICATIONS OF CORIOLIS FORCE

The Coriolis force comes into play due to the rotation of the earth that causes a deflection in the direction of trade winds, gives a rotatory effect to cyclones, brings about erosion of the right bank of

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rivers and other geographical effects. Some of them are:

(*i*) **Direction of trade winds.** Due to heating of earth's surface near the equator, the air in contact with it also gets heated and rises up. The cooler air from the north or the south rushes towards the equator. In the northern hemisphere the air does not follow the north-south direction but is deflected towards the west due to Coriolis force. The Coriolis force is given by $-2m(\vec{\omega} \times \vec{v})$. The direction of ω the angular velocity vector is along the axis of the earth upward from the north pole. The wind which is flowing from north to south tangential to earth's surface has a velocity component perpendicular to the axis and directed away from it. An application of the rules for vector product of two vectors indicates that this force will deflect the moving particles to their **right** *i.e.*, towards the west in the northern hemisphere giving rise to a north-west trade wind. For the same reason the deflection of the air rushing to the equator from the south in the southern hemisphere will be towards its **right** *i.e.*, towards the east thus giving rise to a south-east trade wind.

(*ii*) **Cyclones.** Whenever a region of low pressure arises in the northern hemisphere the air from the surrounding areas rushes towards it. The Coriolis force deflects this rushing air towards its right giving rise to a clock wise rotation round the low pressure zone. The process continues till the thrust due to pressure gradient is balanced by the thrust due to Coriolis force. At the equator the horizontal component of Coriolis force $2m\omega v \sin \lambda$ which is effective in deflecting the wind = 0 as $\lambda = 0$. There are, therefore, *no cyclones produced on the equator*.

(*iii*) **Greater erosion of the right bank of rivers.** The water of the rivers flowing north to south or south to north experiences a Coriolis force towards its right with the result that the *right bank of the river is eaten away more rapidly and is steeper than the left bank*. It should be noted that in the northern hemisphere for a river-flowing from north to south the right bank is to the west and for a river flowing from south to north the right bank is to the east.

(*iv*) **Gulf stream.** The warm Gulf stream flows from south to north and deflects towards the east thereby producing a temperature effect on the climate of some countries of Europe.

(*v*) **Firing of a missile.** The speed of a missile is very large. Therefore, while firing a missile the deviation due to Coriolis force is taken into account in calculating the direction of the target.

1.25 FOUCAULT'S PENDULUM

Foucault's pendulum. It is a simple device for conveniently detecting even slow rotation of the earth and provides a direct experimental evidence that the earth rotates about its axis from *West to East* once in 24 hours.

Construction. It consists of a very heavy mass (28 kg in the original experiment performed by Foucault in Paris in 1851) suspended by a very long wire (70 metres) so that the time period of the pendulum was large (about 17 seconds). The attachment of the upper end of the wire allows the pendulum to swing with equal freedom in any direction so that its period of oscillation in any plane is exactly the same. Such a pendulum once set oscillating continues to oscillate for a fairly long time.

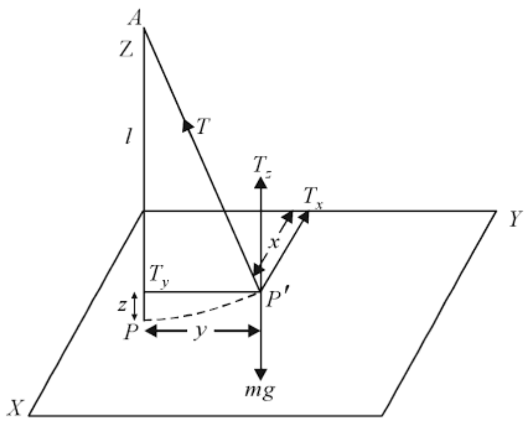

Fig. 1.15

Working. When the pendulum is started it swings in a definite vertical plane. The plane of oscillation is observed to precess around the vertical axis during a period of several hours.

$$m\ddot{z} = T_z - mg + 2m\omega \cos\lambda\, \dot{x} \qquad ...(iii)$$

Let x, y, z be the co-ordinates of the point P' and l the length of Foucault's pendulum then

$$\frac{T_x}{T} = \frac{-x}{l} \text{ or } T_x = -T\,\frac{x}{l}$$

$$\frac{T_y}{T} = \frac{-y}{l} \text{ or } T_y = -T\,\frac{y}{l}$$

$$\frac{T_z}{T} = \frac{l-z}{l} \text{ or } T_z = T\,\frac{l-z}{l}$$

As the amplitude of the pendulum is small z = 0

$\therefore$ $\qquad T_z = T$ (approximately)

Again as $\qquad z = 0,\ \dot{z} = 0$ and $\ddot{z} = 0$

Substituting $\qquad T_z = T$ and $\ddot{z} = 0$ in equation (*iii*), we have

$$0 = T - mg + 2m\omega \cos\lambda\dot{x}$$

The quantity $2m\omega \cos\lambda\dot{x}$ is the vertical component of Coriolis force. It is a very small quantity and can be neglected as compared to mg. $\quad\therefore\ T = mg$

Substituting $\qquad T = mg$ in equation (*i*) and (*ii*), we have

$$m\ddot{x} = -mg\,\frac{x}{l} + 2m\omega \sin\lambda\,\dot{y} \qquad ...(iv)$$

and

$$m\ddot{y} = -mg\,\frac{y}{l} - 2m\,\omega \sin\lambda\,\dot{x} \qquad ...(v)$$

The time-period of the pendulum is given by $t = 2\pi\sqrt{\frac{l}{g}}$ or $\frac{2\pi}{t} = \sqrt{\frac{g}{l}}$

Let $\qquad \frac{2\pi}{t} = \omega_0 \quad$ then $\frac{g}{l} = \omega_0^2$

Substituting $\frac{g}{l} = \omega_0^2$ in (*iv*) and (*v*), we get

$$\ddot{x} = -\omega_0^2\ x + 2\omega \sin\lambda\,\dot{y} \qquad ...(vi)$$

$$\ddot{y} = -\omega_0^2\ y - 2\omega \sin\lambda\,\dot{x} \qquad ...(vii)$$

Multiplying (*vii*) by $i = \sqrt{-1}$ and adding to (*vi*), we get

$$\ddot{x} + i\ddot{y} = -\omega_0^2\ [x + iy] - 2i\omega \sin\lambda\ [\dot{x} + i\dot{y}]$$

or

$$(\ddot{x} + i\ddot{y}) + 2i\omega \sin\lambda\ (\dot{x} + i\dot{y}) + \omega_0^2\ (x + iy) = 0$$

Put $\qquad x + iy = u \quad$ then $\dot{u} = \dot{x} + i\dot{y}$ and $\ddot{u} = \ddot{x} + i\ddot{y}$

Hence $\qquad \ddot{u} + 2i\ \omega \sin\lambda\dot{u} + \omega_0^2\, u = 0$

Substituting $\qquad \omega \sin\lambda = \omega_z$, we get $\ddot{u} + 2i\ \omega_z\ \dot{u} + \omega_0^2\, u = 0 \qquad ...(viii)$

The above equation in the operator form is put as $(D^2 + 2i\ \omega_z\ D + \omega_0^2)\ u = 0$

where $\qquad D = \frac{d}{dt}$ and $D^2 = \frac{d^2}{dt^2}$

or $\qquad D^2 + 2i\omega_z D + \omega_0^2 = 0$

The two roots of the above equation are $D = \dfrac{-2i\omega_z \pm \sqrt{4(-1)\omega_z^2 - 4\omega_0^2}}{2}$

$= -i\omega_z \pm i\omega_1$

where $\omega_1^2 = \omega_z^2 + \omega_0^2$

or $D_1 = -i(\omega_z - \omega_1)$ and $D_2 = -i(\omega_z + \omega_1)$

The general solution of equation (*viii*) is

$$u = Ae^{-i(\omega_z - \omega_1)t} + Be^{-i(\omega_z + \omega_1)t} = \left(Ae^{+i\omega_1 t} + Be^{-i\omega_1 t}\right)e^{-i\omega_z t} \quad ...(ix)$$

where A and B are arbitrary constants.

We have stated above that $\omega_1^2 = \omega_0^2 + \omega_z^2$

where $\omega_0 = \dfrac{2\pi}{t} = \sqrt{\dfrac{l}{g}}$ = angular velocity of oscillation of the pendulum and ω_z the Z-component $\omega \sin\lambda$ of the angular velocity of rotation of the earth which is very small as compared to ω_0.

$\therefore$ ω_z^2 can be neglected as compared to ω_0^2.

Hence $\omega_1^2 = \omega_0^2 + \omega_z^2 = \omega_0^2$ (approximately) or $\omega_1 = \omega_0$

Substituting $\omega_1 = \omega_0$ in equation (*ix*) we get $u = \left(Ae^{+i\omega_0 t} + Be^{-i\omega_0 t}\right)e^{-i\omega_z t}$ ...(*x*)

Considering $\omega_z = 0$ we get $u = Ae^{+i\omega_0 t} + Be^{-i\omega_0 t}$

The above equation gives the trajectory of the path traced out by the bob.

Now $Ae^{+i\omega_0 t} = A\cos\omega_0 t + iA\sin\omega_0 t$

and $Be^{-i\omega_0 t} = B\cos\omega_0 t - iB\sin\omega_0 t$

$\therefore$ $Ae^{+i\omega_0 t} + Be^{-i\omega_0 t} = (A+B)\cos\omega_0 t + i(A-B)\sin\omega_0 t$

But $u = x + iy$

$\therefore$ $(A+B)\cos\omega_0 t = x$ and $(A-B)\sin\omega_0 t = y$

Squaring and adding we get $\dfrac{x^2}{(A+B)^2} + \dfrac{y^2}{(A-B)^2} = \cos^2\omega_0 t + \sin^2\omega_0 t = 1$

which is the equation of an ellipse with its centre at the origin. Thus we find the trajectory of the bob of a Foucault's pendulum is elliptical.

Assumptions

The following assumptions have been made in the treatment of Foucault's pendulum to show that the trajectory of the bob of the pendulum is elliptical.

(*i*) The amplitude of the pendulum is small.

(*ii*) As ω the angular velocity of earth's rotation is small the effect of centrifugal force $(mr\omega^2)$ being proportional to ω^2 is neglected.

(*iii*) The pendulum oscillates very approximately in the X-Y plane and hence the component of $\vec{r}$ along Z-axis *i.e.*, $\vec{z} = 0$

(*iv*) The vertical component of Coriolis force $2m\omega\cos\lambda\ \dot{x}$ is a very small quantity and is neglected as compared to mg.

SOLVED EXAMPLES

Example 1.1 The path of a projectile is defined by the equation $r = 3t - t^2/30$ and $\theta^2 = 1600 - t^2$. Find its velocity and acceleration after 30 sec.

Solution. Given $r = 3t - \dfrac{t^2}{30}$

At $t = 30\ s\ ;\quad r = 3 \times 30 - \dfrac{30^2}{30} = 60\ m$

$\therefore$ $r = 60$ m

$$\dot{r} = \frac{dr}{dt} = 3 - \frac{t}{15}$$

At $t = 30\,s;\quad \dot{r} = 3 - \dfrac{30}{15} = 1\ \text{ms}^{-1}$

$\therefore$ $\dot{r} = 1\ \text{ms}^{-1}$

$$\ddot{r} = \frac{d(\dot{r})}{dt} = -\frac{1}{15}$$

$\therefore$ $\ddot{r} = -\dfrac{1}{15}\ \text{ms}^{-2}$

Given $\theta^2 = 1600 - t^2$

At $t = 30$ s $\quad \theta^2 = 1600 - 900 = 700$

$\therefore$ $\theta^2 = 700$

and $\theta = \sqrt{1600 - t^2}$

At $t = 30$ s; $\quad \theta = \sqrt{1600 - 900} = 10\sqrt{7}$

$\therefore$ $\theta = 10\sqrt{7}$ rad

Differentiating $\theta^2 = 1600 - t^2$

we get $2\theta\dot{\theta} = -2t$

$\therefore$ $\dot{\theta} = -\dfrac{t}{\theta}.$

At $t = 30\ s;\ \dot{\theta} = -\dfrac{30}{10\sqrt{7}} = -\dfrac{3}{\sqrt{7}}$

$\therefore$ $\dot{\theta} = -\dfrac{3}{\sqrt{7}}$ rad sec^{-1}

Differentiating $\dot{\theta} = -\dfrac{t}{\theta}$, we get

$$\ddot{\theta} = -t \times -1\theta^{-2}\ \dot{\theta} - \frac{1}{\theta} = \frac{t}{\theta^2}\dot{\theta} - \frac{1}{\theta}$$

At $t = 30\ s,\ \ddot{\theta} = -\dfrac{30}{700} \times \dfrac{3}{\sqrt{7}} - \dfrac{1}{10\sqrt{7}}$

$$= -\frac{9}{70\sqrt{7}} - \frac{1}{10\sqrt{7}} = -\frac{16}{70\sqrt{7}}$$

$\therefore$ $\ddot{\theta} = -\dfrac{16}{70\sqrt{7}}$ rad sec^{-2}

Now velocity $\vec{v} = \dot{r}\,\hat{r} + r\dot{\theta}\hat{\theta}$

$$= \hat{r} + 60 \times \frac{-3}{\sqrt{7}}\,\hat{\theta} = \hat{r} - \frac{180}{\sqrt{7}}\,\hat{\theta}$$

Also acceleration $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$

or $$\vec{a} = \left(-\frac{1}{15} - 60 \times \frac{9}{7}\right)\hat{r} + \left(60 \times \frac{-16}{70\sqrt{7}} + 2 \times 1 \times \frac{-3}{\sqrt{7}}\right)\hat{\theta}$$

$$= -\left(\frac{8107}{105}\right)\hat{r} - \left(\frac{96}{7\sqrt{7}} + \frac{6}{\sqrt{7}}\right)\hat{\theta}$$

$$= -\left(\frac{8107}{105}\right)\hat{r} - \left(\frac{138}{7\sqrt{7}}\right)\hat{\theta} = -\left[\frac{8107}{105}\,\hat{r} + \frac{138}{7\sqrt{7}}\,\hat{\theta}\right]$$

Example 1.2 A point moving in a plane has co-ordinates $x = 3$, $y = 4$ and has components of speed $\dot{x} = 5$ m/sec, $\dot{y} = 8$ m/sec at some instant of time. Find the components of speed in polar co-ordinates r, θ along directions $\hat{r}$ and $\hat{\theta}$.

Solution. Component of speed along $\hat{r}$

$$|\vec{v}_r| = \frac{\dot{x}x + \dot{y}y}{(x^2 + y^2)^{1/2}} = \frac{5 \times 3 + 8 \times 4}{(9 + 16)^{1/2}} = 9.4 \text{ m/s}$$

Component of speed along $\hat{\theta}$

$$|\vec{v}_\theta| = \frac{\dot{y}x - \dot{x}y}{(x^2 + y^2)^{1/2}} = \frac{8 \times 3 - 5 \times 4}{(9 + 16)^{1/2}} = 0.8 \text{ m/s}$$

Example 1.3 A moving particle has co-ordinates $(6t + 3)$, $8t$, 5 m in frame S at any time t. The frame S' is moving relative to S with a velocity $3\hat{i} + 4\hat{j}$ m/s. Find the co-ordinates and velocity of the particle in frame S'.

Solution. Galelian transformation equations for space co-ordinates are

$$x' = x - v_x t \quad y' = y - v_y t \quad z' = z - v_z t$$

Now $$x = 6t + 3 \quad y = 8t \quad z = 5; \quad v_x = 3 \quad v_y = 4 \quad v_z = 0$$

$\therefore$ $$x' = x - v_x t = 6t + 3 - 3t = 3t + 3$$

$$y' = y - v_y t = 8t - 4t = 4t\,;\; z' = z - v_z t = 5 - 0 = 5$$

$\therefore$ Co-ordinates of the particle in S' are $3t + 3$, $4t$, 5.

Velocity. $x' = 3t + 3 \quad \therefore \quad \frac{dx'}{dt'} = \frac{dx'}{dt} \cdot \frac{dt}{dt'} = \frac{dx'}{dt}$ $\quad \ldots\left[\because \begin{array}{l} t' = t \\ dt' = dt \end{array}\right]$

Hence $$\frac{dx'}{dt} = 3 \text{ or } \vec{u}'_x = 3\hat{i}$$

Similarly $$\frac{dy'}{dt'} = \frac{dy'}{dt} = 4 \quad \text{or} \quad \vec{u}'_y = 4\hat{j} \qquad \frac{dz'}{dt'} = \frac{dz'}{dt} = 0$$

$\therefore$ Velocity in frame S', $\vec{u}' = \vec{u}'_x + \vec{u}'_y + \vec{u}'_z = 3\hat{i} + 4\hat{j}$

Example 1.4 The position vector of a point in the frame S' moving with constant velocity of 10 cm/sec along X-axis is given by (11, 9, 8) cm. Calculate the position with respect to the frame S if the two frames were coincident only $\frac{1}{2}$ sec. earlier. (*P.U.* 2001, 2000)

Solution. As the position co-ordinates are given in the frame S' which is moving with constant velocity of $v = 10$ cm s^{-1} along $+X$-axis, the position co-ordinates in the frame S are given by inverse Galelian transformations

$$x = x' + vt; \quad y = y'; \quad z = z'; \quad t = t'$$

Here $\quad x' = 11 \text{ cm} \quad y' = 9 \text{ cm} \quad z' = 8\text{cm} \quad t' = \frac{1}{2} \text{ sec.}$

$\therefore \quad x = x' + vt = 11 + 10 \times \frac{1}{2} = 16 \text{ cm}; \ y' = 9 \text{ cm}; \ z' = 8 \text{ cm}$

$\therefore$ The position co-ordinates of the point in the frame S are (16, 9, 8) cm.

Example 1.5 A frame S' is moving with velocity $5\hat{i} + 7\hat{j}$ m/s relative to an inertial frame S. A particle is moving with velocity $(t + 5)\hat{i} + 9\hat{j}$ m/s with respect to S. Find the acceleration of the particle in the frame S'.

Solution. Velocity of the particle in frame S, $\vec{u} = (t + 5)\hat{i} + 9\hat{j}$ ms^{-1}

Velocity of frame S' relative to S, $\quad \vec{v} = 5\hat{i} + 7\hat{j}$

According to Galelian transformation equations for velocity,

Velocity of the particle with respect to frame S',

$$\vec{u'} = \vec{u} - \vec{v} = (t + 5)\hat{i} + 9\hat{j} - (5\hat{i} + 7\hat{j}) = t\hat{i} + 2\hat{j} \text{ ms}^{-1}$$

$\therefore$ Acceleration of the particle in the frame S', $\vec{a'} = \frac{d\vec{u'}}{dt'} = \frac{d\vec{u'}}{dt}$ $\quad$ ...[$\because dt' = dt$]

$$= \frac{d}{dt}[t\hat{i} + 2\hat{j}] = \hat{i} \qquad \therefore \quad \vec{a'} = \hat{i} \text{ ms}^{-2}$$

Example 1.6 Calculate the fictitious force and total force acting on a mass of 5 kg in a frame of reference moving (*i*) vertically downward and (*ii*) vertically upward with an acceleration of 5 metres/sec^2. Acceleration due to gravity = 9.8 metres/sec^2. (*H.P.U.* 2001; *Pbi. U.*,2000)

Solution. The acceleration due to gravity = 9.8 metres/sec^2

Take the earth to be an inertial frame of reference and upward direction as positive, (downward direction negative)

$$m = 5 \text{ kg/m} \quad \vec{a_0} = 5 \text{ metres/sec}^2$$

$\therefore$ Weight of the body $\quad \vec{F_i} = mg = 5 \times (-9.8) = -49$ Newton

(The negative sign indicates that the weight is acting downward.)

(*i*) **Motion downward.** The fictitious force acting on the mass during downward motion

$$\vec{F_0} = m\vec{a_0} = -5(-5) = 25 \text{ Newton (upward)}$$

$\therefore$ Total force $\quad \vec{F'} = \vec{F_i} - \vec{F_0} = -49 + 25 = -24$ Newton (downward)

(*ii*) **Motion upward.** The fictitious force acting on the mass during upward motion

$$\vec{F_0} = m\vec{a_0} = -5(5) = -25 \text{ Newton (downward)}$$

$\therefore$ Total force $\quad \vec{F'} = \vec{F_i} - \vec{F_0} = -49 - 25 = -74$ Newton (downward)

Example 1.7 Calculate the total force acting on a freely falling body of mass 5 kg with reference to a frame moving with a downward acceleration of 2 metres per second².

Solution. The force $\vec{F}'$ acting on a body in a non-inertial frame is given by $\vec{F}' = \vec{F}_i - \vec{F}_0$ where $\vec{F}_i$ is the force on the same body in an inertial frame and $\vec{F}_0$ is the fictitious force due to the accelerated motion of the non-inertial frame. As the body is falling freely, downward force on it in the inertial frame of the earth $\vec{F}_i = 0$

$$\therefore \qquad \vec{F}' = -\vec{F}_0 \quad \text{or} \quad \vec{F}' = -m\vec{a}_0$$

where $\vec{a}_0$ is the acceleration of the non-inertial frame and m the mass of the body.

As the reference frame is moving downward with an acceleration of 2 m/s² $\vec{a}_0 = -2 \text{ ms}^{-2}$

$$\therefore \qquad \vec{F}' = -m\vec{a}_0 = -(-2 \times 5) = +10 \text{ N}$$

The positive sign indicates that the fictitious force is acting upward.

Example 1.8 A frame of reference is moving with an acceleration of 5 m/sec² downward. Find the apparent force and total force acting on a body of mass 10 kg falling freely relative to the frame.

Solution. Apparent force = Total force = – (– 5 × 10) = 50 N (upward)

Example 1.9 Calculate the effective weight of an astronaut ordinarily weighing 60 kg when his rocket moves vertically upward with 5 g acceleration.

Solution. As the rocket moves vertically upward with an acceleration 5 g, it is a non-inertial frame and therefore the total force on the astronaut is given by $\vec{F}' = \vec{F}_i - \vec{F}_0$

where $\vec{F}_i$ is the force on the astronaut in an inertial frame and $\vec{F}_0$ is the fictitious force on the astronaut due to the acceleration of the rocket.

$$\text{Now } \vec{F}_i = 60 \text{ kg. wt} = 60\, g \text{ N}$$

$$\text{and} \qquad \vec{F}_0 = -m\vec{a}_0 = -60 \times 5\, g \text{ N} = -300\, g \text{ N}$$

$\therefore$ Effective weight of the astronaut $\vec{F}' = \vec{F}_i - \vec{F}_0 = 60\, g - (-300\, g)$

$$= 360\, g \text{ N} = 360 \text{ kg}$$

Example 1.10 How much faster than its present speed should the earth rotate so that bodies lying on the equator may fly off into space?

Solution. The centrifugal force acting on a body of mass m at the equator is equal to $m R\omega^2$ where R is the radius of the earth and ω its angular velocity. If g is the acceleration due to gravity when the earth is at rest, then g' the acceleration when the earth has an angular velocity ω is given by

$$g' = g - R\omega^2 \cos^2 \lambda = g - R\omega^2 \qquad [\because \lambda = 0, \cos \lambda = 1 \text{ and } \cos^2 \lambda = 1]$$

For the body to fly off into space the new angular speed ω' should be such that

$$g - R\omega'^2 = 0 \text{ or } \frac{R\omega'^2}{g} = 1 \qquad ...(i)$$

$$\text{but} \qquad \frac{R\omega^2}{g} = \frac{1}{291} \qquad ...(ii)$$

Dividing (i) by (ii), we have $\dfrac{\omega'^2}{\omega^2} = 291$ or $\dfrac{\omega'}{\omega} = \sqrt{291} = 17.06$

Solution. Velocity of the bullet $\vec{v} = 500 \text{ ms}^{-1}$

Co-latitude $\phi = 60°$ $\therefore$ Latitude $\lambda = (90 - \phi) = 30°$ and $\sin\lambda = \sin 30 = \frac{1}{2}$

The X-axis is taken towards the East, Y-axis towards the North and Z-axis in the vertically upward direction

Coriolis acceleration is given by $\vec{a}_a = -2\vec{\omega} \times \vec{v}_a$

Horizontal component of Coriolis acceleration $(\vec{a}_a)_H = 2\omega \sin\lambda\,(v_y \hat{i} - v_x \hat{j})$

and Vertical component of Coriolis acceleration $(\vec{a}_a)_V = 2\omega\, v_x \cos\lambda\, \hat{k}$

(*a*) Bullet fired Eastward. Then $v_x = \vec{v} = 500 \text{ ms}^{-1}$; $v_y = 0$

$\therefore$ Horizontal component of Coriolis acceleration

$$(\vec{a}_a)_H = -2\,\omega \sin\lambda v\,\hat{j}$$
$$= 2 \times \frac{2\pi}{24 \times 60 \times 60} \times \frac{1}{2} \times 500 = 0.036 \text{ ms}^{-1}$$

along $-\hat{j}$ *i.e.*, $-Y$ direction or towards South.

Vertical component of Coriolis acceleration

$$(\vec{a}_a)_V = 2\omega v_x \cos\lambda\,\hat{k}$$
$$= 2 \times \frac{2\pi}{24 \times 60 \times 60} \times \frac{\sqrt{3}}{2} \times 500 = 0.036\sqrt{3}$$

along $+\hat{k}$ *i.e.*, $+Z$ direction or vertically upwards.

$\therefore$ Magnitude of the resultant acceleration

$$|\vec{a}_a| = |(\vec{a}_a)_H|^2 + |(\vec{a}_a)_V|^2 = \sqrt{(0.036)^2 + (0.036 \times \sqrt{3})^2} = 0.072 \text{ ms}^{-1}$$

(*b*) Bullet fired northwards. Then $v_y = \vec{v} = 500 \text{ ms}^{-1}$; $v_x = 0$

$\therefore$ Horizontal component of coriolis acceleration $(\vec{a}_a)_H = 2\omega \sin\lambda\, v\,\hat{i} = 0.036 \text{ ms}^{-1}$ along $\hat{i}$ *i.e.*, $+X$ direction

Vertical component of coriolis acceleration $(\vec{a}_a)_V = 2\omega\, v_x \cos\lambda\,\hat{k} = 0$

$\therefore$ Resultant acceleration $= 0.036 \text{ ms}^{-1}$ along $+X$ direction.

Example 1.14 Calculate the magnitude and direction of Coriolis acceleration of a rocket moving with a velocity of 2 km S^{-1} at 60° South latitude.

Solution. For a body moving in a vertical direction, Coriolis force is given by

$$\vec{F}_{cor} = -2m(\vec{\omega} \times \vec{v}) = -2m \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \omega_y & \omega_z \\ 0 & 0 & \dot{z} \end{vmatrix}$$

$$\therefore \qquad \vec{F}_{cor} = -2m\omega_y\, \dot{z}\,\hat{i}$$

For a rocket moving vertically *upwards* at 60° *South* latitude (λ)

$$\dot{z} = +v \quad \text{and} \quad \omega_y = -\omega\cos\lambda \qquad \therefore \quad \vec{F}_{cor} = +2m\omega v \cos\lambda\,\hat{i}$$

Hence rate of rotation $= \dfrac{2\pi}{48}$ radians per hour

In turning from NS to NE-SW direction the pendulum turns through $45° = \dfrac{\pi}{4}$ radian as shown.

$$\therefore \text{ Time taken } = \frac{\dfrac{\pi}{4}}{2\pi/48} = \frac{48}{8} = 6 \text{ hours}$$

Example 1.19 Show that at latitude θ, the plane of oscillation of Foucault's pendulum rotates through $2\pi \sin \theta$ every day. Explain this physically on a pole of the earth.

Solution. The period of rotation of a Foucault's pendulum at a latitude θ is given by

$$T = \frac{2\pi}{\omega \sin \theta}$$

where $\dfrac{2\pi}{\omega} = 24$ hours. $\quad \therefore \quad T = \dfrac{24}{\sin \theta}$ hours

In T hours the plane rotates through 2π radians.

$\therefore$ In 24 hours the plane will rotate through an angle

$$\phi = \frac{2\pi}{T} \times 24 = \frac{2\pi}{24} \sin \theta \times 24 = 2\pi \sin \theta \text{ radian}$$

At the poles $\qquad \theta = 90°, \ \sin \theta = \sin 90° = 1$

$\therefore \qquad \phi = 2\pi$ radians

Thus the plane of oscillation of a Foucault's pendulum rotates through 2π radians in 24 hours at the poles which means that the period of rotation of the earth about its own axis is 24 hours.

EXERCISE CH-1

LONG QUESTIONS

1. State Newtons laws of motion. Show that Newton's first law of motion is simply a special case of the second law. Discuss the limitations of Newtons laws of motion.
(*Nagpur Uni.* 2007, 2006, 2003)

2. (*i*) State and explain Newton's second law of motion.
(*ii*) State the limitations of Newton's laws of motion.
(*iii*) Derive the equations for components of a velocity, Acceleration and force in a cartesian coordinate system. (*Nagpur Uni.* 2007)

3. A particle is moving along a curve in a plane. Derive expression for its radial and transverse components of velocity and acceleration. (*Nagpur Uni.* 2007, 2006)

4. A particle is moving along a curve in a plane. Derive expression for the radial and transverse component of velocity and acceleration. Prove that for the motion of a particle in a plane

$$\vec{v} = \dot{r}\hat{r} + \boldsymbol{r}\dot{\theta}\hat{\theta} \quad \text{and} \quad \vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (\ddot{r}\theta + 2r\ddot{\theta})\,\hat{\theta}$$

where the letters have their usual meaning. (*Cal. Uni.* 2003)

5. For planar motion $x = r \cos \theta$; $y = r \sin \theta$. Prove that

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} \quad \text{and} \quad r\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r}$$

6. (*i*) What are Galelian transformations?

(*ii*) Derive Galelian transformation equations for two inertial frames.

(*iii*) State and prove Galelian invariance. (*Pbi. Uni.* 2001; *K.U.* 2001; *Calicut U.* 2003; *P.U.* 2000; *G.N.D.U.* 2000; *Guwahati U.* 2000)

7. (*a*) What are Galelian transformations?

(*b*) Prove that Newton's laws of motion are invariant under Galelian transformations.

(*c*) What are the quantities that are invariant under Galelian transformations?

(*P.U.* 2000; 2003; *M.D.U.* 2003; *Pbi. Uni.* 2002)

8. Show that the laws of conservation of Momentum and Energy are invariant to Galelian transformations. (*P.U.* 2000, *G.N.D.U.* 2004, *Pbi. U.* 2001)

9. (*a*) Show that a frame of reference having a uniform translatory motion (or moving with constant velocity) relative to an inertial frame is also inertial.

(*P.U.*, 2001)

(*b*) Show that the force on the particle is the same in two frames connected by Galelian transformations. Hence justify that if the relation $\vec{F} = m\vec{a}$ is used to define force observers in all frames of reference would agree on the magnitude and direction of the force $\vec{F}$ independent of relative velocity of reference frames. (*Guwahati.U., 2000*)

10. (*a*) Show that a frame of reference having a uniform translatory motion relative to an inertial frame is also inertial. (*P.U.* 2001)

(*b*) Show that the force on the particle is the same in two frames connected by Galelian transformations. (*Guwahati* 2000)

11. (*i*) Explain the terms non-inertial frame of reference and fictitious force.

(*ii*) Calculate the total force and fictitious force acting on a body in a non-inertial frame. Give two examples. (*Pbi. U.* 2000, 2001; *Guwahati U.* 2000; *Calicut U.* 2003)

12. Distinguish between inertial and non-inertial frames of reference. Give one example of each. Is earth an inertial frame ? Give reasons.

(*Nag. U.* 2009, 2008, 2007, 2006; *H.P.U.*, 2003, 2001, 2000; *Bhopal U.*, 2004; *Bang.U.*, 2004; *Osm.U.*, 2004; *M.D.U.*, 2003; *P.U.*, 2004, 2001; *Guwahati U.*, 2000)

13. Prove that a frame of reference moving with uniform translational acceleration with respect to an inertial frame is a non-inertial frame. (*P.U.* 2000)

14. (*a*) What are different types of fictitious forces in a uniformly rotating frame of reference? (*G.N.D.U.*, 2004)

(*b*) Prove that Coriolis force owes its existence to the motion of the particle with respect to a rotating frame of reference. (*P.U.*, 2001, 2000; *Pbi.U.*, 2000)

15. What is Newtonian principle of relativity? Discuss with examples. Why should laws of nature be the same in all inertial frames of reference? (*H.P.U.*, 2000)

16. (*i*) Obtain relation for displacement in spherical polar coordinates.

(*ii*) Drawing a neat diagram prove that for motion of a particle in space $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\,\hat{\theta} + r\sin\theta\dot{\phi}\,\hat{\phi}$ where the symbols have their usual meanings.

(*iii*) Give the physical significance of each of them. (*H.P.U.* 2001, *P.U.* 2003,2000)

17. Starting from the relationship between spherical polar co-ordinates and cartesian co-ordinates, find the value of v_r, v_θ and v_ϕ in terms of x, y, z and $\dot{x}, \dot{y}, \dot{z}$. *(P.U. 2001)*

18. A reference frame '*a*' rotates with respect to another reference frame '*b*' with an angular velocity $\vec{\omega}$. If the position, velocity and acceleration of a particle in frame '*a*' are represented by $\vec{r}, \vec{v}_a$ and $\vec{a}_a$ show that acceleration of the particle in frame '*b*' is given by $\vec{a}_b$ where

$$\vec{a}_b = \vec{a}_a + 2\,\vec{\omega} \times \vec{v}_a + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \frac{d\vec{\omega}}{dt} \times \vec{r}$$

Explain the physical significance of various terms in the above expression and show that Coriolis and centrifugal forces are consequences of rotation of frame of reference.
(*P.U.*, 2004, 2001; *G.N.D.U.*, 2004; *H.P.U.*, 2001; *Nagpur U.* 2006, *Guwahati U.*, 2000; *Bhopal U.*, 2004)

19. (*a*) Taking earth to be a rotating frame of reference discuss the effect of centrifugal force due to rotation on the value of *g*. Show that the effect is maximum at the equator and minimum at the poles.

(*b*) Hence find the angle through which a plumb line will be deflected at a place in latitude λ. Calculate the deflection for $\lambda = 45°$.

20. Define Coriolis force. Discuss the effect of Coriolis force on a particle moving on the surface of the earth. Calculate the horizontal and vertical components of Coriolis force. (*Pbi.U.*, 2003)

21. (*a*) What will be the direction of Coriolis force in Northern and Southern hemisphere.

(*b*) How much faster than its present speed should the earth rotate so that bodies lying on the equator may fly off into the space. (*P.U.* 2001, 2000; *Pbi. U.* 2000)

[Ans. $\frac{\omega'}{\omega} = 17.06$]

22. Discuss the effect of Coriolis force due to rotation of the earth on the setting up cyclones, trade winds and describe other geographical effects of this force.
(*H.P.U.* 2003, *P.U.* 2004, 2003, 2002, 2001, 2000; *Bhopal U.* 2004)

23. (*a*) Describe the construction of a Foucault's pendulum. Show that the bob of the pendulum describes an ellipse which rotates with an angular velocity $\omega \sin \lambda$ where λ is the latitude of the place. (*H.P.U.*, 2000; *P.U.*, 2000; *Calicut U.* 2003; *Kolkata U.* 2004)

(*b*) Show that the rotation of the plane of oscillation of the Foucault's pendulum is a direct proof of the rotation of the earth about its own axis. (*H.P.U.*, 2000)

(*c*) State the assumptions made in the treatment of Foucault's pendulum. (*Pbi.U., 2000*)

SHORT QUESTIONS

1. What was Newton's notion of space.

2. State and explain Newton's laws of motion. (*Nag. Uni.* 2009, 2007, 2006)

3. 'Newton's first law is simply a special case of the second law.' Explain.

4. What are the limitations of Newton's laws of motion. (*Nag. Uni.* 2007, 2003)

5. Obtain an expression for radial and transverse component of velocity of a particle moving in rectangular axes in a plane. (*Nag. Uni. 2007, 2006*)

6. Derive the law of conservation of linear momentum from Newton's laws of motion. (*Nag. U.* 2003)
7. What is 'Newtonian principle of relativity?' Discuss with example. (*H.P.U.* 2000)
8. Is earth an inertial frame? Explain.
9. Distinguish between inertial and non-inertial frame of reference.
10. Explain Fictitious or Pseudo force.
11. Calculate the total force and pseudo force acting on a body in an non-inertial frame.
12. Starting from the expression for velocity $v = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r \sin \theta\dot{\phi}\hat{\phi}$ obtain an expression for acceleration in spherical polar co-ordinates. (*H.P.U.* 2000; *P.U.* 2004, 2003)
13. A reference frame rotates with respect to another frame with angular velocity $\vec{\omega}$. Find the relation for acceleration. (*Nag. Uni.* 2006)
14. Can a particle be in equilibrium in a non-inertial frame? (*Pbi. U.*, 2003)
15. For planar motion derive a relationship between acceleration as expressed in cartesian and polar co-ordinates. (*H.P.U.* 2001, 2000)
16. Give the physical significance of fictitious force.
17. Explain 'Coriolis force owes its existence to the motion of a particle with respect to a rotating frame of reference.
18. Mention applications of Coriolis force and Explain.
19. What will be the direction of Coriolis force in Northern and Southern hemispheres? (*P.U.*, 2001, 2000; *Pbi. U.*, 2000)
20. Describe how a Foucault's pendulum is used as a device to illustrate that the earth is not an inertial frame. (*Kolkata U.* 2002)
21. Discuss the effect of rotation of the earth on the value of *g*.
22. Discuss the Foucault's pendulum and its uses.
23. Discuss the assumptions in Foucault's pendulum experiment.
24. What is the effect of Coriolis force on a moving particle on surface of earth.
25. Discuss the effect of Centrifugal force on acceleration due to gravity (g).
26. Is rocket a system of variable mass? Explain. (*Nag. Uni.* 2008)
27. Give an example of a non-inertial frame explaining why it is non-inertial. (*Kolkata U.* 2001)
28. Discuss the effect of Coriolis force in the formation of cyclones. (*Kolkata U.* 2001)
29. Discuss rotating frames and Coriolis force, state some effects of Coriolis force. (*D.A.U. Agra,* 2003)

NUMERICALS

1. The position vector of a point is given by

$$\vec{r} = \left(\frac{4}{3}t^3 - 2t\right)\hat{i} + t^2\hat{j}.$$

Find the velocity and acceleration of the point at $t = 3$ sec. The distance is measured in metres. (**Ans.** 34.5 ms^{-1}, 24.1 ms^{-2})

2. The motion of a particle is described by the equations $x = 4 \sin 2t$, $y = 4 \cos 2t$, $z = 6t$. Find velocity and acceleration of the particle. (*H.P.U.* 2001) (**Ans.** 10 ms^{-1}, 18 ms^{-2})

3. The polar coordinates of a point are $(r, \theta, \phi) = 8, 30°, 45°$. Find the Cartesian coordinates of the same point. (*Pbi. U.* 2003, *H.P.U.* 2000) (**Ans.** $2\sqrt{2}, 2\sqrt{2}, 4\sqrt{3}$)

4. The Cartesian coordinates of a point are (1, 0, 0). Find the spherical polar coordinates of this point. (*Pbi.* 2002) (**Ans.** $1, \frac{\pi}{2}, 0$)

5. If the earth were to cease rotating about its axis, what will be the change in value of g at a place of latitude 45° assuming the earth to be a sphere of radius 6.38×10^8 cm.

(**Ans.** 1.686 cm/sec^2)

6. The motion of a particle can be expressed in terms of the following parametric equations:

$$x = 5t^2 - 7; \; y = 7 \cos t \text{ and } z = 3 \sin t.$$

Find the magnitude of instantaneous velocity and acceleration of the particle.

$$\left[\textbf{Ans. } |v| = \sqrt{9 + 40 \sin^2 t + 100\, t^2} \quad |a| = \sqrt{109 + 40 \cos^2 t}\right]$$

7. A particle moving in the X-Y plane has co-ordinates $(4t, 4t^2)$ m at any instant, find the velocity of the particle. (*P.U.*, 2000)

[**Ans.** $(4\hat{i} + 8\hat{j})$]

8. The motion of a particle is observed for 10 sec. and is found to be in accord with the following equation

$$r = R \text{ (a constant)}, \; \theta = \frac{\pi}{12} t \text{ and } \phi = \pi t$$

Find the velocity and acceleration of the particle at an arbitrary time $t < 10$ sec.

Hint. $v = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r \sin$

and $a = (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2 \theta)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta} - r \sin\theta \cos\theta\dot{\phi}^2)\hat{\theta}$

$$+ (2\dot{r}\dot{\phi}\sin\theta + 2r \cos\theta\dot{\theta}\dot{\phi} + r \sin\theta\ddot{\phi})\hat{\phi}$$

$$[\textbf{Ans. } |\vec{v}| = \left(\frac{\pi R}{12}\right)\left[1 + 144 \sin^2 \frac{\pi t}{12}\right]^{1/2}$$

$$|\vec{a}| = \frac{\pi^2 R}{144}\left(577 + 204485 \sin^2 \frac{\pi t}{12}\right)^{1/2}]$$

9. Calculate the time required for the plane of vibration of a Foucault's pendulum to rotate once at a latitude 45°. How much time will it take to rotate through 60° at that place and at a place of latitude 60°? [**Ans.** $24\sqrt{2}$ hrs.; $4\sqrt{2}$ hrs.; $\frac{8}{\sqrt{3}}$ hrs.]

10. A man weights 70 kg. What would be his weight in lift moving (*i*) upward (*ii*) downward, with acceleration 25% of g ? [**Ans.** (*i*) 87.5 kg (*ii*) 52.5 kg]

11. Calculate the fictitious force and total force acting on a freely falling body of mass 18 kg with respect to a frame moving with a downward acceleration of 6 m/sec^2. [**Ans.** 108 N]

12. A particle is thrown vertically upwards with a velocity 70 m/sec at a place with latitude 60°. Find how far from the original position will it land. [**Ans.** 0.00643 m]

2

MOTION UNDER A CENTRAL FORCE

INTRODUCTION

In the previous chapter we have seen how Galileo's understanding of inertia led Newton to formulate his three laws of motion which gave qualitative and quantitative measure of a force. Different types of forces which actually arise in nature and also of those types of forces which can not be explained by Newton's laws. Basically there are four fundamental types of forces depending upon the interaction of matter. These are (*i*) gravitational force due to gravitational interaction (*ii*) electromagnetic force due to electromagnetic interaction (*iii*) strong interaction and (*iv*) weak interaction. The latter two forces are the *short-range* forces (of the order of the size of the nucleus, *i.e.* 10^{-13} cm). Gravitational force is a long range force (also electrostatic force) and the interaction is the weakest. Still it holds the earth together with the sun at its centre in our solar system, several stars (like sun) in a galaxy, several galaxies in a milky-way and so on in the universe.

All the celestial bodies in the universe are bound by the gravitational force. Gravitational force is a conservative force and is an example of a central force. The first systematic study was made by Ptolemy in 100 AD on the assumption of geocentric universe; *i.e.* every body in the universe revolves around the earth. This idea was discarded in 1543 AD by Nicolaus Copernicus suggesting *heliocentric universe*, *i.e.* bodies in the universe revolve around the sun, to which Galileo and Bruno supported. In 1569, Tycho Brahe made careful observations of heavenly bodies for 30 years but failed to formulate. Kepler who was assistant to Tycho Brahe carried out the work and proved the correctness of heliocentric theory. He established three important laws about planetary motion around the sun.

2.1 CENTRAL FORCE

A central force is that force which is always directed away or towards a fixed centre and the magnitude of which is a function only of the distance from the centre taken as origin.

In spherical polar co-ordinates we express the central force as $\vec{F}(r) = F(r)\,\hat{r}$ where $\hat{r}$ is a unit vector along $\vec{r}$ and $\vec{F}(r)$ is a function of r.

Characteristics. The characteristics of central force between two bodies are

(*i*) The magnitude of the force depends only on the distance between the centres of the two bodies.

(*ii*) It acts along the line joining the centres of the two bodies.

(*iii*) A central force is a **conservative force** *i.e.*, it is the gradient of some scalar potential U or $\vec{F} = -\text{ grad } U = -\vec{\nabla} U$.

Examples. Familiar examples of central force are :

(*i*) Gravitational force of attraction between two masses.

If m_1 and m_2 are two isolated masses at a distance r apart, then the gravitational force of attraction on the mass m_1 due to the mass m_2 is given by $\vec{F}_{12} = -\frac{Gm_1 m_2}{r^2}\hat{r} = F(r)\hat{r}$

where $F(r) = -\frac{Gm_1 m_2}{r^2} = -\frac{c}{r^2}$ and $c = Gm_1 m_2$

(*ii*) Electrostatic force of attraction or repulsion between two charges.

If $+q_1$ and $+q_2$ are two isolated point charges at a distance r apart, then the electrostatic force of repulsion on the charge $+q_1$ due to the charge $+q_2$ in vacuum is given by

$$\vec{F}_{12} = K\frac{q_1 q_2}{r^2}\hat{r} = F(r)\hat{r}$$

where $F(r) = \frac{Kq_1 q_2}{r^2} = +\frac{c}{r^2}$ and $c = Kq_1 q_2$

In S.I. units $K = \frac{1}{4\pi\varepsilon_0}$, where ε_0 is the permitivity of free space.

Non-Central Force. A non-central force is that force which does not simply depend upon the distance between the centres of the two interacting bodies but also on other parameters such as their *spin* and *relative orientation*.

Characteristics. The characteristics of non-central forces are :

(*i*) They are short range forces *i.e,* the force acts only when the interacting particles are very close to each other.

(*ii*) Non-central forces do not necessarily act along the line joining the centres of the two bodies.

(*iii*) A non-central force is non-conservative and can not be derived from some scalar potential *i.e.* they are not the gradient of some scalar function.

Examples. Familiar examples of non-central forces are :

(*i*) **Weak forces** called into play in β-decay and decay processes where the decay products are *leptons* (electrons, positrons, *neutrinos*, μ-mesons etc.) are non-central. Weak forces are non-zero only when the inter acting particles just overlap.

(*ii*) **Strong nuclear forces** between proton-proton (*p-p* interaction), proton-neutron (*p-n* interaction) and neutron-neutron (*n-n* interaction) are non-central as these are due to the exchange of π^+, π^- and π^0 mesons respectively.

2.2 CONSERVATIVE FORCE

(*i*) A force $\vec{F}$ acting on a particle is a conservative force if the curl of the force is zero. *i.e.*

$$\vec{\nabla} \times \vec{F} = 0 \quad \text{... } (i)$$

(*ii*) If V is a scalar function of co-ordinates of the particle then

$$\vec{\nabla} \times (-\vec{\nabla} V) = 0 \quad \text{... } (ii)$$

From (*i*) and (*ii*), we have

$$\vec{F} = -\vec{\nabla} V$$

$\therefore$ *A force acting on a particle is conservative if it is given by the gradient of a scalar function V.* This scalar function is called the *potential energy* of the particle.

(*iii*) If the force $\vec{F}$ acting on the particle displaces it from the position A to B, then work done by the force

$$W = \int_A^B \vec{F} \cdot \vec{dr}$$

If $\vec{F}$ is a conservative force $\vec{F} = -\vec{\nabla} V$ and $W = \int_A^B -\vec{\nabla} V \cdot d\vec{r}$

Now $$\vec{\nabla} V = \frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k} \text{ and } d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$\therefore$ $$\vec{\nabla} V \cdot d\vec{r} = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = dV$$

$\therefore$ $$W = \int_A^B -\vec{\nabla} V \cdot d\vec{r} = -\int_A^B dV = V_A - V_B \qquad \text{... (iii)}$$

i.e., the work done depends only on the value of potential function V at the initial and final positions but is independent of the path joining A and B.

Hence '*A force $\vec{F}$ acting on a particle is conservative if the work done in taking the particle from one point to the other depends only on the initial and final positions but is independent of the actual path*'.

Non-conservative force. A force $\vec{F}$ acting on a particle is non-conservative if

(*i*) The curl of the force does not vanish *i.e.* $\vec{\nabla} \times \vec{F} \neq 0$

or (*ii*) It cannot be expressed as the gradient of a scalar function and

or (*iii*) The work done in moving the particle from one point to another depends upon the actual path of displacement

Examples. (*a*) Conservative forces (*i*) The gravitational force between two masses

(*ii*) The coulomb force between two stationary charges.

(*b*) Non - conservative forces (*i*) Exchange forces in nuclear Physics

(*ii*) Forces between moving charges.

2.3 CENTRAL FORCE:A CONSERVATIVE FORCE

A central force is conservative force and *a conservative force is the negative gradient of scalar potential*. Therefore a central force is negative gradient of scalar potential. It is proved as under.

A central force is represented as $\vec{F} = F(r)\,\hat{r}$ where $F(r)$ is a function of r only and $\hat{r}$ a unit vector along $\vec{r}$. Suppose the force acts on a particle and displaces it through a small distance $d\vec{r}$, then work done by the force

$$dW = \vec{F} . \vec{dr} \qquad \text{... (i)}$$

$\therefore$ Work done in moving the particle from the position $A = r_1$ to $B = r_2$ is given by

$$W = \int_A^B \vec{F}.\vec{dr} = \int_A^B F\ (r)\ \hat{r}.\vec{dr} = \int_A^B F\ (r)\ \hat{r}.dr\ \hat{r}$$

$$= \int_A^B F\ (r)\ dr = [V]_A^B$$

As $F\ (r)$ is a function of r only, its integral (V) is also a function of r *i.e.*

$$W = [V]_A^B = V_B - V_A \qquad \text{... }(ii)$$

Thus the work done depends only on the values of the integral V at $r = r_1$ at A, (V_A) and at $r = r_2$ at B, (V_B) *i.e.* at the two end points and not on the actual path followed.

Hence a central force is conservative force.

Conservative Force as Negative Gradient of Scalar Potential

If $\vec{F}$ is a conservative force, then according to the definition of potential energy

$$\int_A^B \vec{F}.\vec{dr} = V_A - V_B = [-V]_A^B = -\int_A^B dV$$

$$\therefore \qquad - dV = \vec{F}.\vec{dr} \quad \text{and} \quad - V = \int \vec{F}.\vec{dr} \qquad \text{...}(iii)$$

The negative sign simply indicates that the work done by the conservative force is equal to the corresponding decrease in potential energy.

In Cartesian co-ordinates

$$\vec{dr} = dx\ \hat{i} + dy\ \hat{j} + dz\ \hat{k}$$

and

$$\vec{F} = F_x\ \hat{i} + F_y\ \hat{j} + F_z\ \hat{k}$$

According to relation (*iii*)

$$- V = \int \vec{F}.\vec{dr}$$

$$= \int (F_x\ \hat{i} + F_y\ \hat{j} + F_z\ \hat{k}).(dx\ \hat{i} + dy\ \hat{j} + dz\ \hat{k})$$

$$= \int F_x\ dx + \int F_y\ dy + \int F_z\ dz \qquad \text{...}(iv)$$

Partially differentiating V with respect to x, y and z, we get

$$- V = -\left[\frac{\partial V}{\partial x}\ dx + \frac{\partial V}{\partial y}\ dy + \frac{\partial V}{\partial z}\ dz\right] \qquad \text{...}(v)$$

Comparing (*iv*) and (*v*), we have

$$F_x = \frac{-\partial V}{\partial x}; F_y = \frac{-\partial V}{\partial y}; F_z = \frac{-\partial V}{\partial z}$$

$$\therefore \qquad \vec{F} = -\left[\frac{\partial V}{\partial x}\ \hat{i} + \frac{\partial V}{\partial y}\ \hat{j} + \frac{\partial V}{\partial z}\ \hat{k}\right]$$

$$= -\left(\hat{i}\ \frac{\partial}{\partial x} + \hat{j}\ \frac{\partial}{\partial y} + \hat{k}\ \frac{\partial}{\partial z}\right) V$$

$$= -\vec{\nabla} V = -\text{ grad } V \qquad \ldots (vi)$$

Thus a conservative force can be expressed as the negative gradient of scalar potential function V.

2.4 ANGULAR MOMENTUM CONSERVED IN CENTRAL FORCE FIELD

Consider a particle of reduced mass m moving about the origin O. The force acting on the particle is a central force. It is, therefore given by

$$\vec{F} = \vec{F}(r)\,\hat{r}$$

where $\hat{r}$ is a unit vector along the direction of $\vec{r}$ and $\vec{r}$ is the position vector of the mass m.

The torque acting on a particle subjected to a central force is zero. This is proved as under.

Torque is defined as the vector product $\vec{r} \times \vec{F}$ where $\vec{F}$ is the force acting on the particle and $\vec{r}$ its vector distance from the origin.

$$\therefore \qquad \text{Torque } \vec{\tau} = \vec{r} \times \vec{F} = |\vec{r}|\,\hat{r} \times F(r)\,\hat{r}$$

$$= |\vec{r}|\,F(r)\,(\hat{r} \times \hat{r}) = 0 \qquad [\because \hat{r} \times \hat{r} = 0]$$

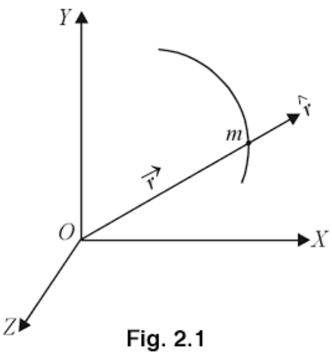

Fig. 2.1

Now torque $\vec{\tau}$ is the time rate of change of angular momentum $\vec{J}$.

or
$$\vec{\tau} = \frac{d\vec{J}}{dt}$$

If
$$\vec{\tau} = 0, \quad \frac{d\vec{J}}{dt} = 0 \text{ or } J = \text{a constant}$$

Hence *the angular momentum of a particle under a central force always remains constant and is therefore, a constant of motion.* In other words, '*in a central force field the angular momentum of a particle is conserved.*'

2.5 CENTRAL FORCE IS CONFINED IN A PLANE

The angular momentum of a particle of mass m moving with a velocity $\vec{v}$ (having a linear momentum $\vec{p} = m\vec{v}$) is given by

$$\vec{J} = \text{moment of linear momentum} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

where $\vec{r}$ is the position vector of the particle with respect to the centre of the force lying at O, the origin of the co-ordinate system. For a central force the angular momentum $\vec{J}$ remains constant.

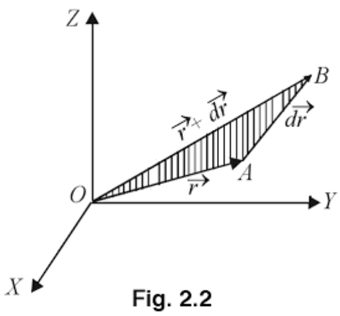

Fig. 2.2

When $\vec{J}$ remains constant its magnitude as well as direction remains the same. The direction of $\vec{J}$ is perpendicular to the plane containing the vectors $\vec{r}$ and $\vec{v}$. Hence *the path of the particle under a central force must always lie in one plane i.e.*, the plane containing the vectors $\vec{r}$ and $\vec{v}$. If we take the direction of angular momentum $\vec{J}$ along the Z–axis,, the motion of the particle will take place in X-Y plane as $\vec{r}$ and $\vec{v}$ will both lie in this plane. (Fig. 2.2).

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Hence we conclude that motion under a central force is confined in a plane.

The angular momentum of the earth as it moves in its orbit round the sun under a central gravitational force remains constant in direction as well as in magnitude. Hence the orbit lies in a plane containing the vector $\vec{v}$ the velocity of the earth and the vector $\vec{r}$ representing its vector distance from the sun.

2.6 LAW OF EQUAL AREAS

Let m be the reduced mass of the particle and $\vec{r}$ its instantaneous position vector at the point A with respect to the centre of force lying at O, the origin of the co-ordinate system. If the position vector changes to $\vec{r} + \vec{dr}$ at B in time dt, then $\vec{AB} = (\vec{r} + \vec{dr}) - \vec{r} = \vec{dr}$

The area $OAB = \vec{dA}$ swept by the radius vector $\vec{r}$ in time dt is given by $\vec{dA} = \frac{1}{2}\vec{r} \times \vec{dr}$

The rate at which the radius vector $\vec{r}$ sweeps the area is known as areal velocity.

$\therefore$ Areal velocity,
$$\frac{\vec{dA}}{dt} = \frac{1}{2}\vec{r} \times \frac{\vec{dr}}{dt} = \frac{1}{2m}\vec{r} \times \frac{m\,\vec{dr}}{dt}$$
$$= \frac{1}{2m}\vec{r} \times m\vec{v} = \frac{1}{2m}\vec{r} \times \vec{p} = \frac{\vec{J}}{2m} = \text{a constant}$$

as the angular momentum $\vec{J}$ is a constant of motion and m is also a constant.

Thus the areal velocity of the radius vector for a particle under a central force is constant or the radius vector sweeps area at a constant rate.

2.7 ACCELERATION OF A PARTICLE IN POLAR CO-ORDINATE

The acceleration of a particle in plane polar co-ordinates (r, θ) is given by
$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\theta} = \vec{a_r} + \vec{a_\theta}$$
where $\vec{a_r}$ is the *radial acceleration* and $\vec{a_\theta}$ the *transverse acceleration*.

Writing the two independently we have $\vec{a_r} = (\ddot{r} - r\dot{\theta}^2)\,\hat{r}$ and $\vec{a_\theta} = (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\theta}$

Multiplying the above relations by m the mass of the particle, we have

$$m\vec{a_r} = m\,(\ddot{r} - r\dot{\theta}^2)\,\hat{r} = F\,(r)\,\hat{r} \qquad ...(i)$$

and
$$m\vec{a_\theta} = m\,(r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\theta} = F\,(\theta)\,\hat{\theta} \qquad ...(ii)$$

where $F\,(r)\,\hat{r}$ gives the *radial force* along the direction of increasing $\vec{r}$ and $F\,(\theta)\hat{\theta}$ the *transverse force* along the direction of increasing θ.

From equation (i), we have $m\ddot{r} - mr\dot{\theta}^2 = F(r)$ ... (iii)

This equation is known as *radial equation of motion.*

In the case of a central force $F\,(\theta)\hat{\theta}$ which represents the component of force depending upon angle is zero as the force only acts along the direction of r and is denoted as $F\,(r)\hat{r}$. To prove that $F\,(\theta)\hat{\theta} = 0$ for a central force we have,

The magnitude of angular momentum $J = |\vec{J}| = mr^2\,\dot{\theta}$ ...(iv)

$$\therefore \qquad \frac{d|\vec{J}|}{dt} = 2mr\,\dot{r}\dot{\theta} + mr^2\,\ddot{\theta} = mr\,(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$$

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as in the case of a central force $\vec{J}$ = a constant and therefore $\frac{d|\vec{J}|}{dt} = 0$

Hence equation (*ii*) can be stated as $F(\theta)\hat{\theta} = m\,(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} = 0$

or $$mr\ddot{\theta} + 2m\,\dot{r}\,\dot{\theta} = 0 \qquad ...(v)$$

This equation is known as *equation of motion of* θ *co-ordinate.*

2.8 LAW OF CONSERVATION OF ENERGY

According to the radial equation of motion

$$F(r) = m\ddot{r} - mr\dot{\theta}^2 \qquad ...(i)$$

Also magnitude of angular momentum $J = mr^2\dot{\theta}$ or $\dot{\theta} = \frac{J}{mr^2}$

Substituting this value of $\dot{\theta}$ in Eq. (*i*), we get $m\ddot{r} - \frac{J^2}{mr^3} = F(r)$

A central force is also a conservative force, and a conservative force is represented as negative gradient of potential or $F(r) = -\frac{dU}{dr}$ $\quad \therefore\ m\ddot{r} - \frac{J^2}{mr^3} = -\frac{dU}{dr}$

or $$m\ddot{r} = -\frac{d}{dr}\left[U + \frac{J^2}{2mr^2}\right]$$

Multiplying both sides of the equation by $\dot{r}$ we get $m\dot{r}\ddot{r} = -\frac{d}{dr}\left[U + \frac{J^2}{2mr^2}\right]\dot{r}$

or $$\frac{d}{dt}\left[\frac{1}{2}m\dot{r}^2\right] = -\frac{d}{dr}\left[U + \frac{J^2}{2mr^2}\right]\frac{dr}{dt} = -\frac{d}{dt}\left[U + \frac{J^2}{2mr^2}\right]$$

$$\therefore \frac{d}{dt}\left[\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + U\right] = 0 \text{ or } \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + U = \text{a constant} \qquad ...(ii)$$

as $$\frac{J^2}{2mr^2} = \frac{(mr^2\dot{\theta})^2}{2mr^2} = \frac{1}{2}mr^2\dot{\theta}^2$$

Now in Eq. (*ii*) $\frac{1}{2}m\dot{r}^2$ represents the kinetic energy of translation, $\frac{1}{2}mr^2\dot{\theta}^2$ the energy due to centripetal force and U the potential energy. In other words the left hand side of the equation gives the total energy of the system. Since it is equal to a constant it proves the law of conservation of energy directly from the equations of motion using the expression for acceleration of a particle moving in a plane in polar co-ordinates. $\quad \therefore\ \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + U = E \qquad ...(iii)$

where E is the total energy.

(*i*) **Relation between *r* and *t*.** From the above equation, we have

$$\dot{r} = \frac{dr}{dt} = \left[\frac{2}{m}\left(E - U - \frac{J^2}{2mr^2}\right)\right]^{1/2} \text{ or } dt = \frac{dr}{\left[\frac{2}{m}\left(E - U - \frac{J^2}{2mr^2}\right)\right]^{1/2}} \qquad ...(iv)$$

Hence $t = \int \frac{dr}{\left[\frac{2}{m}\left(E - U - \frac{J^2}{2mr^2}\right)\right]^{1/2}}$

where $r = r_0$ at $t = 0$. This equation gives the relation between r and t.

(*ii*) **Relation between θ and *t*.** To find the relation between θ and t, we have

$$mr^2\dot{\theta} = mr^2 \frac{d\theta}{dt} = J \qquad \therefore \; d\theta = \frac{J}{mr^2}\, dt \qquad ...(v)$$

or $$\theta = \int \frac{J}{mr^2}\, dt + \theta_0$$ where θ_0 is the value of θ at $t = 0$

This equation gives the relation between θ and t.

2.9 NATURE OF MOTION UNDER CENTRAL FORCE

A central force obeying inverse square law is, in general, represented as

$$F(r) = \pm \frac{c}{r^2}.$$

The positive sign being taken for a repulsive force and negative sign for an attractive force. The corresponding potential energy is given by

$$U = -\int F(r)\, dr = -\int + \frac{c}{r^2}\, dr = +\frac{c}{r} \qquad ...(i)$$

for a repulsive force and

$$U = -\int F(r)\, dr = -\int - \frac{c}{r^2}\, dr = -\frac{c}{r} \qquad ...(ii)$$

for an attractive force.

The total energy of a particle moving under a central force is given by

$$E = \frac{1}{2} m\dot{r}^2 + \frac{J^2}{2mr^2} + U \qquad ...(iii)$$

$$= \frac{1}{2} m\dot{r}^2 + \frac{J^2}{2mr^2} \pm \frac{c}{r} \qquad ...(iv)$$

where $\frac{c}{r}$ is the potential energy due to inverse square law force, $\frac{J^2}{2mr^2} = \frac{1}{2} mr^2\dot{\theta}^2$ the potential energy associated with centrifugal force and $\frac{1}{2} m\dot{r}^2$ the radial kinetic energy.

$\therefore$ Effective potential energy $U' = U + \frac{J^2}{2mr^2}$

Hence equation (*iii*) can be written as $E = \frac{1}{2} m\dot{r}^2 + U'$ ...(*v*)

1. Repulsive force. $U = +\frac{c}{r}$. As $\frac{J^2}{2mr^2}$ is a always positive U' is a positive quantity and hence E is positive.

2. Zero force. For no force $U = 0$, $U' = \frac{J^2}{2mr^2}$ is positive and E is positive.

3. Attractive force. For an attractive force $U = -\frac{c}{r}$ and $U' = \frac{J^2}{2mr^2} - \frac{c}{r}$.

The variation of $\frac{J^2}{2mr^2}$, with r is shown in Fig. 2.3 [curve (*i*)] and the variation of $-\frac{c}{r}$ with r is shown in [curve (*ii*)].

As U is a negative quantity, U' can have a positive, zero or negative value. For $\frac{c}{r} < \frac{J^2}{2mr^2}$, U' is positive; for $\frac{c}{r} = \frac{J^2}{2mr^2}$, U' is zero; and for $\frac{c}{r} > \frac{J^2}{2mr^2}$; U' is negative. The variation of U', the effective potential energy with r is shown in Fig. 2.3 [curve (*iii*)].

For *negative* values of U' the total energy E is positive for $U' < \frac{1}{2}m\dot{r}^2$, E is zero for $U' = \frac{1}{2}m\dot{r}^2$ and E is negative for $U' > \frac{1}{2}m\dot{r}^2$.

Hence three cases arise corresponding to total energy E positive, zero or negative.

Case 1: Total energy *E* positive. Suppose the particle has a total positive energy $E = E_1$. The particle comes from infinity, where $r = \infty$ and $U' = \frac{J^2}{2mr^2} - \frac{c}{r} = 0$ and $E = E_1 = \frac{1}{2}m\dot{r}^2$

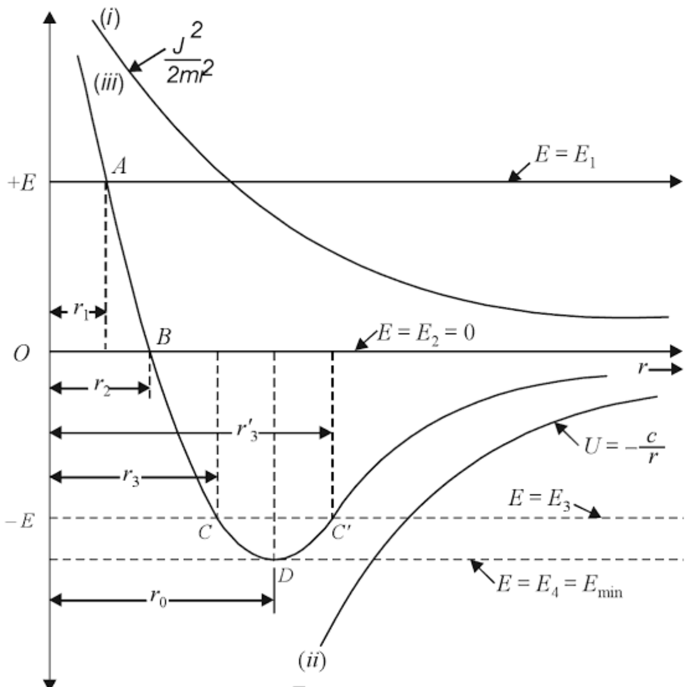

Fig. 2.3

As the particle travels from infinity towards the other mass U' first becomes negative, reaches a minimum (negative) value and then goes on increasing till $U' = E$, for $r = r_1$ at A as shown in Fig. 2.3 [curve (*iii*)]. The particle cannot come closer than r_1, because then U' will be greater than E_1 and $E_1 - U'$ will have a negative value but $E_1 - U' = \frac{1}{2}m\dot{r}^2$, the radial kinetic energy which must always be positive. Thus the region to the right of A is entirely accessible to the particle and the region to the left of A is strictly forbidden. The point A at which the radial kinetic energy vanishes is called a *turning point*. Hence such a motion will be unbounded *i.e.*, the particle will comefrom infinity go upto r_1 and return back to infinity. The path is a *hyperbola*.

Case 2: Total energy *E* zero. Suppose the particle has total energy $E = E_2 = 0$. In this case its energy line coincides with r axis and there is only one turning point at B where $OB = r_2$, the value of

r_2 being greater than r_1. The motion of the particle is still unbounded. The path is a *parabola*.

Case 3: Total energy *E* negative. In this case we shall deal with two different situations.

(*i*) Suppose the particle has a total negative energy $E = E_3$. The total energy line for E_3 cuts the curve for U' at two points C and C' corresponding to radial distance r_3 and r'_3. Thus there are two turning points. At these points $U' = \frac{1}{2} m\dot{r}^2$, the radial kinetic energy. The regions $r < r_3$ and $r > r_3'$ are forbidden and the motion of the particle is bounded in the region $r_3 < r < r_3'$ the path is an *ellipse*.

(*ii*) Suppose the particle has a total negative energy $E = E_4$ so that total energy line for E_4 meets the curve for U'at the minima D corresponding to radial distance r_0. The value of r at which U' has a minimum value can be calculated by equating $\frac{dU'}{dr} = 0$ or $\frac{d}{dr}\left\{\frac{J^2}{2mr^2} - \frac{c}{r}\right\} = 0$

or $$\frac{-J^2}{mr^3} + \frac{c}{r^2} = 0 \qquad \text{or} \qquad r = \frac{J^2}{mc} \qquad ...(vi)$$

$\therefore$ Minimum value of $$r = r_0 = \frac{J^2}{mc}$$

The minimum total energy corresponding to r_0 is given by

$$E_{\min} = U'_{\min} = \frac{J^2}{2mr_0^2} - \frac{c}{r_0} = \frac{mc^2}{2J^2} - \frac{mc^2}{J^2} = -\frac{1}{2}\frac{mc^2}{J^2}$$

Hence when $E = E_4 = U'_{\min}$ the motion is possible only at one radius r_0. The orbit is a *circle* of radius $r_0 = \frac{J^2}{mc}$

When $E = U'_{\min}$, $\frac{J^2}{mr^3} = \frac{c}{r^2}$ [from relation (*vi*)]

but $\frac{c}{r^2}$ is the central force obeying inverse square law $= F(r)$

$$\therefore \quad F(r) = \frac{J^2}{mr^3} = \frac{(mr^2\dot{\theta})^2}{mr^3} = mr\dot{\theta}^2$$

i.e., the force is just equal to the centrifugal force required for circular motion.

Hence when E lies between $-\frac{mc^2}{2J^2}$ and zero the motion is elliptic with *two turning points* at $r = r_3$ and $r = r'_3$. The distances r_3 and r_3' are called *apsidal distances*.

For energies less than $E_4 = E_{\min}$ no solution of the problem is possible and the condition cannot be realised physically. Thus *for bounded motion the total energy E must be negative* but not less than

$$E_{\min} = -\frac{1}{2}\frac{mc^2}{J^2}$$

In other words

For bounded motion $E < 0 > -\frac{1}{2}\frac{mc^2}{J^2}$. ... (*vii*)

2.10 DIFFERENTIAL EQUATION OF MOTION UNDER CENTRAL FORCE

The acceleration of a particle in plane polar co-ordinates is given by

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} = \vec{a}_r + \vec{a}_\theta$$

The inverse square law force is a central force and may be represented by $F(r)\hat{r}$ where $F(r)$ is a function depending upon r. For such a force the component of acceleration along the direction of increasing θ is zero *i.e.*, $\vec{a}_\theta = 0$

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If m is the reduced mass, then

$$F(r)\,\hat{r} = m\vec{a}_r = (m\ddot{r} - mr\dot{\theta}^2)\,\hat{r}$$

or $$F(r) = m\ddot{r} - mr\dot{\theta}^2$$

The *magnitude* of angular momentum $J = mr^2\,\dot{\theta}$ or $\dot{\theta} = J/mr^2$

$\therefore$ $$mr\dot{\theta}^2 = \frac{J^2 mr}{m^2 r^4} = \frac{J^2}{mr^3}$$

Hence $$F(r) = m\ddot{r} - \frac{J^2}{mr^3}$$

or $$m\ddot{r} = F(r) + \frac{J^2}{mr^3} \qquad ...(i)$$

or $$\ddot{r} - \frac{J^2}{m^2 r^3} = \frac{F(r)}{m}$$

The relation (i) gives the radial equation of motion of a particle of reduced mass m under a central force $F(r)$.

Differential equation of motion. To get the differential equation of the orbit and hence the shape of the trajectory of the particle moving under inverse square law force it is convenient to put $r = \dfrac{1}{u}$

So that $$\dot{r} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -\frac{1}{u^2}\,\dot{\theta}\,\frac{du}{d\theta} \qquad ...(ii)$$

As stated above $$J = mr^2\,\dot{\theta} \quad \therefore\ r^2\,\dot{\theta} = \frac{1}{u^2}\,\dot{\theta} = \frac{J}{m}$$

Substituting this value of $\dfrac{1}{u^2}\,\dot{\theta}$ in (ii), we have $\dot{r} = -\dfrac{J}{m}\dfrac{du}{d\theta}$

Differentiating again we get $\ddot{r} = \dfrac{d}{dt}(\dot{r}) = -\dfrac{J}{m}\dfrac{d}{dt}\left(\dfrac{du}{d\theta}\right) = -\dfrac{J}{m}\dfrac{d\theta}{dt}\cdot\dfrac{d}{d\theta}\left(\dfrac{du}{d\theta}\right)$

$$= -\frac{J}{m}\,\dot{\theta}\,\frac{d^2u}{d\theta^2} \qquad ...(iii)$$

Substituting $\dot{\theta} = \dfrac{J}{mr^2}$ in (iii) we have

$$\ddot{r} = -\frac{J^2}{m^2r^2}\frac{d^2u}{d\theta^2} = -\frac{J^2u^2}{m^2}\frac{d^2u}{d\theta^2}$$

Substituting the value of $\ddot{r}$ and $\dot{r}$ in the equation of motion given in (i), we get

$$m\left[-\frac{J^2u^2}{m^2}\frac{d^2u}{d\theta^2}\right] = F\left(\frac{1}{u}\right) + \frac{J^2}{m}u^3$$

or $$\frac{d^2u}{d\theta^2} = -u - \frac{m}{J^2u^2}F\left(\frac{1}{u}\right) \qquad ...(iv)$$

which is the *differential equation of motion*.

[**Note.** For $J = 0$, the above equation is absurd but since $J = mr^2\dot{\theta}$; $J = 0$ means $\dot{\theta} = 0$ or θ = constant which gives the equation of the straight line passing through the origin.]

Attractive inverse square force. For an inverse square law *attractive* force, say the force of gravitation $$F(r) = F(r)\,\hat{r} = -\left(\frac{Gm_1m_2}{r^2}\right)\hat{r} = -\frac{c}{r^2}\,\hat{r}$$

or $$F(r) = -\frac{c}{r^2} \quad \text{or} \quad F\left(\frac{1}{u}\right) = -cu^2$$

Taking the case of gravitational attractive force and substituting $F\left(\frac{1}{u}\right) = -cu^2$ in (*iv*), we have

$$\frac{d^2u}{d\theta^2} + u = \frac{mc}{J^2}$$

This equation can be put in the form $$\frac{d^2}{d\theta^2}\left(u - \frac{mc}{J^2}\right) + \left(u - \frac{mc}{J^2}\right) = 0 \qquad \text{...}(v)$$

as $\frac{mc}{J^2}$ = a constant. To find a solution of the above equation put $y = u - \frac{mc}{J^2}$

$$\therefore \qquad \frac{dy}{d\theta} = \frac{du}{d\theta} \quad \text{and} \quad \frac{d^2y}{d\theta^2} = \frac{d^2u}{d\theta^2}$$

Substituting in (*v*) we get $\frac{d^2y}{d\theta^2} + y = 0$

This is the differential equation of simple harmonic motion. Its solution is

$$y = A\cos(\theta - \theta_0)$$

where A and θ_0 are constants to be determined from boundary conditions.

Hence $$u - \frac{mc}{J^2} = A\cos(\theta - \theta_0) \quad \text{or} \quad u = \frac{mc}{J^2} + A\cos(\theta - \theta_0) \qquad \text{...}(vi)$$

In terms of r, we can put the above equation as

$$\frac{1}{r} = \frac{mc}{J^2} + A\cos(\theta - \theta_0) \qquad \text{...}(vii)$$

If we orient our co-ordinate system such that θ_0 in the above equation is zero, then

$$\frac{1}{r} = \frac{mc}{J^2} + A\cos\theta = \frac{mc}{J^2}\left\{1 + \frac{J^2A}{mc}\cos\theta\right\} \qquad \text{...}(viii)$$

This gives the polar (r, θ) equation of the orbit of a particle moving under an inverse square law attractive force.

Comparing equation (*viii*) with the general equation of a conic in polar co-ordinates

$$\frac{1}{r} = \frac{1}{l}(1 + \varepsilon\cos\theta)$$

where l = a constant, we find that it represents a conic section with $l = \frac{J^2}{mc}$ and $\varepsilon = \frac{AJ^2}{mc}$.

The quantity ε is called the eccentricity. Thus the trajectory of the particle is given by

$$\frac{1}{r} = \frac{mc}{J^2}(1 + \varepsilon\cos\theta) \qquad \text{...}(ix)$$

The shape of the trajectory of the particle, therefore, depends upon the value of ε and hence on J and A.

(*a*) For $\varepsilon > 1$ the path is a *hyperbola*.

(*b*) For $\varepsilon = 1$ the path is a *parabola*.

(*c*) For ε having a value between 0 and 1. $(0 < \varepsilon < 1)$ the path is an *ellipse*.

(*d*) For $\varepsilon = 0$ the path is a *circle*.

2.11 POLAR EQUATION UNDER CENTRAL FORCE

As the particle moves under the action of a force $F = \frac{c}{r^2}$ about the fixed centre, the motion is under a *central force* obeying *inverse square law*. The force is $F = \frac{c}{r^2}$. It can be $F = +\frac{c}{r^2}$ (repulsive force) or $F = -\frac{c}{r^2}$ (attractive force).

The polar equation of a particle under the attractive (gravitational) force has been derived in **Article 2.10** Eq. (*viii*) and is

$$\frac{1}{r} = \frac{mc}{J^2}\left\{1 + \frac{J^2 A}{mc}\cos\theta\right\}$$

where J is the angular momentum of the particle and A a constant.

For repulsive force $F = +\frac{c}{r^2}$. Substituting in Eq. (*iv*) **Article 2.10** the value of $F(r) = +\frac{c}{r^2}$

or $$F\left(\frac{1}{u}\right) = +cu^2 \text{ we have}$$

$$\frac{d^2u}{d\theta^2} = -u - \frac{mc}{J^2} = -\left[u + \frac{mc}{J^2}\right] \quad \text{or} \quad \frac{d^2u}{d\theta^2} + \left[u + \frac{mc}{J^2}\right] = 0$$

or $$\frac{d^2}{d\theta^2}\left(u + \frac{mc}{J^2}\right) + \left(u + \frac{mc}{J^2}\right) = 0 \qquad \left[\because \frac{mc}{J^2} = \text{a constant}\right]$$

Proceeding as in **Article 2.10** Eq. (*v*) we get the solution of the above equation as

$$u + \frac{mc}{J^2} = A\cos(\theta - \theta_0) \quad \text{or} \quad u = A\cos(\theta - \theta_0) - \frac{mc}{J^2}$$

$$= \frac{mc}{J^2}\left[\frac{AJ^2}{mc}\cos(\theta - \theta_0) - 1\right] = \frac{mc}{J^2}[\varepsilon\cos(\theta - \theta_0) - 1]$$

where $$\varepsilon = \frac{AJ^2}{mc}.$$

If we orient our co-ordinate system such that θ_0 in the above equation is zero, then

$$u = \frac{mc}{J^2}(\varepsilon\cos\theta - 1)$$

or $$\frac{1}{r} = \frac{mc}{J^2}(\varepsilon\cos\theta - 1)$$

This is the *polar* (r, θ) equation of the orbit under a central force given by $F = +\frac{c}{r^2}$.

2.12 SHAPE OF ORBIT TRACED UNDER CENTRAL ATTRACTIVE AND REPULSIVE INVERSE SQUARE FORCE

2.12.1 Central Attractive Inverse Square Force

The polar equation of the orbit traced by a particle moving under central *attractive* inverse square force is given by

$$\frac{1}{r} = \frac{mc}{J^2} + A\cos(\theta - \theta_0) \qquad \text{[See **Article 2.10** Eq. (*vii*)]}$$

where m is the reduced mass of the particle, c the force constant, J the angular momentum, A and θ_0 constants of motion. To find the value of A we shall find the turning points of the trajectory of the particle.

Maximum and minimum value of *r*. In the above equation if $(\theta - \theta_0) = 0$, $\cos(\theta - \theta_0) = 1$, then $\frac{1}{r}$ has the maximum value [*i.e.*, r has the minimum value].

$$\therefore \qquad \left(\frac{1}{r}\right)_{\max} = \frac{mc}{J^2} + A \qquad ...(i)$$

If $(\theta - \theta_0) = \pi$, $\cos(\theta - \theta_0) = -1$, then $\frac{1}{r}$ has the minimum value [*i.e.*, r has the maximum value].

$$\therefore \qquad \left(\frac{1}{r}\right)_{\min} = \frac{mc}{J^2} - A \qquad ...(ii)$$

Turning points. The total energy E of the particle under a central force is given by

$$E = \frac{1}{2} m\dot{r}^2 + \frac{J^2}{2mr^2} + U$$

where $\frac{1}{2} m\dot{r}^2$ is the kinetic energy, $\frac{J^2}{2mr^2} = \frac{1}{2} mr^2\dot{\theta}^2$ is the energy due to centripetal force and U the potential energy. For a gravitational force of attraction *obeying* inverse square law

$$U = -\frac{c}{r} \text{ where } c = G\, m_1\, m_2 \qquad \therefore\ E = \frac{1}{2} m\dot{r}^2 + \frac{J}{2mr^2} - \frac{c}{r}$$

For an electrostatic force of repulsion between two like charges in S.I. units

$$U = +\frac{c}{r} \text{ where } c = \frac{1}{4\pi\varepsilon_0} q_1 q_2 \qquad \therefore E = \frac{1}{2} m\dot{r}^2 + \frac{J^2}{2mr^2} + \frac{c}{r} \qquad ... (iii)$$

Now, because E, the total energy is a constant, we can evaluate it at any convenient point on the orbit. We take the *turning points* where r is a maximum or a minimum. At these points

$$\frac{dr}{dt} = \dot{r} = 0$$

Thus the turning point is defined as a point at which the radial velocity $\dot{r}$ of the particle is zero.

In such a case the kinetic energy of the particle given by $\frac{1}{2} m\dot{r}^2 = 0$ and total energy $E = \frac{J^2}{2mr^2} - \frac{c}{r}$ is the *effective potential energy* because $\frac{J^2}{2mr^2} = \frac{1}{2} mr^2\dot{\theta}^2$ is the energy due to centripetal force and $\frac{c}{r}$ the energy due to gravitational *attractive* force.

Thus a turning point may also be defined as that point at which the total energy E of the particle is equal to its effective potential energy.

Value of A for attractive inverse square force. Considering the case of an *attractive* inverse square force, at the turning points

$$E = \frac{J^2}{2mr^2} - \frac{c}{r} \qquad \text{or} \qquad \frac{J^2}{2m}\frac{1}{r^2} - c\,\frac{1}{r} - E = 0 \qquad ... (iv)$$

This is an equation in $\frac{1}{r}$ and its two roots give the value of $\left(\frac{1}{r}\right)_{\max}$ and $\left(\frac{1}{r}\right)_{\min}$. The two roots are

$$\frac{1}{r} = c \pm \frac{\sqrt{c^2 + 4\dfrac{J^2}{2m}E}}{\dfrac{J^2}{m}}$$

$$\therefore \qquad \left(\frac{1}{r}\right)_{\max} = \frac{mc}{J^2} + \left[\left(\frac{mc}{J^2}\right)^2 + \left(\frac{2mE}{J^2}\right)\right]^{1/2} \qquad ...(v)$$

and $$\left(\frac{1}{r}\right)_{\min} = \frac{mc}{J^2} - \left[\left(\frac{mc}{J^2}\right)^2 + \left(\frac{2mE}{J^2}\right)\right]^{1/2} \qquad ...(vi)$$

Comparing the values of $\left(\frac{1}{r}\right)_{\max}$ with that in Eq. (*i*), we have

$$\frac{mc}{J^2} + A = \frac{mc}{J^2} + \left[\left(\frac{mc}{J^2}\right)^2 + \left(\frac{2mE}{J^2}\right)\right]^{1/2}$$

$$\therefore \qquad A^2 = \left(\frac{mc}{J^2}\right)^2 + \frac{2mE}{J^2} = \frac{m^2c^2}{J^4} + \frac{2mE}{J^2} \qquad ...(vii)$$

If we orient our co-ordinate system such that θ_0 in Eq. $\frac{1}{r} = \frac{mc}{J^2} + A\cos(\theta - \theta_0)$ is zero, then

$$\frac{1}{r} = \frac{mc}{J^2}\left(1 + \frac{AJ^2}{mc}\cos\theta\right) \qquad ...(viii)$$

Comparing equation (*viii*) with the general equation of conic in polar co-ordinates

$$\frac{1}{r} = \frac{1}{l}(1 + \varepsilon\cos\theta)$$

where l = a constant, we find that it represents a conic section with $l = \frac{J^2}{mc}$ and $\varepsilon = \frac{AJ^2}{mc}$.

The quantity ε is called the *eccentricity* and its value is given by

$$\varepsilon^2 = \frac{A^2J^4}{m^2c^2} = 1 + \frac{2J^2E}{mc^2} \qquad \therefore\ \varepsilon = \left[1 + \frac{2J^2E}{mc^2}\right]^{1/2} \qquad ...(ix)$$

and the trajectory of the particle is given by

$$\frac{1}{r} = \frac{mc}{J^2}(1 + \varepsilon\cos\theta) \qquad ...(x)$$

The shape of the trajectory of the particle depends upon the value of ε and hence on the relationship between the total energy E and its angular momentum J.

(*i*) If $E > 0$ (+ ve); $\varepsilon > 1$ the path is a *hyperbola*.

(*ii*) If $E = 0$; $\varepsilon = 1$ the path is a *parabola*.

(*iii*) If $E < 0$ (– ve); $\varepsilon < 1$ the path is an *ellipse*.

(*iv*) If $E = -\frac{mc^2}{2J^2}$; $\varepsilon = 0$ the path is a *circle*, as shown in Fig. 2.4.

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Number of turning points. When the shape of the orbit is a hyperbola or parabola, there is only one turning point. When the shape of the orbit is elliptic or circular there are two turning points.

2.12.2 Central Repulsive Inverse Square Force

For a *repulsive* force

$$E = \frac{1}{2} m\dot{r}^2 + \frac{J^2}{mr^2} + \frac{c}{r}$$

all the quantities on the right hand side are positive.

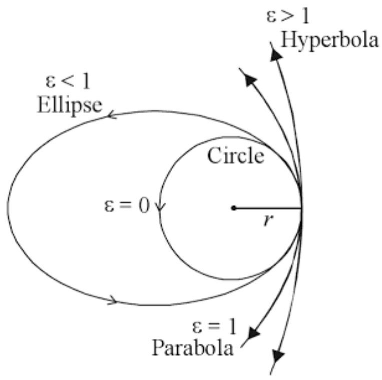

Fig. 2.4

Therefore $E > 0$ and the trajectory of the body under the influence of a repulsive force is always a hyperbola. The most familiar example is the path of a positively charged α-particle moving under the influence of a positively charged nucleus.

Conclusion. Hence we conclude that

For an **attractive** force $E = \frac{1}{2} m\dot{r}^2 + \frac{J^2}{mr^2} - \frac{c}{r}$

The first two quantities on the right hand side are positive and the last quantity is negative.

$\therefore$ $E >=< 0$ according as $\frac{1}{2} m\dot{r}^2 + \frac{J^2}{2mr^2} >=< \frac{c}{r}$ and hence accordingly the trajectory will be a hyperbola, a parabola or an ellipse.

For $E = -\dfrac{mc^2}{2J^2}$ the trajectory is a circle. ... (*xi*)

For repulsive force the trajectory is always a hyperbola.

2.13 SHAPE AND SIZE OF ELLIPTIC ORBIT

A planet revolves round the Sun in an elliptic orbit under the gravitational attractive inverse square law force.

The polar equation of the orbit is given by $\frac{1}{r} = \frac{mc}{J^2}(1 + \varepsilon \cos \theta)$

or $$r = \frac{J^2 / mc}{(1 + \varepsilon \cos \theta)} \qquad ...(i)$$

where J is the angular momentum and m the mass of the planet, c a constant $= GMm$, G being the gravitational constant and M the mass of the sun. ε is the eccentricity of the orbit given by $\varepsilon = \left(1 + \dfrac{2J^2E}{mc^2}\right)^{1/2}$. As the orbit of the planet is elliptic the eccentricity $\varepsilon < 1$ and E the total energy is **negative.**

The value of r is a minimum when $\cos\theta = +1$

$$\therefore \qquad r_{\min} = \frac{J^2}{mc\,(1 + \varepsilon)}$$

At this point the planet is closest to the Sun and this point is called *perihilion* as shown at A in Fig. 2.5.

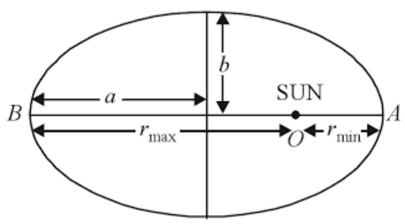

Fig. 2.5

The value of r is a maximum when $\cos\theta = -1$

particle moves and G, the gravitational constant.

$$\therefore \qquad E = -\frac{mc^2}{2J^2}(1-\varepsilon^2) \qquad \ldots (i)$$

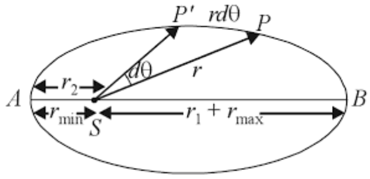

Fig. 2.6

In the case of a planet revolving round the Sun, if $\varepsilon < 1$ the total energy of the system (planet and the sun) is **negative**, the planet remains bound to the attracting centre, the Sun. As the planet is bound to the Sun and cannot escape from it, it moves around the Sun in a closed elliptic orbit. This establishes Kepler's first law of planetary motion.

In the case of the motion of a planet round the sun, the sun is at one focus which is taken as the centre of the co-ordinate system of the ellipse. Thus each planet moves in an ellipse with sun at its focus as shown in Fig. 2.6. The point A where the planet is closest to the sun is called *perihelion* and the point B where it is farthest from the sun is called *apehelion*.

(b) **Second law.** Suppose a planet P is moving in an elliptic orbit as shown in Fig. 2.6. If it moves from P to P' in a small time dt the area swept out by the radius vector is the area of the figure SPP'. If dt is infinitesimally small. PP' is a straight line $= rd\theta$ and SPP' is a triangle.

$$\text{The area of the triangle } SPP' = dA = \frac{1}{2}r.rd\theta = \frac{1}{2}r^2\,d\theta$$

This is the area swept out in a time dt.

$$\therefore \text{ Rate at which area is swept } = \frac{dA}{dt} = \frac{1}{2}r^2.\frac{d\theta}{dt} = \frac{1}{2}r^2\,\dot{\theta} \qquad \ldots (ii)$$

The angular momentum $J = mr^2\,\dot{\theta}$ and is a constant under a **central force.**

$$\therefore \quad r^2\dot{\theta} = \frac{J}{m} = \text{a constant under gravitational force or } \frac{1}{2}r^2\dot{\theta} = \frac{J}{2m} = \text{a constant.}$$

Hence $\dfrac{dA}{dt} = \dfrac{1}{2m}J$ = a constant which verifies the second law that *the radius vector joining the sun to the planet sweeps out equal areas in equal intervals of time*. In other words, *the areal velocity of a planet around the sun is constant*.

(c) **Third law.** As proved above $\dfrac{dA}{dt} = \dfrac{J}{2m}$ or $dA = \dfrac{J}{2m}dt$ or $\displaystyle\int_0^T dA = \int_0^T \frac{J}{2m}dt$

where T is the time period of one full revolution.

Integrating we get $A = \dfrac{JT}{2m}$ where A = area of the ellipse

Now the area of the ellipse $= \pi ab$

where a = semi-major axis and b = semi-minor axis.

$$\therefore \qquad \frac{JT}{2m} = \pi ab \quad \text{or} \quad T = \frac{2\pi mab}{J}$$

$$\text{Now} \qquad b = a\sqrt{1-\varepsilon^2}$$

$$\therefore \qquad T = \frac{2\pi ma^2\sqrt{1-\varepsilon^2}}{J} \quad \text{or} \quad T^2 = \frac{4\pi^2 m^2 a^4 (1-\varepsilon^2)}{J^2} \qquad \ldots (iii)$$

As the origin is taken as the focus S, $2a = r_{max} + r_{min}$

Now $$\frac{1}{r} = \frac{mc}{J^2}\left(1 + \frac{AJ^2}{mc}\cos\theta\right) = \frac{mc}{J^2}(1 + \varepsilon\cos\theta) \quad \text{... [Eq. (}viii\text{) Article 2.12]}$$

$\therefore$ $$\left(\frac{1}{r}\right)_{\max} = \frac{mc}{J^2}(1 + \varepsilon)$$

or $$(r)_{\min} = \frac{J^2}{mc}\frac{1}{1+\varepsilon}$$

Similarly $$r_{\max} = \frac{J^2}{mc}\frac{1}{1-\varepsilon}$$

Hence $$2a = \frac{J^2}{mc}\left[\frac{1}{1+\varepsilon} + \frac{1}{1-\varepsilon}\right]$$

or $$a = \frac{J^2}{2mc}\left[\frac{2}{1-\varepsilon^2}\right] = \frac{J^2}{mc\,(1-\varepsilon^2)}$$

Now from (*iii*) we have $$T^2 = \frac{4\pi^2 m^2 a^4\,(1-\varepsilon^2)}{J^2} = \frac{4\pi^2 m^2 a^3}{J^2}\cdot\frac{J^2\,(1-\varepsilon^2)}{mc\,(1-\varepsilon^2)} = \frac{4\pi^2 m a^3}{c}$$

Hence $$\frac{T^2}{a^3} = \frac{4\pi^2 m}{c} = \text{a constant} \quad \text{...(}iv\text{)}$$

Thus the square of the period of revolution of the planet about the sun divided by the cube of major axis of the orbit is a constant which is Kepler's third law.

or *The square of time period of revolution of a planet is proportional to the cube of semi-major axis of the orbit.*

$$T^2 \propto a^3 \quad \text{...(}v\text{)}$$

2.15 NEWTON'S LAW OF GRAVITATION

Every body in the universe attracts every other body with a force which varies directly as the product of the masses of the two bodies and inversely as the square of the distance between them.

Mathematically, $$F = -G\frac{m_1 m_2}{r^2} \quad \text{...(}i\text{)}$$

where m_1 is the mass of one body, m_2 that of the other, r the distance between their centres, G *the* constant of gravitation and F the force with which the bodies attract each other. The negative sign indicates that it is an attractive force.

Gravitational constant. If in the above equation $m_1 = m_2 = 1$ and $r = 1$

then $$F = G \text{ (numerically)}$$

Therefore, the *gravitational constant is equal to the force of attraction between two bodies each of unit mass lying at a unit distance apart.* The value of G is 6.66×10^{-11} newton m^2/kg^2 in S.I. units.

Dimensions of *G* From relation (*i*)

$$G = \frac{Fr^2}{m_1 m_2} \text{ (numerically)}$$

$\therefore$ Dimensions of G are $$\frac{M^1 L^1 T^{-2} L^2}{M^2} = [M^{-1} L^3 T^{-2}]$$

2.16 GRAVITATIONAL FIELD

The space around a body within which its gravitational force of attraction can be experienced is called the gravitational field.

The intensity of the gravitational field or gravitational attraction at a point is the force experienced by a unit mass placed at the point.

The intensity of the gravitational field at a point distant r from a point mass m, $\vec{E} = -\frac{Gm}{r^2}\hat{r}$ since a unit mass lying there is attracted by this force. The negative sign indicates that the force is directed towards the mass m.

2.17 GRAVITATIONAL POTENTIAL

The work done in moving a unit mass from infinity to any point in the gravitational field of a body is called the gravitational potential at the point due to the body.

Gravitational potential at a point in a gravitational field may also be defined as the potential energy of unit mass placed at that point.

The difference of gravitational potential between two points in a gravitational field is the work done in moving a unit mass from one point to the other along the gravitational force of attraction.

Relation between gravitational field and potential. If there are two points A and B lying very near each other at a distance dr in a gravitational field F acting in the direction indicated, then the work done in moving a unit mass from B to A along the direction of the force $= F\,dr$. If dV is the difference of gravitational potential between the point A and B, then

Fig. 2.7

$$V_A - V_B = dV = -F\,dr; \qquad \therefore\; E = -\frac{dV}{dr} \qquad \text{... }(i)$$

Hence the intensity of gravitational field at any point is equal to the potential gradient or the negative of the space rate of change of gravitational potential.

Gravitational potential due to a point mass. Consider a point A at a distance r from a particle of mass m, then

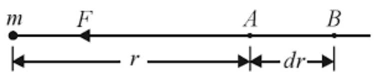

Fig. 2.8

Force experienced by a unit mass at A

$$F = -\frac{GM}{r^2} \text{ along } Am$$

$\therefore$ Difference of gravitational potential between two points A and B at a distance dr apart

$$V_A - V_B = dV = -F dr = -\left(\frac{-Gm}{r^2}\right)dr = \frac{Gm}{r^2}dr$$

Hence potential at $$A = \int_{\infty}^{r} dV = \int_{\infty}^{r} \frac{Gm}{r^2}dr \text{ or } V = GM\left[-\frac{1}{r}\right]_{\infty}^{r} = -\frac{Gm}{r} \qquad \text{... }(ii)$$

2.18 GRAVITATIONAL POTENTIAL DUE TO SPHERICAL SHELL

(*i*) *Point outside the shell.* Consider a point P outside the spherical shell at a distance r from its centre O (Fig. 2.9).

Let the radius of the spherical shell be a and ρ its mass per unit area of the surface.

Join OP. Draw two planes very close to each other perpendicular to OP and cutting the shell in CD and EF respectively.

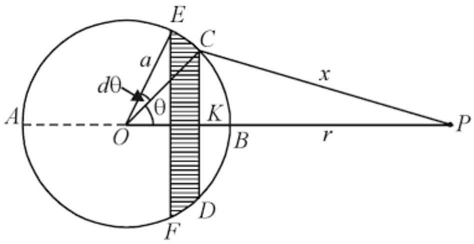

Fig. 2.9

Then these planes cut a ring or slice from the shell of radius CK and thickness CE as shown separately in Fig. 2.10.

Join OC and OE and let the angle BOC be θ and the small angle COE be $d\theta$.

$\therefore$ Thickness of the shell $CE = a\, d\theta$

Radius of the shell $CK = a \sin\theta$

Circumference of the shell $= 2\pi.\, CK = 2\pi a \sin\theta$

Hence surface area of the slice $= 2\pi a \sin\theta\; a\, d\theta = 2\pi a^2 \sin\theta\, d\theta$

Mass of the slice $= 2\pi a^2 \sin\theta\, d\theta\, \rho$

Every point on the rim of this slice or ring is at a distance CP, *i.e.*, x from P

$\therefore$ Potential at P due to the ring

$$dV = -\frac{G 2\pi a^2 \rho \sin\theta\, d\theta}{x} \qquad ...(i)$$

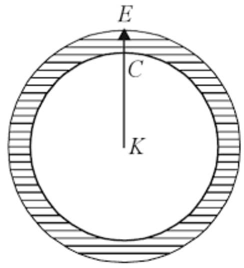

Fig. 2.10

To find the value of x.

In triangle OCP, $x^2 = a^2 + r^2 - 2ar\cos\theta$

Differentiating, we get $2x dx = 2ar \sin\theta\, d\theta$

[$\because$ a and r are constant]

$$\therefore \qquad x = \frac{ar \sin\theta\, d\theta}{dx}$$

Substituting the value of x in (i), we get

$$dV = -\frac{G\, 2\pi a^2 \rho \sin\theta\, d\theta}{ar \sin\theta\, d\theta}\, dx = -\frac{2\pi a \rho G}{r}\, dx \qquad ...(ii)$$

Integrating equation (ii) for the whole shell between the limits

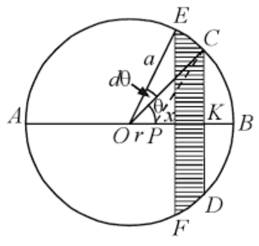

Fig. 2.11

$PB = (r - a)$ and $PA = (r + a)$, we get

$$V = -\int_{r-a}^{r+a} \frac{2\pi a \rho G}{r}\, dx$$

$$= -\frac{2\pi a \rho G}{r}[x]_{r-a}^{r+a} = \frac{4\pi a^2 \rho G}{r}$$

But $4\pi a^2$ is the surface area of the whole shell.

Therefore m the mass of the whole shell $= 4\pi a^2 \rho$

or $\qquad$ Potential $V = -\dfrac{Gm}{r}$ $\qquad$ ... (iii)

Hence the mass of the whole shell behaves as if it were concentrated at the centre.

(ii) *Potential on the surface of the shell.* For a point on the surface of the shell $r = a$

$\therefore$ $\qquad$ Potential $V = -\dfrac{Gm}{a}$ $\qquad$ ...(iv)

(iii) *Point inside the shell.* Proceeding exactly in the same way to find the potential at P due to the ring $CDEF$, we get $dV = -\dfrac{2\pi a \rho G}{r}\, dx$ $\qquad$ ...(v)

Now $\qquad\qquad OP = r$ as shown in Fig 2.11

$\therefore$ Integrating equation (*iii*) between the limit $PB = (a - r)$ and $PA = (a + r)$, we get

$$V = -\int_{a-r}^{a+r} \frac{2\pi a \rho G}{r}\, dx$$

$$= -\frac{2\pi a\rho G}{r}[x]_{a-r}^{a+r} = -\frac{2\pi a\rho G}{r}\, 2r = -4\pi a \rho G$$

Multiplying and dividing by a, we get $V = -\dfrac{4\pi a^2 \rho G}{a}$

Substituting $4\pi a^2 \rho = m$, we have $V = -\dfrac{Gm}{a}$ ... (*vi*)

This value is the same as at a point on the surface of the shell.

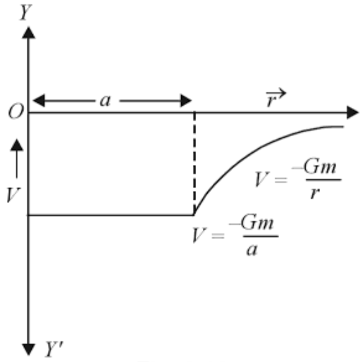

Fig. 2.12

Hence the potential at every point within a spherical shell is the same as on the surface of the shell itself.

2.19 GRAVITATIONAL FIELD DUE TO SPHERICAL SHELL

(*i*) *Point outside the shell.* The gravitational field, F is negative space rate of the change of gravitational potential V or $F = -\dfrac{dV}{dr}$

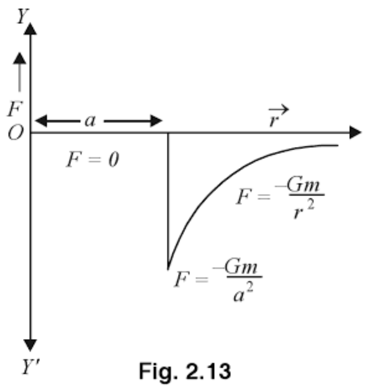

Fig. 2.13

Since the gravitational potential at a point outside the shell is given by $V = -\dfrac{Gm}{r}$

$\therefore$ Field $\qquad F = -\dfrac{dV}{dr} = -\dfrac{Gm}{r^2}$

Hence the intensity of gravitational field varies inversely as the square of distance of the point from the centre of the shell and is directed towards the centre.

Vectorially $\qquad \vec{F} = -\dfrac{Gm}{r^2}\hat{r}$

where $\hat{r}$ is a unit vector in the direction of increasing r.

(*ii*) *Point on the surface of shell.* On the surface $r = a$

$$\therefore \qquad F = -\frac{Gm}{a^2}$$

(*iii*) *Point inside the shell.* In this case $V = -\dfrac{Gm}{a}$

$$\therefore \qquad F = -\frac{dV}{dr} = 0 \qquad \left[\because \frac{Gm}{a} = \text{constant}\right]$$

Hence the intensity of gravitational field inside the shell is zero.

2.20 GRAVITATIONAL POTENTIAL DUE TO SOLID SPHERE

(*i*) *Point outside the sphere.* Consider a point P outside the sphere at a distance r from its centre O (Fig. 2.14).

Divide the sphere into a large number of thin spherical shells concentric with the sphere, of masses m_1, m_2, m_3 ......, etc. respectively.

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Now potential at a point outside the shell is given by

$V = -\dfrac{Gm}{r}$

where r is the distance of the point form the centre of the shell.

$\therefore$ Potential at P due to all the shells is

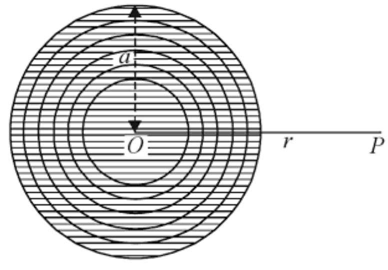

Fig. 2.14

$$V = -\left[\frac{Gm_1}{r} + \frac{Gm_2}{r} + \frac{Gm_3}{r} + ...\right]$$

$$= -\frac{G}{r}(m_1 + m_2 + m_3 + ...) = -\frac{GM}{r} \quad ... (1)$$

where M is the mass of the solid sphere.

Hence in the case of a solid sphere also the whole mass can be supposed to be concentrated at its centre.

(*ii*) *Point on the surface.* For a point on the surface of the sphere of radius a,

$$r = a \text{ and } V = -\frac{GM}{a} \quad ... (2)$$

(*iii*) *Point inside the sphere.* Consider a point P inside the sphere at a distance r from its centre O. Let a be the radius and ρ the density of the sphere.

With O as centre and radius OP draw a sphere. Then the point P lies on the surface of the solid sphere of radius r and inside the spherical shell of internal radius r and external radius a.

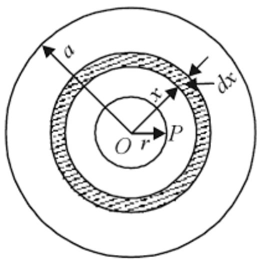

Fig. 2.15

Volume of the inner solid sphere $= \dfrac{4}{3}\pi r^3$

$\therefore$ Mass of the inner solid sphere $= \dfrac{4}{3}\pi r^3 \rho$

Hence potential at P due to the inner solid sphere

$$V_1 = -\frac{4}{3}\frac{\pi r^3 \rho G}{r} = -\frac{4}{3}\pi r^2 \rho G$$

To find the potential due to the outer spherical shell, draw two concentric spheres with radii x and $x + dx$ respectively forming a thin spherical shell of thickness dx.

Now surface area of the spherical shell $= 4\pi x^2$

Volume of the shell $= 4\pi x^2 dx$ $\quad\therefore$ Mass of the shell $= 4\pi x^2 dx\rho$

Remembering that the potential at any point within a spherical shell is the same as on the surface, we have

$$\text{Potential at } P \text{ due to the shell} = -\frac{4\pi x^2 \, dx \, \rho G}{x} = -4\pi x dx \, \rho G \quad ...(i)$$

$\therefore$ Potential V_2 at P due to the shell of internal radius r and external radius a is obtained by integrating equation (*i*) between the limits $x = r$ and $x = a$.

$$\therefore \quad V_2 \int_r^a -4\pi\rho G x dx = -4\pi\rho G\left[\frac{x^2}{2}\right]_r^a = -4\pi\rho G\left(\frac{a^2}{2} - \frac{r^2}{2}\right) = -2\pi\rho G(a^2 - r^2)$$

$$\therefore \quad \text{Total potential at } P = V_1 + V_2 = -\left[\frac{4}{3}\pi r^2 \rho G + 2\pi\rho G\,(a^2 - r^2)\right]$$

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$$= -2\pi\rho G\left(\frac{2}{3}r^2 + a^2 - r^2\right) = -\frac{2}{3}\pi\rho G\,(3a^2 - r^2) = -\frac{4}{3}\pi a^3\,\rho G\,\frac{(3a^2 - r^2)}{2a^3}$$

But $\frac{4}{3}\pi a^3\rho = M$ the mass of the sphere

$\therefore$ Potential $V = -GM\,\dfrac{3a^2 - r^2}{2a^3}$ ... (3)

(*iv*) *Point at the centre of the sphere.* At the centre of the sphere $r = 0$

$\therefore$ Potential $V = -\dfrac{3}{2}\dfrac{GM}{a}$ ... (4)

The variation of gravitational potential due to a sphere at a point outside it, on the surface and inside it is shown in Fig. 2.16

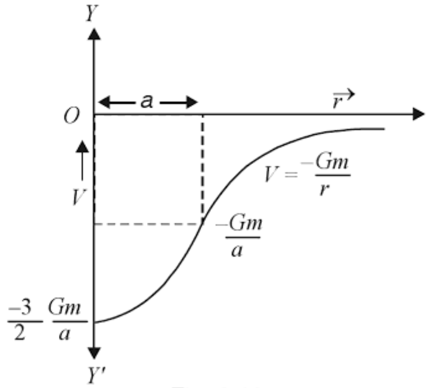

Fig. 2.16

2.21 GRAVITATIONAL FIELD DUE TO SOLID SPHERE

(*i*) *Point outside the sphere.* The gravitational field $F = -\dfrac{dV}{dx}$

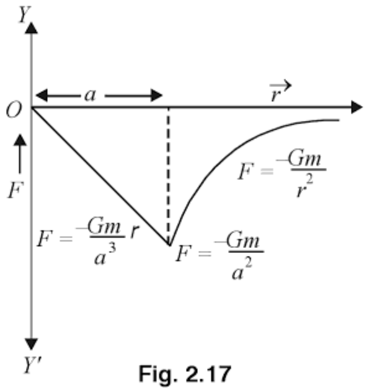

Fig. 2.17

Since the gravitational potential at a point outside the sphere is given by $V = -\dfrac{GM}{r}$

$\therefore$ $F = -\dfrac{dV}{dr} = -\dfrac{GM}{r^2}$ ... (*i*)

(*ii*) *Point on the surface.* For a point on the surface of the sphere $r = a$

$\therefore$ $F = -\dfrac{GM}{a^2}$ ... (*ii*)

(*iii*) *Point inside the sphere.* For a point inside the sphere

$$V = -GM\,\frac{3a^2 - r^2}{2a^3}$$

$\therefore$ $F = -\dfrac{dV}{dr} = -\dfrac{GM}{2a^3}\,2r = -\dfrac{GM}{a^3}r$ ... (*iii*)

Hence the gravitational field at a point inside the solid sphere is proportional to its distance from the centre.

The variation of gravitational field due to a sphere for a point outside the sphere, on the surface and inside it is shown in Fig. 2.17

2.22 GRAVITATIONAL POTENTIAL DUE TO THICK SPHERICAL SHELL

(*i*) *Inside the shell.* Consider a point P inside the shell bounded by two concentric spheres of radii a and b respectively, at a distance r from the centre O(Fig. 2.18).

With O as centre and radius x and $x + dx$ draw two concentric spheres enclosing a thin spherical shell of thickness dx

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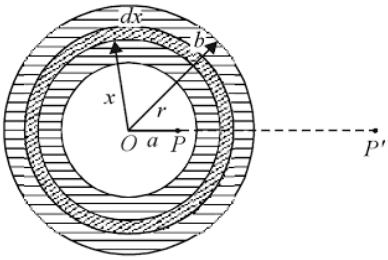

Fig. 2.18

The volume of the thin shell = $4\pi x^2 dx$

$\therefore$ Mass of the thin shell = $4\pi x^2 dx\rho$

$\therefore$ Potential at P due to the thin shell

$$= -\frac{G4\pi x^2 dx\rho}{x} = -4\pi\rho Gx\,dx$$

since the potential inside a shell is the same as on its surface.

Hence potential V at P due to the shell formed by two spheres of radii a and b is given by

$$V = \int_a^b -4\pi\rho Gx dx = -4\pi\rho G\left[\frac{x^2}{2}\right]_a^b = -2\pi\rho G(b^2 - a^2) \quad ...(i)$$

(*ii*) *Outside the shell.* When the point P lies outside the shell at a distance r from the centre, the potential at P due to the thin shell $= \dfrac{G4\pi x^2\, dx\rho}{r}$

since the mass behaves as if it were concentrated at the centre.

Hence potential V at P' due to the shell bounded by two spheres of radii a and b is given by

$$V = \int_a^b -\frac{4\pi\rho G}{r}x^2\, dx = -\frac{4\pi\rho G}{r}\left[\frac{x^3}{3}\right]_a^b = -\frac{4\pi\rho G}{3r}(b^3 - a^3) \quad ...(ii)$$

But $\dfrac{4}{3}\pi(b^3 - a^3)\rho = M$ the mass of the whole shell Hence $V = -\dfrac{GM}{r}$ ...(*iii*)

(*iii*) *Between the two surfaces.* Let the point P lie between the two surfaces at a distance r from the centre O. With O as centre draw a sphere of radius r, as shown in Fig. 2. 19.

As the point P lies just *inside* the spherical shell of internal radius r and external radius b, the potential at $P = -\ 2\pi\rho G\ (b^2 - r^2)$ [Compare with equation (*i*)]

As the point P lie, just *outside* the spherical shell of internal radius a and external radius r the potential at $P = -\dfrac{4\pi\rho G}{3r}(r^3 - a^3)$ [Compare with equation (*ii*)]

$$\text{Total Potential at } P = -\left[2\pi\rho G\,(b^2 - r^2) + \frac{4\pi\rho G}{3r}(r^3 - a^3)\right]$$

$$= -\left[2\pi\rho G\left(b^2 - r^2 + \frac{2}{3}r^2 - \frac{2}{3}\frac{a^3}{r}\right)\right] = -\left[2\pi\rho G\left(b^2 - \frac{r^2}{3} - \frac{2a^3}{3r}\right)\right] \quad ...(iii)$$

2.23 GRAVITATIONAL FIELD (*attraction*)

(*i*) *Inside the shell:*

Gravitational potential $V = -\,2\pi\rho G\,(b^2 - a^2)$

$\therefore$ Gravitational field $F = -\dfrac{dV}{dr} = 0$

as $2\pi\rho G\,(b^2 - a^2)$ is constant quantity.

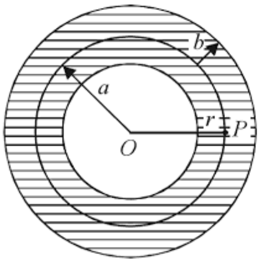

Fig. 2.19

(*ii*) *Outside the shell :*

Gravitational potential $V = -\dfrac{4\pi\rho G}{3r}(b^3 - a^3)$

$\therefore$ Gravitational field $F = -\dfrac{dv}{dr} = -\dfrac{4\pi\rho G}{3r^2}(b^3 - a^3) = -\dfrac{GM}{r^2}$ ... (*iv*)

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(*iii*) *Between the two surfaces:*

Gravitational potential $$V = -2\pi\rho G\left(b^2 - \frac{r^2}{3} - \frac{2a^3}{3r}\right)$$

Gravitational field $$F = -\frac{dV}{dr} = 2\pi\rho G\left(-\frac{2r}{3} + \frac{2a^3}{3r^2}\right) = -\frac{4\pi\rho G}{3r^2}(r^3 - a^3) \qquad \text{... }(v)$$

Therefore the gravitational field is the same as if the part of the shell lying outside the point P were not present.

2.24 GRAVITATIONAL SELF-ENERGY OF A BODY

Every body possesses gravitational self-energy due to its mass. The self-energy of a body is equal to the work done by gravitational force in bringing infinite small masses that are initially at infinite distance from one another to a point where the body is formed. The gravitational self-energy of a body is negative because the gravitational force is attractive.

Suppose a body consists of N particles each of mass m_1, m_2, m_3, ... etc., then

$$U = -\frac{G}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{m_i m_j}{r_{ij}}$$

2.25 GRAVITATIONAL SELF-ENERGY OF A UNIFORM SOLID SPHERE

Let M be the mass and R the radius of a uniform solid sphere. A solid sphere may be considered to have been built up by formation of successive concentric spherical shells of increasing radii from zero to R as shown in Fig. 2.20.

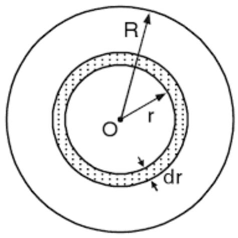

Fig. 2.20

Let r be the radius of a solid spherical core, dr the thickness of the surrounding spherical shell, and ρ the density of the material.

The gravitational potential energy of the shell in the gravitational field of the core is given by

$$dU = -G\frac{(\text{mass of the core}) \times (\text{mass of the shell})}{r}$$

$$= -G\cdot\frac{1}{r}\left(\frac{4}{3}\pi r^3\rho\right)\times(4\pi r^2 dr\cdot\rho)$$

$$= -\frac{1}{3}G\,(4\pi\rho)^2 r^4\,dr$$

The self-energy U_s of the solid sphere, then

$$U_s = \int dU = -\int_0^R \frac{1}{3}G\,(4\pi\rho)^2 r^4\,dr$$

$$= -\frac{1}{3}G(4\pi\rho)^2\int_0^R r^4\,dr$$

$$= -\frac{1}{3}G(4\pi\rho)^2\left(\frac{R^5}{5}\right)$$

$$= -\frac{1}{3\times 5}G\times 9\left(\frac{4}{3}\pi R^3\rho\right)^2\cdot\frac{1}{R}$$

$$= -\frac{3}{5}\left(\frac{GM^2}{R}\right) \quad \text{... }(i)$$

where $\qquad M = \frac{4}{3}\pi R^3\rho$

Equation (i) gives the gravitational self-energy of a uniform solid sphere.

2.26 GRAVITATIONAL SELF ENERGY OF GALAXY

There are number of galaxies in the universe and each galaxy contains large number of stars, planets and other matter. Our own galaxy is estimated to contain 1.6×10^{11} stars including the solar system. All the objects in the galaxy are bounded by the gravitational force. Similarly gravitational force also exists between various galaxies.

Let the Galaxy be made of N number of stars, each of mass M_1, M_2, M_3, ... M_N. During its formation, let star of mass M_2 be brought from infinity to distance R_{12} near star M_1, then P.E. of star of mass M_1 will be

$$u_1 = -\frac{GM_1M_2}{R_{12}}$$

When star of mass M_3 is brought at distance R_{13}, it will contribute to P.E. of M_1 as

$$u_2 = -\frac{GM_1M_3}{R_{13}} \qquad \text{and so on.}$$

Thus total P.E. of star of mass M_1 will be

$$U_1 = u_1 + u_2 + \text{.........} + u_s$$

$$= -\frac{GM_1M_2}{R_{12}} - \frac{GM_1M_3}{R_{13}} - \frac{GM_1M_4}{R_{14}} \text{............} - \frac{GM_1M_N}{R_{1N}}$$

$$= -GM_1\left(\frac{M_2}{R_{12}} + \frac{M_3}{R_{13}} + \text{.......} + \frac{M_N}{R_{1N}}\right)$$

$$= -GM_1\sum_{K=2}^{N}\left(\frac{M_K}{R_{1K}}\right) \quad \text{... }(i)$$

This summation excludes the term $k = 1$

Similarly, the P.E. of M_2 will be

$$U_2 = -GM_2\sum_{K=1}^{N}\left(\frac{M_K}{R_{2K}}\right), \text{ excluding term } k = 2 \quad \text{... }(ii)$$

....

....

$$U_N = -GM_N\sum_{K=1}^{N}\left(\frac{M_K}{R_{NK}}\right), \text{ excluding term k = N} \quad \text{... }(iii)$$

In summing U_1, U_2, ... U_N, it is found each term appears twice, Hence, the sum should be divided by 2, to find total P.E. of the galaxy.

$$\therefore \qquad U = \frac{1}{2}[U_1 + U_2 + U_3 + ... + U_N]$$

$$= \left(-\frac{G}{2}\right)\left[M_1 \sum_{\substack{K=2\\K\neq 1}}^{N} \frac{M_K}{R_{1K}} + M_2 \sum_{\substack{K=1\\K\neq 2}}^{N} \frac{M_K}{R_{2K}} + ...\right]$$

$$= \left(-\frac{G}{2}\right)\left[\sum_{j=1}^{N} M_j \sum \frac{M_K}{R_{jK}}\right]_{j\neq K}$$

$$= \left(-\frac{G}{2}\right)\left[\sum_{j=1}^{N} \sum_{K=1}^{N} \frac{M_j M_K}{R_{jK}}\right]_{j\neq K} \qquad ...(iv)$$

As the distance between two stars is same

$R_{jk} = R$

$$\text{U} = \left(-\frac{G}{2R}\right)\sum_{j=1}^{N} M_j \sum_{k=1}^{N} M_k \text{, where } j \neq k$$

$$= \left(-\frac{G}{2R}\right)\left[M_1 \sum_{\substack{K=2\\K\neq 1}}^{N} M_K + M_2 \sum_{\substack{K=1\\K\neq 2}}^{N} M_K + M_3 \sum_{\substack{K=1\\K\neq 3}}^{N} M_K + ...\right]$$

$$= \left(-\frac{G}{2R}\right)[M_1(M_2 + M_3 + ... + M_N) + M_2(M_1 + M_3 + ... + M_N)$$

$$+ .... + M_N (M_1 + M_2 + ... + M_{N-1})]$$

In each bracket there are $(N - 1)$ terms and these are N terms. Also $M_1 = M_2 = M_3 = ... = M$,

Hence,

$$U = -\left(\frac{G}{2R}\right)M^2(N-1)N$$

Since N is very large, $(N - 1)$ is taken nearly equal to N.

Thus, the gravitation energy of the Galaxy is

$$U = -\frac{GM^2N^2}{2R} \qquad ...(v)$$

2.27 GAUSS'S AND POISON'S EQUATIONS

From a point of mass M, the gravitational field $\vec{E}$ at a distance r from M is given by

$$E = -\frac{GM}{r^2} \qquad ...(i)$$

The flux ϕ of the gravitational field is the field through the surface of the sphere of radius r.

$$\phi = -\frac{GM}{r^2} \times 4\pi r^2 = -4\pi GM \qquad ...(ii)$$

Since the flux is due to normal component, the normal component of $\vec{E}$ is $-\frac{GM}{r^2}\cos\theta$ and the flux $d\phi$ through an element dA is

$$d\phi = -\frac{GM}{r^2}\cos\theta\, dA = \vec{E}\cdot d\vec{A} = \hat{n}\cdot E\, dA$$

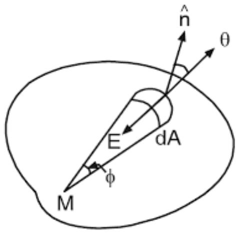

Fig. 2.21

∴ The total gravitational flux enclosed by the closed surface is

$$\phi = \int \vec{E}\cdot dA = \int \hat{n}\cdot\vec{E}\, dA$$

$$\phi = \int \vec{E}_1\cdot dA + \int \vec{E}_2\cdot d\vec{A} + \ldots\ldots = -4\pi G(M_1 + M_2 + \ldots\ldots)$$

$$\therefore \qquad \phi = -4\pi GM$$

where M is sum of all masses enclosed by the surface.

This represents the **Gauss's law.**

Now if curl $\vec{E} = \nabla\times\vec{E} = 0.$ We can define the potential as

$$E = -\text{grad } V = -\nabla V \qquad \ldots (iii)$$

Now $\qquad \phi = \int_S \hat{n}\cdot\vec{E}\, dA$

But $\qquad \phi = -4\pi GM.$

∴ For a continuous mass distributions,

$$\phi = -\int_m 4\pi G\, dm = -\int_\tau 4\pi G\rho\, d\tau \qquad \ldots (iv)$$

where $dm = \rho d\tau$ and ρ is mass density.

$$\therefore \qquad \int_S \hat{n}\cdot\vec{E}\, dA = -\int_\tau 4\pi G\rho\, d\tau \qquad \ldots (v)$$

Applying Gauss's divergence theorem to the L.H.S., we get

$$\int_S \hat{n}\cdot\vec{E}\, dA = -\int_\tau \nabla\cdot\vec{E}\, d\tau \qquad \ldots (vi)$$

Substituting we have

$$\int_\tau \nabla\cdot\vec{E}\, d\tau = -\int_\tau 4\pi G\rho\, d\tau \qquad \ldots (vii)$$

or $$-\int_\tau (\nabla\cdot E + 4\pi G\rho)\, d\tau = 0 \qquad \ldots (viii)$$

Since this is for any volume, we get

$$\nabla\cdot\vec{E} = -4\pi G\rho.$$

By definition of potential, we have $\vec{E} = -\nabla V$ where V is the potential

$$\therefore \qquad \nabla\cdot\vec{E} = \nabla\cdot(-\nabla V) = -4\pi G\rho \qquad \ldots (ix)$$

or $$\nabla^2 V = 4\pi G\rho \qquad \ldots (x)$$

In Cartesian form

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{dz^2} = 4\pi G\rho \qquad \ldots (xi)$$

This is called the **Poisson's equation**

SOLVED EXAMPLES

Example 2.1 What is a conservative force ? Show that the force $\vec{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$ is a conservative force. (*Pbi. U.*, 2003)

Solution. Conservative force. *A force is said to be conservative if the curl of the force is zero.*

or $$\vec{\nabla} \times \vec{F} = 0$$

Given $$\vec{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$$

$$\therefore \qquad \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx)\right] + \hat{j}\left[\frac{\partial}{\partial z}(yz) - \frac{\partial}{\partial x}(xy)\right] + \hat{k}\left[\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz)\right]$$

$$= \hat{i}\,(x - x) + \hat{j}\,(y - y) + \hat{k}\,(z - z) = 0$$

Since $\vec{\nabla} \times \vec{F} = 0$, force $\vec{F}$ is conservative.

Example 2.2 The earth is moving round the sun under gravitational force and its orbit has semi-major axis 1.495 × 10^8 km. When the earth passes closest to the sun at its perihilion its distance is 1.47 × 10^8 km and its orbital velocity is 0.303 kms^{-1}. Find the velocity of the earth at the apehilion and its angular velocities at the two points.

Solution. As the motion of the earth round the sun is under gravitational force which is a central force, the angular momentum of the earth at the two positions, the perihilion and the apehilion is conserved.

or Angular momentum at apehilion = angular momentum at perihilion

or $$mv_{ap}\, r_{ap} = mv_{peri}\, r_{peri}$$

Now $$r_{peri} = 1.47 \times 10^8 \text{ km} \qquad ...(i)$$

Semi major axis $a = 1.495 \times 10^8$ km $\qquad \therefore$ Major axis $= 2a = 2.990 \times 10^8$ km

Now $$r_{ap} + r_{peri} = 2a$$

$$\therefore \qquad r_{ap} = 2a - r_{peri} = 2.990 \times 10^8 - 1.47 \times 10^8 = 1.520 \times 10^8 \text{ km}$$

Substituting in (*i*), we have $v_{ap} \times 1.52 \times 10^8 = 0.303 \times 1.47 \times 10^8$

$\therefore$ Velocity of the earth at apehilion $$v_{ap} = \frac{0.303 \times 1.47 \times 10^8}{1.52 \times 10^8} = 0.293 \text{ kms}^{-1}$$

The angular velocity at the perihilion $$\omega_{peri} = \frac{v_{peri}}{r_{peri}} = \frac{0.303}{1.47 \times 10^8} = 0.206 \times 10^{-8} \text{ rad s}^{-1}$$

The angular velocity at the apehilion $$\omega_{ap} = \frac{v_{ap}}{r_{ap}} = \frac{0.293}{1.520 \times 10^8} = 0.193 \times 10^{-8} \text{ rad s}^{-1}$$

Example 2.3 The motion of a particle under the influence of a central force is described by $r = a \sin\theta$. Find an expression for the force.

Solution. The motion of the particle is given by

$$r = a \sin\theta$$

$\therefore \qquad u = \frac{1}{r} = \frac{1}{a \sin \theta} = \frac{\operatorname{cosec} \theta}{a}$

Hence $\qquad \frac{du}{d\theta} = -\frac{\operatorname{cosec} \theta \cot \theta}{a}$

and $\qquad \frac{d^2u}{d\theta^2} = -\frac{1}{a}(-\operatorname{cosec} \theta \operatorname{cosec}^2 \theta - \operatorname{cosec} \theta \cot \theta \cot \theta)$

$$= \frac{1}{a} \operatorname{cosec} \theta (\operatorname{cosec}^2 \theta + \cot^2 \theta)$$

The differential equation of motion of the orbit of a particle moving under a central force is given by

$$\frac{d^2u}{d\theta^2} = -u - \frac{m}{J^2u^2} F\left(\frac{1}{u}\right)$$

$$\therefore \qquad \frac{m}{J^2u^2} F\left(\frac{1}{u}\right) = -u - \frac{d^2u}{d\theta^2}$$

$$= -\frac{1}{a}[\operatorname{cosec} \theta + \operatorname{cosec} \theta (\operatorname{cosec}^2 \theta + \cot^2 \theta)]$$

$$= -\frac{1}{a} \operatorname{cosec} \theta (1 + \operatorname{cosec}^2 \theta + \cot^2 \theta)$$

$$= -\frac{2}{a} \operatorname{cosec}^2 \theta = -2a^2u^3$$

$$\therefore \qquad F\left(\frac{1}{u}\right) = \frac{-(2J^2 u^5 a^2)}{m} = \frac{-2J^2 a^2 u^5}{m} \qquad \therefore F(r) = \frac{-2J^2a^2}{m}\left(\frac{1}{r^5}\right)$$

This is the required force law.

Example 2.4 Calculate the period of revolution of Neptune round the sun given that the diameter of the orbit is 30 times the diameter of the earth's orbit round the sun, both orbits being assumed to be circular.

Solution. Let a_1 and a_2 be the mean radii of the orbit of the earth and Neptune respectively.

$$\therefore \qquad \frac{a_2}{a_1} = 30$$

Period of revolution of earth T_1 = one year.

Let T_2 be the period of revolution of Neptune. Then according to Kepler's third law

$$\frac{T_2^2}{T_1^2} = \left(\frac{a_2}{a_1}\right)^3 \qquad \therefore T_2^2 = T_1^2 \left(\frac{a_2}{a_1}\right)^3 = 30 \times 30 \times 30$$

or $\qquad T_2 = 30\sqrt{30}$ years = 164.3 years

Example 2.5 A sphere of mass 19 kg. is attracted by another sphere of mass 150 kg. when their centres are separated by a distance 0.28 m with a force equal to the weight of 0.25 mg. Calculate the gravitational constant. If the distance is halved what would be the new force in Newton? Assume g = 9.8 ms^{-2}.

Solution. Here $\qquad m_1$ = 19 kg; m_2 = 150 kg; r = 0. 28 m

Force $\qquad F$ = 0.25 mg. wt = $0.25 \times 10^{-6} \times 9.8$ N

Now $\qquad F = \frac{Gm_1 m_2}{r^2}$ or $G = \frac{Fr^2}{m_1 m_2}$

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$$= \frac{0.25 \times 10^{-6} \times 9.8 \times (0.28)^2}{19 \times 150} = 6.74 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2}$$

When the distance is halved, the force F' between the two masses becomes 4 times as the force is inversely proportional to the square of the distance between the centres of the two masses.

$$\therefore \quad F' = 4F = 4 \times 0.25 \times 10^{-6} \times 9.8 = 9.8 \times 10^{-6} \text{ Newton.}$$

Example 2.6 Find the mass of sun from the following data : Radius of earth's orbit r = 1.5 × 10⁸ km, G = 6.67 × 10⁻¹¹ Nm² kg⁻². (*Gharwal U.* 2000)

Solution. The earth revolves round the sun in more or less circular orbit of radius r. The gravitational force of attraction between the sun of mass M and earth of mass m is balanced by the centripetal force $mr\omega^2$, where ω is the angular velocity of earth

$$\therefore \quad \omega = \frac{2\pi}{365 \times 24 \times 3600} = 1.99 \times 10^{-7} \text{ rad s}^{-1}$$

$$r = 1.5 \times 10^8 \text{ km} = 1.5 \times 10^{11} \text{ m}$$

Now
$$\frac{GMm}{r^2} = mr\omega^2$$

$$\therefore \quad M = \frac{r^3\omega^2}{G} = \frac{(1.5 \times 10^{11})^3 \times (1.99 \times 10^{-7})^2}{6.67 \times 10^{-11}} = 2.004 \times 10^{30} \text{ kg}$$

Example 2.7 If the mass of sun is 2 × 10³⁰ kg, distance of earth from the sun is 1.5 × 10¹¹ m and period of revolution of the former around the latter is 365.3 days, find the value of *G*.

Solution. The force of attraction F between the sun of mass M and earth of mass m separated by a distance r is given by $F = \frac{GMm}{r^2}$

This force is balanced by the centripetal force $mr\omega^2$ where w is the angular velocity of earth

$$\therefore \quad \frac{GMm}{r^2} = mr\omega^2 \quad \text{or} \quad G = \frac{r^3\omega^2}{M}$$

Now $M = 2 \times 10^{30}$ kg, $r = 1.5 \times 10^{11}$ m, $G = ?$

$$\omega = \frac{2\pi}{365.3 \times 24 \times 3600} = 11.991 \times 10^{-7} \text{ rad s}^{-1}$$

$$\therefore \quad G = \frac{(1.5 \times 10^{11})^3 \times (1.991 \times 10^{-7})^2}{2 \times 10^{30}} = 6.688 \times 10^{-11} \text{ Nm}^2\text{-kg}^{-2}$$

Example 2.8 A satellite revolves in a circular orbit at a height of 200 km from the surface of earth. If the period of revolution of satellite is 90 mts, G = 6.66 × 10⁻¹¹ Nm² kg⁻² and mean radius of earth is 6 × 10⁶ m, calculate the average density of earth.

Solution. Distance of satellite from the centre of earth

$$R_1 = 6 \times 10^6 + 0.2 \times 10^6 = 6.2 \times 10^6 \text{ m}$$

If M is the mass of earth and m that of satellite, then $\frac{GMm}{R_1^2} = mR_1\,\omega^2$

or $$M = \frac{R_1^3\,\omega^2}{G} = \frac{4}{3}\pi R^3 \rho$$

or $$\rho = \frac{3R_1^3\,\omega^2}{4\pi R^3 G} = \frac{3\times(6.2\times10^6)^3}{4\pi\times(6\times10^6)^3\times6.66\times10^{-11}}\times\left(\frac{2\pi}{90\times60}\right)^2 = 5.355\times10^3\ \text{kg m}^{-3}$$

Example 2.9 The radius of earth is 6.637×10^6 m, its mean density 5.57×10^3 kg m^{-3} and gravitational constant 6.66×10^{-11} N m^2 kg^{-2}. Calculate the earth's surface potential.

Solution. Considering the earth to be a homogeneous sphere, the magnitude of gravitational potential on its surface =

$$\frac{GM}{r} = \frac{G\cdot\frac{4}{3}\pi r^3\rho}{r} = \frac{4G\pi r^2\rho}{3}$$

$$= \frac{4\times6.66\times10^{-11}\times\pi\times(6.637\times10^6)^2\times5.57\times10^3}{3}$$

$$= 6.845\times10^7\ \text{J/kg}$$

Example 2.10 Show that gravitational intensity and potential at any point on the surface of the earth are 'g' and 'gR' respectively assuming the earth to be a uniform sphere of radius 'R'.

Solution. The intensity of gravitational field on the surface of a sphere $= -\dfrac{GM}{R^2}$

The negative sign indicates that the field is directed towards the centre of the earth.

Now a body of mass m lying on the earth's surface is attracted by the earth with a force

$$mg = \frac{GMm}{R^2} \text{ or } g = \frac{Gm}{R^2}$$

Hence the intensity of gravitational field on the surface of the earth = g (numerically).

The gravitational potential on the surface of a sphere $= -\dfrac{GM}{R} = -R\dfrac{GM}{R^2} = gR$ (numerically).

Example 2.11 The earth's mass is 80 times that of the moon and their diameters are 12800 km and 3200 km respectively. What is the value of g on the moon? g on earth is 9.8 ms^{-2}.

Solution. The acceleration due to gravity on the surface of a sphere is given by $g = \dfrac{GM}{R^2}$ where M is the mass and R the radius of the sphere. Taking the earth and moon to be spheres, acceleration due to gravity on the surface of the earth

$$g = G\frac{80\,m}{(64\times10^5)^2}\ \text{ms}^{-2}$$

where m is the mass of the moon and 64×10^5 m the radius of the earth.

Acceleration due to gravity on the surface of the moon

$$g' = \frac{Gm}{(16\times10^5)^2}$$ where 16×10^5 m is the radius of the moon.

$$\therefore \quad \frac{g'}{g} = \frac{(64\times10^5)^2}{80\times(16\times10^5)^2} = \frac{1}{5} \qquad \therefore\ g' = 9.8\times\frac{1}{5} = 1.96\ \text{ms}^{-2}$$

Example 2.12 Two bodies of mass M_1 and M_2 are kept separated by a distance d. Prove that at the point where the value of gravitational intensity produced by them is zero, the potential is

$$V = -\frac{G}{d}(M_1 + M_2 + 2\sqrt{M_1 M_2}).$$

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Solution. Let the point where the gravitational intensity produced by the two bodies is zero, be at a distance x from mass M_1, then

$$-\frac{GM_1}{x^2} + \frac{GM_2}{(d-x)^2} = 0 \quad \text{or} \quad \frac{GM_1}{x^2} = \frac{GM_2}{(d-x)^2}$$

$$\therefore \quad \frac{M_2}{M_1} = \frac{(d-x)^2}{x^2} \quad \text{or} \quad \frac{\sqrt{M_2}}{\sqrt{M_1}} = \frac{(d-x)}{x} = \frac{d}{x} - 1$$

$$\text{or} \quad \frac{d}{x} = \frac{\sqrt{M_2}}{\sqrt{M_1}} + 1 \quad \text{or} \quad x = \frac{d\sqrt{M_1}}{\sqrt{M_2} + \sqrt{M_1}}$$

Gravitational potential at the same point

$$= -\frac{GM_1}{x} - \frac{GM_2}{d-x} = -\left[\frac{GM_1(\sqrt{M_2} + \sqrt{M_1})}{d\sqrt{M_1}} + \frac{GM_2}{d - \dfrac{d\sqrt{M_1}}{\sqrt{M_2} + \sqrt{M_1}}}\right]$$

$$= -\frac{G}{d}\left[\sqrt{M_1}\,(\sqrt{M_2} + \sqrt{M_1}) + \frac{M_2(\sqrt{M_2} + \sqrt{M_1})}{\sqrt{M_2}}\right]$$

$$= -\frac{G}{d}[\sqrt{M_1 M_2} + M_1 + M_2 + \sqrt{M_1 M_2}]$$

$$\text{or} \quad V = -\frac{G}{d}[M_1 + M_2 + 2\sqrt{M_1 M_2}]$$

Example 2.13 Find the gravitational self-energy of the sun. [Given : $G = 6.67 \times 10^{-11}$ Nm^2 / kg^2; mass of the sun = 2×10^{30} kg; radius of the sun = 7×10^8 m.] (*Nagpur Uni.* 2008)

Solution. Gravitational self-energy

$$U_s = -\frac{3}{5}\left(\frac{GM^2}{R}\right)$$

$$= -\frac{3}{5} \times \frac{6.67 \times 10^{-11} \times (2 \times 10^{30})}{7 \times 10^8}$$

$$= -2.29 \times 10^{41} \text{ J}$$

The gravitational self-energy is negative, because the gravitational force is attractive.

EXERCISE CH.2

LONG QUESTIONS

1. (*a*) What are central and non-central forces? Give three characteristics of each. Give two examples of central and non-central forces.
(*Pbi. U.* 2001; *P.U.* 2000; *G.N.D.U.* 2000; *H.P.U.* 2003, *Luck. U.* 2001; *Purvanchal U.* 2006, 2005; *Kerala U.* 2001; *Gharwal U.* 2000; *Osm. U.* 2004; *Nagpur U.* 2009)

(*b*) (*i*) Why gravitational and Coulomb forces are called inverse square law forces? (*H.P.U.*, 2000)

(*ii*) Why nuclear force is called non-central force? (*H.P.U.*, 2001)

2. (*a*) Prove that a central force is a conservative force and a conservative force can be expressed as negative gradient of potential or as-grad V where V is potential energy.

(*Purvanchal U.* 2007, 2005; *M.D.U.* 2003; *G.N.D.U.* 2004, 2003; *Meerut U.* 2002)

(*b*) Give two examples of conservative and non-conservative forces.

(*H.P.U.* 2001; *G.N.D.U.* 2000; *Guwahati U.* 2000)

3. A particle moves under a central force. Show that (*i*) its orbit lies in a plane and (*ii*) the radius vector from the centre of the force to the particle sweeps area at a constant rate.

(*Meerut U.* 2002; *P.U.* 2001; *G.N.D.U.* 2003)

4. If $\vec{r}$ is the radius vector joining a particle of mass m with centre of force and $\vec{A}$ the area swept by the radius vector, show that $d\vec{A} = \frac{1}{2}\vec{r} \times d\vec{r}$ and $\frac{d\vec{A}}{dt} = \frac{J}{2m}$ where J is the angular momentum of the particle about the centre of force. (*P.U.* 2004)

5. (*a*) State the expression for acceleration of a particle moving in a plane in polar co-ordinates and derive (*i*) the radial equation of motion (*ii*) equation of motion of θ-coordinates.

(*b*) Prove the law of conservation of energy in central motion.

6. State the relation showing total energy for a particle moving under a central force. Hence derive a relation between (*i*) r and t (*ii*) θ and t and (*iii*) θ and r.

7. (*a*) Discuss the nature of orbital motion under a central force field when the force obeys inverse square law and is (*i*) repulsive (*ii*) zero and (*iii*) attractive.

(*b*) What is potential energy curve? Making use of potential energy curve explain the nature of motion when total energy is (*i*) positive (*ii*) zero (*iii*) negative but greater than minimum value and (*iv*) minimum.

(*Utkal U.* 2003; *P.U.* 2004, 2003, 2000; *G.N.D.U.* 2000)

8. Establish the differential equation of motion under a central force and deduce its solution for attractive inverse square force field. (*H.P.U.* 2003, 2001)

9. (*a*) A particle of mass m traces a circle of radius r under attractive inverse square force $-\frac{c}{r^2}$. Show that the energy of the particle at any point on the circle is $-\frac{c}{2r}$.

(*b*) Show that the differential equation of motion of a particle of mass m under the influence of a central isotropic force can be written as

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{J^2u^2} F\left(\frac{1}{u}\right)$$

where $u = \frac{1}{r}$, (r, θ) are the plane polar co-ordinates of the particle and J the angular momentum.

10. (*a*) Prove that the shape of the orbit traced by a particle moving under attractive inverse square force depends on the angular momentum and total energy of the particle. What are turning points ? What is the number of turning points in an elliptic orbit?

(*P.U.* 2001, *G.N.D.U.* 2000; *Pbi. U.* 2003)

(*b*) What will be the shape of the orbit of a particle moving under repulsive inverse square force? Explain.

11. (*a*) State and prove Kepler's laws of planetary motion.

(*D.A.U. Agra* 2008, 2003; *Nag. U.* 2007, 2008; *Calicut U.*, 2003)

(*b*) Show that the areal velocity of a planet round the sun is constant.

12. (*a*) Show that the radius vector joining the Sun to a planet sweeps out equal areas in equal intervals of time. (*Purvanchal U.* 2005)

(*b*) Show that the square of the time period of revolution of a planet is proportional to the cube of semi-major axis of the orbit. (*Bhopal U.* 2004; *Osm. U.* 2004; *P.U.* 2001, 2000, *Gharwal U.* 2000; *G.N.D.U.* 2002, 2001, 2000; *Pbi. U.* 2003, 2000; *Indore U.* 2001; *Kerala U.* 2001; *Meerut U.* 2002)

13. (*a*) Obtain an expression for the gravitational potential and attraction due to a thin uniform spherical shell at a point (*i*) outside (*ii*) at the surface and (*iii*) inside the shell.

(*Nagpur U.* 2008; *Meerut U.* 2005, 2001; *Gharwal U.* 2000; *Agra U.* 2005, 2003)

(*b*) Graphically represent the variation of potential with distance due to a thin spherical shell. (*Kerala U.* 2001)

14. Find the intensity of gravitational field due to a thin spherical shell at a point (*i*) external to the shell (*ii*) at the surface of the shell and (*iii*) inside the shell.

(*Nagpur U.*, 2008, 2001; *Gharwal U.*, 2000; *Guwahati. U.* 2000)

15. (*a*) Define gravitational potential.

(*b*) Derive an expression for the gravitational potential at a point (*i*) outside (*ii*) on the surface and (*iii*) inside a solid sphere.

(*Kerala U.* 2001; *Guwahati U.* 2000; *Indore U.* 2001; *Meerut U.* 2003, 2000; *M.S. U. Tiruneveli*, 2007; *Purvanchal U.* 2004; *D.A.V. Agra,* 2008; 2006)

(*c*) Hence find gravitational field (attraction) at these points and show that it is proportional to the distance from the centre of the sphere for a point inside it.

(*Indore U.* 2001; *Meerut U.* 2000, *Kerala. U.* 2001)

16. The gravitational potential at a point at a distance r from the centre of a solid sphere is given by $V = -\dfrac{GM\,(3a^2 - r^2)}{2a^3}$ where M is the mass and a the radius of the sphere. Find the field intensity at this point. (*Meerut U.*, 2003)

17. Find the gravitational potential and attraction due to a spherical shell bounded by spheres of radii a and b at a point (*i*) inside the shell, (*ii*) outside the shell and (*iii*) between the two surfaces. (*Arga U.* 2007; *Cal. U.*, 2003)

18. (*a*) Explain the terms

(*i*) Gravitational Field

(*ii*) Gravitational intensity,

(*iii*) Gravitational Potential. (*Nagpur U.* 2008, 2007, 2006)

(*b*) What will be gravitational potential and intensity of a thin spherical shell of mass 10 kg and radius 0.1 m at a point 0.2 m outside of its surface ? ($G = 6.67 \times 10^{-11}$ Nm^2/kg^2).

(*Nagpur*, 2008, 2004) [**Ans.** -1.66×10^{-8} N/kg]

19. A particle moving under a central force describes spiral orbit given by $r = a$ exp. $(b\theta)$ where a and b are some constants. Obtain the force law. (*Pbi. U.* 2000, *G.N.D.U.* 2002)

20. A particle follows a spiral orbit given by $r = c\theta^2$ under an unknown force law. Prove that such an orbit is possible in a central field. Also find the form of the force law.

21. Show that the gravitational energy of a galaxy is given by $U_s = -\dfrac{GN^2M^2}{2R}$ where N = Number of stars; M = Mass of each star. R = Average distance between each pair of stars.

(*Nagpur,* 2003)

22. (*a*) Explain the term Gravitational self-energy of a Galaxy.

(*b*) Obtain an expression for Gravitational self-energy of a Galaxy in terms of number of stars in Galaxy, mass of each star and average distance between each pair of stars. (*Nagpur Uni.* 2009)

SHORT QUESTIONS

1. When a particle moves under a central force, prove that the angular momentum of a particle is conserved. (*Calicut U.* 2003; *Meerut U.* 2005, 2003; *Purvanchal U.* 2005; *D.A.U. Agra* 2008)

2. Derive the polar equation of the orbit of a particle of mass m moving under the action of a force field $F = \frac{c}{r^2}$ about a fixed centre. (*P.U.*, 2001)

3. A planet moving round the sun is suddenly stopped. Find the time taken by the planet to fall into the sun in terms of the period of revolution of the planet around the sun.

4. What parameters determine the shape and size of an elliptic orbit of a planet? Explain.

5. State Newton's law of gravitation. What is meant by gravitational constant? What are its dimensions? (*Nag. U.* 2007, 2006)

6. Explain the terms gravitational field and gravitational potential. Find the relation between them. (*Nagpur U.*, 2007, 2003, 2001; *Meerut U.* 2003, 2002, 2000, *Agra,* 2005; *M.S.U. Tiruneveli*, 2007)

7. Explain the concept of self energy of a body. Deduce an expression for gravitational self energy of any uniform solid sphere. (*Nagpur*, 2006, 2003; *Agra U.* 2007, 2004)

8. Find the force field associated with the potential energy $V = Ae^{\alpha(x+y+z)}$ where A and α are constants. (*H.P.U.* 2003)

9. Prove that all ellipses with the same major axes have the same energy.

10. The equation of the orbit of a particle of mass m moving under the action of a central force field about a fixed centre is $r = \frac{1}{2\theta}$. Find the force law. (*P.U.*, 2000)

11. What is central force? Show that the motion of a particle under central force is always confined to a single plane, if the motion of the particle is not parallel to the force direction.

(*Nagpur U.* 2007, *Kolkata U.* 2002)

12. Two particles having masses M and m respectively are initially at rest an infinite distance apart and attract each other according to the law of gravitation. Show that their velocity of approach $v = \sqrt{\frac{2G(M+m)}{a}}$ where a is their separation.

13. Why gravitational and Coulomb's forces are called central forces?

14. Why nuclear forces are called non-central forces?

15. Show that a conservative force can be expressed as $\vec{F} = -\text{grad}\ U$, where U is potential energy. (*Meerut U.*, 2000; *Purvanchal U.*, 2005)

16. Show that work done in a conservative field around a closed path is zero. (*Nag. U.* 2007)

17. What are Kepler's laws of planetary motion? Deduce them from Newton's law of gravitation. (*Meerut, U.* 2005)

18. Deduce Newton's law of gravitation from Kepler's laws.
(*Guwahati U.* 2007; *Nagpur U.* 2005)

19. Derive an expression for gravitational field and potential due to a solid sphere.
(*M.S.U. Tiruneveli*, 2007)

20. State Kepler's laws of planetary motion and Newton's laws of gravitation.
(*Nagpur Uni.* 2009, 2007)

21. Define gravitational potential and derive an expression for it at a point at *a* distance *r* from a body of mass *m*. (*Nagpur Uni.* 2009)

NUMERICALS

1. The minimum and maximum distance of a comet from the Sun are 7×10^{10} and 1.4×10^{12} m respectively. If the speed of the comet at the nearest point is 6×10^4 m/s, calculate the speed at the farthest point. (*Gharwal. U.*, 2000) [**Ans.** $V = 3 \times 10^3$ m/s]

Hint: $7 \times 10^{10} \times 6 \times 10^4 = 1.4 \times 10^{12}$ V

2. A sphere of mass 19 kg is attracted by another sphere of mass 150 kg when their centres are seperated by a distance 0.28 m with a force equal to the weight of 0.25 mg. Calculate the gravitational constant. If the distance is halved what would be the new force in Newton? Assume $g = 9.8$ ms^{-2}.

3. Suppose the earth is revolving round the Sun in a circular orbit of radius one astronomical unit (1.5×10^8 km). Find the mass of the Sun. $G = 6.67 \times 10^{-11}$ Nm^2 kg^{-2}.
(*Gharwal U.* 2000) [**Ans.** 2.004×10^{30} kg]

4. If the mass of the Sun is 1.5×10^{11} m and period of revolution of the Earth around the Sun is 365.3 days, find the value of G. (**Ans.** 6.688×10^{-11} Nm^2/kg^2.)

5. Show that gravitational potential at the centre of a solid sphere is 3/2 times that on its surface.

6. A satellite revolves round a planet in an elliptical orbit. Its maximum and minimum distances from the planet are 1.5×10^7 m and 0.5×10^7 m respectively. If the speed of the satellite at the farther point is 5×10^3 ms^{-1}. Calculate the speed at the nearest point.
(**Ans.** 15×10^{-3} ms^{-1})

7. A spherical mass of 20 kg situated at the surface of the earth is attracted by another mass of 150 kg with a force equal to the weight of 0.25 mg when the centres of masses are 30 cm apart. Calculate the mass and mean density of the earth assuming the radius of the earth to be 6×10^5 cm.

Hint: $0.25 \times 10^{-3}\ g = \dfrac{Gm_1m_2}{r^2}$ and $M = \dfrac{g}{G}R^2$. Calculate $\dfrac{g}{G}$ from the first relation and substitute in the second. [**Ans.** $M = 4.8 \times 10^{27}$ gm and mean density = 5.31 gm/cc]

8. The moon describes a circular orbit of radius 3.8×10^5 kilometres about the earth in 27 days and the earth describes a circular orbit of radius 1.5×10^8 kms round the Sun in 365 days. Determine the mass of the Sun in terms of the earth.

Hint: $\dfrac{GM}{d^2} = d\left(\dfrac{2\pi}{T_1}\right)^2$ or

$T_1^2 = \dfrac{4\pi^2}{GM}d^3$ Similarly $T_2^2 = \dfrac{4\pi^2}{GE}\cdot x^3$ Hence $\dfrac{M}{E} = \dfrac{d^3}{x^3}\cdot\dfrac{T_2^2}{T_1^2}$ [**Ans.** $3.3666 \times 10^5\ E$]

9. A sphere of mass 40 kg is attracted by another sphere of mass 80 kg when the centres are 0.3 m apart with a force equal to the weight of 0.25 mg. Calculate the gravitational constant G. [**Ans.** 6.89×10^{-11} Nm^2 kg^{-2}]

10. Calculate the gravitational energy of the galaxy composed of 1.6×10^{11} stars each of mass 2×10^{30} kg. The average distance between a pair of stars is 10^{21} m. ($G = 6.67 \times 10^{-11}$ Nm^2/kg^2) (*Nag. U.*, 2009, 2003) [**Ans.** -34.15×10^{50} J]

11. A satellite is orbiting very close to a planet of density 8×10^3 kg m^{-3}. If gravitational constant $G = 6.67 \times 10^{-11}$ Nm^2 kg^{-2}, find the time period of the satellite.

[**Ans.** 1.167 hour]

12. Calculate the gravitational potential and intensity of gravitational field of a thin spherical shell of mass 10 kg and radius 0.1 m at a point 0.1 m outside from the surface.

(*Nag. U.*, 2008, 2004) [**Ans.** -1.66×10^{-8} N/kg]

13. The period of revolution of phobos around Mars is 0.319 days and its mean distance from the Mars is 9.519×10^6 m. Calculate the mass of Mars (*i*) in kg and (*ii*) in units of earth's mass. Mass of the earth is 5.977×10^{24} kg.

[**Ans.** 67.33×10^{23} kg, 0.1127 times the mass of the earth]

14. Two satellites A and B of the same mass are orbiting the earth at altitudes R and $3R$ respectively, where R is the radius of the earth. Taking their orbits to be circular, obtain the ratios of their kinetic and potential energies.

$$\left[\textbf{Ans. } \frac{K.E.\ of\ A}{K.E.\ of\ B} = \frac{mMG}{4R} : \frac{mMG}{8R}\ i.e. \text{ as } 2:1\right]$$

15. Prove that force $\vec{F} = (2xy + z^2)\,\hat{i} + x^2\hat{j} + 2xz\hat{k}$ is a conservative force.

(*Meerut U.* 2000)

16. Show that $\vec{F} = (y^2 - x^2)\,\hat{i} + 3xy\,\hat{j}$ is a non-conservative force. (*Meerut U.* 2006)

3 CONSERVATION OF MOMENTA AND MECHANICS OF CENTRE OF MASS

INTRODUCTION

In the previous chapter we have considered a single particle upon which number of forces act. The path of the particle, its velocity, energy etc. can be determined by various equations, so far established. In this chapter, we will discuss the effect of different forces acting upon number of particles seperated from one another. To describe the motion we will have to write number of equations describing the motion of individual particle and solve these equations. This process is tedious and almost impossible as the particles involved are very very large in number. Such group of particles in collective motion is called 'system of particles'. The distance between the particles and mass of the system may or may not remain constant, when the group *i.e.* system of particles is in motion. The motion of such a system can be studied by considering the concept of 'Centre of mass' and is discussed here. Before, studing the motion of centre of mass, we will discuss first the prinicple of conservation of linear and angular momenta.

3.1 LINEAR MOMENTUM

The linear momentum of a body is defined as the product of its mass and linear velocity. If m is the mass of the body and $\vec{v}$ its velocity, then

Linear momentum $\qquad \vec{p} = m\vec{v}$

It is a *vector* quantity. Its units are kg ms^{-1}.

Principle of conservation of linear momentum. It states "*The total linear momentum of a system of particles free from the action of external forces and subjected only to their mutual interaction remains constant, no matter how complicated the forces are.*"

Mathematically, according to Newton's second law of motion $\vec{F} = \frac{d\vec{p}}{dt}$ where $\vec{F}$ is the applied external force acting on the system. If for an isolated system the external force is absent, $\vec{F} = 0$

$$\therefore \qquad \frac{d\vec{p}}{dt} = 0 \text{ or } \vec{p} = \text{a constant}$$

i.e., the total momentum of the system remains constant.

Example. When a bullet is fired from a gun, the bullet of mass m moves forward with a velocity $\vec{v}$ and the gun of mass M kicks backward with a velocity $\vec{V}$.

Momentum of the bullet in the forward direction $= m\vec{v}$

Momentum of the gun in the backward direction $= -M\vec{V}$

Now $$m\vec{v} = -M\vec{V} \quad \text{or} \quad m\vec{v} + M\vec{V} = 0$$

This shows that the total momentum of the bullet and the gun which was zero before the bullet was fired remains the same even after firing the bullet *i.e.*, the linear momentum is conserved.

3.2 LAW OF CONSERVATION OF LINEAR MOMENTUM

According to Newton's first law of motion "*A body remains at rest or continues to move with a uniform velocity if no external force is acting on it.*" According to Newton's second law of motion, "*The rate of change of momentum of a body is proportional to the force acting on it.*" Mathematically, if a particle of mass m is moving with a velocity $\vec{v}$ its linear momentum $\vec{p} = m\vec{v}$

When a force $\vec{F}$ is applied to it, the momentum changes at the rate $\frac{d\vec{p}}{dt}$

$$\therefore \qquad \vec{F} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt}$$

Clearly when $$\vec{F} = 0, \ \frac{d\vec{p}}{dt} = 0 \ \text{or} \ \vec{p} = m\vec{v} = \text{a constant}$$

i.e., a particle continues to move with a constant velocity. *Thus Newton's first law of motion is only a special case of second law.*

Now consider a system of two particles which are acted only by forces of interaction like Coulomb's forces or gravitational forces and *no external forces act on the system.* Let $\vec{F}_{12}$ be the force exerted by the first particle on the second, known as **action** and $\vec{F}_{21}$ the force exerted by the second on the first, known as **reaction**, then according to Newton's third law of motion, *action and reaction being equal and opposite.* $$\vec{F}_{12} = -\vec{F}_{21}$$

According to Newton's second law of motion

$$\vec{F}_{12} = \frac{d\vec{p}_2}{dt} = \frac{d(m_2\vec{v}_2)}{dt} = m_2\frac{d\vec{v}_2}{dt}$$

where m_2, $\vec{v}_2$ and $\vec{p}_2$ are the mass, velocity and momentum of the second particle.

Similarly $$\vec{F}_{21} = \frac{d\vec{p}_1}{dt} = \frac{d}{dt}(m_1\vec{v}_1) = m_1\frac{d\vec{v}_1}{dt}$$

where m_1, $\vec{v}_1$ and $\vec{p}_1$ are the mass, velocity and momentum of the first particle.

$$\because \qquad \vec{F}_{12} = -\vec{F}_{21} \qquad \vec{F}_{12} + \vec{F}_{21} = 0$$

or $$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = 0 \quad \text{or} \quad \frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = 0$$

Integrating, we have $\vec{p}_1 + \vec{p}_2 = \text{a constant}$ or $m_1\vec{v}_1 + m_2\vec{v}_2 = \text{a constant}$

The quantity $\vec{p}_1 + \vec{p}_2 = m_1\vec{v}_1 + m_2\vec{v}_2$ represents the total linear momentum of the system. Hence we conclude that

'If Newton's second and third law of motion hold good, the total linear momentum of the system of two particles remains constant'. The above law can be extended to a system of three or more interacting particles. The law of conservation of linear momentum is, therefore, a basic law and is stated as under.

"The total linear momentum of a system of particles free from the action of external forces and subjected only to their mutual interaction remains constant, no matter how complicated the forces are."

3.3 ANGULAR MOMENTUM

The angular momentum of a particle about a fixed point is defined as the moment of its linear momentum about that point. It is measured by the vector product of linear momentum $\vec{p} = m\vec{v}$ of the particle and its vector distance $\vec{r}$ from the fixed point in the inertial frame.

$\therefore$ Angular momentum $\quad \vec{J} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$

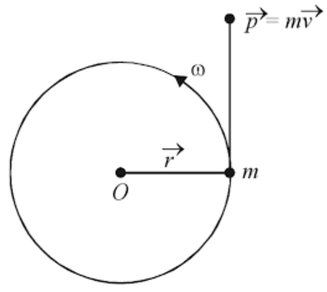

Fig. 3.1

Angular momentum being the vector product of two vectors is obviously a *vector* quantity. Its direction is perpendicular both to $\vec{r}$ and $\vec{p}$ or $\vec{v}$ as given by the right hand screw rule.

Angular momentum of a particle. In the case of a particle of mass m describing a circle of radius r with a uniform angular velocity of magnitude ω and linear velocity $v = r\omega$, the magnitude of the angular momentum $= mvr = mr^2\omega$ and its direction is perpendicular to the plane of the circle (Fig. 3.1).

In a general case also, the magnitude of the angular momentum of a particle is given by

$$|\vec{J}| = r\,(mv) = r\,(mr\omega) = mr^2\omega$$

$$= mr^2 \frac{d\theta}{dt} = mr^2\,\dot{\theta} \qquad ...(i)$$

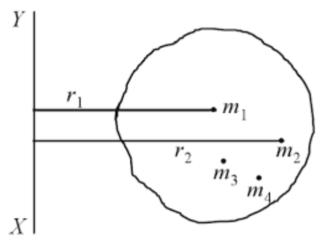

Fig. 3.2

Angular momentum of a rigid body. *The sum of the moments of the linear momentum of all the particles of a rotating rigid body about the axis of rotation is called its angular momentum.*

When a body is free to rotate about an axis the angular velocity of all the particles, at whatever distance they may be, is the same. Since the distance of the various particles from the axis of rotation is not the same their linear velocities will be different.

Consider the particles m_1, m_2... of the rigid body lying at distances r_1, r_2... from the axis of rotation XY and having magnitudes of linear velocities v_1, v_2 ... respectively fig. (3.2). If ω is the magnitude of angular velocity, then

Linear velocity of the particle $m_1 = v_1 = r_1\omega$

$\therefore$ Magnitude of linear momentum of the particle $m_1 = m_1 v_1 = m_1 r_1 \omega$

Hence the magnitude of moment of linear momentum or angular momentum of the particle m_1 about the axis of rotation $= m_1 v_1 r_1 = m_1 r_1^2 \omega$

Similarly the moment of linear momentum or angular momentum of the particle m_2 about the axis of rotation $= m_2 v_2 r_2 = m_2 r_2^2 \omega$

$\therefore$ Sum of the magnitudes of moments of linear momentum or angular momentum of all the particles $= m_1 r_1^2 \omega + m_2 r_2^2 \omega + ...$

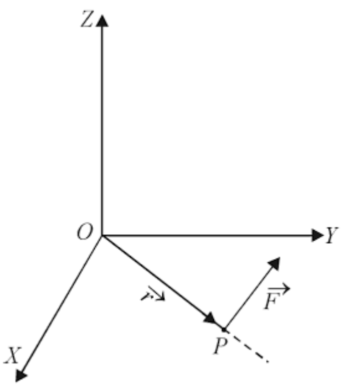

Fig. 3.3

or $$J = \Sigma\, m\, r^2\omega = I\omega$$

where $I = \Sigma mr^2$ = moment of inertia of the rigid body about the axis of rotation and J is the magnitude of the angular momentum.

Vectorially $\vec{J} = I\,\vec{\omega}$

Hence the angular momentum of a rigid body is the product of its moment of inertia and angular velocity.

Units. The unit of angular momentum is Kg m^2 s^{-1} or Joule sec.

Torque. *If a force $\vec{F}$ acts on a particle at a point P whose position with respect to the origin O of the inertial frame is given by the displacement vector $\vec{r}$, the torque $\vec{\tau}$ on the particle with respect to the origin O is defined as $\vec{\tau} = \vec{r} \times \vec{F}$.*

Torque is a vector quantity. Its direction is normal to the plane formed by $\vec{r}$ and $\vec{F}$ and the sense is given by the right hand rule for the cross product of two vectors.

Units. The unit of torque is Newton metre (Nm).

Torque as the time rate of change of angular momentum. The angular momentum.

$$\vec{J} = \vec{r} \times \vec{p}$$

Differentiating this expression with respect to time, we get

$$\frac{d\vec{J}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

But $$\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\,\vec{v} = m(\vec{v} \times \vec{v}) = 0$$

and $$\frac{dp}{dt} = \text{rate of change of linear momentum}$$
$$= \vec{F} \text{ the force acting on the particle}$$

$$\therefore \quad \frac{d\vec{J}}{dt} = \vec{r} \times \vec{F}$$

But $\vec{r} \times \vec{F}$ is the torque $\vec{\tau}$ acting on the particle

$$\therefore \quad \text{Torque} = \frac{d\vec{J}}{dt} = \vec{\tau} = \vec{r} \times \vec{F}$$

Hence torque is also defined as the time rate of change of angular momentum.

3.4 TORQUE ON A RIGID BODY

Torque is defined as the time rate of change of angular momentum. For a rigid body, Angular momentum $\vec{J} = I\,\vec{\omega}$

$$\therefore \quad \text{Torque } \tau = \frac{d\vec{J}}{dt} = I\frac{d\vec{\omega}}{dt} = I\,\vec{\alpha}$$

where $\vec{\alpha}$ is the angular acceleration.

∴ *Torque on a rigid body is the product of moment of inertia and angular acceleration.*

Torque $$\vec{\tau} = I\,\vec{\alpha} = I\,\frac{d\vec{\omega}}{dt}$$

If no torque acts on a body $\vec{\tau} = 0$

$\therefore \quad I\dfrac{d\vec{\omega}}{dt} = 0$ As I cannot be zero, $\dfrac{d\omega}{dt} = 0$ or $\vec{\omega}$ = is a constant.

$\therefore$ Angular velocity remains conserved.

Relation between Force and Torque. Torque is defined as the time rate of change of angular momentum.

$\therefore$ Torque $\qquad \vec{\tau} = \dfrac{d\vec{J}}{dt}$

but $\qquad \vec{J} = \vec{r} \times \vec{p}$

Now $\qquad \dfrac{d\vec{J}}{dt} = \dfrac{d}{dt}(r \times p) = \dfrac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \dfrac{d\vec{p}}{dt}$

But $\qquad \dfrac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = m(\vec{v} \times \vec{v}) = 0$

$\therefore \qquad \vec{\tau} = \vec{r} \times \dfrac{d\vec{p}}{dt} = \vec{r} \times \vec{F} \qquad \left[\because \vec{F} = \dfrac{d\vec{p}}{dt}\right]$

$\therefore$ Torque is the vector product of position vector $\vec{r}$ and force $\vec{F}$.

3.5 LAW OF CONSERVATION OF ANGULAR MOMENTUM

It states that "*When the total external torque acting on a system of particles is zero, the total angular momentum of the system remains constant*".

When there is no external torque acting on a system of particles $\vec{\tau} = 0$

Now torque $\tau = \dfrac{d\vec{J}}{dt}$ where $\vec{J}$ is the total angular momentum of the system.

$\therefore \qquad \dfrac{d\vec{J}}{dt} = 0 \quad$ or $\quad \vec{J}$ = a constant.

i.e. when there is no external torque acting on a system of particles, the total angular momentum of the system remains constant.

3.6 EXAMPLES OF CONSERVATION OF ANGULAR MOMENTA

(*i*) Suppose a person carrying heavy weights in his hands stretched out is standing on a rotating platform. When the person suddenly folds his arms, his moment of Inertia I decreases. As angular momentum $\vec{L} = I\vec{\omega}$ is constant, the angular velocity $\vec{\omega}$ of his body which is the same as the angular velocity of the rotating platform increases.

(*ii*) The angular speed of the inner layers of whirlwind (or tornado) is always very high. In a whirlwind, air from nearby regions gets concentrated in a small space thereby decreasing the moment of inertia I. As angular momentum $\vec{L} = I\vec{\omega}$ = constant, $\vec{\omega}$ increases and attains very high values as as I decreases.

(*iii*) The angular velocity of revolution of a planet around the Sun in an elliptic orbit increases when the planet comes closer to the Sun and vice-versa., in order to conserve angular momenta.

(*iv*) As the motion of the earth around the Sun is under gravitational force which is a central force, the angular momentum of the earth moving round the Sun is conserved. When the earth is at the *perihilion i.e.* closest to the Sun, the angular velocity is *maximum* and when the earth is at the *apehilion i.e.* farthest away from the Sun its angular velocity is *minimum*.

(*v*) During the course of the performance an ice-skater (or a ballet dancer) takes advantage of the principle of conservation of angular momentum by stretching out its arms and legs or vice-versa. On

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doing so the moment of inertia increases / decreases and hence angular velocity decreases/increases.

(*vi*) The shape of galaxy.

Galaxies are physical systems, containing anything from 10^9 to 10^{12} stars with a lot of free gas, bound by gravitational attraction, and free to rotate about their own axes of symmetry. The contraction of the gas in a galaxy continues in order to conserve its angular momentum until equilibrium is attained when the inward pull due to gravitational force on it just balanced by the outward pull due to centrifugal force, *i.e.*,

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

3.7 ANGULAR MOMENTUM REMAINS CONSTANT UNDER A CENTRAL FORCE

From the relation $\vec{\tau} = \frac{d\vec{J}}{dt} = \vec{r} \times \vec{F}$ we find that the torque acting on the particle is zero and hence the angular momentum remains conserved if (*i*) its position vector $\vec{r} = 0$ (*ii*) the force $\vec{F}$ applied to it is zero but if $\vec{r} \neq 0$ and $\vec{F} \neq 0$ the third condition that must be satisfied for $\vec{J}$ = constant is, (*iii*) the direction of both $\vec{r}$ and $\vec{F}$ should be the same *i.e.*, the line of action of the force passes through the fixed or the reference point. The third condition is satisfied in the case of a *central force i.e.*, a force which is always directed towards or away from a fixed point.

Consider a particle subjected to a central force depending upon the distance from the fixed point. Such a force can be represented as $\vec{F} = f(r)\hat{r}$

where $f(r)$ is some scalar function of the distance and $\hat{r}$ is a unit vector along $\vec{r}$ given by $\frac{\vec{r}}{|\vec{r}|}$.

$$\therefore \quad \vec{\tau} = \frac{d\vec{J}}{dt} = \vec{r} \times \vec{F} = r \times f(r)\hat{r} = f(r)\,(\vec{r} \times \hat{r}) = f(r)\left(\vec{r} \times \frac{\vec{r}}{|\vec{r}|}\right) = 0 \quad [\because \vec{r} \times \vec{r} = 0]$$

$\therefore$ Torque acting on a particle under the influence of a central force = 0

$$\because \qquad \vec{\tau} = \frac{d\vec{J}}{dt} = 0 \quad \therefore \quad \vec{J} = \text{a constant}$$

Hence the angular momentum of a particle under the influence of a central force always remains constant.

3.8 ISOTROPY AND ROTATIONAL INVARIANCE OF SPACE

Space is *isotropic i.e.*, if we move in any direction from a point there is nothing to distinguish one direction from another. For example, if we keep a particle P_1 fixed and move another particle P_2 on the surface of a sphere with P_1 as centre the interaction energy between the two will be the same for all positions of P_2 on the surface of the sphere. *Isotropy* thus means *rotational* invariance of the properties of free space.

The term rotational invariance of space further implies that the potential energy of interaction between two particles is an *invariant i.e.*, does not change if position co-ordinates of both the particles are rotated through the same angle about an arbitrary axis.

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3.9 CONSERVATION OF ANGULAR MOMENTUM AND ROTATIONAL INVARIANCE (ISOTROPY OF SPACE)

The space simultaneously possesses the property of

(*i*) Linear uniformity and (*ii*) Rotational invariance

Therefore the form of potential energy function $U(r_1, r_2) = U(r)$ where

$r = \left|\vec{r}\right| = \left|\vec{r}_1 - \vec{r}_2\right|$, $\vec{r}_1$ and $\vec{r}_2$ being the space co-ordinates of the two interacting particles P_1 and P_2 respectively, satisfies both the conditions stated above. The potential U depends upon the magnitude of the separation of the two points $\left|\vec{r}_1 - \vec{r}_2\right|$ but is independent of the actual value of the space co-ordinates and their direction.

The force on the particle $P_1 = \vec{F}_{21} = -\left(\frac{\partial U}{\partial r_1}\right)\hat{r} = -\left(\frac{\partial U}{\partial r}\right)\left(\frac{\partial r}{\partial r_1}\right)\hat{r}$

The force on the particle $P_2 = \vec{F}_{12} = -\left(\frac{\partial U}{\partial r_2}\right)\hat{r} = -\left(\frac{\partial U}{\partial r}\right)\left(\frac{\partial r}{\partial r_2}\right)\hat{r}$

where $\hat{r}$ is a unit vector along $\vec{r}$ *i.e.*, the direction along which either force of interaction acts.

Two cases arise

(*i*) $r_1 > r_2$

In such a case $\quad r = r_1 - r_2 \quad \therefore \frac{\partial r}{\partial r_1} = 1$ and $\frac{\partial r}{\partial r_2} = -1$

$\therefore \quad \vec{F}_{21} = -\left(\frac{\partial U}{\partial r}\right)\hat{r}$ and $\vec{F}_{12} = +\left(\frac{\partial U}{\partial r}\right)\hat{r}$

Hence $\quad \vec{F}_{12} = -\vec{F}_{21}$

(*ii*) $r_2 > r_1$

In such a case $\quad r = r_2 - r_1 \quad \therefore \frac{\partial r}{\partial r_1} = -1$ and $\frac{\partial r}{\partial r_2} = +1$

$\therefore \quad \vec{F}_{21} = +\left(\frac{\partial U}{\partial r}\right)\hat{r}$ and $\vec{F}_{12} = -\left(\frac{\partial U}{\partial r}\right)\hat{r}$

Hence again $\quad \vec{F}_{12} = -\vec{F}_{21}$

Thus the potential function $U(r)$ leads to Newton's third law of the motion whether

$$r_1 > r_2 \text{ or } r_2 > r_1.$$

As proved above

$$\vec{F}_{21} = \pm\left(-\frac{\partial U}{\partial r}\right)\hat{r} \quad \text{and} \quad \vec{F}_{12} = \pm\left(-\frac{\partial U}{\partial r}\right)\hat{r}$$

If $-\frac{\partial U}{\partial r} = F(r)$, $\vec{F}_{12}$ as well as $\vec{F}_{21}$ can be expressed as $\pm F(r)\,\hat{r}$. In other words, the force will depend only on r the distance between the centres of the two particles. It is, therefore, a **central** force. Thus we find that rotational invariance of space requires motion under a central force.

Now the torque $\vec{\tau}$ for a force about the centre of the force is given by

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times F(r)\hat{r} = F(r)\vec{r} \times \hat{r} = F(r)\, r\left[\hat{r} \times \hat{r}\right] = 0$$

$\therefore$ For such a force, torque $\tau = 0$

But torque is the rate of change of angular momentum $\vec{J}$ $\quad \therefore\ \vec{\tau} = \dfrac{d\vec{J}}{dt} = 0$

Integrating we get, $\vec{J}$ = a constant. This proves the law of conservation of angular momentum.

Thus the rotational invariance or isotropy of space shows that

(*i*) There is a central force between the two particles interacting with each other.

(*ii*) $\vec{F}_{21} = -\vec{F}_{12}$ *i.e.*, Newton's third law of motion holds good.

These two properties, on combination, show that the angular momentum is a constant. *Hence the rotational invariance of space leads to the principle of conservation of angular momentum.*

3.10 ANGULAR IMPULSE

If an unbalanced external torque $\vec{\tau}$ is applied to a body rotating with an angular velocity $\vec{\omega}_1$ for a time t so that the angular velocity changes to $\vec{\omega}_2$ then

Angular acceleration $\quad \vec{\alpha} = \dfrac{\text{Change in angular velocity}}{\text{time}}$

$$= \frac{\vec{\omega}_2 - \vec{\omega}_1}{t} \qquad ...(i)$$

But torque $\quad \vec{\tau} = I\vec{\alpha} = \dfrac{I\vec{\omega}_2 - I\vec{\omega}_1}{t}$ $\quad$ or

or $\quad \vec{\tau}\, t = I\vec{\omega}_2 - I\vec{\omega}_1$

The product of the torque and the time for which it acts is called **angular impulse.** It is measured by the total change in angular momentum that the rotating body undergoes.

If the torque acts for a small time dt and brings about a small change in angular velocity $d\omega$, then,

Angular impulse $\quad \vec{\tau}\, dt = I\, d\vec{\omega}$

Angular momentum and linear momentum. Linear momentum is the product of mass and linear velocity, whereas angular momentum is the product of moment of inertia and angular velocity,

Linear momentum $\quad \vec{p} = m\vec{v} \qquad$ Angular momentum $\vec{J} = I\vec{\omega}$

Angular impulse and linear impulse. Angular impulse is the product of the torque and the time. It is also the total change in angular momentum. Linear impulse is the product of force and time.

It is also the total change in linear momentum.

Linear impulse $\vec{F}\, dt = m\, d\vec{v} \qquad$ Angular impulse $\vec{\tau}\, dt = I\, d\vec{\omega}$

3.11 HOMOGENEITY AND ISOTROPY OF TIME

Time. According to Newton '*Absolute true and mathematical time of itself and from its own nature, flows equally without relation to anything external and is otherwise called duration.*'

Properties. (*i*) One dimensionality. Time is one dimensional. It flows only in one direction. It is, therefore, specified by a single variable t. It is independent of space.

(*ii*) Homogeneity. Time is homogeneous *i.e.*, it flows uniformly. The results of an experiment do not change when we change the time of an experiment. In other words, '*The result of an experiment is independent of the change in the origin of time*'. This property is known as homogeneity of time.

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(iii) **Isotropy.** Theoretically time is isotropic *i.e.*, the laws of Physics remain unaltered by changing $+t$ to $-t$.

3.12 CONSERVATION OF ENERGY FROM HOMOGENEITY OF TIME AND NEWTON'S SECOND LAW OF MOTION

Time is homogeneous *i.e.*, it flows uniformly.

The total energy E of a system is given by $E = U + K$

where U is the potential energy and K the kinetic energy.

The Coulomb force between two charges q_1 and q_2, a distance r apart is given by

$$\vec{F}_c = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 \cdot q_2}{r^2}\right)\hat{r} = k\frac{q_1 \cdot q_2}{r^2}\,\hat{r}$$

where k is a constant $= 1/4\pi\varepsilon_0$.

Similarly the gravitational force between two masses m_1 and m_2, a distance r apart, is also given by

$$\vec{F}_g = G\left(\frac{m_1 . m_2}{r^2}\right)\hat{r}$$

where G is the gravitational constant.

We find that in the above expressions for force, time does not appear *explicitly*. By *explicit* dependence on time we mean that time should occur as such directly in the expression for force.

Now, for a conservative system $\vec{F} = -\left(\frac{\partial U}{\partial r}\right)\hat{r}$

In other words, U is a function of $\vec{F}$ and $\vec{r}$. If time flows uniformly, the forces acting at a point do not depend explicitly on time *i.e.* force $\vec{F}$ is a function of $\vec{r}$ only. Since U is a function of $\vec{F}$ and $\vec{r}$ it is also function of $\vec{r}$ only. The principle of homogeneity of time, therefore, states $\frac{\partial U}{\partial t} = 0$.

The expression for kinetic energy $K = \frac{1}{2}mv^2$ also does not depend explicitly on time.

$$\therefore \qquad \frac{\partial K}{\partial t} = 0$$

If total energy E is a function of $\vec{r}$ and t, then $E = E(\vec{r}, t)$

and
$$dE = \frac{\partial E}{\partial r}dr + \frac{\partial E}{\partial t}dt = \frac{\partial}{\partial t}(U + K)\,dr + \frac{\partial}{\partial r}(U + K)\,dt$$

$$= \left(\frac{\partial U}{\partial r} + \frac{\partial K}{\partial r}\right)dr + \left(\frac{\partial U}{\partial t} + \frac{\partial K}{\partial t}\right)dt = \left[\left(\frac{\partial U}{\partial r}\right) + \left(\frac{\partial K}{\partial r}\right)\right]dr \quad ...(i)$$

as
$$\frac{\partial U}{\partial t} = 0 \quad \text{and} \quad \frac{\partial K}{\partial t} = 0$$

From (i), we have
$$\frac{dE}{dt} = \left(\frac{\partial U}{\partial r} + \frac{\partial K}{\partial r}\right)\frac{dr}{dt}$$

Now
$$\frac{\partial U}{\partial r} = -F \quad \text{and} \quad \frac{\partial K}{\partial r} = \frac{\partial}{\partial r}\left(\frac{1}{2}mv^2\right) = mv\frac{\partial v}{\partial r}$$

Since v does not depend explicitly on time. $\frac{\partial v}{\partial r} = \frac{dv}{dr}$

$$\therefore \qquad \frac{\partial K}{\partial r} = mv\frac{dv}{dr} = m\frac{dr}{dt}\cdot\frac{dv}{dr} = m\frac{dv}{dt} = ma$$

Hence
$$\frac{dE}{dt} = (-F + ma)\frac{dr}{dt}$$

The expression within brackets is zero as $F = ma$, according to Newton's second law of motion.

$$\therefore \qquad \frac{dE}{dt} = 0 \text{ or } E = \text{a constant.}$$

Thus we find that the homogeneity of time (*i.e.*, if time flows uniformly) and Newton's second law of motion lead to the principle of conservation of energy.

3.13 CENTRE OF MASS

For a system of n particles, suppose there is one point for the system where entire mass of the system can be assumed to be concentrated. If all the external forces are assumed to act on this point, then its motion is similar to that of the system upon which the same external forces acting. The point is called "the centre of mass" of the system.

Suppose we have a system of total mass M consisting of n particles or mass points of masses $m_1, m_2, \ldots\ldots m_i \ldots\ldots m_n$ whose position vectors are represented by $\vec{r}_1, \vec{r}_2 \ldots\ldots \vec{r}_i \ldots\ldots \vec{r}_n$ respectively, then the centre of mass of the system is defined as the point whose position vector $\vec{R}$ is given by

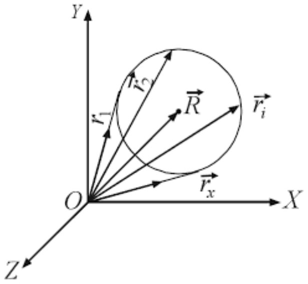

Fig. 3.4

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_i\vec{r}_i + \ldots + m_n\vec{r}_n}{m_1 + m_2 + \ldots\ldots + m_i + \ldots + m_n}$$

$$= \frac{\sum m_i\vec{r}_i}{\sum m_i} = \frac{\sum m_i\vec{r}_i}{M}$$

as $\Sigma\, m_i = M$ the total mass of the system. If the centre of mass coincides with the origin of the system, then

$$\vec{R} = 0 \text{ vector, } \Sigma m_i\vec{r}_i = 0 \text{ vector}$$

The centre of mass is thus defined as a point in space such that the vector sum of the moments of the mass points around it is zero.

Velocity of Centre of Mass. Let us consider the motion of the system consisting of n particles and total mass M assuming that the mass of the system remains constant *i.e.,* no mass enters or leaves the system.

$$\therefore \qquad M\vec{R} = \Sigma m_i\vec{r}_i = m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + \ldots + m_i\vec{r}_i + \ldots\ldots + m_n\vec{r}_n$$

Differentiating with respect to time, we get

$$M\frac{d\vec{R}}{dt} = m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + \ldots + m_i\frac{d\vec{r}_i}{dt} + \ldots\ldots + m_n\frac{d\vec{r}_n}{dt}$$

But
$$\frac{d\vec{R}}{dt} = \vec{V} \text{ the velocity of centre of mass}$$

and
$$\frac{d\vec{r}_1}{dt} = \vec{v}_1,\ \frac{d\vec{r}_2}{dt} = \vec{v}_2 \ldots\ldots \frac{d\vec{r}_i}{dt} = \vec{v}_i \text{ and } \frac{d\vec{r}_n}{dt} = v_n$$

which represent the velocities of individual particles.

$$\therefore \qquad M\vec{V} = m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots\ldots + m_i\vec{v}_i + \ldots\ldots + m_n\vec{v}_n = \Sigma m_i\vec{v}_i \qquad \ldots(i)$$

The velocity of centre of mass is given by

$$\vec{V} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots\ldots + m_i\vec{v}_i + \ldots\ldots + m_n\vec{v}_n}{M} = \frac{\Sigma m_i\vec{v}_i}{M} \qquad \ldots(ii)$$

From relation (*i*) we also find the *vector sum of the linear momenta of the individual particles i.e., the total linear momentum of the system is equal to the product of the total mass of the system and the velocity of centre of mass.*

In the absence of any external force, the total linear momentum of the system $\vec{P}$ is conserved

$$\therefore \qquad M\vec{V} = \Sigma m_i\vec{v}_i = \vec{P} = \text{a constant}$$

or $\qquad \vec{V}$ = a constant vector

Hence the velocity of the centre of mass of a system remains constant if no external force is applied to it.

Acceleration. Differentiating equation (*i*) with respect to time *t*, we have

$$M\frac{d\vec{V}}{dt} = m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} + \ldots\ldots + m_i\frac{d\vec{v}_i}{dt} + \ldots\ldots + m_n\frac{d\vec{v}_n}{dt}$$

or

$$M\vec{a} = m_1\vec{a}_1 + m_2\vec{a}_2 + \ldots\ldots + m_i\vec{a}_i + \ldots + m_n\vec{a}_n$$

where $\vec{a}$ is the acceleration of the centre of mass and $\vec{a}_1, \vec{a}_2$ etc. are the accelerations of individual particles.

The acceleration of the centre of mass is given by

$$\vec{a} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + \ldots\ldots + m_i\vec{a}_i + \ldots + m_n\vec{a}_n}{M} = \frac{\Sigma m_i\vec{a}_i}{M} \qquad \ldots(iii)$$

According to Newton's second law of motion

$$m_1\vec{a}_1 = \vec{F}_1,\ m_2\vec{a}_2 = \vec{F}_2 \ldots\ldots m_i\vec{a}_i = \vec{F}_i \ldots\ldots\ldots m_n\vec{a}_n = \vec{F}_n$$

The external forces acting on different particles

$$\therefore \qquad M\vec{a} = \vec{F}_1 + \vec{F}_2 + \ldots + \vec{F}_i + \ldots + \vec{F}_n$$

Thus the product of the total mass of a system and the vector acceleration of the centre of mass is equal to the vector sum of all the external forces acting on the individual particles of the system.

When no external force is acting

$$\therefore \qquad \vec{F}_1 + \vec{F}_2 + \ldots\ldots + \vec{F}_i + \ldots\ldots + \vec{F}_n = \Sigma\vec{F}_i = 0 \quad \therefore M\vec{a} = 0 \text{ or } \vec{a} = 0 \text{ vector}$$

Hence in the absence of an external force, the acceleration of the centre of mass is zero and therefore the velocity is a constant vector.

3.14 TOTAL LINEAR MOMENTUM ABOUT THE CENTRE OF MASS

Let C be the centre of mass of a number of particles of mass $m_1, m_2 \ldots\ldots m_i \ldots\ldots m_n$ and $\vec{r}_1, \vec{r}_2 \ldots\ldots \vec{r}_i \ldots\ldots \vec{r}_n$ their position vectors with respect to the origin O, then

Position vector of centre of mass

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots\ldots + m_i\vec{r}_i + \ldots\ldots + m_n r_n}{m_1 + m_2 + \ldots\ldots + m_i + \ldots + m_n} = \frac{\Sigma m_i\vec{r}_i}{\Sigma m_i} = \frac{\Sigma m_i\vec{r}_i}{M} \qquad \ldots(i)$$

where $\qquad M = m_1 + m_2 + \ldots\ldots + m_i + \ldots\ldots + m_n = \Sigma m_i$

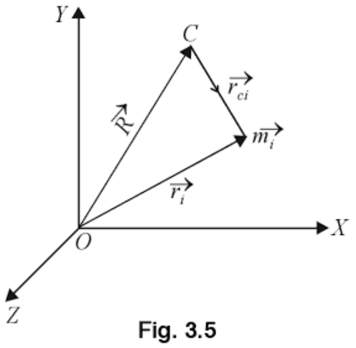

Fig. 3.5

Consider one of the particles of mass m_i having position vector $\vec{r}_i$ then its position with respect to the centre of mass C, $\vec{r}_{ci}$ is given by $\vec{R} + \vec{r}_{ci} = \vec{r}_i$

The position of all other particles with respect to the centre of mass C is also given by similar relations.

Substituting $\vec{r}_i = \vec{R} + \vec{r}_{ci}$ in Equation (*i*), we get

$$M\vec{R} = \Sigma m_i\,(\vec{R} + \vec{r}_{ci}) = \Sigma m_i\,\vec{R} + \Sigma m_i\,\vec{r}_{ci} = M\vec{R} + \Sigma m_i\,\vec{r}_{ci}$$

or $$\Sigma m_i\,\vec{r}_{ci} = 0 \qquad ...(ii)$$

i.e., the sum of the products of mass and position vector of all the particles about the centre of mass is zero.

Differentiating relation (*ii*) with respect to t we get $\Sigma m_i \dfrac{d\vec{r}_{ci}}{dt} = \Sigma m_i\,\vec{v}_{ci} = 0$

because $\dfrac{d\vec{r}_{ci}}{dt} = \vec{v}_{ci}$ = velocity of the particle of mass m_i relative to centre of mass. $\Sigma m_i\,\vec{v}_{ci}$, therefore gives the total linear momentum of all the particles about the centre of mass.

Thus the total linear momentum of the system of particles about the centre of mass is zero.

Note. The centre of mass frame is therefore, sometimes called zero momentum frame.

3.15 SYSTEM OF TWO PARTICLES

(*i*) Position vector of centre of mass. The position of centre of mass for a number of particles is mathematically given by

$$\vec{R} = \frac{\Sigma m_i\,\vec{r}_i}{\Sigma m_i}$$

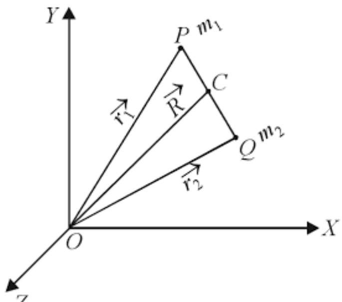

Fig. 3.6

For two mass points m_1 and m_2, $\vec{R} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$ ...(*i*)

where $\vec{r}_1$ and $\vec{r}_2$ are the vector distances of the particles of mass m_1 and m_2 respectively from the origin O.

If, however the centre of mass lies at the origin of the co-ordinate system as in Fig. 3.6 (A)

$$\vec{R} = 0$$

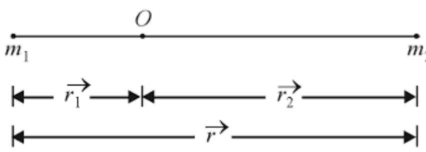

Fig. 3.6(A)

$$\therefore \quad m_1\vec{r}_1 + m_2\vec{r}_2 = 0 \text{ or } \frac{m_1}{m_2} = \frac{-\vec{r}_2}{\vec{r}_1}$$

Thus the centre of mass divides the line joining the two masses in the inverse ratio of the masses i.e., the heavier mass lies nearer the centre of mass of the two particle system. It should be noted that if O is the origin and $\vec{r}_2$ is positive, $\vec{r}_1$ must be negative.

(*ii*) Velocity of centre of mass. Rewriting equation (*i*) we have $m_1\vec{r}_1 + m_2\vec{r}_2 = (m_1 + m_2)\,\vec{R}$

Differentiating with respect to time we have $m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2 = (m_1 + m_2)\,\dot{\vec{R}}$

or Velocity of centre of mass $$\dot{\vec{R}} = \frac{m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2}{m_1 + m_2} \qquad ...(ii)$$

or
$$\vec{V} = \frac{m_1 \vec{v_1} + m_2 \vec{v_2}}{m_1 + m_2} \qquad ...(iii)$$

where
$$\vec{V} = \dot{\vec{R}}, \quad \vec{v_1} = \dot{\vec{r_1}} \text{ and } \vec{v_2} = \dot{\vec{r_2}}.$$

(*iii*) Acceleration of centre of mass. Differentiating equation (*ii*) again with respect to time, we have the acceleration of the centre of mass

$$\ddot{\vec{R}} = \frac{m_1 \ddot{\vec{r_1}} + m_2 \ddot{\vec{r_2}}}{m_1 + m_2} \text{ or } \vec{a} = \frac{m_1 \vec{a_1} + m_2 \vec{a_2}}{M} \qquad ...(iv)$$

where
$$\vec{a} = \ddot{\vec{R}}, \quad \vec{a_1} = \ddot{\vec{r_1}}, \quad \vec{a_2} = \ddot{\vec{r_2}} \text{ and } M = m_1 + m_2$$

(*iv*) Linear momentum of centre of mass. From equation (*iii*) we have

$$(m_1 + m_2)\vec{V} = m_1 \vec{v_1} + m_2 \vec{v_2}$$

or
$$M\vec{V} = m_1 \vec{v_1} + m_2 \vec{v_2} \qquad [\text{where } M = m_1 + m_2]$$

or
$$\vec{P} = \vec{p_1} + \vec{p_2}$$

where $\vec{P}$ is the linear momentum of the centre of mass and $m_1 \vec{v_1}$ and $m_2 \vec{v_2}$ the linear momentum of masses m_1 and m_2 respectively.

Hence the linear momentum of a system of two particles is equal to linear momentum of the centre of mass.

3.16 EQUATION OF MOTION OF CENTRE OF MASS

Consider two mass points m_1 and m_2 as shown in Fig. 3.7. If c.m. is the centre of mass for these two mass points and $\vec{R}$ the radius vector for the centre of mass, then

$$\vec{R} = \frac{m_1 \vec{r_1} + m_2 \vec{r_2}}{m_1 + m_2}$$

where $\vec{r_1}$ is the radius vector for mass point m_1 and $\vec{r_2}$ that for mass point m_2.

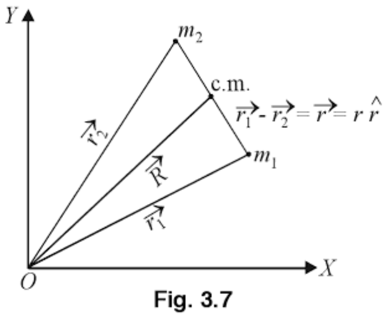

Fig. 3.7

Equivalent one body problem. Suppose there is no *external* force acting on the system and the only forces are those of mutual interaction, then the velocity of the centre of mass is a *constant.* As the centre of mass must be on the line joining m_1 and m_2, the force on m_1 due to m_2 as well as the force on m_2 due to m_1 are both directed towards the centre of mass. Hence these forces are *central forces.*

If we denote the force on m_1 as $\vec{F_{12}} = F(r)\hat{r}$ the force on m_2 being equal and opposite will be denoted by $\vec{F_{21}} = -F(r)\hat{r}$.

$\therefore$
$$m_1 \frac{d^2 \vec{r_1}}{dt^2} = \vec{F_{12}} = F(r)\hat{r} \text{ or } \frac{d^2 \vec{r_1}}{dt^2} = \frac{1}{m_1} F(r)\hat{r} \qquad ...(i)$$

and
$$m_2 \frac{d^2 \vec{r_2}}{dt^2} = \vec{F_{21}} = -F(r)\hat{r} \text{ or } \frac{d^2 \vec{r_2}}{dt^2} = -\frac{1}{m_2} F(r)\hat{r} \qquad ...(ii)$$

Subtracting (*ii*) from (*i*) we have $\dfrac{d^2 \vec{r_1}}{dt^2} - \dfrac{d^2 \vec{r_2}}{dt^2} = \left(\dfrac{1}{m_1} + \dfrac{1}{m_2}\right) F(r)\hat{r}$

Now $\quad \dfrac{d^2 \vec{r_1}}{dt^2} - \dfrac{d^2 \vec{r_2}}{dt^2} = \dfrac{d^2}{dt^2}(\vec{r_1} - \vec{r_2}) = \dfrac{d^2 \vec{r}}{dt^2} \qquad [\because \vec{r} = \vec{r_1} - \vec{r_2}]$

If we put $\quad \dfrac{1}{\mu} = \dfrac{1}{m_1} + \dfrac{1}{m_2} = \dfrac{m_1 + m_2}{m_1 m_2}$ or $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$

then
$$\frac{d^2 \vec{r}}{dt^2} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) F(r)\hat{r} = \frac{1}{\mu} F(r)\hat{r} \text{ or } \mu \ddot{\vec{r}} = F(r)\hat{r} \qquad ...(iii)$$

i.e., the system behaves as a single particle of mass $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$

Here μ is known as the **reduced mass** of the system and acts at a point known as *centre of mass* which divides the line joining the two masses in the inverse ratio of the masses.

The relation $\mu \ddot{\vec{r}} = F(r)\hat{r}$ gives the **'equation of motion'** of the centre of mass of a particle having mass equal to reduced mass μ at a vector distance $\vec{r}$ from one of the particles to the other and shows that two separate equations of motion (*i*) and (*ii*) have been reduced to a single equation involving reduced mass.

We have thus reduced the two body problem to a one body problem.

3.17 MOTION OF REDUCED MASS UNDER INVERSE SQUARE FORCE

The most familiar example of inverse square force is the gravitational force. For forces of gravitational attraction between the two mass points m_1 and m_2

$$\vec{F}(r) = -\frac{G m_1 m_2}{r^2}\hat{r} \quad \text{or} \quad \mu \frac{d^2 \vec{r}}{dt^2} = \mu \ddot{\vec{r}} = -\frac{G m_1 m_2}{r^2}\hat{r}$$

or
$$\frac{m_1 m_2}{m_1 + m_2}\ddot{\vec{r}} = -\frac{G m_1 m_2}{r^2}\hat{r}$$

$\therefore$
$$\ddot{\vec{r}} = -\frac{GM}{r^2}\hat{r} \text{ where } M = m_1 + m_2$$

This is clearly the equation of motion of a particle of **unit** mass at a vector distance r (equal to the distance between the two particles) from a fixed mass $M = m_1 + m_2$ exerting a force of attraction on it. Further the acceleration of one mass with respect to the other mass will appear to be the same but in opposite direction whether the observer is at mass m_1 or m_2.

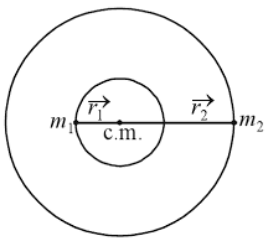

Fig. 3.8

Motion in centre of mass frame. In the absence of any external forces acting on the system, its total linear momentum is conserved. Hence the velocity of centre of mass remains constant in an inertial frame. However, in the centre of mass frame the velocity of centre of mass is *zero.* The two particles or bodies move around the centre of mass in such a way that the vector distance $\vec{r}$ between them remains constant. Hence to an observer at the centre of mass the heavier mass seems to describe a circle with smaller radius and the lighter mass seems to describe a circle with larger radius because $\dfrac{r_2}{r_1} = -\dfrac{m_1}{m_2}$.

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3.18 CENTRE OF MASS AND REDUCED MASS OF TWO PARTICLES

For two particles or mass points m_1 and m_2 lying at vector distances $\vec{r}_1$ and $\vec{r}_2$ from the origin, the position $\vec{R}$ of centre of mass is given by

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \qquad \text{[See **Article 3.15. Eq.** (}i\text{)]}$$

The centre of mass of two particles lies on the line joining the two particles.

When the centre of mass lies at the origin $\vec{R} = 0$ and

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0 \text{ or } \frac{m_1}{m_2} = -\frac{\vec{r}_2}{\vec{r}_1}$$

Thus the centre of mass is a point which divides the line joining the two particles in the inverse ratio of masses.

Reduced mass. When there is no external force acting on the system of two particles and the only forces are those of mutual interaction then the system behaves as a single particle of mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$.

In such a case μ is known as **reduced mass** of the system. *It acts at a point known as centre of mass which divides the line joining the two particles in the inverse ratio of masses.*

Importance in Physics. (*i*) The importance of centre of mass and reduced mass lies in the fact that instead of having two separate equations of motion

$$m_1 \ddot{\vec{r}}_1 = \vec{F}_{12} = \vec{F}(r)\hat{r} \quad \text{and} \quad m_2 \ddot{\vec{r}}_2 = \vec{F}_{21} = -\vec{F}(r)\hat{r}$$

we have a single equation of motion involving reduced mass μ given by $\mu \ddot{\vec{r}} = F(r)\hat{r}$ where $\vec{r} = (\vec{r}_1 - \vec{r}_2)$ thereby reducing the two body problem to a one body problem.

(*ii*) When the centre of mass is taken as the origin of the co-ordinate system $\vec{R} = 0$, the velocity of centre of mass $V = \dot{\vec{R}} = \frac{d\vec{R}}{dt} = 0$ and linear momentum $\vec{P} = \mu\vec{V}$ of the system is also = 0; μ being the reduced mass. This fact is made use of in having a *centre of mass frame of reference* for the study of collision phenomenon between particles.

3.19 INTERACTING TWO BODY SYSTEM REDUCED TO ONE BODY PROBLEM

(*i*) Change in Total Energy. Consider two bodies of masses m_1 and m_2 at positions having radius vectors $\vec{r}_1$ and $\vec{r}_2$ respectively, then

Position vector of $m_1 = \vec{r}_1$ $\quad \therefore$ Velocity of $m_1 = \dot{\vec{r}}_1$

Position vector of $m_2 = \vec{r}_2$ $\quad \therefore$ Velocity of $m_2 = \dot{\vec{r}}_2$

Distance between m_1 and $m_2 = \vec{r}_1 - \vec{r}_2 = \vec{r} = r\hat{r}$

The gravitational force of attraction $\vec{F}(r)\hat{r}$ between the two masses m_1 and m_2 is given by

$$F(r)\hat{r} = -\frac{G m_1 m_2}{r^2}\hat{r}$$

Taking magnitudes only, we have $F(r) = -\frac{G m_1 m_2}{r^2}$

The gravitational potential energy U is, therefore, given by

$$U = -\int F(r)dr = -\int -\frac{Gm_1m_2}{r^2}dr = \frac{-Gm_1m_2}{r} + K$$

where K is the constant of integration.

For $$r = \infty,\ U = 0 \quad \therefore\ K = 0$$

Hence $$U = \frac{-Gm_1m_2}{r} = \frac{-Gm_1m_2}{\left|\vec{r}_1 - \vec{r}_2\right|}$$

i.e, the gravitational potential energy between two mass points varies inversely as the distance between them.

Hence total energy of the two body system E_2 = Kinetic energy of m_1 + Kinetic energy of m_2 + Potential energy of interaction due to gravitational force between them.

$$\therefore \qquad E_2 = \frac{1}{2} m_1 \left|\dot{\vec{r}}_1\right|^2 + \frac{1}{2} m_2 \left|\dot{\vec{r}}_2\right|^2 - \frac{G\,m_1m_2}{\left|\vec{r}_1 - \vec{r}_2\right|} \qquad ...(i)$$

If we reduce the two body system to an *equivalent one body system,* then

Effective reduced mass $\mu = \dfrac{m_1m_2}{m_1 + m_2}$

$\therefore$ Total energy of the equivalent one body system E_1 = Kinetic energy of effective (reduced) mass + Potential energy of interaction due to gravitational force

$$\therefore \qquad E_1 = \frac{1}{2}\mu\left|\dot{\vec{r}}\right|^2 - \frac{Gm_1m_2}{\left|\vec{r}_1 - \vec{r}_2\right|} = \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}\left|\dot{\vec{r}}_1 - \dot{\vec{r}}_2\right|^2 - \frac{G\,m_1m_2}{\left|\vec{r}_1 - \vec{r}_2\right|}$$

$$= \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}\left[\left|\dot{\vec{r}}_1\right|^2 + \left|\dot{\vec{r}}_2\right|^2 - 2\dot{\vec{r}}_1 \cdot \dot{\vec{r}}_2\right] - G\frac{m_1m_2}{\left|\vec{r}_1 - \vec{r}_2\right|} \qquad ...(ii)$$

$\therefore$ Change in total energy

$$E_1 - E_2 = \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}\left[\left|\dot{\vec{r}}_1\right|^2 + \left|\dot{\vec{r}}_2\right|^2 - 2\dot{\vec{r}}_1 \cdot \dot{\vec{r}}_2\right] - \frac{Gm_1m_2}{\left|\vec{r}_1 - \vec{r}_2\right|}$$

$$-\frac{1}{2} m_1\left|\dot{\vec{r}}_1\right|^2 - \frac{1}{2} m_2\left|\dot{\vec{r}}_2\right|^2 + \frac{G\,m_1m_2}{\left|\vec{r}_1 - \vec{r}_2\right|}$$

$$= \frac{1}{2}\frac{1}{m_1 + m_2}\left[m_1\,m_2\left|\dot{\vec{r}}_1\right|^2 + m_1m_2\left|\dot{\vec{r}}_2\right|^2 - 2\,m_1m_2\,(\dot{\vec{r}}_1 \cdot \dot{\vec{r}}_2)\right]$$

$$-\frac{1}{2}\frac{1}{m_1 + m_2}\left[m_1^2\left|\dot{\vec{r}}_1\right|^2 + m_1m_2\left|\dot{\vec{r}}_1\right|^2 + m_2^2\left|\dot{\vec{r}}_2\right|^2 + m_1m_2\left|\dot{\vec{r}}_2\right|^2\right]$$

$$= -\frac{1}{2}\frac{1}{m_1 + m_2}\left[m_1^2\left|\dot{\vec{r}}_1\right|^2 + m_2^2\left|\dot{\vec{r}}_2\right|^2 + 2m_1m_2\,(\dot{\vec{r}}_1 \cdot \dot{\vec{r}}_2)\right]$$

$$= -\frac{1}{2}\frac{1}{m_1 + m_2}(m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2)^2 \qquad ...(iii)$$

The position of centre of mass of the two particle system is given by $\vec{R} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$.

$\therefore$ Velocity of centre of mass $\vec{V} = \dfrac{d\vec{R}}{dt} = \dot{\vec{R}} = \dfrac{m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2}{m_1 + m_2}$

Substituting $\left|m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2\right|^2 = (m_1 + m_2)^2\left|\dot{\vec{R}}\right|^2$ in (*iii*), we have

$$E_1 - E_2 = -\frac{1}{2}(m_1 + m_2)\left|\vec{\dot{R}}\right|^2$$

But $\frac{1}{2}(m_1 + m_2)\left|\vec{\dot{R}}\right|^2$ is the kinetic energy of the centre of mass.

As $E_1 - E_2$ is negative, *the total energy of the system decreases by an amount equal to the kinetic energy of the centre of mass.*

(*ii*) Change in Angular Momentum. The angular momentum of the system = sum of the angular momenta of the two masses

$$\therefore \qquad \vec{J}_2 = \vec{r}_1 \times (m_1 \vec{\dot{r}}_1) + \vec{r}_2 \times (m_2 \vec{\dot{r}}_2) \qquad \text{... }(iv)$$

The angular momentum of the equivalent one body system

$$\vec{J}_1 = \vec{r} \times \mu\, \vec{\dot{r}} = (\vec{r}_1 - \vec{r}_2) \times \frac{m_1 m_2}{m_1 + m_2}(\vec{\dot{r}}_1 - \vec{\dot{r}}_2)$$

$$= \frac{m_1 m_2}{m_1 + m_2}(\vec{r}_1 - \vec{r}_2)(\vec{\dot{r}}_1 - \vec{\dot{r}}_2) \qquad \text{... }(v)$$

$\therefore$ Change in angular momentum

$$\vec{J}_1 - \vec{J}_2 = \frac{m_1 m_2}{m_1 + m_2}(\vec{r}_1 \times \vec{\dot{r}}_1 - \vec{r}_1 \times \vec{\dot{r}}_2 - \vec{r}_2 \times \vec{\dot{r}}_1 + \vec{r}_2 \times \vec{\dot{r}}_2) - (m_1 \vec{r}_1 \times \vec{\dot{r}}_1 + m_2 \vec{r}_2 \times \vec{\dot{r}}_2)$$

$$= \frac{1}{m_1 + m_2}\left[-m_1 m_2\, \vec{r}_1 \times \vec{\dot{r}}_2 - m_1 m_2\, \vec{r}_2 \times \vec{\dot{r}}_1\right] + \left(\frac{m_1 m_2}{m_1 + m_2} - m_1\right)\vec{r}_1 \times \vec{\dot{r}}_1 + \left(\frac{m_1 m_2}{m_1 + m_2} - m_2\right)\vec{r}_2 \times \vec{\dot{r}}_2$$

$$= \frac{1}{m_1 + m_2}\Big[-m_1 m_2\, \vec{r}_1 \times \vec{\dot{r}}_2 - m_1 m_2\, \vec{r}_2 \times \vec{\dot{r}}_1 + (m_1 m_2 - m_1^2 - m_1 m_2)\,\vec{r}_1 \times \vec{\dot{r}}_1 + (m_1 m_2 - m_1 m_2 - m_2^2)\,\vec{r}_2 \times \vec{\dot{r}}_2\Big]$$

$$= \frac{1}{m_1 + m_2}\left[-m_1 m_2\, \vec{r}_1 \times \vec{\dot{r}}_2 - m_1 m_2\, \vec{r}_2 \times \vec{\dot{r}}_1 - m_1^2\, \vec{r}_1 \times \vec{\dot{r}}_1 - m_2^2\, \vec{r}_2 \times \vec{\dot{r}}_2\right]$$

$$= -\frac{1}{m_1 + m_2}\left[m_1 \vec{r}_1 \times m_2 \vec{\dot{r}}_2 + m_2 \vec{r}_2 \times m_1 \vec{\dot{r}}_1 + m_1 \vec{r}_1 \times m_1 \vec{\dot{r}}_1 + m_2 \vec{r}_2 \times m_2 \vec{\dot{r}}_2\right]$$

$$= -\frac{1}{m_1 + m_2}\left[(m_1 \vec{r}_1 + m_2 \vec{r}_2) \times (m_1 \vec{\dot{r}}_1 + m_2 \vec{\dot{r}}_2)\right] \qquad \text{...}(vi)$$

The angular momentum of the centre of mass is given by $\vec{R} \times (m_1 + m_2)\vec{\dot{R}}$. But

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \quad \text{and} \quad \vec{\dot{R}} = \frac{m_1 \vec{\dot{r}}_1 + m_2 \vec{\dot{r}}_2}{m_1 + m_2}$$

$$\therefore \text{ Angular momentum of centre of mass} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \times (m_1 + m_2)\,\frac{m_1 \vec{\dot{r}}_1 + m_2 \vec{\dot{r}}_2}{m_1 + m_2}$$

$$\frac{1}{m_1 + m_2}\left[(m_1 \vec{r}_1 + m_2 \vec{r}_2) \times (m_1 \vec{\dot{r}}_1 + m_2 \vec{\dot{r}}_2)\right] \qquad \text{...}(vii)$$

Comparing (*vii*) and (*vi*) we find that *the angular momentum of the system decreases by an amount equal to the angular momentum of the centre of mass.*

3.20 ROCKET

Rocket is a device which is used to take a satellite at the desired height, as well as to place it into an orbit.

Principle: Rocket propulsion is based on the principle of *conservation of momentum*. A rocket carries both, the fuel (like liquid hydrogen or liquid paraffin) and a suitable oxidiser (like oxygen, hydrogen peroxide or nitric acid) which burn in combustion chamber within the rocket. The large quantity of heat of combustion produced greatly raises pressure inside the chamber, resulting the burnt up gases (like CO, steam etc.) rushing out through an orifice at the back or trail end of the rocket. In consequence, the rocket is propelled forward (Newton's third law of motion). This is because the momentum lost by the jet of fuel gases must be equal to the momentum gained by the rocket.

Single and Multistage Rocket. In order to escape the rocket from the earth's gravitational field, it must attain an escape velocity 11.2 km/sec., or orbiting velocity 8 km/sec. to be able to orbit around the earth close to its surface. To achieve this the mass ratio M_0/M must be 270 or slightly more (Here M_0 is initial mass of rocket and M that after time t). In either cases the rocket has to encounter and overcome the resistance of air. This means that a little more than 270 kg. of the fuel will be consumed for every 1 kg loss of mass of the rocket in terms of burnt fuel. Even today, no single stage rocket is capable of such a performance. Therefore we have to use *multistage i.e.* two, three or even more stage rocket to achieve the desired requirement.

In multistage rocket- When the first stage is ignited, takes the satellite to some height when the fuel is completely burnt up. When the job of first stage is over, it gets detached and is discarded. Immediately the second stage takes over the task of producing further acceleration. The second stage gets detached and the third stage takes over the further job and so on. The velocity of the rocket, thus goes on increasing at each stage and finally a horizontal thurst puts the satellite in a orbit around the earth, on reaching to a desired height.

Theory. To derive the relation for the propulsion, let us suppose that rocket along with its fuel has a mass M and is moving with a velocity $\vec{V}$ at any instant. Let this velocity be considered with respect to some *inertial frame of reference* which is assumed to be the earth in this discussion. After a time Δt, let the mass of fuel ejected be ΔM with an exhaust velocity $\vec{V}_e$ with respect to the *moving rocket*. The velocity $\vec{V}_e$ is constant and being *downward* relative to the rocket is an intrinsic *negative quantity*. The velocity of the ejected gases with respect to the earth is given by $\vec{V}_0 = \vec{V} + \vec{V}_e$.

As mass ΔM has been fired out of the rocket, the mass of rocket decreases to $(M - \Delta M)$ after the exhaust and the velocity increases to $(\vec{V} + \Delta\vec{V})$ where $\Delta\vec{V}$ is a small increase in velocity.

In the inertial frame of earth

Initial momentum of rocket with fuel $\vec{p}_1 = M\vec{V}$

Final momentum of rocket and exhaust gases

$$\vec{p}_2 = (M - \Delta M)(\vec{V} + \Delta\vec{V}) + \Delta M\ (\vec{V} + \vec{V}_e)$$

$$\therefore \quad \Delta\vec{p} = \vec{p}_2 - \vec{p}_1 = (M - \Delta M)\ (\vec{V} + \Delta\vec{V}) + \Delta M\ (\vec{V} + \vec{V}_e) - M\vec{V}$$

$$= M\,\Delta\vec{V} - \Delta M\,\Delta\vec{V} + \Delta M\,\vec{V}_e$$

or
$$\Delta\vec{p} = M\Delta\vec{V} + \Delta M\,\vec{V}_e \qquad ...(i)$$

neglecting $\Delta M\,\Delta\vec{V}$ being the product of two very small quantities as $\Delta M \to 0$ and $\Delta V \to 0$

Dividing (i) by Δt and taking limits when $\Delta t \to 0$, we have $\dfrac{d\vec{p}}{dt} = M\dfrac{d\vec{V}}{dt} + \vec{V}_e\dfrac{dM}{dt}$

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We have considered ΔM the mass of the ejected gases as a positive quantity but as the mass of the rocket decreases with time, the expression $\frac{dM}{dt}$ is also an *intrinsic negative quantity* for the rocket system. Taking magnitudes, we get

$$\frac{dp}{dt} = M\frac{dV}{dt} + V_e\left(-\frac{dM}{dt}\right)$$

According to Newton's second law of motion as applied to the total system consisting of the rocket and ejected gases

$$\frac{d\vec{p}}{dt} = \vec{F}_e$$

where $\vec{F}_e$ is the external force acting on the system.

$$\therefore \qquad \vec{F}_e = M\frac{d\vec{V}}{dt} - \vec{V}_e\frac{dM}{dt}$$

or

$$M\frac{d\vec{V}}{dt} = \vec{F}_e + \vec{V}_e\frac{dM}{dt} \qquad ...(ii)$$

$M\frac{d\vec{V}}{dt}$ gives the net force acting on the rocket.

Thrust on the rocket. If the rocket is moving in a region *outside the influence of the gravitational pull of the earth* and there is no air resistance or any other external force acting on the rocket, then $F_e = 0$.

Hence force acting on the rocket is given by $M\frac{d\vec{V}}{dt} = \vec{V}_e\frac{dM}{dt}$ ...(*iii*)

The quantity $V_e\frac{dM}{dt}$ is the *reaction force* exerted on the rocket by the exhaust of gases and gives the **thrust**.

$\therefore$ Rocket thrust $= \vec{V}_e\frac{dM}{dt}$

As both $\vec{V}_e$ and $\frac{dM}{dt}$ are intrinsic negative quantities $\vec{V}_e\frac{dM}{dt}$ is positive *i.e. the thrust on the rocket is in upward direction.*

Near the earth the upward thrust on the rocket is opposed by the external force of gravitation $M\vec{g}$ = the weight of the rocket (neglecting friction etc.)

$$\therefore \qquad \vec{F}_e = M\vec{g}$$

Hence

$$M\frac{d\vec{V}}{dt} = \vec{V}_e\frac{dM}{dt} + M\vec{g} \qquad ...(iv)$$

$M\vec{g}$ is a negative quantity as g acts in the downward direction and has a magnitude $-g$.

In order to have a large thrust

(*a*) The velocity of exhaust $\vec{V}_e$, should be large and

(*b*) $\frac{dM}{dt}$, the rate at which the fuel is burnt and ejected should be large.

Equation of motion. To get the equation of motion of a rocket which is *far away* from the earth, we have, rewriting equation (*iii*) $d\vec{V} = \vec{V}_e \dfrac{dM}{M}$ ...(*v*)

If M_0 is the initial mass of the rocket and fuel, $\vec{V}_0$ the initial velocity of the rocket and $\vec{V}$ and M the instantaneous values of velocity and mass of the rocket respectively, then integrating equation (*v*)

We have $$\int_{v_0}^{v} d\vec{v} = \vec{V}_e \int_{M_0}^{M} \frac{dM}{M}$$

or $$\left[\vec{V}\right]_{v_0}^{v} = \vec{V}_e \left[\log_e M\right]_{M_0}^{M} \text{ or } \vec{V} - \vec{V}_0 = \vec{V}_e \log_e \frac{M}{M_0}$$

or $$\vec{V} = \vec{V}_0 + \vec{V}_e \log_e \frac{M}{M_0}$$

Taking magnitudes, we get $V = V_0 - V_e \log_e \dfrac{M}{M_0}$ or $V = V_0 + V_e \log_e \dfrac{M_0}{M}$ ...(*vi*)

as direction of V_e is opposite to that of V.

Maximum (burnt out) velocity. If M_f is the mass of the (empty) rocket at *burnt out i.e.*, when the entire fuel has been exhausted and $\vec{V}_f$ is the maximum or final velocity called *burnt out velocity*, then from equation (*vi*), we have

$$\vec{V}_f - \vec{V}_0 = -\vec{V}_e \log_e \frac{M_0}{M_f} \quad ...(vii)$$

The negative sign indicates that the velocity imparted to the rocket $\vec{V}_f$ is in a direction opposite to the velocity of the ejected gases. If the rocket starts from rest $\vec{V}_0 = 0$

$$\therefore \quad \vec{V}_f = -\vec{V}_e \log_e \frac{M_0}{M_f} \quad ...(viii)$$

3.21 COLLISION

When two bodies are approaching each other, a force comes into play between them for a finite time and brings about a measurable change in their velocities, momenta and energy according to the respective laws of conservation, a collision is said to have taken place.

It should, however, be clearly understood that in Physics, collision does not necessarily mean physical contact between the two bodies.

During collision, the force of some interaction comes into play between the two colliding particles or systems for a finite small time which brings about a change in their relative motions. If the collision acts for a time t_0 to $t_0 + \Delta t$, then a time t less than t_0 is known as *time before collision* and a time t greater than $t_0 + \Delta t$ is called *time after collision.*

The collision is termed **scattering** if the nature of particles does not change after collision. Familiar examples of collision or scattering are the deflection of a comet as it passes near the solar system and the deflection of an α-particle by an atomic nucleus. The study of collision is of particular importance in atomic and nuclear physics. The bodies involved may be atoms, nuclei or various elementary particles such as electrons, protons etc.

There are two types of collisions.

(1) Elastic Collision and (2) Inelastic Collision

(1) Elastic scattering (or elastic collision). A collision (scattering) is said to be an elastic collision if (*i*) the final particles after collision are the same as the initial particles before collision,

(*ii*) the sum of the kinetic energies of the particles after collision is the same as the sum of the kinetic energies of the particle before collision.

(2) Inelastic scattering (or inelastic collision). A collision is said to be an inelastic collision if

(*i*) The final particles after collision are the same as the initial particles before collision.

(*ii*) The sum of the kinetic energies of the particles after collision is either more or less than the sum of the kinetic energies of the particles before collision.

A collision is said to be perfectly inelastic if the particles stick permanently together on impact and the loss of kinetic energy is *maximum*, consistent with the law of conservation of momentum.

Conservation of linear momentum. Consider two particles of masses m_1 and m_2 moving with velocities $\vec{u_1}$ and $\vec{u_2}$ respectively before collision in an inertial frame of reference. After collision, their velocities become $\vec{v_1}$ and $\vec{v_2}$ respectively. When no external force acts on the particles their *total linear momentum is conserved.* $\therefore\ m_1\vec{u_1} + m_2\vec{u_2} = m_1\vec{v_1} + m_2\vec{v_2}$

This relation holds good both for elastic as well as inelastic collisions.

Conservation of kinetic energy. In the case of an elastic collision the kinetic energy is also conserved. $\therefore\ \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$

Examples of elastic collision are

(*i*) Collision between molecules of gases according to kinetic theory,

(*ii*) Collisions between atoms and nuclei and

(*iii*) Collisions between particles like electrons, protons, α-particles etc.

Two types of inelastic collisions. In the case of *inelastic* collisions there may be an increase or decrease of kinetic energy giving rise to two types of inelastic collisions *i.e.,* (*i*) **endo-ergic** and (*ii*) **exoergic.**

Endoergic collision. For bodies of *macroscopic* size the loss of kinetic energy occurs as heat, sound etc. but in the case of atoms, molecules etc., the atoms may absorb a part of the kinetic energy and move into an excited state. The kinetic energy of the particles is then reduced and we have

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + E$$

where E is the excitation energy. Such a collision in which the kinetic energy of the final particles is less than the kinetic energy of the initial particles is known as **endo-ergic** collision.

Exoergic collision. If, however, the atoms are already in the excited state and after collision come down to the normal state, the excitation energy adds up to the final kinetic energy and we have

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 + E = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

Such a collision in which the kinetic energy of the final particles is more than the kinetic energy of the initial particles is known as **exoergic** collision.

Reaction. A collision (or scattering) is said to be a reaction if

(*i*) the final particles after collision (known as *products*) are entirely different from the initial particles (known as *reactants*) before collision.

(*ii*) the sum of the kinetic energies of the particles after collision is different from the sum of the kinetic energies of the particles before collision.

In all the above cases the law of conservation of linear momentum and the law of conservation of angular momentum holds good. It is because in each case the interacting particles being isolated, no external torque acts on the system.

Another point to be carefully noted is that in the case of inelastic collision and reaction, only the sum of the kinetic energies after collision is different from the sum of the kinetic energies before collision but the total energy of the system, including kinetic energy, potential energy and any other form of energy remains conserved.

Advantages of Studying Collision Process. Various types of interactions and forces are operating in nature at microscopic as well as macroscopic levels.A study of collision process *i.e.* relative motion between the two interacting particles helps us to understand the basic nature and characteristics of these interactions and forces. This is done by measuring the initial and final energy and linear as well as angular momentum of the participating particles in accordance with the laws of conservation of energy and momentum.

The deflection of the path of a comet passing near the solar system and the elliptic path of planets within the solar system helps us to understand the nature of gravitational forces.

Collision between molecules of gases according to Kinetic theory leads to understanding of inter-atomic and intermolecular forces.

Scattering of α-particles in Rutherford scattering, scattering of high energy photons in Compton scattering, nuclear reactions involving high energy electrons, protons, neutrons, deutrons and α- particles, scattering of slow neutrons by nuclei and emission of α and β particles from radioactive nuclei help us to understand the nature of nuclear forces, properties of the nucleus and atomic structure.

3.22 LABORATORY AND CENTRE OF MASS FRAMES OF REFERENCE

In the study of collisions between two particles we shall come across the laboratory system or laboratory frame of reference and the centre of mass system or centre of mass frame of reference.

A reference frame is the space determined by a rigid body regarded as the base. The rigid body is supposed to extend in all directions as far as necessary. A point in space is located by the three co-ordinates taken with respect to the origin of the reference system.

If the origin of the reference system is a point rigidly fixed to the laboratory it is known as the **laboratory frame.**

The laboratory frame is inertial so long as earth is taken to be an inertial frame.

Centre of mass system (Frame of reference). *If the origin of the reference system is a point rigidly fixed to the centre of mass of a system of particles on which no external force is acting it is known as the* **centre of mass frame of reference.**

In the centre of mass reference frame the position vector of the centre of mass $\vec{R} = 0$ as the centre of mass is itself the origin of the reference system.

$\therefore$ The velocity of centre mass $\vec{V} = \dfrac{d\vec{R}}{dt} = 0$

and the linear momentum $\vec{P} = M\vec{V}$ of the system is also = 0. Hence it is known as a *zero momentum frame.*

Advantages of studying collision process in centre of mass system. (*i*) In the absence of any external force the velocity of the centre of mass is a *constant.* In other words, the centre of mass reference frame moves with a constant velocity with respect to the laboratory frame. *Hence the centre of mass frame is also an inertial frame.*

Various physical quantities measured in the two systems are related to each other by Galelian transformations provided the velocity of centre of mass is small as compared to the velocity of light.

(*ii*) A system of two particles requires six co-ordinates to describe the motion in the laboratory system. Three co-ordinates are required to describe the motion of centre of mass and three more co-ordinates are required to describe the relative motion. But in the centre of mass frame we require only three co-ordinates as the centre of mass is itself at rest in this frame.

The discussion of a collision process, therefore, becomes much simpler in the centre of mass frame of reference than in the laboratory frame.

3.23 PERFECTLY ELASTIC COLLISION IN ONE DIMENSION

(*a*) Laboratory Frame: Let m_1 and m_2 be the masses of the two particles $\vec{u}_1$ and $\vec{u}_2$ and $\vec{v}_1$, $\vec{v}_2$ their respective velocities before and after an elastic one dimensional collision *i.e., a head on collision along the line joining their centres,* then

According to the principle of conservation of linear momentum

$$m_1 \vec{u}_1 + m_2 \vec{u}_2 = m_1 \vec{v}_1 + m_2 \vec{v}_2 \qquad ...(i)$$

and according to the law of conservation of energy

$$m_1 u_1^2 + m_2 u_2^2 = m_1 v_1^2 + m_2 v_2^2 \qquad ...(ii)$$

Rewriting equations (*i*) and (*ii*) and taking magnitudes only we have

$$m_1 (u_1 - v_1) = m_2 (v_2 - u_2) \qquad ...(iii)$$

and $$m_1 (u_1^2 - v_1^2) = m_2 (v_2^2 - u_2^2) \qquad ...(iv)$$

Dividing (*iv*) by (*iii*) we have $u_1 + v_1 = v_2 + u_2$ or $u_1 - u_2 = -(v_1 - v_2)$ ...(*v*)

This shows that *in an elastic one dimensional collision the relative velocity with which the two particles approach each other before collision is equal to the relative velocity with which they recede away from each other after collision.*

Velocity after collision. From equation (*v*) we have $v_1 = v_2 + u_2 - u_1$

and $$v_2 = v_1 + u_1 - u_2$$

Substituting the value of v_2 in (*iii*), we have

$$m_1 (u_1 - v_1) = m_2 (v_1 + u_1 - u_2 - u_2)$$

or $$v_1 (m_1 + m_2) = (m_1 - m_2) u_1 + 2m_2 u_2$$

or $$v_1 = \frac{m_1 - m_2}{m_1 + m_2} u_1 + \frac{2m_2}{m_1 + m_2} u_2 \qquad ...(vi)$$

Similarly $$v_2 = \frac{m_2 - m_1}{m_1 + m_2} u_2 + \frac{2m_1}{m_1 + m_2} u_1 \qquad ...(vii)$$

Special cases. Case (*i*): One of the colliding particles is initially at rest. Let m_2 be initially at rest, then $u_2 = 0$.

Hence $$v_1 = \frac{m_1 - m_2}{m_1 + m_2} u_1 \text{ and } v_2 = \frac{2m_1}{m_1 + m_2} u_1 \qquad ...(viii)$$

Case (*ii*): The particles have the same mass. In such a case $m_1 = m_2$

Substituting in (*vi*), we have $v_1 = u_2$ and substituting in (*vii*), we have $v_2 = u_1$

Hence in one dimensional elastic collision of two particles of equal mass, the particles simply interchange their velocities after collision.

If m_2 is also initially at rest, then $u_2 = 0 \therefore v_1 = 0$ and as before $v_2 = u_1$ ... (*viii a*)

Hence *the first particle of mass m_1 comes to rest after collision and the second particle of mass m_2 acquires the initial velocity of the first.*

Case (*iii*): The particle at rest is very massive. If m_2 is very heavy as compared to m_1 and $u_2 = 0$, then

$m_1 = 0,\ m_1 - m_2 = -m_2$ and $m_1 + m_2 = m_2 \ \therefore v_1 = -u_1$ and $v_2 = 0$

This shows that *when a very light particle collides against a very massive particle at rest, the heavy particle continues to remain at rest and the velocity of the light particle is reversed.*

A familiar example of this is the dropping of a steel ball on an equally hard horizontal surface on the ground. This is in fact a collision between the light ball and the massive ground at rest. The velocity of the ball is reversed on impact. This is judged from the fact that the ball rises to the same height from which it was dropped.

Case (*iv*): Particle at rest is very light. If the particle at rest is very light

$$m_2 = 0; \quad m_1 - m_2 = m_1 \text{ and } m_1 + m_2 = m_1$$

Substituting in relation (*viii*) we have $v_1 = u_1$ and $v_2 = 2u_1$

This shows *that the velocity of the heavy particle remains almost the same after collision and the light particle acquires nearly twice the velocity of the heavy particle.*

(*b*) Centre of Mass Frame: When no external force is acting, the velocity of centre of mass is given by

$$\vec{V}_{\text{cm}} = \frac{m_1 \vec{u}_1 + m_2 \vec{u}_2}{m_1 + m_2}$$

As the collision is one dimensional, therefore taking magnitudes only $V_{\text{cm}} = \dfrac{m_1 u_1 + m_2 u_2}{m_1 + m_2}$ Velocity of the particle of mass m_1 before collision relative to centre of mass frame according to Galelian transformations is given by

$$\vec{u}_1' = \vec{u}_1 - \vec{V}_{\text{cm}} = \vec{u}_1 - \frac{m_1 \vec{u}_1 + m_2 \vec{u}_2}{m_1 + m_2}$$

$$= \frac{m_1 \vec{u}_1 + m_2 \vec{u}_1 - m_1 \vec{u}_1 - m_2 \vec{u}_2}{m_1 + m_2} = \frac{m_2 (\vec{u}_1 - \vec{u}_2)}{m_1 + m_2}$$

Taking magnitudes only $u_1' = \dfrac{m_2 (u_1 - u_2)}{m_1 + m_2}$ ... (*ix*)

Velocity of the particle of mass m_2 before collision relative to centre of mass frame according to Galelian transformations is given by

$$\vec{u}_2' = \vec{u}_2 - \vec{V}_{\text{cm}} = \vec{u}_2 - \frac{m_1 \vec{u}_1 + m_2 \vec{u}_2}{m_1 + m_2}$$

$$= \frac{m_1 \vec{u}_2 + m_2 \vec{u}_2 - m_1 \vec{u}_1 - m_2 \vec{u}_2}{m_1 + m_2} = \frac{m_1 (\vec{u}_2 - \vec{u}_1)}{m_1 + m_2}$$

Taking magnitudes only $u_2' = \dfrac{m_1 (u_2 - u_1)}{m_1 + m_2}$ ... (*ix*) (*a*)

Velocity after collision. The velocity of the particle of mass m_1 after collision relative to centre of mass frame according to Galelian transformations is given by $\vec{v}_1' = \vec{v}_1 - \vec{V}_{cm}$

Taking magnitudes only

$$v_1' = v_1 - V_{\text{cm}} = \frac{(m_1 - m_2)u_1 + 2m_2 u_2}{m_1 + m_2} - \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2}$$

$$= \frac{m_1 u_1 - m_2 u_1 + 2m_2 u_2 - m_1 u_1 - m_2 u_2}{m_1 + m_2} = \frac{-m_2 (u_1 - u_2)}{m_1 + m_2} \quad ...(x)$$

The velocity of the particle of mass m_2 after collision in the centre of mass frame according to

Galelian transformations is given by

$$\vec{v}_2' = \vec{v}_2 - \vec{V}_{cm}$$

Taking magnitudes only

$$v_2' = v_2 - V_{cm} = \frac{(m_2 - m_1)u_2 + 2m_1u_1}{m_1 + m_2} - \frac{m_1u_1 + m_2u_2}{m_1 + m_2}$$

$$= \frac{m_2u_2 - m_1u_2 + 2m_1u_1 - m_1u_1 - m_2u_2}{m_1 + m_2} = \frac{-m_1(u_2 - u_1)}{m_1 + m_2} \quad \ldots (xi)$$

The centre of mass is at rest before and after collision relative to the centre of mass reference frame.

3.24 PERFECTLY INELASTIC COLLISION IN ONE DIMENSION

A collision is said to be perfectly inelastic if the two particles stick together after collision. We shall discuss the problem in the laboratory frame as well as in the centre of mass frame.

(*i*) Laboratory frame. Let m_1 and m_2 be the masses and $\vec{u}_1$ *and* $\vec{u}_2$ the velocities of two particles before collision. Since the two particles stick together on impact let their velocity after collision be V.

Initial linear momentum of mass m_1 moving with a velocity $\vec{u}_1 = m_1\vec{u}_1$. Initial linear momentum of mass m_2 at rest = 0

Let the combined mass $(m_1 + m_2)$ move with a velocity $\vec{v}$ in the initial direction, then

Final linear momentum of the system = $(m_1 + m_2)\vec{v}$

According to the principle of conservation of linear momentum

$$m_1\vec{u}_1 = (m_1 + m_2)\vec{v} \qquad \therefore\ \vec{v} = \frac{m_1}{m_1 + m_2}\vec{u}_1$$

Kinetic energy of m_1 before collision $= \frac{1}{2}m_1u_1^2$

Kinetic energy of m_2 before collision = 0 [$\because$ m_2 is at rest]

$\therefore$ Total kinetic energy before collision $T_1 = \frac{1}{2}m_1u_1^2$

Kinetic energy of combined mass $(m_1 + m_2)$ after collision $T_2 = \frac{1}{2}(m_1 + m_2)v^2$

$$= \frac{1}{2}(m_1 + m_2)\frac{m_1^2u_1^2}{(m_1 + m_2)^2} = \frac{1}{2}\frac{m_1^2}{m_1 + m_2}u_1^2$$

$$\therefore \qquad \frac{T_2}{T_1} = \frac{1}{2}\frac{m_1^2u_1^2}{m_1 + m_2} \times \frac{1}{\frac{1}{2}m_1u_1^2} = \frac{m_1}{m_1 + m_2}$$

As $m_1 < m_1 + m_2$, $T_2 < T_1$ *i.e., the final kinetic energy after collision is less than the kinetic energy before collision in the lab system.*

Decrease in energy. The decrease in energy $E = T_1 - T_2 = \frac{1}{2}m_1u_1^2 - \frac{1}{2}\frac{m_1^2u_1^2}{m_1 + m_2}$

$$= \frac{1}{2}\left[\frac{m_1^2u_1^2 + m_1m_2u_1^2 - m_1^2u_1^2}{m_1 + m_2}\right] = \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}u_1^2 = \frac{1}{2}\mu u_1^2 \quad \ldots (i)$$

where $\mu = \dfrac{m_1m_2}{m_1 + m_2}$ is the reduced mass.

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(*ii*) **Centre of Mass Frame.** Velocity of centre of mass $\vec{V}_{c.m.} = \dfrac{m\vec{u_1}}{m_1 + m_2}$ $\quad [\because \vec{u_2} = 0]$

Velocity of m_1 before collision in C. M. System $= \vec{u_1'} = \vec{u_1} - \vec{V}_{c.m.}$

$$= u_1 - \frac{m_1}{m_1 + m_2}\vec{u_1} = \frac{m_2}{m_1 + m_2}\vec{u_1}$$

Velocity of m_2 before collision in C. M. System $\vec{u_2'} = \vec{u_2} - \vec{V}_{c.m.} = -\vec{V}_{c.m.} = -\dfrac{m_1}{m_1 + m_2}\vec{u_1}$

After collision the two particles stick together and the combined mass ($m_1 + m_2$) moves with a velocity equal to the velocity of centre of mass with respect to the lab. system and is at rest with respect to the centre of mass system itself. Hence according to the principle of conservation of linear momentum $m_1\vec{u_1'} + m_2\vec{u_2'} = 0$ $\quad$ [$\because$ The final linear momentum = 0]

or $\quad m_1\vec{u_1'} = -m_2\vec{u_2'}$

Initial kinetic energy of m_1 in C.M. system $= \dfrac{1}{2}m_1u_1'^2$

Initial kinetic energy of m_2 in C.M. system $= \dfrac{1}{2}m_2u_2'^2$

$\therefore$ Total initial kinetic energy in C.M. system $T_1' = \dfrac{1}{2}m_1u_1'^2 + \dfrac{1}{2}m_2u_2'^2$

Final kinetic energy of the combined mass ($m_1 + m_2$) = $T_2' = 0$

[$\because$ ($m_1 + m_2$) is at rest w.r.t. C.M. system]

Decrease in kinetic energy. The decrease in kinetic energy

$$E = T_1' - T_2' \qquad [\because T_2' = 0]$$

$$= \frac{1}{2}m_1u_1'^2 + \frac{1}{2}m_2u_2'^2$$

$$= \frac{1}{2}m_1\left(\frac{m_2u_1}{m_1 + m_2}\right)^2 + \frac{1}{2}m_2\left(\frac{m_1u_1}{m_1 + m_2}\right)^2$$

$$= \frac{1}{2}\frac{m_1m_2u_1^2}{(m_1 + m_2)^2}[m_1 + m_2] = \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}u_1^2$$

$$= \frac{1}{2}\mu u_1^2 \qquad ...(ii)$$

where $\quad \mu = \dfrac{m_1m_2}{m_1 + m_2}$ is the reduced mass.

Comparing relations (*i*) and (*ii*), we find that the decrease in kinetic energy after collision is the same in the lab. system as well as in the C. M. systems.

This decrease in K.E. may appear as excitation energy of the scattered particle.

It may also be noted that decrease in energy of the combined mass in C.M. system is equal to the initial kinetic energy of the particles in the same system.

3.25 VELOCITIES OF PARTICLES IN ELASTIC COLLISION

(In Centre of mass frame and laboratory frame)

Consider a particle of mass m_1 moving with a velocity $\vec{u}_1$ in the laboratory frame and let it suffer a perfectly elastic collision with a particle of mass m_2 at rest.

The velocity of centre of mass of a system of two particles relative to the laboratory frame is given by

$$\vec{V}_{cm} = \frac{m_1 \vec{u}_1 + m_2 \vec{u}_2}{m_1 + m_2}$$

Since we have assumed the mass m_2 to be initially at rest

$$\vec{u}_2 = 0 \quad \text{and} \quad \vec{V}_{cm} = \frac{m_1}{m_1 + m_2}\vec{u}_1$$

This shows that $\vec{V}_{cm}$ and $\vec{u}_1$ have the same direction. Let $\vec{u}_1'$ and $\vec{u}_2'$ be the initial velocities of the particles of mass m_1 and m_2 before collision *in the centre of mass frame,* then according to Galelian transformation equations

$$\vec{u}_1' = \vec{u}_1 - \vec{V}_{cm} = \vec{u}_1\left(1 - \frac{m_1}{m_1 + m_2}\right) = \frac{m_2}{m_1 + m_2}\vec{u}_1$$

and

$$\vec{u}_2' = \vec{u}_2 - \vec{V}_{cm} = -\vec{V}_{cm} = -\frac{m_1}{m_1 + m_2}\vec{u}_1 \qquad [\because \vec{u}_2 = 0]$$

We also have $\vec{u}_1' - \vec{u}_2' = \vec{u}_1 - \vec{u}_2$ *i.e.,* the relative velocity between the two particles in the laboratory frame and centre of mass frame is the same.

Taking magnitude only

$$u_1' = \frac{m_2}{m_1 + m_2}u_1 \qquad ...(i)$$

$$u_2' = -\frac{m_1}{m_1 + m_2}u_1 \qquad ...(ii)$$

Relation between final velocities and initial velocities in centre of mass frame.

Let $\vec{v}_1'$ and $\vec{v}_2'$ be the final velocities of the particles of mass m_1 and m_2 after collision in the centre of mass frame, then $\vec{v}_1' = \vec{v}_1 - \vec{V}_{cm}$ and $\vec{v}_2' = \vec{v}_2 - \vec{V}_{cm}$ where $\vec{v}_1$ and $\vec{v}_2$ are the final velocities of m_1 and m_2 in the laboratory frame.

In the centre of mass frame, the centre of mass is always at rest, therefore, the total linear momentum before and after collision is not only conserved but is also equal to zero.

$$\therefore \qquad m_1\vec{u}_1' + m_2\vec{u}_2' = 0 \text{ and } m_1\vec{v}_1' + m_2\vec{v}_2' = 0$$

From the above relation we get $\vec{u}_1' = -\dfrac{m_2}{m_1}\vec{u}_2'$ and $\vec{v}_1' = -\dfrac{m_2}{m_1}\vec{v}_2'$

Taking magnitudes only

$$u_1' = -\frac{m_2}{m_1}u_2' \qquad ...(iii)$$

and

$$v_1' = -\frac{m_2}{m_1}v_2' \qquad ...(iv)$$

The negative signs indicate that $\vec{u}_1'$ and $\vec{u}_2'$ act along the same straight line in opposite directions. Similarly $\vec{v}_1'$ and $\vec{v}_2'$ also act along the same straight line in opposite direction as shown in Fig. 3.9 (*a*) and (*b*).

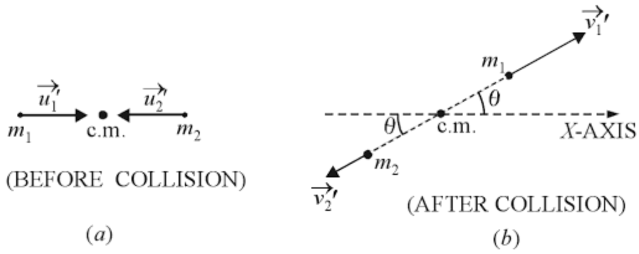

Fig. 3.9

In other words, in the C.M. system the two particles move towards each other before collision and away from each other after collision.

As collision is elastic, the kinetic energy is also conserved.

$$\therefore \qquad \frac{1}{2}m_1 u_1'^2 + \frac{1}{2}m_2 u_2'^2 = \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2 \qquad ...(v)$$

Substituting the values of u_1' and u_2' from (*iii*) and (*iv*) in equation (*v*), we have

$$\frac{1}{2}m_1 \frac{m_2^2}{m_1^2}u_2'^2 + \frac{1}{2}m_2 u_2'^2 = \frac{1}{2}m_1 \frac{m_2^2}{m_1^2}v_2'^2 + \frac{1}{2}m_2 v_2'^2$$

$$\text{or} \qquad u_2'^2\left[\frac{m_2^2}{2m_1} + \frac{1}{2}m_2\right] = v_2'^2\left[\frac{m_2^2}{2m_1} + \frac{1}{2}m_2\right]$$

$$\therefore \qquad u_2' = v_2' \qquad ...(vi)$$

$$\text{Similarly} \qquad u_1' = v_1' \qquad ...(vii)$$

In other words, in an elastic collision in the centre of mass frame, the magnitudes of the velocities of the particles do not change *i.e.,* **there is only a change in direction.**

3.26 APPLICATION OF ELASTIC COLLISION

The theory of elastic collision in one dimension is applicable is *nuclear reactor*. In a nuclear reactor, hydrogen is used as a moderator (a stationary target), for slowing down the neutrons, produced by the fission of uranium atoms, so as to enable them to produce more fissions.

$${}_{92}U^{235} + {}_0n^1 \rightarrow {}_{92}U^{236} \rightarrow {}_{56}Ba^{236} + {}_{56}Kr^{92} + 3\ {}_0n^1 + \text{Energy Released (200 MeV)}$$

This is obvious because the hydrogen nucleus (*i.e.* the proton) has nearly the same mass as that of neutron and hence in a head-on collision with a hydrogen nucleus at rest the neutrons are thus almost brought to rest, or greatly slowed down [special case (ii), eq. (viii a), article 3.23]. If a massive nucleus like that of lead were to be used as the *target*, the neutrons will simply bounce back from it with the same velocity with which they impinge on it.

3.27 PERFECTLY ELASTIC COLLISION IN TWO DIMENSIONS. (*i.e.* Scattering)

(*i*) **Laboratory Frame:** Consider a particle of mass m_1 moving with a velocity $\vec{u_1}$ in the laboratory frame (*incident particle*) and let it have an *elastic* collision with a particle of mass m_2 at rest (*target particle*). After collision, the incident particle (now called the *scattered particle*) moves with a velocity $\vec{v_1}$ making an angle θ_1 with the initial direction and the target particle of mass m_2 moves with a velocity $\vec{v_2}$ making an angle θ_2 with the initial direction of motion of m_1 (Fig. 3.10).

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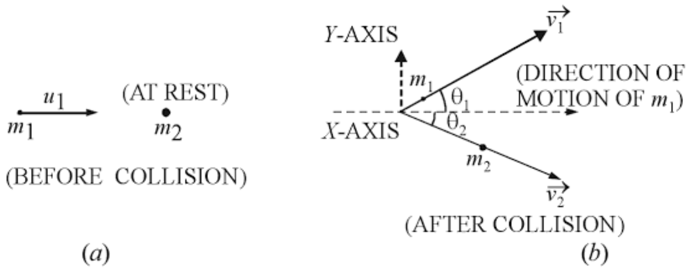

Fig. 3.10

Angle of scattering. The angle θ_1 is known as *angle of scattering. It is defined as the angle between the initial direction and final direction of the incident particle* (or scattered particle) *after it has gone far away.*

Recoil angle. The angle θ_2 is known as *recoil angle. It is defined as the angle between the direction of target* (*or recoil*) *particle after collision and initial direction of the incident particle.*

Suppose the initial path of the particle m_1 is along the X-axis and also the plane containing $\vec{u_1}$ and $\vec{v_1}$ is the XY plane. Then according to the principle of conservation of momentum

$$m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2 = m_1 u_1 \qquad ...(i)$$

and $$m_1 v_1 \sin\theta_1 - m_2 v_2 \sin\theta_2 = 0 \qquad ...(ii)$$

because before collision, the y-component of momentum is zero.

As the collision is perfectly elastic, the total kinetic energy is also conserved.

$$\therefore \qquad \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 u_1^2 \qquad ...(iii)$$

The three equations (*i*), (*ii*) and (*iii*) contain four unknown quantities v_1, v_2, θ_1 and θ_2. To find a solution of these equations at least one of these must be known. Let us suppose θ_1 is known. For simplicity, we further assume that $m_1 = m_2 = m$. Hence equations (*i*), (*ii*) and (*iii*) become

$$v_1 \cos\theta_1 + v_2 \cos\theta_2 = u_1 \qquad ...(iv)$$

$$v_1 \sin\theta_1 - v_2 \sin\theta_2 = 0 \qquad ...(v)$$

$$v_1^2 + v_2^2 = u_1^2 \qquad ...(vi)$$

Rearranging we get

$$u_1 - v_1 \cos\theta_1 = v_2 \cos\theta_2 \qquad ...(vii)$$

$$v_1 \sin\theta_1 = v_2 \sin\theta_2 \qquad ...(viii)$$

$$u_1^2 - v_1^2 = v_2^2 \qquad ...(ix)$$

Squaring and adding equations (*vii*) and (*viii*), we get

$$u_1^2 + v_1^2 \cos^2\theta_1 - 2u_1 v_1 \cos\theta_1 + v_1^2 \sin^2\theta_1 = v_2^2 \cos^2\theta_2 + v_2^2 \sin^2\theta_2$$

or $$u_1^2 + v_1^2 - 2u_1 v_1 \cos\theta_1 = v_2^2$$

But $$v_2^2 = u_1^2 - v_1^2 \quad \therefore \quad u_1^2 + v_1^2 - 2u_1 v_1 \cos\theta_1 = u_1^2 - v_1^2$$

or $$2v_1^2 = 2u_1 v_1 \cos\theta_1 \quad \therefore \quad v_1 = u_1 \cos\theta_1 \qquad ...(x)$$

Velocity of scattered particle. Equation (*x*) gives the value of v_1 the *velocity of scattered particle* in terms of known quantities u_1, the *velocity of incident particle* and θ_1 the *angle of scattering.*

$$\therefore \qquad v_1 = u_1 \cos\theta_1 \qquad ...(x)\,(a)$$

Velocity of recoil particle. Substituting $v_1 = u_1 \cos\theta_1$ in equation (*ix*), we get

$$u_1^2 - u_1^2 \cos^2\theta_1 = v_2^2$$

or $$v_2^2 = u_1^2(1 - \cos^2\theta_1) = u_1^2 \sin^2\theta_1$$

$\therefore$ $$v_2 = u_1 \sin\theta_1 \qquad ...(xi)$$

This relation gives the values of v_2 the *velocity of recoil* particle in terms of u_1, the *velocity of incident particle* and θ_1 the *angle of scattering.*

Angle of recoil. Dividing Eq. (*viii*) by Eq. (*vii*), we get

$$\tan\theta_2 = \frac{v_1 \sin\theta_1}{u_1 - v_1 \cos\theta_1}$$

Substituting $v_1 = u_1 \cos\theta_1$, we have

$$\tan\theta_2 = \frac{u_1 \cos\theta_1 \sin\theta_1}{u_1 - u_1 \cos^2\theta_1} = \frac{u_1 \cos\theta_1 \sin\theta_1}{u_1(1 - \cos^2\theta_1)} = \cot\theta_1 = \tan(\pi/2 - \theta_1)$$

or $$\tan\theta_2 = \tan(\pi/2 - \theta_1) \quad \therefore\ \theta_2 = \pi/2 - \theta_1 \qquad ...(xii)$$

This relation gives the value of θ_2 the *angle of recoil* in terms of θ_1, the *angle of scattering.*

Relation (*xii*) can be put in the form $\theta_1 + \theta_2 = \pi/2$.

This relation shows that if the incident particle collides with a target particle of equal mass at rest in the laboratory frame, both the particles move in directions perpendicular to each other after an elastic collision.

(*ii*) Centre of Mass Frame

(*a*) Relation between scattering angle θ_1 in the laboratory frame and θ in the centre of mass frame. Consider a particle of mass m_1 moving with a velocity $\vec{u_1}$ in the laboratory frame and let it collide with a particle of mass m_2 at rest, the collision being perfectly elastic. After collision the incident particle moves with a velocity $\vec{v_1}$ making *scattering angle* θ_1 with the initial direction and the target particle of mass m_2 moves with a velocity $\vec{v_2}$ making *recoil angle* θ_2 with the initial direction of motion of m_1. The initial path of m_1 is along the X-axis and the plane containing $\vec{u_1}$ and $\vec{v_1}$ is the X-Y plane as shown in Fig. 3.10 (*b*).

Let $\vec{v_1'}$ and $\vec{v_2'}$ be the final velocities of the particles m_1 and m_2 after collision in the centre of mass frame making an angle θ with the X-axis as shown in Fig. 3.10 (*b*), then

$$\vec{v_1'} = \vec{v_1} - \vec{V}_{cm} \text{ and } \vec{v_2'} = \vec{v_2} - \vec{V}_{cm} \text{ As } \vec{u_2} = 0,\ \vec{V}_{cm} = \frac{m_1 \vec{u_1}}{m_1 + m_2}$$

i.e., $\vec{V}_{cm}$ and $\vec{u_1}$ have the same direction along X-axis. Therefore $\vec{V}_{cm}$ has no component along the Y-axis. The y-component of the final velocity of the first particle of mass m_1 is the same in both the frames.

$\therefore$ $$v_1 \sin\theta_1 = v_1' \sin\theta \qquad ...(xiii)$$

As the centre of mass has a velocity $\vec{V}_{cm}$ along X-axis with respect to the Laboratory frame.

$\therefore$ $$v_1 \cos\theta_1 = v_1' \cos\theta + V_{cm} \qquad ...(xiv)$$

Dividing (*xiii*) by (*xiv*), we have

$$\tan\theta_1 = \frac{v_1' \sin\theta}{v_1' \cos\theta + V_{cm}} = \frac{\sin\theta}{\cos\theta + \dfrac{V_{cm}}{v_1'}}$$

but $$\vec{V}_{cm} = \frac{m_1}{m_1 + m_2} u_1$$

and $$v_1' = \frac{m_2}{m_1 + m_2} u_1$$ [For proof **See article 3.25**]

Dividing, we get

$$\frac{V_{cm}}{v_1'} = \frac{m_1}{m_2}$$

$$\therefore \qquad \tan\theta_1 = \frac{\sin\theta}{\cos\theta + \dfrac{m_1}{m_2}} \qquad ...(xv)$$

This gives the relation between scattering angle θ_1 in the laboratory frame and scattering angle θ in the centre of mass frame.

Special cases. (*i*) when $m_1 << m_2$. In this case m_1/m_2 can be neglected in relation (*iii*), and we get

$$\tan\theta_1 = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

Thus if the incident particle is very light as compared to the target particle, the angles of scattering for the incident particle in the laboratory and C.M. system are very nearly equal.

(*i*) **When $m_1 = m_2$.** In this case $m_1/m_2 = 1$

Hence $$\tan\theta_1 = \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin\theta/2\ \cos\theta/2}{1+2\cos^2\theta/2 - 1} = \tan\theta/2 \text{ or } \theta_1 = \theta/2 \qquad ...(xvi)$$

Thus if the incident and target particle are of equal masses, the angle of scattering in the laboratory system is half the angle of scattering in the C.M. system.

In other words, when θ takes the values from 0 to π, θ_1 varies from 0 to $\pi/2$. Hence all the particles in the laboratory frame are scattered in the *forward hemisphere* only.

(*iii*) **When $m_1 > m_2$.** The maximum value of $\tan\theta_1$ *i.e.*

$$\tan\theta_{1(max)} = \left[\frac{m_1^2 - m_2^2}{m_2^2}\right]^{-1/2}$$

from which we get $$\sin\theta_{1(max)} = \frac{m_2}{m_1},$$

When $m_1 > m_2$ *i.e.* $\frac{m_1}{m_2} > 1$ or $\frac{m_2}{m_1} < 1$, θ_1 has a finite positive value. Therefore, all particles in the laboratory frame are scattered in a *forward cone.*

(*b*) Relation between recoil angle θ_2 in the laboratory frame and scattering angle θ in the centre of mass frame. The y-component of the final velocity of the particle of mass m_2 is the same in both the frames.

$$\therefore \qquad v_2 \sin\theta_2 = v_2' \sin\theta$$

As the centre of mass has a velocity $\vec{V}_{cm}$ along X- axis with respect to laboratory frame,

$$v_2 \cos\theta_2 = \vec{V}_{cm} - v_2' \cos\theta$$

$$\therefore \qquad \tan\theta_2 = \frac{v_2'\sin\theta}{\vec{V}_{cm} - v_2'\cos\theta} = \frac{\sin\theta}{\dfrac{\vec{V}_{cm}}{v_2'} - \cos\theta}$$

Now $$v_2' = u_2' \text{ and } \vec{u}_2' = -\vec{V}_{cm}$$

or $$\frac{|\vec{V}_{cm}|}{|\vec{u}_2'|} = \frac{\vec{V}_{cm}}{u_2'} = \frac{\vec{V}_{cm}}{v_2'} = 1$$

Hence
$$\tan\theta_2 = \frac{\sin\theta}{1-\cos\theta}$$
$$= \frac{2\sin\theta/2\,\cos\theta/2}{1-1+2\sin^2\theta/2} = \cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$$
$$\therefore \qquad \theta_2 = \frac{\pi-\theta}{2} \text{ or } 2\theta_2 = \pi-\theta \qquad ...(xvii)$$

The relation $2\theta_2 = \pi - \theta$ is independent of m_1 and m_2 and hence independent of their velocities and energies.

Special case. When $m_1 = m_2$. The relation between scattering angle θ_1 in the Laboratory frame and the angle θ in the centre of mass frame is given by
$$\tan\theta_1 = \frac{\sin\theta}{\cos\theta + \frac{m_1}{m_2}}$$

If the two particles are of the same mass $m_1 = m_2$ then according to relation (*iv*)
$$\theta_1 = \theta/2 \text{ or } \theta = 2\theta_1$$

Substituting in (*xvii*) we get
$$2\theta_2 = \pi - 2\theta_1 \text{ or } \theta_2 = \pi/2 - \theta_1 \text{ or } \theta_1 + \theta_2 = \pi/2 \qquad ...(xviii)$$

In other words in the laboratory frame the two particles of the same mass will move at right angles to each other after collision, if one of these were at rest before collision.

SOLVED EXAMPLES

Example 3.1 Radioactive $_8O^{15}$ decays into $_7N^{15}$ by emitting a positron and a neutrino. The positron and the neutrino are observed to move at right angles to each other and carry momenta 2×10^{-22} and 5×10^{-23} kg m sec^{-1} respectively. Find the momentum of the recoiling nucleus.

Solution. Let the positron move along the $+X$ direction and the neutrino along $+Y$ direction with momenta $\vec{p}_1 = 2 \times 10^{-22} = 20 \times 10^{-23}$ kg m sec^{-1} and $\vec{p}_2 = 5 \times 10^{-23}$ kg m sec^{-1} respectively. If $\vec{P}$ is the momentum of the recoiling nucleus, then according to the principle of conservation of linear momentum
$$\vec{p}_1 + \vec{p}_2 + \vec{P} = 0 \quad \text{or} \quad \vec{P} = -(\vec{p}_1 + \vec{p}_2)$$

Now
$$\left|\vec{p}_1 + \vec{p}_2\right| = \sqrt{(20\times10^{-23})^2 + (5\times10^{-23})^2}$$
$$= 5\sqrt{17}\times10^{-23} \text{ kg m sec}^{-1}$$

and its direction is given by

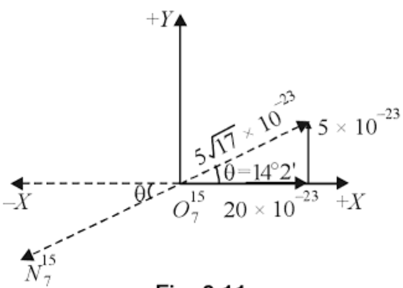

Fig. 3.11

$$\tan\theta = \frac{5\times10^{-23}}{20\times10^{-23}} = \frac{1}{4} = 0.25$$

or
$$\theta = 14°2'$$

$\therefore$ $\vec{P} = 5\sqrt{17}\times10^{-23}$ kg m sec^{-1} and it makes an angle 14°2′ with $-X$ direction or (180° + 14°2′) = 194°2′ with $+X$ direction.

Example 3.2 A body at rest explodes and breaks up into three pieces. Two pieces of equal mass fly off perpendicular to one another with the same speed of 30 m/sec. The third piece has three times the mass of each of the other pieces. Find the magnitude and direction of the veloc-

ity immediately after explosion.

Solution. As the body is at rest, initial linear momentum = 0. Let $\vec{p}_1$ and $\vec{p}_2$ be the linear momentum of the two pieces of equal mass and $\vec{P}$ the linear momentum of the heavy mass, then according to the principle of conservation of linear momentum

$$\vec{p}_1 + \vec{p}_2 + \vec{P} = 0 \quad \text{or} \quad \vec{P} = -(\vec{p}_1 + \vec{p}_2)$$

As the two pieces of equal mass fly off perpendicular to one another let one of them go towards $+X$ direction and the other in $+Y$ direction and m be their mass, then

$$\vec{p}_1 = 30\,m\hat{i} \quad \text{and} \quad \vec{p}_2 = 30\,m\,\hat{j}$$

$$\therefore \quad \vec{P} = -[30m\hat{i} + 30m\,\hat{j}]$$

Let $\vec{v}$ be the velocity of the heavy mass $3m$.

$$\therefore \quad \vec{P} = 3m\vec{v} = -[30m\hat{i} + 30m\,\hat{j}]$$

$$\text{or} \quad v = -10\hat{i} - 10\,\hat{j}$$

$\therefore$ Magnitude of

$$\vec{v} = \sqrt{(-10)^2 + (-10)^2} = 10\sqrt{2}\ m/\text{sec}.$$

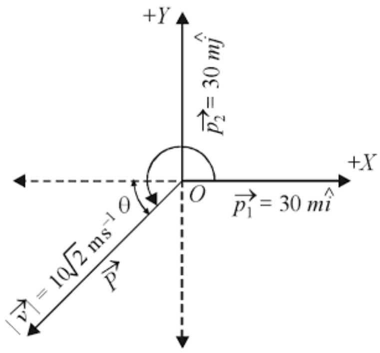

Fig. 3.12

Direction of $\vec{v}$ is given by $\tan\theta = \dfrac{-10}{-10} = +1$

or $\theta = 45°$ with $-X$ direction or $180 + 45° = 225°$ with $+X$ direction.

Example 3.3 A torque of 1 Nm is applied to a wheel of mass 10 kg and radius of gyration 50 cm. What is the resulting acceleration?

Solution. Here mass of the wheel, M = 10 kg

Radius of gyration, K = 50 cm = 0.5 m

Moment of inertia of the wheel, I = MK^2

$= 10 \times 0.5 \times 0.5 = 2.5 \text{ kg m}^2$

Torque, τ = 1 Nm

Now, $\tau = I\alpha$, where α is the angular acceleration

$$\therefore \quad \alpha = \frac{\tau}{I} = \frac{1}{2.5} = 0.4 \text{ rad s}^{-2}$$

Example 3.4 A particle of mass *m* moving in a circular orbit of radius *r* has angular momentum *J* about its centre. Calculate the kinetic energy of the particle in terms of *J*, *m* and *r*.

Solution. The angular momentum of a particle of mass m moving in a circular orbit of radius r is given by

$$J = mvr \qquad ...(i)$$

If E is the kinetic energy of the particle, then $E = \frac{1}{2}mv^2$.

From equation (*i*), we get $v = \dfrac{J}{mr}$ or $v^2 = \dfrac{J^2}{m^2r^2}$

$$\therefore \quad K.E. = \frac{1}{2}mv^2 = \frac{1}{2}m\frac{J^2}{m^2r^2} = \frac{J^2}{2mr^2}$$

Example 3.5 Two bodies of different masses are moving with the same kinetic energy of translation. Which has greater momentum?

Solution. Let m_a and v_a be the mass and velocity of mass A and m_b, v_b that of mass B with $m_a > m_b$, then

$$\text{K.E. of mass } A = \frac{1}{2} m_a v_a^2 \text{ and K.E. of mass } B = \frac{1}{2} m_b v_b^2$$

Now $$\frac{1}{2} m_a v_a^2 = \frac{1}{2} m_b v_b^2 \quad \therefore \quad \frac{m_a}{m_b} = \frac{v_b^2}{v_a^2} \text{ or } \sqrt{\frac{m_a}{m_b}} = \frac{v_b}{v_a}$$

Linear momentum of $A, p_a = m_a v_a$

Linear momentum of $B, P_b = m_b v_b$

$$\therefore \quad \frac{p_a}{p_b} = \frac{m_a v_a}{m_b v_b} = \frac{m_a}{m_b}\sqrt{\frac{m_b}{m_a}} = \sqrt{\frac{m_a}{m_b}} \quad \text{but } m_a > m_b$$

$$\therefore \quad p_a > p_b$$

i.e., the body A with greater mass has greater momentum.

Example 3.6 Centre of mass is at P(1, 1, 1) when system consists of particles of masses 2, 3, 4 and 5 kg. If the centre of mass shifts to Q (2, 2, 2) on removing 5 kg, what was its position?

Solution. The position vector of centre of mass is given by

$$\vec{R} = \frac{m_1 \vec{r_1} + m_2 \vec{r_2} + m_3 \vec{r_3} + m_4 \vec{r_4} + \ldots}{m_1 + m_2 + m_3 + m_4 + \ldots\ldots}$$

1st case $$\vec{R_1} = (\hat{i} + \hat{j} + \hat{k}) = \frac{2\vec{r_1} + 3\vec{r_2} + 4\vec{r_3} + 5\vec{r_4}}{2+3+4+5} \quad \ldots(i)$$

2nd case $$\vec{R_2} = (2\hat{i} + 2\hat{j} + 2\hat{k}) = \frac{2\vec{r_1} + 3\vec{r_2} + 4\vec{r_3}}{2+3+4} \quad \ldots(ii)$$

From (*i*) we have $$14\hat{i} + 14\hat{j} + 14\hat{k} = 2\vec{r_1} + 3\vec{r_2} + 4\vec{r_3} + 5\vec{r_4} \quad \ldots(iii)$$

From (*ii*) we have $$18\hat{i} + 18\hat{j} + 18\hat{k} = 2r_1 + 3\vec{r_2} + 4\vec{r_3} \quad \ldots(iv)$$

Subtracting (*iv*) from (*iii*), we get $-4\hat{i} - 4\hat{j} - 4\hat{k} = 5\vec{r_4}$

$$\therefore \quad \vec{r_4} = -\frac{4}{5}\hat{i} - \frac{4}{5}\hat{j} - \frac{4}{5}\hat{k}$$

$\therefore$ Co-ordinates of the mass of 5 kg are $\left(-\frac{4}{5}, -\frac{4}{5}, -\frac{4}{5}\right)$.

Example 3.7. Two bodies of masses 2 g and 10 g have position vectors $(3\hat{i} + 2\hat{j} - \hat{k})$ and $(\hat{i} - \hat{j} + 3\hat{k})$ respectively. Find the position vectors and the distance of centre of mass from the origin.

Ans. The position of centre of mass of two particles is given by $\vec{R} = \dfrac{m_1 \vec{r_1} + m_2 \vec{r_2}}{m_1 + m_2}$

$$\therefore \quad \vec{R} = \frac{2(3\hat{i} + 2\hat{j} - \hat{k}) + 10(\hat{i} - \hat{j} + 3\hat{k})}{2 + 10}$$

$$= \frac{16\hat{i} - 6\hat{j} + 28\hat{k}}{12} = \frac{4}{3}\hat{i} - \frac{1}{2}\hat{j} + \frac{7}{3}\hat{k}$$

Distance of centre of mass from the origin

$$\left|\vec{R}\right| = \sqrt{\left(\frac{4}{3}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{7}{3}\right)^2} = \sqrt{\frac{16}{9} + \frac{1}{4} + \frac{49}{9}} = 2.73 \text{ units.}$$

Example 3.8 The distance between Carbon and Oxygen atom in '*CO*' molecule is 1.12 Å. Find the centre of mass of '*CO*' molecule with respect to '*C*' atom.

Solution. Let us take the centre of mass as the origin of the co-ordinate system and axis of '*CO*' molecule along *X*-axis, then

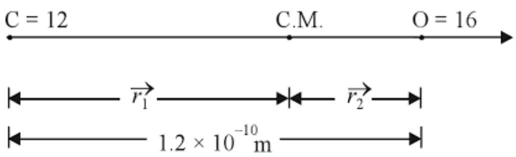

Fig. 3.13

Position of Carbon atom of mass 12

$$= \vec{r}_1 = -r_1\hat{i}$$

Position of Oxygen atom of mass 16

$$= \vec{r}_2 = +(1.12 \times 10^{-10} - r_1)\hat{i}$$

As centre of mass is taken as the origin of the co-ordinate system

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = 0$$

or $$m_1\vec{r}_1 + m_2\vec{r}_2 = 0$$

$\therefore$ $$-12r_1\hat{i} + 16(1.12 \times 10^{-10} - r_1)\hat{i} = 0$$

or $$28r_1\hat{i} = 16 \times 1.12 \times 10^{-10}\ \hat{i}$$

$\therefore$ $$r_1 = \frac{16 \times 1.12 \times 10^{-10}}{28} = 0.64 \times 10^{-10} \text{ m} = 0.64 \text{ Å}$$

As the position of Carbon atom with respect to centre of mass = $-r_1\hat{i}$, the position of centre of mass with respect to '*C*' atom $= +r_1 i = +0.64 \times 10^{-10}$ m.

Example 3.9 Two masses constrained to move in a horizontal plane collide. Given initially $m_1 = 85$ gms, $m_2 = 200$ gms; $u_1 = 6.48$ cms/sec and $u_2 = -6.78$ cms/sec, find the veloctiy of centre of mass.

Solution. The velocity of centre of mass is given by $\vec{V}_{cm} = \dfrac{m_1\vec{u}_1 + m_2\vec{u}_2}{m_1 + m_2}$

$\therefore$ $$\vec{V}_{cm} = \frac{85 \times 6.48 + 200 \times (-6.78)}{85 + 200} = 2.82 \text{ cm/sec}$$ in the direction of motion of m_2.

Example 3.10 Two particles each of mass 2 kg are moving with velocities $2\hat{i} + 4\hat{j}$ m/s and $5\hat{i} + 6\hat{j}$ m/s respectively. Find the kinetic energy of the system relative to centre of mass.

Solution. Given $m_1 = m_2 = 2$ kg.; $u_1 = 3\hat{i} + 4\hat{j}$; $u_2 = 5\hat{i} + 6\hat{j}$

Velocity of centre of mass $$\vec{V}_{cm} = \frac{m_1\vec{u}_1 + m_2\vec{u}_2}{m_1 + m_2} = \frac{2(3\hat{i} + 4\hat{j}) + 2(5\hat{i} + 6\hat{j})}{2 + 2} = 4\hat{i} + 5\hat{j}$$

Velocity of m_1 in centre of mass frame $\vec{u}_1' = \vec{u}_1 - \vec{V}_{cm}$

or $$\vec{u}_1' = 3\hat{i} + 4\hat{j} - 4\hat{i} - 5\hat{j} = -\hat{i} - \hat{j}$$

Velocity of m_2 in centre of mass $\vec{u}_2' = \vec{u}_2 - \vec{V}_{cm}$

or $$\vec{u}_2' = 5\hat{i} + 6\hat{j} - 4\hat{i} - 5\hat{j} = \hat{i} + \hat{j}$$

Kinetic energy relative to centre of mass before collision $= \dfrac{1}{2}m_1 u_1'^2 + \dfrac{1}{2}m_2 u_2'^2$

$$= \frac{1}{2} m_1 \left| \sqrt{(-1)^2 + (-1)^2} \right|^2 + \frac{1}{2} m_2 \left| \sqrt{(+1)^2 + (+1)^2} \right|^2$$
$$= 2 + 2 = 4 \text{ Joule.}$$

Example 3.11 Show that the rocket speed is twice the exhaust speed when $\frac{M_0}{M} = e^2$.

Solution. If V_0 is the speed of the rocket and V_e the exhaust speed, then $\frac{V_0}{V_e} = \log_e \frac{M_0}{M}$.

$$\text{Now } \frac{M_0}{M} = e^2 \qquad \therefore \frac{V_0}{V_e} = \log_e e^2 = 2$$

i.e., the rocket speed is twice the exhaust speed.

Example 3.12 A 6000 kg rocket is set for vertical firing. If the gas exhaust speed is 1000 ms^{-1}, how much gas must be ejected each second to supply the thrust needed (*i*) to overcome the weight of the rocket and (*ii*) to give the rocket an initial upward acceleration of 20 ms^{-2}?

Solution. (*i*) Here mass of the rocket and fuel $M = 6000$ kg

Exhaust velocity $V_e = -1000$ ms^{-1} Acceleration due to gravity $g = -9.8$ ms^{-2}

Net force on the rocket $M\frac{dV}{dt} = V_e \frac{dM}{dt} + Mg$

To overcome the weight of the rocket, the resultant force acting on it $M\frac{dV}{dt} = 0$

$$\therefore \quad 0 = -1000 \frac{dM}{dt} - 6000 \times 9.8 \quad \text{or} \quad \frac{dM}{dt} = -6 \times 9.8 = -58.8 \text{ kg s}^{-1}$$

(*ii*) To give the rocket an initial upward acceleration of 20 ms^{-2}, the resultant force

$$M\frac{dV}{dt} = Ma = 20 \times M \quad \therefore 20 \times 6000 = -1000 \times \frac{dM}{dt} - 6000 \times 9.8$$

$$\text{or} \qquad \frac{dM}{dt} = -\frac{6000\,(20 + 9.8)}{1000} = -178.8 \text{ kg s}^{-1}$$

Example 3.13 A rocket of mass 20 kg has 180 kg of fuel. The exhaust velocity of fuel is 1.60 km s^{-1}. Calculate the ultimate vertical speed gained by the rocket when the rate of consumption of fuel is 2 kg s^{-1}.

Solution. If V_f is the final velocity of the rocket when the entire fuel has been exhausted, V_e the exhaust velocity, and rocket starts from rest, then $V_f = -V_e \log_e \frac{M_0}{M_f}$

where M_0 is the initial mass of rocket and fuel and M_f the mass of the empty rocket.

Now $M_0 = 20 + 180 = 200$ kg

Consumption of fuel = 2 kg s^{-1}

Time to consume whole of fuel $= \frac{180}{2} = 90\,\text{s}$

Exhaust velocity $V_e = -1.6 \text{ kms}^{-1} = -1.6 \times 10^3 \text{ ms}^{-1}$

$$\therefore V_f = 1.6 \times 10^3 \times 2.3026 \log_{10} \frac{200}{20} = 3.684 \times 10^3 \text{ ms}^{-1} = 3.684 \text{ km s}^{-1}$$

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Example 3.14 A rocket starts vertically upward with a speed of v_0. Show that its speed at a height h is given by $v_0^2 - v^2 = \dfrac{2gh}{1 + h/R}$ where R is the radius of earth and g is acceleration due to gravity at earth's surface.

Solution. Let m be the mass of the rocket and M that of the earth, then

Kinetic energy of the rocket at earth's surface $= \dfrac{1}{2} m v_0^2$

and Potential energy of the rocket at earth's surface $= -\dfrac{GmM}{R}$

Negative sign indicates that it is a force of attraction.

When the rocket reaches a height h above the surface of earth, its

Kinetic energy $= \dfrac{1}{2} m v^2$ and potential energy $= \dfrac{-GmM}{R+h}$

According to the law of conservation of energy, we have

$$\frac{1}{2} m v_0^2 - \frac{GmM}{R} = \frac{1}{2} m v^2 - \frac{GmM}{R+h}$$

or
$$v_0^2 - v^2 = \frac{2GM}{R} - \frac{2GM}{R+h} = 2GM\left[\frac{1}{R} - \frac{1}{R+h}\right]$$

If g is the acceleration due to gravity on earth's surface, then $g = \dfrac{GM}{R^2}$

or $\quad GM = gR^2 \quad \therefore \; v_0^2 - v^2 = 2R^2 g\left[\dfrac{1}{R} - \dfrac{1}{R+h}\right] = \dfrac{2gh}{1 + h/R}$

EXERCISE CH. 3

LONG QUESTIONS

1. What is linear momentum? State and explain the principle of conservation of linear momentum with examples. (*Nagpur U.* 2009, 2007)

2. (*a*) Define angular momentum $\vec{J}$ and torque $\vec{\tau}$. What are their units? Show that torque is given by the time rate of change of angular momentum. (*Meerut U.* 2002; *M.D.U.* 2000; *Kerala U.* 2001; *Agra U.* 2007, 2005; *Nagpur U.* 2009)

   (*b*) If no torque acts on a body will its angular velocity remain conserved? (*Nagpur U.* 2008)

3. (*a*) Establish the vector relation between the force and the torque.

   (*b*) Establish the relation between torque applied and angular acceleration.

4. (*a*) State and explain the law of conservation of angular momentum. Illustrate with examples. What are the consequences of the law of conservation of angular momentum? (*G N.D.U.* 2001; *Luck. U.* 2001; *Meerut U.* 2001; *Guwahati U.* 2000; *Bang. U.* 2000; *Nagpur U.* 2008)

   (*b*) Explain why an ice skater always utilises the principle of conservation of angular momentum. (*M.D.U.* 2000)

5. (*a*) What does the term rotational invariance (isotropy) of freee space imply?

(*b*) Show that the property of rotational invariance of space (isotropy) leads to the law of conservation of angular momentum. (*M.D.U.* 2003; *G.N.D.U.* 2004, 2002, 2001, 2000; *Pbi, U.* 2001, 2000, *P.U.* 2000; *H.P.U.* 2003)

6. Define centre of mass. Show that when no external force acts on a body the acceleration of the centre of mass is zero and its velocity is a constant.
(*G.N.D.U.* 2002; *Indore U.* 2001; *Nagpur U.* 2007; *D.A.U. Agra* 2006)

7. What is centre of mass ? Find the total linear momentum of a system of particles about the centre of mass and show that it is zero. (*Nag. U.* 2009, 2007; *Gharwal U.* 2000)

8. (*a*) What is reduced mass? Reduce two body problem to one body problem and obtain equation of motion for equivalent one body problem for two masses.
(*H.P.U.* 2001, 2000; *Luck. U.* 2001; *P.U.* 2000; *Gharwal U.* 2000; *Purvanchal U.* 2005; *Nagpur U.* 2007)

(*b*) Discuss the motion of reduced mass under the influence of inverse square force.
(*H.P.U.* 2000)

(*c*) Distinguish between centre of mass and reduced mass. What is their importance in Physics?

9. Calculate the changes in the values of energy and angular momentum when a two body system interacting through gravitational force is reduced to an equivalent one body problem.

10. (*a*) (*i*) What is the position vector of centre of mass?
(*ii*) What is the velocity and acceleration of centre of mass?

(*b*) Show that the linear momentum of the system of two particles is equal to the linear momentum of the centre of mass. (*Gharwal U.* 2000; *P.U.* 2004; *Pbi. U.* 2003)

11. (*a*) What is a collision? Explain briefly elastic and inelastic scattering. Discuss two types of inelastic scattering. What is the difference between scattering and reaction?

(*b*) What is the advantage of studying a collision process?
(*G.N.D.U.* 2004, 2001; *Osm. U.* 2004; *Nagpur U.* 2003)

12. Discuss the phenomenon of collision in one dimension between two particles when the collision is (*i*) elastic and (*ii*) inelastic in the laboratory frame as well as in the centre of mass frame. (*Madurai U.* 2003; *Calicut U.* 2003)

13. (*a*) Prove that the shape of the orbit traced by a particle moving under attractive inverse square force depends on the angular momentum and total energy of the particle.

(*b*) What will be the shape of the orbit of a particle moving under repulsive inverse square force? Explain.

14. A particle of mass m_1 moving with velocity $\vec{u}_1$ suffers a perfectly inelastic collision with a particle of mass m_2 at rest. Calculate the kinetic energy of the system before and after collision into the laboratory system and centre of mass system. Show that the decrease in kinetic energy is the same in two cases.

15. (*a*) A particle of mass m_1 suffers perfectly elastic collision with another particle of mass m_2 at rest in the laboratory frame of reference. After scattering m_1 and m_2 move at angles θ_1 and θ_2 with respect to the original direction of m_1. Discuss the elastic collision between the two particles in the lab system. (*P.U.* 2000)

(*b*) Show that in the lab system the particles of the same mass will move at right angles to each other after collision if one of them were at rest before collision. (*P.U.* 2000)

16. Deduce the relation between scattering angles in the laboratory and centre of mass frames for particles undergoing elastic collision. Discuss the case when (*i*) $m_1 < m_2$ (*ii*) $m_1 = m_2$ (*iii*) $m_1 > m_2$. Show that when colliding particles are of equal masses, the scattering angle in laboratory system is one half of the scattering angle in centre of mass system.

(*Pbi. U.* 2003, 2001; *P.U.* 2004, 2001; *G.N.D.U.* 2002, 2000)

17. What is a rocket? Describe the principle of a rocket. Why multistage rocket is necessary? Establish the following relation for a rocket

$$V = V_0 + V_e \log_e \frac{M_0}{M}$$

calculate the burnt out velocity when rocket starts from rest.

(*Nagpur U.* 2009, 2007, 2003; *Cal. U.* 2003; *Meerut U.* 2001, 2000; *Agra U.* 2007, 2003)

18. Derive the relation between angle of recoil in laboratory system and angle of scattering in centre of mass system. Prove that in laboratory system two particles of equal masses move at right angles to each other after collision if one of them were at rest before collision. (*Pbi. U.* 2001; *G.N.D.U.* 2000; *P.U.* 2004)

19. A particle of mass m_1 suffers an elastic collision with a particle of mass m_2 initially at rest. If T_1 and T_2 are the kinetic energies of the two particles respectively and T_0 the total energy in the laboratory frame, T_1', T_2' and T_0' the corresponding values in centre of mass frame, show that $T_0 = T_0' + T_{c.m.}$ where $T_{c.m.}$ is the kinetic energy of C.M., and find the value of $\frac{T_0}{T_0'}$ and $\frac{T_1'}{T_2'}$.

20. Two equal masses M each are connected by a rigid rod of negligible mass and length 'a'. The centre of mass of this dumb-bell system is stationary in gravity free space and the system rotates about the centre of mass with angular velocity ω. One of the rotating masses strikes a third stationary mass M which sticks to it. What is the angular momentum of the three mass system about the centre of mass at the instant prior to the collision and at the instant following the collision ? $\left(\textbf{Ans. } \vec{J} = \frac{1}{2} m\, a^2 \omega \hat{k}\right)$

21. (*a*) A particle of mass m traces a circle of radius r under attractive inverse square force $-\frac{c}{r^2}$. Show that the energy of the particle at any point on the circle is $-\frac{c}{2r}$.

(*b*) Show that the differential equation of motion of a particle of mass m under the influence of a central isotropic force can be written as

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{J^2u^2} F\left(\frac{1}{u}\right)$$

where $u = \frac{1}{r}$, (r, θ) are the plane polar co-ordinates of the particle and J the angular momentum.

22. Prove that in centre of mass system the magnitude of the velocities of the particles remain unaltered in elastic collision. (*Pbi. U.* 2001; *P.U.* 2000)

23. (*i*) What is collision? Explain elastic and inelastic collisions.

(*ii*) Discuss the phenomenon of collision, in one dimension, between two particles when collision is perfectly elastic. (*Nagpur Uni.* 2009; *M.S.U. Tiruneveli* 2007) (*iii*) A neutron of mass 1.67×10^{-27} kg and speed 10^5 m/sec collides with a stationary deutron of mass 3.34×10^{-27} kg. The particles do not stick together and no kinetic energy is lost in collision. What is the subsequent speed of each particle.

(*Nagpur U.* 2009, 2006)

[**Ans.** Speed of neutron = 3.33×10^4 m/sec, Speed of deutron = 6.67×10^4 m/sec in opposite direction]

SHORT QUESTIONS

1. Derive the law of conservation of linear momentum from Newton's laws of motion.

2. Illustrate with examples the principle of conservation angular momentum. (*Nagpur U.* 2005)

3. State the principle of conservation of angular momentum. Show that the angular momentum of a particle under the influence of a central force always remains constant. (*G.N.D.U.* 2001)

4. Define angular impulse. Distinguish between momentum, angular impulse and linear impulse.

5. Give Newton's classical definition of time and its properties.

6. What is homogeneity of time? Show that homogeneity of time and Newton's second law of motion result in law of conservation of energy. (*G.N.D.U.* 2004; *P.U.* 2003, 2001, 2000)

7. Show that centre of mass of two particle system divides the line joining the two particles in the inverse ratio of their masses. (*P.U.* 2003; *Pbi. U.* 2003)

8. Explain laboratory and centre of mass systems (or frames of reference). What is the advantage of studying a collision process in centre of mass system? (*G.N.D.U.* 2004, 2001, 2000; *Meerut U.* 2003; *P.U.* 2004, 2001, 2000; *Pbi. U.* 2001, 2000; *Guwahati U.* 2000)

9. Prove that the kinetic energy of the two colliding particles in the centre of mass system are inversely proportional to their masses.

10. Show that in perfectly inelastic collision in Laboratory system there is always loss of kinetic energy. (*G.N.D.U.* 2004)

11. State the relation between velocities in centre of mass system and laboratory system. (*Pbi. U.* 2001)

12. A particle of mass m_1 moving with a velocity u collides head on with a particle of mass m_2 at rest such that after the collision they travel with velocity v_1 and v_2 respectively. If the collision is perfectly elastic one, show that $v_2 = \frac{2m_1u_1}{m_1 + m_2}$. (*Agra U.* 2007; *G.N.D.U.* 2003)

13. Discuss the motion of the top. Calculate the precessional velocity of a top and also show that $\vec{\tau} = \vec{\omega}_p \times \vec{L}$ where $\vec{\tau}$ is torque, $\vec{\omega}_p$ is precessional velocity and $\vec{L}$ is angular momentum. (*Meerut* 2005)

14. Show that the relative velocity between the particles after an elastic head-on collision is equal and opposite to the relative velocity before the collision. (*Agra U.* 2003)

15. Define centre of mass. Explain the position vector and motion of the centre of mass of a system of particles. (*M.S.U. Tiruneveli* 2007)

16. Deduce the equation of motion of the centre of mass. (*Nagpur U.* 2007)

NUMERICALS

1. A flywheel of mass 100 kg and radius of gyration 0.5 m is rotating with a speed of 90 revolutions per minute. Calculate the torque required to bring it to rest in 4 minutes.

(**Ans.** 0.98 Nm)

2. A massive ball comes moving and collides elastically with a comparatively very light stationary ball. Immediately after the collision, what is the approximate ratio of speeds of lighter and massive ball? (*Luck. U.* 2001)

[**Hint.** The velocity of heavy particle remains almost the same after collision and the light particle acquires nearly twice the velocity of heavy particle.

$$\frac{\text{Speed of lighter particle}}{\text{Speed of heavy (massive) particle}} = 2]$$

3. The earth is moving around the Sun under gravitational force and its orbit has semi-major axis of 1.495×10^8 km. When the earth passes closest to the Sun (*i.e.*, at its perhilion) its distance is 1.47×10^8 km and its orbital velocity is 0.303 km s^{-1}. Find earth's velocity at the apehilion and also its angular velocities at the two positions.

[**Hint.** Angular momentum $\vec{J} = r \times \overrightarrow{mv}$ is conserved.]

[**Ans.** v_a = 293 km s^{-1}; $\omega_p = 0.206 \times 10^{-8}$ rad s^{-1}; $\omega_a = 0.193 \times 10^{-9}$ rad s^{-1}]

4. A sand bag of mass 10 kg is suspended with a three metre long weight-less string. A bullet of mass 200 gm is fired with a speed of 20 m/sec into the bag and stays in the bag. Calculate the speed acquired by the bag and maximum displacement of the bag.

[**Ans.** v = 39.2 cm/sec.; Max. displacement = 29°54′]

5. A bomb weighing 50 kg explodes into three pieces in flight when its velocity is $20\hat{i} + 22\hat{j} + 10\hat{k}$ ms^{-1}. Two fragments of the bomb weighing 10 and 20 kg, are found to have velocities $100\hat{i} + 50\hat{j} + 20\hat{k}$ and $30\hat{i} + 20\hat{j} + 10\hat{k}$ ms^{-1}. Find the third piece.

[**Hint.** The mass of third piece = 20 kg. Let its velocity be $(x\hat{i} + y\hat{j} + z\hat{k})ms^{-1}$,

then $50(20\hat{i} + 22\hat{j} + 10\hat{k}) = 10(100\hat{i} + 50\hat{j} + 20\hat{k})$

$$+ 20(30\hat{i} - 20\hat{j} - 10\hat{k}) + 20(x\hat{i} + y\hat{j} + z\hat{k})]$$

[**Ans.** $(-30\hat{i} + 50\hat{j} - 25\hat{k})\,\text{ms}^{-1}$]

6. A grind stone weighing 40 kg has a radius of 1.2 m. Starting from rest it acquires a speed of 150 r.p.m. in 12 sec. Calculate the torque acting on it.

[**Hint.**
$$I = \frac{1}{2}MR^2 = \frac{1}{2} \times 40 \times 1.2 \times 1.2 = 28.8\ \text{Kg m}^2$$
$$\alpha = \frac{2\pi \times 150}{60 \times 12} = \frac{5\pi}{12}\ \text{rad s}^{-2} \qquad \tau = I\alpha = \frac{28.8 \times 5\pi}{12} = 37.7\ \text{Nm}\ ]$$

7. A body of mass 100 gms moving with a velocity $(8\hat{i} - 6\hat{j} - 10\hat{k})$ cm s^{-1} collides with a body of mass 200 gm moving with velocity $(-10\hat{i} + 6\hat{j} - 8\hat{k})$ cm s^{-1} and attain velocities $(3\hat{i} - 4\hat{j} - 5\hat{k})$ and $(-4\hat{i} + 5\hat{j} - 6\hat{k})$ cm s^{-1} respectively. Find kinetic energy before and after collision in the Lab system. Is the collision elastic ?

[**Hint.** Energy before collision

$$= \frac{1}{2} 100\,(8\hat{i} - 6\hat{j} - 10\hat{k})^2 + \frac{1}{2} 200(-10\hat{i} + 6\hat{j} - 8\hat{k})^2$$

$$= 50 \times 200 + 100 \times 200 = 30{,}000 \text{ ergs}$$

$$\text{Energy after collision} = \frac{1}{2} 100\,(3\hat{i} - 4\hat{j} - 5\hat{k})^2 + \frac{1}{2} 200\,(-4\hat{i} + 5\hat{j} - 6\hat{k})^2$$

$$= 50 \times 50 + 100 \times 77 = 10{,}200 \text{ ergs.}$$

The collision is *inelastic* as the kinetic energy is not conserved.]

8. The world's first artificial satellite was reported to be circling the globe at a distance of 900 km. Calculate its minimum velocity and period of revolution. Radius of the earth = 6.4×10^6 metres and $g = 9.8$ m/sec^2. [**Ans.** (*i*) 7.416 m/sec; (*ii*) 1 hr 43 min 4 sec]

9. Two particles of masses 100 and 300 gm have at a given time positions $2\hat{i} + 5\hat{j} + 13\hat{k}$ and $-6\hat{i} + 4\hat{j} - 2\hat{k}$ cm. respectively and velocities $10\hat{i} - 7\hat{j} - 3\hat{k}$ and $7\hat{i} - 9\hat{j} + 6\hat{k}$ cm/s respectively. Find (*a*) the instanteneous position and (*b*) velocity of centre of mass.

(*Nagpur U.* 2007) [**Ans.** (*a*) $\dfrac{-16\hat{i} + 17\hat{j} + 7\hat{k}}{4}$ m/s, (*b*) $\dfrac{31\hat{i} - 34\hat{j} + 15\hat{k}}{4}$ m/s^2]

10. A 200 kg bomb falling freely explodes and splits into two fragments, a larger one of 160 kg and a smaller one of 40 kg. The former moves away with a speed $\vec{V} = \hat{i} + 2\hat{j} + 3\hat{k}$.

What is the speed of the later? [**Ans.** $-4(\hat{i} + 2\hat{j} + 3\hat{k})$]

11. The centre of mass of three particles of masses 4 gm, 5 gm and 6 gm respectively lies at the point (2, 2, 2). Where should a fourth particle of mass 12 gm be placed so that the centre of mass of the system of four particles lies at (0, 0, 0)? (*M.S.U. Tiruneveli* 2007)

[**Ans.** At the point (– 2.5, – 2.5, – 2.5)]

12. In a head on collision, a particle with an initial speed v_0 strikes a stationary particle of the same mass. Find the velocities v_1 and v_2 of the two particles respectively after the collisions if half of the original kinetic energy is lost. (*D.A.U. Agra* 2006)

4

DYNAMICS OF RIGID BODY

INTRODUCTION

The inability of a body to change by itself its state of rest or of uniform motion along a straight line (Newtons first law of motion) is an inherent property of every body and is known as *inertia*. The greater the mass, larger is its inertia. Therefore, mass is taken as a measure of inertia for linear motion. In the similar manner, a body free to rotate about its axis opposes any change in its state of rest or uniform rotation. In other words, it possesses inertia for rotational motion *i.e.* the body opposes the torque or moment of the couple applied to it, hence the name 'moment of inertia'. Moment of Inertia (M.I.) of any rigid body depends upon the distribution of its constituent particles about its axis of rotation. Thus, mass in linear motion is analogous to its moment of inertia in rotational motion.

4.1 INERTIA

According to Newton's first law of motion a body at rest will remain at rest and a body moving with uniform velocity in a straight line will continue to do so unless an external force is applied to it. This property of a body by virtue of which it is unable to change its state of rest or of uniform motion in a straight line by itself is known as **"inertia".** For translatory motion the value of inertia depends only on the mass of the body. The greater is the mass greater is the inertia.

4.2 MOMENT OF INERTIA

In order to define moment of inertia of a body, we will find its kinetic energy of rotation.

Kinetic energy of rotation. For translatory motion kinetic energy depends upon mass m and velocity v and is given by $\frac{1}{2}mv^2$.

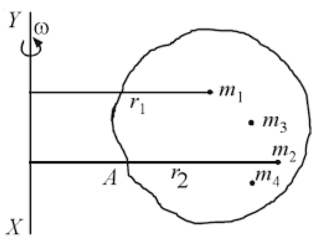

Fig. 4.1

When a body rotates about an axis, the kinetic energy of its rotation is determined not only by its mass m and angular velocity ω, but also depends upon the position of the axis about which it rotates and the distribution of mass about this axis.

If a body A rotates about an axis XY (Fig. 4.1) with an angular velocity ω, all its particles have the same angular velocity, but as they are at different distances from the axis of rotation, their linear velocities are different. Let the linear velocities of the particles of mass m_1, m_2...; distant r_1, r_2... from the axis of rotation be v_1, v_2... respectively. The kinetic energy of the body is, therefore, equal to the sum of the kinetic energies of the various particles and is given by

$$\text{Total K.E.} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \ldots$$

Since $v = r\omega$

$$\therefore \text{Total K.E.} = \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + \frac{1}{2} m_3 r_3^2 \omega^2 + \ldots$$

$$= \frac{1}{2} (\Sigma mr^2) \omega^2 = \frac{1}{2} \omega^2 \Sigma mr^2 = \frac{1}{2} I\omega^2$$

where I is the moment of inertia of the body about the axis XY and is equal to Σmr^2.

We have expressed the moment of inertia as Σmr^2.

Hence *the moment of inertia of a body about an axis is defined as the sum of the products of the mass and the square of the distance of the different particles of the body from the axis of rotation.*

In the above case we have seen that K.E. of rotation $= \frac{1}{2} I\omega^2$

If $\omega = 1$, then $I = 2 \times$ kinetic energy.

Hence moment of inertia may also be defined as twice the kinetic energy of rotation of a body when its angular velocity is unity.

Radius of gyration. If the entire mass of the body is supposed to be concentrated at a point such that the kinetic energy of rotation is the same as that of the body itself, then the distance of that point from the axis of rotation is called the *radius of gyration* of the body about that axis. If k denotes the radius of gyration and M the mass of the body supposed to be concentrated at the point, then we have

$$K.E. = \frac{1}{2} I\omega^2 = \frac{1}{2} \Sigma mr^2 \omega^2 = \frac{1}{2} Mk^2 \omega^2$$

$$\therefore \qquad Mk^2 = \Sigma mr^2 = mn \left(\frac{r_1^2 + r_2^2 + \ldots}{n} \right)$$

where n is the number of particles each of mass m into which the given mass M is divided.

$$\text{Now} \qquad M = mn \qquad \therefore k = \left(\frac{r_1^2 + r_2^2 + \ldots}{n} \right)^{1/2}$$

Hence *the radius of gyration is the square root of the mean square distance of the particles of the body from the axis of rotation.*

According to the definiton of radius of gyration given above the dimensions of k are those of length $[L^1]$ alone.

Now moment of inertia $I = Mk^2$ $\therefore$ Dimensions of $I = [M^1 L^2]$

Units of moment of inertia. In S.I. units the moment of inertia is expressed as kg-m^2.

Moment of inertia - a scalar. Moment of inertia is a *scalar* quantity because the value of I about a given axis remains unchanged by reversing its direction of rotation about that axis. In other words it has no direction. The total moment of inertia of a number of bodies about a given axis is equal to the sum of their individual moments of inertia about that axis.

4.3 PHYSICAL SIGNIFICANCE OF M.I.

Moment of inertia plays the same role in rotatory motion as mass does in linear motion, *i.e.*, moment of inertia is an analogue of mass in linear motion.

According to Newton's first law of motion, a body continues in its state of rest or of uniform motion in a straight line unless some external force acts upon it. This property of matter is known as

inertia. A body always resists an external force tending to change its state of rest or of linear motion. *Greater the mass of the body greater is the force required to produce a given linear acceleration.*

Similarly bodies possess rotational inertia, *i.e.*, a body free to rotate about an axis opposes any change in its state of rest or of rotation. *Greater the moment of inertia of a body greater is the couple required to produce a given angular acceleration*.

The moment of inertia depends not only on the mass of a body but also on the distribution of mass about the axis of rotation.

If a solid disc and a wheel have the same mass, wheel will have a greater moment of inertia as the mass in it is distributed at larger distances from the axis of rotation passing through the centre. The analogy between the moment of inertia in rotational motion and mass in linear motion will be clear from the similarity in the relation for momentum, force, impulse, energy and work as illustrated below:

4.4 ANALOGY BETWEEN TRANSLATORY MOTION AND ROTATORY MOTION

It is noticed that there is an anology between various physical quantities in translatory motion and rotatory motion. The following table gives this analogy in these two types of motions.

Sr. No.	*Translatory motion*	*Rotatory motion*
1.	Mass, m	Moment of Inertia, I
2.	Displacement, S	Angular displacement, θ
3.	Velocity, V	Angular velocity, ω
4.	Linear acceleration, a	Angular acceleration, α
5.	Linear momentum = mv	Angular momentum = $I\,\omega$
6.	Force = ma	Torque or moment of the couple = I × angular acceleration = $I\,\alpha$
7.	Impulse = $m\,(v_2 - v_1)$	Angular impulse = $I\,(\omega_2 - \omega_1)$
8.	kinetic energy = $\frac{1}{2}mv^2$	Rotational K.E. = $\frac{1}{2}I\omega^2$
9.	Work = Force × distance	Work = Couple × angular displacement

4.5 THEOREM OF PERPENDICULAR AXIS

It states that the moment of inertia of a plane lamina about an axis perpendicular to the plane of the lamina is equal to the sum of the moments of inertia of the lamina about the two axes at right angles to each other, in its own plane intersecting each other at the point where the perpendicular axis passes through it.

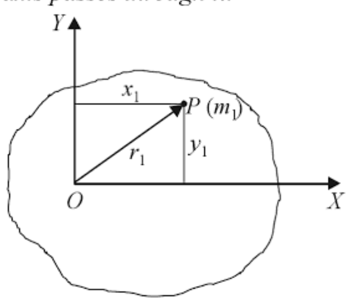

Fig. 4.2

To put the above in mathematical form let I_x and I_y be the moments of inertia about the two axes perpendicular to each other in the plane of the lamina then the moment of inertia I about a line passing through the point of intersection and perpendicular to its plane is given by $I = I_x + I_y$.

Let OX and OY be the two perpendicular axes in the plane of the lamina. Let m_1 be the mass of a particle distant r_1 from an axis through O perpendicular to the plane XOY. The distance of this particle from the Y-axis is x_1 and that from the X-axis is y_1.

Moment of inertia of this particle about the X-axis = $m_1y_1^2$ and moment of inertia of this particle about the Y-axis = $m_1x_1^2$.

If we divide the whole lamina into a number of particles of masses m_1, m_2, m_3... etc. at distances r_1, r_2, r_3 ... etc. so that the corresponding distances are y_1, y_2, y_3 ...from the X-axis and x_1, x_2, x_3 ... from the Y-axis, then

Moment of inertia of the lamina about X-axis

$$I_x = m_1y_1^2 + m_2y_2^2 + ... = \Sigma my^2$$

and the moment of inertia of the lamina about the Y-axis

$$I_y = m_1x_1^2 + m_2x_2^2 + ... = \Sigma mx^2$$

∴ Moment of inertia of the lamina about a perpendicular axis through O

$$\begin{aligned} I &= m_1r_1^2 + m_2r_2^2 ... = \Sigma mr^2 \\ &= m_1 (x_1^2 + y_1^2) + m_2 (x_2^2 + y_2^2) + ... \\ &= m_1x_1^2 + m_2x_2^2 + ... + m_1y_1^2 + m_2y_2^2 + ... \\ &= \Sigma mx^2 + \Sigma my^2 = I_x + I_y \end{aligned}$$

4.6 THEOREM OF PARALLEL AXES

It states that the moment of inertia of a body about any axis is equal to the sum of its moment of inertia about a parallel axis through its centre of gravity and the product of its mass and the square of the distance between the two axes.

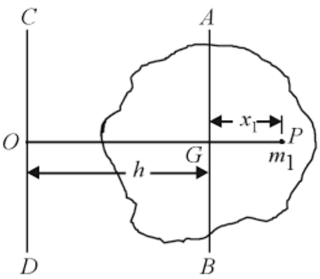

Fig. 4.3

Let CD be an axis in the plane of the paper and AB a parallel axis through G the centre of mass of the body. The perpendicular distance between the two axes is h. Let M be the mass of the body and m_1 the mass of the element at P distant x_1 from AB.

Moment of inertia of m_1 about $CD = m_1 (x_1 + h)^2$

$$\begin{aligned} &= m_1 (x_1^2 + h^2 + 2x_1h) \\ &= m_1x_1^2 + m_1h^2 + 2m_1x_1h \end{aligned}$$

∴ Moment of inertia of the body about CD

$$I = \Sigma m_1x_1^2 + \Sigma m_1h^2 + 2\Sigma m_1x_1h$$

If I_g is the moment of inertia of the body about AB, an axis through G, then $\Sigma m_1x_1^2 = I_g$

∴ $$I = I_g + Mh^2 + 2h\Sigma m_1x_1$$

Now Σm_1x_1 is the sum of the moments of all the particles about AB passing through G the centre of gravity. Since the body is balanced about the centre of mass G, therefore the algebraic sum of all the moments about G is zero.

∴ $$\Sigma m_1x_1 = 0$$

Hence $$I = I_g + Mh^2$$

4.7 MOMENT OF INERTIA OF A CIRCULAR DISC ABOUT AN AXIS THROUGH ITS CENTRE PERPENDICULAR TO ITS PLANE

Let M be the mass of the disc and R its radius. Consider an elementary ring of radius x and width dx as shown in Fig. 4.4. Its area is equal to the product of the circumference and width *i.e.*, $2\pi xdx$.

$$\text{Mass per unit area} = \frac{M}{\pi R^2}$$

$\therefore$ Mass of the element $= \dfrac{M}{\pi R^2} 2\pi x dx = \dfrac{2M}{R^2} x dx$

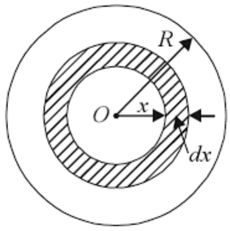

Fig. 4.4

Moment of inertia of the element about an axis through its centre perpendicular to its plane

$$= \frac{2M}{R^2} x dx \; x^2 = \frac{2M}{R^2} x^3 \; dx$$

Hence moment of inertia of the whole disc about this axis

$$I = \frac{2M}{R^2} \int_0^R x^3 \, dx = \frac{2M}{R^2} \left[\frac{x^4}{4} \right]_0^R = \frac{1}{2} MR^2$$

(*a*) **Moment of inertia of a disc about its diameter.** The moment of inertia of a circular disc about an axis perpendicular to its plane and passing through its centre is given by

$$I = \frac{1}{2} MR^2$$

where M is the mass and R its radius.

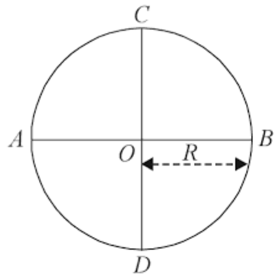

Fig. 4.5

Now consider two perpendicular diameters AB and CD of the circular disc as in Fig. 4.5. Since all the diameters are symmetrical the moment of inertia of the disc about one diameter is the same as that about any other diameter.

If I_1 and I_2 are the moment of inertia of the disc about two axes perpendicular to each other, then applying the *principle of perpendicular axis*, the moment of inertia I of the disc about an axis perpendicular to the plane of the disc through O.

$$I = I_1 + I_2$$

Since the two diameters are symmetrical with respect to the disc $I_1 = I_2$

$$\therefore \quad I = 2I_1 \quad \text{or} \quad I_1 = \frac{I}{2} = \frac{MR^2}{2} \times \frac{1}{2} = \frac{MR^2}{4}$$

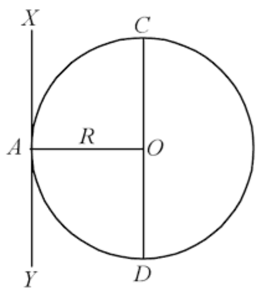

Fig. 4.6

(*b*) **Moment of inertia about a tangent.**

Moment of inertia of a disc about a diameter CD, $I_1 = \dfrac{MR^2}{4}$

Applying the principle of parallel axis, the moment of inertia I about the tangent XY at A. (Fig. 4.6)

$$I = I_1 + M \times OA^2 = \frac{MR^2}{4} + MR^2 = \frac{5}{4} MR^2$$

4.8 MOMENT OF INERTIA OF A SOLID CYLINDER

(*i*) **About its axis of Symmetry**

A cylinder is a thick circular disc or it may be considered to be a combination of a number of thin discs each of mass m and radius R placed one above the other. The moment of inertia of the cylinder will be equal to the sum of the moments of inertia of each of the discs about an axis through the centre and perpendicular to the plane.

$$\text{Moment of inertia of one disc} = \frac{1}{2} mR^2$$

$$\therefore \quad \text{Moment of inertia of the cylinder} = \Sigma \frac{1}{2} mR^2 = \frac{1}{2} R^2 \; \Sigma m = \frac{1}{2} MR^2$$

where M is the mass of the cylinder.

(*ii*) **Moment of inertia of the solid cylinder about an axis passing through its centre and perpendicular to its own axis of symmetry.** Let M be the mass of the cylinder, R its radius and l its length then its mass per unit length $= \frac{M}{l}$.

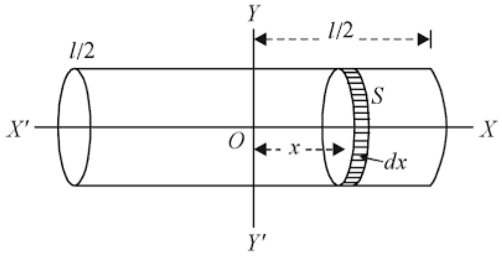

Fig. 4.7

Let YY' be the axis passing through its centre and perpendicular to its own axis XX' (Fig. 4.7).

Consider a thin slice S of thickness dx at a distance x from O.

Mass of the disc $S = \frac{M}{l}\, dx$

Moment of inertia of this disc about any diameter $= \frac{M}{l}\, dx \times \frac{R^2}{4}$

The moment of inertia of this disc about a parallel axis YY'; by the principle of parallel axis $= \frac{M}{l}\, dx\, \frac{R^2}{4} + \frac{M}{l}\, dx\, x^2$

The moment of inertia of the cylinder about the axis YY' is obtained by integrating the above expression within the limits $x = +\frac{l}{2}$ and $x = -\frac{l}{2}$.

$$\therefore \quad I = \frac{M}{l}\int_{-l/2}^{+l/2}\left[\frac{R^2}{4} + x^2\right] dx = \frac{M}{l}\left[\frac{R^2}{4}\,x + \frac{x^3}{3}\right]_{-l/2}^{+l/2} = \frac{M}{l}\left[\frac{R^2}{4}\left(\frac{l}{2}+\frac{l}{2}\right) + \left(\frac{l^3}{24}+\frac{l^3}{24}\right)\right]$$

$$= \frac{M}{l}\left[\frac{R^2 l}{4} + \frac{l^3}{12}\right] = M\left[\frac{R^2}{4} + \frac{l^2}{12}\right]$$

4.9 MOMENT OF INERTIA OF A THIN ROD

(*i*) The moment of inertia of a solid cylinder of mass M, length l and radius R about an axis passing through its centre and perpendicular to its own axis of symmetry is given by

$$I = M\left(\frac{R^2}{4} + \frac{l^2}{12}\right)$$

In the case of a thin rod $R = 0$

$$\therefore \qquad I = M\,\frac{l^2}{12}$$

and radius of gyration $\qquad k = \left(\frac{l^2}{12}\right)^{1/2}$

If this rod rotates about the above axis with constant angular speed ω, then its

$$\text{Kinetic energy} = \frac{1}{2} I\omega^2 = \frac{1}{2}\,\frac{Ml^2}{12}\,\omega^2 = \frac{1}{24}\,Ml^2\omega^2$$

and $\qquad$ Angular momentum $= I\omega = \frac{Ml^2}{12}\,\omega = \frac{1}{12}\,Ml^2\omega$

(*ii*) Moment of inertia of a thin uniform rod of mass M and length l about an axis passing through its centre and perpendicular to its length $= M\,\frac{l^2}{12}$

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∴ Moment of inertia about an axis passing through one end and perpendicular to its length by the *principle of parallel axes,* $I = M\left[\frac{l^2}{12} + \left(\frac{l}{2}\right)^2\right] = M\left[\frac{l^2}{12} + \frac{l^2}{4}\right] = M\,\frac{l^2}{3}$

4.10 M.I. OF AN ANNULAR RING

(*a*) About an axis passing through its centre and perpendicular to its plane. An annular ring or an annular disc is an ordinary disc having a concentric circular hole in it.

Let R_1 and R_2 be the radii of the inner and the outer discs and let M be its mass.

∴ Area of the face of the disc $= \pi\,(R_2^{\,2} - R_1^{\,2})$

∴ Mass per unit area $= \dfrac{M}{\pi\,(R_2^{\,2} - R_1^{\,2})}$

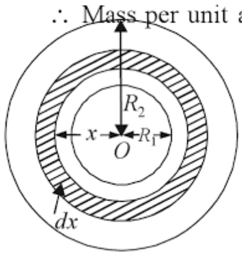

Fig. 4.8

Consider a coaxial disc or a ring of inner radius x and outer radius $(x + dx)$, then

Face area of the ring $= 2\pi x . dx$

∴ Mass of the ring $= \dfrac{M}{\pi(R_2^{\,2} - R_1^{\,2})}\,.\,2\pi x dx = \dfrac{2Mx\,dx}{(R_2^2 - R_1^2)}$

∴ Moment of inertia of this ring about an axis passing through O and perpendicular to its plane

$$= \frac{2Mxdx}{R_2^{\,2} - R_1^{\,2}}\,x^2 = \frac{2Mx^3}{R_2^{\,2} - R_1^{\,2}}\,dx$$

The moment of inertia of the whole annular disc is obtained by integrating the above expression between the limits $x = R_1$ and $x = R_2$.

∴ M.I. of the annular disc about an axis through O and perpendicular to the plane

$$= \frac{2M}{(R_2^{\,2} - R_1^{\,2})}\int_{R_1}^{R_2} x^3\,dx = \frac{2M}{(R_2^{\,2} - R_1^{\,2})}\left[\frac{x^4}{4}\right]_{R_1}^{R_2}$$

$$= \frac{2M}{(R_2^{\,2} - R_1^{\,2})}\left[\frac{R_2^{\,4} - R_1^{\,4}}{4}\right] = \frac{M\,(R_2^{\,2} + R_1^{\,2})}{2} \qquad \text{... }(i)$$

(*b*) About its diameter. Since the annular ring is symmetrical, the moment of inertia of the annular disc about one diameter is equal to the moment of inertia about another diameter. If I is the moment of inertia about any diameter, then the sum of the moments of inertia about two perpendicular diameters, by the principle of perpendicular axes, will be equal to the moment of inertia about an axis through O and perpendicular to its plane, *i.e.,* $\dfrac{M\,(R_2^{\,2} + R_1^{\,2})}{2}$

$$\therefore \quad I + I = \frac{M\,(R_2^{\,2} + R_1^{\,2})}{2} \quad \text{or} \quad I = \frac{M\,(R_2^{\,2} + R_1^{\,2})}{4} \qquad \text{... }(ii)$$

(*c*) M.I. of a Thin Ring. The M.I. of a thin ring made up of a thin wire about an axis passing through its centre and perpendicular to its plane, is obtained by putting $R_1 = R_2 = R$, say in Eq. (i),

We have
$$I = \frac{M\,(2R^2)}{2} = MR^2 \qquad \text{... }(iii)$$

4.11 MOMENT OF INERTIA OF A HOLLOW CYLINDER

(*i*) **About its own Axis**

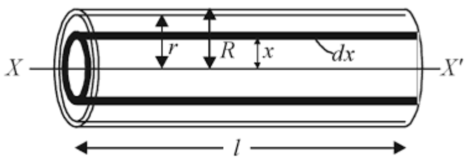

Fig. 4.9

Let R and r be the external and the internal radii respectively of the cylinder of length l, and mass M, then

Area of the face of the cylinder $= \pi\,(R^2 - r^2)$

Volume of the cylinder $= \pi\,(R^2 - r^2)\,l$

$$\text{Mass per unit volume } = \frac{M}{\pi(R^2 - r^2)\,l}$$

Imagine that the cylinder is made up of a large number of thin coaxial cylinders. Let one of such cylinders be of radius x and thickness dx, then

Face area of such a cylinder $= 2\pi x dx$

and Volume of such a cylinder $= 2\pi x dx l$

$$\therefore \quad \text{Mass of such a cylinder} = \frac{M}{\pi(R^2 - r^2)l} \times 2\pi x.dx.l = \frac{2Mxdx}{(R^2 - r^2)}$$

The moment of inertia of such a cylinder about its axis $XX' = \dfrac{2Mxdx}{(R^2 - r^2)}\,x^2 = \dfrac{2M}{(R^2 - r^2)}\,x^3 dx$

Hence moment of inertia of the cylinder about its axis can be obtained by integrating the above expression between the limits $x = r$ and $x = R$.

$$\therefore \text{Moment of inertia } I = \int_r^R \frac{2M}{(R^2 - r^2)}\,x^3.\,dx = \frac{2M}{(R^2 - r^2)}\left[\frac{x^4}{4}\right]_r^R = \frac{2M}{R^2 - r^2}\left[\frac{R^4 - r^4}{4}\right]$$

$$= \frac{M\,(R^2 + r^2)}{2}$$

Alternative method. If we suppose the hollow cylinder to be made up of a number of annular rings of mass m each placed one above the other, then

$$\text{M.I.} = \Sigma m\,\frac{R^2 + r^2}{2} = M\,\frac{R^2 + r^2}{2}$$

(*ii*) **About an axis through the centre and perpendicular to its own axis**

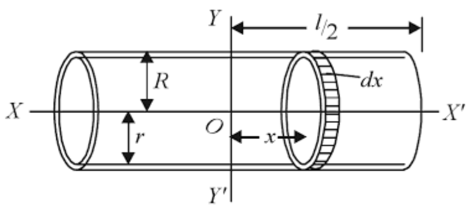

Fig. 4.10

Let M, R, r and l be the mass, external radius, internal radius and length of the cylinder respectively, then

$$\text{Mass per unit volume} = \frac{M}{\pi\,(R^2 - r^2)\,l}$$

Imagine the cylinder to be made of a large number of annular discs of external and internal radii R and r respectively, placed one above the other.

Consider one such disc of thickness dx at a distance x from the axis YY', then

Volume of the disc $= \pi\,(R^2 - r^2)\,dx$

$$\therefore \quad \text{Mass of the disc} = \frac{M}{\pi(R^2 - r^2)l} \times \pi\,(R^2 - r^2)\,dx = \frac{M}{l}\,dx$$

Moment of inertia of the annular disc of external and internal radii R and r respectively about its own diameter $= \text{Mass} \times \dfrac{(R^2 + r^2)}{4}$

or M.I. about its diameter $= \dfrac{M}{l}\, dx\, \dfrac{(R^2 + r^2)}{4}$

Hence by the principle of parallel axes the moment of inertia of the disc about the axis YY' is given by

$$\frac{M\,(R^2 + r^2)}{4l}\, dx + \frac{M}{l}\, dx\, x^2$$

Then moment of inertia I of the hollow cylinder can be obtained by integrating the above expression between the limits

$$x = +\frac{l}{2} \text{ and } x = -\frac{l}{2}.$$

$$\therefore \quad I = \int_{-l/2}^{+l/2} \frac{M\,(R^2 + r^2)}{4l}\, dx + \int_{-l/2}^{+l/2} \frac{M}{l}\, x^2 dx$$

$$= \frac{M\,(R^2 + r^2)}{4l}\, [x]_{-l/2}^{+l/2} + \frac{M}{l}\left[\frac{x^3}{3}\right]_{-l/2}^{+l/2} = \frac{M\,(R^2 + r^2)}{4l}\left[\frac{l}{2} + \frac{l}{2}\right] + \frac{M}{l}\left[\frac{l^3}{24} + \frac{l^3}{24}\right]$$

$$= M\left[\frac{(R^2 + r^2)}{4} + \frac{l^2}{12}\right]$$

4.12 M.I. OF A SOLID SPHERE

(*i*) **About its diameter:** A section of a sphere of radius r through the centre O is shown in Fig. 4.11. Let AB be the diameter about which the moment of inertia is to be found. Consider a thin circular slice of thickness dx at a distance x from the centre, then

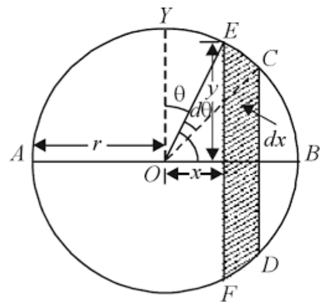

Fig. 4.11

Radius of the slice, $y = \sqrt{r^2 - x^2}$

$\therefore$ Volume of slice $= \pi\,(r^2 - x^2)\, dx$

Let m be the mass per unit volume of the sphere, then

Mass of slice, $M_1 = m\pi\,(r^2 - x^2)\, dx$

Moment of inertia of the slice (disc) about AB

$$= \frac{1}{2} M_1 y^2 = \frac{1}{2} m\pi\,(r^2 - x^2)\, dx\,(r^2 - x^2)$$

$$= \frac{1}{2} m\pi\,(r^2 - x^2)^2\, dx$$

Considering all such elementary discs, the moment of inertia of the sphere about the diameter AB is given by

$$I = 2\int_0^r \frac{1}{2} m\pi\,(r^2 - x^2)^2\, dx = m\pi \int_0^r (r^4 + x^4 - 2r^2x^2)\, dx$$

$$= m\pi\left[r^4 x + \frac{x^5}{5} - \frac{2r^2x^3}{3}\right]_0^r = m\pi\left[r^5 + \frac{r^5}{5} - \frac{2r^5}{3}\right] = \frac{8m\pi r^5}{15}$$

Now mass of the sphere $M = \dfrac{4}{3}\pi r^3\, m$

$$\therefore \quad I = \frac{8m\pi r^5}{15} = \frac{4}{3}\pi r^3 m \times \frac{2}{5} r^2 = \frac{2}{5} Mr^2$$

or Moment of inertia of the sphere about a diameter $I = \dfrac{2}{5} Mr^2$

(ii) **Moment of inertia of the sphere about a tangent.** The tangent to the sphere at any point is parallel to one of its diameters and is at a distance equal to r from the centre. Therefore, by the principle of parallel axis, the moment of inertia of the sphere about a tangent

$$= \frac{2}{5} Mr^2 + Mr^2 = \frac{7}{5} Mr^2$$

4.13 M.I. OF A SPHERICAL SHELL (HOLLOW SPHERE) (*about its diameter*)

Consider a section of a spherical shell through its centre O. Let r be its radius and M its mass, then

Surface area of the shell = $4\pi r^2$ $\quad\therefore$ Mass per unit area $= \dfrac{M}{4\pi r^2}$

Now consider a thin element of the shell bounded by two parallel planes EF and CD at a distance x and $x + dx$ respectively from O. The slice has a radius equal to y and width EC (not dx).

Area of the thin element of the shell = circumference × width = $2\pi y \times EC$

$$= 2\pi y\, rd\theta = 2\pi\, ryd\theta \qquad ...(i)$$

It is clear from the figure that,

$$y = r\cos\theta \text{ and } x = r\sin\theta$$

Differentiating, we have $dx = r\cos\theta d\theta = y\, d\theta$

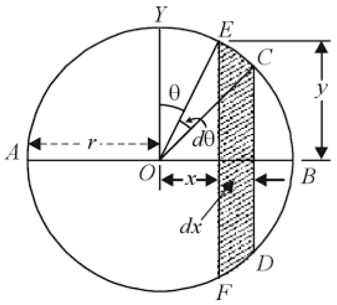

Fig. 4.12

Substituting $dx = y\, d\theta$ in (i), we have

Area of the element = $2\pi r\, dx$

$$\therefore \text{ Mass of the element } = \frac{M}{4\pi r^2} 2\pi r\, dx = \frac{M}{2r} dx$$

Now $y^2 = r^2 - x^2$

Moment of inertia of the slice about a diameter AB

$$= \frac{M}{2r} dx\, (r^2 - x^2)$$

$\therefore$ Moment of inertia of the shell about a diameter

$$= \int_{-r}^{+r} \frac{M}{2r}(r^2 - x^2)\, dx = \frac{M}{2r}\left[\int_{-r}^{+r} r^2 dx - \int_{-r}^{+r} x^2 dx\right]$$

$$= \frac{M}{2r}\left[r^2 x\right]_{-r}^{+r} - \frac{M}{2r}\left[\frac{x^3}{3}\right]_{-r}^{+r} = \frac{M}{2r}\left[(r^3 + r^3) - \left(\frac{r^3}{3} + \frac{r^3}{3}\right)\right] = \frac{M}{2r}\left[2r^3 - \frac{2}{3}r^3\right]$$

$$= \frac{M}{2r}\,\frac{4}{3} r^3 = \frac{2}{3} Mr^2$$

4.14 MOMENT OF INERTIA OF A SOLID SPHERE (Using M.I. of spherical shell)

(*i*) About its diameter

To calculate the moment of inertia of a solid sphere from a knowledge of the moment of inertia of the shell, suppose the sphere is divided into a number of concentric shells of thickness dx each. Now consider a shell at a distance x from the centre.

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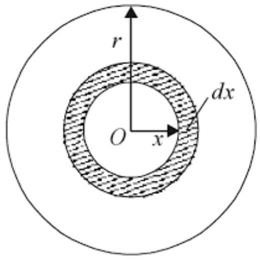

Fig. 4.13

Volume of the shell = $4\pi x^2\, dx$

Volume of the sphere $= \frac{4}{3}\pi r^3$

Mass of the sphere = M

$\therefore$ Mass per unit volume $= \frac{3M}{4\pi r^3}$

Hence mass of the shell $= \frac{3}{4}\frac{M}{\pi r^3}\, 4\pi x^2 dx = \frac{3M}{r^3}\, x^2 dx$

Moment of inertia of the shell about a diameter

$$= \frac{2}{3}\text{ mass} \times (\text{radius})^2 = \frac{2}{3}\frac{3M}{r^3}\, x^2 dx.\, x^2 = \frac{2M}{r^3}\, x^4 dx$$

Moment of inertia of the sphere about a diameter $= \frac{2M}{r^3}\int_0^r x^4 dx = \frac{2M}{r^3}\left[\frac{x^5}{5}\right]_0^r = \frac{2}{5}Mr^2$

(*ii*) About a tangent. The distance between a tangent and a diameter of the sphere = r

Applying principle of parallel axes, we have

Moment of inertia of the sphere about a tangent

$$= \frac{2}{5}Mr^2 + Mr^2 = \frac{7}{5}Mr^2.$$

4.15 MOMENT OF INERTIA OF A THICK SHELL

Consider a thick shell or a hollow sphere of external radius R and internal radius r. To find the moment of inertia suppose it is divided into a number of concentric thin shells of thickness dx each.

Consider a shell at a distance x from the centre, then

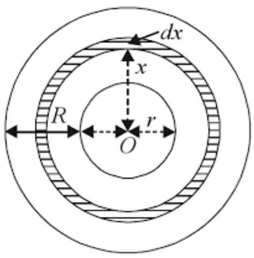

Fig. 4.14

Volume of the shell = $4\pi x^2 dx$

Total volume of the hollow sphere of external radius R and internal radius $r = \frac{4}{3}\pi\,(R^3 - r^3)$

If M is the mass of the hollow sphere, then

Mass per unit volume $= \frac{3M}{4\pi(R^3 - r^3)}$

$\therefore$ Mass of the shell $= \frac{3M}{4\pi(R^3 - r^3)}\, 4\pi x^2\, dx = \frac{3Mx^2\, dx}{(R^3 - r^3)}$

Moment of inertia of the thin shell about a diameter $= \frac{2}{3}\frac{3M}{(R^3 - r^3)}\, x^2\, dx\, x^2 = \frac{2M}{(R^3 - r^3)}\, x^4 dx$

Moment of inertia of the hollow sphere of external radius R and internal radius r

$$= \frac{2M}{(R^3 - r^3)}\int_r^R x^4 dx = \frac{2M}{(R^3 - r^3)}\left[\frac{R^5 - r^5}{5}\right]$$

$$= \frac{2}{5}M\left(\frac{R^5 - r^5}{R^3 - r^3}\right)$$

4.16 M.I. OF A CONE

(*i*) About its vertical axis: Let M be the mass of the cone, h its vertical height and R the radius of its base.

Volume of the cone = $\frac{1}{3}\pi R^2 h$ $\quad \therefore$ Mass per unit volume = $\frac{3M}{\pi R^2 h}$

Imagine that the cone is made up of a number of discs parallel to the base placed one above the other. Consider one such disc of radius r and thickness dx at a distance x from the vertex A.

If α is half the vertical angle of the cone, then $r = x \tan\alpha$

Also $\tan\alpha = \frac{R}{h}$ or $r = \frac{xR}{h}$

Now the volume of the disc = $\pi r^2 dx$

$\therefore$ Mass of the disc $= \frac{\pi r^2 dx\, 3M}{\pi R^2 h} = \frac{3Mr^2}{R^2 h}dx$

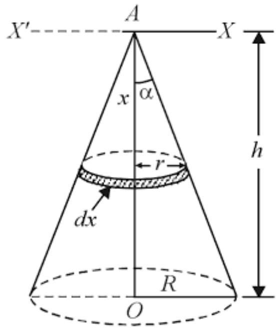

Fig. 4.15

The moment of inertia of the disc about the axis AO passing through the centre and perpendicular to its plane

$$= \frac{\text{Mass of the disc} \times (\text{radius})^2}{2} = \frac{3Mr^2}{R^2h}dx\,\frac{r^2}{2} = \frac{3M}{2R^2h}r^4\,dx$$

But $\quad r = \frac{xR}{h}$

$$\therefore \quad \text{M.I. of the disc} = \frac{3M}{2R^2h}\,\frac{R^4x^4}{h^4}\,dx = \frac{3MR^2}{2h^5}x^4\,dx$$

Hence the moment of inertia of the whole cone about the vertical axis AO will be the integral of the above expression between the limits $x = 0$ and $x = h$.

$$\therefore \text{ M.I. of the cone about its vertical axis } I = \frac{3MR^2}{2h^5}\int_0^h x^4 dx = \frac{3MR^2}{2h^5}\left[\frac{x^5}{5}\right]_0^h = \frac{3MR^2}{10}$$

(*ii*) **About an axis through its vertex and parallel to its base:** Now, again if we consider a disc at a distance x from the vertex of the cone, then its moment of inertia about its diameter

$$= \frac{\text{Mass of the disc} \times (\text{radius})^2}{4} = \frac{3Mr^2}{R^2h}dx\,\frac{r^2}{4}$$

Applying principle of parallel axis, the moment of inertia of this disc about a parallel axis XX' through the vertex of the cone, is given by $I = \frac{3Mr^4}{4R^2h}dx + \frac{3Mr^2}{R^2h}dx\,x^2$

Substituting the value of $r = \frac{xR}{h}$, we get

$$\text{Moment of inertia of disc} = \frac{3Mx^4R^4}{4R^2h^5}dx + \frac{3MR^2}{R^2h^3}x^4dx$$

$\therefore$ Moment of inertia of the cone about an axis XX' parallel to its base and passing through its vertex

$$= \int_0^h \frac{3MR^2}{4h^5}x^4\,dx + \int_0^h \frac{3M}{h^3}x^4\,dx = \frac{3MR^2}{4h^5}\left[\frac{x^5}{5}\right]_0^h + \frac{3M}{h^3}\left[\frac{x^5}{5}\right]_0^h = \frac{3MR^2}{20} + \frac{3Mh^2}{5}$$

4.17 M.I. OF A RECTANGULAR LAMINA

Let M be the mass of the rectangular lamina of sides a and b as shown. Consider an axis AB parallel to the side a and passing through G, the centre of mass. Consider a small strip of breadth dx parallel to the side a at a distance x from the axis AB.

Area of the strip = adx

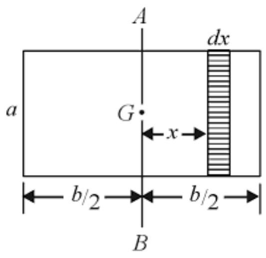

Fig. 4.16

Mass per unit area of the lamina $= \dfrac{M}{ab}$

$\therefore$ Mass of the strip $= \dfrac{M}{ab} \times a dx = \dfrac{M}{b}\, dx$

Moment of inertia of the strip about the axis $AB = \dfrac{M}{b} x^2\, dx$

Moment of inertia of the lamina about the axis AB is obtained by integrating the above expression between the limits

$$x = \frac{b}{2} \text{ and } x = -\frac{b}{2}.$$

$$\therefore \qquad I_y = \int_{-b/2}^{b/2} \frac{M}{b} x^2\, dx = \frac{M}{b}\left[\frac{x^3}{3}\right]_{-b/2}^{b/2} = \frac{M}{b}\left[\frac{b^3}{24} + \frac{b^3}{24}\right] = \frac{M}{b}\frac{b^3}{12} = M\frac{b^2}{12}$$

Similarly the moment of inertia of the lamina about an axis parallel to the side b, $I_x = M\dfrac{a^2}{12}$

Applying the principle of perpendicular axis, the moment of inertia about an axis passing through G and perpendicular to the plane is given by $I = I_x + I_y = \dfrac{Ma^2}{12} + \dfrac{Mb^2}{12} = M\dfrac{a^2 + b^2}{12}$

4.18 MOMENT OF INERTIA OF A RECTANGULAR BAR.

(*i*) About an axis through its C.G. and perpendicular to its length. Consider a rectangular bar of length l, breadth b and thickness a having a mass M.

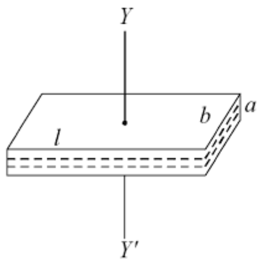

Fig. 4.17

Suppose it is divided into a number of thin laminas of mass m each placed one above the other as shown. The M.I. of a lamina about the axis YY' perpendicular to its plane and passing through its centre

$$= m\frac{l^2 + b^2}{12}$$

$\therefore$ M.I. of the rectangular bar about the axis YY'

$$I_y = \Sigma m\frac{l^2 + b^2}{12} = M\frac{l^2 + b^2}{12}$$

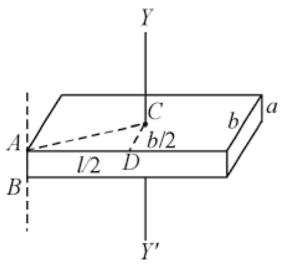

Fig. 4.18

(*ii*) About an axis perpendicular to its length and passing through one of its sides. The M.I. of the bar about a parallel axis through the corner AB as shown in Fig. 4.18

$$= I_y + M \times AC^2$$

But $\quad AC^2 = AD^2 + CD^2 = \left(\dfrac{l}{2}\right)^2 + \left(\dfrac{b}{2}\right)^2$

M.I. about the axis $AB = M\left[\dfrac{l^2 + b^2}{12}\right] + M\left[\dfrac{l^2}{4} + \dfrac{b^2}{4}\right]$

$$= M\frac{l^2 + b^2}{3}$$

For a bar of **square cross-section** $l = b$

$$\therefore \qquad \text{M.I.} = \frac{2}{3}Mb^2$$

4.19 KINETIC ENERGY OF A BODY ROLLING ON A HORIZONTAL PLANE

When a solid spherical ball rolls on a table a point on its surface rotates about an axis passing through its centre and perpendicular to its plane of rotation. The ball also moves forward. It, therefore, possesses both angular velocity and linear velocity due to which it has a rotational kinetic energy and a translational kinetic energy.

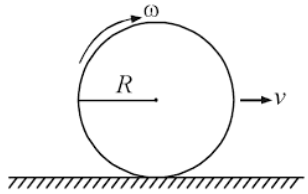

Fig. 4.19

$$\text{Energy due to linear motion} = \frac{1}{2} Mv^2 \qquad \dots (i)$$

where M is the mass of the spherical ball and v its linear velocity.

$$\text{Energy due to rotation} = \frac{1}{2} I\omega^2 \qquad \dots (ii)$$

where I is the moment of inertia of the spherical ball and ω is angular velocity.

$$\therefore \quad \text{Total energy,} \quad E = \frac{1}{2} Mv^2 + \frac{1}{2} I\omega^2$$

$$= \frac{1}{2} Mv^2 + \frac{1}{2} MK^2\omega^2$$

$$= \frac{1}{2} Mv^2 + \frac{\frac{1}{2} MK^2v^2}{R^2}$$

$$= \frac{1}{2} Mv^2 \left[\frac{K^2}{R^2} + 1\right] \qquad \dots (iii)$$

Special cases (*i*) For a spherical ball of radius *R*,

$$I = \frac{2}{5} MR^2$$

$$\therefore \quad I = Mk^2 = \frac{2}{5} MR^2 \quad \text{or} \quad k^2 = \frac{2}{5} R^2$$

$$\therefore \quad \frac{k^2}{R^2} = \frac{2}{5}$$

Substituting in Eq. (*iii*), we get

$$E = \frac{1}{2} Mv^2 \left[\frac{2}{5} + 1\right]$$

$$= \frac{7}{10} Mv^2 \qquad \dots (iv)$$

(*ii*) For a circular disc of radius *R*,

$$I = \frac{MR^2}{2} = Mk^2 \quad \text{or} \quad k^2 = \frac{R^2}{2}$$

$$\frac{k^2}{R^2} = \frac{1}{2}$$

Substituting in Eq. (*iii*), we get

$$E = \frac{1}{2} M v^2 \left[\frac{1}{2}+1\right]$$

$$= \frac{3}{4} M v^2 \qquad \text{... } (v)$$

4.20 ACCELERATION OF A BODY ROLLING DOWN ON INCLINED PLANE

Let a body of radius r and mass M roll without sliping down a smooth inclined plane having an angle of inclination θ. Suppose v is the velocity acquired by the body after traversing a distance S along the plane. Let ω be the angular velocity about an axis through the centre of the body.

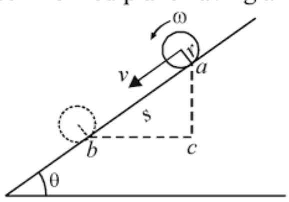

Fig. 4.20

Now, in one revolution the body moves a distance $2\pi r$ whereas a point on the rim turns through 2π radians in the same time.

$\therefore$ Linear velocity $v = r\omega$ or $\omega = \dfrac{v}{r}$

If I is the moment of inertia and k the radius of gyration of the body about the axis of rotation, then the total kinetic energy of the body consists of

(*i*) Energy of translation $= \dfrac{1}{2} Mv^2$ and (*ii*) Energy of rotation $= \dfrac{1}{2} I\omega^2 = \dfrac{1}{2} Mk^2\omega^2$

$\therefore$ Total energy $= \dfrac{1}{2} Mv^2 + \dfrac{1}{2} Mk^2\omega^2$

In moving a distance s from a to b the body comes down vertically a distance

$$ac = s \sin\theta$$

Hence the change in potential energy $= Mgs \sin\theta$

Since there is no slipping, no energy is dissipated and the total gain in kinetic energy is equal to the change in potential energy.

$$\therefore \quad Mgs \sin\theta = \frac{1}{2} Mv^2 + \frac{1}{2} Mk^2 \omega^2 = \frac{1}{2} Mv^2 + \frac{1}{2} Mk^2 \frac{v^2}{r^2} = \frac{1}{2} Mv^2 \left(1 + \frac{k^2}{r^2}\right)$$

$$\text{or} \quad sg \sin\theta = \frac{v^2}{2}\left(1 + \frac{k^2}{r^2}\right)$$

Differentiating with respect to time t, we get

$$\frac{ds}{dt} g \sin\theta = \frac{2v}{2}\frac{dv}{dt}\left(1 + \frac{k^2}{r^2}\right) = v\frac{dv}{dt}\left(1 + \frac{k^2}{r^2}\right)$$

Now $\dfrac{ds}{dt}$ = linear velocity v and $\dfrac{dv}{dt}$ = acceleration a

$$\therefore \quad \frac{dv}{dt} v\left(1 + \frac{k^2}{r^2}\right) = vg \sin\theta \quad \text{or} \quad \frac{dv}{dt} = a = \frac{g \sin\theta}{\left(1 + \dfrac{k^2}{r^2}\right)}$$

This shows that for a given angle of inclination of the plane θ the acceleration is inversely proportional to $\left(1 + \dfrac{k^2}{r^2}\right)$. Thus (*i*) Greater the value of k as compared to r the smaller is the acceleration of the body moving down an inclined plane and hence greater is the time taken in rolling down the plane.

(*ii*) The acceleration is independent of the mass of the body.

Special cases (*i*) **Cylinder.** The moment of inertia of a cylinder about its axis of symmetry about which it rolls $= \frac{1}{2} Mr^2$

$$\therefore \qquad I = \frac{1}{2} Mr^2 = Mk^2 \text{ or } k^2 = \frac{r^2}{2} \text{ and } \frac{k^2}{r^2} = \frac{1}{2}$$

$$\text{Hence} \qquad a = \frac{g \sin \theta}{1 + \frac{k^2}{r^2}} = \frac{g \sin \theta}{1 + \frac{1}{2}} = \frac{2}{3} g \sin \theta$$

(*ii*) **Solid sphere.** The moment of inertia of a solid sphere about a diameter about which it rolls $= \frac{2}{5} Mr^2$

$$\therefore \qquad I = \frac{2}{5} Mr^2 = Mk^2 \text{ or } k^2 = \frac{2}{5} r^2 \text{ and } \frac{k^2}{r^2} = \frac{2}{5}$$

$$\text{Hence} \qquad a = \frac{g \sin \theta}{1 + \frac{k^2}{r^2}} = \frac{g \sin \theta}{1 + \frac{2}{5}} = \frac{5}{7} g \sin \theta$$

(*ii*) **Hollow sphere.** The moment of inertia of a hollow sphere about a diameter about which it rolls $= \frac{2}{3} Mr^2$

$$\therefore \qquad I = \frac{2}{3} Mr^2 = Mk^2 \text{ or } k^2 = \frac{2}{3} r^2 \text{ and } \frac{k^2}{r^2} = \frac{2}{3}$$

$$\text{Hence} \qquad a = \frac{g \sin \theta}{1 + \frac{k^2}{r^2}} = \frac{g \sin \theta}{1 + \frac{2}{3}} = \frac{3}{5} g \sin \theta$$

As an example for $g = 9.8 \text{ ms}^{-2}$ and $\theta = 30°$; $\sin \theta = \frac{1}{2}$, we have

For cylinder $a = \frac{2}{3} \times \frac{1}{2} \times 9.8 = 3.27 \text{ ms}^{-2}$

For solid sphere $a = \frac{5}{7} \times \frac{1}{2} \times 9.8 = 3.5 \text{ ms}^{-2}$

For hollow sphere $a = \frac{3}{5} \times \frac{1}{2} \times 9.8 = 2.94 \text{ ms}^{-2}$

4.21 PRODUCTS OF MOMENT OF INERTIA

When a rigid body moves with one point stationary, the total angular momentum about that point is

$$\vec{J} = \sum_i \vec{r_i} \times \vec{p}_i = \sum_i \vec{r_i} \times m_i \vec{v_i} = \sum_i m_i \vec{r_i} \times \vec{v_i} \qquad \ldots (i)$$

where $\vec{r_i}$ and $\vec{v_i}$ are the radius vector and velocity vector respectively of the i th particle relative to the given point. Since $\vec{r_i}$ is a fixed vector to the body, the velocity $\vec{v_i}$ with respect to the space set of axes solely form the rotational motion of the body about the fixed point.

We know, $$\vec{v}_i = \vec{\omega} \times \vec{r}_i \qquad \text{... }(ii)$$

Substituting Eq. (*ii*) in Eq. (*i*), we have

$$\vec{J} = \sum_i m_i [\vec{r}_i \times (\vec{\omega} \times \vec{r}_i)] \qquad \text{... }(iii)$$

Expanding the tripple cross product using

$$[\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}\,(\vec{A} \cdot \vec{C}) - \vec{C}\,(\vec{A} \cdot \vec{B})]$$

We write Eq. (*iii*) as

$$\vec{J} = \sum_i m_i\,[\vec{\omega}\,(\vec{r}_i \cdot \vec{r}_i) - \vec{r}_i\,(\vec{r}_i \cdot \vec{\omega})]$$

$$= \sum_i m_i\,[\vec{\omega}\, r_i^2 - \vec{r}_i\,(\vec{r}_i \cdot \vec{\omega})]$$

$$= \sum_i m_i\, r_i^2\, \vec{\omega} - m_i\, \vec{r}_i\,(\vec{r}_i \cdot \vec{\omega}) \qquad \text{... }(iv)$$

Expanding angular momentum J, in cartesian co-ordinates, Eq. (*iv*) can be written as

$$J_x = \omega_x \sum_i m_i r_i^2 - \omega_x \sum_i m_i x_i^2 - \omega_y \sum_i m_i x_i y_i - \sum_i \omega_z m_i x_i z_i$$

$$= \omega_x \sum_i m_i (r_i^2 - x_i^2) - \omega_y \sum_i m_i x_i y_i - \omega_z \sum_i m_i x_i z_i \qquad \text{... }(v)$$

With similar equations for other components J_y and J_z of $\vec{J}$.

Eq. (*v*), may be written as

$$J_x = I_{xx}\,\omega_x + I_{xy}\,\omega_y + I_{xz}\,\omega_z \qquad \text{... }(vi)$$

where $$I_{xx} = \sum_i m_i\,(r_i^2 - x_i^2) = \sum_i m_i\,(y_i^2 + z_i^2)$$

$$I_{xy} = -\sum_i m_i\, x_i\, y_i$$

$$I_{xz} = -\sum_i m_i\, x_i\, z_i$$

By analogy, we can write for J_y and J_z from eq. (*vi*) as

$$J_y = I_{yx}\,\omega_x + I_{yy}\,\omega_y + I_{yz}\,\omega_z$$
$$J_z = I_{zx}\,\omega_x + I_{zy}\,\omega_y + I_{zz}\,\omega_z$$

In general, $$J_j = \sum_k I_{jk}\,\omega_k \qquad \text{... }(vii)$$

where j and k takes values x, y, z

Such nine coefficients I_{xx}, I_{xy}, ... etc. are the nine elements of the transformation (3 × 3) matrix, *i.e.* nine quantities I_{jk} form a *symmetric tensor*

$$\Pi = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \qquad \text{... }(viii)$$

In vector notation, we have

$$\vec{J} = \Pi \cdot \vec{\omega} \qquad \text{... } (ix)$$

The diagonal elements are known as "*Moments of inertia coefficients*", while the off-diagonal elements are named as "*Products of M.I.*"

4.22 PRINCIPAL MOMENTS

The methods of matrix algebra enables us to show that for any point in a rigid body one can find a set of cartesian axes for which the inertia tensor will be diagonal and the axes are called "*Principal axes*" and the corresponding diagonal elements I_1, I_2, I_3 are known as "*Principal moments of inertia*". In practice, the principal *M.I.*, being the eigen values of *I*, are found as roots of the folowing determinant

$$\begin{vmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - 1 \end{vmatrix} = 0 \qquad \text{... } (x)$$

The above equation is cubic in *I* whose three roots are the desired principal moments.

4.23 EULER'S EQUATIONS OF MOTION

For a rotational motion about a fixed point or about the centre of mass, the direct Newtonian approach leads to a set of equations known as *Euler's equations of motion*. Let us consider the equation of motion for rotational motion for a fixed coordinate system with origin at centre of mass is

$$\text{Torque,} \quad \vec{\tau}_j = \frac{d\vec{J}_j}{dt} \quad \text{where} \quad J_j = \sum_k I_{jk}\,\omega_k \qquad \text{... } (xi)$$

The moment and product of inertia I_{jk} are with respect to a fixed coordinate frame. By choosing *body-fixed* coordinate system simplifies analysis of motion. Hence, moments and products of inertia both are time-independent.

Now using, relation, $$\frac{d\vec{r}}{dt} = \frac{\delta \vec{r}}{\delta t} + \vec{\omega} \times \vec{r},$$

Eq. (*xi*), may be written as

$$\tau_j = \frac{\delta J_j}{\delta t} + (\vec{\omega} \times \vec{J})_j \qquad \text{... } (xii)$$

If we select the axes of rotation in the body where all the products of inertia vanish

i.e. $\quad I_{ij} = 0 \quad$ for $\quad I \neq j$

Axes for which $I_{ij} = 0$, $i \neq j$ are called the *principal axes of the body*,

$$J_1 = I_{11}\,\omega_1 = I_1\,\omega_1$$
$$J_2 = I_{22}\,\omega_2 = I_2\,\omega_2$$
$$J_3 = I_{33}\,\omega_3 = I_3\,\omega_3$$

where I_1, I_2, I_3 are the principal M.I.

From Eq. (*xii*) in cartesian co-ordinates, the Euler's equations of motion of a rigid body as

$$\left.\begin{aligned} \tau_1 &= I_1\,\dot{\omega}_1 - \omega_2\,\omega_3\,(I_2 - I_3) \\ \tau_2 &= I_2\,\dot{\omega}_2 - \omega_3\,\omega_1\,(I_3 - I_1) \\ \tau_3 &= I_3\,\dot{\omega}_3 - \omega_1\,\omega_2\,(I_1 - I_2) \end{aligned}\right\} \qquad \text{... } (xiii)$$

Eq. (*xii*) and (*xiii*) are called Euler's equations of motion for a rigid body with one point fixed.

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SOLVED EXAMPLES

Example 4.1 Prove that $J^2 = 2\,E\,I$, where J, E, and I are the angular momentum, kinetic energy of rotation and moment of inertia respectively.

Solution. The angular momentum of a rotating body $J = I\omega$ where I is the moment of inertia of the body about the axis of rotation and ω the angular velocity.

Kinetic energy of rotation $E = \frac{1}{2} I\omega^2$

$$\therefore \quad J^2 = I^2\omega^2 = 2 \times \frac{1}{2} I\omega^2 \times I = 2EI$$

Example 4.2 The inter-molecular distance between two atoms of hydrogen molecule is 0.77 Å and mass of the proton is 1.67×10^{-27} Kg. Calculate the moment of inertia of the molecule.

Solution. The hydrogen molecule consists of two atoms of hydrogen which are point masses each $m = 1.67 \times 10^{-27}$ kg separated by a distance 0.77 Å = 0.77×10^{-10} m. The molecule has a moment of inertia only about an axis perpendicular to the line joining the two atoms.

Hence the moment of inertia of the hydrogen molecule about an axis passing through the mid-point of the line joining the two atoms and perpendicular to it

$$I = 2m\left(\frac{l}{2}\right)^2 = \frac{ml^2}{2} = \frac{1.67 \times 10^{-27}\,(0.77 \times 10^{-10})^2}{2}$$

$$= 0.495 \times 10^{-47} \text{ kg m}^2$$

Example 4.3 A flywheel of mass 500 kg, radius 1 metre makes 500 revolutions per minute. Assuming the mass to be concentrated along the rim, calculate the energy of the flywheel.

Solution. Mass of the flywheel $M = 500$ kg (*Nagpur U. 2007*)

As the mass is concentrated along the rim, distance of the mass from the axis or radius of gyration

$$k = 1\text{m}$$

Angular velocity $\omega = 500$ rev/min. $= \frac{500 \times 2\pi}{60}$ rad/sec

Moment of inertia $I = Mk^2 = 500 \times 1 = 500$ kg-m^2

$$\therefore \text{ K.E. of flywheel} = \frac{1}{2} I\omega^2 = \frac{1}{2} Mk^2\omega^2 = \frac{1}{2} \times 500 \times \left(\frac{500 \times 2\pi}{60}\right)^2$$

$$= 6.871 \times 10^5 \text{ J.}$$

Example 4.4 Prove from first principles that out of an infinite number of straight lines which may be drawn parallel to a given direction the moment of inertia of a body is least about the one passing through its centre of gravity.

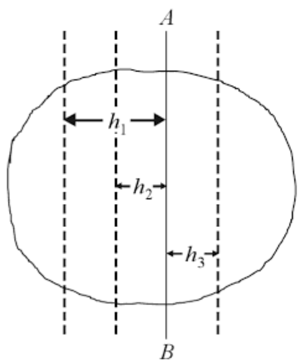

Fig. 4.21

Solution. According to the principle of parallel axes, the moment of inertia I of a body about an axis is equal to the sum of its moment of inertia about a parallel axis through its centre of gravity I_g and the product of its mass M and the square of the distance h between the two axes *i.e.*, $I = I_g + Mh^2$

Hence for a number of axes which are all parallel to each other at distances h_1, h_2, h_3 etc. from the axis AB passing through the centre of gravity the moment of inertia is respectively given by

$$(I_g + Mh_1^2),\ (I_g + Mh_2^2)$$

and so on. The value of h^2 is always positive whether h is towards the left or right of AB. Hence Mh^2 is a positive quantity.

The least value of I is obtained when $h = 0$ *i.e.,* when the axis passes through the centre of gravity.

Example 4.5 Assuming earth to be a sphere of uniform density 5520 kg. m^{-3} and radius 6400 km, calculate the M.I. about its axis of rotation.

Solution. Density of earth $\rho = 5520\ \text{kgm}^{-3}$

Radius of earth $R = 6400\ \text{km} = 6400 \times 10^3\ \text{m}$

$\therefore$ Mass of earth $M = \frac{4}{3}\pi R^3\rho = \frac{4}{3}\pi\,[6400 \times 10^3]^3 \times 5520 = 6.06 \times 10^{24}\ \text{kg}$

Moment of inertia of the earth about its axis $= \frac{2}{5} MR^2$

$$= \frac{2}{5} \times 6.06 \times 10^{24} \times [6400 \times 10^3]^2 = 9.93 \times 10^{37}\ \text{kg m}^2$$

Example 4.6 Moment of inertia of a bigger solid sphere about its diameter is *I*. 64 smaller, equal spheres are made out of bigger sphere. What will be the moment of inertia of such smaller sphere about its diameter?

Solution. Moment of inertia of the bigger sphere $I = \frac{2}{5} MR^2$

where M is its mass and R its radius. Let ρ be the density of the material of the sphere, then

$$M = \frac{4}{3}\pi R^3 \rho$$

Let m be the mass of the smaller sphere, then $m = \frac{M}{64}$

If r is the radius of the smaller sphere, then

$$m = \frac{4}{3}\pi r^3 \rho = \frac{1}{64} \times \frac{4}{3}\pi R^3 \rho$$

or
$$r^3 = \frac{R^3}{64} \quad \text{or} \quad r = \frac{R}{4}$$

$\therefore$ Moment of inertia of the smaller sphere $= I_s = \frac{2}{5} mr^2$

$$= \frac{2}{5}\frac{M}{64}\frac{R^2}{16} = \frac{1}{1024} \times \frac{2}{5} MR^2 \quad \text{or} \quad I_s = \frac{I}{1024}$$

Example 4.7 Calculate the radius of gyration of a solid sphere rotating about its diameter if its radius is 5.0 cm. (*M.D.U.* 2003)

Solution. M.I of solid sphere $= \frac{2}{5} Mr^2 = Mk^2$

$$\therefore \quad k^2 = \frac{2}{5} r^2 = \frac{2}{5} \times 5^2 = \frac{2}{5} \times 25 = 10\ \text{cm}^2$$

or radius of gyration $k = \sqrt{10}\ \text{cm}$

Example 4.8 A hollow steel sphere has its inner and outer radii 5 cm and 12 cm respectively. Calculate its moment of inertia about a diameter. Density of steel is 7.8 × 10^3 kg m^{-3}.

Solution. Outer radius $R = 12\ \text{cm} = 0.12\ \text{m}$, Inner radius $r = 5\ \text{cm} = 0.05\ \text{m}$

Mass of hollow sphere $M = \frac{4}{3}\pi (R^3 - r^3)\,\rho$

M.I. of hollow sphere $I = \frac{2}{5} M \left(\frac{R^5 - r^5}{R^3 - r^3}\right) = \frac{2}{5} \times \frac{4}{3} \pi (R^3 - r^3).\ \rho \left(\frac{R^5 - r^5}{R^3 - r^3}\right)$

$= \frac{8}{15} \times 3.142 \times 7.8 \times 10^3\ (R^5 - r^5)$

$= 13.07 \times 10^3\ (0.12^5 - 0.05^5) = 0.3211\ \text{kg m}^2$

Example 4.9 The flat surface of a hemisphere of radius *r* is cemented to one flat surface of a cylinder of radius *r* and length *L*. If the total mass is *M*, show that the M.I. of combination about the axis of the cylinder will be

$$Mr^2 \frac{(L/2 + 4r/15)}{(L + 2r/3)}.$$

Solution. Suppose m is the mass of cylinder and m' that of the hemisphere and ρ the density of the material, then

M.I. of cylinder about its axis $= \frac{mr^2}{2} = \pi r^2 L\rho \times \frac{r^2}{2} = \frac{\pi r^4 L\rho}{2}$

M.I. of hemisphere about the axis of cylinder $= \frac{2}{5} m'r^2 = \frac{2}{5} \times \frac{2}{3} \pi r^3 \rho r^2 = \frac{4}{15} \pi r^5 \rho$

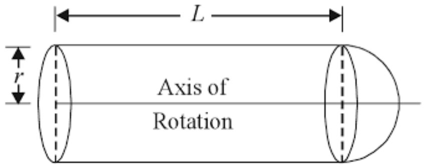

Fig. 4.22

M.I. of the combination about the axis of the cylinder,

$$I = \pi r^4\ \rho\ (L/2 + 4r/15)$$

Now $\quad M = m + m' = \pi r^2 L\rho + \frac{2}{3} \pi r^3 \rho$

$$= \pi r^2 \rho\ (L + 2/3\ r)$$

Hence $\quad I = Mr^2 \frac{(L/2 + 4r/15)}{(L + 2/3r)}$

Example 4.10 (*a*) A circular disc of mass *M* and radius *r* is set rolling on a table. If ω is the angular velocity show that its total energy *E* is given by $\frac{3}{4} Mr^2 \omega^2$.

(*b*) A flat circular disc of mass 0.05 kg and diameter 0.02 m rolls on its edge on a smooth horizontal surface with a velocity 0.05 ms^{-1}. Calculate its total energy.

Solution. (*a*) When a circular disc rolls on a table a point on its circumference rotates about an axis passing through its centre and perpendicular to its plane. In addition, the point moves forward. In other words it possess an angular velocity and a linear velocity. Therefore, it possesses two kinds of energies:

(*i*) Energy due to linear motion, and (*ii*) Energy due to rotation about an axis through its centre.

If v is the linear velocity, the energy due to linear motion $= \frac{1}{2} Mv^2$

If ω is the angular velocity, then the kinetic energy due to rotational motion $= \frac{1}{2} I\omega^2$

where I is the moment of inertia of the disc about an axis perpendicular to its plane and passing through its C.G.

Now $I = \frac{1}{2} Mr^2 \quad \therefore$ K.E. due to rotation $= \frac{1}{2} \times \frac{1}{2} Mr^2\omega^2 = \frac{1}{4} Mr^2 \omega^2$

$\therefore$ Total K.E. $= \frac{1}{2} Mv^2 + \frac{1}{4} Mr^2\omega^2$

Now $v = r\omega \qquad \therefore$ K.E. $= \frac{1}{2} Mr^2 \omega^2 + \frac{1}{4} Mr^2 \omega^2 = \frac{3}{4} Mr^2 \omega^2$

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In terms of linear velocity, $\text{K.E.} = \frac{3}{4} Mv^2$

(*b*) $M = 0.05$ kg, $v = 0.05$ ms^{-1}

$\therefore$ Total kinetic energy $= \frac{3}{4} Mv^2 = \frac{3}{4} \times 0.05 \times (0.05)^2$

$= 93.75 \times 10^{-6}$ kg m^2 s^{-2} or Joule

Example 4.11 An annular disc of mass 0.2 kg and radii 0.2 m and 0.25 m rolls such that the centre has a velocity of 0.5 ms^{-1}. Calculate its kinetic energy.

Solution. Here $M = 0.2$ kg, $v = 0.5$ ms^{-1}

$\therefore$ K.E. due to linear motion $= \frac{1}{2} mv^2 = \frac{1}{2} \times 0.2 \times (0.5)^2 = 0.025$ J

M.I. of annular disc about the axis of rotation $I = \frac{1}{2} M \ (R_2^2 + R_1^2)$

Now $R_2 = 0.25$ m and $R_1 = 0.20$ m

$$\therefore \qquad I = \frac{1}{2} \times 0.2\ (0.25^2 + 0.2^2) = 1.025 \times 10^{-2} \text{ kg m}^2$$

Angular velocity $\omega = \frac{v}{R_2} = \frac{0.5}{0.25} = 2 \text{ rad s}^{-1}$

$\therefore$ K.E. of rotation $= \frac{1}{2} I\omega^2 = \frac{1}{2} \times 1.025 \times 10^{-2} \times 4 = 2.05 \times 10^{-2}$ J

Total K.E. = 0.025 + 0.0205 = 0.0455 J

Example 4.12 A solid sphere of mass 0.1 kg and radius 2.5 cm rolls without slipping with a uniform velocity of 0.1 ms^{-1} along a straight line on a horizontal table. Calculate its total energy.

Solution. As the sphere rolls without slipping it has both energy of translation and rotation.
Here $M = 0.1$ kg, $r = 2.5$ cm $= 0.025$ m, $v = 0.1$ ms^{-1}

$$\therefore \qquad \omega = \frac{v}{r} = \frac{0.1}{0.025} = 4 \text{ rad s}^{-1}$$

K.E. of translation $= \frac{1}{2} Mv^2 = \frac{1}{2} \times 0.1 \times 0.1 \times 0.1 = 0.0005$ J

K.E. of rotation $= \frac{1}{2} I\omega^2 = \frac{1}{2} \times \frac{2}{5} Mr^2 \omega^2 = \frac{1}{5} \times 0.1 \times (.025)^2 \times 16 = 0.0002$ J

Total energy = 0.0005 + 0.0002 = 0.0007 J = 7×10^{-4} J.

Example 4.13 How can a solid sphere be distinguished from a hollow sphere, the two being identical in all respects?

OR

A solid sphere and a hollow sphere of the same mass and same radius are allowed to roll down an inclined plane from the same position. Which one will come down with greater acceleration? Justify your answer.

Solution. Let the external radius of the solid sphere as well as that of the hollow sphere be R, then

Moment of inertia of the solid sphere $I = Mk_1^2 = \frac{2}{5} MR^2 \quad \therefore \ k_1^2 = \frac{2}{5} R^2$

If r is the internal radius of the hollow sphere of the same mass, then

Moment of inertia of the hollow sphere $= Mk_2^2 = \frac{2}{5} M \frac{(R^5 - r^5)}{(R^3 - r^3)}$

$$\therefore \quad k_2^2 = \frac{2}{5}\frac{(R^5 - r^5)}{(R^3 - r^3)} = \frac{2}{5}R^2\left[\frac{1 - r^5/R^5}{1 - r^3/R^3}\right]$$

As r is less than R, $\frac{r}{R}$ is a fraction less than 1 $\quad \therefore \left(\frac{r}{R}\right)^5 < \left(\frac{r}{R}\right)^3$

Hence $1 - \left(\frac{r}{R}\right)^5 > 1 - \left(\frac{r}{R}\right)^3 \quad$ or $\quad k_2^2 > k_1^2$

Thus we see that the value of k^2 for a hollow sphere is greater than that for a solid sphere of the same mass and external radius. Therefore a hollow sphere, when allowed to move along an inclined plane, will have a smaller acceleration and will move slower than the solid sphere because acceleration is inversely proportional to $\left(1 + \frac{k^2}{r^2}\right)$. Hence the time taken to move down a certain distance by the hollow sphere will be greater than that for the solid sphere. This is how these can be distinguished from each other.

Example 4.14 A body of radius *R* and mass *m* is rolling horizontally without slipping with speed *v*. It rolls up a hill to a maximum height $h = 3v^2/4g$. Neglecting friction, find the moment of inertia of the body. What can be the shape of the body?

Solution. Kinetic energy due to linear motion $= \frac{1}{2}mv^2$

Kinetic energy due to rotation $= \frac{1}{2}I\omega^2 = \frac{1}{2}I\frac{v^2}{R^2}$

$\therefore$ Total kinetic energy $= \frac{1}{2}mv^2 + \frac{1}{2}Iv^2/R^2$

As the body rolls up the hill to a height h, the whole of the K.E. is converted into potential energy given by $mgh = mg\frac{3v^2}{4g} = \frac{3}{4}mv^2$

$$\therefore \frac{1}{2}mv^2 + \frac{1}{2}Iv^2/R^2 = \frac{3}{4}mv^2 \quad \text{or} \quad \frac{1}{2}Iv^2/R^2 = \frac{1}{4}mv^2 \quad \text{or} \quad I = \frac{1}{2}mR^2$$

The shape of the body is, therefore, a circular disc of radius R.

Example 4.15 A flat thin uniform disc of radius *a* has a hole of radius *b* in it at a distance *c* from the centre of the disc {*c* < (*a* – *b*)}. If the disc were free to rotate about a smooth circular rod of radius *b* passing through the hole, calculate the moment of inertia about the axis of rotation.

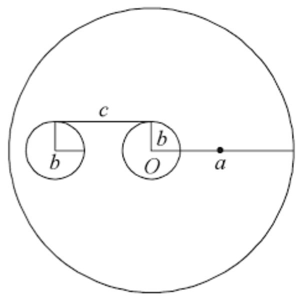

Fig. 4.23

Solution. Let M be the mass of the disc of radius a and having a hole of radius b at a distance c from the centre of the disc.

If the hole were supposed to be at the centre, then the moment of inertia of the disc about an axis through O and perpendicular to the plane of the disc

$$I_0 = \frac{M(a^2 + b^2)}{2}$$

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Applying the principle of parallel axes, moment of inertia about an axis, passing through the centre of a circle of radius b at a distance c from the centre of the disc

$$I = I_0 + Mc^2$$

$$= \frac{M(a^2 + b^2)}{2} + Mc^2$$

EXERCISE CH. 4

LONG QUESTIONS

1. What is inertia? Obtain an expression for the kinetic energy of a rotating body. (*Kerala U.*, 2001; *M.D.U.*, 2003)

2. Explain Moment of Inertia and Radius of Gyration. Give the dimensions and units of moment of inertia. Is it a vector or scalar quantity? (*Kerala U.*, 2001; *M.D.U.*, 2003, 2001; *Gauhati U.* 2007; *Nagpur U.*, 2009, 2007, 2003; *Purvanchal U.* 2005)

3. Prove that $J^2 = 2EI$ where J, E and I are the angular momentum, kinetic energy of rotation and moment of inertia respectively. (*M.D.U.*, 2000)

4. State and prove the theorem of perpendicular axis for moment of inertia. (*Meerut U.*, 2001; *Gauhati U.*, 2007; *Nagpur U.* 2009, 2008; *D.A.U. Agra*, 2008, 2007, 2004; *Purvanchal U.* 2005)

5. State and prove theorem of parallel axes for moment of inertia. (*M.D.U.*, 2003; *Meerut U.*, 2002, 2001; *Gharwal U.*, 2000; *Cal. U.*, 2003, *Nagpur U.* 2007, *Purvanchal U.* 2006, 2005, 2004; *Agra U.* 2006)

6. Determine the M.I. of circular disc about an axis through its centre perpendicular to its plane. (*Gauhati U.* 2007; *Madurai U.*, 2003)

7. Calculate the moment of inertia of a solid cylinder about

(*i*) axis of cylindrical symmetry. (*Gauhati U.* 2000; *Gharwal U.* 2000)

(*ii*) about the axis passing through its centre and perpendicular to its own axis of symmetry. (*M.D.U.*, 2002; *Purvanchal U.* 2004)

8. Calculate the moment of inertia of a thin rod of mass M and length l about an axis passing through its centre and perpendicular to its length. If this rod rotates about the above axis with a constant angular speed ω, determine its angular momentum and kinetic energy.

9. Obtain an expression for the moment of inertia of an annular ring about an axis (*a*) passing through the centre and perpendicular to its plane. (*b*) about its diameter.

10. Obtain an expression for the moment of inertia of a hollow cylinder about its own axis of symmetry.

11. Derive an expression for the moment of inertia of a hollow cylinder about an axis through the centre and perpendicular to its own axis. (*K.U.* 2000)

12. Derive the formula for the moment of inertia of a uniform solid sphere (*i*) about its diameter and (*ii*) about its tangent. (*M.D.U.* 2001; *K.U.* 2000; *Purvanchal U.* 2006; *Meerut* 2004)

13. (*a*) Calculate the moment of inertia of a thin spherical shell (hollow sphere) about a diameter.

(*b*) Hence derive the moment of inertia of (*i*) a solid sphere about a diameter and a tangent and (*ii*) a thick shell about an axis through the centre.

(*Meerut U.* 2003; *K.U.* 2001; *Kerala U.* 2001; *M.D.U.* 2003, 2001; *Gauhati U.* 2000)

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14. Obtain an expression for the moment of inertia of a solid cone (*i*) about its vertical axis and (*ii*) about an axis through its vertex and parallel to its base.

15. Calculate the moment of inertia of a rectangular lamina about an axis perpendicular to its plane and passing through its centre of gravity.

16. Find an expression for the moment of inertia of a rectangular solid bar of length l about an axis perpendicular to its length and passing through its centre of gravity.
(*K.U.*, 2002, 2000)

17. Deduce an expression for the moment of inertia of a rectangular bar about an axis perpendicular to the length of the bar and passing through one of its sides. Hence find the moment of inertia of a square bar about the same axis. (*M.D.U.* 2000; *Kerala U.* 2001)

18. A solid spherical ball rolls on a table. Find the ratio of its translational and rotational kinetic energies and the total energy of the spherical ball. What fraction of the total energy is rotational? (*Kerala U.* 2001; *Gauhati U.* 2000)

19. Derive an expression for the acceleration of a body rolling down a smooth inclined plane without slipping. What are the values of acceleration for a cylinder, solid sphere and hollow sphere of the same radius? (*K.U.*, 2002, 2001; *M.D.U.* 2003, 2001, *Gharwal U.*, 2001; *Bang.U.*, 2000)

20. A sphere, a solid cylinder, a spherical shell and a ring of the same mass and radius are allowed to roll down from rest simultaneously on an inclined plane from the same height without slipping. Prove that the sphere reaches down first, then the cylinder, thereafter the shell and the last the ring.

21. (*i*) Define products of moment of inertia, principal axes and principal moment of inertia

(*ii*) Derive Euler's equation of motion of a rigid body.

(*iii*) Using law of parallel axes, calculate the moment of inertia of a disc of mass 0.2 kg and radius 0.5 m about an axis passing through its edge and perpendicular to the plane of the disc. (*Nagpur U.* 2008) **[Ans.** 0.075 kg m^2]

SHORT QUESTIONS

1. What is the physical significance of Moment of inertia? (*Nagpur U.*, 2007, 2003; *M.D.U.*, 2002; *Gharwal U.*, 2000; *Indore U.*, 2001; *Gauhati U.* 2007, 2000)

2. Moment of inertia plays the same role in rotation as mass does in translation. Justify.

3. Determine the moment of inertia of a circular disc. (*a*) about a diameter (*b*) about a tangent.

4. Find the moment of inertia of a thin uniform rod about an axis passing through one end and perpendicular to its length. (*Kerala* 2001)

5. What must be the relation between l and R if the moment of inertia of the cylinder about its axis is to be the same as the moment of inertia about the equatorial axis?

[Ans. $l = \sqrt{3}R$]

6. What do you mean by product of inertia, principal moment of inertia and principal axes of inertia? (*Nagpur U.* 2007, 2008)

7. Derive Euler's equation of motion of a rigid body.
(*Nagpur U.* 2007, 2008; *D.A.U. Agra* 2008, 2007, 2005, 2004)

8. How can a solid-sphere be distinguish from a hollow sphere, the two being identical in all respects?

9. A solid sphere and a hollow sphere of the same mass and same radius are allowed to roll down an inclined plane from the same position. Which one will come down with greater acceleration? Justify your answer. [**Ans.** Solid sphere, because acceleration is proportional to $R^2/(k^2 + R^2)$ for a given angle of inclination θ]

10. A solid sphere and a hollow sphere have the same mass and the same radius. Will they have the same moment of inertia about their principal axis? Explain.
(*Purvanchal* 2005) [**Ans.** Hollow sphere has larger M.I.]

11. On what factors does radius of gyration depend. (*Nagpur U.* 2008)

NUMERICAL QUESTIONS

1. Centres of four solid spheres of diameter $2a$ and mass m make a square of side b. Calculate the moment of inertia of the system about one side of the square.

$$\left[\textbf{Ans. } \frac{2}{5}\,\mathrm{m}\,(4a^2 + 5b^2)\right]$$

2. State the expression for the moment of inertia of a uniform cylinder of length l and radius R about an axis through its centre and normal to its length. If the above moment of inertia is to be a minimum, determine the ratio $\frac{l}{R}$, when the mass of the cylinder is kept constant and show that the ratio is $\sqrt{3} : \sqrt{2}$.

3. A solid cylinder (*a*) rolls, (*b*) slides from rest down an inclined plane. Neglect friction and compare the velocities in both cases when the cylinder reaches the bottom of the inclined plane.

$$\left[\textbf{Ans. } \frac{v_1}{v_2} = \sqrt{\frac{2}{3}} = 0.8166\right]$$

4. The flat surface of a hemisphere of radius r is cemented to one flat surface of a cylinder of radius r and length L. If the total mass is M, show that the M.I. of combination about the axis of the cylinder will be

$$Mr^2\,\frac{(L/2 + 4r/15)}{(L + 2r/3)}$$

5. A fly wheel in the form of a solid disc of 5000 kg and 1 m radius is rotating making 120 r.p.m. Compute (*i*) K.E. and (*ii*) angular impulse if the wheel is brought to rest in 2 seconds. [**Ans.** 1.974×10^5 J, 1.571×10^4 N ms^{-1}]

6. A uniform thin bar of mass 3 kg and length 0.9 m is bent to make an equilateral triangle. Calculate the moment of inertia about an axis passing through the centre of mass and perpendicular to the plane of the triangle. [**Ans.** 0.6225 kg m^2]

7. A solid cylinder of diameter 8 cm and mass 0.25 kg rolls down an inclined plane rising 3 in 20 without slipping. Find the acceleration and total energy of the cylinder after 5 sec. [**Ans.** 0.98 ms^{-2}; 4.5 J]

8. Two thin discs each of mass 0.1 kg and radius 0.05 m are placed at either end of a rod 0.2 m long and 0.01 m in diameter. What is the moment of inertia of the system about an axis passing through the centre of the rod and perpendicular its length? Density of the material of the rod is 7.8×10^3 kg m^{-3}. [**Ans.** 2.5342×10^{-3} kg m^2]

9. Determine the moment of inertia of the earth, assuming it to be a uniform sphere of radius 6400 km and mass 6×10^{24} kg. [**Ans.** 98.3×10^{35} kg m^2]

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10. A flywheel of mass 500 kg and 2 metres diameter makes 500 revolutions per minute. Assuming the mass to be concentrated at the rim calculate the angular velocity, the energy and the moment of inertia of the flywheel.

[**Ans.** 50 $\pi/3$ rad/sec; 68.57 × 10^4 Joule; 500 kg m]

11. A thin hollow cylinder open at both ends and weighing 96 kg (*a*) slides with a speed of 10 m/sec without rotating, (*b*) rolls with the same speed without slipping. Compare the kinetic energies of the cylinder in the two cases.

Hint. As the cylinder is thin $\therefore R = r = a$ and $I = \dfrac{M(R^2 + r^2)}{2} = Ma^2$

(*a*) K.E. $= \dfrac{1}{2}Mv^2$

(*b*) K.E. $= \dfrac{1}{2}Mv^2 + \dfrac{1}{2}I\omega^2$

$$= \frac{1}{2}Mv^2 + \frac{1}{2}Ma^2\left(\frac{v}{a}\right)^2 = \frac{1}{2}Mv^2 + \frac{1}{2}Mv^2 = Mv^2$$

[**Ans.** (*a*) 4800 Joule (*b*) 9600 Joule]

12. A diatomic molecule consists of two atoms of masses m_1 and m_2 kg. The two atoms are at a constant distance of x metres. Calculate the moment of inertia of the system about an axis passing through the centre of gravity of the system and perpendicular to the line joining the atoms.

$$\left[\textbf{Ans. } I = m_2^2 x^2 \left(1 - \frac{1}{m_1 + m_2}\right)\right]$$

13. Four spheres each of mass M and radius R are placed at four corners of a square of side a. Find moment of inertia of this arrangement about any side of square.

(*Purvanchal* 2006) [**Ans.** $\dfrac{2}{5}$M$(4\text{R}^2 + 5a^2)$]

5 OSCILLATIONS

SIMPLE HARMONIC MOTION

INTRODUCTION

Motion of a clock pendulum, balance wheel of a clock, mass attached to a suspended spring, vibrations of prongs of tunning fork, up and down motion of needle of sewing machine, a bar magnet suspended in uniform magnetic field are some of the familiar examples of simple harmonic motion (S.H.M.).

Every oscillatory motion is simple harmonic in character and continues due to the interaction of *inertia* and *elasticity*. When an oscillator is displaced from its position of equilibrium by application of a force and thus doing work on it, a restoring force comes into play tending to bring it back to its equilibrium position. According to Hooke's law, this restoring force is proportional to displacement and depends upon the *elastic properties* or *elasticity* of the system. As soon as the restoring force tries to bring the system back to its equilibrium position, the property of *inertia* opposes this change in velocity. Further, when the system reaches the equilibrium position, it overshoots the mark and moves beyond the mean position again due to inertia of motion. The motion continues till the deforming force due to inertia developes brings the system to rest. The restoring force again then sets the oscillator into motion back towards the equilibrium position. Throughout the process, the total energy remains conserved only the conversion of P.E. to K.E. and vice-versa takes place within a limit set by potential well.

5.1 OSCILLATORY AND SIMPLE HARMONIC MOTIONS

Oscillatory motion. A motion which repeats itself after regular intervals of time is called a *periodic motion*.

If a body in periodic motion executes to and fro motion about a fixed reference point it is said to have an oscillatory motion.

The term oscillatory motion is not restricted only to '*displacement*' of a mechanical oscillator, but it may be any physical quantity. For example in electrical system an oscillatory variation of charge, current or voltage may take place.

Simple harmonic motion. *When a body moves such that its acceleration is always directed towards a certain fixed point and varies directly as its distance from that point, the body is said to execute simple harmonic motion.*

For such a motion to take place the force acting on the body should be directed towards the fixed point and should also be proportional to the *displacement i.e.*, the distance from the fixed point. The function of the force is to bring the body back to its equilibrium position and hence this force is often known as *restoring force.*

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It should be noted that all the periodic motions are not S.H.M. For example: the motion of the earth around the sun and the motion of the moon around the earth are periodic but not simple harmonic.

However, the reverse is true; *i.e.* all simple harmonic motions are periodic. A S.H.M. has a definite frequency and hence has fixed time period (T) which makes it a periodic motion.

5.2 POTENTIAL WELL

Consider a particle oscillating in a conservative force field. The variation of potential energy about its equilibrium (*i.e.* mean) position is explained by potential energy curve as shown in fig. 5.1. For simplicity, we have considered motion only in one dimensional, restricted to only the x-axis, say. The nature of the curve obtained between the potential energy (U) and the position (x) of the particle, with well-marked maxima (A and C) and minimum potential energy (U_{min}) at its equilibrium position B. Since the system is conservative, its K.E. must be maximum at B, so as to move it towards either A or C.

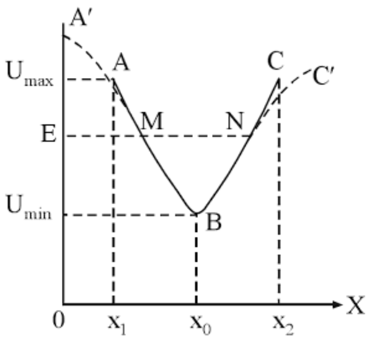

Fig. 5.1

As the particle, executing S.H.M., moves away from B, it comes under the action of *linear restoring force*, F = – grad U *i.e.* $F = - dU/dx$, (motion restricted to the x-axis only). Thus, the slope of the curve (*i.e.* dU/dx) at any given point gives the value of the force acting on the particle at that point. Its K.E. decreases further and correspondingly the P.E. increases, so that total energy E = T + U at any point remaining constant. This continues till it reaches to A (or C). At A, whole of its energy is in potential form equal to E. The K.E. is reduced to zero. Therefore, the particle stops here momentarly and then starts moving back towards the equilibrium position B. Its P.E. converts into K.E. At B, again the particle has minimum potential energy (U_{min}) and its K.E. becomes maximum. Point B is known as position of unstable equilibrium. Hence it continues to move further, towards C (or A) and so on. Thus, the particle remains confined to the region ABC, oscillating simple harmonically between the turning points A and C. Such a region bounded by curve ABC is called a *potential well* or a *potential valley* and always exists about the point of minimum potential energy (U_{min}) *i.e.* the equilibrium position for a particle executing S.H.M.

Actual shape of the potential well curve is as shown by dotted curve A′ B C′.

5.3 DIFFERENTIAL EQUATION OF S.H.M.

Consider a particle of mass m executing simple harmonic motion. If y be the displacement of the particle from equilibrium position at any instant t, the restoring force F acting on the particle is given by

$$F \propto y \quad \text{or} \quad F = -sy$$

where s is the *force constant* of proportionality or *stiffness or spring constant*. The negative sign is used to indicate that the direction of the force is opposite to the direction of increasing displacement.

Force constant s is defined as the restoring force per unit displacement

or $$s = \frac{F}{y}.$$

Its unit is *Newton per metre*.

If $\frac{d^2y}{dt^2}$ is the acceleration of the particle at time t, then

$$m\frac{d^2y}{dt^2} = -sy \quad \text{or} \quad \frac{d^2y}{dt^2} + \frac{s}{m}y = 0$$

Substituting $\frac{s}{m} = \omega^2$, we get $\frac{d^2y}{dt^2} + \omega^2 y = 0$ ...(*i*)

This is the general *differential* equation of motion of a *simple harmonic oscillator*.

5.4 SOLUTION OF DIFFERENTIAL EQUATION

The differential equation representing S.H.M. is given by

$$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

Multiply by $2\frac{dy}{dt}$ we get

$$2\frac{dy}{dt}\frac{d^2y}{dt^2} + \omega^2\, 2y\frac{dy}{dt} = 0$$

Integrating, we have

$$\left(\frac{dy}{dt}\right)^2 = -\omega^2 y^2 + C \qquad \text{...}(ii)$$

where C is the constant of integration.

When the displacement is maximum *i.e.*, at $y = a$

where a is the amplitude of the oscillating particle, $\frac{dy}{dt} = 0$

i.e., the particle is momentarily at rest in the *extreme* position and begins its journey in the backward direction.

Substituting $y = a$ and $\frac{dy}{dt} = 0$ in equation (*ii*), we have

$$C = a^2\omega^2$$

Substituting this value of C, in Eq. (*ii*) we get

$$\left(\frac{dy}{dt}\right)^2 = \omega^2 (a^2 - y^2)$$

or

$$\frac{dy}{dt} = \omega\sqrt{a^2 - y^2} \qquad \text{...}(iii)$$

This equation gives the *velocity* of the particle executing simple harmonic motion at a time t, when the displacement = y

$$\therefore \quad \frac{dy}{\sqrt{a^2 - y^2}} = \omega\, dt$$

Integrating, we have

$$\sin^{-1}\frac{y}{a} = \omega t + \phi$$

or

$$y = a\sin(\omega t + \phi) \qquad \text{...}(iv)$$

where ϕ is another constant of integration.

The term $(\omega t + \phi)$ represent the total *phase* of the particle at time t and ϕ is known as the *initial phase* or *phase constant*. If the time is recorded from the instant when $y = 0$ and increasing then $\phi = 0$.

Other Solutions. The equation $y = a \sin(\omega t + \phi)$ is just one solution of the differential equation $\frac{d^2 y}{dt^2} + \omega^2 y = 0.$

An equally valid solution of this equation is

$$y = a \cos(\omega t + \phi) \qquad \text{...}(v)$$

Expanding $y = a \sin(\omega t + \phi)$, we have

$$y = a \cos\phi \sin\omega t + a \sin\phi \cos\omega t$$
$$= A \sin\omega t + B \cos\omega t \qquad \text{...}(vi)$$

which is another valid solution, in which

$$A = a \cos\phi \text{ and } B = a \sin\phi$$

i.e., $$a = \sqrt{A^2 + B^2} \text{ and } \phi = \tan^{-1}\left(\frac{B}{A}\right)$$

Similarly by expanding $y = a \cos(\omega t + \phi)$, we have

$$y = a \cos\phi \cos\omega t - a \sin\phi \sin\omega t$$
$$= A \sin\omega t + B \cos\omega t$$

where $$A = -a \sin\phi \text{ and } B = a \cos\phi$$

i.e., $$a = \sqrt{A^2 + B^2} \text{ and } \phi = \tan^{-1}\left(\frac{-A}{B}\right)$$

The various general solutions of differential equation

(*i*) In **sine – cosine form** are

$$y = a \sin(\omega t + \phi)$$
$$y = a \cos(\omega t + \phi)$$
$$y = A \sin\omega t + B \cos\omega t$$

(*ii*) **Exponential form.** We can put differential equation (*i*) in the operator form by substituting

$$\frac{d}{dt} = D$$

or $$\frac{d^2}{dt^2} = D^2 \text{ and we get}$$

$$D^2 y + \omega^2 y = 0$$

or $$D^2 = -\omega^2$$

$\therefore$ $$D = \pm i\omega$$

Hence the general solution of equation (*i*), becomes

$$y = Ae^{i\omega t} + Be^{-i\omega t}$$

In order that this solution may give a real value of y, A and B must be complex conjugates of each other *i.e.*,

$$A = a + ib \text{ and } B = a - ib$$

A second form of the general solution of equation (*i*) is

$$y = a\, e^{i(\omega t + \phi)}$$

It has 'two constants a and ϕ and satisfies differential equation (*i*).

Thus the solutions of differential equation (*i*) in **exponential form** are

$$y = Ae^{i\omega t} + Be^{-i\omega t}$$

and $$y = a\, e^{i(\omega t + \phi)}$$

All these alternative ways of writing the solution of differential equation (*i*) have their own advantages. For a particular problem we select the form most convenient for the purpose. We shall use the solution $y = a \sin(\omega t + \phi)$ in general.

Simple Harmonic Motion is Sinusoidal or Co-sinusoidal. A simple harmonic motion can be represented by the relations

$$y = a \sin(\omega t + \phi) \quad \text{...}(i)$$

or

$$y = a \cos(\omega t + \phi) \quad \text{...}(ii)$$

where y is the displacement at a time t, a the amplitude, ω the angular frequency and ϕ the phase constant.

From Eq. (i), we have

$$\frac{dy}{dt} = a\omega \cos(\omega t + \phi) \quad \text{and} \quad \frac{d^2y}{dt^2} = -a\omega^2 \sin(\omega t + \phi)$$
$$= -\omega^2 y \quad \text{...}(iii)$$

Similarly from Eq. (ii), we have

$$\frac{dy}{dt} = -a\omega \sin(\omega t + \phi) \quad \text{and} \quad \frac{d^2y}{dt^2} = -a\omega^2 \cos(\omega t + \phi)$$
$$= -\omega^2 y \quad \text{...}(iv)$$

Thus according to relation (iii) and (iv) the acceleration $\left(\frac{d^2y}{dt^2}\right)$ is proportional to displacement y and is directed towards the mean position which proves that the motion represented by Eq. (i) and (ii) is simple harmonic.

This is why a simple harmonic motion is called sinusoidal or co-sinusoidal.

5.5 VELOCITY OF SIMPLE HARMONIC OSCILLATOR

The displacement of a simple harmonic oscillator at any instant of time t is given by

$$y = a \sin(\omega t + \phi) \quad \text{... }(i)$$

The velocity is defined as the time rate of change of displacement.

$$\therefore \text{ Velocity } v = \frac{dy}{dt} = \dot{y} = a\omega \cos(\omega t + \phi)$$
$$= a\omega \sin\left(\omega t + \phi + \frac{\pi}{2}\right) \quad \text{... }(ii)$$

As

$$\sin(\omega t + \phi) = \frac{y}{a}, \quad \cos(\omega t + \phi) = \sqrt{1 - \frac{y^2}{a^2}} = \sqrt{\frac{a^2 - y^2}{a}}$$

$$\therefore \quad v = \omega\sqrt{a^2 - y^2} \quad \text{... }(iii)$$

Maximum velocity. The velocity of the oscillator is maximum when $\sin\left(\omega t + \phi + \frac{\pi}{2}\right) = 1$

$$\therefore \quad v_{max} = a\omega$$

The value of $v = v_{max}$ when $y = 0$ *i.e.*, the particle executing S.H.M. is in its mean position.

(b) Comparing equations (i) and (ii), we find that the velocity of simple harmonic oscillator at any instant of time t *leads* the displacement by a phase difference $\frac{\pi}{2}$ radian (or 90°) *i.e.*, the two are in *quadrature.*

The velocity varies harmonically with the same frequency ω.

5.6 ACCELERATION OF SIMPLE HARMONIC OSCILLATOR

Acceleration is defined as the time rate of change of velocity.

Now velocity $\qquad v = a\,\omega \cos(\omega\, t + \phi)$ ...(*i*)

$\therefore$ Acceleration $\qquad = \dfrac{dv}{dt} = \ddot{y} = -a\omega^2 \sin(\omega t + \phi)$

$\qquad = a\,\omega^2 \sin(\omega\, t + \phi + \pi)$ ...(*ii*)

Maximum acceleration. The acceleration of the oscillator is maximum when $\sin(\omega\, t + \phi + \pi) = 1$ and is given by $\left(\dfrac{d^2 y}{dt^2}\right)_{max} = a\omega^2$.

Phase relationship between displacement, velocity and acceleration

The velocity $v = a\omega\cos(\omega t + \phi) = a\omega\sin\left(\omega t + \phi + \dfrac{\pi}{2}\right)$ ... (*iii*)

Comparing (*ii*) and (*iii*), we find that the acceleration of a simple harmonic oscillator leads the velocity by $\dfrac{\pi}{2}$ radian in phase.

The displacement $y = a \sin(\omega\, t + \phi)$ ...(*iv*)

Comparing (*ii*) and (*iv*), we find that the acceleration of a simple harmonic oscillator leads the displacement by π radian or (180°) in phase *i.e.*, the acceleration and displacement are in *antiphase*.

5.7 GRAPHICAL REPRESENTATION OF DISPLACEMENT, VELOCITY AND ACCELERATION

Let the displacement of a particle executing S.H.M. be

$$y = A \sin(\omega\, t + \phi)$$

$\therefore$ Velocity $\qquad v = \dfrac{dy}{dt} = A\omega\cos(\omega t + \phi)$

and acceleration $\qquad a = \dfrac{d^2 y}{dt^2} = -A\omega^2 \sin(\omega t + \phi)$

Hence for $\quad y = 0, \quad v = +A\,\omega, \quad a = 0$

$\qquad y = +A, \quad v = 0, \quad a = -\omega^2 A$

$\qquad y = -A, \quad v = 0, \quad a = +\omega^2 A$

The same is shown graphically as under

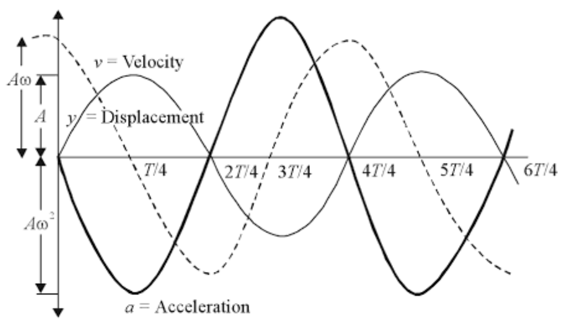

Fig. 5.2

5.8 PERIOD AND FREQUENCY OF S.H.M.

Periodic time. A simple harmonic motion is represented by the equation

$$y = a \sin (\omega t + \phi)$$

In this equation if we increase t by $\frac{2\pi}{\omega}$, then

$$y = a \sin \left[\omega \left(t + \frac{2\pi}{\omega} \right) + \phi \right]$$
$$= a \sin [\omega t + 2\pi + \phi]$$
$$= a \sin (\omega t + \phi)$$

i.e., the displacement of the particle after a time $T = \frac{2\pi}{\omega}$ is the same.

Hence T gives the periodic time of the simple harmonic oscillator.

$\therefore$ $$T = \frac{2\pi}{\omega}$$

and Frequency $$n = \frac{1}{T} = \frac{\omega}{2\pi}$$

Thus $\omega = 2\pi n$ = angular velocity of the harmonic oscillator. The acceleration of a simple harmonic oscillator is given by the relation

$$\frac{d^2 y}{dt^2} = -\omega^2 y$$

Neglecting the negative sign, we have

$$\omega^2 = \frac{d^2 y}{dt^2} \bigg/ y = \frac{\text{Acceleration}}{\text{Displacement}}$$

$\therefore$ $$\omega = \sqrt{\frac{\text{Acceleration}}{\text{Displacement}}}$$

and $$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{\text{Displacement}}{\text{Acceleration}}}$$

Also $$n = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{\text{Acceleration}}{\text{Displacement}}}.$$

5.9 ENERGY OF A SIMPLE HARMONIC OSCILLATOR

Energy of a simple harmonic oscillator. The total energy of a simple harmonic oscillator at any time t is the sum of its kinetic energy and potential energy at that instant of time.

$\therefore$ $$E_{total} = K.E. + P.E.$$

Kinetic energy. The general equation of a simple harmonic oscillator is given by

$$y = a \sin (\omega t + \phi)$$

where y is the displacement of the oscillator at any time t, a is the amplitude, ω the angular velocity and ϕ the initial phase or epoch.

$\therefore$ The velocity at any instant

$$v = \frac{dy}{dt} = \dot{y} = a\omega \cos (\omega t + \phi)$$

$\therefore$ Kinetic energy of the oscillator $= \frac{1}{2} m \left(\frac{dy}{dt} \right)^2 = \frac{1}{2} m \dot{y}^2$

$$= \frac{1}{2} m a^2 \omega^2 \cos^2 (\omega t + \phi)$$

Potential energy. The potential energy is equal to the amount of work done in overcoming the restoring force from the mean position through a displacement y.

Now, acceleration $= \frac{d^2 y}{dt^2} = -a\omega^2 \sin(\omega t + \phi)$

$$= -\omega^2 y$$

The negative sign indicates that the acceleration is directed towards the mean position.

$\therefore$ Restoring force $= m\omega^2 y = sy$ where s is the force constant of proportionality or *stiffness*.

Hence total work done by the force through a displacement y

$$= \int_0^y sy\,dy = \frac{1}{2} sy^2 = \frac{1}{2} m\omega^2 y^2$$

or $$P.E. = \frac{1}{2} sy^2 = \frac{1}{2} m\omega^2 y^2 = \frac{1}{2} m\omega^2 a^2 \sin^2 (\omega t + \phi)$$

Total Energy. $$E = K.E. + P.E. = \frac{1}{2} m\dot{y}^2 + \frac{1}{2} sy^2$$

$$= \frac{1}{2} ma^2\omega^2 \cos^2 (\omega t + \phi) + \frac{1}{2} ma^2 \omega^2 \sin^2 (\omega t + \phi)$$

$$= \frac{1}{2} ma^2 \omega^2$$

but $$\omega^2 = \frac{s}{m} \qquad (\because m\,\omega^2 = s)$$

Hence $$E = \frac{1}{2} sa^2$$

Also $\omega = 2\pi n$ $\quad \therefore E = \frac{1}{2} ma^2 4\pi^2 n^2 = 2ma^2\pi^2 n^2$

Thus total energy of the harmonic oscillator is a constant and proportional to the square of the amplitude.

Maximum K.E. The maximum value of $K.E.$ for $\cos(\omega t + \phi) = 1$ is given by

$$K.E. \text{ (max)} = \frac{1}{2} ma^2\omega^2$$

Maximum P.E. The maximum value of $P.E.$ for $\sin(\omega t + \phi) = 1$ is given by

$$P.E. \text{ (max)} = \frac{1}{2} ma^2\omega^2$$

$\therefore$ $K.E.$ (max) $= P.E.$ (max) $=$ Total energy $E = \frac{1}{2} m a^2\omega^2$

K.E. and P.E. at half-amplitude point.

Total energy $= \frac{1}{2} m a^2\omega^2$

Potential energy $= \frac{1}{2} m\omega^2 y^2$

At $y = \frac{a}{2}$, the mid point of amplitude

$\therefore$ Potential energy $= \frac{1}{2} m\omega^2 \frac{a^2}{4} = \frac{1}{8} m\omega^2 a^2 = \frac{1}{4} \times \frac{1}{2} m a^2\omega^2$

$\therefore$ Kinetic energy = Total energy – Potential energy

$$= \frac{1}{2}ma^2\omega^2 - \frac{1}{8}ma^2\omega^2 = \frac{3}{8}ma^2\omega^2 = \frac{3}{4}\times\frac{1}{2}ma^2\omega^2$$

$\therefore$ $\frac{1}{4}$ of the total energy is potential and $\frac{3}{4}$ is kinetic. P.E. and K.E. are equal at $y = \pm a/\sqrt{2}$ (Refer Example 5.8)

Graphical representation of energy. The variation of total energy, kinetic energy and potential energy of a harmonic oscillator with displacement y is shown in Fig. 5.3.

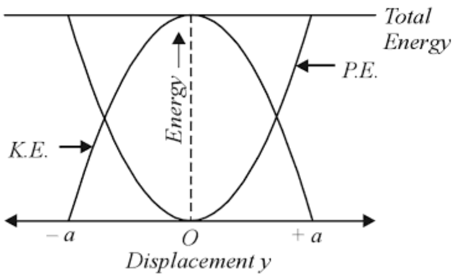

Fig. 5.3

(*i*) Kinetic energy. The kinetic energy is given by $\frac{1}{2}m\dot{y}^2$ where $\dot{y}$ is the velocity of the oscillator. The velocity and hence the kinetic enrgy is zero at the extreme positions $y = \pm a$ and maximum in the mean position $y = 0$. The velocity and hence the *K.E.* decreases as the oscillator moves away from the mean position and finally becomes zero at the extreme positions as shown by the curve marked *K.E.*

(*ii*) Potential energy. The potential energy is given by $\frac{1}{2}sy^2$ where s is the *stiffness*. For $y = 0$, *P.E.* = 0 and for $y = \pm a$, the *P.E.* is maximum $= \frac{1}{2}sa^2$. Thus potential energy goes on increasing as the oscillator move away from its mean position and becomes maximum at the extreme positions $y = \pm a$ as shown by the curve marked *P.E.*

(*iii*) Total energy. The total energy is given by $\frac{1}{2}sa^2$ which is a constant. This is represented by the straight line marked '*Total energy*' parallel to the displacement axis.

Thus, the total energy of a particle, at any point executing S.H.M., remains constant *i.e.* remains conserved.

5.10 EXAMPLES OF SIMPLE HARMONIC OSCILLATORS

A spring and mass system, simple pendulum, compound pendulum, torsion pendulum, Helmholtz resonator, LC circuit, vibrations of a magnet, two masses connected by a spring, bifilar oscillations are some of the examples of simple harmonic oscillator.

5.10.1 Spring and Mass System. (Loaded Spring)

Consider a spring whose upper end is fixed to a rigid support and the lower end is attached is a mass m. (fig. 5.4) Suppose the spring be extended through a distance l, (say), due to the weight mg of the mass, a linear restoring force = $s \times l$ at once comes into play in the opposite direction. The equilibrium position is attained, when these two forces balance each other, *i.e.*

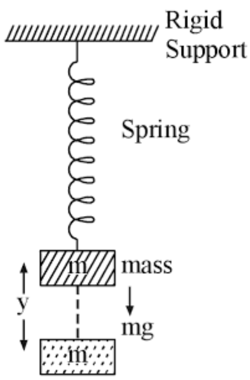

Fig. 5.4

$$mg = s \times l$$

$$\therefore \quad s = mg/l$$

where s is force constant or stiffness of the spring. If the mass be now pulled down through a distance y from this equilibrium position, the linear restoring force,

$$F = mg - s\,(l + y)$$

$$= mg - \frac{mg}{l}(l + y)$$

$$= mg - mg - \frac{mg}{l} \cdot y$$

$$= -\frac{mg}{l} \cdot y$$

$$= -sy$$

If $\frac{d^2y}{dt^2}$ be the acceleration set up in the spring then,

$$m\frac{d^2y}{dt^2} = -sy$$

$$\frac{d^2y}{dt^2} = -\frac{s}{m} \cdot y - \mu y$$

or

$$\frac{d^2y}{dt^2} + \mu y = 0$$

where $\mu = \frac{s}{m}$ = acceleration per unit displacement. This is the differential equation of the mass suspended from the spring.

$$\therefore \qquad \frac{d^2y}{dt^2} \propto -y$$

Thus, linear acceleration is directly proportional to its linear displacement and is directed oppositely to it. Hence, mass m attached to a spring executes S.H.M. and its time period is given by

$$T = 2\pi\sqrt{\frac{1}{\mu}} = 2\pi\sqrt{\frac{m}{s}}$$

and frequency,

$$n = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{s}{m}}$$

Stiffer Spring

When we use a stiffer spring of spring constant s' such that $s' > s$

then

$$T' = 2\pi\sqrt{\frac{m}{s'}}$$

As $s' > s$, T' is $< T$, the time period will decrease.

Dimensions of Force Constant

Consider a mass m attached to a spring executing S.H.M. If y is the displacement of the mass from its equilibrium position at any instant of time t, then the restoring force F acting on the mass is given by

$$F = -sy$$

where s is the *force constant* or *stiffness* of the spring.

$\therefore$ Dimensions of s are $= \frac{F}{y} = \frac{Force}{Displacement} = \frac{\left[M^1 L^1 T^{-2}\right]}{\left[L^1\right]} = \left[M^1 T^{-2}\right]$

Stiffness is expressed in newton per metre (Nm^{-1}).

5.10.2 Simple Pendulum

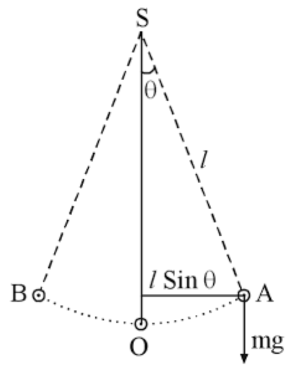

Fig. 5.5

A Simple Pendulum: *A simple pendulum is a heavy particle (ideally, a point mass) suspended by an inextensible and weightless string from a rigid support.*

Let S be the point of suspension and O, the mean or equilibrium position of the pendulum *i.e.* bob. Suppose the bob is displaced through an angle θ. Then, the weight mg acting downward exerts a torque $-\,mgl \sin\theta$ about the point of suspension tending to bring it back to O. The – ve sign indicates that the torque is oppositely directed to the displacement (θ).

If $\dfrac{d^2\theta}{dt^2}$ be the acceleration of the bob towards O, and I be its $M.I.$ about the point of suspension and is perpendicular to the plane of paper, the moment of force *i.e.* the torque acting on the bob is also given by $I \cdot \dfrac{d^2\theta}{dt^2}$.

Therefore,
$$I \cdot \frac{d^2\theta}{dt^2} = -\,mgl \sin\theta$$

∴
$$\frac{d^2\theta}{dt^2} = -\frac{mgl}{I}\theta$$

Since $M.I.$ of the bob about the point of suspension is ml^2, we have

$$\frac{d^2\theta}{dt^2} = -\frac{mgl}{ml^2}\cdot\theta = \frac{-g}{l}\cdot\theta = -\mu\cdot\theta$$

where $\mu = \dfrac{g}{l}$, the acceleration per unit displacement

∴
$$\frac{d^2\theta}{dt^2} \;\alpha\; -\theta$$

Thus, the angular acceleration of the bob is directly proportional to its angular displacement and is directed towards its mean position (O); (– ve sign).

Hence, the pendulum executes S.H.M. and its time period will be

$$T = 2\pi\sqrt{\frac{1}{\textit{Angular acceleration per unit angular displacement}}}$$

$$= 2\pi\sqrt{\frac{1}{\mu}}$$

$$T = 2\pi\sqrt{\frac{l}{g}} \qquad \ldots (i)$$

Equation (i) gives period of the simple pendulum.

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Minimum time period. The time period of the compound pendulum is **minimum** when the distance of the point of suspension from *C. G.* is equal to the radius of gyration.

To prove this put the relation for the time period

$$T = 2\pi\sqrt{\frac{k^2 + l^2}{lg}} \quad \text{as}$$

$$T^2 = \frac{4\pi^2}{g}\left(\frac{k^2 + l^2}{l}\right) = \frac{4\pi^2}{g}\left(\frac{k^2}{l} + l\right)$$

Differentiating this expression with respect to l, we get

$$2T\frac{dT}{dl} = \frac{4\pi^2}{g}\left(-\frac{k^2}{l^2} + 1\right)$$

For the value of time period T to be minimum $\frac{dT}{dl} = 0$

or $$1 - \frac{k^2}{l^2} = 0 \quad \text{or} \quad l = \pm k.$$

Further, $\frac{d^2T}{dl^2}$ comes out to be positive, if $l = +k$

$$\therefore \qquad T_{\min} = 2\pi\sqrt{\frac{\frac{k^2}{k} + k}{g}} = 2\pi\sqrt{\frac{k+k}{g}}$$

$$T_{\min} = 2\pi\sqrt{\frac{2k}{g}} \qquad \ldots (vi)$$

5.10.4 Torsion Pendulum

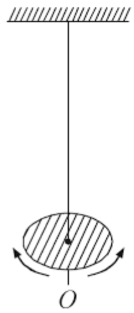

Fig. 5.7

A heavy body, like a cylinder or a disc, fastened at its mid-point to a fairly long and thin wire, suspended from a rigid support, constitutes a torsional pendulum.

The disc (or cylinder) is now turned through the some angle θ. Due to this, a restoring torsional couple = $C\theta$ is produced in the wire, which tends to bring it back into its original condition, where C is restoring couple per unit angular twist of the wire. If I is *M.I.* of the disc (or cylinder) about the wire as axis, passing through its center and $\frac{d^2\theta}{dt^2}$, its angular acceleration, the couple acting on it is also given by $I\frac{d^2\theta}{dt^2}$

Hence, $$I\frac{d^2\theta}{dt^2} = -C\theta$$

where – ve sign indicates that the restoring couple or torque is oppositely directed to the angular displacement θ,

$$\frac{d^2\theta}{dt^2} = -\left(\frac{C}{I}\right)\theta = -\mu\theta$$

where $\frac{C}{I} = \mu$, the angular acceleration per unit angular displacement.

$$\therefore \qquad \frac{d^2\theta}{dt^2} \propto -\theta$$

Thus, the angular acceleration is directly proportional to its angular displacement and is directed oppositely to it.

Hence the torsion pendulum executes an angular S.H.M. and its time period will be

$$T = 2\pi\sqrt{\frac{1}{\mu}} = 2\pi\sqrt{\frac{I}{C}}.$$

Merits of Torsion Perdulum: (1) In torsional pendulum, no approximations have been made in deriving the relation for T, unlike in the case of a simple pendulum or a compound pendulum (like θ should be very small etc.). Hence the time period of a torsional pendulum, remains unaffected (*i.e.* oscillations remain isochronous) even if the amplitude be large. This is because the *C.G.* of the suspended body, instead of moving in an arc, remains fixed in its position (2) The time period of the pendulum is quite independent of the value of g.

5.10.5 Helmholtz Resonator

A resonator is a device to analyse a complex note of sound *i.e.* to determine what particular frequencies are present in the given note. For this, the resonator should exhibit a sharp resonance *i.e.* it should resound with a note of one and only one frequency, namely its own natural frequency. Thus, resonators of different frequencies or varying frequencies are used detect the unknown note.

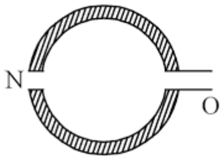

Fig. (*a*) Spherical

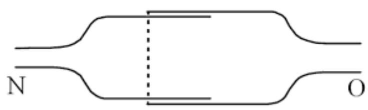

Fig. (*b*) Cylindrical

Fig. 5.8

Helmoholtz showed that "Volume resonators" have high sharpness of resonance if the resonator is a large vessel, spherical or cylindrical, of glass or metal, containing air, with a narrow neck N, through which it communicates with the outside air, and receives a complex note to be analysed. A narrow aperture O, at the opposite end to be plugged into the ear.

Theory : Let l be the length of the neck,

A be the area of cross-section

ρ be the density of air in vessel

Then, the mass of air piston or air plug in the neck

$= lA\rho$

If this air plug is to be forced 'inwards' through a small displacement x, the decrease in the volume of air inside the vessel is $-\delta v$

$$-\delta v = -A \cdot x$$

resulting in a slight increase in pressure δp. Then, if k is the volume elasticity of air and V be the total volume of the vessel, then

$$k = -\frac{\delta p}{\delta v/V} = -\delta p\frac{V}{dv}$$

$$\therefore \qquad \delta p = -k\frac{\delta v}{V}$$

∴ force acting on the air plug outwards

$$= A \cdot \delta p$$

$$= -A \cdot k \cdot \frac{\delta v}{V}$$

$$= -Ak\frac{Ax}{V}$$

$$= -\frac{kA^2 x}{V}$$

If $\frac{d^2x}{dt^2}$ be the acceleration of the air plug, the force acting on it also

$$= \text{mass} \times \frac{d^2x}{dt^2}$$

$$= lA\rho \cdot \frac{d^2x}{dt^2}$$

$$\therefore \qquad lA\rho \cdot \frac{d^2x}{dt^2} = -\frac{kA^2x}{V}$$

$$\therefore \qquad \frac{d^2x}{dt^2} = -\frac{kA^2}{V} \times \frac{1}{lA\rho} \times x = -\frac{kA}{Vl\rho}x = -\mu x$$

Where $\mu = \frac{kA}{Vl\rho}$ = acceleration per unit displacement.

$$\therefore \quad \frac{d^2x}{dt^2} \alpha - x$$

Thus, air plug or air piston executes S.H.M. and its time period will be

$$T = 2\pi\sqrt{\frac{1}{\mu}} = 2\pi\sqrt{\frac{Vl\rho}{kA}}$$

But velocity of sound in air, $v = \sqrt{\frac{k}{\rho}}$

$$\therefore \qquad T = \frac{2\pi}{v}\sqrt{\frac{Vl}{A}}$$

$$\therefore \qquad \text{Frequency } n = \frac{1}{T} = \frac{v}{2\pi}\sqrt{\frac{A}{Vl}}$$

Thus, frequency of resonator depends upon the total volume (V) of the vessel, length l and area (A) of cross section of neck.

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5.10.6 Inductance-Capacitance or LC Circuit

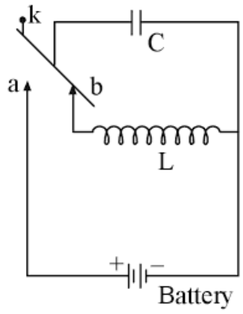

Fig. 5.9

Differential equation of an electrical oscillator. Consider an electrical circuit in which an inductor of inductance L and negligible resistance is connected to a capacitor of capacitance C. When the capacitor is first charged by a battery, the electrostatic energy resides in the dielectric medium of the capacitor. As the capacitor begins to discharge through the inductor, a current begins to flow through the inductor coil. This sets up a magnetic field around the coil. As the capacitor discharges the electrostatic energy of the capacitor is converted into magnetic energy of the inductor. There is thus a continuous exchange of energy between the inductor and the capacitor. As the resistance is negligible, there is no dissipation of energy in the form of heat etc.

If L is the co-efficient of self inductance of the coil and $\frac{dI}{dt}$ the rate of growth of current, then voltage across the inductor

$$= -L\frac{dI}{dt} = -L\frac{d^2q}{dt^2}$$

and voltage across the capacitor $= \frac{q}{C}$

As there is no source of *e.m.f.* in the circuit

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = 0$$

or $$\frac{d^2q}{dt^2} + \frac{q}{LC} = 0 \quad \text{or} \quad \frac{d^2q}{dt^2} + \mu q = 0 \qquad \text{... }(i)$$

This equation is a differential equation of S.H.M. for an electrical circuit.

where $\mu = \frac{1}{LC}$, a constant of the electrical circuit.

From eq. (*i*), $\frac{d^2q}{dt^2} \alpha - q$, Hence the discharge of the condenser is oscillatory. Its period is

$$T = 2\pi\sqrt{\frac{1}{\mu}} = 2\pi\sqrt{LC} \qquad \text{... }(ii)$$

and frequency is $$n = \frac{1}{T} = \frac{1}{2\pi}\cdot\frac{1}{\sqrt{LC}} \qquad \text{... }(iii)$$

Since the equation (*i*) is the differential equation of S.H.M. for an electrical circuit, its solution is

$$q = q_0 \sin(\omega t + \phi) \qquad \text{... }(iv)$$

where q_0 is the maximum value of the charge and is known as the *amplitude of the charge*.

Electrical oscillator in practice. The inductance L always has some ohmic resistance due to which there is a continuous loss of energy and the amplitude of S.H.M. slowly dies down to zero. If, however corresponding amount of energy is supplied to the circuit from an external source the amplitude of S.H.M. can be maintained. This is done by using a valve or transistor circuit. In this way simple harmonic electrical oscillations can be realised in practice, e.g. frequency generator.

5.10.7 Vibration of a Magnet

If a bar magnet is freely suspended in a uniform magnetic field $\vec{B}$, it remains in equilibrium, with its axis parallel to the field.

Suppose that $\vec{M}$ is the magnetic dipole moment of the magnet and $\vec{B}$ is a induction of magnetic field. The magnet is then given a small angular displacement θ, a restoring couple acts on the magnet tending to bring it back to the equilibrium position and is given by restoring couple, $\vec{T} = \vec{M} \times \vec{B}$

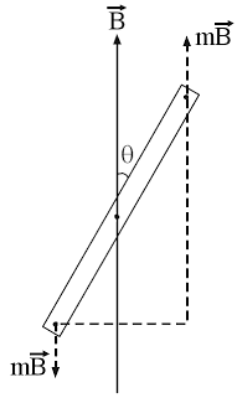

Fig. 5.10

The magnitude of restoring couple will be $T = -MB \sin\theta$

–ve sign indicates that torque is oppositely directed to its displacement θ.

If θ is very small, Sin θ ≅ θ

$$\therefore \qquad T = -MB\,\theta$$

The couple produces angular acceleration $\frac{d^2\theta}{dt^2}$. If I is $M.I.$ of the magnet then moment of restoring couple is also given by

$$T = M.I. \times \text{angular acceleration}$$
$$= I \times \alpha$$
$$= I \times \frac{d^2\theta}{dt^2} \qquad \left(\because \alpha = \frac{d^2\theta}{dt^2}\right)$$

Equating,

$$\therefore \qquad I \times \frac{d^2\theta}{dt^2} = -MB\,\theta$$

$$\therefore \qquad \frac{d^2\theta}{dt^2} = -\frac{MB}{I}\cdot\theta = -\mu\theta$$

Where $\frac{MB}{I} = \mu$ = acceleration per unit angular displacement

$$\therefore \qquad \frac{d^2\theta}{dt^2} \propto -\theta$$

Hence, the magnet executes S.H.M. Its period wil be

$$T = 2\pi\sqrt{\frac{1}{\mu}} = 2\pi\sqrt{\frac{I}{MB}}$$

5.10.8 Two-Body Harmonic Oscillator

A system of two masses (bodies) connected by a spring, so that both are free to oscillate simple harmonically along the length of the spring constitutes a "two-body harmonic oscillator" or a coupled oscillator.

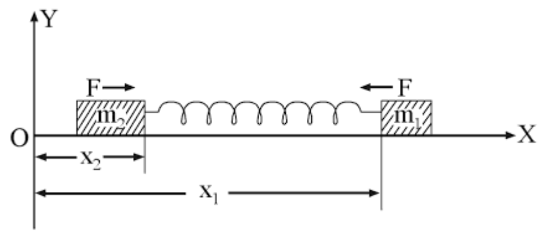

Fig. 5.11

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Examples:

Diatomic molecules like H_2, CO, HCl etc. can oscillate along their respective axes of symmetry. Electro magnetic forces which couple two atoms in a di-atomic molecule, act like a tiny and massless spring.

Let us consider, the general case of two-body oscillator consisting of two masses m_1 and m_2 connected by a spring of force constant c and are free to oscillate on a frictionless surface.

Let the normal length of the spring be l and let, at any given instant, the co-ordinate of the two ends of the spring be x_1 and x_2

Extension of the spring, $x = (x_1 - x_2) - l$

Where x is positive, if the spring is stretched; x is zero, if the spring has its normal length and x is –ve, if the spring is compressed. Here, we assume x to be +ve.

The forces (F) exerted by the spring on the two masses are equal in magnitude but opposite in sign, the magnitude of each being cx.

Let $\frac{d^2x_1}{dt^2}$ be the acceleration of mass m_1 and $\frac{d^2x_2}{dt^2}$ be the acceleration of mass m_2, then

$$m_1 \frac{d^2x_1}{dt^2} = -cx \qquad \text{... }(i)$$

and

$$m_2 \frac{d^2x_2}{dt^2} = +cx \qquad \text{... }(ii)$$

Multiplying eq. (i) by m_2 and eq. (ii) by m_1, and subtracting, we get

$$m_1m_2 \frac{d^2x_1}{dt^2} - m_1m_2 \frac{d^2x_2}{dt^2} = -m_2cx - m_1cx$$

$$m_1m_2 \frac{d^2}{dt^2}(x_1 - x_2) = -cx\,(m_1 + m_2)$$

$$\left(\frac{m_1m_2}{m_1 + m_2}\right)\frac{d^2(x_1 - x_2)}{dt^2} = -cx$$

Now l being a constant and,

putting $\left(\frac{m_1m_2}{m_1 + m_2}\right) = \mu$, called the reduced mass of the system (being smaller than either of the two masses m_1 and m_2), we have

$$\frac{d^2x}{dt^2} = -\frac{c}{\mu}\cdot x$$

$$\therefore \quad \frac{d^2x}{dt^2} \propto -x \qquad \left(\frac{c}{\mu} \text{ being constant}\right)$$

Thus, the system oscillates simple harmonically with a time-period

$$T = 2\pi\sqrt{\frac{1}{c/\mu}} = 2\pi\sqrt{\frac{\mu}{c}}$$

And frequency,

$$n = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{c}{\mu}}$$

This gives the frequency of oscillation.

5.10.9 Bifilar Oscillations

Bifilar Oscillations (with parallel threads): If a heavy and uniform bar or cylinder (or any rigid body), be suspended horizontally by means of two equal, vertical, flexible and inelastic threads, equidistant from its center of gravity, the arrangement constitutes bifilar suspension.

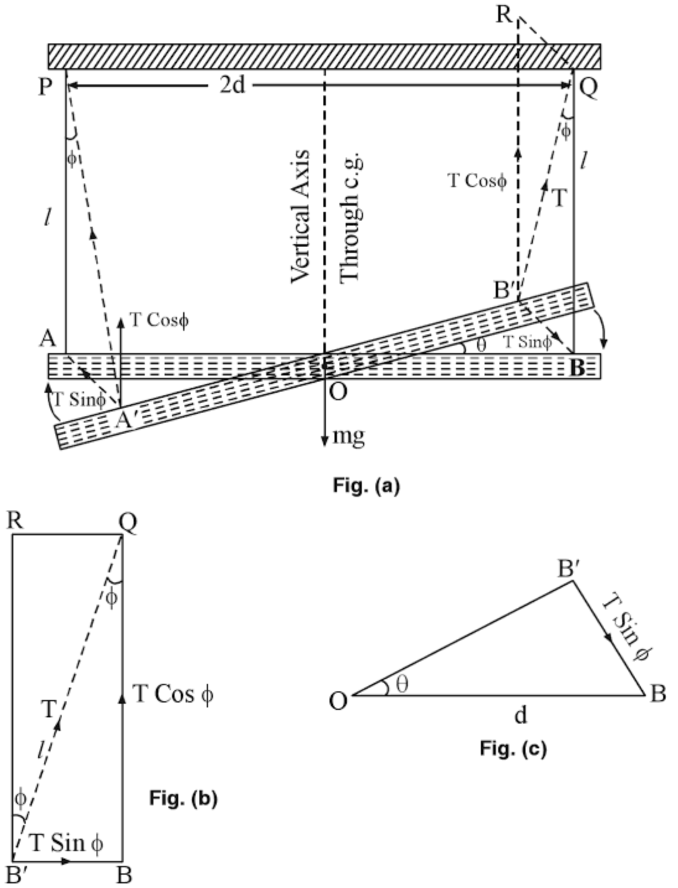

Fig. (a)

Fig. (b)

Fig. (c)

Fig. 5.12

On being displaced a little in its own plane *i.e.* horizontal plane and then released, the bar or cylinder executes S.H.M. about the vertical axis through its center of gravity.

Let AB [fig. (*a*)] represents the equilibrium position of a cylinder of mass m with its c.g. at O where its weight mg acts vertically downwards. Let the two suspension threads PA and QB be parallel to each other, and distance $2d$ apart and let the length of each thread be l.

Now, If the cylinder by displaced a little into the position A′ B′ through a small angle θ, the suspension threads take up the positions PA' and QB' at an angle ϕ with their original positions, where ϕ is small.

Let T be the tension in each thread acting upwards along it. Then, resolving it into its two rectangular components, we have (*i*) component T Cos ϕ, acting vertically upwards and (*ii*) component T Sin ϕ, acting horizontally along $B'B$ or $A'A$.

Vertical components support the weight of the cylinder Hence.

$$2\,T \cos\phi = mg$$

$$T \cos\phi = mg/2$$

And, since ϕ is small, Cos $\phi \cong 1$

So that $$T = \frac{mg}{2}$$

The components T Sin ϕ acting at A' and B' being equal, opposite and parallel constitute a couple, tending to bring the cylinder back to its original position. Since $A'A$ and $B'B$ are practically at right angles to $A'B'$, we have

$$\text{Moment of the restoring couple} = T \text{ Sin } \phi \cdot 2d$$
$$= T \cdot \phi \cdot 2d$$

Now, $$\theta = \frac{BB'}{OB} = \frac{BB'}{d}$$

$$\therefore \quad BB' = \theta \cdot d$$

And, $$\phi = \frac{BB'}{OB'} = \frac{BB'}{l} = \frac{\theta \cdot d}{l}$$

$$\therefore \text{ Moment of the restoring couple} = T \cdot \frac{\theta d}{l} \cdot 2d$$
$$= \frac{mg}{2} \cdot \frac{\theta d}{l} \cdot 2d$$
$$= \frac{mgd^2}{l} \cdot \theta$$

But, the moment of the restoring couple $= I \cdot \dfrac{d^2\theta}{dt^2}$

where I is M.I. of the cylinder about the Vertical axis through its c.g. and $\dfrac{d^2\theta}{dt^2}$ is its angular acceleration.

$$\therefore \quad I \cdot \frac{d^2\theta}{dt^2} = -\frac{mgd^2}{l}\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{mgd^2}{Il} \cdot \theta = -\mu\theta$$

where $\dfrac{mgd^2}{I \cdot l} = \mu$, angular acceleration per unit angular displacement.

$$\therefore \quad \frac{d^2\theta}{dt^2} \alpha - \theta$$

Hence, the bifilar oscillations are S.H.M and its, time period is given by

$$T = 2\pi\sqrt{\frac{1}{\mu}} = 2\pi\sqrt{\frac{I \cdot l}{mgd^2}}$$

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$$\therefore \qquad T = 2\pi \frac{1}{d} \cdot \sqrt{\frac{Il}{mg}}$$

OR, if we put $I = mk^2$, where k is radius of gyration of the cylinder about the vertical axis through O, we have

$$T = 2\pi \frac{1}{d} \cdot \sqrt{\frac{mk^2 \cdot l}{mg}} = 2\pi \frac{1}{d} \cdot \sqrt{\frac{k^2 \cdot l}{g}}$$

$$T = 2\pi \frac{k}{d} \cdot \sqrt{\frac{l}{g}}$$

This equation gives the time period of the bifilar oscillator having parallel threads.

5.11 SIMILARITY BETWEEN MECHANICAL AND ELECTRICAL OSCILLATIONS

There is a similarity between mechanical and electrical oscillations as discussed below:

The equation of motion of a simple harmonic mechanical oscillator is given by

$$\frac{d^2y}{dt^2} + \frac{s}{m} y = 0 \qquad \ldots(i)$$

where y is the displacement, s the force constant of proportionality or stiffness and m the mass of oscillator.

The angular frequency is given by

$$\omega^2 = \frac{s}{m} \quad \text{or} \quad \omega = \sqrt{\frac{s}{m}}$$

or frequency $$n = \frac{1}{2\pi}\sqrt{\frac{s}{m}}.$$

The equation for displacement is given by

$$y = a \sin(\omega t + \phi)$$

where a and ϕ are constants, known as amplitude and initial phase angle.

The total energy of the mechanical oscillator

$$E = \frac{1}{2} m v^2 + \frac{1}{2} s y^2 = \frac{1}{2} m \dot{y}^2 + \frac{1}{2} s y^2$$

where $\frac{1}{2} m \dot{y}^2$ is the *K.E.* and $\frac{1}{2} sy^2$ the *P.E.*

The equation of motion of a simple harmonic electrical oscillator is given by

$$\frac{d^2q}{dt^2} + \frac{1}{LC} q = 0$$

where q is the charge, L the inductance and C the capacitance of the electrical circuit.

The angular frequency is given by

$$\omega = \sqrt{\frac{1}{LC}}$$

or frequency $$n = \frac{1}{2\pi\sqrt{LC}}$$

The charge on the capacitance varies harmonically and is represented by an equation similar to displacement equation *i.e.*,

$$q = q_0 \sin(\omega t + \phi)$$

where q_0 is the amplitude of charge and ϕ the phase difference.

The current $I = \frac{dq}{dt}$ corresponds to velocity $v = \frac{dy}{dt}$ and is given by

$$I = \omega q_0 \cos(\omega t + \phi).$$

The voltage across the capacitor $V = \frac{q}{C} = \frac{q_0}{C} \sin(\omega t + \phi)$

Both I and V, therefore, vary harmonically with the same angular velocity ω.

The total energy of an electrical oscillator is the sum of the magnetic energy and electric energy.

The magnetic energy can be calculated from the current I and potential $V = L\frac{dI}{dt}$ across the inductance and is given by

$$\int VI\, dt = \int L\frac{dI}{dt} I\, dt = \int LI dI = \frac{1}{2}LI^2 = \frac{1}{2}L\dot{q}^2$$

Compare it with kinetic energy in a mechanical oscillator given by $\frac{1}{2}m\dot{y}^2$. Thus mass in a mechanical circuit corresponds to inductance in an electrical circuit and velocity to electric current.

The electrostatic energy can be calculated from the voltage across the capacitor and is given by

$$\frac{1}{2}CV^2 = \frac{1}{2}C\left(\frac{q}{C}\right)^2 = \frac{1}{2}\frac{q^2}{C}$$

Compare it with potential energy in a mechanical oscillator given by $\frac{1}{2}sy^2$. Thus stiffness s in a mechanical circuit corresponds to $\frac{1}{C}$ in an electrical circuit or pliability $\frac{1}{s}$ corresponds to capacitance C.

5.12 ROLE OF L AND C IN ELECTRICAL OSCILLATOR

An oscillating electrical circuit consists of an inductance L and a capacitance C. In the electrical oscillator it is the charge on the capacitance that oscillates. In other words, *the charge on the capacitor is the harmonically varying quantity which gives rise to electrical oscillations*. In an electrical oscillator *charge* corresponds to *displacement* in a mechanical oscillator. The inductance L is the electrical counterpart of mass m (inertia) and the reciprocal of capacitance ($1/C$) is the counter part of stiffness s. The frequency is given by

$$n = \frac{1}{2\pi\sqrt{LC}}$$

We can realise an LC circuit in practice and obtain simple harmonic oscillations if the circuit has *zero* (ohmic) resistance and there is no loss of energy. In practice, however, it is not possible to have a circuit with zero resistance. Hence energy is supplied to the LC circuit to make up for this small inevitable loss by electronic devices like thermionic valves or transistors.

SOLVED EXAMPLES

Example 5.1 A mass of 1 kg is attached to a spring of stiffness constant 16 Nm^{-1}. Find its natural frequency. (*Nagpur U.* 2008; *H.P.U.* 2001)

Solution. Here stiffness constant $s = 16$ Nm^{-1}, mass $m = 1$ kg

$\therefore$ Frequency $$n = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{16}{1}} = \frac{2}{\pi} = 0.64 \text{ Hz}$$

Example 5.2 A particle of mass 100 gm is placed in a field of potential $U = 5x^2 + 10$ ergs / gm. Find the frequency. (*Meerut U.* 2001)

Solution. Here potential energy $U = 5x^2 + 10$ ergs/gm ; $m = 100$ gm

Now $$F = -\frac{dU}{dx} \qquad \therefore \quad F = -\frac{d}{dx}(5x^2+10) = -10\,x$$

Taking x as the displacement, we have $$F = m\,\frac{d^2x}{dt^2}$$

$$\therefore \qquad m\,\frac{d^2x}{dt^2} = -10\,x \quad \text{or} \quad \frac{d^2x}{dt^2} = -\frac{10}{m}\,x$$

This is the equation of a simple harmonic motion with force constant $s = 10$ dyne/cm.

$$\therefore \quad \text{Frequency} \quad n = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{10}{100}} = \frac{1}{2\pi\sqrt{10}} = 0.05\ \text{Hz}$$

Example 5.3 A spring whose force constant is 80 N/m, hangs vertically supporting a 1 kg mass at rest. Find the distance by which the mass should be pulled down so that on being released it may pass the equilibrium position with a velocity of 1 m/s.

Solution. For a loaded spring

$$\omega = \sqrt{\frac{c}{m}}$$

Here, $c = 80$ N/m, $m = 1$ kg

$$\therefore \qquad \omega = \sqrt{\frac{80}{1}} = 8.944$$

Let a be the distance by which the mass has to be pulled down (amplitude of vibration)

$$v = a\omega$$

or $$a = \frac{v}{\omega}$$

Here $v = 1$ m/s and $\omega = 8.944$

$$\therefore \qquad a = \frac{1}{8.944} = 0.1118\,\text{m}$$

Example 5.4 (*a*) An oscillatory motion of a body is represented by $y = ae^{i\omega t}$ where symbols have usual meaning. Show that the motion is simple harmonic.

(*b*) The displacement of a moving particle at any time t is given by $y = a\cos\omega t + b\sin\omega t$ show that the motion is simple harmonic.

Solution. (***a***) Given $y = ae^{i\omega t}$

Differentiating with respect to 't' we get

$$\frac{dy}{dt} = ae^{i\omega t} \times i\,\omega = i\,a\,\omega\,e^{i\omega t}$$

Differentiating again, we get

$$\frac{d^2y}{dt^2} = i\,a\,\omega\,e^{i\omega t} \times i\,\omega = -\,a\,\omega^2 e^{i\omega t} = -\,\omega^2 y$$

or $$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

This is differential equation of S.H.M. Hence $y = ae^{i\omega t}$ represents a S.H.M.

(*b*) $y = a \cos \omega t + b \sin \omega t$

$\therefore$ $$\frac{dy}{dt} = -a\omega \sin \omega t + b\omega \cos \omega t$$

and $$\frac{d^2y}{dt^2} = -a\omega^2 \cos \omega t - b\omega^2 \sin \omega t$$
$$= -\omega^2 (a \cos \omega t + b \sin \omega t)$$
$$= -\omega^2 y$$

or $$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

Hence $y = a \cos \omega t + b \sin \omega t$ is the equation of simple harmonic motion.

Example 5.5 A particle executes S.H.M. of period 10 sec. and amplitude 5 cm. Calculate the maximum amplitude of velocity. (*Gauhati U.* 2002)

Solution. Here displacement amplitude a = 5 cm; Time period T = 10 sec.

$\therefore$ Angular frequency $\omega = \frac{2\pi}{T} = \frac{2\pi}{10} s^{-1}$

Maximum velocity $= a\omega = \frac{5 \times 2\pi}{10}$ cm/s

$= \pi$ cm/sec or 3.14 cm/sec.

Example 5.6 Calculate the displacement to amplitude ratio for a S. H. M. when K. E. is 90% of total energy.

Solution. If m is the mass of the particle executing S.H.M., a the amplitude and ω the angular velocity, then

Total energy $= \frac{1}{2} m a^2 \omega^2$

Let y be the displacement when *K. E.* = 90% of total energy,

As *K.E.* is 90% of total energy,

Potential energy = Total energy – Kinetic energy = 10% of total energy

Now potential energy $= \frac{1}{2} m\omega^2 y^2$

$$\therefore \quad \frac{\frac{1}{2} m\omega^2 y^2}{\frac{1}{2} m\omega^2 a^2} = \frac{y^2}{a^2} = \frac{10}{100} = 0.1$$

or $$\frac{\text{Displacement}}{\text{Amplitude}} = \frac{y}{a} = \sqrt{0.1} = 0.316$$

Example 5.7 What is the ratio of kinetic energy at displacement one fourth to one third of the amplitude in case of simple harmonic motion?

Solution. If m is the mass of the particle executing S.H.M., a the amplitude and ω the angular velocity, then

Total energy $$= \frac{1}{2} m\omega^2 y^2$$

When the displacement is y, Potential energy $= \frac{1}{2} m\omega^2 y^2$

$\therefore$ Kinetic energy = Total energy – Potential energy $= \frac{1}{2} m\omega^2 (a^2 - y^2)$

At displacement $y = \frac{a}{4}$, Kinetic energy $E_1 = \frac{1}{2} m\omega^2 \left(a^2 - \frac{a^2}{16}\right) = \frac{15}{32} m\omega^2 a^2$

At displacement $y = \frac{a}{3}$, Kinetic energy $E_2 = \frac{1}{2} m\omega^2 \left(a^2 - \frac{a^2}{9}\right) = \frac{8}{18} m\omega^2 a^2$

$$\therefore \qquad \frac{E_1}{E_2} = \frac{15}{32} \times \frac{18}{8} = \frac{135}{128} = 1.055$$

Example 5.8 A simple harmonic oscillator is characterised by $y = a \cos \omega t$. Calculate the displacement at which kinetic energy is equal to its potential energy.

(*Nagpur U.* 2003; *Pbi.U.*, 2000)

Or

At what displacement from the mean position the total energy of a simple harmonic oscillator is half kinetic and half potential. (*G.N.D.U.*, 2001)

Solution. Let y be the displacement at which kinetic energy of the simple harmonic oscillator is equal to its potential energy.

Now $\quad y = a \cos \omega t \quad \therefore \frac{dy}{dt} = -a\omega \sin \omega t$

The kinetic energy of a simple harmonic oscillator is given by

$$K.E. = \frac{1}{2} m v^2 = \frac{1}{2} m \left(\frac{dy}{dt}\right)^2 = \frac{1}{2} m (-a\omega \sin \omega t)^2$$

$$= \frac{1}{2} m a^2 \omega^2 \sin^2 \omega t = \frac{1}{2} m a^2 \omega^2 (1 - \cos^2 \omega t)$$

$$= \frac{1}{2} m\omega^2 (a^2 - a^2 \cos^2 \omega t) = \frac{1}{2} m\omega^2 (a^2 - y^2)$$

$$P.E. = \frac{1}{2} s y^2 \text{ where } s = \text{stiffness and } \frac{s}{m} = \omega^2 \text{ or } s = m\omega^2$$

$$\therefore \qquad P.E. = \frac{1}{2} m\omega^2 y^2$$

when $\quad K.E. = P.E.$

$$\frac{1}{2} m\omega^2 (a^2 - y^2) = \frac{1}{2} m\omega^2 y^2$$

or $\quad a^2 - y^2 = y^2 \quad$ or $\quad a^2 = 2y^2$

$\therefore$ Displacement $\quad y = \pm \frac{a}{\sqrt{2}}$

In other words, for a displacement $y = \pm \frac{a}{\sqrt{2}}$ from the mean position $P.E. = \frac{1}{4} m\omega^2 a^2$, $K.E. = \frac{1}{4} m a^2 \omega^2$ and total energy $= \frac{1}{2} m\omega^2 a^2$

i.e., half of the total energy is kinetic and half potential.

Example 5.9 A hollow sphere is filled with water, used as pendulum bob. If water trickles out slowly through a hole made at the bottom, how will the time period be effected?

Solution. The time period of a simple pendulum is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

where l is the length of the pendulum and g the acceleration due to gravity.

When the water trickles out slowly through a hole made at the bottom of the hollow sphere used as bob of the pendulum, the mass of the bob goes on slowly decreasing. As time period does not depend upon the mass of the bob, there is no change in time period.

Example 5.10 A uniform spring of force constant *S* is cut into two pieces whose lengths are in the ratio of 1:3. Calculate the force constant of each piece.

Solution. Here force = F

Increase in length = l

$$\therefore \qquad s = \frac{F}{l}$$

When the springs are cut in the ratio 1:3, Increase in length for the first piece for force will be $\frac{l}{4}$, and increase in length for the second piece will be $\frac{3l}{4}$.

$$\therefore \qquad s_1 = \frac{F}{l/4} = 4\left[\frac{F}{l}\right]$$

$$\text{or,} \qquad s_1 = 4s \qquad \ldots (i)$$

$$\text{and} \qquad s_2 = \frac{F}{3l/4} = \frac{4}{3}\left[\frac{F}{l}\right]$$

$$\therefore \qquad s_2 = \frac{4}{3}s \qquad \ldots (ii)$$

Example 5.11 A lift is ascending at acceleration of 3m/s^2. What is the period of oscillation of a simple pendulum of length one metre suspended in the lift?

Solution. The time period of a simple pendulum is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

where l is the length of a simple pendulum and g is acceleration due to gravity.

As the lift is ascending with an acceleration of 3ms^{-2}, acceleration due to gravity $g' = 9.8 + 3 = 12.8$ ms^{-2}

Given : length of pendulum, l = 1m.

$\therefore$ Time period of the simple pendulum suspended in the lift

$$T' = 2\pi\sqrt{\frac{l}{g'}}$$

$$= 2\pi\sqrt{\frac{1}{12.8}}$$

$$= 1.756 \text{ sec}$$

Example 5.12 A spring of force constant 1200 N/m is mounted on a horizontal table. A mass of 3 kg is attached to the free end of the spring which is pulled sideways to a distance of 2 cm and released. What is (*a*) the frequency of oscillation of the mass, and (*b*) the max. acceleration of the mass. (*Nagpur U.* s/2007)

Solution. (*a*) Period, $$T = 2\pi\sqrt{\frac{m}{s}} = 2\pi\sqrt{\frac{3}{1200}} = 2\pi\sqrt{\frac{1}{400}}$$

$\therefore$ Frequency, $$n = \frac{1}{T} = \frac{1}{2\pi}\sqrt{400} = \frac{20}{2\times 3.14} = 3.185 \text{ Hz.}$$

(*b*) Acceleration, $$\frac{d^2x}{dt^2} = -\frac{c}{m}x$$

$$= -\frac{1200}{3}\times 0.02 \qquad (\because x = 0.02 \text{ m})$$

$$= -8 \text{ m/s}$$

Example 5.13 The particle performing S.H.M. has a mass 2.5 gm and frequency of vibration 10 Hz. It is oscillating with an amplitude of 2 cm. Calculate the total energy of the particle. (*Nagpur U.* 2009, *s*/2007)

Solution. Total energy, $$E_T = \frac{1}{2} m\,\omega^2 a^2.$$

$$\omega = 2\pi n = 2\pi \times 10 = 20\,\pi\, s^{-1}$$

$$a = 2 \text{ cm} = 0.02 \text{ m}$$

$$m = 2.5 \text{ gm} = 2.5 \times 10^{-3} \text{ kg}$$

$\therefore$ $$E_T = \frac{1}{2}\times(20\pi)^2\times(2.5\times10^{-3})^2$$

$$= \frac{1}{2}\times 400(3.14)^2\times 6.25\times10^{-6}$$

$$= 200 \times 9.86 \times 6.25 \times 10^{-6}$$

$$= 12324.5 \times 10^{-6}$$

$$= 12.3245 \times 10^{-3} \text{ J}$$

Example 5.14 The amplitude of a simple harmonic oscillator is doubled. How does this effect the time period, total energy and maximum velocity of the oscillator.

(*P.U.*, 2003; *Pbi.*, *U.* 2000)

Solution. When the displacement y of a simple harmonic oscillator is given by $y = A\sin(\omega t + \phi)$

velocity $v = \dfrac{dy}{dt} = A\omega\cos(\omega t + \phi)$

and acceleration $a = \dfrac{d^2y}{dt^2} = -A\omega^2\sin(\omega t + \phi) = -\omega^2 y$

Now time period $T = 2\pi \sqrt{\dfrac{\text{Displacement}}{\text{Acceleration}}} = 2\pi \sqrt{\dfrac{y}{\omega^2 y}} = \dfrac{2\pi}{\omega}$ (ignoring the negative sign)

As $\dfrac{2\pi}{\omega}$ is a constant, the time period does not depend upon amplitude but remains constant .

Maximum velocity $v_{max} = A\,\omega$ when $\cos(\omega t + \phi) = 1$

When amplitude A is doubled, the maximum velocity is also doubled.

Total energy $= \dfrac{1}{2} m A^2 \omega^2$

When the amplitude A is doubled, the total energy becomes four times, because energy $\propto A^2$.

Example 5.15 Show that for a particle executing S.H.M. the average value of kinetic and potential energy is the same and each is equal to half the total energy.

(*Nagpur U.* 2009; *Pbi.U.*, 2003; *Luck. U.* 2002)

Solution. Average kinetic energy. Kinetic energy at any instant

$$= \frac{1}{2} m v^2 = \frac{1}{2} m \dot{y}^2 = \frac{1}{2} m \left(\frac{dy}{dt}\right)^2$$

For a S.H.M. $\quad y = a \sin(\omega t + \phi)$

and $\quad \dfrac{dy}{dt} = a\,\omega \cos(\omega t + \phi)$

$\therefore$ Instantaneous $\quad K.E. = \dfrac{1}{2} m a^2 \omega^2 \cos^2(\omega t + \phi)$

If T is the time period, then

Average $K.E.$

$$= \langle K.E. \rangle = \frac{1}{T} \int_0^T \frac{1}{2} m a^2 \omega^2 \cos^2(\omega t + \phi)\, dt$$

$$= \frac{m a^2 \omega^2}{2T} \int_0^T \cos^2(\omega t + \phi)\, dt$$

$$= \frac{m a^2 \omega^2}{2T} \int_0^T \frac{1}{2}[1 + \cos 2(\omega t + \phi)]\, dt$$

$$= \frac{m a^2 \omega^2}{4T} \left[\int_0^T dt + \int_0^T \cos 2(\omega t + \phi)\, dt\right]$$

$$= \frac{m a^2 \omega^2}{4T} T \qquad \left[\because \int_0^T \cos(2\omega t + \phi)\, dt = 0\right]$$

$$= \frac{1}{4} m a^2 \omega^2 \qquad ...(i)$$

Average potential energy. Instantaneous potential energy is given by

$$P.E. = \frac{1}{2} s y^2 = \frac{1}{2} s a^2 \sin^2(\omega t + \phi)$$

$$\therefore \text{Average } P.E. = \langle P.E. \rangle = \frac{1}{T} \int_0^T \frac{1}{2} s a^2 \sin^2(\omega t + \phi)]\, dt$$

$$= \frac{s a^2}{2T} \int_0^T \frac{1}{2}[1 - \cos 2(\omega t + \phi)]\, dt$$

$$= \frac{sa^2}{4T}\left[\int_0^T dt - \int_0^T \cos 2(\omega t + \phi)\, dt\right]$$

$$= \frac{sa^2}{4T} T = \frac{1}{4} sa^2 \qquad \left[\because \int_0^T \cos 2\,(\omega t + \phi)\, dt = 0\right]$$

But $\omega^2 = \dfrac{s}{m}$ $\quad \therefore\ s = m\omega^2$

$\therefore$ Average $\quad P.E. = \dfrac{1}{4} m a^2 \omega^2$ ...(*ii*)

Total Energy. The total energy $= \dfrac{1}{2} m a^2 \omega^2$ ...(*iii*)

Thus it is clear from Equations (*i*), (*ii*) and (*iii*) that the average kinetic energy of a harmonic oscillator is equal to the average potential energy and is equal to *half* the total energy *i.e.*,

$$< K.E. > = < P.E. > = \frac{1}{2}\ E_{\text{Total}}$$

Example 5.16 A spring when compressed by 10 cm develops a restoring force of 10 N. A body of mass 4 kg is attached to it. Calculate the compression of the spring due to the weight of the body and calculate the period of oscillation. (*Pbi. U.*, 2001)

Solution. Here restoring force F = 10 N; displacement y = 10 cm = 0.1 m

$\therefore$ Force constant $\quad s = \dfrac{F}{y} = \dfrac{10}{0.1} = 100 \text{ Nm}^{-1}$

Mass attached m = 4 kg

$\therefore$ Force applied F = 4 kg ωt = 4 × 9.8 = 38.2 N

$\therefore$ Displacement $\quad y' = \dfrac{F'}{s} = \dfrac{39.2}{100} \text{m} = 39.2 \times 10^{-2} \text{ m} = 39.2 \text{ cm}$

Time period $\quad T = 2\pi \sqrt{\dfrac{m}{s}} = 2\pi \sqrt{\dfrac{4}{100}} = \dfrac{2}{5}\pi = 1.26$ sec.

Example 5.17 The displacement of a simple harmonic oscillator is given by

$$x = a \sin(\omega\, t + \phi)$$

If the oscillations started at time $t = 0$ from a position x_0 with velocity $\dot{x} = v_0$, show that

$$\tan\phi = \frac{\omega x_0}{v_0} \text{ and } a = \left[x_0^2 + \frac{v_0^2}{\omega^2}\right]^{\frac{1}{2}}.$$

Solution. Given $\quad x = a \sin(\omega\, t + \phi)$; at $t = 0$, $x = x_0$

$\therefore \quad x_0 = a \sin\phi$

Also $\quad \dot{x} = a\,\omega \cos(\omega t + \phi)$; at $t = 0$, $\dot{x} = v_0$

$\therefore \quad v_0 = a\,\omega \cos\phi$

Hence $\quad \tan\phi = \dfrac{\sin\phi}{\cos\phi} = \dfrac{x_0}{a} \times \dfrac{a\omega}{v_0} = \dfrac{\omega x_0}{v_0}$

and $\quad a^2 \sin^2\phi + a^2 \cos^2\phi = x_0^2 + \dfrac{v_0^2}{\omega^2}$

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or $$a = \left[x_0^2 + \frac{v_0^2}{\omega^2} \right]^{\frac{1}{2}}$$

Example 5.18. A man stands on a platform which vibrates simple harmonically in a vertical direction at a frequency of 5 Hertz. Show that the mass loses contact with the platform when the displacement exceeds 10^{-2} metres.

Solution. The mass loses contact with the platform when the upward force acting on it exceeds its weight *mg*.

The mass vibrates simple harmonically given by

$$x = a \sin(\omega t + \phi)$$

$\therefore$ Acceleration $\ddot{x} = a\,\omega^2 \sin(\omega t + \phi) = -\,\omega^2 x$

Upward force $= -\,m\,\omega^2 x$

In the limiting case $mg = m\omega^2 x$

or $$x = \frac{g}{\omega^2} = \frac{9.81}{4\pi^2 5^2} = .01\,\text{m} = 10^{-2}\ \text{metre}$$

Example 5.19 A body executing S. H. M. has velocities 80 cm/s and 60 cm/s when displacements are 3 cm and 4 cm respectively. Calculate the amplitude of vibration and the time taken to travel 2.5 cm from positive extremity of the oscillation.

Solution. Velocity $v = \omega\sqrt{a^2 - y^2}$

where ω = angular velocity; a = amplitude and y = displacement from the mean position

$\therefore$ $$80\ \text{cm}\,\text{s}^{-1} = \omega\sqrt{a^2 - 3^2} \qquad ...(i)$$

$$60\ \text{cm}\,\text{s}^{-1} = \omega\sqrt{a^2 - 4^2} \qquad ...(ii)$$

$\therefore$ $$\frac{80}{60} = \frac{4}{3} = \frac{\sqrt{a^2 - 3^2}}{\sqrt{a^2 - 4^2}} \quad \text{or} \quad \frac{16}{9} = \frac{a^2 - 9}{a^2 - 16}$$

or $a = 5$ cm

Substituting $a = 5$ cm in (*i*) we get

$$80 = \omega\sqrt{5^2 - 3^2}$$

or $\omega = 20$ rad. s^{-1}

Now $y = a \sin \omega t$

$\therefore$ Time taken to reach the positive extremity is given by

$$5 = 5 \sin \omega t$$

or $\sin \omega t = 1$

or $\omega t = \pi / 2$

$\therefore$ $$t = \frac{\pi}{2\omega} = \frac{\pi}{40} s$$

Distance of the point 2.5 cm from positive extremity, from the mean position = 5 – 2.5 = 2.5 cm

Time taken to reach a point 2.5 cm from the mean position is given by

$$2.5 = 5 \sin \omega t$$

or $$\sin \omega t = \frac{1}{2}$$

or $$\omega t = \pi / 6$$

$\therefore$ $$t = \frac{\pi}{6\omega} = \frac{\pi}{120} \text{ sec}$$

Time taken to travel from a point 2.5 cm from the positive extremity to the positive extremity

$$= \frac{\pi}{40} - \frac{\pi}{120} = \frac{\pi}{60} = .052 \text{ sec}$$

Hence time taken to travel from positive extremity to a point 2.5 cm away = .052 sec.

EXERCISE CH.5

LONG QUESTIONS

1. Derive a general differential equation of motion of a simple harmonic oscillator and obtain its various solutions. (*Nagpur U.* 2006)

2. (***a***) Find an expression for the velocity of a simple harmonic oscillator. (*Nagpur U.* 2008, 2003; *Bang. U.*, 2000)

(***b***) Velocity of simple harmonic oscillator at any time t leads the displacement by a phase angle $\frac{\pi}{2}$ radian. Explain why? (*Nagpur U.* 2003)

3. (***a***) Find an expression for the acceleration of a simple harmonic oscillator. (*Nagpur U.* 2008)

(***b***) Show that for the body executing simple harmonic motion the acceleration leads the velocity by $\frac{\pi}{2}$ and displacement by π.

4. Show the phase relation between the displacement velocity and acceleration diagrammatically, given displacement $y = A \sin (\omega t + \phi)$.

5. (***a***) Find an expression for the periodic time and frequency of a simple harmonic oscillator.

(***b***) A mass m suspended from a spring of stiffness s executes S.H.M. Set up the differential equations of motion and calculate its time period. If we use a stiffer spring without changing the mass how will the period of oscillation change? (*Gauhati U.*, 2000)

6. (***a***) Derive an expression for the total energy of a harmonic oscillator and show that the mechanical energy remains conserved. (*Meerut U.*, 2002; *Gauhati U.* 2007; *Nagpur U.* 2006, 2007; *Purvanchal U.* 2005)

(***b***) When the displacement is one half of the maximum amplitude, what fraction of the total energy is kinetic and what fraction is potential in simple harmonic motion?

7. What is a compound pendulum? How does it differs from simple pendulum. Derive an expression for its time period. What is the condition for the time period to be minimum?

(*Nagpur U.* 2009, 2005, 2006; *P.U.*, 2004, 2003)

8. Show that an oscillating loaded spring exhibits S.H.M. Find its periodic time. (*Nagpur U.* 2006)

9. Show that oscilations of torsion pendulum are S.H.M. Find its period. What are the merits of torsion pendulum. (*Nagpur U.* 2009, s/2006, s/2007)

10. What is Helmholtz resonator? Show that the oscillations are simple harmonic. Find its frequency. (*Nagpur U.* 2007, 2008; *Meerut* 2005)
11. Discuss the *LC* circuit and show that the charge on capacitor oscillates simple harmonically. Find the frequency of oscillation. Can it be realised in practice? (*Agra U.* 2007, 2004)
12. Deduce the expression for period of a vibrating magnet in a uniform magnetic field.
13. What is meant by a two-body harmonic oscillator? Obtain an expression for its time period. (*Nagpur U.* 2006, 2007, 2008)
14. Show that bifilar oscillations are S.H.M. Derive the expression for periodic time of a bifilar oscillator with parallel threads. (*Nagpur U.* s/2003, 2007, 2008)
15. Derive the differential equation of S.H.M. for an electrical circuit. Can it be realised in practice?
16. Explain simple harmonic motion and discuss its characteristics.
17. Define simple harmonic motion. Show that a uniform circular motion is equivalent to two simple harmonic motion at right angles to each other.
18. Show that the time period of oscillations of a loaded spring is

$$T = 2\pi\sqrt{\frac{Mx}{mg}}$$

19. What is a compound pendulum? Derive expression for its time period. Explain the term : length of an equivalent simple pendulum. (*Purvanchal U.* 2004)
20. Derive the equation for simple harmonic motion from energy considerations.

SHORT QUESTIONS

1. Differentiate between simple harmonic motion and oscillatory motion. Define Simple harmonic motion. Discuss potential well. (*Nagpur U.* 2009; *H.P.U.*, 2001; *Bang. U.*, 2000; *Gauhati U.*, 2000)
2. Explain how interaction of inertia and elasticity account for simple harmonic motion. (*P.U.*, 2002)
3. Enumerate the characteristics of simple harmonic motion. (*Purvanchal U.* 2005)
4. Give some examples of simple harmonic oscillators.
5. Derive an expression for time period of oscillations of a loaded spring. (*Nagpur U.* 2009)
6. Show that a simple pendulum executes S.H.M. Find its period.
7. What are the dimensions of force constant of vibrating spring? (*Pbi. U.* 2002, *H.P.U.* 2000)
8. Discuss the points of similarities between mechanical and electrical oscillations.
9. What oscillates in a simple harmonic electrical oscillator? Can we realise in practice?
10. Name the periodic motion which is not oscillatory.
11. Are all periodic motions simple harmonic? Is the reverse true? Explain. (*P.U.* 2004; *Pbi. U.* 2003)
12. Explain the terms: restoring force and force constant.
13. A hollow sphere is filled with water and used as a pendulum bob. If water is allowed to flow slowly through a hole at the bottom, how will the period change?

14. Describe bifilar oscillations and bifilar suspension. (*Nag. U.*, 2009, 2008, 2007)

15. Show that the total energy of a simple harmonic oscillator is directly proportional to the square of its amplitude. (*D.A.U. Agra*, 2008)

16. Express the kinetic and potential energies as a function of its total energy of a harmonic oscillator when its displacement is half the amplitude. (*Agra U.* 2005, 2003)

NUMERICAL QUESTIONS

1. A time $t = 0$, *a* train of waves has the form

$$y = 4\sin 2\pi\left(\frac{x}{100}\right)$$

The velocity of the wave is 30 cm/s. Find the equation given the wave form at time $t =$ 2 sec.

2. A particle vibrates simple harmonically with an amplitude of 13 cm. The period of oscillation is 2π seconds. Calculate the velocity of the vibrating particle at the instant the displacement is 5 cm. Also calculate the frequency of oscillation.

[**Ans.** (*i*) 12 cm/s (*ii*) $\frac{1}{2\pi}$ Hertz]

3. A spring is hung vertically and loaded with a mass of 400 gms made to oscillate. Calculate (*i*) the time period, and (*ii*) the frequency of oscillation. When the spring is loaded with 100 gms it extends by 5 cm. [**Ans.** (*i*) 0.898 s (*ii*) 1.11 Hz]

4. The scale of a spring balance reading 0 – 5 kg is 12.5 cm. A body suspended from the balance oscillates with a frequency $5/\pi$ Hertz. Calculate the mass of the body attached to the spring. [**Ans.** $m = 3.92$ kg]

5. A particle executing S.H.M. along a straight line has a velocity 4 cms^{-1} at a distance 3 cm from the mean position and 3 cms^{-1} at a distance of 4 cm from it. Find the time it takes to travel 2.5 cm from positive extremity of the oscillation. [**Ans.** 2.19 cm]

6. Two masses of 1.0 kg and 3.0 kg respectively are connected by a spring of force constant 250 N/m and are placed on a smooth table. They are slightly pulled apart and released. What is the frequency of the two-body system? [**Ans.** 2.90 Hz]

7. A particle is making simple harmonic motion along x-axis. If at a distance x_1 and x_2, the velocities of particle are v_1 and v_2. Calculate its time period. (*Agra* 2003)

8. The equation of a transverse wave is $y = x_0 \cos 2\pi\,(vt - x/\lambda)$ if $\lambda = \pi x_0$, then prove that the maximum velocity of the particle will be twice of the velocity of wave. (*Agra* 2003)

9. A spherical Helmholtz resonator of capacity 10 litres has a narrow neck of length 0.01 m and radius 0.01 m. If the velocity of sound in air is 340 m/sec, calculate the frequency of the note to which it sharply responds. (*Nagpur U.* 2008) [**Ans.** 95.92 Hz]

10. A uniform circular disc of radius 10 cm oscillates in a vertical plane about a horizontal axis. Find the distance of the axis of rotation from the centre for which the period is minimum. Also find minimum time period. (*Nagpur Uni.* 2009)

[**Ans.** $l = 7.07$ m, 0.75 sec]

[**Hint:** $I = \text{Mk}^2 = \frac{1}{2}\text{MR}^2;\ k = \text{R}/\sqrt{2};\ l = k$

$$T_{\min} = 2\pi\sqrt{\frac{2\text{R}/\sqrt{2}}{g}} = 2\pi\sqrt{\frac{1.414\text{R}}{g}}\ ]$$

6

LISSAJOUS' FIGURES

INTRODUCTION

In the previous chapter, we have discussed simple harmonic oscillators and estimation of their period and frequency. In this chapter, we will study the resultant of two S.H.M. s acting on a particle in the same direction as well as in the perpendicular directions. Former gives the phenomenon of beats while the later the phenomenon of Lissajous' figures. Both these phenomena are useful in physics in many applications. Lissajous' figures produced by two S.H.M.s may be mechanical or alternating voltages acting at right angles to each other.

6.1 RESULTANT OF TWO S.H.M.S IN THE SAME DIRECTION (SAME PERIOD BUT DIFFERING IN PHASE)

Consider two simple harmonic waves of same period (T) *i.e.* same wavelength λ. Let a and b be the amplitudes of the two waves and phase difference ϕ. The displacement of a particle at any instant t due to each wave is given by

$$y_1 = a \sin \frac{2\pi}{\lambda}(vt - x) \qquad \text{... (1)}$$

$$y_2 = b \sin \left[\frac{2\pi}{\lambda}(vt - x) + \phi\right] \qquad \text{... (2)}$$

Where v is the velocity of propagation of each wave.

The resultant displacement of the particle due to the superposition of the two waves,

$$\begin{aligned} y &= y_1 + y_2 \\ &= a \sin \frac{2\pi}{\lambda}(vt - x) + b \sin\left[\frac{2\pi}{\lambda}(vt - x) + \phi\right] \\ &= a \sin \frac{2\pi}{\lambda}(vt - x) + b\left[\sin \frac{2\pi}{\lambda}(vt - x)\cos\phi + \cos\frac{2\pi}{\lambda}(vt - x)\sin\phi\right] \\ &= \left[\sin\frac{2\pi}{\lambda}(vt - x)\right](a + b\cos\phi) + \left[\cos\frac{2\pi}{\lambda}(vt - x)\right](b\sin\phi) \qquad \text{... (3)} \end{aligned}$$

Let $a + b\cos\phi = A\cos\theta$ ... (4)

$b\sin\phi = A\sin\theta$ ... (5)

Dividing (5) by (4), we have

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$$\tan\theta = \frac{b\sin\phi}{a+b\cos\phi} \qquad \text{... (6)}$$

Squaring and adding (4) and (5), we have

$$A = \sqrt{a^2 + b^2\cos^2\phi + 2ab\cos\phi + b^2\sin^2\phi}$$

$$= \sqrt{a^2 + b^2 + 2ab\cos\phi} \qquad \text{... (7)}$$

The resultant displacement y from eq. (3), is given by

$$y = (A\cos\theta)\sin\frac{2\pi}{\lambda}(vt-x) + A\sin\theta\cdot\cos\frac{2\pi}{\lambda}(vt-x)$$

$$= A\sin\left[\frac{2\pi}{\lambda}(vt-x)+\phi\right] \qquad \text{... (8)}$$

Hence, the resultant wave has the same time period (*i.e.* same wavelength and frequency) but has different amplitude and phase.

Case 1: Constructive Interference.

When $\phi = 0, 2\pi, \ldots\ldots\, n(2\pi)$ etc.

$\tan\theta = 0$ and $A = a + b$

i.e. the resultant vibration is in phase with the component vibrations.

Hence, if the phase difference between the two waves is zero ($\phi = 0$), they reinforce each other.

If $a = b$, then $A = a + b = 2a$

Intensity of sound $I = A^2 = 4a^2$

Therefore, the intensity of sound is maximum at those points, where the two waves differ in phase by $n(2\pi)$ or the path differences is $n\lambda$ (here $n = 0, 1, 2, 3$ ..... etc) This gives constructive interference.

Case 2: Destructive Interference.

When $\phi = \pi, 3\pi, 5\pi, \ldots\ldots\ldots\ldots (2n+1)\pi$ etc.

$$A = a - b$$

If $a = b$, then $A = 0$

And Intensity of sound $I = A^2 = 0$

Therefore, the intensity sound is minimum (nearly zero) at those points, where the two waves differ in phase by $(2n + 1)\pi$ or the path difference $(2n + 1)\lambda/2$ (here $n = 0, 1, 2$, ... etc.)

This gives destructive interference.

6.2 CONDITIONS FOR INTERFERENCE OF SOUND

The following conditions are necessary for interference of sound waves:

(1) The two sound sources must be *coherent*. It means the phase difference between the two sources must remain constant so that the phase difference between the two wave trains at any point does not change with time.

(2) The two sources must emit waves of the same frequency and amplitude.

(3) The two wave trains must travel in the same direction. *i.e.* parallel to each other.

Condition (1) is necessary so that the positions of maximum and minimum intensity of sound are distinct. If the difference between a and b is large, the minimum intensity positions are not distinct.

Condition (2) is necessary to have sustained maxima and minima. If the phase difference between the two waves continously changes, the maximum and minimum intensity positions are not fixed.

Condition (3) is necessary because with increase in obliquity between the waves, the resultant intensity decreases.

6.3 BEATS

When two sounding bodies of nearly the same frequency and amplitude are sounded together, the resultant sound consists of alternate maxima and minima. *This phenomenon in which waxing and vaning of sound at regular intervals is heard, is called beats.*

The number of beats heard per second is equal to the difference in frequency between the two sounding bodies. However, beats, are heard only when the difference in frequency is not more than ten.

Analytical treatment of beats:

Consider two wave trains of slightly different frequencies of values n and m. The displacement of a particle due to these waves is given by

$$Y_1 = a \sin 2\pi\, nt$$

and $$Y_2 = a \sin 2\pi\, mt$$

When these two waves superimpose the resultant displacement (Y) is given by

$$\begin{aligned} Y &= Y_1 + Y_2 = a \sin 2\pi\, nt + a \sin 2\pi\, mt \\ &= 2a \cos 2\pi \left(\frac{n-m}{2}\right) t \ \sin 2\pi \left(\frac{n+m}{2}\right) t\ . \\ &= A \sin 2\pi \left(\frac{n+m}{2}\right) t \end{aligned}$$

This equation represents a periodic vibration of amplitude.

Case 1 : The amplitude will be maximum, $A = \pm 2a$

when $\cos 2\pi \left(\frac{n-m}{2}\right) t = \pm 1$

Or $\pi (n-m)t = k\pi$ where $k = 0, 1, 2, 3$ ...... etc.

$\therefore \quad t = \dfrac{k}{n-m} = 0, \dfrac{1}{n-m}, \dfrac{2}{n-m}, \dfrac{3}{n-m}$, ... and so on.

$\therefore$ Time interval between two consecutive maxima $= \dfrac{1}{n-m}$

Frequency of maxima $= (n - m)$

Case 2 : The amplitude will be minimum, $A = 0$

when $\cos 2\pi \left(\frac{n-m}{2}\right) t = 0$

Or $\pi (n-m)\, t = k\pi + \dfrac{\pi}{2}$ where $k = 0, 1, 2, 3$, ... etc.

$$\therefore \quad t = \frac{k}{n-m} + \frac{1}{2(n-m)}$$

$$= \frac{1}{2(n-m)}, \frac{3}{2(n-m)}, \frac{5}{2(n-m)}, \ldots\ldots\ldots$$

and so on

Time interval between two consecutive minima $= \dfrac{1}{(n-m)}$

$\therefore$ Frequency of minima $= (n - m)$

Since one maximum and one minimum of sound constitutes a beat, the number of beats = $n - m$, per second.

Thus, the number of beats produced per second is equal to the difference in the frequencies of the two sounding bodies.

6.4 LISSAJOUS' FIGURES

When a particle is acted upon simultaneously by two simple harmonic motions (S.H.M.s) at right angles to each other, the resultant path traced out by the particle is called *Lissajous' figures*. The shape of the resultant path *i.e.* figure depends on the time period, phase difference and the amplitude of the two S.H.M.s. The two S.H.M.s may be mechanical (using two tuning forks) or alternating voltages (using two electrical a.c. sources) acting simultaneously at right angles to each other.

6.5 RESULTANT OF TWO S.H.M.S AT RIGHT ANGLES (SAME PERIOD BUT DIFFERING IN AMPLITUDE AND PHASE)

Let the two simple harmonic motions of equal frequencies (equal period and wavelength) be along the X and Y axes. If their displacements at any time t, be x and y, and are represented by

$$x = a \sin \omega t \quad \ldots (1)$$

and

$$y = b \sin (\omega t + \phi) \quad \ldots (2)$$

From eq. (1), $\sin \omega t = \dfrac{x}{a}$ $\quad \therefore \quad \cos \omega t = \sqrt{1 - \dfrac{x^2}{a^2}}$

From eq. (2),

$$\frac{y}{b} = \sin (\omega t + \phi)$$

$$= \sin \omega t \cdot \cos \phi + \cos \omega t \sin \phi$$

Substituting the values of $\sin \omega t$ and $\cos \omega t$, we have

$$\frac{y}{b} = \frac{x}{a} \cos \phi + \sqrt{\left(1 - \frac{x^2}{a^2}\right)} \sin \phi$$

$$\left(\frac{y}{b} - \frac{x}{a} \cos \phi\right) = \sqrt{\left(1 - \frac{x^2}{a^2}\right)} \cdot \sin \phi$$

Squaring both sides, we get,

$$\left(\frac{y}{b} - \frac{x}{a} \cos\phi\right)^2 = \left(1 - \frac{x^2}{a^2}\right) \cdot \sin^2 \phi$$

$$\frac{y^2}{b^2} - \frac{2xy}{ab} \cos \phi + \frac{x^2}{a^2} \cos^2 \phi + \frac{x^2}{a^2} \sin^2 \phi - \sin^2 \phi = 0$$

$$\frac{y^2}{b^2} - \frac{2xy}{ab}\cos\phi + \frac{x^2}{a^2}(\cos^2\phi + \sin^2\phi) = \sin^2\phi$$

$$\boxed{\frac{y^2}{b^2} - \frac{2xy}{ab}\cos\phi + \frac{x^2}{a^2} = \sin^2\phi} \quad \text{... (3)}$$

This represents the resultant motion of the particle, which is in general an ellipse inclined to the axes of co-ordinates.

Important cases :

Case (*i*) When $\phi = 0$, then $\sin\phi = 0^c$ and $\cos\phi = 1$.

Substituting in equation (3), we get

$$\frac{y^2}{b^2} - \frac{2xy}{ab}\cos\phi + \frac{x^2}{a^2} = 0$$

$$\therefore \quad \frac{y^2}{b^2} - \frac{2xy}{ab} + \frac{x^2}{a^2} = 0$$

$$\therefore \quad \left(\frac{y}{b} - \frac{x}{a}\right)^2 = 0$$

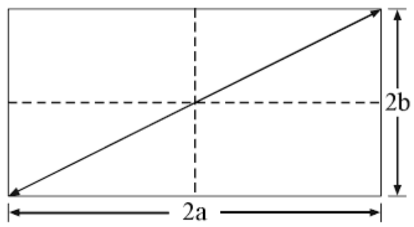

Fig. 6.1

Thus, the resultant motion is a pair of coincident straight lines lying in quadrants I and III of rectangles:as shown in fig (6.1). The straight lines are inclined to the x-axis at an angle θ given by

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Case (*ii*): When $\phi = \frac{\pi^c}{4}$, equation (3) reduces to

$$\frac{y^2}{b^2} - \frac{2xy}{ab}\cdot\frac{1}{\sqrt{2}} + \frac{x^2}{a^2} = \left(\frac{1}{\sqrt{2}}\right)^2$$

$$\frac{y^2}{b^2} - \frac{\sqrt{2}xy}{ab} + \frac{x^2}{a^2} = \frac{1}{2}$$

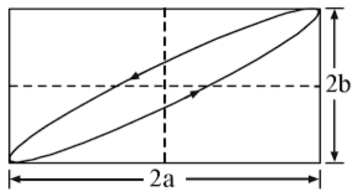

Fig. 6.2

This represents an oblique ellipse as shown in fig (6.2)

Case (*iii*): When $\phi = \frac{\pi^c}{2}$,

Substituting in eq. (3), we get

$$\frac{y^2}{b^2} - \frac{2xy}{ab}\cos 90° + \frac{x^2}{a^2} = \sin^2 90°$$

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1$$

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

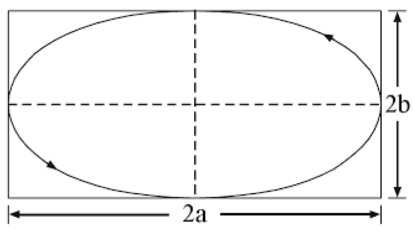

Fig. 6.3

This represents an ellipse whose major and minor axes coincide with the co-ordinate axes: as shown in fig (6.3)

Special Case : If the amplitudes are equal *i.e.* $a = b$, then

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$

$$x^2 + y^2 = a^2$$

This represents a circle radius a.

Hence the resultant motion is circular as shown in fig (6.4)

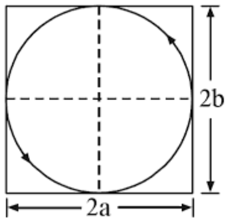

Fig. 6.4

Case (*iv*): When $\phi = \frac{3\pi^c}{4}$, the equation (3) reduces to

$$\frac{y^2}{b^2} - \frac{2xy}{ab}\cos 135° + \frac{x^2}{a^2} = \sin^2 135°$$

$$\frac{y^2}{b^2} - \frac{2xy}{ab}\left(\frac{-1}{\sqrt{2}}\right) + \frac{x^2}{a^2} = \left(\frac{1}{\sqrt{2}}\right)^2$$

$$\therefore \qquad \frac{y^2}{b^2} + \frac{\sqrt{2}xy}{ab} + \frac{x^2}{a^2} = \frac{1}{2}$$

This equation represents an oblique ellipse lying in quadrants II and IV as shown in fig (6.5)

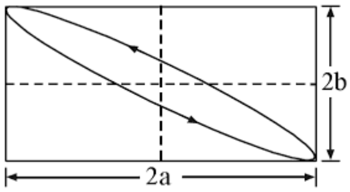

Fig. 6.5

Case (*v*): When $\phi = \pi^C$, Eq. (3) reduces to

$$\frac{y^2}{b^2} - \frac{2xy}{ab}\cos 180° + \frac{x^2}{a^2} = \sin^2 180°$$

$$\therefore \qquad \frac{y^2}{b^2} + \frac{2xy}{ab} + \frac{x^2}{a^2} = 0$$

$$\therefore \qquad \left(\frac{y}{b} + \frac{x}{a}\right)^2 = 0$$

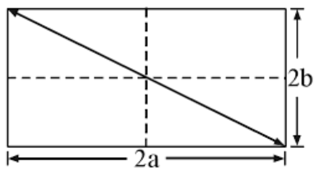

Fig. 6.6

This equation represents a pair of coincident straight lines lying in quadrants II and IV, as shown in fig (6.6)

After this, the whole cycle is repeated in the reverse order as shown below (fig 6.7).

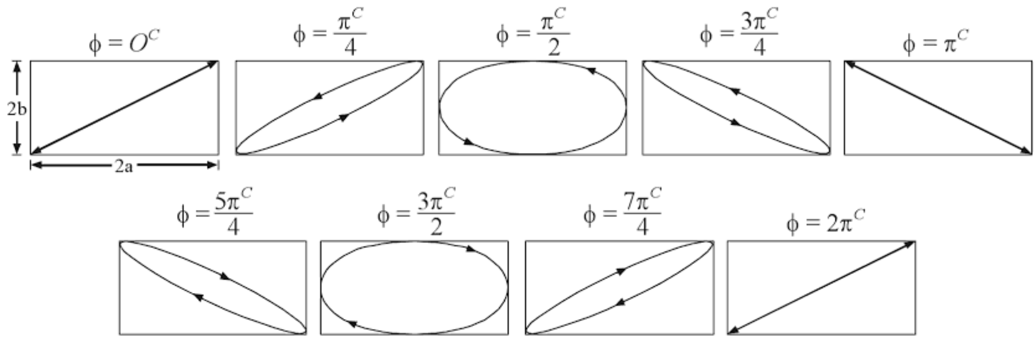

Fig. 6.7

These figures are called as Lissajous' figures

6.6 RESULTANT OF TWO S.H.M.S AT RIGHT ANGLES WITH FREQUENCY RATIO 1:2

Let us consider two S.H.M.s perpendicular to each other having different amplitudes and phase and frequencies in the ratio 1:2.

Let the first SHM be along X-axis and the second along the Y-axis. Since their frequencies are in the ratio 1:2, their angular velocities ($\omega = 2\pi n$) will be in the ratio 1:2.

The displacements due to the two S.H.M.s at any time t can be represented as

$$x = a \sin \omega t \quad \text{... (1)}$$

$$y = b \sin (2\omega t + \phi) \quad \text{... (2)}$$

From eq. (1),

$$\sin \omega t = \frac{x}{a} \quad \text{and} \quad \cos \omega t = \sqrt{\left(1 - \frac{x^2}{a^2}\right)}$$

From eq. (2),

$$\begin{aligned} \frac{y}{b} &= \sin (2\,\omega t + \phi) \\ &= \sin 2\,\omega t \cos \phi + \cos 2\,\omega t \cdot \sin \phi \\ &= 2 \sin \omega t \cdot \cos \omega t \cdot \cos \phi + (1 - 2 \sin^2 \omega t) \sin \phi \end{aligned}$$

Substituting the values of $\sin \omega t$ and $\cos \omega t$, we have

$$\frac{y}{b} = 2\frac{x}{a}\sqrt{\left(1 - \frac{x^2}{a^2}\right)} \cdot \cos \phi + \sin \phi - 2\frac{x^2}{a^2} \sin \phi$$

$$\left(\frac{y}{b} - \sin \phi\right) + \frac{2x^2}{a^2} \sin \phi = \frac{2x}{a}\sqrt{\left(1 - \frac{x^2}{a^2}\right)} \cdot \cos \phi$$

Squaring both sides, we get

$$\begin{aligned} \left(\frac{y}{b} - \sin \phi\right)^2 + \frac{4x^4}{a^4}\sin^2 \phi + \frac{4x^2}{a^2} \sin \phi \left(\frac{y}{b} - \sin \phi\right) &= \frac{4x^2}{a^2}\left(1 - \frac{x^2}{a^2}\right) \cos^2 \phi \\ &= \frac{4x^2}{a^2}\cos^2 \phi - \frac{4x^4}{a^4} \cdot \cos^2 \phi \end{aligned}$$

$$\begin{aligned} \left(\frac{y}{b} - \sin \phi\right)^2 + \frac{4x^2 y}{a^2 b}\sin \phi - \frac{4x^2}{a^2}\sin^2 \phi + \frac{4x^4}{a^4}\sin^2 \phi &= \frac{4x^2}{a^2}\cos^2 \phi - \frac{4x^4}{a^4}\cos^2 \phi \end{aligned}$$

$$\left(\frac{y}{b} - \sin \phi\right)^2 + \frac{4x^2 y}{a^2 b}\sin \phi - \frac{4x^2}{a^2}(\sin^2 \phi + \cos^2 \phi) + \frac{4x^4}{a^4}(\sin^2 \phi + \cos^2 \phi) = 0$$

$$\left(\frac{y}{b} - \sin \phi\right)^2 + \frac{4x^2 y}{a^2 b}\sin \phi - \frac{4x^2}{a^2} + \frac{4x^4}{a^4} = 0 \quad \text{... (3)}$$

Equation (3) gives the resultant motion of the particle.

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Some important cases :

Case 1: When $\phi = 0^C$, Substituting in eq. (3), we get

$$\left(\frac{y}{b} - \sin 0^C\right)^2 + \frac{4x^2y}{a^2b}\sin 0^C - \frac{4x^2}{a^2} + \frac{4x^4}{a^4} = 0$$

$$\frac{y^2}{b^2} - \frac{4x^2}{a^2} + \frac{4x^4}{a^4} = 0$$

$$\frac{y^2}{b^2} + \frac{4x^2}{a^2}\left(\frac{x^2}{a^2} - 1\right) = 0$$

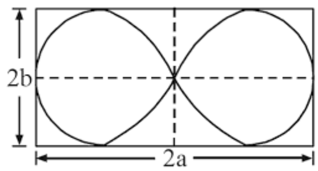

Fig. 6.8

The resultant motion is shown by fig (6.8)

Case 2: When $\theta = \left(\frac{\pi}{2}\right)^C$, Substituting in equation (3), we get

$$\left(\frac{y}{b} - \sin\frac{\pi}{2}\right)^2 + \frac{4x^2y}{a^2b}\sin\frac{\pi}{2} - \frac{4x^2}{a^2} + \frac{4x^4}{a^4} = 0$$

$$\left(\frac{y}{b} - 1\right)^2 + \frac{4x^2y}{a^2b} - \frac{4x^2}{a^2} + \frac{4x^4}{a^4} = 0$$

$$\left(\frac{y}{b} - 1\right)^2 + \frac{4x^2}{a^2}\left(\frac{y}{b} - 1\right) + \frac{4x^4}{a^4} = 0$$

$$\left[\left(\frac{y}{b} - 1\right) + \frac{2x^2}{a^2}\right]^2 = 0$$

$$\left(\frac{y}{b} - 1\right) + \frac{2x^2}{a^2} = 0$$

$$\frac{y}{b} = 1 - \frac{2x^2}{a^2}$$

$$\frac{y}{b} = \frac{a^2 - 2x^2}{a^2}$$

$$\therefore \quad y = -\frac{b}{a^2}(2x^2 - a^2)$$

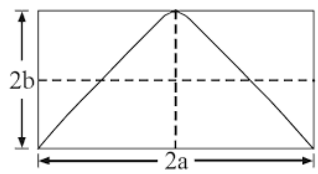

Fig. 6.9

This equation represents a parabola as shown in fig. (6.9). The Lissajous' figures obtained under the action of two S.H.M.s having different amplitudes, phases and frequencies in the ratio 1:2 are as shown in fig 6.10

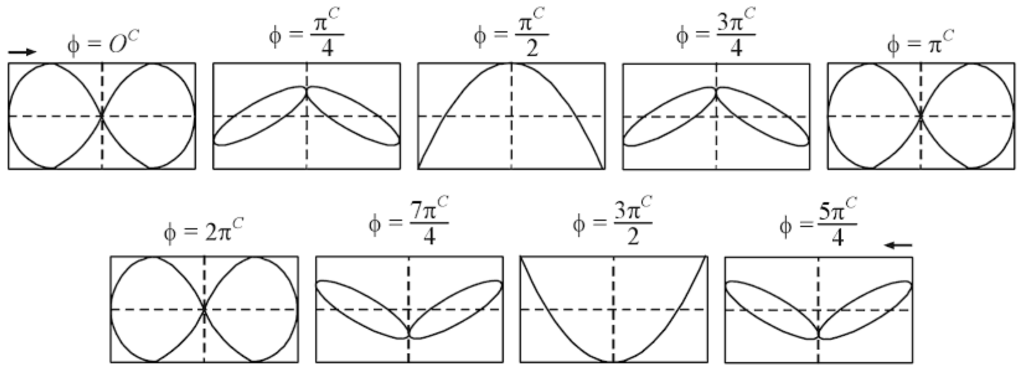

Fig. 6.10

6.7 GRAPHICAL METHOD FOR TRACING LISSAJOUS' FIGURES

The nature of the resultant path or the curve traced out depends upon

(*i*) the amplitudes,

(*ii*) the periods (or frequencies), and

(*iii*) the phase difference between the two component vibrations.

(*i*) Amplitudes different, periods same and phase difference $\frac{\pi}{4}$ (45°) . Let the simple harmonic vibration along the *X*-axis have an amplitude *a* and the vibration along the *Y*-axis an amplitude *b*.

Draw two circles of reference of radii *a* and *b* equal to the amplitude *b*.

Draw two circles of reference of radii *a* and *b* equal to the amplitudes of the corresponding simple harmonic motions taking place along the *X*-axis and the *Y*-axis respectively. In the circle of radius *a* draw a diameter *XX′* parallel to the *X*-axis and in the circle of radius *b* draw a diameter *YY′* parallel to the *Y*-axis. As the *periods* are *equal* divide both the circles into an equal number of parts, say 8, so that each part is travelled in the *same* time. Draw lines through these points perpendicular to the lines *XX′* and *YY′* respectively, so as to enclose a rectangle *PQRS* as shown in Fig. 6.11. It is supposed that the particle O_1 vibrating along *XX′* (*X*-vibration) first begins its journey towards the right and the particle O_2 vibrating along *YY′* (*Y*-vibration) first begins its journey in the *upward direction*.

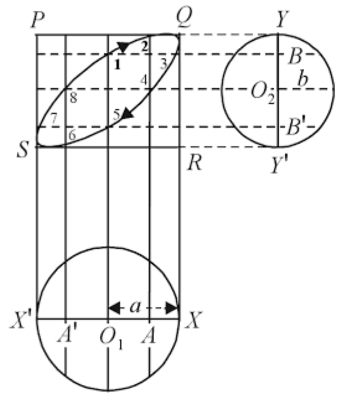

Fig. 6.11

Let the *Y*-vibration be ahead of the *X*-vibration by an angle $\pi/4$. It means that when the particle O_1 starts from its mean position along the *X*-axis, the particle O_2 has already completed 1/8th of its vibration along the *Y*-axis, *i.e.*, it is at *B*. The resultant position of a particle when both the vibrations act on it is represented by the point marked 1.

When the particle O_1 reaches *A* the particle O_2 reaches *Y* and the resultant position is represented by the point marked 2. Proceeding in this manner the positions as indicated by the points 3, 4....8 are marked. Joining all these points by a free hand curve an oblique ellipse as shown in fig. 6.11 is obtained.

(*ii*) Amplitudes different, periods in the ratio of 1 : 2 and phase difference zero. Let the simple harmonic vibration along the *X*-axis have an amplitude *a* and the vibration along the

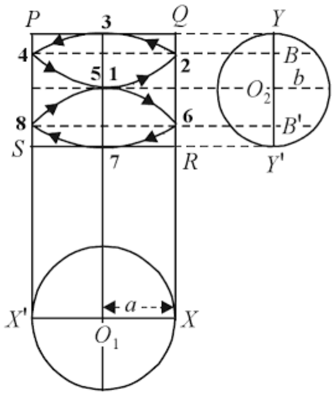

Fig. 6.12

Y-axis an amplitude b. The period of the Y-vibration is taken to be *double* that of the X-vibration.

Draw two circles of reference of radii a and b equal to the amplitudes of the corresponding simple harmonic motions taking place along the X-axis and Y-axis respectively. In the circle of radius a draw a line XX' parallel to the X-axis and in the circle of radius b draw a line YY' parallel to the Y-axis. As the periods are in the ratio 2 : 1 divide the circle YY' into 8 equal parts and the circle XX' into 4 equal parts (*i.e.*, in the ratio of their corresponding periods) so that each part is travelled in the same time. Draw lines through these points perpendicular to the lines XX' and YY' respectively so as to enclose a rectangle $PQRS$ as shown in Fig. 6.12.

Since the two vibrations are in phase the particle O_1 and O_2 start simultaneously from their mean positions and the resultant position of a particle when both the vibrations act on it is represented by the point 1. When the particle O_1 is at X the particle O_2 is at B and the resultant position is represented by the point 2. Similarly when the particle O_1 is in position O_1, X' and again at O_1 the particle O_2 is in the position Y, B and O_2 respectively. The resultant position is represented by the points 3, 4 and 5 respectively. The particle O_1 now begins its second vibration and when, it is at X, O_1 and X', the particle O_2 is at B', Y' and B'. The resultant position is represented by the points 6, 7 and 8 respectively. Joining all these points by a free hand curve the figure as shown in Fig. 6.12 is obtained.

***(iii)* Amplitude different, periods in the ratio of 1 : 2 and phase difference a quarter of the smaller period.** Draw two circles of reference of radii a and b and divide the circle YY' into *eight* equal parts and the circle XX' into *four* equal parts, *i.e.*, in the ratio of their corresponding periods as explained in (*ii*).

Since the X-vibration (of shorter period) is ahead of the Y-vibration by a quarter period, the particle O_1 will be in the position X when the particle O_2 starts from its mean position. Therefore, the resultant position of a particle when both the vibrations act on it is represented by the point 1. When the particle O_1 comes back to the mean position, the particle O_2 is at B and the resultant position is represented by the point 2. Similarly when the particle O_1 is in position X', O_1 and back again at X, the particle O_2 is in the position Y, B and back again to O_2 respectively. The resultant position is represented by the points 3, 4 and 5 respectively. The particle O_1 now begins its second vibration and when it is at O_1, X, O_1 and X', the particle O_2 is at B', Y', B' and O_2 respectively. The resultant is represented by the points 6, 7, 8 and 1 respectively. Joining all these points by a free hand curve, the figure of a parabola symmetrical about the X-axis as shown in Fig. 6.13 is obtained.

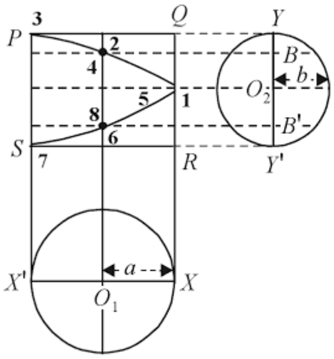

Fig. 6.13

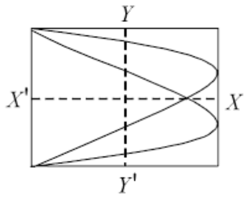

Fig. 6.14

If however, amplitudes are different, periods in the ratio 1 : 2 and the phase, difference is *one eight* of the smaller period, then proceeding exactly as in (*iii*), we get the curve as shown in Fig. 6.14

6.8 EXPERIMENTAL ARRANGEMENT TO DEMONSTRATE FORMATION OF LISSAJOUS FIGURES BY OPTICAL METHOD

The experimental arrangement is shown in Fig. 6.15. A and B are two tuning forks arranged such that the vibrations of A are in a vertical plane and those of B in a horizontal plane. M_1 and M_2 are small plane mirrors attached to their prongs. A narrow beam of light from a point sources S is made convergent by means of a convex lens L. The beam is reflected in turn from M_1 and M_2 and finally is received on a screen.

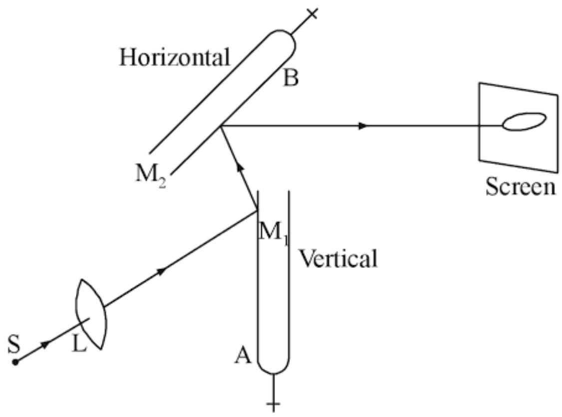

Fig. 6.15

When A alone vibrates a vertical straight line is obtained on the screen and when B alone vibrates a horizontal straight line is obtained. But when both A and B are made of vibrate simultaneously a resultant curve is obtained on the screen. If the frequencies are exactly equal the resultant curve will be a straight line or a circle or an ellipse. The shape of the curve depends on:

(*i*) the amplitudes of vibrations.

(*ii*) the ratio of the frequencies of the vibrations.

(*iii*) the phase difference between the two vibrations.

With the help of Lissajous' figures we can compare the frequencies of two tuning forks and if the frequency of one of them is known the frequency of the other can be determined.

Let us consider two tuning forks A and B. Suppose the frequency of A is exactly *known* to be n_1 and the frequency n_2 of B is *unknown*. Suppose n_2 is nearly equal to n_1. The two tuning forks are arranged to produce Lissajous figures. The S.H.Ms. due to the tuning forks may be represented by.

$$x = a \sin 2\pi n_1 t \qquad \text{... (1)}$$

$$y = b \sin 2\pi n_2 t = \text{b} \sin [2\pi n_1 t + 2\pi n_2 t - 2\pi n_1 t]$$

$$= b \sin [2\pi n_1 t + 2\pi(n_2 - n_1)t] \qquad \text{... (2)}$$

when $n_2 > n_1$,

or,
$$y = b \sin [2\pi n_1 t - 2\pi(n_1 - n_2)t] \qquad \text{... (3)}$$

when $n_1 > n_2$

If the figures go through a cycle of changes after a definite time interval t, then

$$2\pi (n_2 - n_1)t = \pm 2\pi$$

or
$$(n_2 - n_1)t = \pm 1 \qquad \text{... (4)}$$

Now we consider the following two cases:

(1) If $n_2 > n_1$, then $(n_2 - n_1)t = +1$

or $n_2 = n_1 + \frac{1}{t}$ ... (5)

(2) If $n_2 < n_1$, then $(n_1 - n_2)t = 1$

or $n_2 = n_1 - \frac{1}{t}$ ... (6)

Now to find whether $n_2 > n_1$ or $n_2 < n_1$, a small quantity of wax is applied to one of the prongs of the tuning, fork B and the new time interval t for a cycle of changes of Lissajous figures is determined. On applying wax to B its frequency decreases. Suppose its frequency decreases to n'_2.

If $n_2 > n_1$ the difference $(n'_2 - n_1)$ will be *less* then $(n_2 - n_1)$, *i.e.* $(n'_2 - n_1) < (n_2 - n_1)$. Hence in this case $t' > t$.

If $n_2 < n_1$ the difference $(n_1 - n'_2)$ will be greater then $(n_1 - n_2)$, *i.e.* $(n_1 - n'_2) > (n_1 - n_2)$. Hence in this case $t' < t$.

Thus on applying a small quantity of wax to the fork of unknown frequency n_2

(*i*) if we find that $t' > t$, then $n_2 > n_1$ and therefore n_2 is given by

$$(n_2 - n_1)\ t = 1$$

or
$$n_2 = n_1 + \frac{1}{t}$$

(*ii*) if we find that $t' < t$, then $n_2 < n_1$ and therefore n_2 is given by

$$(n_1 - n_2)\ t = 1$$

or
$$n_2 = n_1 - \frac{1}{t}$$

6.9 FORMATION OF LISSAJOUS' FIGURE BY USING C.R.O

A very convenient method for obtaining Lissajous' figures is by using a cathode ray oscilloscope (C.R.O).

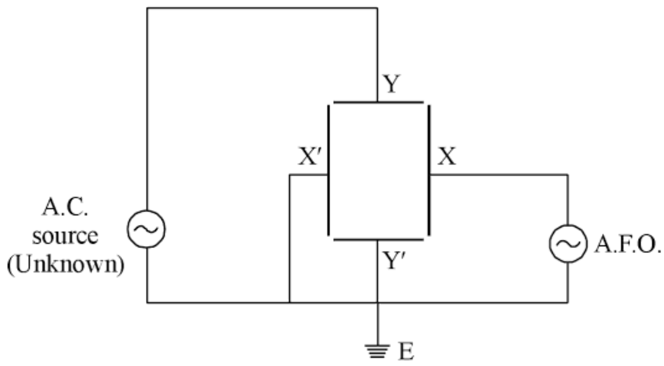

Fig. 6.16

An electrical experiment is shown in fig. 6.16 to obtain Lissajous' figures. A source of a.c. voltage of unknown frequency is connected to the vertical input terminals $Y - Y'$ of the CRO. Audio frequency oscillator (A.F.O.) of known frequency is connected to the horizontal input terminals $X - X'$. By adjusting the phase and frequency of unknown source, a steady Lissajous' figure, a

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straight line, an ellipse, a circle, a figure of eight etc. is obtained on the screen of CRO. This electrical arrangement is used to determine unknown frequency of the A.C. source.

The unknown frequency f of any. a.c. source can be determined by using the formula

$$f = \frac{N_h}{N_v} \cdot f_0$$

where f_0 is known frequency of another a.c. source (say A.F.O.); N_h is number of horizontal cuts and N_v is number of vertical cuts. Suppose the Lissajous' figure is figure of eight (8), this has $N_h = 1$, $N_v = 2$ and $f_0 = 100$ Hz, then unknown frequency, $f = \frac{1}{2} \times 100 = 50$ Hz.

SOLVED EXAMPLES

Example 6.1 Two tuning forks produce 4 beats per second when sounded together. One of them is in unison with 1.20 m length of wire and the other with 1.25 m of it. Calculate the frequencies of the forks.

Solution. Let n_1 and n_2 be the frequencies of the forks, then

$$n_1 - n_2 = 4 \qquad \ldots (i)$$

If the string is stretched by a tension T and has mass per unit length m, then

$$n_1 = \frac{1}{2 \times 1 \cdot 20}\sqrt{\frac{T}{m}} \quad \text{and} \quad n_2 = \frac{1}{2 \times 1 \cdot 25}\sqrt{\frac{T}{m}}$$

$$\therefore \qquad \frac{n_1}{n_2} = \frac{1 \cdot 25}{1 \cdot 20} = \frac{25}{24}$$

$$\therefore \qquad n_1 = n_2 \times \frac{25}{24}$$

Substituting in eq. (i), we get

$$n_2 \times \frac{25}{24} - n_2 = 4$$

$$\therefore \qquad n_2 = 96 \text{ Hz}$$

$$\text{and} \qquad n_1 = 96 + 4 = 100 \text{ Hz}$$

Example 6.2 Calculate the velocity of sound in a gas in which the waves of wavelength 50 cm and 50.5 cm produce 6 beats per second.

Solution. Let v be the velocity of sound in the gas.

$$\lambda_1 = 50 \text{ cm}, \quad \text{and} \quad \lambda_2 = 50.5 \text{ cm}$$

$$= 0.5 \text{ m}, \qquad = 0.505 \text{ m}$$

$$\therefore \qquad n_1 = \frac{v}{\lambda_1} = \frac{v}{0.500}, \qquad n_2 = \frac{v}{0.505}$$

$$\therefore \qquad n_1 - n_2 = 6, \qquad \frac{v}{0.500} - \frac{v}{0.505} = 6$$

$$(0.505 - 0.500)\, v = 6 \times 0.500 \times 0.505$$

$$v = \frac{6 \times 0.500 \times 0.505}{0.005}$$

$$v = 303 \text{ ms}^{-1}$$

Example 6.3 Lissajous' figures are produced with two forks whose frequencies are approximately in the ratio in the ratio 1 : 1. The figures take 15 seconds to go through a cycle of change. On loading slightly one tuning fork with wax, the figures take 10 seconds for the cycle of changes. If the frequency of the other tuning fork is 256 Hz, find the frequency of the first tuning fork before and after loading.

Solution. Given $n_1 = 256$ Hz, $t = 15$ sec., $t' = 10$ sec.

Let n_2 be the unknown frequency before loading and n'_2 the frequency after loading.

(*i*) *Before loading:* Since $t' < t$ the diff. of n_1 and n'_2 is greater than that of n_1 and n_2. And since after loading a tuning fork its frequency decreases *i.e.* $n'_2 < n_2$

$$\therefore \quad n_2 < n_1$$

Hence n_2 before loading is given by

$$(n_1 - n_2)\, t = 1$$

or

$$n_2 = n_1 - \frac{1}{t}$$

$$= 256 - \frac{1}{15} = 256 - 0.067$$

$$= 255.933 \text{ Hz.}$$

(*ii*) *After loading :* $n_2 = n_1 - \dfrac{1}{t'} = 256 - \dfrac{1}{10} = 255.9$ Hz

Example 6.4 Lissajous' figures are produced with two tuning forks whose frequencies are approximately in the ratio 2 : 1. It takes 6 seconds to go through a cycle of changes. On loading slightly, the fork of the higher pitch with wax, the period of cycle is raised to 10 seconds. If the frequency of the lower fork is 150, what is the frequency of the other fork before and after loading?

Solution. Given $\dfrac{n_1}{n_2} \approx \dfrac{2}{1}$, $\quad n_2 = 150$ Hz.

$t = 6$ sec., $\quad t' = 10$ sec.

Let n_1 be frequency of the tuning fork of higher pitch before loading and n'_1 the frequency after loading.

(*i*) *Frequency before loading :* Since $t' > t$ the difference n'_1 and $2n_2$ is *less* than that of n_1 and $2n_2$. And since after loading tuning fork its frequency decreases *i.e.* $n'_1 < n_1$

$$\therefore \quad n_1 > 2n_2$$

Hence n_1 before loading is given by $(n_1 - 2n_2)\, t = 1$

or

$$n_1 = 2n_2 + \frac{1}{t}$$

$$= 2 \times 150 + \frac{1}{6}$$

$$= 300.16 \text{ Hz.}$$

(*ii*) *Frequency after loading :*

$$n'_1 = 2n_2 + \frac{1}{t'}$$

$$= 2 \times 150 + \frac{1}{10}$$

$$= 300.1 \text{ Hz.}$$

EXERCISE CH. 6

LONG QUESTIONS

1. Obtain the resultant of two simple harmonic motions of the same period executing in the same direction but differing in phase and amplitude. (*Nag. U. S*-2003; *Agra U.* 2005)

2. What are beats? Show that the number of beats produced is equal to the difference in the frequencies of the two sounding bodies.

3. Calculate the resultant of two simple harmonic vibrations of the same frequency acting along the same line but differing in phase. What is the amplitude when the phase difference is $\frac{\pi}{2}$? (*Bhopal U.* 2004)

4. A particle is subjected simultaneously to two S.H.M. of the same period but of different amplitudes and phases in perpendicular directions. Find the expression for the resultant motion. For what condition the path may be a straight line, ellipse or circle? Discuss the different important cases. (*Calicut U.* 2003; *Merrut U.*, 2003, 2002, 2000; *Nagpur U.*, 2003, 2001; *C.U.*, 2002)

5. Calculate the resultant of two simple harmonic vibrations at right angles when their periods are in the ratio of 2 : 1 and there is a phase difference 0 or $\pi/2$.

(*Nagpur U.* 2009; *Meerut U.*, 2001; *Agra U.* 2004)

6. (*i*) What are Lissajous figures?

(*ii*) Obtain an expression for the resultant of two S.H.M.s perpendicular to each other having different amplitudes and phase and frequencies in the ratio 1 : 2

(*D.A.U. Agra*, 2008; *Nagpur U.* 2009)

7. (*a*) Show that the resultant of two simple harmonic motions at right angles to each other and having equal periods and amplitudes but phase difference 90° is a circle.

(*Agra U.* 2007; *Nagpur U.*, 2003, 2002)

(*b*) Two mutually perpendicular S.H.M.s are represented by equations $x = 4 \sin \omega t$ and $y = 3 \cos \omega t$. Find the semi-major and semi-minor axis of an ellipse formed by their superposition. (*Nagpur U.*, 2001)

8. What are Lissajous figures? How will you trace graphically the Lissajous figures? When (*i*) the periods are equal and the phase difference is $\frac{\pi}{4}$ and (*ii*) the periods are in the ratio of 2 : 1 and phase difference is (*a*) zero and (*b*) a quarter of the smaller period.

(*Purvanchal U.* 2006, 2004; *Meerut* 2005)

9. Describe the experimental arrangement to obtain Lissajous' figures, using electrical experiment. (*Nagpur U.* 2005)

10. (*a*) What are uses of Lissajous' figures?

(*b*) Describe a method to demonstrate Lissajous' figures?

(*c*) Obtain the equation for the resultant motion of two S.H.M.s at right angles to each other with phase difference of (*i*) zero and (*ii*) $3\pi/4$ radian. (*Nagpur U.* 2003)

11. (***i***) A particle is subjected simultaneously to two S.H.M.s at right angles to each other having equal period but different amplitude and phase difference ϕ. Find the resultant motion.

(***ii***) Under what conditions will the path be (*a*) an ellipse, and (*b*) a straight line?

(***iii***) Show that the resultant of two S.H.M.s at right angles to each other, having equal period and equal amplitude, but phase difference of 90° is a circle. (*Nagpur U.* 2007)

SHORT QUESTIONS

1. Discuss the necessary conditions for interference of sound waves.

2. Use of Lissajous' figures to determine the unknown Frequency of a Tuning Fork. (*Purvanchal U.* 2006)

3. What are beats? (*Nagpur U.* 2009)

4. What are Lissajous' figures? Mention their uses. (*Nagpur U.* 2007)

5. Explain, how Lissajous' figures are useful to determine the unknown frequency of tuning fork. (*Nagpur U.* 2008)

6. Describe various methods for demonstrating the formation of Lissajous' figures. What are the uses of Lissajous' figures? (*Purvanchal U.* 2006)

NUMERICAL QUESTIONS

1. Two tuning forks A and B of frequencies approximately in the ratio 1:1, produce Lissajous' figures. The figures take 5 sec. to go through a complete cycle of changes. On loading slightly the tuning fork A with wax the figure takes 10 sec. to go through the cycle. If the frequency of A is 320 Hz, calculate the frequency of B.

[**Hint:**

On loading A, $t' > t \;\therefore\; n_A > n_B \;\therefore\; n_A = n_B + \frac{1}{t} \;\therefore\; n_B = n_A - \frac{1}{t} = 320 - \frac{1}{5} = 319.8$ Hz]

2. Two tuning forks A and B are employed to produce Lissajous' figures. It is found that the figure symmetrical ellipse recures with the same direction of rotation after an interval of 10 sec. On slightly loading B with wax, the same figure recures after an interval of 16 sec. If the frequency of A is 320 Hz, find the frequency of B before and after loading.

[**Ans.** 320.1 Hz, 320.0625 Hz]

3. Two tuning forks produce Lissajous' figures. The figures change from a parabola to figure of eight and again to a parabola, the whole cycle occupying 6 seconds. If the frequency of one is 100, find the possible frequencies of the other. $\left[\textbf{Ans. } 50 \pm \frac{1}{12}, 200 \pm \frac{1}{6}\right]$

4. In an electrical experiment on Lissajous figures, if the frequency of known a.c. source is 200 Hz and the Lissajous figure obtained has a shape . Find the frequency of Unknown source.

[**Hint.** Here, $N_h = 3$, $N_v = 1$, $f_0 = 200$ Hz

$\therefore$ Unknown frequency, $f = \frac{N_h}{N_v} \times f_0 = \frac{3}{1} \times 200 = 600$ Hz.]

5. The Lissajous figure in the case of two tuning forks is a figure having three loops. Find the ratio of their frequencies. (*Meerut U.* 2005) [**Ans.** 3 : 1]

7

DAMPED, FORCED HARMONIC OSCILLATOR

INTRODUCTION

In an ideal simple harmonic motion (S.H.M.), the displacement follows a sinusoidal curve. The amplitude of oscillations remain constant *for an infinite time*. This is because there is no loss of energy and thus total energy remains constant. Such oscillations are called **"Free oscillations."** However, in actual practice, the simple harmonic motion always experiences frictional or resistive forces due to which energy of free oscillator is continuously lost and consequently the amplitude of vibration decreases gradually and ultimately the body comes to rest. Hence, decay of amplitude with time is called 'damping'. Such oscillations are called **"damped harmonic oscillations."**

In order to maintain the amplitude constant, an external periodic force is applied. These forced vibrations initially gains the frequency equal to its natural frequency and then after short time, the oscillator acquires the frequency of the impressed periodic force. In this chapter, we will discuss amplitude resonance, quality factor and energy considerations of forced harmonic oscillator.

7.1 DAMPING AND DAMPING FORCE

The amplitude of a vibrating string, a sounding tuning fork and an oscillating pendulum goes on gradually decreasing and ultimately these bodies stop vibrating, It is because some energy is inevitably lost due to resistive or viscous forces. For example, in the case of a simple pendulum, energy is lost due to friction at the supports and resistance of air. Thus, oscillations get *damped*. Those simple harmonic vibrations where amplitude decreases with the passage of time are called damped simple harmonic vibrations. The damping force is found to be proportional to the velocity of the vibrating body and can be proved as under:

Consider a body executing simple harmonic motion is given by

$$y = a \sin \omega t$$

and acted upon by a *small* damping force f.

As a general case suppose that $\vec{f}$ is a function of displacement, velocity and acceleration and is given by

$$\vec{f} = A + B\vec{y} + C\frac{d\vec{y}}{dt} + D\frac{d^2\vec{y}}{dt^2}$$

The work done by the damping force $\vec{f}$ in displacing the body through a small displacement $d\vec{y}$ is given by

$$dW = \vec{f} \,.\, d\vec{y} = \vec{f} \,.\, \frac{d\vec{y}}{dt}\, dt = \vec{f} \,.\, \vec{v}\, dt$$

If T is the time period of the vibrating body, then

$$\text{Work done per cycle } W = \int_0^T dW$$

$$= \int_0^T \left(A + B\vec{y} + C\,\frac{d\vec{y}}{dt} + D\,\frac{d^2\vec{y}}{dt^2} \right) \cdot \left(\frac{d\vec{y}}{dt} \right) dt$$

As $y = a \sin \omega t$; $\dfrac{dy}{dt} = a\omega \cos \omega t$ and $\dfrac{d^2 y}{dt^2} = -a\omega^2 \sin \omega t$

$$\therefore \quad W = \int_0^T A a\omega \cos \omega t\, dt + \int_0^T (B - D\omega^2) a^2 \omega \sin \omega t \cos \omega t\, dt$$

$$+ \int_0^T C a^2 \omega^2 \cos^2 \omega t\, dt$$

Now $\displaystyle\int_0^T A a\omega \cos \omega t\, dt = Aa[\sin \omega t]_0^T = 0$

$$\int_0^T (B - D\omega^2)\, a^2 \omega \sin \omega t \cos \omega t\, dt = \frac{(B - D\omega^2) a^2 \omega}{2} \int_0^T \sin 2\omega t\, dt$$

$$= \frac{(B - D\omega^2) a^2 \omega}{2} \left[-\frac{\cos 2\omega t}{2\omega} \right]_0^T = 0$$

and $\displaystyle\int_0^T C a^2 \omega^2 t^2 \cos^2 \omega t\, dt = C a^2 \omega^2 \int_0^T \frac{1 + \cos 2\omega t}{2}\, dt$

$$= \frac{1}{2} C a^2 \omega^2 \left[\int_0^T dt + \int_0^T \cos 2\omega t\, dt \right]$$

$$= \frac{1}{2} C a^2 \omega^2 T = \pi\, C a^2 \omega \qquad \left[\because \int_0^T \cos 2\omega t\, dt = 0 \text{ and } \int_0^T dt = T\right]$$

The damping force being a frictional force is a non-conservative force and work done against it over a full time period cannot be zero.

Hence work is done only *by* that term of the damping force which is *proportional to velocity* of the vibrating body, the work done due to all other terms being zero. In other words, *the damping force is independent of displacement as well as acceleration and depends only upon velocity*.

7.2 DIFFERENTIAL EQUATION OF MOTION OF A DAMPED HARMONIC OSCILLATOR.

A damped harmonic oscillator experiences two forces:

(*i*) The restoring force proportional to displacement y given by $-sy$ where s is a constant known as *stiffness constant* or *spring constant*. The negative sign shows that the direction of the restoring force is opposite to that of displacement, and

(*ii*) A retarding force proportional to velocity given by $-r\,\dfrac{dy}{dt}$ where r is another constant known as *damping constant*. The negative sign again shows that the retarding force also acts opposite to the direction of motion of the body.

If m is the mass of the vibrating particle and its acceleration $\frac{d^2y}{dt^2}$ when it has a displacement y and is moving with a velocity $\frac{dy}{dt}$, then

$$m\frac{d^2y}{dt^2} = -r\frac{dy}{dt} - sy$$

$$m\frac{d^2y}{dt^2} + r\frac{dy}{dt} + sy = 0 \qquad \text{... }(i)$$

Dividing by m, we get

$$\frac{d^2y}{dt^2} + \frac{r}{m}\frac{dy}{dt} + \frac{s}{m}y = 0$$

Putting $\frac{r}{m} = 2b$ and $\frac{s}{m} = \omega_0^2$

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega_0^2 y = 0 \qquad \text{... }(ii)$$

Here b is called the *damping co-efficient* and $2b$ gives the *force due to resistance of the medium per unit mass per unit velocity.*

Equation (*ii*) is known as differential equation of a damped simple harmonic oscillator.

7.3 SOLUTION OF DIFFERENTIAL EQUATION OF DAMPED S.H.M.

The differential equation representing a damped harmonic motion is given by

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega_0^2 y = 0 \qquad \text{... }(i)$$

where $2b = \frac{r}{m}$ and $\omega_0 = \sqrt{\frac{s}{m}}$

To solve the above differential equation, let us assume the solution

$$y = C\,e^{\lambda t} \qquad \text{... }(ii)$$

where C and λ are constants

$$\therefore \quad \frac{dy}{dt} = C\lambda\, e^{\lambda t} \quad \text{and} \quad \frac{d^2y}{dt^2} = C\lambda^2\, e^{\lambda t}$$

Substituting in Eq. (*i*), we get

$$C\lambda^2 e^{\lambda t} + 2bC\lambda\, e^{\lambda t} + \omega_0^2\, C\, e^{\lambda t} = 0$$

$$C\, e^{\lambda t}\,[\lambda^2 + 2b\lambda + \omega_0^2] = 0$$

$$\therefore \quad \lambda^2 + 2b\lambda + \omega_0^2 = 0 \qquad \text{... }(iii) \qquad (\because C\, e^{\lambda t} \neq 0)$$

This equation is quadratic in λ. Hence it has two roots.

$$\lambda = \frac{-2b \pm \sqrt{4b^2 - 4\omega_0^2}}{2} = -b \pm \sqrt{b^2 - \omega_0^2}$$

$$\therefore \quad \lambda_1 = -b + \sqrt{b^2 - \omega_0^2} \quad \text{and} \quad \lambda_2 = -b - \sqrt{b^2 - \omega_0^2}$$

Therefore, the general solution of equation (*i*) will be

$$y = A\, e^{\lambda_1 t} + B\, e^{\lambda_2 t}$$

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$$= A \cdot e^{\left(-b+\sqrt{b^2-\omega_0^2}\right)t} + B\, e^{\left(-b-\sqrt{b^2-\omega_0^2}\right)t}$$

$$= A \cdot e^{-bt}\left[A\, e^{\left(\sqrt{b^2-\omega_0^2}\right)t} + Be^{\left(-\sqrt{b^2-\omega_0^2}\right)t}\right]$$

$$y = e^{-bt}\,[\,A \cdot e^{\alpha t} + B\, e^{-\alpha t}] \qquad \text{... }(iv)$$

where A and B are constants and $\alpha = \sqrt{b^2 - \omega_0^2}$

To evaluate A and B:

The constants A and B are determined from the initial conditions. Suppose at $t = 0$, $y = y_0$ and velocity $= \dfrac{dy}{dt} = 0$

Substituting in Equation (iv), we get

$$y_0 = A + B \qquad \text{or} \qquad A + B = y_0 \qquad \text{... }(v)$$

Differenting Equation (iv), we get

$$\frac{dy}{dt} = -b\, e^{-bt}(A\, e^{\alpha t} + B\, e^{-\alpha t}) + e^{-bt}(A\,\alpha\, e^{\alpha t} - B\,\alpha\, e^{-\alpha t})$$

$$= -b\, e^{-bt}(A\, e^{\alpha t} + B\, e^{-\alpha t}) + e^{-bt}\cdot\alpha\,(A\, e^{\alpha t} - B\, e^{-\alpha t})$$

Substituting $t = 0$, and $\dfrac{dy}{dt} = 0$, we have

$$0 = -b\,(A + B) + \alpha\,(A - B)$$

$$\alpha\,(A - B) = b\,(A + B) = by_0$$

$$\therefore \qquad A - B = \frac{b}{\alpha}\, y_0 \qquad \text{... }(vi)$$

Adding and subtracting equations (v) and (vi), we get

$$A = \frac{y_0}{2}\left(1 + \frac{b}{\alpha}\right) \qquad \text{... }(vii)$$

$$B = \frac{y_0}{2}\left(1 - \frac{b}{\alpha}\right) \qquad \text{... }(viii)$$

Substituting these values in Equation (iv), we get

$$y = e^{-bt}\left[\frac{y_0}{2}\left\{\left(1 + \frac{b}{\alpha}\right)e^{\alpha t} + \left(1 - \frac{b}{\alpha}\right)e^{-\alpha t}\right\}\right]$$

$$= y_0 e^{-bt}\left[\frac{1}{2}\left\{e^{\alpha t} + \frac{b}{\alpha}\, e^{\alpha t} + e^{-\alpha t} - \frac{b}{\alpha}\, e^{-\alpha t}\right\}\right]$$

$$= y_0 \cdot e^{-bt}\left[\frac{1}{2}\left\{(e^{\alpha t} + e^{-\alpha t}) + \frac{b}{\alpha}\,(e^{\alpha t} - e^{-\alpha t})\right\}\right]$$

$$\therefore \qquad y = y_0 \cdot e^{-bt}\left[\left(\frac{e^{\alpha t} + e^{-\alpha t}}{2}\right) + \frac{b}{\alpha}\left(\frac{e^{\alpha t} - e^{-\alpha t}}{2}\right)\right] \qquad \text{... }(ix)$$

where $b = \dfrac{r}{2m}$, and $\alpha = \sqrt{b^2 - \omega_0^2} = \sqrt{\dfrac{r^2}{4m^2} - \dfrac{s}{m}}$

Equation (*ix*) is the general solution of a damped harmonic oscillator.

This equation gives the displacement (y) of a particle of mass m executing damped harmonic motion. The nature of motion depends upon the relative values of b and ω_0. The following three important cases arise:

Case 1 : Aperiodic or dead beat motion.

When $b^2 > \omega_0{}^2$. This is the case of *heavy damping*. In this case, $\sqrt{b^2 - \omega_0^2}$ is real, since $b^2 - \omega_0^2$ is positive. Equation (*ix*) shows that the displacement y decreases from its initial value y_0 and it tends to zero when $t \to \infty$. In this case no oscillations occur. Hence such a motion is called *aperiodic* or *overdamped*.

Under very heavy damping, the particle passes its equilibrium position at the most once before returning asymptotically to rest. This is met with a dead beat moving coil galvanometer or a pendulum vibrating in a viscous liquid like oil. The decrease of y with time t is shown by curve (a) in Fig. (7.1).

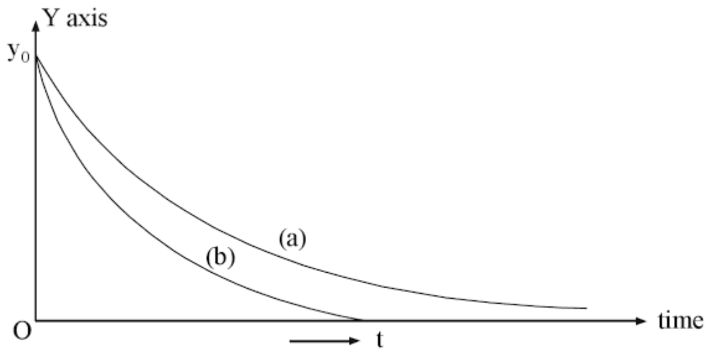

Fig. 7.1

Case 2: Critically damped motion.

When $b^2 = \omega_0^2$ or $\alpha = \sqrt{b^2 - \omega_0^2} = 0$

If we substitute this value in eq. (*ix*), the second term has the indeterminate form $\dfrac{o}{o}$. Thus, in this case, equation (*ix*) does not represent the solution *i.e.* solution breaks down. Let us, however, consider the case when $\sqrt{b^2 - \omega_0^2} \neq 0$ but very small quantity, say h. Therefore, $\sqrt{b^2 - \omega_0^2} = h$

Substituting in eq. (*ix*), we get

$$y = y_0 \cdot e^{-bt}\left[\left(\frac{e^{ht} + e^{-ht}}{2}\right) + \frac{b}{h}\left(\frac{e^{ht} - e^{-ht}}{2}\right)\right]$$

$$= y_0\, e^{-ht}\left[\cos h(ht) + \frac{b}{h}\sin h\,(ht)\right]$$

$$= y_0\, e^{-bt}\left[\left(1 + \frac{h^2t^2}{2\,!} + \frac{h^4t^4}{4\,!} + \ldots\right) + \frac{b}{h}\left(ht + \frac{h^3t^3}{3\,!} + \frac{h^5t^5}{5\,!} + \ldots\right)\right]$$

Now since h is very small, the terms containing h_2 and higher powers are negleted.

$$\therefore \qquad y = y_0 e^{-bt}(1 + bt) = y_0 (1 + bt) e^{-bt} \qquad \ldots (x)$$

This equation shows that at $t = 0$, $y = y_0$ and as t increases y decreases as shown by curve (b). The motion is neither overdamped nor oscillatory and is said to be critically damped. The motion is neither overdamped nor oscillatory and is said to be critically damped. In other words, it is a transition stage between dead beat motion and the damped oscillatory motion.

Case 3. Oscillatory damped S.H.M.

When $b^2 < \omega_0^2$ (light damping). In this case $b^2 - \omega_0^2 = \dfrac{r^2}{4m^2} - \dfrac{s}{m}$ is a negative quantity. Hence $\sqrt{b^2 - \omega_0^2}$ is an imaginary quantity.

Let $\sqrt{b^2 - \omega_0^2} = i\,\omega'$ or $\omega' = \sqrt{\omega_0^2 - b^2} = \sqrt{\dfrac{s}{m} - \dfrac{r^2}{4m^2}}$

Substituting, the general solution (ix) becomes

$$y = y_0 e^{-bt}\left[\left(\frac{e^{i\omega' t} + e^{-i\omega' t}}{2}\right) + \frac{b}{\omega'}\left(\frac{e^{i\omega' t} + e^{-i\omega' t}}{2i}\right)\right]$$

$$= y_0 e^{-bt}\left[\cos\omega' t + \frac{b}{\omega'}\sin\omega' t\right]$$

Now, let $\dfrac{b}{\omega'} = \cot\phi$

So that $\tan\phi = \dfrac{\omega'}{b} = \dfrac{\sqrt{\omega_0^2 - b^2}}{b}$

$\therefore \quad \sin\phi = \dfrac{\omega'}{\omega_0} = \dfrac{\sqrt{\omega_0^2 - b^2}}{\omega_0}$

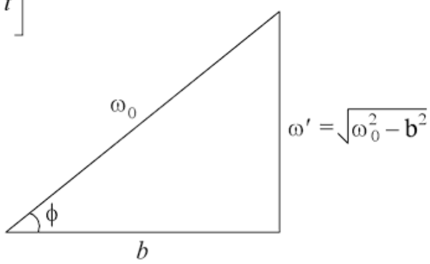

Fig. 7.2

Hence, from equation (xii), we get

$$y = y_0 e^{-bt}\left[\cos\omega' t + \frac{\cos\phi}{\sin\phi}\cdot\sin\omega' t\right]$$

$$= \frac{y_0 e^{-bt}}{\sin\phi}[\sin\phi\cos\omega' t + \cos\phi\cdot\sin\omega' t]$$

$$y = \frac{y_0\,\omega_0\,e^{-bt}}{\omega'}\cdot\sin(\omega' t + \phi)$$

$$= A_0 e^{-bt}\sin(\omega' t + \phi)$$

$$= A\sin(\omega' t + \phi) \qquad \ldots (xiii)$$

where, $\quad \omega_0 = \sqrt{\dfrac{s}{m}}$

$$b = \frac{r}{2m}$$

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$$\omega' = \sqrt{\omega_0^2 - b^2} = \sqrt{\frac{s}{m} - \frac{r^2}{4m^2}} \quad \text{and} \quad A_0 = \frac{y_0\,\omega_0}{\omega'}\;\frac{y_0\,\omega_0}{\sqrt{\omega_0^2 - b^2}}$$

From equation (*xiii*), we come to the following conclusions (1) The motion of the particle is oscillatory, the displacement y varies as a sine curve as shown in Fig. 7.3.

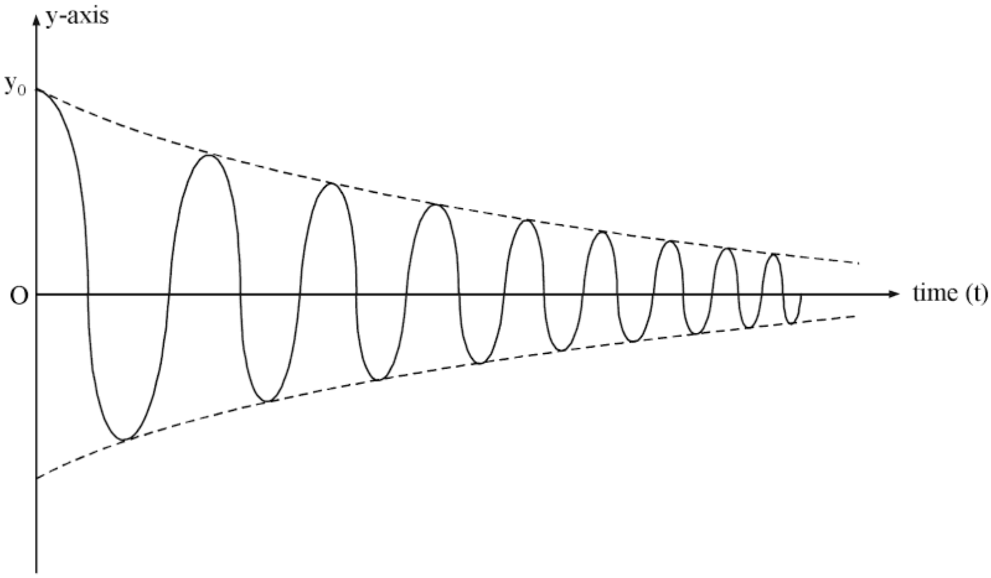

Fig. 7.3

(2) The amplitude of the S.H.M. is

$$A = \frac{y_0\omega_0}{\sqrt{\omega_0^2 - b^2}}\,e^{-bt} \qquad \ldots (xiv)$$

Thus amplitude A decreases exponentially with time (Fig. 7.3)

Therefore the motion is called damped oscillatory.

(3) The periodic time t and the frequency f of the damped S.H.M. are

$$T = \frac{2\pi}{\omega'} = \frac{2\pi}{\sqrt{\frac{s}{m} - \frac{r^2}{4m^2}}} \qquad \ldots (xv)$$

$$f = \frac{1}{2\pi}\sqrt{\frac{s}{m} - \frac{r^2}{4m^2}} \qquad \ldots (xvi)$$

The frequency of natural undamped oscillations ($r \rightarrow 0$) is

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{s}{m}} \qquad \ldots (xvii)$$

Thus, f is less than f_0. The difference depends upon the damping factor r. Larger the value of r, higher is the difference between f_0 and f.

7.4 DISCHARGE OF A CAPACITOR THROUGH INDUCTANCE AND RESISTANCE

Consider an electrical circuit containing a resistance R, an inductance L and a capacitance C as shown in Fig. 7.4 connected to a battery and a key K. When the key K is pressed downward the capacitor C is charged. Let q_0 be the charge on it. When the key K is released the capacitor circuit is completed through the inductance L and resistance R. The capacitor slowly loses its charge due to which there is a current I at any instant. The current varies at the rate $\frac{dI}{dt}$. If q is the charge on the capacitor at any instant t, then

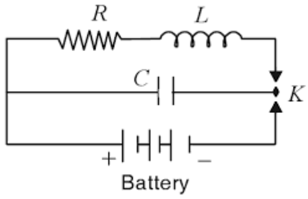

Fig. 7.4

Potential difference across the capacitor $C = \frac{q}{C}$

Potential difference across the resistor $R = RI$

Back $e.m.f$ in the inductor $L = -L\frac{dI}{dt}$

$$\therefore \qquad \frac{q}{C} + RI = -L\frac{dI}{dt} \quad \text{or} \quad \frac{q}{C} + RI + L\frac{dI}{dt} = 0$$

But as $\quad I = \frac{dq}{dt}$ and $\frac{dI}{dt} = \frac{d^2q}{dt^2} \qquad \therefore \; L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$

or
$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{I}{LC}q = 0 \qquad ...(i)$$

Put $\quad \frac{R}{L} = 2b$ and $\frac{1}{LC} = \omega_0^2$, then

$$\frac{d^2q}{dt^2} + 2b\frac{dq}{dt} + \omega_0^2 q = 0 \qquad ...(ii)$$

This equation is similar to equation (ii) of article 7.2.

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega_0^2 y = 0$$

where the displacement y has been replaced by charge q.

Proceeding as in article 7.3, we obtain a general solution of the above equation (ii) as

$$\therefore \qquad q = q_0 \cdot e^{-bt}\left[\frac{(e^{\alpha t} + e^{-\alpha t})}{2} + \frac{b}{\alpha}\cdot\frac{(e^{\alpha t} - e^{-\alpha t})}{2}\right] \qquad ...(iii)$$

Here, $\qquad b = \frac{R}{2L}$

and $\qquad \alpha = \sqrt{b^2 - \omega_0^2} = \sqrt{\left(\frac{R^2}{4L^2}\right) - \frac{1}{LC}}$

The following cases arises :

Case 1: Heavy damping. When $b^2 > \omega_0^2$ then $\sqrt{b^2 - \omega_0^2}$ is real. The charge decreases rapidly with time. The discharge is known as aperiodic, dead-beat or non-oscillatory as shown by curve (a) in figure 7.5.

Case 2: Critical damping. When $b^2 = \omega_0^2$ *i.e.* $\frac{R^2}{4L^2} = \frac{1}{LC}$

The quantity under the root sign of $\sqrt{b^2 - \omega_0^2}$ is zero.

This represents a critically damped discharge, *i.e.* it is neither dead-beat nor oscillatory. This is shown by curve (b) in fig. 7.5.

Case 3: Light damping. When $b^2 < \omega_0^2$ *i.e.* $\frac{R^2}{4L^2} < \frac{1}{LC}$ then $\sqrt{b^2 - \omega_0^2}$ is imaginary and the discharge is oscillatory as shown by curve (c) in fig. 7.5.

Proceeding as in article 7.3 (case 3), we obtain the equation for oscillatory charge as

$$q = \frac{q_0\, \omega_0\, e^{-bt}}{\omega'} \sin(\omega' t + \phi)$$

or

$$q = A_0\, e^{-bt} \sin(\omega' t + \phi)$$

or

$$q = A \sin(\omega' t + \phi) \qquad \ldots (iv)$$

where,

$$A_0 = \frac{q_0\, \omega_0}{\omega'} = \frac{q_0 \cdot \omega_0}{\sqrt{\omega_0^2 - b^2}}$$

$$\omega_0 = \sqrt{\frac{1}{LC}} \quad \text{and}$$

$$b = \frac{R}{2L}$$

From equation (iv), we conclude that

(1) The variation of charge with time is oscillatory. The charge varies as a sine curve as shown by curve c in fig. 7.5.

(2) **Time Period :** The time period of the oscillations is given by

$$T = \frac{2\pi}{\omega'} = \frac{2\pi}{\sqrt{\omega_0^2 - b^2}} = \frac{2\pi}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}$$

(3) **Frequency :** The frequency of the damped oscillation is given by

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\omega_0^2 - b^2}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

If R is negligible,

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC}} = \frac{1}{2\pi\sqrt{LC}}$$

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(4) **Amplitude :** The amplitude of the oscillatory charge is

$$A = \frac{q_0 \, \omega_0}{\sqrt{\omega_0^2 - b^2}} \cdot e^{-bt}$$

or,
$$A = A_0 \, e^{-bt}$$

where
$$A_0 = \frac{q_0 \, \omega_0}{\sqrt{\omega_0^2 - b^2}} \quad \text{and} \quad b = R/2L, \quad \omega_0 = \sqrt{\frac{1}{LC}}$$

$\therefore$
$$A = A_0 \, e^{-\left(\frac{R}{2L}\right)t}$$

Higher the values of R, smaller is the value of the factor $e^{-\left(\frac{R}{2L}\right)t}$. Hence as R increases, the amplitude corresponding to a particular value of t, decreases exponentially and becomes zero as $t \to \infty$.

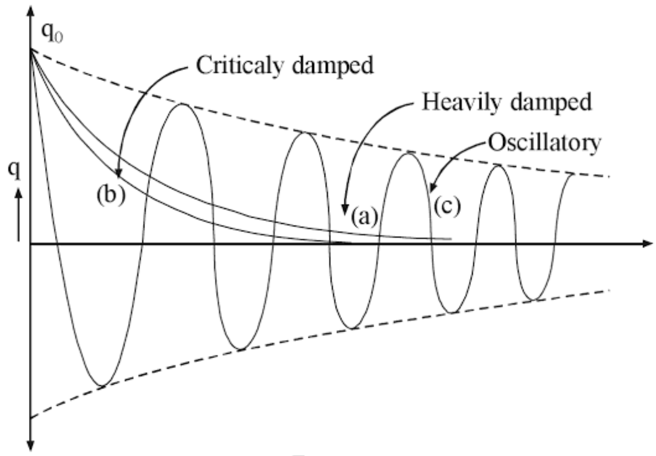

Fig. 7.5

The frequency of damped oscillations is given by

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

As R increases, the factor $\frac{R^2}{4L^2}$ increases and the quantity under the root sign decreases. Hence as resistance increases, its frequency decreases.

7.5 USE OF CRITICAL DAMPING IN DESIGNING ELECTRICAL INSTRUMENTS

The general solution of the differential equation of a damped harmonic motion is

$$y = e^{-bt}\left[A \cdot e^{\left(\sqrt{b^2 - \omega_0^2}\right)t} + B \cdot e^{\left(-\sqrt{b^2 - \omega_0^2}\right)t}\right]$$

where A and B are arbitrary constants, the values of which can be determined from boundary conditions $b = \frac{r}{2m}$ and $\omega_0^2 = \frac{s}{m}$ where r is the damping constant, m the mass of the oscillator and s the stiffness of the spring.

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Critical damping : When $b^2 = \omega_0^2$

i.e. $$b^2 - \omega_0^2 = \frac{r^2}{4m^2} - \frac{s}{m} = 0$$

The displacement is $y = (A + B)\, e^{-bt}$

This is the limiting case of behaviour shown by the differential equation of motion of a damped simple harmonic oscillator.

If $b < \omega_0$, then $b^2 - \omega_0^2$ is a *negative* quantity and the particle under goes an *oscillatory* damped simple harmonic motion. For $b = \omega_0$, the motion is neither over damped nor oscilatory. It is said to be *critically damped*. For $b > \omega_0$, $b^2 - \omega_0^2$ is positive and the motion is *over damped* or *non-oscillatory*. The property of critical damping is used in designing electrical measuring instruments like *dead beat* and *ballistic galvanometers*.

Dead beat galvanometer. A dead beat galvanometer is used for measuring *steady current*. In such a galvanometer the motion of the moving coil should be non-oscillatory. For non-oscillatory behaviour $b^2 > \omega_0^2$. To increase the value of b and therefore, the electromagnetic damping the coil is wound over a metallic core. The eddy currents set up in the coil, metallic frame and the metal core create a very high damping and make the motion of the coil dead beat. Ammeter and voltmeter are basically dead beat galvanometer and use the same designing that of galvanometer.

Ballistic galvanometer. Such a galvanometer is used for measuring charge which flows not as a steady current but as a sudden discharge. The motion of the moving coil should be oscillatory and there should be least damping *i.e.*, $b^2 < \omega_0^2$. The coil is, therefore wound over a non-metallic core to reduce electromagnetic damping. The time period, therefore, becomes large logarithmic decrement.

7.6 WORK DONE AGAINST RESISTIVE FORCES IN A DAMPED S.H.M.

The instantaneous total energy in case of a simple harmonic oscillator is the sum of its kinetic and potential energies and is given by

$$E = \text{K.E.} + \text{P. E.} = \frac{1}{2} m \left(\frac{dy}{dt}\right)^2 + \frac{1}{2} sy^2$$

where m is the mass and s the force constant of proportionality or stiffness of the oscillator.

$$\therefore \quad \frac{dE}{dt} = \frac{d}{dt}\left[\frac{1}{2} m \left(\frac{dy}{dt}\right)^2 + \frac{1}{2} sy^2\right]$$

$$= m \frac{dy}{dt} \cdot \frac{d^2 y}{dt^2} + sy \frac{dy}{dt}$$

$$= \frac{dy}{dt}\left[m \frac{d^2 y}{dt^2} + sy\right] \qquad ...(i)$$

For mechanical damped harmonic motion

$$m \frac{d^2 y}{dt^2} + r \frac{dy}{dt} + sy = 0$$

$$\therefore \quad m \frac{d^2 y}{dt^2} + sy = - r \frac{dy}{dt}$$

$\therefore$ Damping or retarding force $F_d = r \dfrac{dy}{dt} = -\left[m \dfrac{d^2 y}{dt^2} + sy\right]$

Substituting in equation (*i*), we have

$$\frac{dE}{dt} = -F_d \frac{dy}{dt} = -\frac{F_d dy}{dt} = -\frac{dW}{dt}$$

because $F_d\, dy$ gives the work done dW for a small displacement dy and $-\frac{dW}{dt}$ gives the work done per unit time or the rate of doing work against resistive force. Thus in the case of a damped oscillator,

Rate of loss of energy $\frac{dE}{dt}$ = Rate of doing work against the resistive force $\frac{dW}{dt}$.

7.7 EXPONENTIAL DECAY OF ENERGY OF DAMPED S.H.M.

The equation of motion of a damped simple harmonic oscillator is given by

$$y = \frac{y_0 \omega_0}{\sqrt{\omega_0^2 - b^2}} e^{-bt} \sin\left(\sqrt{\omega_0^2 - b^2}\, t + \phi\right) \qquad ...(i)$$

where $\omega_0^2 = \frac{s}{m}$ and $2b = \frac{r}{m}$; s being the constant of proportionality for the restoring force given by sy, r the constant of proportionality for the retarding or resistive force given by $r\frac{dy}{dt}$, m the mass of the oscillator and ϕ, the phase angle given by

$$\tan\phi = \frac{\sqrt{\omega_0^2 - b^2}}{b}$$

The amplitude of the oscillations is given by

$$A = \frac{y_0\ \omega_0}{\sqrt{\omega_0^2 - b^2}} e^{-bt} = A_0\, e^{-bt}$$

where $$A_0 = \frac{y_0\ \omega_0}{\sqrt{\omega_0^2 - b^2}} \text{ when } t = 0 \text{ and } e^{-bt} = 1$$

The total energy for damped motion $$E = \frac{1}{2} m A^2 \omega^2$$

$$= \frac{1}{2} m\omega^2\ A_0^2\, e^{-2bt} = E_0\, e^{-2bt}$$

where $$E_0 = \frac{1}{2} m\omega^2 A_0^2$$

It is clear from the equation $A = A_0 e^{-bt}$ and $E = E_0 e^{-2bt}$ that the amplitude of vibration and total energy goes on decreasing exponentially with time as the terms e^{-bt} as well as e^{-2bt} decay exponentially with time.

7.8 LOGARITHMIC DECREMENT

From Fig. (7.5) it is clear that the amplitude in case of a damped harmonic motion goes on decreasing progressively. The amplitude of the damped oscillations at any time t is given by

$$A = A_0\, e^{-bt}$$

where $b = \frac{r}{2m}$, r being the *damping constant* and m the mass of the oscillator. Let us consider the oscillations to start from the mean position, then after a time $t = \frac{T}{4}$ where T is the time period, the oscillating particle goes to the extreme position. Let the first amplitude be denoted by A_1, then

$$A_1 = A_0\, e^{-bT/4}$$

The particle will come to the mean position and then go to the extreme position on the *other* side, again come back to the mean position and go to extreme position on the *same* side after a time T *i.e.*, $\left(T + \frac{T}{4}\right)$ from start.

Let the second amplitude be A_2, then

$$A_2 = A_0\, e^{-b\left(T + \frac{T}{4}\right)}$$

Similarly the successive amplitudes of the 3rd, 4th, 5th etc. oscillations will be given by

$$A_3 = A_0\, e^{-b\left(2T + \frac{T}{4}\right)}$$

$$A_4 = A_0\, e^{-b\left(3T + \frac{T}{4}\right)}$$

....

....

$$A_n = A_0\, e^{-b\left((n-1)T + \frac{T}{4}\right)}$$

Hence
$$\frac{A_1}{A_2} = \frac{A_2}{A_3} = \frac{A_3}{A_4} = \ldots\ldots = \frac{A_{n-1}}{A_n} = e^{bT}$$

Taking natural logarithms (to the base e), we have

$$\log_e \frac{A_1}{A_2} = \log_e \frac{A_2}{A_3} = \ldots = bT = \lambda \text{ (say)}$$

where λ is the *logarithmic decrement*.

It is defined as the natural logarithm of the ratio between the two successive amplitudes on the same side of the mean position of a damped oscillation separated by one full period.

Knowing the value of logarithmic decrement λ, we can find the value of A_0.

To find the value of A_0. The amplitude A_0 which could be obtained if damping force were absent can be calculated from the relation

$$A_1 = A_0\, e^{-\frac{bT}{4}} = A_0\, e^{-\frac{\lambda}{4}} \qquad \therefore A_0 = A_1 e^{\lambda/4} = A_1\left[1 + \frac{\lambda}{4}\right]$$

neglecting square and higher powers of λ as the logarithmic decrement λ is a very small quantity as compared to 1.

We shall calculate the relation for logarithmic decrement separately for a mechanical oscillator and an electrical oscillator.

For a mechanical oscillator. For a mechanical oscillator the time period

$$T = \frac{2\pi}{\sqrt{\omega_0^2 - b^2}} = \frac{2\pi}{\omega'}$$

where $2b = \frac{r}{m}$, r being the *damping constant* and m the mass of the oscillator; $\omega_0^2 = \frac{s}{m}$, s being the *stiffness constant*.

$\therefore$ Logarithmic decrement $\lambda = bT = \dfrac{r}{2m}T$

$$= \frac{r}{2m}\frac{2\pi}{\omega'} = \frac{\pi r}{m\omega'}$$

$$= \frac{\pi r}{m\sqrt{\omega_0^2 - b^2}} = \frac{\pi r}{m\sqrt{\dfrac{s}{m} - \dfrac{r^2}{4m^2}}}$$

Damping co-efficient *b*. As $\lambda = bT$, the logarithmic decrement gives a method of finding the damping co-efficient b. As $b = \dfrac{r}{2m}$, the value of r the constant of proportionality for retarding force can also be determined.

For electrical oscillator. For electrical oscillator also

$$T = \frac{2\pi}{\sqrt{\omega_0^2 - b^2}} = \frac{2\pi}{\omega'}$$

where $2b = \dfrac{R}{L}$, R being the resistance and L the inductance and $\omega_0^2 = \dfrac{1}{LC}$, C being the capacitance of the electrical circuit.

$\therefore$ Logarithmic decrement $\lambda = bT = \dfrac{R}{2L}T$

$$= \frac{R}{2L}\frac{2\pi}{\omega'} = \frac{\pi R}{L\omega'}$$

$$= \frac{\pi R}{L\sqrt{\omega_0^2 - b^2}} = \frac{\pi R}{L\sqrt{\dfrac{1}{LC} - \dfrac{R^2}{4L^2}}}$$

Damping co-efficient *b*. As $\lambda = bT$, the logarithmic decrement gives a method of finding the damping co-efficient b and hence $\dfrac{R}{2L}$.

Experimental determination. Starting from $A = A_0\, e^{-bt}$, we have proved that

$$\frac{A_1}{A_2} = \frac{A_2}{A_3} = \frac{A_3}{A_4} \;.....\; = \frac{A_{n-1}}{A_n} = e^{bt}$$

$$\therefore \qquad \log_e \frac{A_{n-1}}{A_n} = bT = \lambda$$

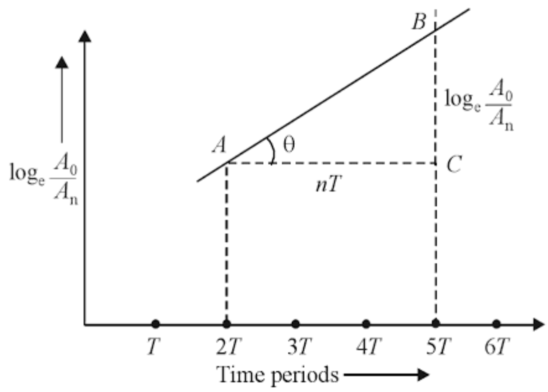

Fig. 7.6

where λ is the logarithmic decrement. The value of λ is determined by comparing two amplitudes which are n periods apart so that

$$\frac{A_0}{A_n} = e^{n\lambda}$$

or
$$\log_e \frac{A_0}{A_n} = n\lambda$$

If, therefore, a graph is plotted between $\log_e \frac{A_0}{A_n}$ and n, it is a straight line.

Thus to find λ experimentally, the values of amplitudes are found at each time period interval. A graph is plotted between time period intervals T, $2T$, $3T$ ... etc. and the corresponding value of $\log_e \frac{A_0}{A_1}, \log_e \frac{A_0}{A_2}, \log_e \frac{A_0}{A_3} \ldots$ etc.

The graph is a straight line as shown in Fig. 7.6.

$$\therefore \qquad \lambda = \frac{\log_e \frac{A_0}{A_n}}{n} = \frac{BC}{AC} = \tan\theta$$

7.9 PHYSICAL SIGNIFICANCE OF LOGARITHMIC DECREMENT

The logarithmic decrement of a damped oscillatory system gives the measure of the rate at which the amplitude of vibration decays.

As logarithmic decrement $\lambda = \frac{\pi r}{m\omega'} = \frac{r}{2m}T$, it gives a method of evaluating *r the damping constant.*

7.10 RELAXATION TIME

The relaxation time is defined as the time in which the amplitude of the damped oscillations falls to $\frac{1}{e}$ *of its original value.*

If A_0 is the original amplitude of a damped oscillator, which becomes $A = \frac{A_0}{e}$ in time t_r, then t_r is said to be the *relaxation time*. As $1/e = 0.368$, the amplitude of a damped oscillator falls to 36.8% of its initial value in a time t_r = relaxation time. Higher is the relaxation time slower is the rate of fall of amplitude and hence slower is the rate of dissipation of energy.

The amplitude of the damped oscillator at any time t is given by

$$A = A_0 e^{-bt}$$

where b is the *damping coefficient*.

If t is the relaxation time, then

$$A = A_0 e^{-bt_r} = A_0 \frac{1}{e} = A_0 e^{-1}$$

$$\therefore \qquad e^{-bt_r} = e^{-1}$$

or
$$bt_r = 1$$

or relaxation time $t_r = \dfrac{1}{b}$ ...(*i*)

It is clear from equation (*i*) that *relaxation time varies inversely as the damping coefficient*.

(*i*) **Relaxation time of mechanical oscillator.** For a mechanical oscillator $2b = \dfrac{r}{m}$ where r is the damping constant.

$$\therefore \qquad b = \frac{r}{2m}$$

Hence relaxation time for a mechanical oscillator $= \dfrac{1}{b} = \dfrac{2m}{r}$

Thus relaxation time is inversely proportional to damping constant.

(*ii*) **Relaxation time of electrical oscillator.** For an electrical oscillator

$$q = q_0 e^{-bt}$$

$$\therefore \text{ For } \qquad t = t_r = \frac{1}{b}, \quad q = q_0 e^{-1} = \frac{q_0}{e}$$

For an electrical oscillator $2b = \dfrac{R}{L}$ where R is the resistance and L the inductance of the circuit. Hence relaxation time for an electrical oscillator

$$= \frac{1}{b} = \frac{2L}{R}$$

Thus relaxation time is inversely proportional to the resistance of the oscillator circuit.

Relation between logarithmic decrement and relaxation time. Relaxation time $t_r = \dfrac{1}{b}$ and logarithmic decrement $\lambda = bT$

$$\therefore \qquad \lambda = \frac{T}{t_r}$$

Thus the logarithmic decrement may also be defined as the ratio of the time period of vibration T to the modulus of decay, (or relaxation time).

A knowledge of relaxation time and the time period therefore, helps in the determination of logarithmic decrement according to the relation

$$\lambda = \frac{T}{t_r}$$

7.11 QUALITY FACTOR OR '*Q*' VALUE

The quality factor measures the rate of decay of energy of the damped harmonic oscillator. For an undamped simple harmonic oscillator the total energy (kinetic + potential) at any time is given by

$$E = \frac{1}{2} m \omega_0^2 A^2$$

In the case of damped oscillations the work done against the retarding or resistive force is not stored as potential energy. As a result, as the particle oscillates, the total energy goes on decreasing. The amplitude of the damped oscillations is given by

$$A = A_0 e^{-bt}$$

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and ω_0 is replaced by $\sqrt{\omega_0^2 - b^2} = \omega'$.

$\therefore$ Energy of the damped oscillator is given by

$$E = \frac{1}{2} m\omega'^2 A_0^2 e^{-2bt} = E_0 e^{-2bt} \qquad ...(i)$$

where $$E_0 = \frac{1}{2} m\omega'^2 A_0^2$$

When $$t = \frac{1}{2b}$$

$$E = E_0 e^{-2b/2b} = E_0 e^{-1} = \frac{E_0}{e}$$

$\therefore$ In time $t = \dfrac{1}{2b} = \dfrac{1}{2}\dfrac{2m}{r} = \dfrac{m}{r}$

where r is the damping constant and m the mass of the oscillator, the energy of the oscillator decays to $\dfrac{1}{e} E_0$ from E_0. During this time the oscillator would have vibrated through an angle θ given by

$$\theta = \omega' t = \omega' \times \frac{m}{r} = \frac{\omega' m}{r} \text{ radians.}$$

The quantity $\dfrac{\omega' m}{r} = Q$ gives the *quality factor or Q-value.*

Definition: *Thus quality factor is defined as the angle (in radians) through which the damped oscillator oscillates as its energy decays to* $\dfrac{E_0}{e}$ *from* E_0.

$\therefore$ Quality factor for a **mechanical** oscillator

$$Q = \frac{\omega' m}{r} = \frac{\omega'}{2b} \qquad ...(ii)$$

For an **electrical** oscillator $2b = \dfrac{R}{L}$ where L is the inductance and R the resistance of the electrical circuit.

$\therefore$ Quality factor for an **electrical** oscillator

$$= \frac{\omega'}{2b} = \frac{\omega' L}{R}$$

Quality factor in terms of energy. The energy of the damped oscillator according to equation (i) is given by

$$E = E_0 e^{-2bt}$$

$\therefore$ Power dissipation P = Rate of loss of total energy with time $= -\dfrac{dE}{dt}$ is given by

$$P = -\frac{dE}{dt} = 2b E_0 e^{-2bt} = 2bE$$

or $$-dE = 2b E\, dt = 2\frac{r}{2m} E\, dt = \frac{r}{m} E\, dt$$

If $dT = T$ the time period of the oscillator, then

$$dT = T = \frac{2\pi}{\omega'}$$

Hence loss of energy during the time of one time period or loss of energy per cycle

$$-dE_T = \frac{r}{m} ET = \frac{r}{m} E \frac{2\pi}{\omega'}$$

$$\therefore \qquad \frac{\omega' m}{r} = -\frac{2\pi E}{d E_T}$$

But $\frac{\omega' m}{r}$ is the quality factor as proved in (*ii*).

$\therefore$ Quality factor $Q = 2\pi \times \frac{\text{Energy of the oscillator}}{\text{Energy lost per cycle}}$

The quality factor $Q = \frac{\omega'}{2b}$ where $2b = \frac{r}{m}$ for a mechanical oscillator and $2b = \frac{R}{L}$ for an electrical oscillator.

The quality factor Q is *large* if the damping coefficient b is *small* and Q is *small* if the damping coefficient b is *large*. The quality factor, therefore, represents the efficiency of the oscillator.

Hence lower the damping higher is the quality factor.

7.12 EFFECT OF Q ON FREQUENCY OF OSCILLATOR

The value of ω' is given by $\omega' = \sqrt{\omega_0^2 - b^2}$, when the damping is small (or in other words the quality factor Q is large) b^2 is negligible as compared to ω_0^2 and hence $\omega' = \omega_0$.

$\therefore$ Quality factor $Q = \frac{m\omega'}{r} = \frac{m\omega_0}{r} = \frac{m}{r}\left(\frac{s}{m}\right)^{1/2}$

where $\sqrt{\frac{s}{m}} = \omega_0$, s being the spring constant and m the mass of a S.H.M. oscillator.

As $\frac{m}{r}\left(\frac{s}{m}\right)^{1/2} = a$ constant, the quality factor is a constant. As $\omega' = \omega_0 =$ *a constant, a large quality factor has little or no effect on the frequency.*

7.13 AMPLITUDE, ENERGY, LOGARITHMIC DECREMENT AND RELAXATION TIME IN TERMS OF Q

(*i*) **Amplitude.** The amplitude A at any time t is given by

$$A = A_0 e^{-bt}$$

and $Q = \frac{\omega'}{2b}$ $\qquad \left[\because b = \frac{\omega'}{2Q}\right]$

Substituting we have

$$A = A_0 e^{-\frac{\omega' t}{2Q}}$$

(*ii*) **Energy.** The energy of the oscillator

$$E = \frac{1}{2} m\omega'^2 A_0^2 e^{-2bt}$$

$$= \frac{1}{2} m\omega'^2 A_0^2 e^{-2\frac{\omega'}{2Q}t} = E_0 e^{-\frac{\omega' t}{Q}} \qquad \left[\because E_0 = \frac{1}{2} m\omega'^2 A_0^2\right]$$

$$\therefore \qquad E = E_0 e^{-\omega' t/Q}$$

(*iii*) **Logarithmic decrement.** The logarithmic decrement

$$\lambda = bT = \frac{b2\pi}{\omega'} = \frac{\pi}{Q} \qquad \left[\because \; Q = \frac{\omega'}{2b}\right]$$

(*iv*) **Relaxation time.** The relaxation time

$$t_r = \frac{1}{b} = \frac{2Q}{\omega'} \qquad \left[\because \; Q = \frac{\omega'}{2b}\right]$$

7.14 QUANTITATIVE MEASUREMENT OF DAMPING EFFECT

The equation of motion of a damped simple harmonic oscillator is given by

$$m\frac{d^2y}{dt^2} + r\frac{dy}{dt} + sy = 0$$

or

$$\frac{d^2y}{dt^2} + \frac{r}{m}\frac{dy}{dt} + \frac{s}{m}y = 0$$

or

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega_0^2 y = 0$$

where

$$2b = \frac{r}{m} \text{ and } \omega_0^2 = \frac{s}{m}$$

The factor $2b$ is defined as the damping force per unit mass per unit velocity and is called the *damping coefficient* or *damping factor* and is a quantitative measure of damping. The value of b can be determined experimentally by the following methods:

(*i*) **By logarithmic decrement.** For a damped oscillator the amplitude decays exponentially with time in accordance with the relation

$$A = A_0 e^{-bT}$$

If $A_1, A_2 \ldots. A_{n-1}, A_n$ are the successive amplitudes, then

$$\frac{A_1}{A_2} = \frac{A_2}{A_3} \ldots. = \frac{A_{n-1}}{A_n} = e^{bt}$$

$\therefore$

$$\log_e \frac{A_{n-1}}{A_n} = bT = \lambda$$

λ is a constant known as logarithmic decrement. The value of λ is determined by comparing two amplitudes which are n periods apart so that

$$\frac{A_0}{A_n} = e^{n\lambda}$$

or

$$n\lambda = \log_e \frac{A_0}{A_n}$$

The value of b is calculated from the value of λ and T which can be determined experimentally.

(*ii*) **By relaxation time.** The value of damping coefficient b can be determined by finding the relaxation time t_r *i.e.*, the time in which the amplitude of a damped oscillator falls to $1/e$ of its original value by using the relation $b = \frac{1}{t_r}$.

The value of t_r is experimentally determined by plotting a graph between amplitude and time and finding from the graph the value of time $t = t_r$ in which the amplitude falls to $1/e = 0.368$ of its original value.

(*iii*) **By quality factor *Q*.** The quality factor Q is given by

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$$Q = \frac{\omega' m}{r} = \frac{\omega'}{2b}$$

But $\omega' = \frac{2\pi}{T}$ where T is the time period.

$$\therefore \qquad Q = \frac{2\pi}{T}\frac{1}{2b} = \frac{\pi}{bT}$$

from which we have damping co-efficient

$$b = \frac{\pi}{QT}.$$

The quality factor Q and time period T can be determined experimentally.

7.15 EFFECT OF DAMPING ON NATURAL FREQUENCY

The decay of amplitude of a damped harmonic oscillator with time is called damping.

In the case of real vibratory systems damping depends upon (*i*) *resistive or viscous forces e.g., resistance of air* (*ii*) *structural conditions e.g.*, friction at the supports of a vibratory system like a simple pendulum and (*iii*) *velocity* — The damping force is independent of displacement or acceleration but is proportional to velocity.

The effect of damping on natural frequency is discussed below:

The frequency of natural undamped oscillations of a mechanical oscillator is given by

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{s}{m}}$$

where s is a constant known as *stiffness constant* or *spring constant* and m the mass of vibrating particle.

The frequency of damped oscillations is given by

$$f' = \frac{1}{2\pi}\sqrt{\frac{s}{m} - \frac{r^2}{4m^2}}$$

where r is also a constant known as *damping constant*. Thus f' is less than f_0 *i.e.* the presence of damping reduces the natural frequency of the oscillator. The difference depends upon the damping factor. Larger is the value of r higher is the difference between f' and f_0 and smaller is the frequency of the damped oscillator as compared to the natural frequency of the oscillator.

7.16 DRIVEN (OR FORCED) HARMONIC OSCILLATOR

When the harmonic oscillator oscillates with its natural frequency in a medium like air, its oscillations get damped, *i.e.* the amplitude decreases exponentially with time to zero. The other damping forces experienced may be friction, viscosity etc. If the amplitudes of oscillations are to be maintain indefinetly, energy must be supplied externally. Such oscillations of the body under the action of external periodic force are known as *forced oscillations* and the oscilator is called as *forced harmonic oscillator* or *driven harmonic oscillator.*

The frequency of external periodic force applied not necessarily the same that of its natural frequency. Initially the body tries to have oscillations equal to its natural frequency, but after some initial erratic movements, it ultimately settles down to oscillations with the driving frequency *i.e.* the frequency of the driving force.

7.17 DIFFERENTIAL EQUATION FOR FORCED HARMONIC OSCILLATOR

Let $F = F_0\, e^{i\,\omega t}$ be an external periodic force applied to a damped harmonic oscillator. Here F_0 is the maximum value of the applied force and ω its angular frequency, is applied to the oscillating system. Then, the differential equation for motion of forced (*i.e.* driven) harmonic oscillator is

$$m\frac{d^2y}{dt^2} + r\frac{dy}{dt} + sy = F_0\, e^{i\omega t} \qquad ...(i)$$

The complete solution of this equation will consist of two parts.

(*i*) Transient state. The first part is known as the *transient term* which dies away with time. The transient behaviour persists only for a short interval of time. During this time the natural oscillations of the system will prevail and the system will behave as if no external force is acting.

The displacement of the oscillations is, therefore, given by the solution of the equation

$$m\frac{d^2y}{dt^2} + r\frac{dy}{dt} + sy = 0 \qquad ...(ii)$$

Dividing equation (*ii*) by m and substituting

$$\frac{s}{m} = \omega_0^2 \text{ and } \frac{r}{m} = 2b, \quad \text{we have}$$

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega_0^2\, y = 0.$$

The oscillatory solution of this equation is

$$y = A_0\, e^{-bt} \sin\left(\sqrt{\omega_0^2 - b^2}\; t + \phi\right)$$
$$= A_0\, e^{-bt} \sin(\omega' t + \phi)$$

where $\omega' = \sqrt{\omega_0^2 - b^2}, \quad A_0 = \dfrac{y_0\,\omega_0}{\sqrt{\omega_0^2 - b^2}}$

and $\tan\phi_0 = \dfrac{\sqrt{\omega_0^2 - b^2}}{b}$

The amplitude $A_0\, e^{-bt}$ decays with time due to the presence of the term e^{-bt}. The quantity ω_0 is known as the natural frequency of the oscillator in the absence of dissipative as well as driving forces.

(*ii*) Steady state. The second part is known as the *steady state term* which describes the behaviour of the oscillator after the transient term has died away.

To begin with both the terms contribute to the motion of the oscillator but later on the transient term becomes almost negligible and the ultimate behaviour of the oscillator will be given by the steady state term. During the transient stage the oscillator neither oscillates with its natural frequency nor with the frequency of the impressed force. During the steady state the oscillator oscillates with the frequency of the external driving force.

The external driving force is a periodic force which may be given by $F = F_0 \cos\omega t$ or $F = F_0 \sin\omega t$ or by a complex quantity $F = F_0\, e^{i\omega t}$ as $F_0\, e^{i\omega t} = F_0(\cos\omega t + i\sin\omega t)$ is a combination of the two forces. If the external driving force is $F_0 \cos\omega t$, then it is the real part of the complex quantity $F_0\, e^{i\omega t}$. If the force is $F_0 \sin\omega t$, then it is that part of the complex quantity $F_0\, e^{i\omega t}$ which is preceded by i. We shall, therefore consider a general case of the external force given by $F = F_0\, e^{i\omega t}$. and derive from it the particular cases of $F = F_0 \cos\omega t$ or $F = F_0 \sin\omega t$.

Equation (*i*) may now be written as

$$m\frac{d^2y}{dt^2} + r\frac{dy}{dt} + sy = F_0\, e^{i\omega t} \qquad ...(iii)$$

In the steady state let one particular solution of the equation be

$$y = A\,e^{i\omega t}$$

where A is a complex quantity.

$$\therefore \qquad \frac{dy}{dt} = i\omega A e^{i\omega t}$$

and
$$\frac{d^2y}{dt^2} = -\omega^2 A e^{i\omega t}$$

Substituting in equation (*iii*), we have

$$A\,(-\omega^2 m + i\,\omega\,r + s)\,e^{i\omega t} = F_0\,e^{i\omega t}$$

This relation is true for all values of t.

$$\therefore \qquad A = \frac{F_0}{i\omega r + (s - \omega^2 m)}$$

$$= \frac{-iF_0}{\omega r - i(s - \omega^2 m)}$$

$$= \frac{-iF_0}{\omega\left[r + i\left(\omega m - \frac{s}{\omega}\right)\right]} = \frac{-iF_0}{\omega \vec{Z}_m}$$

where $\vec{Z}_m$ is the mechanical impedance given by

$$\vec{Z}_m = r + i\left(\omega m - \frac{s}{\omega}\right)$$

Putting $r = Z_m \cos\phi$ and $\omega m - \frac{s}{\omega} = Z_m \sin\phi$ where Z_m is the modulus of the complex impedance $\vec{Z}_m$, we get

$$\vec{Z}_m = r + i\left(\omega m - \frac{s}{\omega}\right)$$

$$= Z_m \cos\phi + i\,Z_m \sin\phi$$

or
$$\vec{Z}_m = Z_m\,e^{i\phi}$$

where
$$Z_m = \left|\vec{Z}_m\right| = \left[Z_m^{\,2}\cos^2\phi + Z_m^{\,2}\sin^2\phi\right]^{1/2}$$

or
$$Z_m = \sqrt{r^2 + \left(\omega m - \frac{s}{\omega}\right)^2}$$

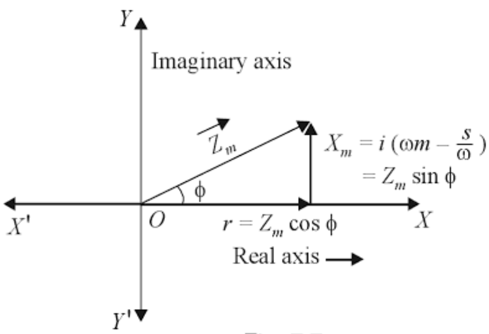

Fig. 7.7

and
$$\tan\phi = \frac{Z_m \sin\phi}{Z_m \cos\phi} = \frac{\omega m - \frac{s}{\omega}}{r} = \frac{X_m}{r} \qquad \text{... }(iii\,A)$$

ϕ being the phase difference between displacement y and applied force F

$$\therefore \qquad y = Ae^{i\omega t} = \frac{-iF_0}{\omega \vec{Z}_m}e^{i\omega t}$$

$$= \frac{-iF_0}{\omega Z_m e^{i\phi}}e^{i\omega t}$$

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$$\therefore \qquad y = \frac{-iF_0}{\omega Z_m} e^{i(\omega t - \phi)} \qquad \ldots (iv)$$

Let us discuss two cases when $F = F_0 \cos \omega t$ and when $F = F_0 \sin \omega t$. We will show that the driven oscillator is always behind the driving force in phase (ϕ), and is explained as under :

***(i)* When Driving force $F = F_0 \cos \omega t$.** If the external driving force is $F_0 \cos \omega t$, then it is the real part of the complex function $F_0 e^{i\omega t}$. Therefore, the steady state term will *only* consist of the real part of equation (*iv*), which may be put in the form

$$y = \frac{-iF_0}{\omega Z_m}[\cos(\omega t - \phi) + i \sin(\omega t - \phi)]$$

$$= \frac{F_0}{\omega Z_m}[-i\cos(\omega t - \phi) + \sin(\omega t - \phi)] \qquad \ldots (v)$$

$$= \frac{-iF_0}{\omega Z_m}\cos(\omega t - \phi) + \frac{F_0}{\omega Z_m}\sin(\omega t - \phi)$$

and the real part of the above equation gives

$$y = \frac{F_0}{\omega Z_m}\sin(\omega t - \phi) \qquad \ldots (vi)$$

$$= \frac{F_0}{\omega Z_m}\cos\left[\omega t - \left(\phi + \frac{\pi}{2}\right)\right]$$

The driving applied force is $F_0 \cos \omega t$ which is the real part of

$$F_0 e^{i\omega t} = F_0 \cos \omega t + i F_0 \sin \omega t$$

Comparing $\qquad y = \dfrac{F_0}{\omega Z_m}\cos\left[\omega t - \left(\phi + \dfrac{\pi}{2}\right)\right]$

with $\qquad F = F_0 \cos \omega t$

We find that the total phase difference between the displacement y and the applied force F is $-\left(\phi + \dfrac{\pi}{2}\right)$ in the steady state. The amplitude is $\dfrac{F_0}{\omega Z_m}$.

***(ii)* When Driving force $F = F_0 \sin \omega t$.** The external driving force $F_0 \sin \omega t$ is that part of the complex function $F_0 e^{i\omega t}$ which is multiplied by vector operator i. Therefore, the steady state term will now *only* consist of that part of equation (*iv*) which is multiplied by i, which from equation (*v*), we get

$$y = \frac{-F_0}{\omega Z_m}\cos(\omega t - \phi) \qquad \ldots (vii)$$

$$= \frac{F_0}{\omega Z_m}\sin\left[\omega t - \left(\phi + \frac{\pi}{2}\right)\right]$$

The driving force is $F_0 \sin \omega t$.

Comparing $\qquad y = \dfrac{F_0}{\omega Z_m}\sin\left[\omega t - \left(\phi + \dfrac{\pi}{2}\right)\right]$ with

$\qquad F = F_0 \sin \omega t$

We again find that the total phase difference between the displacement y and applied force F is $-(\phi + \pi/2)$ in the steady state. The amplitude is also $\dfrac{F_0}{\omega Z_m}$.

Phase difference between driven oscillator and driving force. We have proved above that when the driving force $F = F_0 \cos \omega t$, the displacement $y = \dfrac{F_0}{\omega Z_m}\cos[\omega t - (\phi + \pi/2)]$ and when $F = F_0 \sin \omega t$, $y = \dfrac{F_0}{\omega Z_m}\sin[\omega t - (\phi + \pi/2)]$. In both the cases we find that the driven oscillator is always behind the driving force in phase. The value of the difference in phase is $-(\phi + \pi/2)$.

7.18 CHANGE IN AMPLITUDE WITH FREQUENCY OF DRIVING FORCE

When a periodic driving force $F = F_0 \cos \omega t$ is applied to a damped oscillator and a steady state is reached, the displacement is given by

$$y = \frac{F_0}{\omega Z_m}\sin(\omega t - \phi) \qquad ...(i)$$

where Z_m is the mechanical impedance offered by the oscillator and its magnitude is given by

$$Z_m = \sqrt{r^2 + \left(\omega m - \frac{s}{\omega}\right)^2} \qquad ...(ii)$$

and $$\tan\phi = \frac{\omega m - \dfrac{s}{m}}{r}$$

r being the *damping constant, m* the mass and s the stiffness or *spring constant* of the oscillator.

For proof see **[article 7.17 Eq. (*iii A*)]**

Substituting $2b = \dfrac{r}{m}$ and $\omega_0^2 = \dfrac{s}{m}$ in (*ii*) we get

$$Z_m = \frac{m}{\omega}\sqrt{4b^2\omega^2 + (\omega^2 - \omega_0^2)^2}$$

ω_0 being the natural frequency of the oscillator.

$$\therefore \qquad y = \frac{F_0}{\omega \dfrac{m}{\omega}\sqrt{4b^2\omega^2 + (\omega^2 - \omega_0^2)^2}}\sin(\omega t - \phi)$$

$$= \frac{f_0}{\sqrt{4b^2\omega^2 + (\omega^2 - \omega_0^2)^2}}\sin(\omega t - \phi)$$

where $$f_0 = \frac{F_0}{m} \text{ and } \tan\phi = \frac{\omega m - \dfrac{s}{m}}{r} = \frac{\omega^2 - \omega_0^2}{2b\omega}$$

Substituting $\dfrac{f_0}{\sqrt{4b^2\omega^2 + (\omega^2 - \omega_0^2)^2}} = a$, the *displacement amplitude,* we have

$$y = a \sin(\omega t - \phi) \qquad ...(iii)$$

We shall study the behaviour of displacement amplitude versus driving force frequency in three stages of frequencies *i.e.*, low frequency, resonant frequency and high frequency.

(*i*) **At low driving frequencies** ($\omega << \omega_0$) *i.e.*, the driving force frequency is very much less than the natural frequency of oscillations. The displacement amplitude

$$a = \frac{f_0}{\sqrt{4b^2\omega^2 + (\omega^2 - \omega_0^2)^2}}$$

When $\omega \to 0$, ω^2 can be neglected as compared to $\omega_0^2, 4b^2\omega^2$ as compared to $(\omega_0^2)^2$ and we get

$$a = \frac{f_0}{\omega_0^2} = \frac{F_0/m}{s/m} = \frac{F_0}{s}$$

Thus at very low driving force frequencies, the amplitude of forced oscillations depends upon the driving force F_0 *and stiffness constant s but is independent of frequency.*

Thus the amplitude of forced oscillator and also the impedance is *stiffness controlled.*

(*ii*) At high driving force frequency ($\omega >> \omega_0$) *i.e.*, the driving force frequency is very much greater than the natural frequency of the oscillator. For large values of ω, ω^2 is very much greater than ω_0^2 so that ω_0^2 can be neglected as compared to ω^2 and then

$$a = \frac{f_0}{\sqrt{4b^2\omega^2 + (\omega^2 - \omega_0^2)^2}} = \frac{f_0}{\sqrt{4b^2\omega^2 + \omega^4}} = \frac{f_0}{\omega^2}$$

If b the damping is very small so that $4\,b^2\,\omega^2$ can be neglected as compared to ω^4.

As $\omega \to \infty$; $a \to 0$ *i.e.*, the amplitude almost approaches zero as the driving force frequency becomes very large.

(*iii*) At frequency of displacement resonance ($\omega = \omega_0$) *i.e.*, the driving force frequency is equal to the natural frequency of the oscillator. This is the case of *amplitude resonance*. At this stage the amplitude is given by

$$a = \frac{f_0}{\sqrt{4b^2\omega^2 + (\omega^2 - \omega_0^2)^2}} = \frac{f_0}{2b\omega}$$

Thus at a driving force frequency equal to the resonant frequency the maximum amplitude is inversely proportional to damping coefficient b. For small damping, amplitude is large and for large damping, amplitude is small. Theoretically for $b = 0$, $a = \infty$.

When, however, damping is present the displacement resonance takes place at a frequency $\omega = \sqrt{\omega_0^2 - 2b^2}$ as proved below.

For amplitude a to be a maximum $\frac{da}{d\omega} = 0$

Now displacement amplitude $a = \frac{f_0}{\sqrt{4b^2\omega^2 - (\omega^2 - \omega_0^2)^2}}$

$\therefore$
$$\frac{d}{d\omega}\left[\frac{f_0}{\sqrt{4b^2\omega^2 + (\omega^2 - \omega_0^2)^2}}\right] = \frac{-f_0[4\omega)(\omega^2 - \omega_0^2) + 8b^2\omega]}{2[4b^2\omega^2 + (\omega^2 - \omega_0^2)^2]^{3/2}} = 0$$

$\therefore$
$$4\omega(\omega^2 - \omega_0^2) + 8b^2\omega = 0$$

or
$$\omega^2 - \omega_0^2 = -2b^2$$

$\therefore$
$$\omega^2 = \omega_0^2 - 2b^2$$

or
$$\omega = \sqrt{\omega_0^2 - 2b^2} = \omega_0\sqrt{1 - \frac{2b^2}{\omega_0^2}}$$

$$= \omega_0 \sqrt{1 - \frac{r^2}{2m^2\omega_0^2}}$$

Thus the *maximum displacement amplitude or displacement resonance takes place at a frequency slightly less than* ω_0 *the natural frequency of the oscillator and is given by*

$$\omega = \sqrt{\omega_0^2 - 2b^2} = \omega_0 \sqrt{1 - \frac{r^2}{2m^2\omega_0^2}}$$

In other words the resonant frequency of driving force ω *is slightly less than the natural frequency of displacement* ω_0.

Maximum value of displacement amplitude. The displacement amplitude is given by

$$a = \frac{f_0}{\sqrt{4b^2\omega^2 - (\omega^2 - \omega_0^2)^2}}$$

The displacement amplitude is maximum at frequency of resonance $\omega = \sqrt{\omega_0^2 - 2b^2}$.

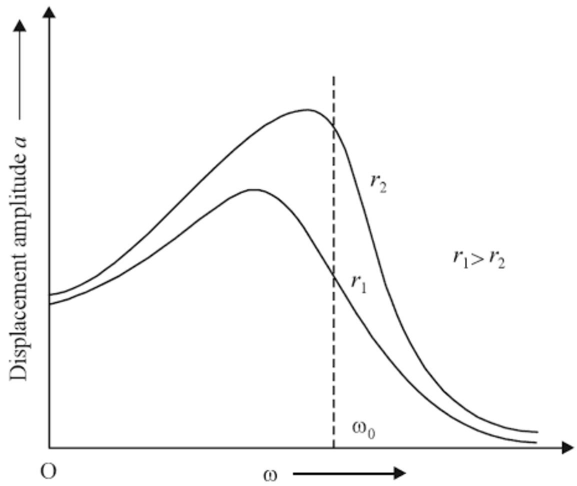

Fig. 7.8

Substituting this value of ω in the expression for a we have

$$a_{max} = \frac{f_0}{\sqrt{4b^2(\omega_0^2 - 2b^2) + (\omega_0^2 - 2b^2 - \omega_0^2)^2}}$$

$$= \frac{f_0}{\sqrt{4b^2(\omega_0^2 - 2b^2) + 4b^2}} = \frac{f_0}{2b\sqrt{\omega_0^2 - b^2}}$$

$$= \frac{f_0}{\frac{r}{m}\sqrt{\omega_0^2 - \frac{r^2}{4m^2}}} = \frac{F_0}{r\sqrt{\omega_0^2 - \frac{r^2}{4m^2}}} = \frac{F_0}{r\omega'}$$

where $\omega' = \sqrt{\omega_0^2 - \frac{r^2}{4m^2}} = \sqrt{\frac{s}{m} - \frac{r^2}{4m^2}}$ = natural frequency of damped oscillator.

Graphical representation. We have seen that amplitude of forced vibrations at low frequencies is $\frac{F_0}{s}$ and almost becomes zero at very high frequencies. The maximum value of displacement amplitude $a_{max} = \frac{F_0}{r\,\omega'}$. Thus *higher the damping smaller is the amplitude at resonance* as shown in Fig. 7.8. As $r_1 > r_2$ the value of a_{max} corresponding to r_1 is less than that corresponding to r_2. The frequency for maximum displacement resonance amplitude $\omega' = \sqrt{\omega_0^2 - \frac{r^2}{4m^2}}$. Hence, as r decreases ω' comes closer to ω_0. (as shown in Fig. 7.8, $r_1 > r_2$). Therefore a_{max} corresponding to r_2 is closer to ω_0 as compared to a_{max} corresponding to r_1.

When the frequency of external periodic force is equal to the natural frequency of the body the external periodic force helps to increase the amplitude of vibration of the body at each step as the external applied force is always in phase with the vibrations of the body. Every new impulse adds to effect of all the previous impulses as the successive impulses always arrive in phase and the accumulated effect is to make the body vibrate with a large amplitude, resulting into resonance.

7.19 COMPARISON OF NATURAL FREQUENCY AND DISPLACEMENT RESONANCE FREQUENCY

The displacement resonance occurs at an angular frequency ω given by

$$\omega = \sqrt{\omega_0^2 - 2b^2} = \sqrt{\frac{s}{m} - \frac{r^2}{2m^2}} \qquad ...(i)$$

$$\left(\text{since } \omega_0^2 = \frac{s}{m} \text{ and } 2b = \frac{r}{m}\right)$$

The natural (angular) frequency of damped oscillations is

$$\omega' = \sqrt{\omega_0^2 - b^2} = \sqrt{\frac{s}{m} - \frac{r^2}{4m^2}} \qquad ...(ii)$$

Comparing (*i*) and (*ii*) we find that $\omega' > \omega$

Hence natural frequency of damped oscillations is greater than the frequency at which displacement resonance occurs.

7.20 VARIATION OF PHASE DIFFERENCE BETWEEN DISPLACEMENT AND DRIVING FORCE FREQUENCY

When a periodic driving force $F = F_0 \cos \omega t$ is applied to a damped oscillator and a steady state is reached the displacement is given by

$$y = \frac{F_0}{\omega Z_m} \sin(\omega t - \phi) \qquad ...(i)$$

Where Z_m is the mechanical impedance offered by the oscillator and its magnitude is given by

$$Z_m = \sqrt{r^2 + \left(\omega m - \frac{s}{\omega}\right)^2}$$

r being the *damping constant*, m the mass and s the *stiffness* or *spring constant* of the oscillator.

$$\tan\phi = \frac{\omega m - \frac{s}{\omega}}{r} = \frac{\omega^2 - \frac{s}{m}}{\frac{r}{m}\omega} = \frac{\omega^2 - \omega_0^2}{2b\omega}$$

where $\omega_0^2 = \frac{s}{m}$ and $2b = \frac{r}{m}$

The driving force $F = F_0 \cos \omega t$ ...(*ii*)

Relation (*i*) can be put in the form

$$y = \frac{F_0}{\omega Z_m} \cos\left[\omega t - \left(\phi + \frac{\pi}{2}\right)\right] \qquad ...(iii)$$

From Eq. (*iii*) and (*ii*), we find that displacement lags behind the driving force by an angle $\left(\phi + \frac{\pi}{2}\right)$. The value of ϕ or $\tan \phi$ varies with ω. Thus phase difference between displacement y and driving force F will also vary with ω. Let the phase difference between y and F be denoted by θ, then

$$\theta = -\left(\frac{\pi}{2} + \phi\right)$$

We shall study the variation of θ (or ϕ) versus ω in three stages of frequencies *i.e.*, low frequency, high frequency and resonant frequency.

(*i*) Low frequency. When $\omega << \omega_0$. At $\omega = 0$

$$\tan\phi = \frac{\omega - \omega_0^2}{2b\omega} = -\infty$$

$\therefore \qquad \phi = -\pi/2$

The total phase difference $\theta = -\left(\frac{\pi}{2} + \phi\right) = -\left(\frac{\pi}{2} - \frac{\pi}{2}\right) = 0$

Thus at $\omega = 0$ the driving force and resulting displacement are in *phase*. As ω increases a slight phase difference is introduced, θ having a small but positive value indicating that the displacement lags the force.

(*ii*) Resonant frequency. When $\omega = \omega_0$ where ω_0 is the natural frequency of undamped oscillations

$$\tan\phi = \frac{\omega^2 - \omega_0^2}{2b\omega} = 0$$

$\therefore \qquad \phi = 0$

$\therefore$ Total phase difference $\theta = -\left(\frac{\pi}{2} + \phi\right) = -\pi/2$

Thus at $\omega = \omega_0$ the resultant displacement lags behind the driving force by an angle $\frac{\pi}{2}$.

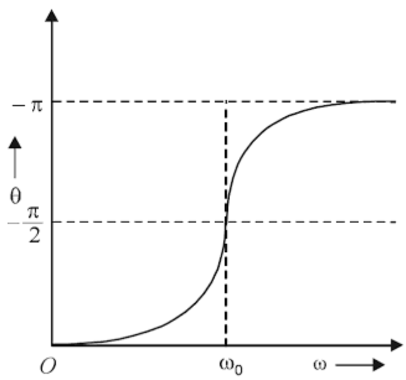

Fig. 7.9

(*iii*) High frequency. When $\omega >> \omega_0$. As $\omega \to \infty$

$$\tan\phi = \frac{\omega^2 - \omega_0^2}{2b\omega} = +\infty$$

$\therefore \qquad \phi = \frac{\pi}{2}$

$\therefore$ Total phase difference $\theta = -\left(\frac{\pi}{2} + \phi\right)$

$$= -\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = -\pi$$

Thus, when ω increases beyond ω_0 the displacement lags behind the driving force by an angle greater than $\frac{\pi}{2}$ and finally when $\omega \to \infty$ the phase angle increases to π *i.e.* displacement lags behind by π.

The variation of total phase difference θ with ω is shown in fig. 7.9.

7.21 VELOCITY OF A FORCED OSCILLATOR

When a periodic driving force $F = F_0\, e^{i\,\omega t}$ is applied to a damped oscillator and a steady state is reached the displacement is given by

$$y = -\frac{i F_0}{\omega Z_m} e^{i(\omega t - \phi)}$$

where $$Z_m = \sqrt{r^2 + \left(\omega \cdot m - \frac{s}{\omega}\right)^2} \qquad \text{... } (i)$$

and $$\tan\phi = \frac{\omega m - s/\omega}{r}$$

r being the *damping constant*, m the mass and s the *stiffness* or *spring constant*.

Equation (i) can be put in the form

$$y = -\frac{i F_0}{\omega Z_m}[\cos(\omega t - \phi) + i\sin(\omega t - \phi)]$$

or $$y = -\frac{i F_0}{\omega Z_m}\cos(\omega t - \phi) + \frac{F_0}{\omega Z_m}\sin(\omega t - \phi) \qquad \text{... } (ii)$$

Put $F = F_0\, e^{i\,\omega t}$ in the form

$$F = F_0(\cos\omega t + i\sin\omega t)$$

or $$F = F_0\cos\omega t + i F_0\sin\omega t \qquad \text{... } (iii)$$

Comparing the real parts of equations (ii) and (iii) we have

when $$F = F_0\cos\omega\, t \qquad \text{...}(iv)$$

$$y = \frac{F_0}{\omega Z_m}\sin(\omega t - \phi) \qquad \text{...}(v)$$

Similarly comparing the imaginary parts of equations (ii) and (iii), we have

when $$F = F_0\sin\omega\, t \qquad \text{...}(vi)$$

$$y = \frac{F_0}{\omega Z_m}\cos(\omega t - \phi) \qquad \text{...}(vii)$$

The velocity of a forced oscillator is defined as the rate of change of displacement with time. We shall calculate the value of velocity

(i) When the driving force is $F_0\cos\omega\, t$ and

(ii) When the driving force is $F_0\sin\omega\, t$.

(*i*) Driving force F_0 cos ω *t*. When the driving force is given by $F = F_0 \cos \omega t$, the displacement of the forced oscillations in the steady state is given by

$$y = \frac{F_0}{\omega Z_m} \sin(\omega t - \phi)$$

∴ velocity
$$v = \frac{dy}{dt} = \frac{F_0}{\omega Z_m} \omega \cos(\omega t - \phi)$$

$$= \frac{F_0}{Z_m} \cos(\omega t - \phi) = v_0 \cos(\omega t - \phi) \quad ...(viii)$$

$$= v_0 \sin\left[(\omega t - \phi) + \frac{\pi}{2}\right] \quad ...(ix)$$

where $v_0 = F_0 / Z_m$

Comparing equation (*ix*) *with* (*v*), *we find that velocity always leads the displacement by a phase angle* $\frac{\pi}{2}$. *Comparing equation* (*viii*) *with* (*iv*), *we find that velocity lags behind the driving force by a phase angle* ϕ.

(*ii*) Driving force F_0 sin ω *t*. When the driving force is $F = F_0 \sin \omega t$, the displacement is given by

$$y = \frac{F_0}{\omega Z_m} \cos(\omega t - \phi)$$

∴ velocity
$$v = \frac{dy}{dt} = \frac{-F_0}{\omega Z_m}(-\omega) \sin(\omega t - \phi)$$

$$= \frac{F_0}{Z_m} \sin(\omega t - \phi) = v_0 \sin(\omega t - \phi) \quad ...(x)$$

$$= v_0 \cos\left[(\omega t - \phi) + \frac{\pi}{2}\right] \quad ...(xi)$$

where $v_0 = \dfrac{F_0}{Z_m}$

Comparing equation (*xi*) *with* (*vii*), *we again find that velocity always leads the displacement by a phase angle* $\frac{\pi}{2}$.

Comparing equation (*x*) with (*vi*) *we again find that velocity lags behind the driving force by a phase angle* ϕ.

7.22 VARIATION OF VELOCITY AMPLITUDE WITH DRIVING FORCE FREQUENCY

We have seen in the above discussion that when the applied force is $F = F_0 \cos \omega t$ the velocity of forced vibrations is $v = v_0 \cos(\omega t - \phi)$ and when the applied force is $F = F_0 \sin \omega t$ the velocity of forced vibrations is $v = v_0 \sin(\omega t - \phi)$ where $v_0 = F_0 / Z_m$.

The amplitude of the velocity in each case

$$v_0 = \frac{F_0}{Z_m} = \frac{F_0}{\sqrt{r^2 + X_m^2}} = \frac{F_0}{\sqrt{r^2 + \left(\omega m - \frac{s}{\omega}\right)^2}} \quad ...(xii)$$

We shall discuss the behaviour of velocity amplitude versus driving force frequency in three stages *i.e.*, low frequency, high frequency and resonant frequency.

(*i*) Low frequency. According to equation (*xii*), the mechanical impedance

$$Z_m = \sqrt{r^2 + \left(\omega m - \frac{s}{\omega}\right)^2}$$

$\left(\omega m - \frac{s}{\omega}\right) = X_m$ is known as mechanical reactance. It consists two parts $\omega\, m$ and $\frac{s}{\omega}$. For *low* frequencies $\frac{s}{\omega}$ is a large quantity. In such a case the impedance Z_m depends upon s and is said to be *stiffness controlled.*

When $\qquad \omega \to 0,\ m\omega \to 0$ and $\frac{s}{\omega} \to \infty$

$$\therefore \qquad Z_m = \left(r^2 + \frac{s^2}{\omega^2}\right)^{\frac{1}{2}} = \frac{s}{\omega}$$

As $\frac{s}{\omega}$ is a very large quantity and r^2 can be neglected as compared to $\frac{s^2}{\omega^2}$.

$$\therefore \qquad v_0 = \frac{F_0}{Z_m} = \frac{F_0}{s/\omega} = F_0 \frac{\omega}{s}$$

when $\qquad \omega \to 0,\ v_0 \to 0$

(*ii*) High frequency. At high frequencies $\omega\, m$ is a large quantity. In such a case the impedance Z_m depends upon m and is said to be *mass controlled* or *inertia controlled.*

When $\qquad \omega \to \infty,\ m\omega \to \infty$ but $\frac{s}{\omega} \to 0$

$$\therefore \qquad Z_m = \left[r^2 + \left(m\omega - \frac{s}{\omega}\right)^2\right]^{1/2} = \left[r^2 + m^2\omega^2\right]^{1/2} = m\omega$$

[because $(m\,\omega)$ is very large, r^2 can be neglected as compared to $m^2\,\omega^2$.]

$$\therefore \qquad v_0 = \frac{F_0}{Z_m} = \frac{F_0}{m\omega}$$

when $\qquad \omega \to \infty, v_0 \to 0$

(*iii*) Resonant frequency. For a frequency ω_0 at which $\omega_0\, m = \frac{s}{\omega_0}$ the reactance $X_m = \omega_0\, m - \frac{s}{\omega_0} = 0, Z_m$ has its minimum value given by $Z_m = r$. It is a real quantity with zero reactance. At this frequency the velocity amplitude has its maximum value $v_0 = \frac{F_0}{Z_m} = \frac{F_0}{r}$. This frequency ω_0 is known as the *frequency* of *velocity resonance*.

Thus at frequency of velocity resonance, the velocity amplitude depends upon the driving force F_0 *and r the resisting force per unit velocity but is independent of frequency.*

Velocity resonance occurs at natural frequency of the oscillator. For velocity resonance $\omega = \omega_0 = \sqrt{\frac{s}{m}}$ = natural frequency of the oscillator. This can be proved as follows.

$\because \quad v_0 = \dfrac{F_0}{Z_m}, v_0$ has maximum value when Z_m is a minimum. Now Z_m is minimum for a frequency given by $\dfrac{dZ_m}{d\omega} = 0.$

As $$Z_m = \left[r^2 + \left(\omega m - \frac{s}{\omega}\right)^2\right]^{\frac{1}{2}}$$

$$\frac{dZ_m}{d\omega} = \frac{1}{2}\left[r^2 + \left(\omega m - \frac{s}{\omega}\right)^2\right]^{-\frac{1}{2}} 2\left(\omega m - \frac{s}{\omega}\right)\left(m + \frac{s}{\omega^2}\right) = 0$$

$\therefore$ $$\left(\omega m - \frac{s}{\omega}\right)\left(m + \frac{s}{\omega^2}\right) = \frac{1}{\omega}\left(\omega m - \frac{s}{\omega}\right)\left(\omega m + \frac{s}{\omega}\right) = 0$$

or $$\omega^2 m^2 - \frac{s^2}{\omega^2} = 0$$

$\therefore$ $$\omega^4 = \frac{s^2}{m^2}$$

or $$\omega = \sqrt{\frac{s}{m}} = \omega_0$$

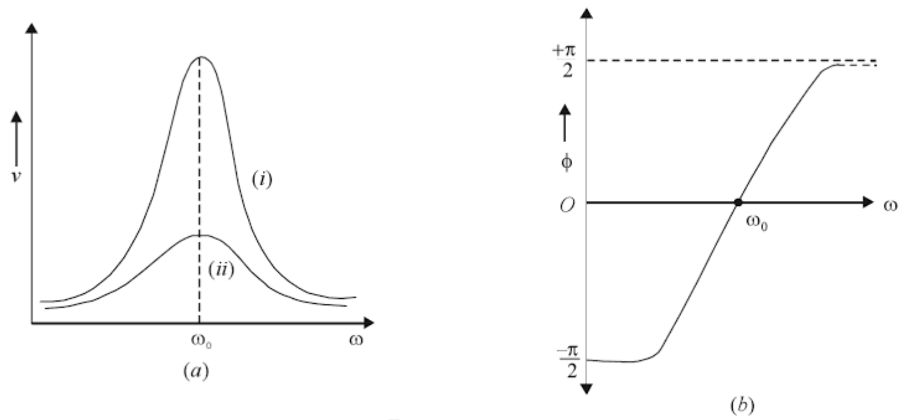

Fig. 7.10

The variation of magnitude of velocity with frequency of a driving force is shown in Fig. 7.10 (*a*).

At resonance $v_0 = \dfrac{F_0}{r}$. The height and sharpness of the peak at resonance depends upon r. For small value of r, the velocity at resonance is very large, curve (*i*) and for large values of r the velocity has a smaller value, curve (*ii*), *i.e.* a flat curve.

7.23 VARIATION OF PHASE DIFFERENCE BETWEEN VELOCITY AND DRIVING FORCE FREQUENCY

The velocity for a forced oscillator is given by

$$v = \frac{F_0}{Z_m}\cos(\omega t - \phi)$$

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When the driving force is given by

$$F = F_0 \cos \omega t$$

The phase difference between velocity and driving force is $-\phi$ *i.e.*, the velocity *lags behind* the driving force by an angle ϕ given by

$$\tan \phi = \frac{\omega m - \dfrac{s}{\omega}}{r}$$

(*i*) **Low frequency.** At low frequencies $\omega \to 0$

$$m\omega \to 0 \text{ and } \frac{s}{\omega} \to \infty$$

$\therefore$ $\tan \phi = -\infty$

or $\phi = -\dfrac{\pi}{2}$

$\therefore$ Phase difference between velocity and driving force $-\phi = +\dfrac{\pi}{2}$. *Hence the resulting velocity leads the driving force by phase angle* $\dfrac{\pi}{2}$.

(*ii*) **High frequency.** At high frequencies $\omega \to \infty$

$\therefore$ $m\omega \to \infty$ and $\dfrac{s}{\omega} \to 0$

$\therefore$ $\tan \phi = +\infty$

or $\phi = +\dfrac{\pi}{2}$

$\therefore$ Phase difference between velocity and driving force $-\phi = -\dfrac{\pi}{2}$. *Hence the resulting velocity lags behind the driving force by phase angle* $\dfrac{\pi}{2}$.

(*iii*) **Resonant frequency.** At resonant frequency $\omega = \omega_0 = \sqrt{\dfrac{s}{m}}$

or $\omega^2 = \dfrac{s}{m}$ $\therefore$ $m\omega = \dfrac{s}{\omega}$

$\therefore$ $\tan \phi = 0$ or $\phi = 0$.

$\therefore$ Phase difference between velocity and driving force $-\phi = 0$. *Hence the resulting velocity is in phase with the driving force.*

Hence as the driving force frequency increases from zero to ω_0 the velocity *lags* behind the force by an angle $\dfrac{\pi}{2}$ which slowly increases to zero when the velocity is in phase with the force. On further increasing the frequency the velocity *leads* the force and as ω approaches ∞ the angle of lead approaches $\dfrac{\pi}{2}$. This is shown in Fig. 7.10 (*b*).

7.24 PHASE DIFFERENCE AT DISPLACEMENT RESONANCE AND VELOCITY RESONANCE

When a periodic force

$$F = F_0 \cos \omega t \qquad ...(i)$$

is applied to a damped oscillator and a steady state is reached, the displacement is given by

$$y = \frac{F_0}{\omega Z_m} \sin(\omega t - \phi) \qquad ...(ii)$$

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where $Z_m = \sqrt{r^2 + \left(\omega m - \frac{s}{\omega}\right)^2}$, r is the *damping constant*, m the mass and s the *stiffness* or *spring constant.*

The phase difference between the driving force and displacement of forced oscillator is given by

$$\tan\phi = \frac{\omega m - s/\omega}{r}$$

Equation (*ii*) can be put in the form

$$y = \frac{F_0}{\omega Z_m}\cos\left[\omega t - (\phi + \pi/2)\right] \qquad ...(iii)$$

Comparing Eqs. (*i*) and (*iii*) we find that displacement lags behind the driving force by an angle $\phi + \pi/2$.

At displacement resonance $\omega m = s/\omega$ or $\omega m - s/\omega = 0$

$\therefore$ $\tan\phi = 0$ or $\phi = 0$

$\therefore$ *At resonance displacement lags behind the driving force by an angle* $\pi/2$.

For the same periodic force $F = F_0 \cos\omega t$ the velocity of the forced oscillator is given by

$$v = \frac{F_0}{Z_m}\cos(\omega t - \phi)$$

and phase difference $\tan\phi = \dfrac{\omega m - s/\omega}{r}$

At velocity resonance again $\omega m = s/\omega$ or $\omega m - s/\omega = 0$

$\therefore$ $\tan\phi = 0$ or $\phi = 0$.

$\therefore$ *At resonance, velocity is in phase with the driving force.*

7.25 DISPLACE RESONANCE OCCURS AT SLIGHTLY LESS FREQUENCY THAN VELOCITY RESONANCE

The velocity resonance occurs at a frequency

$$\omega = \omega_0$$

where ω_0 is the frequency of free oscillations of the forced oscillator.

The displacement resonance takes place at a frequency

$$\omega' = \omega_0\sqrt{1 - \frac{r^2}{2m^2\omega_0^2}}$$

since r, m and ω_0 are all positive quantities (being squares) the displacement resonance occurs at a frequency ω' slightly less than ω the frequency of velocity resonance.

7.26 VARIATION OF ACCELERATION AMPLITUDE WITH DRIVING FORCE FREQUENCY

When the driving force is given by

$$F = F_0 \cos\omega t$$

the displacement of the forced oscillations in the steady state is given by

$$y = \frac{F_0}{\omega Z_m}\sin(\omega t - \phi)$$

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and velocity $$v = \dot{y} = \frac{dy}{dt} = \frac{F_0}{Z_m} \cos(\omega t - \phi)$$

$\therefore$ acceleration $$\ddot{y} = \frac{d^2 y}{dt^2} = -\frac{F_0\,\omega}{Z_m} \sin(\omega t - \phi)$$

where $$Z_m = \sqrt{r^2 + \left(\omega m - \frac{s}{\omega}\right)^2}$$

$\therefore$ Acceleration amplitude $$= \frac{F_0\,\omega}{\left[r^2 + \left(\omega m - \frac{s}{\omega}\right)^2\right]^{1/2}}$$

$$= \frac{F_0}{\left[\frac{r^2}{\omega^2} + \left(m - \frac{s}{\omega^2}\right)^2\right]^{1/2}}$$

Now three cases arise :

(*i*) **At low frequency.** At low frequency $\omega \to 0$, therefore the term s/ω^2 dominates provided r is small. In such a case the oscillator is said to be *stiffness controlled.* When r is small and $\omega \to 0, Z_m = \frac{s}{\omega}$ and hence acceleration amplitude $= \frac{F_0\,\omega}{s/\omega} = \frac{F_0\,\omega^2}{s}$. But $\omega \to 0$

Hence acceleration amplitude is zero.

(*ii*) **At high frequency.** For large values of $\omega, \frac{s}{\omega}$ can be neglected as compared to $\omega\, m$ and r^2 as compared to $\omega^2 m^2$ so that we get $Z_m = \omega\, m$. In such a case the oscillator is said to be *inertia controlled.*

$\therefore$ For large frequencies acceleration amplitude

$$= \frac{F_0\,\omega}{Z_m} = \frac{F_0\,\omega}{\omega\, m} = \frac{F_0}{m}$$

i.e., the acceleration amplitude depends upon the driving force F_0 and mass m but is independent of frequency.

(*iii*) **Maximum value of acceleration amplitude.** The acceleration amplitude becomes maximum when $\frac{\omega}{Z_m}$ is minimum or $\frac{Z_m}{\omega}$ is a maximum *i.e.*

$$\frac{d}{d\omega}\left(Z_m\,\omega^{-1}\right) = \frac{d}{d\omega}\left\{\omega^{-1}\left[r^2 + \left(\omega m - \frac{s}{\omega}\right)^2\right]^{1/2}\right\}$$

$$= \frac{d}{d\omega}\left[r^2/\omega^2 + \left(m - \frac{s}{\omega^2}\right)^2\right]^{1/2}$$

$$= \frac{1}{2}\left[r^2/\omega^2 + \left(m - \frac{s}{\omega^2}\right)^2\right]^{-1/2}\left[\left(-2\frac{r^2}{\omega^3}\right) + 2\left(m - \frac{s}{\omega^2}\right)\left(\frac{2s}{\omega^3}\right)\right]$$

$$= \frac{-\dfrac{r^2}{\omega^3} + \dfrac{2sm}{\omega^3} - \dfrac{2s^2}{\omega^5}}{\left[r^2/\omega^2 + \left(m - \dfrac{s}{\omega^2}\right)^2\right]^{1/2}} = 0$$

or $$-\frac{r^2}{\omega^3} + \frac{2sm}{\omega^3} - \frac{2s^2}{\omega^5} = 0$$

or $$-r^2 + 2sm - \frac{2s^2}{\omega^2} = 0$$

or $$2sm - r^2 = \frac{2s^2}{\omega^2}$$

or $$\omega^2 = \frac{2s^2}{2sm - r^2}$$

or $$\omega = \sqrt{\frac{2s^2}{2sm - r^2}}$$

Thus the acceleration amplitude is maximum when

$$\omega = \sqrt{\frac{2s^2}{2sm - r^2}}$$

Taking this value of ω as ω', we have

$$\text{Maximum acceleration amplitude} = \frac{F_0\,\omega'}{\left[r^2 + \left(\omega' m - \dfrac{s}{\omega'}\right)^2\right]^{1/2}}$$

(*a*) **Maximum acceleration amplitude at high frequency**

The acceleration amplitude $= \dfrac{F_0\,\omega}{Z_m}$ where

$$Z_m = \sqrt{r^2 + \left(\omega m - \frac{s}{\omega}\right)^2}$$

For large value of $\omega, \dfrac{s}{\omega}$ can be neglected as compared to $\omega\, m$, r^2 as compared to $\omega^2\, m^2$ and we get $Z_m = \omega\, m$.

$\therefore$ For large frequencies maximum acceleration amplitude

$$= \frac{F_0\,\omega}{Z_m} = \frac{F_0\, m}{\omega\, m} = \frac{F_0}{m}$$

i.e., the maximum acceleration amplitude depends upon the driving force and the mass m but is independent of frequency.

(*b*) **When damping constant is very small**

The acceleration amplitude resonance frequency is given by

$$\omega' = \sqrt{\frac{2s^2}{2sm - r^2}}$$

When damping constant r is very small, r^2 can be neglected as compared to 2 sm and we get

$$\omega' = \sqrt{\frac{2s^2}{2sm}} = \sqrt{\frac{s}{m}} = \text{natural frequency of forced oscillator.}$$

Hence when damping constant is very small the acceleration amplitude resonance frequency equals the natural frequency of the forced oscillator.

7.27 SHARPNESS OF RESONANCE

The displacement amplitude of a forced oscillator is given by

$$a = \frac{f_0}{\sqrt{4b^2\omega^2 - (\omega^2 - \omega_0^2)^2}}$$

At resonance the frequency of the applied force is equal to the frequency of the oscillator *i.e.*

$$\omega = \omega_0 \text{ or } (\omega^2 - \omega_0^2) = 0$$

$\therefore$ Amplitude of forced vibration at resonance $= \frac{f_0}{2b\omega}$ where f_0 is the applied force per unit mass, $\frac{\omega}{2\pi}$ the frequency of the applied force and b the damping coefficient.

As the frequency of the applied force ω is increased or decreased from its resonant value ω_0 the value of the amplitude always decreases.

When the amplitude at resonance falls rapidly as the frequency ω of the applied force is changed slightly from its resonant value the resonance is said to be *sharp*.

When the amplitude at resonance falls gradually as the frequency ω of the applied force is changed slightly from its resonant value the resonance is said to be flat.

The sharpness of resonance depends upon damping.

Effect of damping on sharpness of resonance.

(*i*) When $b = 0$ *i.e.*, there is no damping, the amplitude at resonance becomes infinite as shown in Fig. 7.11 (*i*).

(*ii*) When damping is small the amplitude is sufficiently large but falls off rapidly as the frequency of the applied force becomes slightly different from the natural frequency of the body as shown in Fig. 7.11 (*ii*).

(*iii*) As the value of b increases the amplitude at resonance decreases, but the curves in Fig. 7.11 (*iii*) and (*iv*) fall of gradually and even when the frequency of the applied force is a little different from the natural frequency of the body the amplitude is nearly the same. *In other words, when damping is small resonance is sharp and when damping is large resonance is flat.*

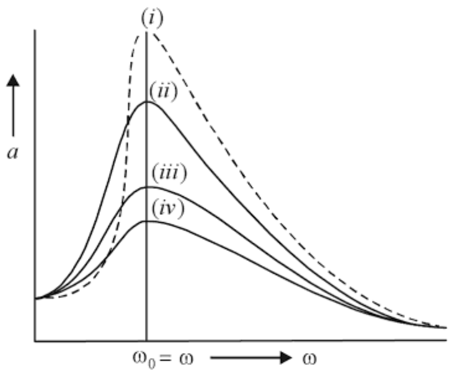

Fig. 7.11

Amplitude. The amplitude of the forced oscillator at resonance becomes very large. This is because resonance is due to the accumulated effect of a number of successive impulses. Each impulse imparts energy to the forced oscillator at the proper time in the right direction *i.e.*, the forced oscillator and the applied force *are in phase*. This happens when there is almost zero damping.

If the damping is large as *in the case of an air column* the natural vibrations die out quickly and by the time the effect of, say, the 10th impulse is coming that of the first almost disappears. It means there is no chance for the impulse to go out of phase and decrease the amplitude. For maximum amplitude in such a case the impressed force should be in phase for 10 vibrations only and slight amount of mistuning will have no effect.

Example :

If the damping is small *as in the case of tuning fork* the natural vibrations die out very slowly and when the 10th impulse comes the effect of the first still persists. The impulses will get out of phase and destroy each other's effect even if there is a small difference in frequency. The tuning, therefore, should be sharp so that the impressed force and the natural vibrations remain practically in phase.

7.28 POWER SUPPLIED BY DRIVING FORCE

When an oscillator is set into forced vibrations by an external driving force some energy is dissipated in every cycle due to damping resistance r. To maintain the steady state of the forced oscillator this energy loss has to be supplied by the driving force.

When a damped simple harmonic oscillator is acted upon by an external periodic driving force $F = F_0 \cos \omega t$, the velocity of the forced oscillator is given by

$$\frac{dy}{dt} = v = \frac{F_0}{Z_m}\cos(\omega t - \phi)$$

where $$Z_m = \sqrt{r^2 + \left(\omega m - \frac{s}{\omega}\right)^2}$$

and $$\tan\phi = \frac{\left(\omega m - \frac{s}{\omega}\right)}{r}$$

The work done by the driving force F in producing a displacement y is given by

$$W = F\,y$$

∴ The instantaneous power supplied to the oscillator

$$P = \frac{dW}{dt} = F\frac{dy}{dt} = Fv$$

= Instantaneous force × instantaneous velocity

$$= F_0 \cos\omega t \times \frac{F_0}{Z_m}\cos(\omega t - \phi)$$

∴ Work done in a time $dt = \dfrac{F_0^2}{Z_m}\cos\omega t\cos(\omega t - \phi)\,dt$

or Work done in a time T where T is the time period of the oscillator

$$= \frac{F_0^2}{Z_m}\int_0^T \cos\omega t\cos(\omega t - \phi)\,dt$$

$\therefore$ Average power over one complete cycle

$$P_{(av)} = \frac{F_0^{\,2}}{Z_m T}\int_0^T [\cos\omega t \cos(\omega t - \phi)\,dt$$

$$= \frac{F_0^{\,2}}{Z_m T}\int_0^T [\cos^2\omega t\cos\phi + \cos\omega t\sin\omega t\sin\phi]\,dt$$

$$= \frac{F_0^{\,2}}{Z_m T}\left[\cos\phi\int_0^T \cos^2\omega t\,dt + \sin\phi\int_0^T \cos\omega t\sin\omega t\,dt\right]$$

$$= \frac{F_0^{\,2}}{2Z_m}\cos\phi$$

$\because$ $$\frac{1}{T}\int_0^T \cos^2\omega t\,dt = \frac{1}{T}\int_0^T \frac{\cos 2\omega t + 1}{2}\,dt$$

$$= \frac{1}{T}\int_0^T \frac{\cos 2\omega t}{2}\,dt + \frac{1}{T}\int_0^T \frac{1}{2}\,dt = \frac{1}{2}$$

because $$\left[\int_0^T \frac{\cos 2\omega t}{2}\,dt = 0\right]$$

and $$\int_0^T (\cos\omega t\sin\omega t)\,dt = \int_0^T \frac{\cos 2\omega t}{2} = 0$$

$\therefore$ $$\sin\phi\int_0^T \cos\omega t\sin\omega t\;dt = 0$$

$\therefore$ Average power supplied by the driving force to the oscillator

$$P_{(av)} = \frac{F_0^{\,2}}{2Z_m}\cos\phi \qquad \ldots (i)$$

The quantity $\cos\phi$ is known as '*power factor*'.

Graphical representation

The average power supplied to the forced oscillator is given by

$$P_{(av)} = \frac{F_0^2}{2Z_m}\cos\phi$$

where $Z_m = \sqrt{r^2 + \left(\omega m - \frac{s}{\omega}\right)^2}$ and $\cos\phi = \frac{r}{Z_m} = \frac{r}{\sqrt{r^2 + \left(\omega m - \frac{s}{\omega}\right)^2}}$

when $\cos\phi = 1$ *i.e.*, when $\left(\omega m - \frac{s}{\omega}\right) = 0,\ \omega^2 = \frac{s}{m} = \omega_0^2$

which is the case of resonance, the average power $P_{(av)}$ is maximum. The average power $P_{(av)}$ depends upon Z_m and in turn on frequency ω. We shall consider the variation of $P_{(av)}$ with ω in three different cases.

1. At low frequency. At low frequency $\omega \to 0$

$$\therefore \quad m\omega \to 0, \frac{s}{\omega} \to \infty$$

$\therefore$ Z_m tends to be very large and hence the average power $P_{(av)}$ is small.

For $\omega = 0$, $Z_m = \infty$ and $P_{(av)} = 0$

2. At high frequency. At high frequency $\omega \to \infty$

$$\therefore \quad m\omega \to \infty, \frac{s}{\omega} \to 0$$

$\therefore$ Z_m again tends to be very large and hence the average power $P_{(av)}$ is small.

3. At resonant frequency. At resonant frequency $\omega = \omega_0$

$$\therefore \quad Z_m = r \text{ and } \cos\phi = \frac{r}{Z_m} = \frac{r}{r} = 1$$

Thus the average power at resonant frequency is maximum and given by

$$P_{(av)} = \frac{F_0^2}{2Z_m} = \frac{F_0^2}{2r}$$

The variation of $P_{(av)}$ with driving force frequency is shown in Fig. 7.12. At $\omega = 0$, $P_{(av)} = 0$ and when ω is very large $P_{(av)}$ again tends to zero. The peak occurs at $\omega = \omega_0$. At this frequency, the power supplied to the forced oscillator by the driving force is maximum and equal to $\frac{F_0^2}{2r}$. Thus height of the peak and sharpness of its peak at resonance is determined by the damping constant r. The variation of $P_{(av)}$ with ω determines the response of the oscillator to the driving force.

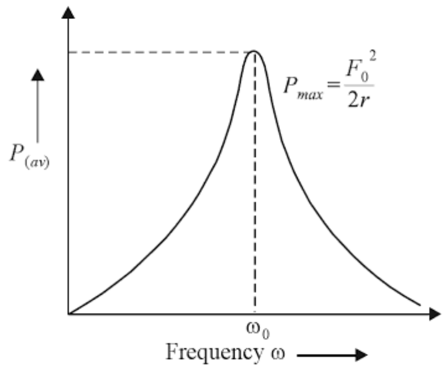

Fig. 7.12

The graph between $P_{(av)}$ and ω is known as *absorption curve of the oscillator.*

The power supplied to an oscillator by the driving force is maximum at resonant frequency *i.e.*, when the frequency of the driving force ω is equal to the natural frequency of oscillation ω_0 of the oscillator. The average power supplied to the oscillator $P_{av} = \frac{F_0^2}{2Z_m} \cos\phi$

where $$Z_m = \sqrt{r^2 + \left(\omega m - \frac{s}{\omega}\right)^2} \text{ and } \cos\phi = \frac{r}{\sqrt{r^2 + \left(\omega m - \frac{s}{\omega}\right)^2}}$$

when $$\omega^2 = \omega_0^2 = \frac{s}{m}, \left(\omega m - \frac{s}{\omega}\right) = 0 \text{ and } \cos\phi = 1,$$

the average power is maximum and given by $\frac{F_0^2}{2Z_m} = \frac{F_0^2}{2r}$.

7.29 POWER SUPPLIED TO FORCED OSCILLATOR

If r is the frictional force per unit velocity, then total frictional force $= rv = r\dfrac{dy}{dt}$.

The rate of working by the oscillator against the frictional force

$$= \text{frictional force} \times \text{instantaneous velocity}$$

$$= \left(r\frac{dy}{dt}\right)\frac{dy}{dt} = r\left(\frac{dy}{dt}\right)^2 = rv^2$$

$$= r\frac{F_0^2}{Z_m^2}\cos^2(\omega t - \phi)$$

as $\quad v = \dfrac{F_0}{Z_m}\cos(\omega t - \phi)$ when driving force is $F = F_0 \cos \omega t$.

The average value of $\cos^2(\omega t - \phi)$ over one period is given by

$$\frac{1}{T}\int_0^T \cos^2(\omega t - \phi)\,dt = \frac{1}{T}\int_0^T \frac{[1 - \cos 2(\omega t - \phi)]}{2}\,dt$$

$$= \frac{1}{T}\left[\frac{1}{2}\int_0^T dt - \frac{1}{2}\int_0^T \cos 2(\omega t - \phi)\right] = \frac{1}{2}$$

$\therefore$ Average loss of power $= \dfrac{1}{2}\dfrac{F_0^2}{Z_m^2}r$ ...(i)

But $\quad \cos\phi = \dfrac{r}{Z_m}$

Substituting in (ii), we have

Average loss of power $= \dfrac{1}{2}\dfrac{F_0^2}{Z_m}\cos\phi$ ...(ii)

Thus, comparing (i) of article (7.28) and (ii) of article (7.29), we have

'*Average power supplied by the driving force to the oscillator is equal to the average power dissipated against the frictional force or* the *average work done per second against the resistive or damping force*'.

Note. The students are well advised to compare the above result with the power in an alternating current circuit which is given by $VI\cos\phi$ where V and I are the r.m.s. values of voltage and current respectively and $\cos\phi$ is the power factor.

As $\quad I = \dfrac{V}{Z_e}$, we have $\quad VI\cos\phi = \dfrac{V^2}{Z_e}\cos\phi$

Moreover $\quad V_{r.m.s.} = \dfrac{V_0}{\sqrt{2}}$

$$\therefore \quad \frac{V^2}{Z_e}\cos\phi = \frac{V_0^2}{2Z_e}\cos\phi$$

Thus applied e.m.f. in the electric circuit corresponds to the applied driving force and electric impedance Z_e to the mechanical impedance Z_m.

Thus, the power supplied by the driving force is not stored in the mechanical oscillator but is used up in doing work against the dissipative frictional resisting forces. The energy supplied by the external periodic force is exactly the same as that lost per cycle due to damping resistance. The energy supplied by the external periodic force helps in the maintenance of the oscillations of forced oscillator. Thus, in the steady state, the amplitude and period of the forced oscillator are so adjusted that the power supplied by the external periodic force is equal to the energy dissipated against the frictional forces.

7.30 ABSORPTION RESONANCE CURVE

The average power supplied to the forced oscillator by the driving force is given by

$$P_{(av)} = \frac{F_0^2}{2Z_m}\cos\phi = \frac{F_0^2 r}{2Z_m^2}$$

where $$Z_m = \sqrt{r^2 + \left(\omega m - \frac{s}{\omega}\right)^2}$$

and $$\cos\phi = \frac{r}{Z_m} = \frac{r}{\sqrt{r^2 + \left(\omega m - \frac{s}{\omega}\right)^2}}$$

The quantity $P_{(av)}$ *i.e.*, the average power thus depends upon Z_m and in turn on ω. It has a maximum value when $\cos\phi = 1$ which gives $\left(\omega m - \frac{s}{\omega}\right) = 0$ so that $\phi = 0$ and $Z_m = r$.

When $\omega m - \frac{s}{\omega} = 0, \omega^2 = \frac{s}{m} = \omega_0^2$ which is the case of *velocity resonance.* Hence $P_{(av)}$ has a maximum value at velocity resonance. It is because when $\phi = 0$, velocity and the applied periodic force are always in phase and hence maximum power is supplied to the oscillator.

$$\therefore \quad P_{(av.max)} = \frac{F_0^2}{2Z_m} = \frac{F_0^2}{2r} \qquad [\because Z_m = r]$$

The variation of $P_{(av)}$ with ω the frequency of the driving force for a forced oscillator is shown in Fig. 7.13. It is known as a $P_{(av)} - \omega$ graph.

The graph between $P_{(av)}$ and ω is called the *absorption resonance curve* of the forced oscillator.

It is so called because the power absorbed by the oscillator is maximum at the frequency of velocity resonance.

Resonance Absorption Band Width. From Fig. 7.13, it is seen that $P_{(av)}$ has a maximum value for $\omega = \omega_0$ as proved above, its value decreases on either side of the maxima.

Let ω_1 and ω_2 be the frequencies for which $P_{(av)} = \frac{1}{2}P_{(av.max)}$.

The condition for $P_{(av)} = \frac{1}{2}P_{(av.max)}$

gives $$\frac{F_0^2 r}{2Z_m^2} = \frac{1}{2}\frac{F_0^2}{2r}$$

$$\therefore \quad Z_m^2 = 2r^2$$

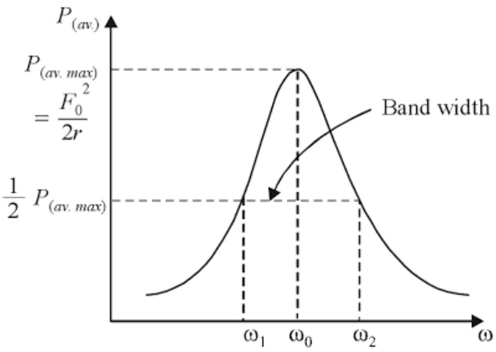

Fig. 7.13

or $$r^2 + \left(\omega m - \frac{s}{\omega}\right)^2 = 2r^2$$

or $$\omega m - \frac{s}{\omega} = \pm r$$

Let ω_1 be less than ω_0 and ω_2 greater than ω_0, then

$$\omega_2 m - \frac{s}{\omega_2} = + r$$

or $$\omega_2{}^2 - \frac{s}{m} = + \frac{r}{m}\omega_2 \qquad ...(i)$$

and $$\omega_1 m - \frac{s}{\omega_1} = - r$$

or $$\omega_1^2 - \frac{s}{m} = - \frac{r}{m}\omega_1 \qquad ...(ii)$$

Subtracting (*ii*) from (*i*), we have

$$\omega_2^2 - \omega_1^2 = \frac{r}{m}(\omega_2 + \omega_1)$$

or $$\omega_2 - \omega_1 = \frac{r}{m}$$

The angular frequency difference $\omega_2 - \omega_1$ is called the bandwidth of the response curve. It is defined as the difference in the angular frequencies below and above the frequency of velocity resonance for which the average power drops to half the maximum value.

The *band width* is shown on a $P - \omega$ graph in Fig. 7.13.

Effect of damping. As band width $(\omega_2 - \omega_1)$ is directly proportional to r the damping constant, the band width increases as the damping constant increases.

7.31 QUALITY FACTOR *Q* IN TERMS OF BAND WIDTH

The quality factor Q of a mechanical oscillator is given by $Q = \frac{m\omega_0}{r}$ where ω_0 is the (angular) frequency of velocity resonance, m the mass and r the damping constant.

Resonance absorption band width $\omega_2 - \omega_1 = \frac{r}{m}$ where ω_1 is less than ω_0 and ω_2 greater than ω_0 for which the average power drops to half the maximum value.

Hence quality factor $$Q = \frac{m\omega_0}{r} = \frac{\omega_0}{\omega_2 - \omega_1} \qquad \left[\because \omega_2 - \omega_1 = \frac{r}{m}\right]$$

$$= \frac{\text{Frequency at resonance}}{\text{Full bandwidth at half maximum power}}$$

$\therefore$ *Q-value of an oscillator is the ratio of frequency of maximum velocity response to the band width at half maximum power.*

Effect of increase in *Q*-value. As $Q = \dfrac{\omega_0}{\omega_2 - \omega_1} = \dfrac{\omega_0}{\text{band width}}$, when Q-value of an oscillator is increased $(\omega_2 - \omega_1)$ *i.e.*, band width decreases because ω_0 is a constant quantity.

As $\quad Q = \dfrac{\omega_0}{\text{band width}}, \qquad \text{band width} = \dfrac{\omega_0}{Q}$

7.32 *Q* IN TERMS OF ENERGY DECAY

The quality factor of a forced oscillator

$$Q = \frac{2\pi \times \text{Average energy stored}}{\text{Energy dissipated per cycle}} = \frac{2\pi E_{av}}{T \times P_{av}}$$

The instantaneous value of energy = $K.E. + P.E.$ at any time.

The
$$K.E. = \frac{1}{2} m \left(\frac{dy}{dt}\right)^2$$

and
$$P.E. = \int_0^y sy\, dy = \frac{1}{2} sy^2$$

∴ Total energy $\quad E = \dfrac{1}{2} m \left(\dfrac{dy}{dt}\right)^2 + \dfrac{1}{2} sy^2$

If the applied driving force is represented by $F = F_0 \cos \omega t$, then for the forced oscillator

$$y = \frac{F_0}{\omega Z_m} \sin(\omega t - \phi)$$

and
$$\frac{dy}{dt} = \frac{F_0}{Z_m} \cos(\omega t - \phi)$$

∴
$$E = \frac{1}{2} m \left(\frac{F_0}{Z_m}\right)^2 \cos^2(\omega t - \phi) + \frac{1}{2}\frac{s}{\omega^2}\left(\frac{F_0}{Z_m}\right)^2 \sin^2(\omega t - \phi)$$

For a complete cycle the average value of $\cos^2(\omega t - \phi)$

$$= \frac{1}{T}\int_0^T \cos^2(\omega t - \phi)\, dt = \frac{1}{2}$$

and the average value of $\sin^2(\omega t - \phi)$

$$= \frac{1}{T}\int_0^T \sin^2(\omega t - \phi)\, dt = \frac{1}{2}$$

∴
$$E_{(av)} = \frac{1}{4} m \left(\frac{F_0}{Z_m}\right)^2 + \frac{1}{4}\frac{s}{\omega^2}\left(\frac{F_0}{Z_m}\right)^2$$

$$= \frac{1}{4} m \left(\frac{F_0}{Z_m}\right)^2 \left[1 + \frac{s}{m\omega^2}\right]$$

$$= \frac{1}{4} m \left(\frac{F_0}{Z_m}\right)^2 \left[1 + \frac{\omega_0^2}{\omega^2}\right]$$

and $$P_{(av)} = \frac{F_0^{\,2}}{2Z_m}\cos\phi = \frac{F_0^{\,2}}{2Z_m}\frac{r}{Z_m} = \frac{F_0^{\,2}}{Z_m^2}\frac{r}{2}$$

$\therefore$ $$Q = 2\pi \frac{E_{(av)}}{T \times P_{(av)}} = \frac{2\pi}{T}\frac{m}{2r}\left(1 + \frac{\omega_0^2}{\omega^2}\right)$$

At resonance $$\omega_0 = \omega \qquad \therefore \quad 1 + \frac{\omega_0^2}{\omega^2} = 2$$

and $$\frac{2\pi}{T} = \omega = \omega_0$$

$\therefore$ $$Q = \frac{m\omega_0}{r}$$

Hence quality factor $Q = \dfrac{2\pi \times \text{Average energy stored}}{\text{energy dissipated per cycle}}$ gives the definition of Q in terms of energy decay.

7.33 FIGURE OF MERIT

It is defined as the ratio of the frequency at velocity resonance (*maximum velocity response*) *to the full bandwidth at half maximum power.*

If ω_0 is the frequency of velocity resonance and $\omega_2 - \omega_1$ the full bandwidth at half maximum power, then

$$\text{Figure of merit} = \frac{\text{Frequency at velocity resonance}}{\text{Full bandwidth at half maximum power}}$$

$$= \frac{\omega_0}{\omega_2 - \omega_1} = \frac{\omega_0 m}{r} = Q \qquad [\because \omega_2 - \omega_1 = r/m]$$

where Q is the quality factor.

Quality factor a measure of sharpness of resonance. The figure of merit (or quality factor) measures the *sharpness of tuning*. Higher the figure of merit for a given resonance frequency smaller is the bandwidth *i.e.*, sharper is the tuning. This fact is used to increase the '*selectivity*' of radio and T.V. sets. The sharpness of response of a circuit allows radio signals (R.F. or V.H.F) to be reproduced without any interference from other signals of frequencies close to it.

Physical significance of *Q*-value

Q-value represents the sharpness of absorption curve. The larger the value of quality factor Q sharper is the peak value of absorption curve.

Q-value also measures the amplification factor. Thus at displacement resonance the displacement at low frequency is amplified by Q times.

***Q* value as Amplification Factor.** The displacement at resonance is given by

$$A_{max} = \frac{F_0}{\omega' r}$$

where $$\omega' = \sqrt{\frac{s}{m} - \frac{r^2}{4m^2}}$$

The amplitude $$A = \frac{f_0}{\sqrt{4b^2\omega^2 + (\omega^2 - \omega_0^2)^2}} \qquad \left[\text{where } f_o = \frac{F_o}{m}\right]$$

when $\omega \to 0$, the amplitude $A_0 = \dfrac{f_0}{\omega_0^2} = \dfrac{f_0}{\frac{s}{m}} = \dfrac{F_0}{s}$.

The ratio of the maximum displacement at resonance to the maximum displacement when the frequency of the applied force approaches zero is known as amplification factor.

Let us study resonance in LCR circuit in detail, along with Q factor. For that we will first write an expression for current in the circuit in the steady state, both when the applied e.m.f. is $E = E_0 \cos \omega t$ and $E = E_0 \sin \omega t$.

7.34 CURRENT IN AN ELECTRIC CIRCUIT

1. Series LCR circuit. Consider an electric oscillator consisting of an electric circuit containing a resistance R, an inductance L and a capacitance C in series as shown in Fig. 7.14 connected to an $A.C.$ source for which the *e.m.f.* is represented by

$$E = E_0 e^{i\omega t}$$

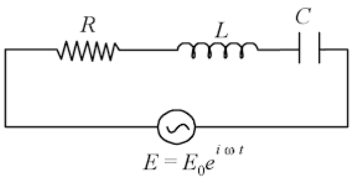

Fig. 7.14

ω being the angular frequency of the applied alternating *e.m.f.* Such a circuit behaves as a driven oscillator. The loss of energy in the resistance is compensated by the supply of energy from the $A.C.$ source just as the loss of energy due to friction in a mechanical driven oscillator is compensated by the driving force.

If I is the current at any instant, then

Fall of potential across the resistance $R = RI$

If the current varies at the rate $\dfrac{dI}{dt}$, then the voltage required to overcome the back *e.m.f.* in the inductance $= L\dfrac{dI}{dt}$

If q is the charge accumulated on the capacitor at that instant, then

Fall of potential across the capacitor $= \dfrac{q}{C}$

Hence the voltage equation at any instant is given by

$$RI + L\frac{dI}{dt} + \frac{q}{C} = E = E_0 e^{i\omega t}$$

Now $$I = \frac{dq}{dt} \text{ and } \frac{dI}{dt} = \frac{d^2q}{dt^2}$$

Hence $$R\frac{dq}{dt} + L\frac{d^2q}{dt^2} + \frac{q}{C} = E_0 e^{i\omega t}$$

or $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E_0\, e^{i\omega t} \qquad ...(i)$$

Steady state. In the steady state let one particular solution of the equation be

$$q = q_0\, e^{i\omega t}$$

where q_0 is a complex quantity.

$$\therefore \qquad \frac{dq}{dt} = i\omega\, q_0\, e^{i\omega t} \quad \text{and} \quad \frac{d^2q}{dt^2} = -\omega^2\, q_0\, e^{i\omega t}$$

Substituting in equation (i), we have

$$\left(-\omega^2 L + i\,\omega R + \frac{1}{C}\right) q_0\, e^{i\omega t} = E_0\, e^{i\omega t}$$

This relation is true for all values of t.

$$\therefore \qquad q_0 = \frac{E_0}{i\,\omega R + \left(\frac{1}{C} - \omega^2 L\right)} = \frac{-E_0}{\omega R - i\left(\frac{1}{C} - \omega^2 L\right)}$$

$$= \frac{-i\,E_0}{\omega\left[R + i\left(L\omega - \frac{1}{C\omega}\right)\right]} = \frac{-i\,E_0}{\omega\, \vec{Z}_e} \qquad ...(ii)$$

where $\vec{Z}_e$ is a complex quantity called the *electric impedance* (or effective resistance) of the circuit, given by

$$\vec{Z}_e = R + i\left(L\omega - \frac{1}{C\omega}\right)$$

R is the *ohmic* resistance of the circuit and $\left(L\omega - \frac{1}{C\omega}\right) = X$ is called the *total reactance*, $L\omega = X_L$ represents the *reactance due to inductance* and $\frac{1}{C\omega} = X_C$ represents the *reactance due to capacitance.*

Now, put $R = Z_e \cos\phi$

and $$\left(L\omega - \frac{1}{C\omega}\right) = Z_e \sin\phi$$

so that $$Z_e = \sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}$$

and $$\tan\phi = \frac{\left(L\omega - \frac{1}{C\omega}\right)}{R}$$

$$\therefore \qquad \vec{Z}_e = Z_e \cos\phi + i\, Z_e \sin\phi$$

$$= Z_e(\cos\phi + i\sin\phi) = Z_e\, e^{i\phi}$$

Substituting in equation (ii), we get

$$q_0 = \frac{-i\,E_0}{\omega\, Z_e\, e^{i\phi}}$$

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But $$q = q_0\, e^{i\omega t}$$

$$\therefore \qquad q = \frac{-\,i\,E_0\,e^{i\omega t}}{\omega\, Z_e\, e^{i\phi}} = \frac{-\,i\,E_0}{\omega\, Z_e}\,e^{i(\omega t - \phi)} \qquad ...(iii)$$

The current I in the electric circuit (similar to velocity v in a mechanical oscillator) is given by

$$I = \frac{dq}{dt}$$

$$\therefore \qquad I = \frac{dq}{dt} = \frac{-i\,E_0\,e^{i(\omega t - \phi)}}{\omega\, Z_e} \times i\,\omega = \frac{E_0}{Z_e}\,e^{i(\omega t - \phi)}$$

$$= I_0\, e^{i(\omega t - \phi)} \qquad ...(iv)$$

where $$I_0 = \frac{E_0}{Z_e}$$

The current I and the applied *e.m.f.* E are not in phase. There is a phase difference of ϕ between applied *e.m.f.* and current.

(*i*) Applied e.m.f. $E = E_0 \cos \omega t$. The applied *e.m.f.* $E = E_0\, e^{i\omega t}$ can be put in the form

$$E = E_0\,(\cos \omega t + i \sin \omega t)$$

$$= E_0 \cos \omega t + i\, E_0 \sin \omega t \qquad ...(v)$$

Thus $E = E_0 \cos \omega t$ is the real part of the complex quantity $E = E_0^{i\omega t}$.

Therefore the steady state solution for the current I will be the real part of equation (*iv*) which may be put in the from

$$I = I_0\, e^{i(\omega t - \phi)} = I_0\,[\cos(\omega t - \phi) + i \sin(\omega t - \phi) \qquad ...(vi)$$

The real part is $I = I_0 \cos(\omega t - \phi)$

Thus when the applied *e.m.f.* is $E = E_0 \cos \omega t$, the current is

$$I = I_0 \cos(\omega t - \phi) = \frac{E_0}{Z_e}\cos(\omega t - \phi) \qquad ...(vii)$$

where $$\tan \phi = \frac{L\omega - \dfrac{1}{C\omega}}{R} \quad \text{and} \quad Z_e = \sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}$$

(*ii*) Applied e.m.f. $E = E_0 \sin \omega t$. When the applied *e.m.f.* is $E = E_0 \sin \omega t$, it is that part of equation (*v*) which is preceded by vector operator i. Therefore, the steady state term for the current I will also be that part of equation (*vi*) which is preceded by vector operator i.

$$\therefore \qquad I = I_0 \sin(\omega t - \phi) = \frac{E_0}{Z_e}\sin(\omega t - \phi)$$

The value of Z_e and ϕ are given in part (*i*) above.

2. Current in a parallel *LC* circuit. In a parallel *L-C* circuit an inductance L having a very low ohmic resistance and a capacitor of capacitance C are connected in parallel. An *A.C.* source of voltage $E = E_0 \sin \omega t$ is connected at the points A and B as shown in Fig. 7.15. Suppose a current i flows through the circuit at any instant. Let i_1 be the current through the inductance L and i_2 through the capacitor C, then

$$i = i_1 + i_2 \qquad ...(i)$$

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Let the current through the inductor L vary at the rate $\frac{di_1}{dt}$, then

$$E - L\frac{di_1}{dt} = 0$$

or
$$L\frac{di_1}{dt} = E = E_0 \sin \omega t$$

$\therefore$
$$i_1 = \frac{E_0}{L}\int \sin \omega t \, dt$$

$$= -\frac{E_0}{\omega L}\cos \omega t$$

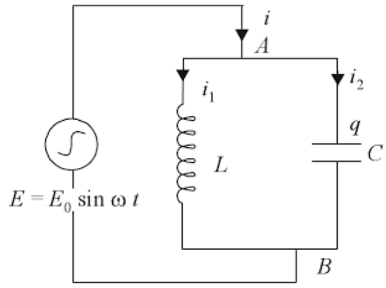

Fig. 7.15

Let the charge on the capacitor at any instant be q, then

$$q = CE = C\,E_0 \sin \omega\, t$$

$\therefore$ Current through the capacitor $i_2 = \frac{dq}{dt} = C\,\omega\,E_0 \cos \omega\, t$

Substituting the values of i_1 and i_2 in (i), we have

$$i = i_1 + i_2 = C\,\omega\,E_0 \cos \omega\, t - \frac{E_0}{\omega L}\cos \omega\, t$$

$$= \left(C\,\omega - \frac{1}{\omega L}\right)E_0 \cos \omega\, t$$

If $C\,\omega = \frac{1}{\omega L}$ the current in the circuit will be zero. This will happen when

$$C\,\omega = \frac{1}{\omega L} \quad \text{or} \quad \omega = \frac{1}{\sqrt{LC}} \quad \text{or} \quad n = \frac{1}{2\pi\sqrt{LC}}$$

i.e., the frequency of the applied $A.C.$ voltage is equal to the natural frequency of the $L.C.$ circuit.

SOLVED EXAMPLES

Example 7.1 A particle of mass 5 kg lies in a potential field $V = 8x^2 + 200$ Joules/kg. Calculate its time period. (*Nagpur U.* 2004)

Solution. Potential Energy (U) of a particle is

$$U = mV$$
$$= m\,(8x^2 + 200)$$

Restoring force (F) acting is given by

$$F = -\frac{dU}{dx}$$

$$= -\frac{d}{dx}(8\text{m}^2 + 200\text{ m})$$

$$= -16\,mx = -kx$$

As $F \propto -x$, the motion is S.H.M.

$\therefore$ Angular frequency, $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{16\,m}{m}} = 4$

and Time period, $T = \frac{2\pi}{\omega} = \frac{2\pi}{4} = 1.57$ sec.

Example 7.2 A mass of 25 × 10^{-2} kg is suspended from the lower end of a vertical spring having a force constant 25 Nm^{-1}. What should be the damping constant of the system so that the motion is critically damped? (*Nagpur U.* 2004, 2001)

Solution. Here $m = 25 \times 10^{-2}$ kg; Force (spring) constant, $s = 25$ Nm^{-1}

Damping constant r = ?

For critical damping $\frac{r^2}{4m^2} - \frac{s}{m} = 0$

or $r^2 = 4ms$ or $r = 2\sqrt{ms} = 2\sqrt{25 \times 10^{-2} \times 25}$

$= 2 \times 25 \times 10^{-1} = 5$ kg s^{-1}

Example 7.3 A mass 25 × 10^{-3} kg is suspended from the lower end of a vertical spring having a force constant 25 N/m. The mechanical resistance of the system is 1.5 Ns/m. The mass is displaced vertically and released. Find whether the motion is oscillatory? If so, calculate its period of oscillation.

Solution. Given : $m = 25 \times 10^{-3}$ kg, $s = 25$ N/m, $r = 1.5$ Ns/m

$$b^2 = \frac{r^2}{4m^2} = \frac{(1.5)^2}{4 \times (25 \times 10^{-3})^2} = \frac{2.25}{25 \times 10^{-4}} = 900$$

and $\omega^2 = \frac{s}{m} = \frac{25}{25 \times 10^{-3}} = 10^3 = 1000$

Since, $b^2 < \omega^2$, the motion is oscillatory.

$\therefore$ Period, $T = \frac{2\pi}{\sqrt{\frac{s}{m} - \frac{r^2}{4m^2}}} = \frac{2 \times 3.14}{\sqrt{1000 - 900}} = \frac{6.28}{\sqrt{100}} = \frac{6.28}{10} = 0.628$ s.

Example 7.4 A mass of 1 kg is suspended from a spring of stiffness constant 25 N m^{-1}. If the undamped (or natural) frequency is $\frac{2}{\sqrt{3}}$ times the damped frequency, calculate the damping factor (or constant).

Solution. Damped frequency $f' = \frac{1}{2\pi}\sqrt{\frac{s}{m} - \frac{r^2}{4m^2}}$

and undamped frequency $f_0 = \frac{1}{2\pi}\sqrt{\frac{s}{m}}$

Now $\frac{f_0}{f'} = \frac{2}{\sqrt{3}}$

or $\frac{\sqrt{\frac{s}{m}}}{\sqrt{\frac{s}{m} - \frac{r^2}{4m^2}}} = \frac{2}{\sqrt{3}}$ or $\frac{\frac{s}{m}}{\frac{s}{m} - \frac{r^2}{4m^2}} = \frac{4}{3}$

or $$\frac{3}{4}\frac{s}{m} = \frac{s}{m} - \frac{r^2}{4m^2}$$

or $$\frac{r^2}{4m^2} = \frac{1}{4}\frac{s}{m}$$

$\therefore$ $$r^2 = sm$$

Given $$s = 25\ \text{Nm}^{-1} \quad \text{and} \quad m = 1\ \text{kg}$$

$$r^2 = 25\,\text{Nm}^{-1}\ \text{kg} = 25\,\text{kg}^2\,\text{s}^{-2} \qquad [\because\ \text{N} = \text{kg m s}^{-2}]$$

$\therefore$ $$r = 5\,\text{kg s}^{-1}$$

Example 7.5 Show that the unit of damping term (or damping coefficient) *b* is s^{-1}.
(*G.N.D.U.* 2001; *H.P.U.* 2002;)

Solution. For a damped harmonic oscillator the equation of motion is

$$m\frac{d^2y}{dt^2} + r\frac{dy}{dt} + sy = 0$$

or $$\frac{d^2y}{dt^2} + \frac{r}{m}\frac{dy}{dt} + \frac{s}{m}y = 0$$

Substituting $\frac{r}{m} = 2b$ and $\frac{s}{m} = \omega^2$, we get

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega^2 y = 0$$

$\therefore$ The unit of $r\frac{dy}{dt}$ is that of force *i.e.*, Newton

$\therefore$ Unit of damping constant r is $\dfrac{\text{Force}}{\text{velocity}} = \text{Nm}^{-1}\,\text{s} = \text{kg m s}^{-2}\,\text{m}^{-1}\,\text{s} = \text{kg s}^{-1}$

Hence unit of damping (coefficient) or term $b = \dfrac{r}{m} = \dfrac{\text{kg s}^{-1}}{\text{kg}} = \text{s}^{-1}$ *i.e.* b has the dimensions of frequency.

Example 7.6 In an oscillatory circuit L = 0.5 H, C = 1.8 μfd what is the maximum value of resistance to be connected so that the circuit may produce oscillations. (*Nagpur U.*, 2006)

Solution. $L = 0.5$ H $\quad C = 1.8\ \mu\text{fd} = 1.8 \times 10^{-6}$ F.

Let R be the maximum resistance for which the discharge is oscillatory. For the circuit to produce oscillations,

$$\frac{R^2}{4L^2} = \frac{1}{LC}$$

or $$R = 2\sqrt{\frac{L}{C}} = 2 \times \sqrt{\frac{0.5}{1.8 \times 10^{-6}}} = 1054\ \text{ohm}$$

Example 7.7 The frequency of an underdamped harmonic oscillator is adjusted to be equal to half the frequency experienced by the oscillator without damping. Calculate the logarithmic decrement of the system.

Solution. Let the frequency of the harmonic oscillator without damping be ω_0 and the frequency of the underdamped harmonic oscillator be ω', then

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$$\omega' = \frac{1}{2}\omega_0$$

Now $\omega' = \sqrt{\omega_0^2 - b^2}$ where b is the *damping coefficient.*

$\therefore$ $\frac{1}{2}\omega_0 = \sqrt{\omega_0^2 - b^2}$ or $\frac{\omega_0^2}{4} = \omega_0^2 - b^2$

or $b^2 = \omega_0^2 - \frac{\omega_0^2}{4} = \frac{3}{4}\omega_0^2$

$$b = \frac{\sqrt{3}}{2}\omega_0 = \sqrt{3}\,\omega'$$

Now logarithmic decrement $\lambda = bT$ and $T = \frac{2\pi}{\omega'}$

$\therefore$ $\lambda = \sqrt{3}\,\omega' . \frac{2\pi}{\omega'} = 2\pi\sqrt{3}$

Example 7.8 A damped vibrating system, starting from rest, reaches a first amplitude of 500 mm which reduces to 50 mm in that direction after 100 oscillations each of period 2.3 sec. Find the damping constant, relaxation time and the correction for the first displacement for damping [$\log_e^{10}$ = 2.3]. (*Indore U.* 2001)

Solution. The damping constant b is given by the relation

$$\log_e \frac{A_1}{A_2} = \log_e \frac{A_2}{A_3} = ... = b$$

where A_1, A_2, A_3 etc. are the successive amplitudes and T the time period. The amplitude n vibrations after the first *i.e.*, $n + 1$ vibration is A_{n+1}, then

$$\log_e \frac{A_1}{A_{n+1}} = \log_e \frac{A_1}{A_2} \times \log_e \frac{A_2}{A_3} \times ... \log_e \frac{A_n}{A_{n+1}} = nkT$$

Now A_1 = 500 mm; A_{101} = 50 mm; n = 100; T = 2.3 sec

$\therefore$ $\log_e \frac{500}{50} = 100b \times 2.3$

or $2.3 \log_{10} 10 = 2.3 \times 100\, b$

or $100\, b = 1$

$\therefore$ Damping constant $b = \frac{1}{100} = .01$

Relaxation time $t_r = \frac{1}{b} = \frac{1}{.01} = 100$ Sec .

Logarithmic decrement $\lambda = bT = .01 \times 2.3 = 0.023$

The amplitude A_0 which could be obtained if the damping force were absent, is given by

$$A_0 = A_1\left(1 + \frac{\lambda}{4}\right) = 500\left(1 + \frac{1}{4} \times .01 \times 2.3\right)$$
$$= 502.875$$

$\therefore$ Correction for the first displacement = 2.875 mm.

Example 7.9 Deduce the frequency and quality factor of an LCR circuit with L = 2mH, C = 5μF and R = 0.2 ohm. (*G.N.D.U.* 2003; *P.U.* 2004, 2003)

Solution. Here $L = 2mH = 2\times 10^{-3}\,H$; $C = 5\mu\text{F} = 5\times 10^{-6}$ F

$R = 0.2\ \Omega = 2\times 10^{-1}\ \Omega$

$$\text{Angular frequency } \omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

$$= \sqrt{\frac{1}{2\times 10^{-3}\times 5\times 10^{-6}} - \frac{2\times 2\times 10^{-2}}{4\times 4\times 10^{-6}}}$$

$$= \sqrt{10^8 - 25\times 10^2} = 10^4 \text{ rad s}^{-1} \qquad \text{(approx)}$$

$$\text{Frequency } \nu = \frac{\omega'}{2\pi} = \frac{10^4}{2\pi} = \frac{10}{2\pi}\times 10^3 = 1.59\times 10^3 = 1590 \text{ Hz}$$

and quality factor $Q = \dfrac{L\omega'}{R} = \dfrac{2\times 10^{-3}\times 10^4}{2\times 10^{-1}} = 10^2 = 100$

Example 7.10 A simple pendulum has a period of 1 sec. and an amplitude of 10°. After 10 complete oscillations its amplitude is reduced to 5°. What is the relaxation time of the pendulum and quality factor?

Solution. For a damped simple harmonic oscillator, the amplitude is given by

$$A = A_0 e^{-bt} = A_0 e^{-\frac{r}{2m}t} \qquad \text{...}(i)$$

Here $A_0 = 10°;\ A = 5°$

after 10 complete oscillations *i.e.*, after a time = 10 × Times period = 10 × 1 = 10 sec. Substituting in (*i*)

$$5 = 10\, e^{-10b}$$

$$\therefore \qquad e^{-10b} = \frac{1}{2} \qquad \therefore \quad e^{10b} = 2$$

$$\text{or} \qquad b = \frac{\log_e 2}{10} = \frac{0.6931}{10} = 0.06931$$

$$\text{The relaxation time} \quad t_r = \frac{1}{b} = \frac{1}{0.06931} = 14.428 \text{ sec}$$

$$\text{Quality factor} \quad Q = \omega\frac{m}{r} = \frac{\omega}{2b} \quad \text{where} \quad \omega = \frac{2\pi}{T} = 2\pi \qquad [\because T = 1s]$$

$$\therefore \qquad Q = \frac{2\pi}{T\times 2b} = \frac{\pi}{bT} = \frac{\pi}{b} = \frac{\pi}{0.06931} = 45.33$$

Example 7.11 A condenser of capacity 1 μF, an inductance of 0.2 Henry and a resistance of 800 ohm are connected in series. Is the circuit oscillatory? If yes, calculate the frequency and quality factor of the circuit. (*G.N.D.U.* 2002; *H.P.U.* 2002)

Solution. Here $C = 1\ \mu\text{F} = 10^{-6}$ F; $L = 0.2$ H; $R = 800\ \Omega$

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For the circuit to be oscillatory $\frac{R^2}{4L^2} < \frac{1}{LC}$

Now $\frac{R^2}{4L^2} = \frac{800 \times 800}{4 \times 0.2 \times 0.2} = 4 \times 10^6$

and $\frac{1}{LC} = \frac{1}{0.2 \times 10^{-6}} = 5 \times 10^6$

As $\frac{R^2}{4L^2} < \frac{1}{LC}$, the circuit is oscillatory.

The angular frequency $\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \sqrt{5 \times 10^6 - 4 \times 10^6}$

$= 10^3$ rad s^{-1}

Hence frequency $f' = \frac{\omega'}{2\pi} = \frac{10^3}{2\pi} = 159$ s^{-1}

and quality factor $Q = \frac{L\omega'}{R} = \frac{0.2 \times 10^3}{800} = 0.25$

Example 7.12 A damped oscillator consists of a mass 200 gm attached to a spring of constant 100 Nm^{-1} and damping constant 5 Nm^{-1}s. It is driven by a force $F = 6 \cos \omega t$ Newton, where $\omega = 30$ s^1. If displacement in steady state is $x = A \sin (\omega t - \phi)$ metre, find A and ϕ. Also calculate the power supplied to the oscillator. (*Pbi.U.*, 2003; *P.U.*, 2001, 2000)

Solution. Amplitude of the driving force $F_0 = 6$ N

Frequency of the driving force $\omega = 30$ s^{-1}

Mass of damped oscillator $m = 200$ gm $= 0.2$ kg

Spring constant $s = 100$ N m^{-1}

Damping constant $r = 5$ N m^{-1} s

When an external force $F = F_0 \cos \omega t$ acts on a damped oscillator the steady state is given by

$$x = A \sin (\omega t - \phi)$$

where $A = \frac{F_0}{\omega Z_m}$, Z_m being the 'impedance' of the mechanical system given by

$$Z_m = \sqrt{r^2 + \left(\omega m - \frac{s}{\omega}\right)^2}$$

and $$\tan \phi = \frac{\omega m - \frac{s}{\omega}}{r}$$

$\therefore$ $$\tan \phi = \frac{30 \times 0.2 - \frac{100}{30}}{5} = 0.534$$

$\therefore$ $$\phi = 28° \, 6' \text{ and } \cos \phi = 0.8821$$

$$Z_m = \sqrt{(5 \times 5) + \left(30 \times 0.2 - \frac{100}{30}\right)^2} = 5.67 \,\text{N m}^{-1}\,\text{s}$$

$$\therefore \qquad A = \frac{F_0}{\omega Z_m} = \frac{6}{30 \times 5.67} = 0.0352\,\text{m} = 3.52\,\text{mm}$$

$$\text{Average power supplied} = \frac{F_0^2}{2Z_m}\cos\phi = \frac{6 \times 6}{2 \times 5.67} \times 0.8821 = 2.8\,\text{watt}$$

Example 7.13 If the resonant (angular) frequency of acoustic system is 280 Hz and half power frequencies are 200 Hz and 360 Hz respectively, calculate the quality factor.

(*Nagpur U.*, 2002)

Solution. Here $\omega_0 = 280$ Hz, $\omega_1 = 200$ Hz, $\omega_2 = 360$ Hz

$$\therefore \quad \text{Quality factor} \quad Q = \frac{\omega_0}{\omega_2 - \omega_1} = \frac{280}{360 - 200} = \frac{280}{160} = 1.75$$

Example 7.14 Light of wavelength 6 × 10^{-5} cm is emitted by an electron in an atom (a damped simple harmonic oscillator) with a quality factor 3 × 10^{6}. Find the width of the spectral line from such an atom from resonance bandwidth.

Solution. The quality factor Q is given by

$$Q = \frac{\omega_0}{\omega_2 - \omega_1}$$

where ω_0 is the angular frequency of maximum velocity response (resonance) and $\omega_2 - \omega_1$ is the bandwidth in terms of angular frequency at half maximum power.

Quality factor in terms of wavelength. The above relation can be put in terms of wavelength as under. Let ν_1, ν_2 and ν_0 be the frequencies and λ_1, λ_2 and λ_0 the wavelengths corresponding to angular frequencies ω_1, ω_2 and ω_0, then

$$\frac{\omega_0}{\omega_2 - \omega_1} = \frac{2\pi\nu_0}{2\pi\nu_2 - 2\pi\nu_1} = \frac{\nu_0}{\nu_2 - \nu_1}$$

$$= \frac{\dfrac{c}{\lambda_0}}{\dfrac{c}{\lambda_2} - \dfrac{c}{\lambda_1}} = \frac{\dfrac{1}{\lambda_0}}{\dfrac{1}{\lambda_2} - \dfrac{1}{\lambda_1}} \qquad \left[\because \nu = \frac{c}{\lambda}\right]$$

$$= \frac{1}{\lambda_0}\frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2} = \frac{\lambda_0}{\lambda_1 - \lambda_2}$$

since $\lambda_1\lambda_2 = \lambda_0^2$ (approx.). Hence $Q = \dfrac{\lambda_0}{\lambda_1 - \lambda_2}$

$$\therefore \quad \text{Bandwidth} \qquad \lambda_1 - \lambda_2 = \frac{\lambda_0}{Q}$$

Now $\lambda_0 = 6 \times 10^{-5}$ cm $= 6 \times 10^{-7}$ m and $Q = 3 \times 10^6$

$$\therefore \quad \text{Bandwidth} = \frac{\lambda_0}{Q} = \frac{6 \times 10^{-7}}{3 \times 10^6} = 2 \times 10^{-13}\,\text{m} = 2 \times 10^{-3}\,\text{Å}$$

Example 7.15 The line width of orange line of Kr^{86} at λ = 6058 Å is found to be 5.50 × 10^{-3} Å. Calculate (*a*) line frequency (*ii*) line width in Hertz.

Solution. Line wavelength λ_0 = 6058 Å = 6058 × 10^{-10} m

$$\therefore \quad \text{Line frequency } \nu_0 = \frac{c}{\lambda_0} = \frac{3\times 10^8}{6058\times 10^{-10}} = 4.95\times 10^{14} \text{ Hertz}$$

Line width in wavelength $\lambda_1 - \lambda_2$ = 5.50 × 10^{-3} Å = 5.50 × 10^{-13} m

$$\therefore \quad \text{Line width in terms of frequency } = \frac{c}{\lambda_1 - \lambda_2}$$

$$= \frac{3\times 10^8}{5.50\times 10^{-13}} = 5.45\times 10^{20} \text{ Hz}$$

Example 7.16 A root mean square voltage of 100 volts is applied to a series *LCR* circuit, having *R* = 10 ohm, *L* = 10 mH and *C* = 1 μ F. Calculate

(*i*) The natural frequency

(*ii*) Current at resonance

(*iii*) *Q* value of the circuit at resonance

(*iv*) Bandwidth of the circuit.

Solution.

$$(i) \text{ Natural frequency } \nu = \frac{1}{2\pi\sqrt{LC}} = 1592 \text{ Hz}$$

$$(ii) \text{ Current at Resonance } I_{max} = \frac{V_{max}}{R} = \frac{\sqrt{2}V_{rms}}{R} = \frac{\sqrt{2}\times 100}{10} = 14.14 \text{ A.}$$

$$(iii)\ Q \text{ value } = \omega_0 \frac{L}{R} = \frac{1}{\sqrt{LC}}\frac{L}{R} = \sqrt{\frac{L}{C}}\frac{1}{R} = \sqrt{\frac{10\times 10^{-3}}{1\times 10^{-6}}}\times\frac{1}{10} = 10$$

(*iv*) Bandwidth $\omega_2 - \omega_1$

$$Q = \frac{\omega_0}{\omega_2 - \omega_1}$$

$$\therefore \qquad \omega_2 - \omega_1 = \frac{\omega_0}{Q} = \frac{1}{\sqrt{LC}}\cdot\frac{1}{Q} = \sqrt{\frac{1}{10\times 10^{-3}\times 10^{-6}}}\times\frac{1}{10}$$

$$= 1000 \text{ rad s}^{-1}$$

Example 7.17 The equation of motion is

$$2\times 10^{-4}\frac{d^2x}{dt^2} + 4\times 10^{-2}\frac{dx}{dt} + 5x = 0.124 \sin 100\ t.$$

where all quantities are in S.I. units.

Find. (*i*) Natural frequency of undamped oscillation.

(*ii*) Mechanical impedance. (*Nagpur U. w*/2009, 2004)

Solution. Comparing with

$$m\frac{d^2x}{dt^2}+r\frac{dx}{dt}+sx = F_0 \sin \omega_0 t$$

we get, $m = 2 \times 10^{-4}$ kg, $r = 4 \times 10^{-2}$ Nsm^{-1}, $s = 5$ Nm^{-1}, $F_0 = 0.124$ N

and $\omega_0 = 100$ rad s^{-1}.

(*i*) Natural frequency of undamped oscillation is

$$F_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{s}{m}}$$

$$= \frac{1}{2\times 3.14}\sqrt{\frac{5}{2\times 10^{-4}}} = \frac{10^2}{6.28}\sqrt{\frac{5}{2}} = 25.16 \text{ Hz}$$

(*ii*) Mechanical impedance (z):

$$\text{Amplitude,} \quad a = \frac{f_0}{\sqrt{4b^2\omega_0^2+(\omega^2-\omega_0^2)^2}} = \frac{F_0}{m\sqrt{4b^2\omega_0^2+(\omega^2-\omega_0^2)}} = \frac{F_0}{z}$$

$\therefore$ Mechanical impedance, $z = m\sqrt{4b^2\omega_0^2+(\omega^2-\omega_0^2)^2}$

Here $\omega^2-\omega_0^2 = \dfrac{s}{m}-\omega_0^2$

$$= \frac{5}{2\times 10^{-4}}-(10^2)^2 = 1.5\times 10^4$$

and $4b^2\omega_0^2 = (2b)^2\omega_0^2 = \left(\dfrac{r}{m}\right)^2\omega_0^2 = \left(\dfrac{4\times 10^{-2}}{4\times 10^{-4}}\right)\times(10^2) = 4\times 10^8$

$\therefore$ $z = 2\times 10^{-4}\sqrt{4\times 10^8+2.25\times 10^8} = 2\times 10^{-4}\sqrt{10^8(4+2.25)}$

$$= 2\times 10^{-4}\times 10^4\times\sqrt{6.25} = 2\times 2.5 = 5 \text{ Nsm}^{-1}$$

Example 7.18 A harmonic oscillator consisting of 50 gm mass attached to a mass less spring has a quality factor 200. If it oscillates with an amplitude of 2 cm in resonance with a periodic force of frequency 20 Hz. Calculate (*i*) the average energy stores in it and (*ii*) the rate of dissipation of energy. (*Nagpur U.* 2007)

Solution. Here, the average energy stored in the oscillator is equal to its maximum potential energy $= \dfrac{1}{2}cx^2 = \dfrac{1}{2}m\omega_0^2 x_0^2$

Here, $m = 50$ gm, $\omega_0 = 2\pi n = 2\pi\times 20 = 40\pi$, and $x_0 = 2$ cm.

(*i*) The average energy stored or absorbed in the oscillator

$$= \frac{1}{2}\times 50\times(40\pi)^2\times(2)^2 = 1.58\times 10^6 \text{ ergs.}$$

(*ii*) By definition, the quality factor

$$Q = 2\pi\frac{\text{Average energy stored}}{\text{Energy dissipated per cycle}}$$

$\therefore$ Energy dissipated per cycle = $2\pi \times \dfrac{\text{Average energy stored}}{Q}$

$$= 2\pi \times \frac{1.58 \times 10^6}{200} = \pi \times 1.58 \times 10^4 \text{ Ergs.}$$

Since there are 20 cycles per cycle, we have energy dissipated per second or rate of dissipated, of energy

$$= 20 \times \pi \times 1.58 \times 10^4 = 9.926 \times 10^5 \cong 10^6 \text{ erg/sec}$$

EXERCISE CH. 7

LONG QUESTIONS

1. What do you mean by damping? Prove that the damping force is independent of acceleration or displacement and is proportional to velocity.
(*H.P.U.* 2003, 2000; *G.N.D.U.* 2000)

2. **(*a*)** What are damped vibrations? Establish the differential equation of motion for a damped harmonic oscillator and obtain an expression for displacement. Discuss the case of heavy damping, and light damping. (*P.U.* 2003; *Nagpur U.* 2002, 2001; *Kerala U.* 2001; *Pbi. U.* 2001; *H.P.U.* 2000; *Purvanchal U.* 2007, 2006, 2004)

(*b*) Using the general solution of equation of damped simple harmonic motion discuss the case of critical damping. (*Nagpur U.* 2001; *P.U.*)

3. Write down the equation of damped simple harmonic oscillator. Find the expression for displacement and discuss when we get oscillatory damped simple harmonic motion.
(*P.U.* 2003; *Luck. U.* 2002; *Nagpur U.* 2001; *Indore U.* 2001; *Purvanchal U.* 2004; *H.P.U.* 2000)

4. Show that $x = (A + Bt)\, e^{\frac{-r}{2m}t}$ is the solution of the differential equation $\dfrac{d^2x}{dt^2} + \dfrac{r}{m}\dfrac{dx}{dt} + \dfrac{s}{m}x = 0$ for the critically damped oscillations.

5. A charged capacitor discharges through an inductance and resistance in series. Discuss possible solutions and represent these by graphs. Deduce the condition under which the discharge is (*i*) critically damped and (*ii*) oscillatory. Obtain expression for the frequency. Does the presence of resistance effect the amplitude and frequency of damped oscillations? (*P.U.*, 2003; *H.P.U.*, 2002, 2000)

6. Define logarithmic decrement and derive a relation for it for a mechanical oscillator and an electrical oscillator. How can you determine experimentally the value of logarithmic decrement and damping co-efficient? (*P.U.* 2003, 2002; *Meerut U.* 2001; *G.N.D.U.* 2004, 2000; *Pbi. U.*, 2000)

7. **(*a*)** In a ballistic galvanometer the coil is wound over a non-metallic frame, comment.
(*Pbi. U.* 2002)

(*b*) Why the coil of a moving coil galvanometer, ammeter and voltmeter is wound over a metallic frame? (*Pbi. U.* 2001)

8. Define relaxation time of damped oscillatory system. Show that it varies inversely as damping constant. Derive an expression for relaxation period of (*i*) a mechanical oscillator (*ii*) electrical oscillator. How is it useful in determining the logarithmic decrement of a system? (*P.U.*, 2004; *H.P.U.*, 2001; *Pbi. U.*, 2000; *G.N.D.U.*, 2001)

9. (***a***) Define quality factor of a damped oscillator. Deduce an expression for it for a mechanical oscillator and an electrical oscillator.

(***b***) Show that (*i*) lower the damping higher will be the quality factor.

(*ii*) For large quality factor damping has little or no effect on the frequency.

(*G.N.D.U.* 2002; *H.P.U.* 2001; *Pbi. U.* 2000)

10. (***a***) What is damping? On what factors the damping depends?

(***b***) What is the effect of damping on the natural frequency of an oscillator?

(*Pbi.U.* 2003)

(***c***) Does viscous damping remain proportional to velocity under all conditions?

(*Pbi.U.* 2000)

(***d***) Write unit of damping constant and damping co-efficient for mechanical and electrical oscillator. (*H.P.U.* 2002)

11. Distinguish between transient and steady state in a forced oscillator. Explain the transient and steady state behaviour of a mechanical oscillator driven by a force

$$F = F_0\, e^{i\,\omega\, t}$$

Discuss the case when $F = F_0 \cos \omega\, t$ and when $F = F_0 \sin \omega\, t$ and show that the driven oscillator is always behind the driving force in phase.

(*Nagpur U.* 2003; *H.P.U.*, 2003, 2002, 2000; *P.U.*, 2004, 2003, 2002, 2000; *G.N.D.U.*, 2003; *Luck.U.*, 2001; *Pbi.U.*, 2002, 2000; *Meerut U.*, 2002)

12. Discuss the behaviour of displacement versus driving force frequency in case of a forced oscillator. Show that (*i*) The displacement at low frequency is independent of frequency (*ii*) The resonant frequency of driving force is slightly less than the natural frequency of the oscillator (*iii*) Maximum amplitude $A_{max} = \dfrac{F_0}{r\,\omega'}$ where r is damping constant and

$\omega' = \sqrt{\dfrac{s}{m} - \dfrac{r^2}{4m^2}}$ (*H.P.U.*, 2003, 2001; *P.U.*, 2002, 2000; *G.N.D.U.*, 2001; *Meerut U.*, 2000)

13. For a mechanical oscillator driven by a force $F = F_0 \cos \omega\, t$, discuss the variation of phase difference between displacement and driving force with driving force frequency. What is the phase difference between displacement and force at very high frequency?

(*G.N.D.U.*, 2002; *H.P.U.*, 2001, 2000)

14. (***a***) Draw a graph between velocity amplitude and driving force frequency of a forced oscillator. (*H.P.U.*, 2001)

(***b***) Show that whereas at resonance displacement lags behind the driving force by $\pi/2$, the velocity is in phase with driving force. (*G.N.D.U.*, 2002)

15. Discuss the variation of acceleration amplitude with driving force frequency of a forced mechanical oscillator and show that

(***i***) Acceleration amplitude at high frequency is frequency independent and

(***ii***) When damping constant is very small the acceleration amplitude resonance frequency equals the natural frequency of the forced oscillator. (*P.U.*, 2003; *G.N.D.U.*, 2003)

16. (***a***) What is absorption resonance curve? Draw it for a forced oscillator. Why is it so called? (*P.U.*, 2001, 2000)

(*b*) What is meant by resonance absorption band width? Define it and show it on a P-ω graph. Explain how the damping of the medium effects the band width?
(*Pbi.U.*, 2003; *Nagpur U.*, 2001; *H.P.U.*, 2000)

17. (*a*) Derive an expression for the Q-value of a forced oscillator in terms of resonance absorption band width. If the Q-value of an oscillator is increased how is band width of the absorption curve affected. Prove that band width $= \frac{\omega_0}{Q}$ where the symbols have their usual meaning. (*Pbi.U.* 2001; *P.U.* 2004, 2001; *Nagpur U.* 2003, 2001; *G.N.D.U.*, 2000)

18. (*a*) Find the Q-value of an oscillator in terms of the energy decay.

(*b*) What is figure of merit of a vibration and how is it defined in terms of bandwidth? The quality factor Q is a measure of sharpness of resonance in case of a forced oscillator. Explain. (*P.U.*, 2000)

19. An electric oscillator is driven by an *e.m.f.* $E = E_0\, e^{i\omega t}$. Derive an expression for the current in the electric circuit in the steady state and find its value when the applied *e.m.f.* is

(*i*) $E = E_0 \cos \omega t$ (*ii*) $E_0 \sin \omega t$ (*P.U.*, 2001, 2000)

20. (*i*) What are damped oscillations?

(*ii*) Obtain differential equation of motion of a damped harmonic oscillator.

(*iii*) Obtain general solution of differential equation of a damped harmonic oscillator.

(*iv*) In an oscillatory circuit, $L = 0.2$ Henry, $C = 0.0012\ \mu F$. What is the maximum value of resistance for the circuit to be oscillatory? (*Nagpur U.* 2006)

21. Obtain an expression for the power dissipated in damped harmonic oscillator.
(*Nagpur U.* 2007)

SHORT QUESTIONS

1. Show that in case of a damped oscillator the loss of energy is equal to the rate of doing work against resistive force. (*G.N.D.U.* 2002, 2000; *P.U.* 2004)
2. Show that the energy of damped vibrations of a damped simple harmonic oscillator decreases exponentially with time. (*G.N.D.U.* 2004, 2003)
3. What is physical significance of logarithmic decrement of a damped oscillatory system?
4. Explain how the conditions of critical damping are used in designing of electrical instruments.
5. Express amplitude, energy, logarithmic decrement and relaxation time in terms of Q the quality factor. (*Pbi. U.* 2002; *H.P.U.* 2000)
6. Discuss the methods (logarithmic decrement, relaxation time and quality factor) for quantitative measurement of damping effect in a damped simple harmonic oscillator.
7. Which is greater the natural frequency of damped oscillations or frequency of displacement resonance?
8. Why is there large amplitude, when frequency of external periodic force is same as natural frequency of the body?
9. Show that the displacement resonance occurs at a frequency slightly less than the frequency of velocity resonance (*P.U.*, 2002; *H.P.U.*, 2001)
10. What is sharpness of resonance? Explain the effect of damping on sharpness of resonance.
(*Meerut U.*, 2005, 2003, 2001; *Nagpur U.*, 2003, 2002; *G.N.D.U.*, 2003; *Luck.*, 2002)

11. Why should two tuning forks be very accurately in unison to show resonance while a tuning fork and an air column require to be only approximately tuned?

12. Derive an expression for the average power supplied to a forced oscillator by an external driving force

$$F = F_0 \cos \omega t$$ (*P.U.*, 2000)

13. Show graphically the variation of average power of a forced oscillator with driving force frequency. (*P.U.*, 2000)

14. Show that in a driven oscillator, the maximum power is absorbed at the frequency of velocity resonance and not at the frequency of amplitude resonance. (*Agra U.* 2006)

15. Is the energy stored in a forced oscillator? Explain and justify.
(*P.U.*, 2003; *Pbi. U.*, 2001, 2000)

16. If the displacement at resonance is given by

$$A_{max} = \frac{F_0}{\omega' r} \quad \text{where} \quad \omega' = \sqrt{\frac{s}{m} - \frac{r^2}{4m^2}}$$

show that the displacement at low frequencies is amplified by a factor Q at displacement resonance. (*P.U.* 2004)

Or

Find Q value of the oscillatory system as an amplification factor.
(*Pbi.U.*, 2002, 2000; *P.U.*, 2004, 2002, 2001; *H.P.U.*, 2000)

17. A parallel L-C circuit is connected to an $A.C.$ source of voltage $E = E_0 \sin \omega t$. Calculate the current in the circuit at any instant.

18. What is meant by damped harmonic oscillations? Give an example.
(*Nagpur U.* 2007, 2008)

19. Define Quality Factor of a harmonic oscillator. Obtain an equation of quality factor for a damped harmonic oscillator. (*Nagpur U.* 2006, 2007, 2008)

20. What is resonance? Explain sharpness of resonance. (*Nagpur U.* 2009, 2008)

21. What is physical significance of Q-value of a forced oscillator?

22. Prove that the band width of the resonance absorption curve defines the phase angle range $\tan \phi = \pm 1$. (*H.P.U.* 2003, 2002, 2000; *P.U.* 2000)

23. What is driven harmonic oscillator? How does it differ from simple and damped harmonic oscillator? (*Nagpur Uni.* 2009)

NUMERICAL QUESTIONS

1. Calculate the band width of an acoustic system having Q = 1.75 and resonant frequency 280 Hz. [**Ans.** 160 Hz] (*Nagpur U.*, 2001, 2006)

2. Find the resonant frequency of an acoustic system having Q equal to 1.60 and half power frequencies equal to 180 Hz and 380 Hz respectively. [**Ans.** 320 Hz] (*Nagpur U.* 2008)

3. Find the frequency of a circuit containing inductance of 100 μH and capacity of 0.01 μF. To which wavelength (of radio wave) its response will be maximum?
[**Ans.** 1890 m] (*P.U.* 2001, 2000)

4. A torsion pendulum of moment of inertia 10^5 gm-cm^2 suffers an angular displacement of 4° under a constant torque of 2000 dynes-cm. What should be the frequency of the periodic torque that would set the pendulum in resonant vibration?
[**Ans.** 9.926×10^5 erg/sec]

5. In an oscillatory circuit, $L = 0.2$ henry, $C = 0.0012$ μF. What is the maximum value of resistance for the circuit to be oscillatory. [**Ans.** Less than 2.582×10^4 ohms]

6. The quality factor of an oscillatory circuit is 100 and its frequency 8 kHz. How will its frequency change if the resistance in the circuit be reduced to zero?

[**Ans.** The frequency will increase by 1 Hz]

8

COUPLED OSCILLATORS

INTRODUCTION

For small oscillations, there is usually a coupling between two or more oscillators. When the motion of one oscillating system influences another, the two are said to be *coupled.*

The two or more oscillators linked together in such a way that an exchange of energy transfer takes place between them are called coupled oscillators.

In a forced oscillator, it is assumed that the external driving force is practically not affected by the oscillations of the driven system, *i.e.* the flow of energy between the driving agency and the driven system is only in one direction-from the driving agency to the driven system. However, in actual practice, there is always some feed back of energy, although small. In this chapter, let us study two coupled oscillators and their normal modes of vibration and extend the idea to N coupled oscillators.

8.1 SOME DEFINITIONS

Normal Coordinates: Normal co-ordinates are those co-ordinates which help us to express the equations of motion of the harmonic oscillators of a coupled system in the form of a set of *linear differential equations with constant co-efficients* and in which each equation contains only *one variable.*

Normal Modes of Vibration. *The manner in which a coupled system oscillates is called a mode.* The mode of a coupled system may be harmonic or non-harmonic.

The harmonic modes of a coupled system are called normal modes.

Normal modes have definite characteristics and are represented by linear differential equations with constant coefficients and only one dependent variable or normal co-ordinates. A normal mode has its own frequency known as *normal frequency*. In each normal mode all the components of the system vibrate with the same normal frequency. The normal modes of vibration are *entirely independent of each other,* since the energy associated with a normal mode is never exchanged with the energy associated with another normal mode. The total energy of the oscillator is equal to the sum of the energies of all the normal modes. If at any time only one mode is excited and vibrates the other modes will always be at rest and unexcited and these will acquire no energy from the vibrating mode.

Degrees of Freedom. A degree of freedom of a system is the *independent way* by which the system may *acquire energy.* A degree of freedom is assigned its own particular normal co-ordinates. The *number* of degrees of freedom and the number of normal co-ordinates of a system is the number of different ways in which the system can acquire energy. Each harmonic oscillator has two degrees of freedom as it may have *both kinetic* as well as *potential energy.*

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The kinetic energy of a simple harmonic oscillator of mass m and having displacement co-ordinate x is given by $\frac{1}{2}m\dot{x}^2 = a\,\dot{x}^2$ where $a = \frac{1}{2}m$. The potential energy is given by $\frac{1}{2}s\,x^2 = b\,x^2$ where s is the stiffness constant.

If the normal modes of a harmonic oscillator are represented by normal co-ordinates X and Y, then the total energy corresponding to the two modes will be

$$E_X = a\dot{X}^2 + bX^2$$

and

$$E_Y = c\dot{Y}^2 + d\,Y^2$$

where a, b, c and d are constants, $a\dot{X}^2$ and $c\dot{Y}^2$ give the kinetic energy and $b\,X^2$ and $d\,Y^2$ the potential energy.

8.2 EQUATION OF MOTION OF STIFFNESS COUPLED SYSTEM OF TWO PENDULUMS

Consider a coupled system of two identical pendulums each having a pendulum bob of mass m suspended by a light weightless, rigid rod of length l. The bobs are connected by a light spring of stiffness s, whose normal length is equal to the distance between the bobs when none of the two pendulums is displaced from its equilibrium position. Such pendulums are known as **stiffness coupled.**

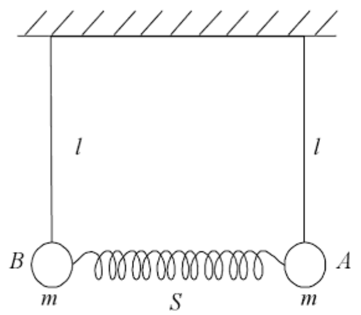

Fig. 8.1

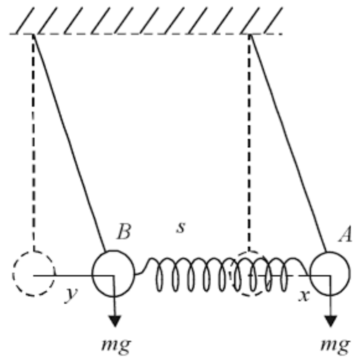

Fig. 8.2

Let the two pendulums A and B be set into vibrations with a *small* amplitude in the plane of the paper and let x and y be the displacements (in the same direction) of the pendulums marked A and B respectively as shown in Fig. 8.1 and Fig. 8.2 then the spring is elongated by a length $(x - y)$ and the corresponding force comes into play is $s\,(x - y)$. The component of the force due to gravity tending to bring the bobs of the pendulums back to its mean position

$$= -\frac{mgx}{l} \text{ for the pendulum } A$$

and

$$= -\frac{mgy}{l} \text{ for the pendulum } B.$$

∴ The equations of motion for the pendulums A and B respectively, are

$$m\ddot{x} = -mg\frac{x}{l} - s\,(x - y) \qquad ...(i)$$

and

$$m\ddot{y} = -mg\frac{y}{l} + s\,(x - y) \qquad ...(ii)$$

The first term in each equation is the normal simple harmonic motion term and the second term is due to the *coupling of the spring*. If $x > y$ the spring is extended beyond its normal length and will, therefore, apply a force against the acceleration of x but in favour of the acceleration of y. Dividing equations (i) and (ii) by m and substituting $\frac{g}{l} = \omega_0^2$, we have

$$\ddot{x} + \frac{g}{l}x = \ddot{x} + \omega_0{}^2 x = -\frac{s}{m}(x - y) \quad ...(iii)$$

and $$\ddot{y} + \frac{g}{l}y = \ddot{y} + \omega_0{}^2 y = -\frac{s}{m}(y - x) \quad ...(iv)$$

Adding (*iii*) and (*iv*), we get

$$\ddot{x} + \ddot{y} + \frac{g}{l}(x + y) = \ddot{x} + \ddot{y} + \omega_0{}^2 (x + y) = 0 \quad ...(v)$$

Subtracting (*iv*) from (*iii*), we get

$$\ddot{x} - \ddot{y} + \frac{g}{l}(x - y) = \ddot{x} - \ddot{y} + \left(\omega_0^2 + \frac{2s}{m}\right)(x - y) = 0 \quad ...(vi)$$

Let us choose two new co-ordinates X and Y so that

$$X = x + y \quad \text{and} \quad Y = x - y$$

then $$\dot{X} = \dot{x} + \dot{y} \quad \text{and} \quad \dot{Y} = \dot{x} - \dot{y}$$

and $$\ddot{X} = \ddot{x} + \ddot{y} \quad \text{and} \quad \ddot{Y} = \ddot{x} - \ddot{y}$$

Substituting the values in (*v*) and (*vi*), we get

$$\ddot{X} + \frac{g}{l}X = \ddot{X} + \omega_0^2 X = 0 \quad ...(vii)$$

and $$\ddot{Y} + \left(\frac{g}{l} + \frac{2s}{m}\right)Y = \ddot{Y} + \left(\omega_0^2 + \frac{2s}{m}\right)Y = 0 \quad ...(viii)$$

Normal co-ordinates. It is seen that whereas equations (*v*) and (*vi*) each have two variables, we find that equations (*vii*) and (*viii*) have only one variable. The motion of the coupled system is thus described in terms of two co-ordinates X and Y. *Each equation of motion is a linear differential equation of a simple harmonic oscillator with constant coefficients with only one variable*. The co-ordinates X and Y are therefore *normal co-ordinates* of the coupled system.

Thus the coupled system of two simple pendulums has *two normal modes*, one described by *normal co-ordinate* X and the other by the *normal co-ordinate* Y.

In phase mode. $Y = 0$. When normal co-ordinate $Y = 0$, $x - y = 0$ *i.e.*, $x = y$ for all times. The motion is completely described by

$$\ddot{X} + \frac{g}{l}X = \ddot{X} + \omega_0^2 X = 0$$

As the relative displacement of the two pendulum always remain the same ($\because x = y$), *the stiffness of the coupling has no effect.* The spring always remains at its normal length. The frequency of oscillations is the same as that of either pendulum in isolation *i. e.*, $\omega_0^2 = \frac{g}{l}$. Both pendulums are always swinging in the same phase. Such vibrations are called *in-phase vibrations* and the corresponding mode is known as *in-phase mode*.

Out of phase mode : $X = 0$. When normal co-ordinate $X = 0$, $x + y = 0$ or $x = -y$ for all times. The motion is completely described by

$$\ddot{Y} + \left(\frac{g}{l} + \frac{2s}{m}\right)Y = \ddot{Y}\left(\omega_0^2 + \frac{2s}{m}\right)Y = 0$$

When the displacement (x) of the pendulum A is *positive* that of B (y) is *negative* but equal to that of A ($x = -y$). At zero displacement the two pendulums will be moving in opposite directions. Thus the periodic motion of the two pendulums will be 180° out of phase. Such vibrations are called *out of phase vibrations* and the corresponding mode is known as *out of phase mode*.

8.3 TOTAL ENERGY OF TWO IDENTICAL STIFFNESS COUPLED PENDULUMS

The equation of motion of a stiffness coupled system of two identical pendulums A and B is given by

$$\ddot{X} + \omega_0^2 X = 0 \quad ...(i)$$

$$\ddot{Y} + \left(\omega_0^2 + \frac{2s}{m}\right)Y = 0 \quad ...(ii)$$

where $X = x + y$ and $Y = x - y$ are the *normal co-ordinates of the system* x and y are the displacement (in the same direction) of the pendulums A and B respectively as shown in Fig. 8.2 and $\omega_0^2 = \frac{g}{l}$ where l is the length of the pendulum and s is the stiffness constant. Solving equations (i) and (ii), we get

$$X = x + y = X_0 \cos(\omega_1 t - \phi_1) \quad ...(iii)$$

and $$Y = x - y = Y_0 \cos(\omega_2 t - \phi_2) \quad ...(iv)$$

X_0 and Y_0 are the *normal mode amplitudes* and $\omega_1 = \omega_0 = \left(\frac{g}{l}\right)^{\frac{1}{2}}$ and

$$\omega_2 = \left(\omega_0^2 + \frac{2s}{m}\right)^{\frac{1}{2}} = \left(\frac{g}{l} + \frac{2s}{m}\right)^{\frac{1}{2}}$$ are *normal mode frequencies.*

To simplify further discussions, let

$$X_0 = Y_0 = 2a$$

and $$\phi_1 = \phi_2 = 0$$

$\therefore$ Equations (iii) and (iv) give

$$X = x + y = 2a \cos \omega_1 t \quad ...(v)$$

and $$Y = x - y = 2a \cos \omega_2 t \quad ...(vi)$$

Adding (v) and (vi), we get

$$2x = X + Y = 2a \cos \omega_1 t + 2a \cos \omega_2 t$$

$$\therefore \quad x = \frac{1}{2}(X + Y) = a\cos\omega_1 t + a\cos\omega_2 t$$

Subtracting (vi) from (v), we get

$$y = \frac{1}{2}(X - Y) = a\cos\omega_1 t - a\cos\omega_2 t$$

The corresponding velocities are given by

$$\dot{x} = -a\omega_1 \sin\omega_1 t - a\omega_2 \sin\omega_2 t$$

and $$\dot{y} = -a\omega_1 \sin\omega_1 t + a\omega_2 \sin\omega_2 t.$$

Now let the system be set in motion by displacing the bob x to the right by a distance $x = 2a$ keeping $y = 0$ and both the bobs be released from rest so that $\dot{x} = \dot{y} = 0$ at a time $t = 0$. The motion of pendulum A is given by

$$x = a\cos\omega_1 t + a\cos\omega_2 t = 2a\cos\frac{(\omega_2 - \omega_1)t}{2}\cos\frac{(\omega_1 + \omega_2)t}{2}$$

and the motion of pendulum B is given by

$$y = a\cos\omega_1 t - a\cos\omega_2 t = -2a\sin\frac{(\omega_2 - \omega_1)t}{2}\sin\frac{(\omega_1 + \omega_2)t}{2}$$

$\dfrac{\omega_2 - \omega_1}{2} = \omega_m$ is called the *modulated* or beat frequency and $\dfrac{\omega_1 + \omega_2}{2} = \omega_a$ is called the *average* frequency.

Hence the modulated amplitude of pendulum $A = A = 2a \cos \omega_m t$

and $x = A \cos \omega_a t$

The modulated amplitude of pendulum $B = B = 2a \sin \omega_m t$

and $y = -B \sin \omega_a t$

If we assume that the spring is very weak and does not store any energy we can consider the modulated amplitude $2a \cos \omega_m t$ and $2a \sin \omega_m t$ to remain constant over one cycle of average frequency.

$\therefore$ Energy of pendulum A ; $E_A = \dfrac{1}{2} m v^2 = \dfrac{1}{2} m (A\omega_a)^2$

$$= \frac{1}{2} m (2a \cos \omega_m t)^2 \omega_a^2 = 2 m a^2 \omega_a^2 \cos^2 \omega_m t$$

Energy of pendulum B; $E_B = \dfrac{1}{2} mv^2 = \dfrac{1}{2} m (B\omega_a)^2$

$$= \frac{1}{2} m (2a \sin \omega_m t)^2 \omega_a^2 = 2 m a^2 \omega_a^2 \sin^2 \omega_m t$$

Hence total energy $E = E_A + E_B = 2 m a^2 \omega_a^2$

Evidently, the total energy $E = E_A + E_B$ is constant as m, a and ω_a are constants.

Also $E_A - E_B = 2 m a^2 \omega_a^2 [\cos^2 \omega_m t - \sin^2 \omega_m t]$

$$= E \cos 2 \omega_m t = E \cos (\omega_2 - \omega_1) t$$

$\therefore$ $$E_A = \frac{1}{2} E [1 + \cos (\omega_2 - \omega_1) t]$$

and $$E_B = \frac{1}{2} E [1 - \cos (\omega_2 - \omega_1) t].$$

This shows that total energy is *constant* but *it flows back* and *forth between the pendulums at the modulated* (or beat) *frequency*.

Thus we see that after drawing aside the bob of the first pendulum by a distance $2a$ and releasing it the pendulum shows the cosine behaviour at a frequency ω_a which is the average of the two normal mode frequencies and its amplitude also varies according to the cosine law at a low frequency of half the difference between the normal mode frequencies. On the other hand pendulum 2 starts from zero and vibrates according to the sine law with the same average frequency and its amplitude builds up to $2a$ and then decays according to sine law at the same low frequency of half the difference between the normal mode frequencies.

The initial configuration $x = 2a$, $y = 0$ can be decomposed into X and Y modes as shown in Fig. 8.3. The X mode *i.e.*, $x = y = a$ so that $X_0 = x + y = 2a$ is known as the '*in phase*' mode and the Y mode *i.e.*, $x = a, y = -a$ so that $Y_0 = x - y = 2a$ is known as the *out of phase* mode.

As the Y-mode has a higher frequency it will gain half a vibration (phase difference π radian) on the X-mode after a number of vibrations and the combination of X and Y mode then will give $x = 0$ and

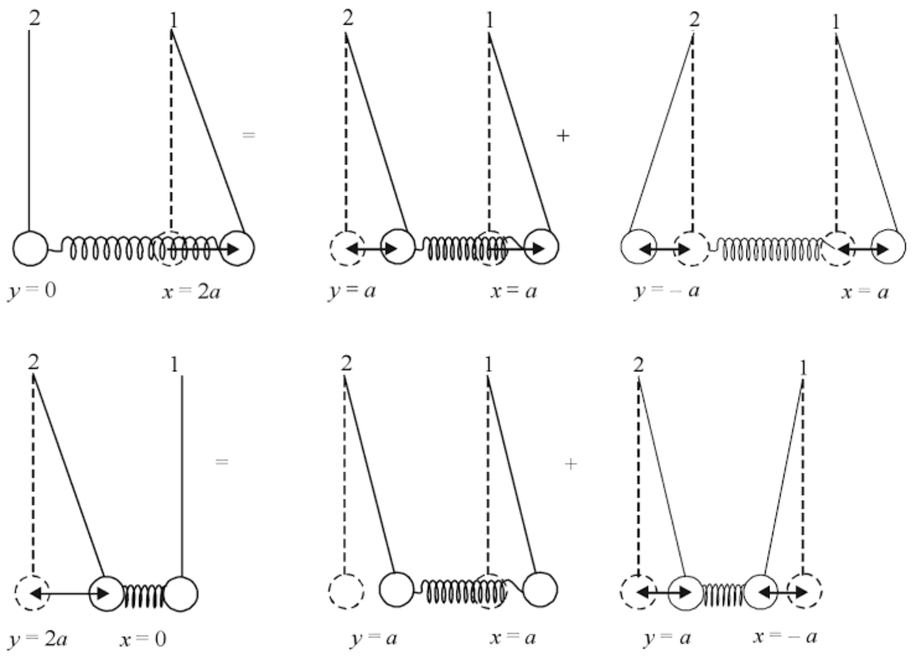

Fig. 8.3

$y = 2a$ as shown in the lower Fig. 8.3. After some time when Y gains another half a vibration $x = 2a$ and $y = 0$. Thus the pendulums only exchange energy the normal modes do not. The '*in phase*' mode remains '*in phase*' mode and '*out of phase mode*' remains '*out of phase*' mode.

8.4 TYPES OF COUPLING

Two oscillators are coupled together to bring about an exchange of energy between them. The common coupling components for mechanical oscillators may be '*stiffness*' or '*mass*' and in case of electrical oscillators it may be '*capacitance*' or '*inductance*'. Capacitance and inductance are energy storing electrical components and coupling through them consumes no power, there by making it possible for the energy transfer to take place over a number of oscillations. Coupling may also be done through the '*resistance*' component but this causes an inevitable loss of energy so that the amplitude goes on rapidly falling.

8.5 FREQUENCY OF OSCILLATION IN-PHASE AND OUT-OF-PHASE MODE

The differential equations of motion of coupled system of two identical (mechanical) oscillators say two simple pendulums in terms of normal co-ordinates X and Y are given by equations (*vii*) and (*viii*) of article 8.2 as

$$\ddot{X} + \frac{g}{l}X = \ddot{X} + \omega_0^2 X = 0$$

and

$$\ddot{Y} + \left(\frac{g}{l} + \frac{2s}{m}\right)Y = \ddot{Y} + \left(\omega_0^2 + \frac{2s}{m}\right)Y = 0$$

where $X = x + y$ and $Y = x - y$; x and y being the displacements of the two (mechanical) oscillators, s the stiffness and m the *mass*.

(*i*) In Phase Mode: In the *in-phase* mode $x = y$ so that $Y = x - y = 0$. In such a case the motion is completely represented by the equation

$$\ddot{X} + \frac{g}{l}X = \ddot{X} + \omega_0^2 X = 0$$

The frequency of oscillations of the coupled system is, therefore, $\omega_0 = \sqrt{\frac{g}{l}}$ which is the same as that of uncoupled oscillators.

Thus *in the in-phase mode the frequency of oscillations of the coupled system is the same as that of the uncoupled oscillators or of either pendulum in isolation.* This is due to the fact that the two pendulums are always oscillating in phase as shown in Fig 8.2 *so that the light spring always has its natural length*.

***(ii)* Out of Phase Mode:** In the *out of phase* mode $x = -y$ so that $X = x + y = 0$. In such a case the motion is completely represented by the equation

$$\ddot{Y} + \left(\omega_0^2 + \frac{2s}{m}\right)Y = 0$$

The frequency of oscillations of the coupled system is, therefore, given by

$$\omega' = \sqrt{\omega_0^2 + \frac{2s}{m}}$$

which is greater than ω_0, the frequency of the uncoupled oscillators.

Thus *in the out of phase mode the frequency of oscillations of the coupled system gets raised.*

This is due to the fact that the two pendulums are always out of phase either in the *extended* or in the *compressed* position as shown in Fig. 8.3 *i.e. extreme positions* and hence the *coupling becomes effective*.

8.6 CHARACTERISTICS OF IN-PHASE AND OUT-OF-PHASE MODES OF VIBRATION

Two oscillating , identical coupled simple pendulums have as we know, two normal modes of vibration

(*i*) In-phase mode and (*ii*) Out of phase mode, each having its own characteristics.

***(a)* Characteristics of in phase mode.** (*i*) The in-phase mode can be excited by displacing the bobs of the two pendulums to the same side by the same amount and then letting them oscillate purely on their own.

(*ii*) The equation of the in-phase mode is

$$\ddot{X} + \omega_0^2 X = 0 \qquad ...(i)$$

It describes the oscillatory behaviour of the system when

$$Y = (x - y) = 0 \quad \text{or} \quad x = y$$

(*iii*) The displacement of both the (bobs) or masses is the *same in magnitude* as well as in *direction* and both the masses continuously oscillate in the *same phase*. It means that both the pendulums pass through their mean position or through either of the extreme positions simultaneously.

(*iv*) The amplitude of the two simple pendulums is the same *i.e. a*.

(*v*) The shape or configuration of the mode is $\frac{x}{y} = +1$

(*vi*) Each pendulum executes simple harmonic oscillation at frequency $\omega_0 = \sqrt{g/l}$ which is the natural frequency of free oscillations of either pendulum in isolation.

(*vii*) In the in-phase mode there is no effect of the stiffness term *s* of the coupling on the motion of the two masses because the spring always has its natural length — it is neither stretched nor compressed.

(*viii*) A solution of Eq. (*i*) gives

$$X = X_0 \cos(\omega_0 t - \phi).$$

If the maximum value of x and y be a, then $X_0 = 2a$ is the *normal mode amplitude*.

(*b*) Characteristics of out of phase mode (*i*) The out of phase mode can be excited by displacing the bobs of the two pendulums in opposite directions by the same amount and then let them oscillate freely on their own.

(*ii*) The equation of the out of phase mode is

$$\ddot{Y} + \left(\omega_0^2 + \frac{2s}{m}\right) Y = 0 \qquad ...(ii)$$

It describes the oscillatory behaviour of the system when

$$X = (x + y) = 0 \quad \text{or} \quad x = -y$$

(*iii*) The displacement of either mass (bob) is equal in magnitude but opposite in direction to the other mass and the two masses continuously oscillate *out of phase* by an angle of 180° or π-radian with respect to each other.

It means that both the pendulums pass through their mean position (in opposite directions) or are at their (opposite) extreme positions simultaneously.

(*iv*) The amplitude of the two simple pendulums is the same *i.e.*, '*a*'

(*v*) The shape or configuration of the mode is $\frac{x}{y} = -1$.

(*vi*) Each pendulum executes simple harmonic oscillations at a frequency

$$\omega' = \sqrt{\frac{g}{l} + \frac{2s}{m}} = \sqrt{\omega_0^2 + \frac{2s}{m}}$$

which is higher than the natural frequency of the free oscillations of either pendulum.

(*vii*) In the out of phase mode the coupling term s dominates the motion of the two masses and raises the frequency of oscillation. It is because the coupling spring is either in the stretched or in the compressed state. Only once in one time period of oscillation the spring acquires its normal length when the masses pass through their mean positions.

(*viii*) A solution of Eq. (*ii*) gives

$$Y = Y_0 \cos(\omega' t - \phi')$$

If the maximum value of x and y be a, then $Y_0 = 2a$ is the normal mode amplitude.

NOTE. X_0 and Y_0 are purely mathematical quantities and do not represent the amplitude of the oscillation of either pendulum which has a value $a = \frac{X_0}{2}$ for the in-phase mode and $a = \frac{Y_0}{2}$ for the out of phase mode.

8.7 GENERAL METHOD OF FINDING NORMAL MODE FREQUENCIES

Consider a system of two coupled simple pendulums of the same length coupled by a spring of stiffness s, then their equations of motion are

$$m\ddot{x} + \frac{mg}{l} x + s(x - y) = 0 \qquad ...(i)$$

and

$$m\ddot{y} + \frac{mg}{l} y - s(x - y) = 0 \qquad ...(ii)$$

where x and y are the displacements of the bobs of the two pendulums respectively from their mean positions; l is the length of each pendulum, m the mass of each pendulum bob as proved in Eq. (*i*) and (*ii*) of article 8.2.

Now, when a coupled system oscillates in a *single normal mode* each component of the system vibrates with the frequency of that mode. Therefore, supposing that the system of the coupled pendulums vibrates only in one of its normal modes, let the frequency be ω, then the solutions of the above equations are

$$x = A\cos\omega t$$

and
$$y = B\cos\omega t$$

where A and B are the displacement amplitude of x and y at the frequency ω.

$$\therefore \quad \ddot{x} = -A\omega^2\cos\omega t \text{ and } \ddot{y} = -B\omega^2\cos\omega t$$

Substituting in (*i*) and (*ii*), we have

$$\left[-m\omega^2 A + \frac{mg}{l}A + s(A-B)\right]\cos\omega t = 0 \qquad ...(iii)$$

and
$$\left[-m\omega^2 B + \frac{mg}{l}B - s(A-B)\right]\cos\omega t = 0 \qquad ...(iv)$$

First normal mode frequency. Adding (*iii*) and (*iv*), we get

$$(A+B)\left(-m\omega^2 + \frac{mg}{l}\right) = 0 \qquad ...(v)$$

This equation is satisfied when $\omega^2 = \dfrac{g}{l}$.

Thus this gives the *first normal mode frequency* (*or frequency of in-phase mode*) $\omega = \sqrt{\dfrac{g}{l}}$.

Second normal mode frequency. Subtracting (*iv*) from (*iii*), we get

$$(A-B)\left(-m\omega^2 + \frac{mg}{l} + 2s\right) = 0 \qquad ...(vi)$$

This equation is satisfied when

$$\omega^2 = \frac{g}{l} + \frac{2s}{m}$$

This gives the *second normal mode*, *frequency*, (or *frequency of out of phase mode*)

$$\omega = \sqrt{\frac{g}{l} + \frac{2s}{m}}$$

In phase conditions. Substituting $\omega^2 = \dfrac{g}{l}$ in (*vi*), we have

$$(A-B)\,2s = 0$$

or
$$A - B = 0$$

$$\therefore \quad A = B$$

which gives the *in phase* condition.

Out of phase condition. Substituting $\omega^2 = \dfrac{g}{l} + \dfrac{2s}{m}$ in (*v*), we have

$$(A+B)\,2s = 0$$

$$\therefore \quad A + B = 0$$

or
$$A = -B$$

which gives the *out of phase* or (antiphase) condition.

8.8 EXCHANGE OF ENERGY BETWEEN TWO NORMAL MODES

The equation of motion of a stiffness coupled system of two identical simple pendulums A and B is given by

$$\ddot{X} + \omega_0^2 X = 0 \qquad ...(i)$$

and $$\ddot{Y} + \left(\omega_0^2 + \frac{2s}{m}\right) Y = 0 \qquad ...(ii)$$

where $X = x + y$ and $Y = x - y$ are the normal co-ordinates of the system. x and y are the displacements (in the same direction) of the pendulums A and B respectively, $\omega_0^2 = \frac{g}{l}$ where l is the length of the pendulum and s the stiffness constant as given by eq. (*vii*) and (*viii*) of article 8.2.

Solving equation (*i*) and (*ii*), we get

$$X = x + y = X_0 \cos(\omega_1 t - \phi_1)$$

and $$Y = x - y = Y_0 \cos(\omega_2 t - \phi_2)$$

where X_0 and Y_0 are the normal mode amplitudes, $\omega_1 = \omega_0 = \left(\frac{g}{l}\right)^{\frac{1}{2}}$ and $\omega_2 = \left(\omega_0^2 + \frac{2s}{m}\right)^{\frac{1}{2}}$ $= \left(\frac{g}{l} + \frac{2s}{m}\right)^{\frac{1}{2}}$ are normal mode frequencies.

Putting $X_0 = Y_0 = 2a$, we have

$$X = 2\,a \cos(\omega_1 t - \phi_1)$$

and $$Y = 2\,a \cos(\omega_2 t - \phi_2)$$

As $X = x + y$, X mode is known as *in-phase* mode. The total energy of the system in X-mode is given by

$$E_x = \frac{1}{2} m\,\omega_1^2 (2a)^2 = 2\,m\,a^2 \omega_1^2$$

As $Y = x - y$, Y-mode is known as *out of phase* mode. The total energy of the system in Y-mode is given by

$$E_y = \frac{1}{2} m\,\omega_2^2 (2a)^2 = 2\,m\,a^2 \omega_2^2$$

Both E_x and E_y do not vary with time. They are constant quantities. Therefore, *we conclude that no exchange of energy takes place from one normal mode to another.* In other words, *normal modes are independent of each other.*

When we say that normal co-ordinates are independent of each other, we mean that between the in phase mode represented by the normal co-ordinate X and the out of phase mode represented by the normal co-ordinate Y there is no exchange of energy and hence the two are independent of each other.

8.9 FREQUENCY OF A TWO BODY COUPLED OSCILLATOR

Consider two masses m_1 and m_2 connected by a spring of stiffness or force constant s. When the two masses are displaced from their equilibrium positions, the spring either contracts or extends, depending upon the displacement of the masses. This causes a linear restoring force to be produced in the spring and both masses begin to vibrate harmonically about their equilibrium position. Such a system is known as a *two body oscillator* or *coupled oscillator*. A familiar example is a diatomic molecule in which the two atoms are connected by some internal force known as '*bond*'.

As no external force acts on the system, the centre of mass either remains stationary or moves with constant velocity. The motion executed is simple harmonic. The system will oscillate with a frequency $\sqrt{\frac{s}{\mu}}$ where $\mu = \frac{m_1 m_2}{m_1 + m_2}$. [For proof refer 5.9.8]

8.10 APPLICATIONS

Inductively coupled circuits. Two electrical circuits are said to be inductively coupled when the magnetic flux due to the current flowing in one circuit threads the second circuit. The two circuits are then said to have a *mutual inductance*. According to Faraday's law of electro-magnetic induction, whenever the magnetic flux in one circuit changes an induced *e.m.f.* is set up in the other which is proportional to the time rate of change of magnetic flux and lasts only for the time the change is taking place. The most familiar example is that of a transformer whose working depends on the mutual induction between its primary and secondary coils.

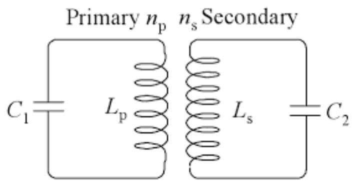

Fig. 8.4

The power source is connected to the primary and the secondary is wound over the primary in the *same sense*.

Two inductively coupled electrical circuits are shown in Fig. 8.4.

Let n_p represent the number of turns in the primary coil. If ϕ is the magnetic flux set up in the *primary* when a unit current flows through a single turn of the primary coil, then assuming that there is no leakage of flux outside the coil,

Flux linked with *each* primary turn $= n_p\phi$

$\therefore$ Total flux linked with primary coil $= n_p \cdot n_p \phi$

or $$L_p = n_p^2 \phi$$

where L_p is the co-efficient of self-induction of the primary. Similarly, if a unit current flowing through a single turn of the secondary coil also produces a magnetic flux ϕ, then

Flux linked with *each* secondary turn $= n_s\phi$

or $$L_s = n_s^2 \phi$$

where L_s is the co-efficient of self-induction of the secondary.

If we suppose that all the lines of magnetic flux due to a unit current in the primary, thread all the turns of the secondary, then

Total flux lines linking the secondary $= n_s (n_p \phi)$

or $$M = n_s n_p \phi = \sqrt{L_p L_s}$$

where M is the co-efficient of mutual induction between the two coils.

Co-efficient of coupling. The above result is true only when there is no leakage of magnetic flux. In practice, however, some leakage of flux does take place and $M < \sqrt{L_p L_s}$.

The ratio $\frac{M}{\sqrt{L_p L_s}} = k$ is called the *co-efficient of coupling*.

For *small* value of k the two circuits have a *loose coupling* and are said to be *lightly coupled*.

8.11 ENERGY TRANSFER BETWEEN TWO ELECTRICALLY COUPLED CIRCUITS

To consider the *energy transfer* between the two inductively coupled circuits as shown in Fig. 8.4, let the two circuits be made to oscillate with a frequency ω and let $I_p = I_1 e^{i\omega t}$ and $I_s = I_2 e^{i\omega t}$ be the currents in the two circuits respectively, then

Rate of change of primary current $= \dfrac{dI_p}{dt}$

$$= \frac{d}{dt} I_1 e^{i\omega t} = i\omega I_1 e^{i\omega t} = i\omega I_p$$

$\therefore$ Induced *e.m.f.* in the primary coil $= -L_p \dfrac{dI_p}{dt} = -i\omega L_p I_p$

and Rate of change of secondary current $= \dfrac{dI_s}{dt}$

$$= \frac{d}{dt} I_2 e^{i\omega t} = i\omega I_2 e^{i\omega t} = i\omega I_s$$

$\therefore$ Induced *e.m.f.* in the secondary coil $= -L_s \dfrac{dI_s}{dt} = -i\omega L_s I_s$

and Potential difference across the capacitor C_1 (Fig. 8.4) in the primary circuit $= \dfrac{\int I_p\, dt}{C_1}$

$$= \frac{\int I_1 e^{i\omega t} dt}{C_1} = \frac{I_1 e^{i\omega t}}{i\omega C_1} = -\frac{i I_p}{\omega C_1}$$

Potential difference across the capacitor C_2 (Fig. 8.4) in the secondary circuit $= \dfrac{\int I_s\, dt}{C_2}$

$$= \frac{\int I_2 e^{i\omega t} dt}{C_2} = \frac{I_2 e^{i\omega t}}{i\omega C_2} = -\frac{i I_s}{\omega C_2}$$

If the two circuits are considered free from resistance and have inductance and capacitance only, then the *e.m.f.* equations for the primary and the secondary respectively are

$$-L_p \frac{dI_p}{dt} + \frac{\int I_p\, dt}{C_1} - M\frac{dI_s}{dt} = 0$$

and
$$-L_s \frac{dI_s}{dt} + \frac{\int I_s\, dt}{C_2} - M\frac{dI_p}{dt} = 0$$

Substituting the values of various quantities, we have

$$i\omega L_p I_p - \frac{i}{\omega C_1} I_p + i\omega M I_s = 0 \qquad \ldots(i)$$

and
$$i\omega L_s I_s - \frac{i}{\omega C_2} I_s + i\omega M I_p = 0 \qquad \ldots(ii)$$

Multiplying (i) by $\dfrac{\omega}{i L_p}$ and (ii) by $\dfrac{\omega}{i L_s}$, we have

$$\omega^2 I_p - \frac{I_p}{L_p C_1} + \frac{M}{L_p}\omega^2 I_s = 0$$

and
$$\omega^2 I_s - \frac{I_s}{L_s C_2} + \frac{M}{L_s}\omega^2 I_p = 0$$

Now $\frac{1}{L_p C_1} = \omega_1^2$ where ω_1 is the *natural frequency* of the *primary circuit* containing inductance L_p and capacitance C_1 and $\frac{1}{L_s C_2} = \omega_2^2$ where ω_2 is the *natural frequency* of the *secondary* circuit containing inductance L_s and capacitance C_2.

Substituting $\frac{1}{L_p C_1} = \omega_1^2$ and $\frac{1}{L_s C_2} = \omega_2^2$, we get

$$I_p (\omega_1^2 - \omega^2) = \frac{M}{L_p} \omega^2 I_s$$

and
$$I_s (\omega_2^2 - \omega^2) = \frac{M}{L_s} \omega^2 I_p$$

Multiplying we have

$$I_p I_s (\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2) = \frac{M^2}{L_p L_s} \omega^4 I_s I_p$$

or
$$(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2) = \frac{M^2}{L_p L_s}\omega^4 = k^2\omega^4$$

where $k = \frac{M}{\sqrt{L_p L_s}}$ is the co-efficient of coupling.

To simplify put $\omega_1 = \omega_2 = \omega_0$, then

$$(\omega_0^2 - \omega^2)^2 = k^2 \omega^4$$

or
$$\omega_0^2 - \omega^2 = \pm k \omega^2$$

$\therefore$
$$\omega = \pm \frac{\omega_0}{\sqrt{1 \pm k}}$$

The negative sign on the right hand side has no meaning as ω cannot be negative.
The positive sign gives two frequencies

$$\omega_a = \frac{\omega_0}{\sqrt{1 + k}} \text{ and } \omega_b = \frac{\omega_0}{\sqrt{1 - k}}.$$

These are the *normal mode frequencies.*

In phase and out of phase normal modes. From the relation

$$I_p \left(\omega_1^2 - \omega^2\right) = \frac{M}{L_p}\omega^2 I_s$$

We have
$$\frac{I_s}{I_p} = \frac{\omega_1^2 - \omega^2}{\omega^2} \frac{L_p}{M} = \frac{\omega_0^2 - \omega^2}{\omega^2} \frac{L_p}{M} \qquad [\because \omega_1 = \omega_0]$$

$$= \left(\frac{\omega_0^2}{\omega^2} - 1\right) \frac{L_p}{M} \qquad ...(iii)$$

Selecting the value of $\omega = \dfrac{\omega_0}{\sqrt{1+k}}$, we have

$$\frac{\omega_0^2}{\omega^2} = 1 + k \text{ or } \frac{\omega_0^2}{\omega^2} - 1 = +k$$

Substituting in Eq. (*iii*), we get

$$\frac{I_s}{I_p} = +k\,\frac{L_p}{M} \qquad \text{... } (iv)$$

It is clear that the right hand side of the above equation is positive *i.e.* I_s and I_p are *in phase*.

Hence $\omega = \dfrac{\omega_0}{\sqrt{1+k}} = \omega_a$ represents the frequency of the *in-phase mode*.

Similarly selecting $\omega = \dfrac{\omega_0}{\sqrt{1-k}}$, we find that

$$\frac{I_s}{I_p} = -k\,\frac{L_s}{M} \qquad \text{... } (v)$$

i.e. I_s and I_p are *out of phase*. Hence $\omega = \dfrac{\omega_0}{\sqrt{1-k}} = \omega_s$ represents the frequency of *out of phase mode*.

Loose and tight coupling. In loose coupling k is small (<<1) and both the systems behave almost independently. In this case ω_a and ω_b are very nearly equal to ω_0. In tight coupling k is large (very nearly equal to unity) so that ω_a and ω_b differ from ω_0 by a large quantity, the peak values of current are displaced and the dip between the peaks is more pronounced. The variation of current amplitude with driving force frequency ω for different values of k is shown in Fig. 8.5. The three cases shown are

(1) for k small (2) for k having intermediate value and (3) for k large.

It may be noted that for k small (loose coupling) band width is negligible *i.e.*, resonance occurs at one frequency. For large k (tight coupling) bandwidth is large *i.e.*, resonance occurs at two widely apart frequencies.

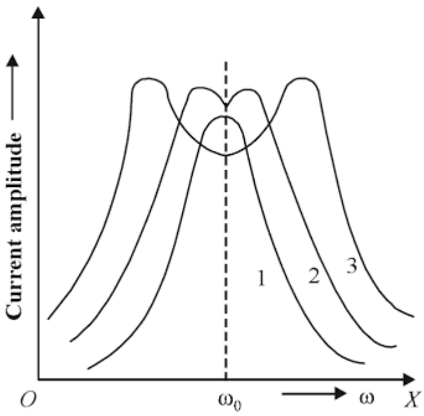

Fig. 8.5

8.12 N-COUPLED OSCILLATORS

We have seen so far a system in which two atoms coupled together by a massless spring, with an analogy of diatomic molecules like O_2, H_2, N_2 etc. However, in a solid, say in a crystal large number of molecules (*viz.* mono-, di-, poly-atomic) arranged in a particular regular manner depending upon its structure *i.e.* simple cubic (s.c.), body-centred cubic (b.c.c.), or face-centred cubic (f.c.c.). A lattice may be regarded as a regular arrangement of atoms which are joined together by elastic springs in three-dimension. At room temperature (or above), the connected atoms start oscillating about their mean position and thereby forms a case of N-coupled oscillator, in which N number of atoms coupled together by massless springs. The lattice may vibrate freely in its normal modes due to its internal energy or may experience forced vibrations under the action of external forces, which may be mechanical or electromagnetic in nature.

Simplest Case: One dimensional monoatomic lattice.

Consider a one-dimensional chain of atoms, each of mass m attached by massless springs as shown in Fig. 8.6. In figure both, the state of equilibrium and state of displacement (disturbed state) are shown.

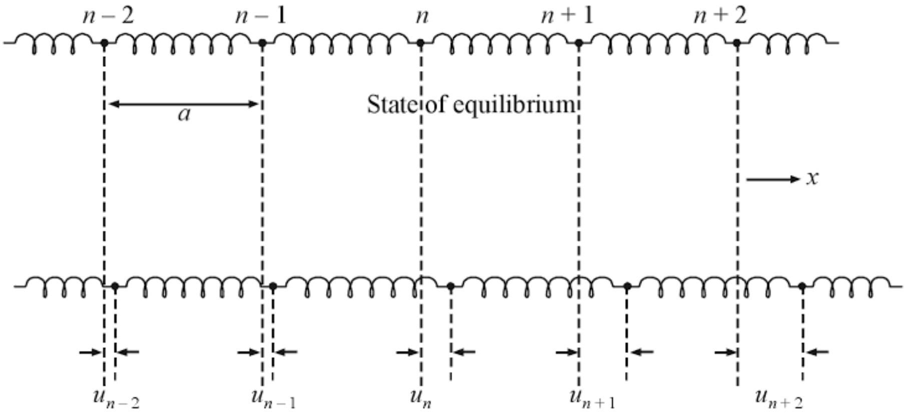

Fig. 8.6

Consider an equilibrium state of the atoms in which the atoms are equally spaced by sites, ... $n-2$, $n-1$, n, $n+1$, $n+2$, ... as shown in figure. Let a be the lattice parameter, (the distance between two atoms in equilibrium state), the x-coordinates of the corresponding atoms are given by .... $(n-2)a$, $(n-1)a$, na, $(n+1)a$, $(n+2)a$, ... In the state of vibratory motion along x-axis, the atoms execute simple periodic motion (S.H.M.) about their mean positions and become source of elastic waves which propogate through the medium. Let the displacements, at any instant of time t, of .... $(n-2)^{th}$, $(n-1)^{th}$, n^{th}, $(n+1)^{th}$, $(n+2)^{th}$ .... from their mean position be .... u_{n-2}, u_{n-1}, u_n, u_{n+1}, u_{n+2}, ... respectively.

Assuming the springs to be ideally elastic, force between any two atoms is directly proportional to its linear displacement. If u is the displacement of spring with spring constant β, the force exerted by the spring on the atom is given by $F = \beta\, u$.

Since n^{th} atom is connected to its neighbours $(n-1)^{th}$ and $(n+1)^{th}$ atoms by two springs, it experiences two opposite forces. The net force on the n^{th} atom will be

$$F = \beta\,(u_{n+1} - u_n) - \beta\,(u_n - u_{n-1})$$

$$= \beta\,(u_{n+1} + u_{n-1} - 2u_n) \quad \text{... } (i)$$

Using Newton's second law of motion, we write

$$m\frac{d^2u_n}{dt^2} = \beta\,(u_{n+1} + u_{n-1} - 2u_n) \quad \text{... } (ii)$$

where $\dfrac{d^2u_n}{dt^2}$ represents the acceleration of the n^{th} atom. Let the solution for this wave function be

$$u_n = u_0\, e^{\,i\,(\omega t - Kna)} \quad \text{... } (iii)$$

where na represents the x-coordinate of n^{th} atom in equilibrium state, $K = 2\pi/\lambda$ is the wave vector or propogation vector and ω is the angular frequency of the wave. Similar quantities for $(n-)^{th}$ and $(n+1)^{th}$ atoms are

$$u_{n-1} = u_0 e^{\,i\,[\omega t - K(n-1)a]} \quad \text{... } (iv)$$

$$u_{n+1} = u_0\, e^{\,i\,[\omega t - K(n+1)a]} \quad \text{... } (v)$$

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From eq. (*iii*), (*iv*) and (*v*), we obtain

$$-m\omega^2 = \beta[e^{iKa/2} - e^{-iKa/2}]^2 \quad \text{... }(vi)$$

Since, $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ or $\sin^2 x = -\dfrac{1}{4}(e^{ix} - e^{-ix})^2$

Substituting in eq. (*vi*),

$$-m\omega^2 = -4\beta \sin^2\left(\frac{Ka}{2}\right)$$

or

$$\omega = \pm\sqrt{\frac{4\beta}{m}} \sin\left(\frac{Ka}{2}\right) \quad \text{... }(vii)$$

If c and ρ denotes the longitudinal stiffness and the mass per unit length of the line respectively, then

$$c = \beta a \quad \text{and} \quad \rho = \frac{m}{a}$$

$\therefore$

$$\omega = \pm\frac{2}{a}\sqrt{\frac{c}{\rho}} \sin\left(\frac{Ka}{2}\right)$$

or

$$\omega = \pm\frac{2}{a} v_s \sin\left(\frac{Ka}{2}\right) \quad \text{... }(viii)$$

where $v_s = \sqrt{c/\rho}$ is a constant for a given lattice.

Eq. (*viii*) gives the angular frequency in terms of v_s, generally referred as velocity of sound waves in solids.

SOLVED EXAMPLES

Example 8.1 One of the pendulums of a coupled oscillator is clamped while the other is free to oscillate. Show that the frequency of the single pendulum is given by

$$\nu = \frac{1}{2\pi}\sqrt{\frac{g}{l} + \frac{s}{m}}\,.$$

Solution. Suppose the two pendulums A and B of length l and having bobs of mass m are coupled together by a light spring of stiffness constant s as shown in Fig. 8.1. Then, as proved in (article 8.2) the equations of motion of the pendulums A and B respectively, when the coupled system is set into oscillation are

$$m\frac{d^2x}{dt^2} = -mg\frac{x}{l} - s(x - y)$$

and

$$m\frac{d^2y}{dt^2} = -mg\frac{y}{l} + s(x - y)$$

When the pendulum B is clamped, $y = 0$ and the equation of motion of A is given by

$$m\frac{d^2x}{dt^2} = -mg\frac{x}{l} - sx$$

or

$$\frac{d^2x}{dt^2} = -\left(\frac{g}{l} + \frac{s}{m}\right)x$$

This is the equation of motion of a simple harmonic oscillator and the angular frequency of oscillation ω_1 is given by

$$\omega_1 = \sqrt{\frac{g}{l} + \frac{s}{m}}$$

and frequency $$\nu_1 = \frac{1}{2\pi}\sqrt{\frac{g}{l} + \frac{s}{m}}\ .$$

When the pendulum A is clamped $x = 0$ and the equation of motion of B is given by

$$m\frac{d^2y}{dt^2} = -\,mg\frac{y}{l} + s(-\,y)$$

or $$\frac{d^2y}{dt^2} = -\left(\frac{g}{l} + \frac{s}{m}\right)y$$

This is again the equation of motion of a simple harmonic oscillator and the angular frequency of oscillation ω_2 is given by $\omega_2 = \sqrt{\frac{g}{l} + \frac{s}{m}}$

and frequency $$\nu_2 = \frac{1}{2\pi}\sqrt{\frac{g}{l} + \frac{s}{m}}$$

Thus whether the pendulum A is clamped or the pendulum B, the frequency of oscillation of the unclamped single pendulum is given by

$$\nu = \frac{1}{2\pi}\sqrt{\frac{g}{l} + \frac{s}{m}}$$

or $$\omega = \sqrt{\frac{g}{l} + \frac{s}{m}}$$

Example 8.2 Two identical pendulums are connected by a light spring attached to their bobs. The mass of each bob is 10 gm and the stiffness constant of the spring is 8×10^{-3} Nm^{-1}. When one pendulum is clamped the period of the other is found to be 1.20 sec. Find the periods of the normal modes. (*H.P.U.*, 2001)

Solution. When one of the pendulums is clamped the angular frequency ω of the other is given by

$$\omega = \sqrt{\frac{g}{l} + \frac{s}{m}}$$

But $\frac{g}{l} = \omega_0^2$ where ω_0 is the normal mode frequency of the in-phase mode.

$$\therefore \qquad \omega = \sqrt{\omega_0^2 + \frac{s}{m}}$$

or $$\omega^2 = \omega_0^2 + \frac{s}{m} \qquad ...(i)$$

Here $T = 1.20$ *sec.* $\therefore$ $$\omega = \frac{2\pi}{T} = \frac{2\pi}{1.20} = 5.237\ s^{-1}$$

$$s = 8 \times 10^{-3}\ \text{Nm}^{-1}\ ; \quad m = 109 = 10^{-2}\ \text{kg}$$

Substituting in Eq. (*i*), we have

$$(5.237)^2 = \omega_0^2 + \frac{8 \times 10^{-3}}{10^{-2}}$$

or $$\omega_0^2 = (5.237)^2 - 0.8 = 26.626$$

$\therefore$ $$\omega_0 = 5.16 \text{ sec}^{-1}$$

and $$T_0 = \frac{2\pi}{\omega_0} = 1.218 = 1.22 \text{ sec.}$$

This is the time period of the in-phase mode.

The normal mode frequency of the out of phase mode is given by

$$\omega_2 = \sqrt{\omega_0^2 + \frac{2s}{m}}$$

$\therefore$ $$\omega_2^2 = 26.626 + \frac{2 \times 8 \times 10^{-3}}{10^{-2}} = 26.626 + 1.6$$

$$= 28.226$$

or $$\omega_2 = 5.312$$

Hence $$T_2 = \frac{2\pi}{\omega_2} = 1.18 \text{ sec.}$$

This is the time period of the out of phase mode.

Example 8.3 The angular vibrational frequency of *CO* molecule is 0.6 × 10^{15} s^{-1}. Calculate the amount of work required for stretching it by 0.5 Å from the equilibrium position.

Solution. Reduced mass of *CO* molecule

$$\mu = \frac{m_1 \times m_2}{m_1 + m_2} = \frac{12 \times 16}{12 + 16} \text{ a.m.u} = 6.85 \times 1.67 \times 10^{-27} \text{ kg}$$

(where 1 a.m.u. = 1.67×10^{-27} kg)

Angular vibrational frequency $\omega = 2\pi n = 0.6 \times 10^{15} \text{ s}^{-1}$

Now $$\omega = \sqrt{\frac{s}{\mu}}$$

$\therefore$ Inter-atomic force constant $s = \omega^2 \mu$

$$= (0.6 \times 10^{15})^2 \times 6.85 \times 1.67 \times 10^{-27}$$

$$= 4.118 \times 10^3 \text{ Nm}^{-1}$$

Work done for stretching by $x = 0.5$ Å $= 0.5 \times 10^{-10}$ m

$$W = \frac{1}{2} sx^2 = \frac{1}{2} \times 4.118 \times 10^3 \times 0.5 \times 0.5 \times 10^{-20} = 5.15 \times 10^{-18} \text{ J}$$

Example 8.4 Sodium chloride molecule vibrates with natural frequency 1.14 × 10^{13} Hz. Calculate the interatomic force constant for the molecule. Given mass of sodium atom = 23 a.m.u. and that of chlorine atom is 35 a.m.u. (1 a.m.u. = 1.67 × 10^{-27} kg).

Solution. The vibrations of sodium chloride molecule are similar to the vibrations of a mechanical system of two masses coupled by a spring of force constant *s*, the frequency of which is given by

$$n = \frac{1}{2\pi}\sqrt{\frac{s}{\mu}}$$

where μ is the reduced mass equal to $\dfrac{m_1 m_2}{m_2 + m_2}$.

∴ Reduced mass of sodium chloride molecule

$$\mu = \frac{23 \times 35}{23 + 35} = 13.88 \text{ a.m.u.} = 13.88 \times 1.67 \times 10^{-27} = 23.18 \times 10^{-27} \text{ kg}$$

Now frequency $n = \dfrac{1}{2\pi}\sqrt{\dfrac{s}{\mu}}$

∴ $$n^2 = \frac{1}{4\pi^2}\frac{s}{\mu}$$

or $$s = 4\pi^2 n^2 \mu = 4 \times \left(\frac{22}{7}\right)^2 \times (1.14 \times 10^{13})^2 \times 23.18 \times 10^{-27}$$

$$= 118.9 \text{ Nm}^{-1}.$$

Example 8.5 In a transformer the mutual inductance of two coils is 0.3 H where as the self inductance of primary and secondary are 0.28 H and 0.36 H respectively. Is the transformer loose or tight coupled. (*G.N.D.U.* 2004, *P.U.*, 2002, 2000; *H.P.U.*, 2003)

Solution. Co-efficient of coupling, $k = \dfrac{M}{\sqrt{L_p L_s}} = \dfrac{0.3}{\sqrt{0.28 \times 0.36}} = 0.95$

As the co-efficient of coupling k is nearly equal to unity, the transformer is almost *tight coupled.*

EXERCISE CH. 8

LONG QUESTIONS

1. (***a***) Explain the meaning of coupled oscillator. (*P.U.*, 2001)

(***b***) Define and explain normal co-ordinates, degrees of freedom and normal modes of vibration of an oscillatory system.

(*Pbi.U.*, 2002; *H.P.U.*, 2000; *P.U.*, 2000; *G.N.D.U.*, 2000)

2. What is meant by coupling of two oscillators? Discuss completely the oscillations of two identical stiffness coupled pendulums and write the equation of motion of the system in different cases in terms of normal co-ordinates, X and Y. (*H.P.U.*, 2002, 2001; *P.U.*, 2002)

3. Derive an expression for the total energy of a stiffness coupled system of identical pendulums and show that the total energy of the system remains constant.

(*H.P.U.*, 2003, 2002)

4. Which different types of coupling are used for coupling two oscillators? Show that in the in phase mode, the frequency of oscillations is the same as of uncoupled oscillators whereas in the out of phase mode, the frequency of oscillations gets raised.

5. Give a general method of finding normal mode frequencies and obtain expression for the normal mode frequencies of stiffness coupled pendulums. (*G.N.D.U.*, 2002, *P.U.*, 2000)

6. (***a***) Do normal modes exchange energy with each other ?

(*b*) Show that normal modes are independent of each other and there is no exchange of energy between two coupled pendulums. What do you mean by the statement that normal co-ordinates are independent of each other? (*P.U.*, 2003, 2000; *Pbi.U.*, 2002, 2000; *H.P.U.*, 2003, 2002, 2000)

7. Explain transfer of energy between two resistance free electric circuits which are inductively coupled. When is the coupling loose or tight? Obtain an expression for the normal mode frequencies and show that they are almost equal for loose coupling. (*P.U.*, 2004, 2003, 2002, 2001, 2000; *Pbi.U.*, 2003, 2002; *H.P.U.*, 2001, 2000)

SHORT QUESTIONS

1. Give the characteristics of the in-phase and out of phase mode of vibration of two identical coupled simple pendulums (oscillators). (*G.N.D.U.*, 2003, 2002, 2001; *Pbi.U.*, 2003; *H.P.U.*, 2000; *P.U.*, 2000,)

2. Explain the inductance coupling of two electrical oscillators and define co-efficient of coupling. (*G.N.D.U.*, 2001; *P.U.*, 2004, 2001; *H.P.U.*, 2001)

3. Discuss coupled oscillations using coupled system of two identical bodies. (*Nagpur U.* 2007)

4. Obtain an expression for the time period of a two body harmonic oscillator. (*Nagpur U.* 2008, 2006)

5. Define and explain Normal co-ordinates, normal modes and degree of vibration of an oscillatory system.

6. Discuss exchange of energy between two normal modes of frequencies of two coupled pendulums.

7. What do you understand by degenerate (normal) modes of vibration? Writing down the equations of motion of two identical simple pendulums connected by light spring, find out the frequency of faster normal modes of vibration. (*Purvanchal U.* 2007)

8. Explain the oscillations of two coupled oscillators. What are different modes of vibration? (*Nagpur U.* 2009)

NUMERICAL QUESTIONS

1. Two masses m_1 and m_2 are coupled by a spring of stiffnes s. The masses are pulled apart and released. Show that their motion is simple harmonic. Prove that the system will oscillate with a frequency

$$\sqrt{\frac{s}{\mu}} \quad \text{where} \quad \mu = \frac{m_1 m_2}{m_1 + m_2}.$$

(*P.U.*, 2001, 2000; *Nagpur U.* 2007, 2008)

[**Hint:** For proof Pl. refer article 5.9.8]

2. Find self inductance of secondary coil of a transformer having tight coupling with co-efficient of coupling 0.96. The self inductance of primary coil is 0.3 H and mutual inductance 0.2 H between the two coils. [**Ans.** L_s = 0.1477 H]

9 MOTION OF CHARGED PARTICLES IN ELECTRIC AND MAGNETIC FIELDS

INTRODUCTION

Charged particles like electrons, protons, ionized atoms and some times bare nuclei are set into motion under the action of electric and magnetic fields, either mutually perpendicular or parallel. Motion of these charged particles offers the best example of *particle dynamics* as idealised in classical mechanics. They are considered as tiny lumps of matter approximating very closely to the concept of point mass which allows us to restrict the energy account to translational kinetic energy and potential energy of the particle. Macroscopic electric and magnetic fields are employed to subject the particles to desired forces and to obtain predictable trajectories covering measurable distances. These forces are sufficiently large enough to dominate the gravitational force. Moreover, the particle trajectories (usually parabolic) are confined to evacuated enclosers so that the particle paths are not distorted by collisions with atmospheric gas molecules. This ensures the uninterrupted travel of particle over the measurable distances. The trajectory of the path traced can be analysed by the application of the laws of classical Newtonian mechanics.

9.1 $\vec{E}$ AS AN ACCELERATING FIELD

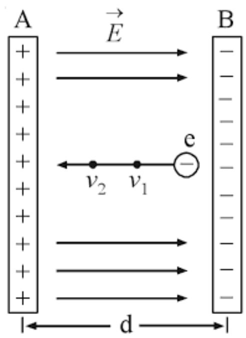

Fig. 9.1

Let us consider the motion of an electron parallel to an uniform electric field $\vec{E}$. For this, consider two plane parallel metal plates A and B seperated by a small distance d and insulated from each other (fig. 9.1). If a dc voltage source is connected between the plates, the plates are charged oppositely and an electric field is produceed in the region between the plates. If the spacing d is small, the electric field $\vec{E}$ will be fairly uniform, except near the edges. The electric field $\vec{E}$ is directed from the positive plate A towards the negative plate B. If the potential difference between the plates A and B is $(V_A - V_B) = V$, the electric field strength E in the region will be

$$E = \frac{V}{d}$$

If the space between the plates is evacuated and an electron of mass m and charge e is placed at rest between the plates and released, the force experienced by an electron due to electric field will be

$$\vec{F} = -e\vec{E} \qquad \ldots (i)$$

The – ve sign indicates that the force $\vec{F}$ accelerates the electron in opposite direction of $\vec{E}$. According to Newton's second law of motion, the acceleration is given by

$$\vec{a} = \frac{\vec{F}}{m}$$

or $$\vec{a} = -\frac{e\vec{E}}{m} \qquad \text{... }(ii)$$

Since, the parameters e, m and $\vec{E}$ are constant here, the acceleration suffered by the electron is constant *i.e.* uniform. This phenomenon is very useful and has an application in the construction as cathode ray osciloscope (CRO), in which an electron beam is accelerated by passing the beam through an electric field.

9.2 PATH DESCRIBED BY ELECTRON IN ELECTRIC FIELD

Suppose a beam of electrons, each of mass m and carrying a negative charge $-e$ is entering the uniform electric field E at point O with velocity v in a horizontal direction as shown in 9.2. The electric field is directed from + ve plate A to – ve plate B, acting right angles to v.

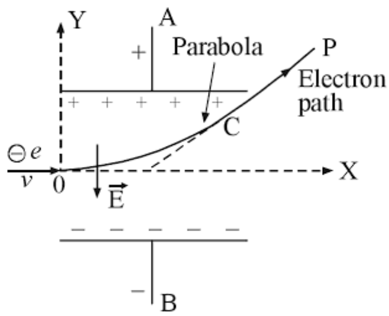

Fig 9.2: Parabolic path of an electron

The electric force acting on the electron is given by

$$F = eE \qquad \text{(in magnitude)}$$

The force F is acting in vertically upward direction, towards the positive plate A. The upward acceleration a_y acquired by the electron will be

$$a_y = \frac{F}{m} = \frac{eE}{m} \qquad \text{... }(i)$$

For the vertical motion the initial upward velocity = 0

∴ At time t, the vertical displacement y is given by

$$y = 0 \times t + \frac{1}{2} a_y t^2$$

$$= \frac{1}{2}\left(\frac{eE}{m}\right)t^2 \qquad \text{... }(ii)$$

In the horizontal direction no force acts on the electron entering the electric field. Therefore, in the electric field, the horizontal velocity v is constant along OX. Hence the horizontal displacement x at time t is given by

$$x = vt \qquad \text{or} \qquad t = x/v \qquad \text{... }(iii)$$

Substituting for t in equation (ii), we have

$$y = \left(\frac{eE}{2mv^2}\right)x^2 \qquad \text{... (iv)}$$

Equation (iv) shows that the path of the electron beam in the electric field is a *parabola* symmetrical about the Y-axis. The beam describes the path OC which is parabolic in nature. When the electron emerges out from the electric field, it travelles along a straigh line CP tangential to the parabola at the exit point C.

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9.3 DISCHARGE TUBE

Michel Faraday in 1836, on the basis of the experiments conducted on electrolysis concluded that there must be some kind of elementary electric charge. In 1891, G.J.Stoney (1826 – 1911) gave the name electron to the elementary electric charge. Sir William Crookes (1832 – 1919), English Physicist, discovered the phenomenon of cathode rays and he suggested in 1879 that the rays consists of streams of negatively charged particles. In 1897, J.J. Thomson confirmed that all these particles are alike and called them electrons He further determined the ratio of the charge e of the electron to its mass m i.e. e/m by using a specially designed discharge tube.

Construction: It is about 30 cm long and 2 to 3 cm in diameter. It has two plane aluminium electrodes: the anode A and cathode K. It is provided with a side-tube which is connected to a pressure gauge G and vacuum pump P through a stop cock S. The pressure of the enclosed gas can be continuously decreased by the pump. The pressure gauge measures the pressure and a desired pressure can be maintained constant by using the stop cock. The anode and the cathode are connected respectively to the positive and negative terminals of the secondary of an induction coil. An induction coil is an apparatus which produces an intermittent unidirectional high potential difference across its secondary coil. For a discharge-tube of about 30 cm long, a potential difference of about 6000 to 15000 V between its electrodes is sufficient for studying the conduction of electricity through the enclosed gas at low pressure.

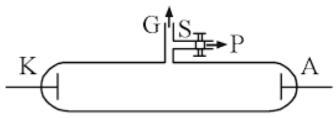

Fig. 9.3 (*a*): Discharge tube

J.J. Thomson in his famous apparatus in order to determine e/m, used a highly evacuated glass enevelope to eliminate the collisions of electrons (generated by the cathode) with air molecules. Two electrodes C and A are provided in the left end of the tube as shown in fig. 9.3 (*b*). A high potential difference applied between the cathode C and anode A causes emission of electron from the surface of the cathode. These electrons are accelerated and collimated by the slits A_1 and A_2 into a narrow parallel beam. This electron pencil beam is further subjected to *electric field and magnetic fields so as to produce a glow on the screen* produced by a pair of deflecting plates. Suitable arrangement is made to apply a fairly uniform magnetic field in the same region. Finally, the electron beam strikes on the fluorescent screen on the opposite side and produces a bright spot.

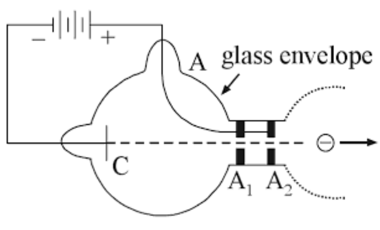

Fig. 9.3 (*b*)

9.4 ELECTRON GUN

An electron gun is a device which produces a narrow beam (also called as pencil beam) of high intensity. It was first designed by V.K. Zworykin in 1933.

Principle. The electron gun makes use of the fact that non-uniform electric fields cause bending of electron paths. Thus, a stream of electrons experiences a change in direction of motion when it travels through a non-uniform electric field. Its path is bent at each equipotential surface in the same way as a light ray is bent at an optical boundary. This leads to the focusing of electrons and the arrangement is called as an **electrostatic lens** or **electron lens**. Two coaxial cylindrical anodes A_1 and A_2 at different positive potentials, forms a electron focussing device and is called an electron lens system.

Construction. The schematic diagram of an electron gun is shown in Fig. 9.4. The electron gun consists of a cathode K, a filament heater F, a control grid G and two anodes A_1 and A_2. The cathode K is a short hollow nickel cylinder and encloses the filament heater F. The front face of the cathode is coated with thoriated tungsten or barium and strontium oxides. The coating helps

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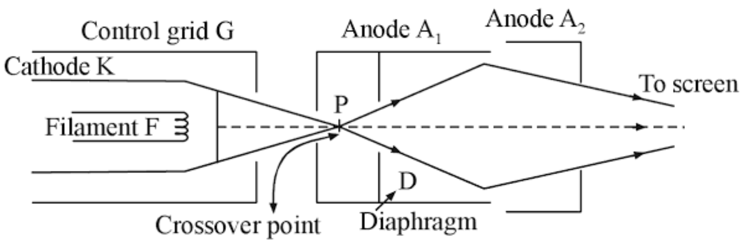

Fig. 9.4 Schematic of an electron gun.

thermionic emission of electrons to occur at moderate temperatures of about 700° to 900°C. The cathode is surrounded by the control grid G, which also is a hollow metal cylinder with a small central aperture in its front face to allow electrons to pass through. Two more short metal cylinders (A_1 and A_2) with central apertures are positioned coaxialy beyond the control grid. The entire assembly is held in an evacuated space so that the electron paths are not disturbed due to collisions with atmospheric molecules. A power supply provides the necessary voltages to the electrodes.

Working. When the power is turned on, the filament heater heats up the cathode. At a temperature characteristic of the cathode material, electrons are emitted from its front surface and they pass through the control grid. The grid is held at a negative potential with respect to the cathode and controls the number of electrons passing through it. If the grid is held at a lesser negative potential, a larger number of electrons pass through it. On the other hand, if the grid is held at a higher negative potential, a smaler number of electrons pass through it. Thus, *the grid with the negative potential on it acts as a gate and regulates the passage of electrons through it*. The anodes A_1 and A_2 are held at positive potential, the potential on A_2 being greater than on A_1.

The cathode, the grid and the first anode A_1 constitute the first electron lens of the system. It is known as **prefocussing lens.**

Electrons emitted from the cathode tend to diverge because they repel each other. The convex equipotential surfaces bend the electron paths toward the centre axis. Consequently, all electrons passing through the aperture in the control grid converge toward a point P just inside the first anode. The point P is located on the axis of the gun and is known as the **cross-over point**.

The area of the point P would be very small compared to the relatively larger cathode surface emitting the electrons. Therefore, the electrons emerging from P can be easily focussed to a fine point better than the electrons emerging from the cathode. Thus, *the role of the first lens is to converge the beam to the cross-over point which then on, acts as a point source of electrons for the second lens.*

The anodes A_1 and A_2 form the second electron lens which draws electrons from the cross-over point and brings them to a fine focus. The diaphragm D in anode A_1 cuts off the wide angle electrons emerging from P. The focal point of the beam is controled by adjusting the potentials on A_1 and A_2. The optical analogy of electron gun is shown in Fig. 9.5.

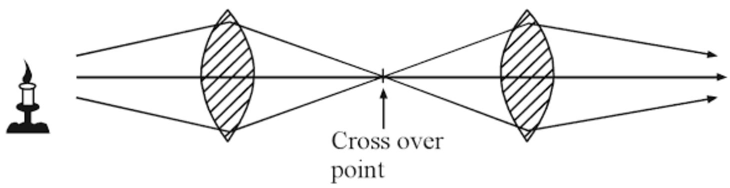

Fig. 9.5. Optical analogy of electron gun.

Applications: The electron gun is used in variety of applications. It is used in a CRO to display wave shapes, in a TV to display pictures, in an electron microscope to obtain a magnified image and in EBM and EWM for machining and welding jobs.

9.5 CATHODE RAY OSCILLOSCOPE (CRO)

The cathode ray oscillograph is an instrument which can record instantaneous values of rapidly varying voltages. It is used to observe various alternating current wave forms and put to other uses in radio-engineering, television and industry. It essentially consists of (*i*) a cathode ray tube (C.R.T.), (*ii*) a sweep circuit or time base circuit, (*iii*) a synchronisation circuit, (*iv*) high and low voltage supplies and (*v*) horizontal and vertical amplifiers, are discussed below :

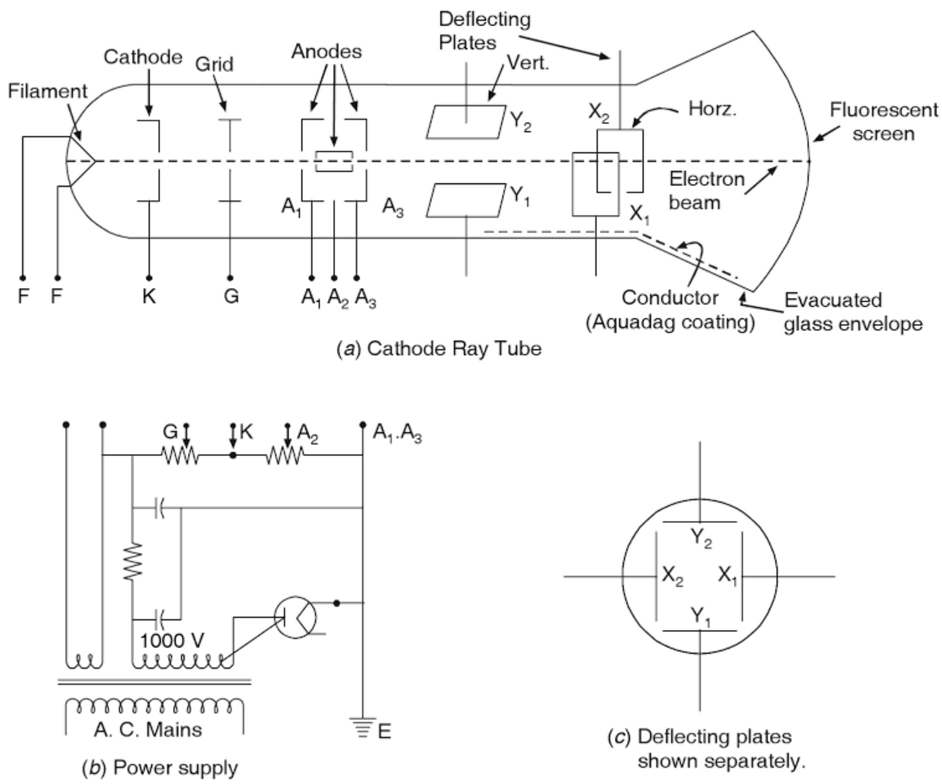

Fig. 9.6

(*i*) **Cathode ray tube.** The cathode ray tube is an evacuated tube of the shape shown in Fig. 9.6 (*a*). Its main parts are an *electron emitter called the cathode* (K), a device to control the intensity called the *grid* (G); accelerating and focusing electrodes (A_1, A_2, A_3) known as anodes; deflecting plates vertical (Y_1, Y_2) and horizontal (X_1, X_2) and a screen which fluoresces when electrons impinge on it.

Electron emitter. The method of obtaining electrons is similar to that used in a thermionic valve. The *cathode K* is indirectly heated by passing a suitable current through the *filament FF* which is electrically insulated but thermally connected to it. The cathode is oxide coated type (with a material like *MgO*) and when heated gives out a copious supply of electrons. (For details refer article 9.4).

Intensity control. The cathode is surrounded by a metal shield in the form of a cylinder with a hole at the end farther from the cathode. The cylinder is given a negative potential with respect to the cathode so that the electrons are repelled from it and any off axis electrons join the axial stream to form a fine beam which passes through the hole. This electrode is called the *grid* (or shield or *modulator*). By varying the *negative* potential of the grid, the emission of electrons from can be varied the cathode can be controlled and thus the *brightness* of the spot on the screen can be varied.

Focus. The beam of electrons from the grid passes through a series of electrodes known as the first, second and third anode (A_1, A_2 and A_3) all of which are at positive potentials with respect to the cathode. These electrodes act like an electron lens system and bring about a sharp focusing of the electron beam on the fluorescent screen. Generally, the anodes A_1 and A_3 are connected together and have the *highest positive potential* with respect to the cathode, while the anode A_2 has a potential lower than that of A_1 and A_3 but higher than that of the cathode. The potential of the anode A_2 can be varied and thus the spot of light brought to a fine focus on the screen.

The whole system consisting of the cathode, the grid and the anode shoots out a fine pencil of fast moving electrons and is, therefore, caled an '*electron gun*'.

Deflection of electron beam. The common form of a cathode ray tube uses electrostatic deflection arrangement. For this purpose two sets of plates X_1X_2 and Y_1Y_2 are mounted between the final anode A_3 and the screen. As shown in Fig. 9.6 (*a*) the plates Y_1Y_2 are horizontal and represent a parallel plate capacitor in which an electric field in the *vertical* (Y) direction is set up when a potential difference is applied to the plates, thereby causing the electron beam to move *up* or *down* according to the polarity of the plates. The plates X_1 and X_2 are vertical and an electric field in the *horizontal* (X) direction is set up when the potential difference is applied to these plates, thereby causing the electron beam to move from left to right or right to left according to the polarity of the plates. If electric fields are simultaneously set up between the vertical and the horizontal deflecting plates, the resultant displacement of the electron wil be in accordance with the vector sum of the vertical and horizontal displacements.

The position of the spot on the screen can, therefore, be adjusted by applying suitable D.C. potentials to each of the two sets of deflecting plates.

One plate of each pair (X_1 and Y_1) is connected to the anode A_3 so that there is no electric field between the plates and the electron gun and the anode A_3 is earthed.

Screen. The screen is coated with a fluorescent material such as zinc orthosilicate known as 'phosphor' and a brilliant spot is visible on the screen due to fluorescence where trhe electron beam strikes. The inside of the neck of the tube is coated with aquadag or some other conducting material, thereby connecting the screen to the final plate A_3. The screen is, therefore, maintained at the same potential as A_3 which keeps the electron beam from A_3 to the screen in free flight as the space through which the electron beam passes is free from all electrostatic (or magnetic) fields except those set up between the deflecting plates. As the electrons strike the screen these '*leak off*' to the anode A_3 and finally through the power supply to the cathode.

Aquadag coating is necessary to remove the electrons striking the fluorescent screen and return the same to the cathode. If the electrons striking the fluorescent screen are not removed from it, these will repel the electrons which arrive on the screen and a stage will soon come when negative charge on the screen will become so large that no more electrons will be able to reach the screen.

The aquadag is not directly connected to the screen. It only collects the secondary electrons emitted by the screen when the electron beam strikes on it and return them to the cathode.

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Willemite – most suitable coating material. In the selection of fluorescent material for a *C.R.T.* screen an important point to be noted is '*persistence*' of its glow after the electrons have ceased to bombard the screen. The most suitable material for this purpose is one which has a persistence less than 0.1 sec. This is highly essential for rapidly changing images such as those displayed on a TV screen, because persistence of vision of the human eye is of that order. Wilemite (Zinc Silicate $ZnSiO_4$) which fluoresces with a green colour is one of the materials which satisfies this condition and is thus most suitable as coating material for *CRT* screen.

***(ii)* Time base.** One of the great advantages of a cathode ray oscilograph is its ability to indicate very short time intervals and to record variations of voltage with time. If an alternating potential difference is applied to the plates Y_1Y_2, the spot on the screen will trace out a vertical line proportional to the peak value of the applied voltage. If the wave form of the alternating voltage is to be obtained the spot must be made to move horizontally along the X-direction as well as move up and down along the Y-axis so that the pattern may spread out on the screen. This is done by connecting the horizontal deflecting plates X_1, X_2 to a source of voltage that rises gradually at constant rate to a maximum value and then suddenly drops back to zero. Such a voltage is said to have a *sawtooth* shape as shown in Fig. 9.7

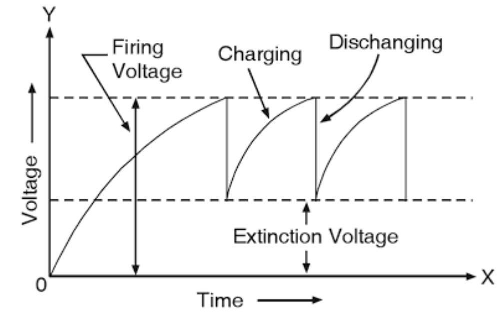

Fig. 9.7

It causes the beam to move horizontally across the screen at a uniform speed and then snaps or sweeps back to its starting point. The electronic circuit producing the sawtooth voltage is called '*sweep circuit*' or '*Time base circuit*'. The name '*time base*' is due to the fact that when a sawtooth wave is applied to the horizontal deflection plates, the X-axis on the screen may be taken to represent '*time*'. A time base is said to be *linear* if it produces a uniform movement of the spot. The sweep circuit or time base enables us to obtain a pattern on the screen which is exactly the same as a curve of varying voltage applied across the vertical deflecting plates Y_1Y_2 as function of time. The only condition that has to be fulfilled is that the period of the sawtooth voltage must be equal to or an exact mulitple of the period of the applied voltage to be studied.

When flyback time is comparable to sweep time. An ideal time base voltage is the one which has the sawtooth wave shape as shown in Fig. 9.8 (*a*).

However, in practice the time base generated in a *CRO* produces the wave form-shown in Fig. 9..8 (*b*). It takes some time for the voltage to fall from the maximum to zero value. This time interval is called *flyback* or *retrace. time.* When the time base voltage has a finite flyback time comparable to *sweep time* the display of the wave form gets distorted and a portion of the wave form corresponding to the flyback time is cut off as shown in Fig. 9.8 (*c*)

The path retraced by the spot during the flyback time is called *retrace path*. This retrace path spoils the good visual effect. To remove this defect, in most of CRO's, the electron beam is blanked off by applying a negative voltage to the *control gird* during the flyback period.

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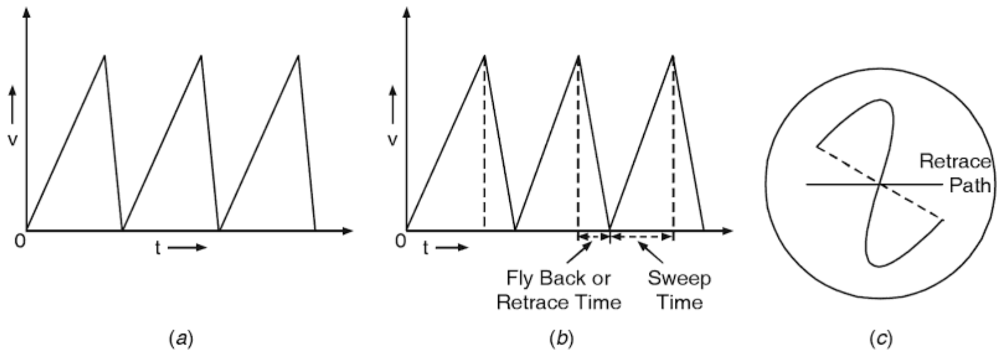

Fig. 9.8

***(iii)* Synchronisation.** When two events occur simultaneously they are said to be synchronised. The input signal applied to the Y-plates, is synchronised with the sweep signal and will appear *stationary* when the beginning of a pulse of the signal wave train is caused to appear at the beginning of the sweep trace. To keep the trace steady a portion of the incoming signal is fed to the sawtooth generator. This serves to *lock* the generator in step with vertical input frequency. In such a case the trace is repeated again and again in the same position and thus appears sationary on the screen.

***(iv)* Power supply.** The power supply provides the high voltage for the cathode ray tube, filament voltage for vacuum tubes (if any), and low voltage for other valves (or transistors) in the circuit and is shown in Fig. 9.6 (*b*).

***(v)* Amplifiers.** To amplify the weak signals so as to increase the deflection in the vertical and horizontal directions without distortion of the incoming signal, the *X* and *Y* plates producing the vertical and horizontal deflections are connected to suitable amplifiers.

Working of CRO. We shall describe the working of a cathode ray oscillograph by giving a method of finding out the wave form of an unknown alternating voltage. The unknown alternating voltage is applied to the *Y* plates as shown in Fig. 9.9. The spot of light will move verticaly up and

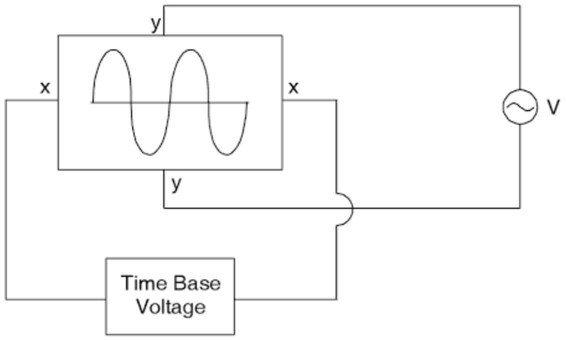

Fig. 9.9

down in a vertical line in response to the voltage applied and will simply trace out a straight line on the screen. When the *time base voltage* is applied to the *X*-plates, it makes the spot move in a horizontal direction and as the spot reaches the end of the screen, it is again made to fly back to the starting point. Since the electron beam is now free to move in two perpendicular directions, actual picture of the wave form will be formed on the screen. A stationary picture is obtained when the time period (or frequency) of the time base voltage is adjusted equal to the time period (or frequency) of the unknown voltage.

Applications and Uses of CRO.

(*i*) The chief use of a cathode ray oscillograph is to demonstrate visually the variations in alternating or fluctuating currents or potentials in electric circuits. The electron beam possesses practically no inertia and immediately responds to any variations.

(*ii*) It is used to measure the phase angle, power factor, frequency of alternating current, inductance and dielectric loss, etc.

(*iii*) It is used to trace the hysteresis loop of iron.

(*iv*) It is used in radio engineering to test the fidelity of response and depth of modulation in wireless sets, to study dynamic valve characteristics, observations on atmospherics, direction finding and detection of reflected wireless waves as in the case of radar.

(*v*) It is used in television receivers.

(*vi*) It is used in industry for the study of mechanical pressure, indicator diagram of internal combustion engines, electro-cardiography, *i.e.*, study of the action of the heart, etc., as a display device.

9.6 ELECTROSTATIC DEFLECTION SENSITIVITY

The deflection sensitivity of an electrostatic deflection cathode ray tube is defined as the amount of spot deflection on the screen in mm per volt potential difference applied between the deflecting plates. It is expressed in mm per volt.

The *dc* or peak to peak *ac* voltage which must be applied across the deflection plates to produce a spot deflection of 1 mm on the screen is called *deflection factor*. It is expressed in volts per mm. Thus deflection sensitivity is reciprocal of deflection factor.

To calculate the expression for electrostatic deflection sensitivity let :

Length of each deflection plate $= l$

Distance between the plates $= d$

Distance between end of plates and screen $= D$

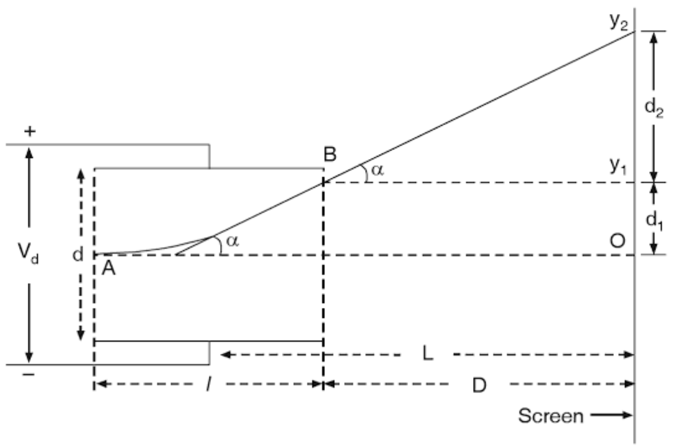

Fig. 9.10

If V_a is the final accelerating anode voltage, and v the velocity of an electron on just entering the deflection plates, then

$$\frac{1}{2} mv^2 = eV_a$$

where e is the charge and m the mass of the electron.

$$\therefore \qquad v^2 = 2\frac{e}{m}V_a \qquad \text{... } (i)$$

If V_d is the voltage apllied between the plates, then

Force acting on the electron towards the upper (positive) plate is given by

$$F = \frac{eV_d}{d}$$

If a is the resulting acceleration of the electron, then

$$a = \frac{e}{m}\frac{V_d}{d}$$

Time taken by the electron to traverse through the length of deflection plates

$$t = \frac{l}{v}$$

Hence upward velocity of the electron on emerging out of the deflection plates

$$v_y = 0 + at = \frac{e}{m}\frac{V_d}{d}\frac{l}{v} \qquad \text{... } (ii)$$

The electron enters the deflecting system at A and leaves at B. The path of the electron between A and B is curved.

Vertical upward displacement d_1 during the time interval t is given by

$$d_1 = 0 + \frac{1}{2}at^2 = \frac{1}{2}\frac{e}{m}\frac{V_d}{d}\frac{l^2}{v^2} = \frac{eV_d l^2}{2mv^2 d} \qquad \text{... } (iii)$$

After emerging from the plates, the electron beam moves in a straight path making an angle α with the original direction and meets the screen at Y_2.

Now $$\tan\alpha = \frac{d_2}{D} = \frac{v_y}{v}$$

or $$d_2 = \frac{Dv_y}{v} = \frac{D}{v}\frac{e}{m}\frac{V_d}{d}\frac{l}{v} = \frac{eV_d Dl}{mv^2 d} \qquad \text{... } (iv)$$

Total deflection $$OY_2 = OY_1 + Y_1\boldsymbol{Y}_2 = d_1 + d_2$$

$$= \frac{eV_d l^2}{2mv^2 d} + \frac{eV_d lD}{mv^2 d} = \frac{eV_d l}{mv^2 d}\left[\frac{l}{2} + D\right] \qquad \text{... } (v)$$

If L is the distance of the screen from the centre of the deflection plates then $L = \frac{l}{2} + D$

Substituting $\frac{l}{2} + D = L$ in relation (v), we have

Total deflection $$= \frac{eV_d lL}{mv^2 d} \qquad \text{... } (vi)$$

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Substituting the value of v^2 from (*i*) in (*vi*), we get

$$\text{Total deflection} = \frac{eV_d lL}{md} \cdot \frac{m}{2V_a e} = \frac{lL}{2d} \frac{V_d}{V_a}$$

$$\text{Hence, deflection sensitivity } S = \frac{\text{Total deflection}}{\text{Voltage between the plates}}$$

$$= \frac{lL}{2d} \frac{V_d}{V_a} \cdot \frac{1}{V_d} = \frac{lL}{2d\,V_a} \text{ m / volt}$$

$$= \frac{500lL}{dV_a} \text{ mm/volt} \qquad \ldots (vii)$$

From relation (*vii*), we find that electrostatic deflection sensitivity is directly proportional to the length of the deflecting plates and to the distance of the screen from the centre of the plates. It is inversely proportional to the distance between the deflecting plates and final anode voltage.

∴ Total spot deflection = Deflection sensitivity × Applid voltage between the plates

Thus we see that the deflection sensitivity can be increased by reducing the anode voltage but this reduces the brightness of the spot. This disadvantage can be overcome by accelerating the beam after it has passed through the deflection system. This process is known as *post acceleration* and is brought about by using an extra electrode caled the *intensifier anode*.

9.7 PATH DESCRIBED BY ELECTRON IN TRANSVERSE MAGNETIC FIELD

Let B be the magnetic induction (*i.e.*, flux density) of a uniform magnetic field acting perpendicular and into the plane of the paper (Fig. 9.11). A beam of electrons each of mass m and carrying negative charge $-e$, moving with velocity v in a horizontal direction enters the magnetic field at O as shown in the figure. Since v and B are at right angles to each other, the magnitude of the magnetic force exerted on an electron is given by

$$F = evB \qquad \ldots (i)$$

The direction of this force is always perpendicular to the plane of v and B. The direction as given by Fleming left hand rule as shown in figure. Since F is always perpendicular to v, it only changes the direction of v and not the speed therefore, the electron describes a *circular path* of radius R in the field. The constant radial force produces the centripetal acceleration of the electron.

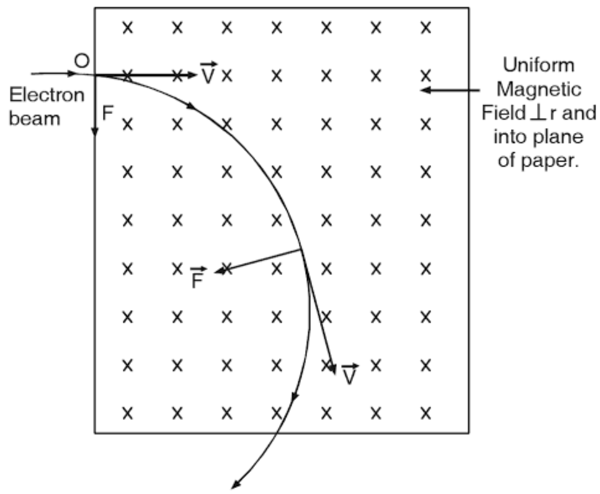

Fig. 9.11

Applying Newton's second law of motion

$$F = \frac{mv^2}{R} \qquad \text{... }(ii)$$

Equating eqn. (*i*) and (*ii*), we get

$$evB = \frac{mv^2}{R} \qquad \text{... }(iii)$$

The radius of the circular path is given by

$$R = \frac{mv}{eB} \qquad \text{... }(iv)$$

or, Speed, $$v = \frac{eBR}{m} \qquad \text{... }(iv)$$

Since the motion is circular, time for one revolution *i.e.*, the period T is given by

$$T = \frac{2\pi R}{v} = \left(\frac{2\pi}{v}\right)\left(\frac{mv}{eB}\right)$$

or $$T = \frac{2\pi m}{eB} \qquad \text{... }(v)$$

and, the frequency of revolution in the orbit is given by

$$f = \frac{eB}{2\pi m} \qquad \text{... }(vi)$$

Kinetic energy of the revolving electron in transverse magnetic field is given by

$$\text{Kinetic Energy} = \frac{1}{2}mv^2$$

$$= \frac{1}{2}m\left(\frac{eBR}{m}\right)^2$$

$$= \frac{e^2B^2R^2}{2m} \qquad \text{... }(vii)$$

Work done. The work done W on the electron by the electric field in moving it from the cathode to anode is given by

$$W = (\text{charge}) \times (\text{potential difference})$$
$$= eV$$

The kinetic energy acquired by the electron is given by

$$E = \frac{1}{2}mv^2$$

Equating, we get

$$eV = \frac{1}{2}mv^2$$

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$$\therefore \qquad \text{Speed } v = \sqrt{\frac{2eV}{m}} \qquad \text{... (viii)}$$

This is the expression for speed acquired by an electron as it reaches the anode.

9.8 MAGNETIC DEFLECTION SENSITIVITY

The deflection sensitivity of a magnetic deflection cathode ray tube is defined as the amount of spot deflection on the screen per unit magnetic flux density produced by the magnetic deflection coils.

The electron beam can be deflected by applying a magnetic field perpendicular to the beam over a short distance along its path. The magnetic field will exert a force on the electrons in a direction at right angles to both; the direction of motion of the electrons and the direction of magnetic field. As a result the electron beam emerging from the magnetic field makes an angle with its original direction. This type of deflection is known as *magnetic deflection.*

To calculate the expression for magnetic deflection sensitivity consider an electron beam AB coming from the electron gun acted upon by a uniform magnetic field of flux denisty B over a length l of its path. The magnetic field is applied perpendicular to the direction of flow of electrons so that the beam describes an arc BC of a circle. The path of the electrons remains circular within the magnetic field and after leaving the magnetic field the electrons move in a direction CD tangential to the circular arc at C. Let r be the radius of the circular arc BC.

If V_a is the accelerating potential and v the velocity attained by the electron, then

$$\frac{1}{2} mv^2 = eV_a$$

or

$$v = \sqrt{\frac{2eV_a}{m}} \qquad \text{... } (i)$$

If B is the magnetic flux density, then Lorentz force acting on the moving electron

$$= B\,ev \qquad \text{... } (ii)$$

This force acts at right angles to the direction of motion of the electron, so that the electron moves in a circular path of radius r and is acted upon by a centripetal force $\dfrac{mv^2}{r}$ ... (iii)

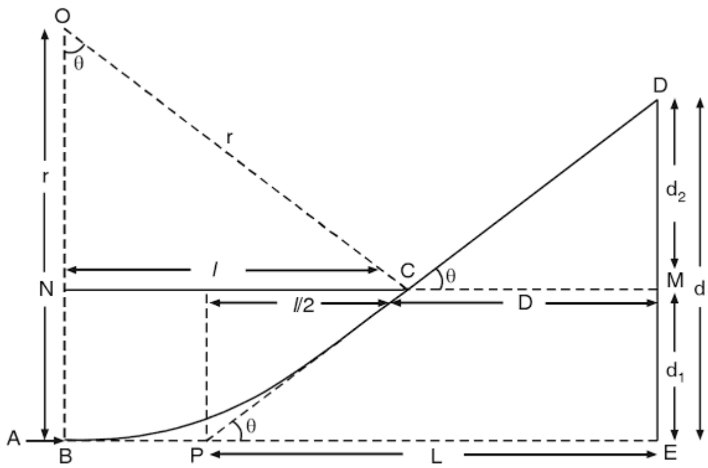


Fig. 9.12

These two forces given by relation (*ii*) and (*iii*) balance each other

$$Bev = \frac{mv^2}{r}$$

or $$r = \frac{mv}{Be}$$

Substituting the value of v from eq. (*i*), we get

$$r = \frac{m}{Be}\sqrt{\frac{2eV_a}{m}} = \frac{1}{B}\sqrt{\frac{2mV_a}{e}} \quad \text{... (iv)}$$

On leaving the magnetic field at the point C the electron beam moves along the straight line CD and hits the screen at the point D. The straight line CD is tangential to the circular path BC at C.

Now, total deflection $$d = ED = EM + MD = d_1 + d_2 \quad \text{... (v)}$$

If O is the centre of the circular arc, θ the angle subtended by the arc at O and r its radius, then

$$d_1 = EM = BN = OB - ON = r - r\cos\theta = r(1 - \cos\theta) \quad \text{... (vi)}$$

Also $$\angle DCM = \angle DPE = \theta.$$

$\therefore$ $$\tan\theta = \frac{d_2}{D}$$

or $$d_2 = D\tan\theta = D\theta \text{ when } \theta \text{ is small}$$

Also when θ is small $$\theta = \sin\theta = \frac{l}{r}$$

$\therefore$ $$d_2 = D\frac{l}{r} \quad \text{... (vii)}$$

and $$\cos\theta = (1 - \sin^2\theta)^{1/2} = 1 - \frac{1}{2}\sin^2\theta = 1 - \frac{\theta^2}{2} = 1 - \frac{l^2}{2r^2}$$

$\therefore$ From eq. (*vi*), $$d_1 = r(1 - \cos\theta) = r\left(1 - 1 + \frac{l^2}{2r^2}\right) = r\frac{l^2}{2r^2} = \frac{l^2}{2r}$$

Hence, total deflection $$d = d_1 + d_2 = \frac{l^2}{2r} + D\frac{l}{r} = \frac{l}{r}\left(\frac{l}{2} + D\right)$$

But $\frac{l}{2} + D = L$, the distance of the screen from the mid point of the field.

$\therefore$ Total deflection $$d = \frac{lL}{r}$$

Substituting the value of r from Eq. (*iv*), we have

$$d = lLB\sqrt{\frac{e}{2mV_a}}$$

$\therefore$ Magnetic deflection sensitivity $$S_m = \frac{d}{B} = lL\sqrt{\frac{e}{2mV_a}} \quad \text{... (viii)}$$

Importance of magnetic deflection sensitivity. The magnetic deflection sensitivity is inversely proportional to $\sqrt{V_a}$, where as electrostatic deflection sensitivity is inversely proportional to V_a. Therefore, electric deflection sensitivity decreases more rapidly with increasing anode voltage than magnetic deflection sensitivity. As the loss of magnetic sensitivity is smaller as V_a is increased for greater spot brightness, it is customary to use magnetic deflection in T.V. system and Radar indicator, where high spot brightness is required.

The electric deflection suffers larger *deflection defocussing i.e.*, as the angle of deflection is increased the spot on the screen tends to be distorted and enlarged but this is much less in the case of magnetic deflection even for large deflection angles. The greater allowable deflection angle decreases the tube length for a given diameter. Hence magnetic deflection is desirable in cathode ray tubes used in television.

Importance of electrostatic deflection sensitivity. The electrostatic deflection requires little power for deflection while large power is consumed in the electromagnets required for magnetic deflection. Hence electrostatic deflection is used in common purpose osciloscopes. Moreover, electrostatic deflection can be conveniently used for high frequencies.

9.9 DUAL BEAM C.R.O.

In a dual (or double) beam *CRO* there are two electron beams produced by the cathode ray tube instead of one.

If the original electron beam leaving the electron gun is split into two separate beams, it is called *split beam C.R.T*

If two separate beams are produced from two different independent electron guns, it is known as *dual gun C.R.T*

The two beams produce two separate spots of light on the flourescent screen of the *C.R.T.* These can be used to display and compare two time related wave forms.

The *double beam C.R.O.* is very convenient to measure phase difference. The two signals, the phase difference between which is to be measured are applied to the two channels of the double beam *C.R.O.* The same trigger is used for the two sweep voltages. The phase difference between the two wave forms displayed on the *CRT* screen can be calculated from the time base. If the two sine wave signals having a time period T are found to attain the *same phase* at times t_1 and t_2 respectively, then phase differene between them

$$\phi = \frac{2\pi}{T}(t_1 - t_2).$$

9.10 MEASUREMENT OF VOLTAGE, PHASE AND FREQUENCY BY CRO.

In laboratory, *CRO* is frequently used to measure voltage (*i.e.*, amplitude), phase and frequency of a unknown source.

***(i)* Measurement of Voltage.** When the XX (X_1, X_2) plates of a *C.R.O* are earthed, its time base circuit switched off and no voltage is applied to the YY (Y_1, Y_2) plates, we get a sharp, bright spot of light at O on the screen. If a *d.c.* voltage is now aplied by connecting Y_2 to the positive and Y_1 to the negative of the *d.c* source, the spot of light is displaced to wards the positive plate Y_2 and appears at O' instead of O. The displacement of the spot OO' is directly proportional to the aplied voltage.

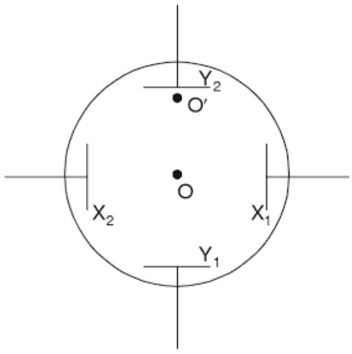


Fig. 9.13

When an *a.c.* voltage is applied to the *YY* plates of the *C.R.O.* keeping *XX* plates earthed the spot keeps on moving up and down due to variation of potential or voltage. But due to persistence of vision a line is traced on the screen. The length of the line corresponds to peak value of *a.c.* voltage and thus gives $2\sqrt{2}$ *r.m.s.* voltage.

If v is the deflection sensitivity of the *CRO i.e.*, voltage required to produce a unit deflection (or a deflection of 1 *mm*) and the displacement of the spot in the first case or the length of line in second case is l (*mm*) then,

unknown voltage $$V = v \times l$$

This should be noted that for *d.c* votage measurement the calibration is also done with *d.c* and for *a.c* voltage measurement the calibration is done with *a.c.*

***(ii)* Measurement of Phase Difference.** Lissajous figures formed on a cathode ray oscilloscope screen can be used to measure the phase difference between two sinusoidal voltages of the same amplitude and frequency.

The two signals are applied simultaneously to the horizontal and vertical deflection plates (X_1, X_2 and Y_1, Y_2). If the two voltages are represented as

$$x = A \sin \omega t \quad \text{... }(i)$$

and $$y = A \sin (\omega t + \phi) \quad \text{... }(ii)$$

where ϕ is the phase angle by which the sinusoidal voltage y leads the voltage x. Eq. (*ii*) cna be put in the form

$$y = A \sin \omega t \cos \phi + A \cos \omega t \sin \phi \quad \text{... }(iii)$$

Now $$x^2 = A^2 \sin^2 \omega t = A^2 (1 - \cos^2 \omega t)$$

$\therefore$ $$A^2 \cos^2 \omega t = A^2 - x^2$$

or $$A \cos \omega t = \sqrt{A^2 - x^2}$$

Substituting in Eq. (*iii*), we have

$$y = x \cos \phi + \sqrt{A^2 - x^2} \sin \phi$$

or $$(y - x \cos \phi)^2 = (A^2 - x^2) \sin^2 \phi$$

or $$y^2 + x^2 \cos^2 \phi - 2xy \cos \phi = A^2 \sin^2 \phi - x^2 \sin^2 \phi$$

or $$y^2 + x^2 - 2xy \cos \phi = A^2 \sin^2 \phi \quad \text{... }(iv)$$

According to Eq. (*iv*), the Lissajous figure is an ellipse shown in Fig. 9.14.

The maximum desplacement A and the vertical displacement y_0 at time $t = 0$ can be measured from the vertical scale of the *C.R.O.*

Putting $$t = 0 \text{ in Eq. } (ii), \text{ we have}$$

$$y_0 = A \sin \phi$$

or $$\sin \phi = \frac{y_0}{A}$$

or phase difference $$\phi = \sin^{-1} \frac{y_0}{A}$$

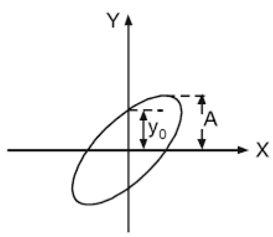


Fig. 9.14

Thus the phase ϕ can be calculated by measuring the value of A and y_0 on the elliptic trace formed on the *C.R.O.* screen.

***(iii)* Measurement of Frequency.** The frequency of a given source can be measured by comparing the frequency of the source with that of a standard oscillator. This is done by obtaining Lissajous figures on the *C.R.O.* screen by connecting the standard frequency oscillator to the vertical deflection plates Y_1, Y_2 and the unknown frequency source to the horizontal deflection plates X_1, X_2. It is assumed that both the oscillators execute simple harmonic motion.

Since simple harmonic motions plotted against time give sinusoidal configurations, two sinusoidal electrical inputs to a cathode ray oscilloscope will give Lissajous pattern on the screen. The exact pattern traced out *i.e.*, the nature of the resultant path depends upon the frequencies (or periods) amplitude and phase relationships of the two inputs.

(1) Equal Frequencies (or periods). Let the two simple harmonic vibrations having the same period take place along the X and Y axes respectively. These can be represented as

$$x = a \sin \omega t$$

and $$y = b \sin (\omega t + \phi)$$

where a and b are the respective amplitudes and the Y-vibrations is ahead of the X-vibration be a phase angle ϕ.

If $$\phi = 0$$

$$x = a \sin \omega t$$

$$y = b \sin \omega t$$

or $$y = \frac{b}{a} x$$

This represents a straight line through the origin as shown in Fig. 9.15 (*i*). The slope is given by

$$\tan \theta = \frac{b}{a}$$

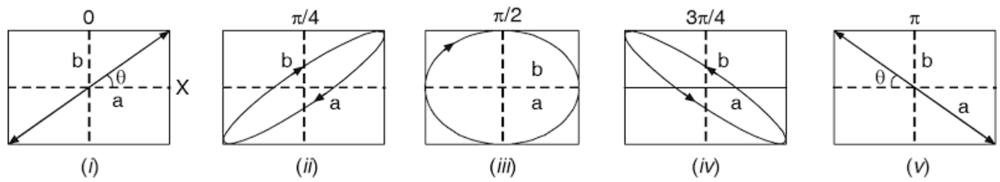


Fig. 9.15

If $\phi = \pi/2$ $$x = a \sin \omega t$$

$$y = b \sin (\omega t + \pi/2) = b \cos \omega t$$

$\therefore$ $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This represents a *symmetrical* ellipse whose major and minor axes coincide with X and Y axes respectively as shown in Fig. 9.15 (*iii*).

If $a = b$ the resultant curve is a circle given by

$$x^2 + y^2 = a^2$$

If $\phi = \pi$ $$x = a \sin \omega t$$

$$y = b \sin (\omega t + \pi) = - b \sin \omega t$$

$\therefore$ $$y = - \frac{b}{a} x$$

This again represents a straight line through the origin as shown is Fig. 9.15 (*v*). The slope is given by

$$\tan\theta = -\frac{b}{a}$$

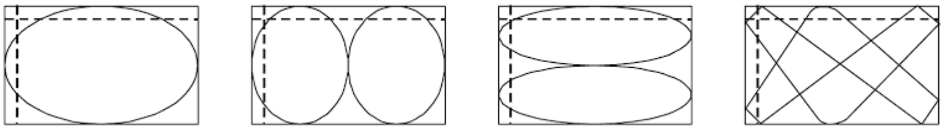
For $\phi = \pi/4$ and $\phi = 3\pi/4$ we get oblique ellipses as shown in Fig. 9.15 (*ii*) and (*iv*)

As ϕ exceeds π the whole cycle is repeated in the *reverse* order.

(2) Frequencies in the ratio of 1 : 2. When the frequency of the Y-vibtation is double that of the X-vibration but the two are in phase ($\phi = 0$), the curve traced out is shown in Fig. 9.16 (*ii*).

If, however, the frequency of the X-vibration is double that of Y-vibration, then the pattern obtained is shown in Fig. 9.16 (*iii*).

In fact, the frequency ratio of the two inpts may be determined from an analyses of the Lissajous figures produced. A simple method is to enclose the Lissajous figures in a rectangle whose sides are paralel to the formation axes of the figure and note the *number of tangency points* along the *vertical* (Y-axis) and the *horizontal* (X-axis). The ratio of the number of tangency points is in the *inverse* ratio of the two frequencies *i.e.*



(*i*) $\frac{\text{Y freq.}}{\text{X freq.}} = 1$ (*ii*) $\frac{\text{Y freq.}}{\text{X freq.}} = 2$ (*iii*) $\frac{\text{Y freq.}}{\text{X freq.}} = \frac{1}{2}$ (*iv*) $\frac{\text{Y freq.}}{\text{X freq.}} = \frac{3}{2}$

Fig. 9.16

$$\frac{\text{Number of tangency points along } X - \text{axis}}{\text{Number of tangency points along } Y - \text{axis}} = \frac{\text{Frequency of } Y - \text{vibration}}{\text{Frequency of } X - \text{vibration}}$$

An alternate method is to count the number of points at which the vertical and horizontal lines cross the figure. In other words

$$\frac{\text{Frequency of } Y - \text{vibration}}{\text{Frequency of } X - \text{vibration}} = \frac{\text{Number of crossing along the } X - \text{axis}}{\text{Number of crossing along the } Y - \text{axis}}$$

The cases of $\frac{Y \text{ freq.}}{X \text{ freq.}} = \frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{2}$ are shown in Fig. 9.16 (*i*), (*ii*) (*iii*) and (*iv*).

As the standard frequency oscillator is connected to the Y-Y plates, the frequency of Y-vibration is known. Hence, the frequency of X-vibration *i.e.*, of the unknown frequency source can be calculated.

9.11 LINEAR ACCELERATOR

For the study of nuclear reactions charged particles having energies of many million electron volts are required. It is difficult to generate direct voltages of the order of 10 million volts chiefly due to insulation difficulties. To obtain linear acceleration of a charged particle in excess of 10 MeV indirect methods are used. The first type is known as 'Dirft tube accelerator' and the second type as 'Wave guide accelerator'. Let us discuss first type *i.e.*, a **linear accelerator.**

A linear accelerator (or a *Linac*) is a device which accelerates charged particles in a straight line by means of oscillating electric field that provides either a series of steady accelerating steps in correct phase at a series of gaps between electrodes or accompanies the charged particles as a travelling wave.

In a linear accelerator, a moderate accelerating potential is applied a number of times so that the charged particles are accelerated along a straight line. A simple form of the linear accelerator is shown in Fig. 9.17. The charged particles or ions travel through an aperture A and move along the axis of a series of coaxial cylindrical electrode 1, 2, 3, 4 etc. These cylindrical electrodes are known as *drift tubes*. The drift tubes are connected to an *A.C.* source of very high frequency say a high frequency (H.F.) oscillator so that alternate tubes have potentials of opposite sign. Thus, in *one-half* cycle if tubes 1 and 3 are positive, 2 and 4 will be negative. After half a cycle the polarities are reversed *i.e.*, 1 and 3 will be negative and 2 and 4 positive.

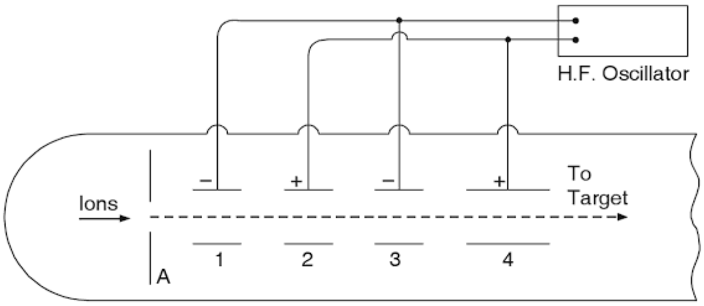


Fig. 9.17

Suppose a positive ion leaves *A* and is accelerated during the half cycle when the drift tube No. 1 is negative with respect to *A*. If *V* is the potential of drift tube 1 with respect to *A*, then velocity v_1 of the ion on reaching the drift tube is given by

$$\frac{1}{2} m v_1^2 = Ve$$

or

$$v_1 = \sqrt{\frac{2Ve}{m}}$$

where *e* is the charge and *m* the mass of the ion. It is supposed that v_1 is small as compared to c the velocity of light so that the change in mass due to relativity effect is negligible. The ions are accelerated in the gap between the tubes but travel with constant velocity in the *field free space* within the tubes themselves. The length of the tube 1 is so adjusted that as the positive ions come out of it, the tube has a positive potential and the next tube No. 2 has a negative potential, *i.e.*, the potentials change sign. The positive ion is again accelerated in the space between the tubes 1 and 2 and on reaching the tube 2 its velocity v_2 is given by

$$\frac{1}{2} m v_2^2 = 2\,Ve$$

or

$$v_2 = \sqrt{2}\sqrt{\frac{2Ve}{m}} = \sqrt{2}\, v_1$$

This shows that v_2 is $\sqrt{2}$ times v_1. In order that this ion on coming out of tube 2 may find tube 3 *just* negative and the tube 2 positive, it must take the *same* time to travel through the tube 2. As the velocity is $\sqrt{2}\, v_1$ the length of tube 2 must be $\sqrt{2}$ times the length of tube 1. For successive accelerations in successive gaps the tubes 1, 2, 3, 4 etc., must have lengths proportional to 1, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$ etc., to a first approximation.

Energy of the ion. If n is the number of gaps that the ion travels in the accelerator and v_n is the final velocity acquired by it, then

Velocity of the ion as it emerges out of the nth tube

$$v_n = \sqrt{n}\sqrt{\frac{2\mathrm{V}e}{m}}$$

$\therefore$ Kinetic energy of the ion $\frac{1}{2}mv_n^2 = n\mathrm{V}e$

The final energy of the ions when they strike the target depends upon the overall length of the accelerator *i.e.*, the total number of gaps and on the energy gained in each gap. The beam striking the target consists of pulses of particles. The number of pulses per second is equal to the frequency of the alternating voltage applied to the drift tubes.

Length of the cylinder. As the ion is accelerated in the gap between two cylindrical electrodes, the time taken by the ion to travel through the cylinder should be equal to half the time period of the high frequency voltage so that each time the ion comes out of the cylinder the polarity changes.

If v_n is the velocity of the ion, the time of passage through the nth cylinder of length L_n

$$t = \frac{L_n}{v_n} = \frac{T}{2} = \frac{1}{2f}$$

where f is the frequency of the oscillating electric field.

$$L_n = \frac{v_n}{2f} = \left(\frac{2nVe}{m}\right)^{1/2}\frac{1}{2f}$$

This equation shows that the length of the successive cylinders has to be increased in order to get a resonance acceleration of the ion at each gap and the length $L_n \propto n^{1/2}$

Limitations :

The limitations of linear accelerrator are :

(*i*) The length of the accelerator becomes inconveniently large and it is difficult to maintain vacuum in such a large chamber.

(*ii*) The ion current is available in the form of pulses of short duration.

Circular Accelerator

A *circular accelrator* is another example of drift tube accelerator. Circular accelerator is a device which can accelerate charged particles by passing them again and again in a radio-frequency (*r. f.*) electric field along a closed path. Familiar examples are (*i*) *cyclotrons* which acclerates protons, deutrons and α-particles and (*ii*) *Betatrons* which accelerates electrons.

Wave Guide Accelerator

Wave Guide Accelerator makes use of electromagnetic radiations (waves) traveling in a wave guide to accelerate charged particles. A wave guide is a hollow pipe (cyclindrical or rectanuglar) of conducting material in which oscillating electro-magnetic fields of the order of 10^5 MHz can be established. If the electro-magnetic wave is fed at one end of the wave guide and there is an electrical conductor at the other end, the advancing and the reflected waves super impose to form *standing* or *stationary* wave. But if the wave gets completely absorbed at the other end the result is a *travelling wave*. For this purpose, the wave guide is terminated at its end by an impedance equal to its characteristic impedance so that the travelling wave is maintained from one end to the other.

To accelerate the charged particle, the speed of the *e.m. wave is synchronised with that of the* particle. The particle, therefore, gains energy from the (axial) *electric field component* of the *e.m. wave* but the *magnetic field component* remains ineffective being at right angles to the path of the particle.

The *R.F.* signal is obtained from a standard master oscillator and is amplified at each feeding station using *Klystron amplifier*.

Acceleration of electrons. To accelerate the electrons a *disc loaded wave guide system* is used. The beam of electrons is injected along the axis of the metallic tube of the wave guide and the *phase velocity* of the *e.m. waves is matched* with that of the electrons.

After the electrons have attained their maximum kinetic energy, they automaticaly emerge out of the accelerator in the form of a fine collimated beam.

Acceleration of protons. Wave guide accelerators can be used for protons. But the design of such accelerator is very different. The electrons being light particles travel most of the time with a velocity very close to the velocity of light. On the other hand, protons gain speed along its total path. As the protons have a *much lower injection velocity*, disc loaded wave guides are not found practicable for phase velocities as low as 0.1 c but in its place *long cylindrical resonant cavity* has to be used.

9.12 ELECTRIC AND MAGNETIC FIELDS IN CROSSED CONFIGURATION

In atomic physics, quite often it becomes necessary to select only those particles (viz. electrons, protons etc.) that have the same velocity, out of the bunch of particles having a wide range of velocities. This is the basis of *velocity selector* or *velocity filter* and achieved by using electric and magnetic fields in crossed configuration. When uniform electric and magnetic fields act perpendicular simultaneously over the same region, the fields are said to be in **crossed configuration.**

When electrons pass through such a region, they are deflected simultaneously by both the fields. Two metal plates seperated by a small distance, when charged set up a uniform electric field E in the y-direction and a uniform magnetic field B is also set up in the same region between the plates in z-direction. Let the magnetic field B is acting into the page. Both these fields are mutually perpendicular and also perpendicular to the direction of motion of electrons, *i.e.*, x-direction. Let the stream of electrons enter the region of crossed configuration with a velocity v [Fig. 9.18].

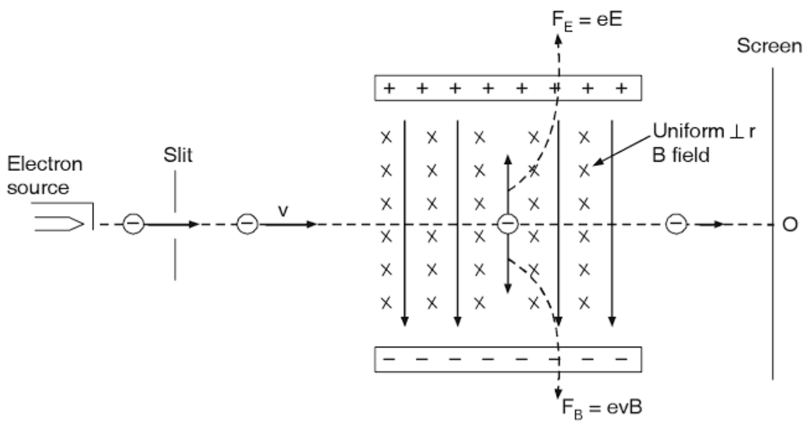


Fig. 9.18 Crossed E and B fields

The electric field E deflects the electrons *upward* with a force

$$F_E = eE \qquad \text{... }(i)$$

and magnetic field B deflects them *downwards* with a force

$$F_B = evB \qquad \text{... }(ii)$$

If the magnitudes of these two fields are adjusted such that the forces acting on the electrons are equal, so that the electrons will not experience any net force. Under this condition

$$eE = evB$$

$$\therefore \qquad v = \frac{E}{B} \qquad \text{... }(iii)$$

The electrons experience a zero net force as the above two forces are equal and opposite thereby balance each other. Therefore, they wil not deviate from their original straight line path *i.e.* x-axis and travel without any change in their velocity v. By knowing the value of E and B, the velocity of electrons and hence their kinetic energy can be determined. In 1897, J.J. Thomson used this method for the determination of electron beam velocity.

9.13 VELOCITY SELECTOR

Velocity selector also known as velocity filter is an electro-optic device which uses the combined effect of two mutual perpendicular fields E and B acting simultaneously over the same region. This crossed-configuration of E and B selects a stream of charged particles of single velocity from a beam of charged particles having a wide range of velocities. [Fig. 9.19].

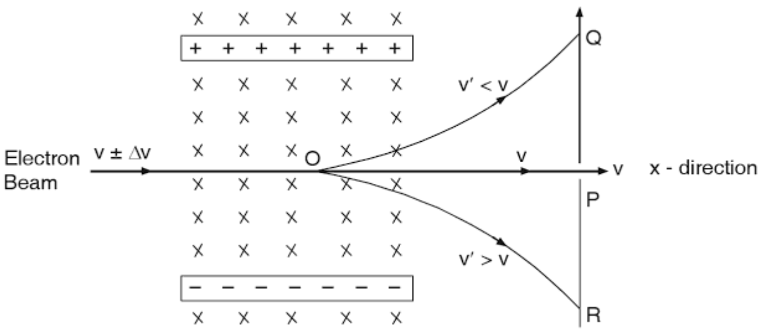


Fig. 9.19

Here the electric field E is produced in verticaly downward direction (– y axis) by a set of charged particle plates and a uniform magnetic field B is aplied perpendicular to it, acting into the paper. If the two fields E and B are adjusted such the electric field force balances the magnetic force B acting on the electrons moving with velocities $v + \Delta v$, then those electrons are not deflected and continue to travel along the straight line, having velocity v. subsequently, they pass through the slit at P. Those electrons moving with velocity lesser than v ($v' < v$) will get deflected upward along OQ and those moving with greater velocity than v ($v' > v$) will get deflected downward along OR. Thus, a strictly homogeneous single velocity electron beam traveling along OP is obtained. This arrangement is, therefore, known as a **velocity selector** or a **velocity filter.** This forms an essential component in Bainbridge mass spectrograph.

9.14 CYCLOTRON

The α and β-particles given out by natural radioactive substances neither possess sufficiently large speeds not their speeds are under control. It was, therefore, felt necessary to accelerate charged particles to very high velocitites by the application of electric and magnetic fields. Cockroft and Walton produced fast moving protons by electronic voltage multiplication device. The best arrangement was, however, made by Lawrence and Livingstone in 1934 and is called a *cyclotron*. This arrangement won the Nobel prize for Lawrence.

Construction. The cyclotron consists of two D-shaped hollow metal segments D_1 and D_2 (called the Dees) lying in a horizontal plane with a smal gap separating them. A magnetic field NS is applied perpendicular to the plane of the paper and D_1 and D_2 are connected to a *high voltage, high frequency* alternating current as shown in Fig. 9.20.

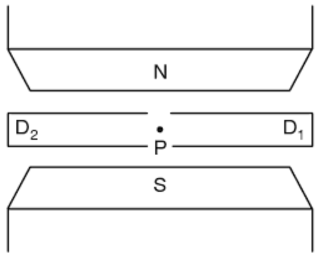


Fig. 9.20

Role of electric and magnetic fields. Simultaneous application of Electric field and magnetic fields plays an important role in the construction of cyclotron.

Electric field. The primary function of the electric field is to provide a potential difference between the dees of the cyclotron *to accelerate the charged particles.* Thus, an alternating electric field having a frequency such that the time taken by the particle to travel through the semi-circular path within the dees is equal to half the time period of the alternating electric field is required.

Magnetic field. The primary function of the magnetic field is to move the charged particle into *a semi-circular path within the dees.*

Principle. The cyclotron is a *magnetic resonance* type *positive ion* accelerator. The charged particle to be accelerated placed at centre point P of the Dee , rapidly passes through an alternating electric field along a closed path, its energy being increased each time. A strong magnetic field is used to control the motion of the particles and to return them periodically to the region of the accelerating electric field. The particle passes definite points of the alternating electric field almost in unison when the field is in the same phase *i.e.,* in *resonance.*

Resonance condition: As we have discussed above, cyclotron is a resonance device. In a cyclotron the value of the magnetic field strength depends upon the frequency of the oscillating electric field applied between the dees. It is so chosen as to give *resonance* between the arival of the charged particle in the gap and the reversal of the voltage between the dees. This is done by adjusting the time taken by the charged particle to describe a semi-circular path equal to half the time period of oscillation of the applied high frequency electric field *i.e.*

$$\frac{\pi}{\omega} = \frac{T}{2}$$

The frequency $$f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{Be}{2\pi m}$$

This gives the *cyclotron resonance condition* for a charged particle of a given value of $\frac{e}{m}$. If this condition is not satisfied there will be no resonance between the arrival of the charged particle in the gap and the revesal of the voltage between the dees. The particle will go out of step and will not be accelerated. This is why it is said that a cyclotron is a *resonance device*.

Theory and working. If a positive ion is generated at a point P, as shown in Fig. 9.21, within the gap at a time when D_1 is at a positive potential and D_2 at a negative potential, it will be accelerated across the gap to D_2 and enter the hollow segment D_2 with a velocity v given by

$$Ve = \frac{1}{2}mv^2$$

where V is the applied voltage and e and m are charge and mass of the particle respectivley. When it is inside the conductor, it will not be acted upon by the electric field but under the influence of the applied magnetic field having a flux density B, it will travel along a circular path, the radius r of which is given by

$$\frac{mv^2}{r} = Bev$$

or
$$r = \frac{mv}{Be}$$

and finally emerges at C in the direction indicated.

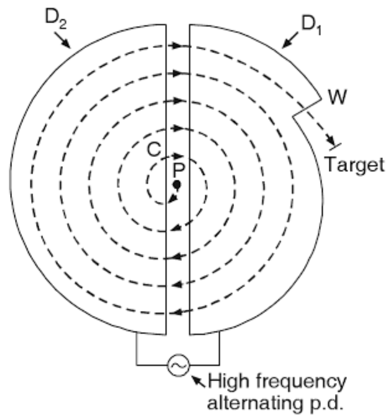


Fig. 9.21

The time taken by the positive ion to travel the semicircular path

$$t = \frac{\pi}{\omega} = \frac{\pi r}{v} = \frac{\pi m}{Be}$$

where ω is the angular velocity of the ion in the circular path and

$$\omega = \frac{Be}{m}$$

The value of t is a constant being independent of the velocity of the ion and the radius in which it travels. If the frequency of the applied voltage is adjusted in such a manner that it is reversed as soon as the particle comes out of D_2, the particle at C will be accelerated across the gap to D_1 and will describe a further circular path in D_1. The radius of this semi-circle as well as speed of the particle will, now, be greater than that in the first case, but as proved above, the time taken by the particle to travel the semi-circular path in D_1 will be the same. Every time the particle emerges out of the Dees, the direction of the voltage is reversed and the particle is accelerated across the gap. The path of the particle will be a spiral and it will finally come out of the Dees in the direction indicated, through the window W.

Maximum kinetic energy of the particle. The final energy E of the charged particle is given by

$$E = \frac{1}{2}mv_{max}^2$$

where v_{max} is the maximum velocity gained by the charged particle in its final orbit of radius r_{max}. Now

$$\frac{mv_{max}^2}{r_{max}} = Bev_{max}$$

$$\therefore \quad v_{max} = \frac{Be}{m} r_{max}$$

Hence $$E = \frac{1}{2} mv_{max}^2 = \frac{1}{2} m \frac{B^2e^2}{m^2} r_{max}^2$$

$$= \frac{1}{2} \frac{B^2e^2r_{max}^2}{m}$$

This relation gives the maximum kinetic energy of the charged particle in terms of *applied magnetic field* and *dee radius.*

The condition for optimal acceleration of the ion in the inter dee gap is that the time taken by the ion to travel the semi-circular path (t) is equal to half the time period (T) of oscillation of the applied high frequency electric field *i.e.*,

$$t = \frac{T}{2}$$

or $$\frac{\pi m}{Be} = \frac{T}{2}$$

or $$T = \frac{2\pi m}{Be}$$

If f is the frequency of the oscillating electric field, then

$$f = \frac{1}{T} = \frac{Be}{2\pi m}$$

This is the basic *cyclotron resonance equation*.

Hence, in terms of f the maximum energy of the charged particle

$$E_{max} = \frac{1}{2} \frac{B^2e^2r_{max}^2}{m}$$

$$= \frac{1}{2} 4\pi^2 m \cdot \frac{B^2e^2}{4\pi^2m^2} r_{max}^2$$

$$= 2\pi^2 m f^2 r_{max}^2$$

The particles are ejected out of the cyclotron as *pulse streams* and not continuously.

With a comparatively small potential difference of the order of 50,000 volts, very fast moving particles can be produced. For example, if the particle makes 200 revolutions before emerging out it will gain a velocity equivalent to a total fall through a potential of

$$2 \times 50000 \times 200 = 20 \text{ million – volts}$$

If heavy hydrogen is used instead of ordinary hydrogen, a beam of high energy deutrons is obtained. As their mass is double they possess greater energy and are more useful as atomic projectiles.

Maximum radius of curvature. If V is the average voltage applied between the dees of a cyclotron and the charged particle crosses the gap between the dees n times to reach the orbit of maximum radius, then energy acquired by the ion having a charge e is given by

$$E_{max} = neV$$

But
$$E_{max} = \frac{1}{2}\frac{B^2e^2r_{max}^2}{m}$$

$\therefore$
$$\frac{B^2e^2r_{max}^2}{2m} = neV$$

or
$$r_{max} = \frac{1}{B}\left(\frac{2mV}{e}\right)^{1/2} n^{1/2}$$

If the values of B and V are kept constant, r_{max} is directly proportional to the square root of the number of times the particles crosses the gap between the dees.

Limitations of the cyclotron

(1) The energy to which a particle can be accelerated in a cycloron is limited due to change in mass with velocity. The mass of a particle, when moving with a velocity v is given by

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

where m_0 is the rest mass and c the velocity of light. As already proved, the time taken by a particle to travel the semi-circular path is $\frac{\pi m}{Be} = \frac{T}{2}$

$\therefore$ Frequency
$$f = \frac{1}{T} = \frac{Be}{2\pi m}$$

$$= \frac{Be\sqrt{1-\frac{v^2}{c^2}}}{2\pi m_0}$$

Hence the frequency of rotation of the charged particle decreases as the velocity increases. As a result it takes a longer time to complete its semi-circular path and the particle continuously goes on lagging behind the applied alternating potential difference till a stage is reached when it can no longer be accelerated further. This discrepancy is optimised by the following two methods:

(*i*) Field variation. The frequency of the ion can be kept constant by taking $B\sqrt{1-\frac{v^2}{c^2}}$ a constant. For this prupose the value of the magnetic field B should increase as the velocity of the ion increases so that the product remains unchanged.

(*ii*) Frequency modulation. In the alternative method, the frequency of the applied $A.C.$ is varied so that it is always equal to the frequency of rotation of the ion. The machine in which the frequency of electric field is kept constant and magnetic field is varied is called **Synchrotron,**

whereas a machine in which magnetic field is kept constant and the frequency of the applied electric field is varied is known as a **Frequency modulated cyclotron** or **Synchro-cyclotron.**

(2) A cyclotron cannot be used to produce high energy electron beams. The reason for the same is that there is an appreciable increase in the mass of the electron at fairly low energies. For example, a 10% increase in the rest mass of the electron takes place at an energy of 50 KeV only. Electron being a very light particle, there is an appreciable increases in its velocity at low energies which is not the case with massive particles like the proton or the α-particle.

(3) Neutron can not be acceleratred by a cyclotron. A neutron carries no charge; it can not be accelerated by the electric field between the two Dee's. It can also be not acted upon by the magnetic field so that its path within the dees cannot be regulated.

9.15 MAGNETIC FIELD FOCUSSING

An appropriate configuration of non-uniform electric field, as seen in article 9.4, is used for focussing of electrons. An electron beam can also be focussed with the help of magnetic field. Depending upon the nature of the magnetic field and the direction of its application, two different types of focussing are obtained.

(*i*) Longitudinal magnetic field focussing, and

(*ii*) Transverse magnetic field focussing.

9.15.1 Longitudinal Magnetic Field Focussing

Charged particles like electrons can be focussed at a point by employing a uniform magnetic field acting along the direction of motion of beam of charged particles; as shown in fig. 9.22. The path of an electron

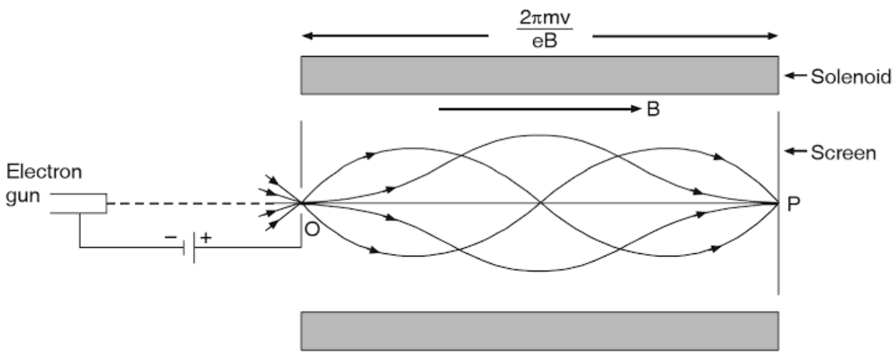


Fig. 9.22: Focusing of charged particles in a longitudinal magnetic field.

in a uniform magnetic field would be a helix if electrons enter at an angle ϕ, the pitch (p) of the helix is given by

$$p = \frac{2\pi mv}{eB}\cos\phi \qquad \ldots (i)$$

Suppose a beam of electrons eneters at point O with a small angle ϕ and with a velocity v. If ϕ is taken too small ($\phi \leq 10°$), cosϕ may be taken as unity. In time T, the charged particles come to focus at point P, the length $OP = l$ is given by

$$l = \frac{2\pi mv}{eB} \qquad \ldots (ii)$$

Therefore, the beam of electron entering the field B making a small angle of divergence ϕ come to focus at a distance l or any point which is located at a distance of integral multiple of l *i.e.* nl.

9.15.2 Transverse Magnetic Field Focussing (180°)

This method is very useful in separating the isotopes†. A transverse uniform magnetic field can also be employed to focus a beam of charged particles. The focussing would be affected after rotating the beam through 180°. The method of semicircular focussing was first employed by Rutherford in 1914. If a beam of *positively charged particles* with the same velocity (using velocity selector) enter a uniform magnetic field acting in a transverse direction [Fig. 9.23 (*a*)] to its motion, and if the angular divergence of the beam is very small (ϕ is small), it gets focussed after completing a semicircle (180°). To seperate and identify the isotopes of an element, its atoms are ionised first then allowed through a slit S into the region of a uniform magnetic field acting transversely as shown in fig. 9.23 (*a*). *The ions belonging to different isotopes* would have different masses.

Therefore, even though having the same velocity possess different momenta and therefore, come to focus as different points. Let the particles are deflected with a radius given by

$$R = \frac{mv}{qB} \qquad [\because \text{Refer Eq. } (iv), \text{ article 9.7}]$$

For the particles having different values of mv, and the beam is not spread, the radius of semicircles will be different as shown in fig. 9.23 (*a*).

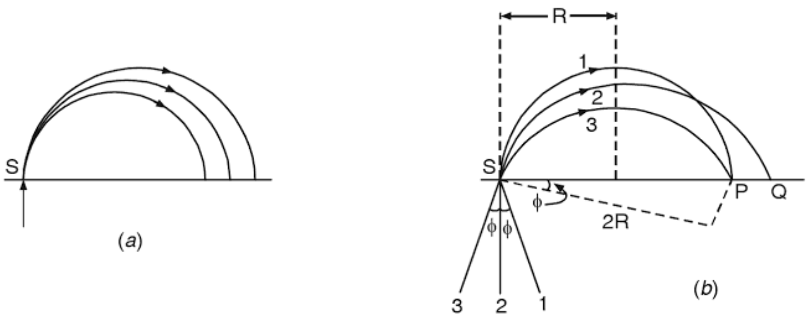


Fig. 9.23

Thus, the particles having the same values of mv, will get focused after 180° deflection at the same point. This is called **momentum selector principle.**

Now, consider that the beam is spread [Fig. 9.23 (*b*)]. The particles moving along path 2 are focussed at Q and particles moving along paths 1 and 3 are focussed at P. The distance

$$PQ = 2R - 2R\cos\phi = 2R(1 - \cos\phi)$$

When ϕ is very small,

$$\cos\phi = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} \ldots\ldots\ldots$$

† **Isotopes:** Isotopes are nuclei having the same atomic (proton) number Z but different atomic mass A. The isotopes of an element contain the same number of protons but different number of neutrons. For example: Oxygen has two isotopes $^{16}O_8$ and $^{17}O_8$. Hydrogen has three isotopes: (*i*) Hydrogen $^1H_1$ (*ii*) Deuterium $^2H_1$ and (*iii*) Tritium $^3H_1$. Neon has two isotopes $^{20}Ne_{10}$, $^{22}Ne_{10}$ Chlorine also has two isotopes $^{35}Cl_{17}$ and $^{37}Cl_{17}$. At present about 297 different isotopes are known

$$= 1 - \frac{\phi^2}{2} \qquad \text{(approximately)}$$

$$\therefore \qquad PQ = 2R\left[1 - \left(1 - \frac{\phi^2}{2}\right)\right]$$

$$= R\phi^2$$

As ϕ is small, ϕ^2 is extremely small and width of focussing PQ is very small. All the particles having same momenta and charge will be focussed between PQ. The particles having different values of momenta and charge will be focussed at different points. Therefore, this method is very useful in seperating the *isotopes*.

9.16 MASS SPECTROGRAPHS

Mass spectrograph is an instrument employed to seperate different isotopes from a stream of positive ions of an element and measures thier individual masses as well as their relative abundances, by using electric and magnetic fields over the same region since the instrument is used to seperate ions of different masses, is called as **mass spectrograph.**

Bainbridge's Mass Spectrograph. A schematic diagram showing the construction of a Bainbridge's mass spectrograph is shown in Fig. 9.24

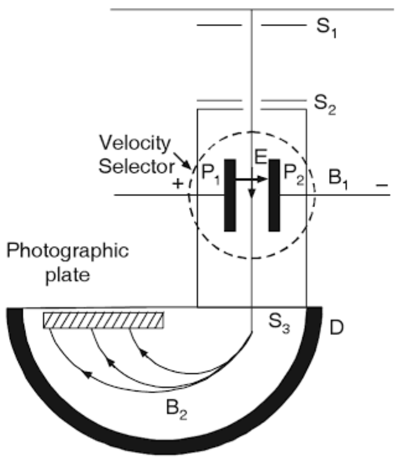


Fig. 9.24

A beam of *positive ions* produced in a discharge tube is collimated into a narrow beam by the two slits S_1 and S_2. After emerging from the slit S_2, the positive ions enter a *velocity selector*.

Velocity selector. It consists of two plates p_1 and p_2 between which a steady electric field E is maintained in a direction at right angles to the ion beam. A magnetic field B_1 is produced by an electromagnet (represented by the dotted circle). The magnetic field B_1 is at right angles to the electric field E as well as the ion beam.

The electric field and the magnetic field of the *velocity selector* are so adjusted that the deflection produced by one is exactly equal and opposite to the deflection produced by the other so that, there is no net deflection for ions having a particular velocity v.

If E and B_1 are the electric field intensity and magnetic induction, then

$$eE = B_1 ev$$

or
$$v = \frac{E}{B_1}$$

Only those ions having this velocity v alone pass through the entry slit S_3 and enter the evacuated chamber D. Thus, all ions entering the chamber have the same velocity.

Working. The positive ions which enter the chamber D are subjected to a strong magnetic field of intensity B_2, perpendicular to the path. The force acting on each ion is $B_2 ev$. As the force acts at right angles to the direction of motion of the ion, the ion moves in a circular path of radius R given by

$$B_2 ev = \frac{Mv^2}{R}$$

or
$$R = \frac{Mv}{B_2 e}$$

$\therefore$
$$\frac{e}{M} = \frac{v}{B_2 R} = \frac{E}{B_1 B_2 R} \qquad \left[\because v = \frac{E}{B_1}\right]$$

As E, B_1 and B_2 are constant

$$\frac{e}{M} \propto \frac{1}{R}$$

After describing the semicircular path the ions strike a photographic plate.

Determination of mass. If e is the same for all ions , $M \propto R$.

Thus, we get a linear mass scale on the photographic plate. The ions of different mass will traverse paths of different radii and strike the photographic plate at different points, thereby giving a typical mass spectrum.

The method is very accurate as the mass scale is linear.

No change in kinetic energy of a charged particle in a magnetic field. When a charged particle having a charge q, and velocity v (kinetic energy = $\frac{1}{2} mv^2$) enters a magnetic field of intensity B, perpendicular to its path the force acting on the charged particle and the direction of the magnetic field. The charged particle will, therefore move in a circular path of radius r, given by

$$Bqv = \frac{mv^2}{r}$$

or
$$r = \frac{mv}{Bq}$$

As the force due to the magnetic field acts in direction at rigtht angles to te direction of motion, the force will only change the direction but will not be able to bring about a change in the magnitude of the velocity. *The kinetic energy of the particle will, therefore, remain unchanged.* The force due to the magnetic field will only make the charged particle move along a circular path with the same velocity and, therefore, the same kinetic energy.

9.17 MOTION OF POSITIVE IONS IN PARALLEL E AND B FIELDS

When an electricaly neutral atom loses one or more electrons, it becomes a positively charged atom and is called a positive ion.

Suppose a beam of such positive ions, each of mass m and charge q moving with a velocity v enters a region in which a uniform electric field of intensity E and a uniform magnetic field of induction B are present. Both the fields E and B are parallel to each other and are at right angle to the direction of motion of the beam (fig. 9.25).

In the region of the fields, an ion experiences the electric force qE in the direction of E and at the same time also experiences a magnetic force qvB in the direction perpendicular to both v and B. In the figure qvB is directed out of the plane of the paper. Therefore, during its passage through the fields, the deflection of the ion due to E is in the y-direction and the deflection due to B is in the x-direction.

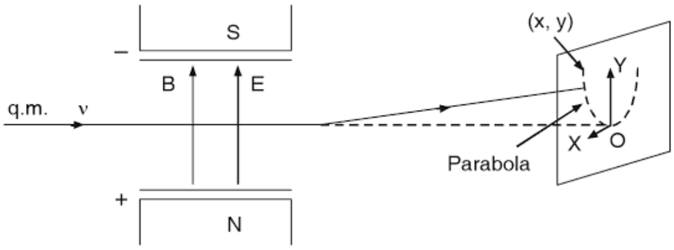


Fig. 9.25

The deflections on the screen are given by

$$S_E = \frac{qElL}{mv^2}$$

and
$$S_B = \frac{qBlL}{mv}$$

where l is the length of the horizontal path through the fields and L is the distance of the screen from the centre of the fields. If x and y are the coordinates of the point on the screen where the deflected beam meets the screen, then

$$y = \frac{qElL}{mv^2} \quad \text{... }(i)$$

$$x = \frac{qBlL}{mv} \quad \text{... }(ii)$$

To eliminate v, we square Eq. (ii) and then dividing by Eq. (i), we get

$$\frac{x^2}{y} = \frac{q^2B^2\,(l\,L)^2}{m^2v^2} \cdot \frac{mv^2}{qElL}$$

$$= \left(\frac{lLB^2}{E}\right)\frac{q}{m}$$

Therefore,
$$x^2 = \left(\frac{lLB^2}{E}\right)\frac{q}{m}\,y \quad \text{... }(iii)$$

This is the equation of a parabola with its vertex at the origin O of the axes on the screen. The parabola is symmetrical about the y-axis. Thus, for constant values of l, L, B and E positive ions having the same charge to mass ratio (q/m), but moving with different velocities, will lie on the parabola represented by Eq. (*iii*).

If each ion has a positive charge equal in magnitude to the electronic charge e, then Eq. (*iii*) is written as

$$x^2 = \left(\frac{lLB^2}{E}\right)\frac{e}{m}y$$

or $$x^2 = C\frac{e}{m}y \qquad \text{... }(iv)$$

where $C = lLB^2/E$

Eq. (*iv*) represents again a parabola, *i.e.* the path traced by a positive charge has parabolic nature. This principle was used by J.J Thomson and he determined $\frac{e}{m}$ by using his very famous Thomson's parabola method.

9.18 THOMSON'S PARABOLA METHOD

(Determination of q/m of positive ions)

J.J. Thomson, in 1911, developed a method of measuring the relative masses of different atoms by employing a parallel configuration of electric field E and magnetic field B simultaneously. The experimental arrangement is shown in fig. 9.26.

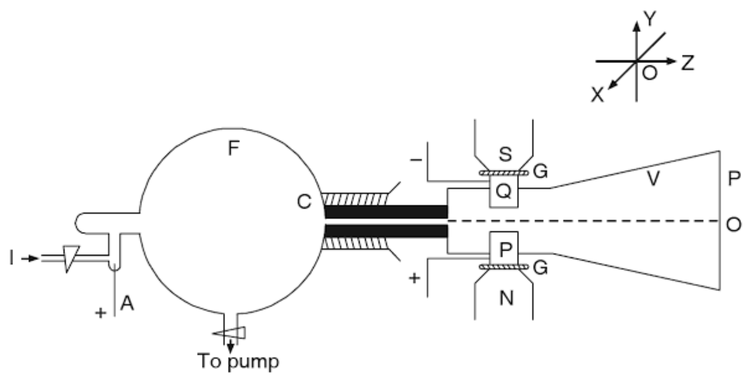


Fig. 9.26

F is a glass flask in which gaseous discharge is produced. The anode A is in a side tube. The cathode C is made of a tube of soft iron faced with aluminium. This tube is fixed to the neck of the flask. Along the axial hole in this tube there is a copper tube of fine bore of about 0.1 mm diameter through which positive ions produced in the flask can pass. I is the inlet for an experimental gas. An electromagnet NS with soft iron pole pieces P and Q provides a strong magnetic field in the y-direction. The pole pieces P and Q are insulated from N and S by thin mica sheets GG. The electric field is set up in the same direction by applying a p.d. between P and Q. Thus, parallel electric and magnetic fields are produced at right angles to the axis of the copper tube. A vertical photographic plate P is at the end of the conical vessel V.

Procedure for *q/m* of the isotopes of an element in gaseous state:

The flask F is evacuated of air to pressure of about 0.001 mm of Hg by means of a vacuum pump. The experimental gas at about the same pressure is now introduced into the flask through the inlet I. A high voltage of about 20000V is applied between the anode A and cathode C. Cathode rays or electrons then flow from C to A and positive ions produced by ionisation of the gas-atoms flow to C. After emerging from C they pass through the region of parallel electric and magnetic fields. Then the deflected ions are incident on the photographic plate P where, they produce traces of parabolas on one side of the y-axis, The fields are then reversed to obtain the parabolic traces on the other side of the y-axis.

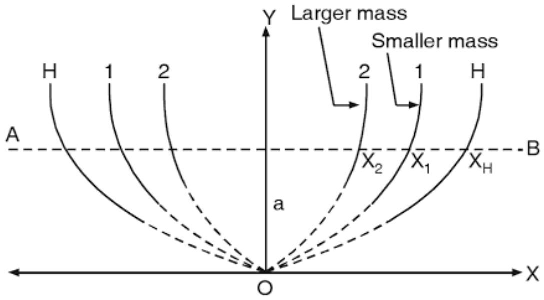


Fig. 9.27

After development of the plate parabolic traces are found on it (Fig. 9.27).

The outermost parabola H is produced by hydrogen ions because they have the greatest value of q/m. (Hydrogen ions are usually present due to the residual air in the flask and the hydrogen parabola is taken as the standard). The parabolas (1) and (2) are produced by the ions of the isotopes of the gas introduced. To find q/m of the ions of the isotopes in terms of q/m of hydrogen ions a straight line AB : y = a; parallel to the x-axis is drawn and the x-coordinates x_H, x_1, x_2 of its point of intersection with the parabolic traces are found (Fig. 9.27). Then for hydrogen, using equation (*iv*) of article 9.17 as

$$x^2 = C\frac{q}{m}y$$

we have

$$x_H^2 = C\frac{q}{m_H}a$$

Therefore, $$\frac{q/m_H}{x_H^2} = \frac{1}{Ca} \quad \text{... }(i)$$

Similarly, for the isotopes

$$\frac{q/m_1}{x_1^2} = \frac{1}{Ca} \quad \text{... }(ii)$$

$$\frac{q/m_2}{x_2^2} = \frac{1}{Ca} \quad \text{... }(iii)$$

where m_H, m_1 and m_2 are the corresponding atomic masses.

From Eq. (*i*), (*ii*) and (*iii*), we have

$$\frac{q/m_1}{x_1^2} = \frac{q/m_2}{x_2^2} = \frac{q/m_H}{x_H^2}$$

Hence $$\frac{q}{m_1} = \frac{x_1^2}{x_H^2}\left(\frac{q}{m_H}\right) \quad \text{... }(iv)$$

and $$\frac{q}{m_2} = \frac{x_2^2}{x_H^2}\left(\frac{q}{m_H}\right) \quad \text{... }(v)$$

Thus the values of q/m of the ions of the isotopes in terms of that of hydrogen ions are found.

Atomic Masses:

If the ionic charge q is the same in each case, then from Eq. (iv) and (v), we have

$$m_1 = \frac{x_H^2}{x_1^2} m_H \quad \text{... }(vi)$$

and $$m_2 = \frac{x_H^2}{x_2^2} m_H \quad \text{... }(vii)$$

This is how, we determine the mass of the isotopes of the desired element.

Discovery of Isotopes

In 1912, Thomson attemped to determine the atomic weight of neon using parabola method. He discovered two parabolas, a strong one corresponding to a mass 20 and a weaker one corresponding to mass 22. Thus, for the first time, Thomson found two kinds of neon atoms, identical in chemical nature and having the same optical spectra but different in mass. In fact, it was shown latter by Thomson's co-worker F.W. Aston that there are three kinds of neon atoms: 90.5% have a mass value of 20, 9.2% have a mass 22 and 0.3% have a mass 21. Thus, neon has three isotopes. Thomson method is an effective experimental tool which provides the firm experiment of the existence of stable isotopes of any desired element.

9.19 MAGNETIC LENS

An axially symmetric magnetic fields have a focusing property when an electron beam passes through them. Such fields are generated by using short solenoids. By encasing the coils in hollow iron shields the magnetic fields are concentrated and improved focusing action is obtained. Such Solenoids are, therefore, called as *thin magnetic lenses.*

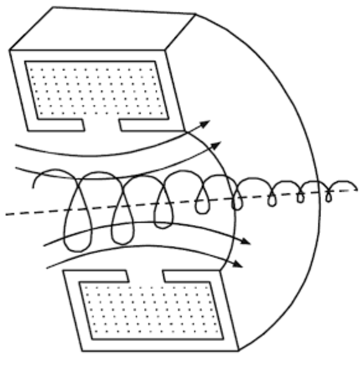


Fig. 9.28

As we know, that an electron traveling in a non-uniform magnetic field describes a helical path of pitch

$$p = \frac{2\pi mv}{eB}.$$

The radius of the helical loop goes on progressively decreasing as the electron approaches to stronger and stronger magnetic field B. Thus, the radii of progressive solenoidal loops goes on shrinking into tighter loops and ultimately $r \to 0$ and the electron beam comes to focus at a point [Fig. 9.28]. In other words, magnetic lens converges the electron beam. However, diverging action is impossible in magnetic lenses.

By adjusting the current through the solenoid and the initial accelerating voltage, the focal distance of magnetic lens can be monitored.

Uses: Magnetic lenses are widely used in electron microscopes and in instruments where such an action is desired.

SOLVED EXAMPLES

Example 9.1 An electron emitted with zero velocity from the hot cathode in a vaccum tube is accelerated by the electric field towards the anode. If the anode is at positive potential of 1000V with respect to the cathode, find the velocity acquired by the electron as it reaches the anode. (e = 1.6 × 10^{-19} C, m = 9.11 × 10^{-31} kg) (*Nagpur Uni.* 2007)

Solution. Given: $V_a = 1000$ V, $e = 1.6 \times 10^{-19}$ C, $m = 9.11 \times 10^{-31}$ kg

$$\frac{1}{2}mv^2 = eV_a$$

$$\therefore \quad v = \sqrt{\frac{2eV_a}{m}} = \sqrt{\frac{2 \times 1.6 \times 10^{-19} \times 1000}{9.11 \times 10^{-31}}}$$

$$= \sqrt{\frac{2 \times 1.6 \times 10^{15}}{9.11}} = 10^7\sqrt{\frac{2 \times 16}{9.11}}$$

$$= \mathbf{1.874 \times 10^7 \text{ m/s}}$$

Example 9.2 Calculate radius of path of an electron is a magnetic field of induction 10^{-4} wb/m^2 perpendicular to its path. (Velocity of the electron = 1.9 × 10^8 m/s, m = 9.1 × 10^{-31}kg, e = 1.6 × 10^{-19} C). (*Nagpur Uni.* 2008)

Solution.

$$R = \frac{mv}{eB} = \frac{9.1 \times 10^{-31} \times 1.9 \times 10^8}{1.6 \times 10^{-19} \times 10^{-4}}$$

$$= \frac{9.1 \times 1.9}{1.6}$$

$$= \mathbf{10.81 \text{ m}}$$

Example 9.3 An electron having kinetic energy 2 × 10^5 eV enters a uniform magnetic field of induction 3 × 10^{-3} Wb/m^2 in a direction perpendicular to the field. Find the radius of the electron path in the field.

Solution. Given $m = 9 \times 10^{-31}$ kg, $e = 1.6 \times 10^{-19}$ C, $1eV = 1.6 \times 10^{-19}$ J,

$B = 3 \times 10^{-3}$ Wb/m^2, K.E. $= 2 \times 10^5 \times 1.6 \times 10^{-19}$ J

$$\frac{1}{2}mv^2 = \text{Kinetic energy, } \kappa \text{ say}$$

$$\therefore \qquad v = \sqrt{\frac{2\kappa}{m}}$$

Radius of the path is given by

$$R = \frac{mv}{eB} = \frac{m}{eB}\sqrt{\frac{2\kappa}{m}}$$

$$= \frac{\sqrt{2\kappa m}}{eB}$$

$$= \frac{\sqrt{2 \times 2 \times 10^5 \times 1.6 \times 10^{-19} \times 9 \times 10^{-31}}}{1.6 \times 10^{-19} \times 3 \times 10^{-3}}$$

$$= \frac{\sqrt{4 \times 1.6 \times 9 \times 10^{-45}}}{1.6 \times 3 \times 10^{-22}} = \frac{\sqrt{4 \times 16 \times 9 \times 10^{-46}}}{48 \times 10^{-23}}$$

$$= \frac{24}{48}$$

$$= \mathbf{0.5\ m}$$

Example 9.4 A beam of electrons moving horizontally with a velocity of 10^7 m/s enters midways a uniform electric field between two horizontal parallel plates 5 cm long and 1.8 cm apart. On emerging from the plates the beam just grazes the edge of the positive plate. Calculate the potential difference V applied between them (Given $e/m = 1.8 \times 10^{11}$ C/kg.)

(*Nagpur Uni.* 2007)

Solution. $v = 10^7$ m/s At $x = 5 \times 10^{-2}$ m, $y = \dfrac{1.8 \times 10^{-2}}{2} = 9 \times 10^{-3}$m

and $d = 1.8 \times 10^{-2}$ m

The equation of the path between the plates is

$$y = \frac{eE}{2mv^2}x^2 = \frac{1}{2}\left(\frac{e}{m}\right)E\left(\frac{x}{v}\right)^2$$

$$\therefore \qquad E = 2\left(\frac{m}{e}\right)\left(\frac{v}{x}\right)^2 y \qquad \text{... (1)}$$

The p.d is given by

$$V = Ed$$

$$= 2\left(\frac{m}{e}\right)\left(\frac{v}{x}\right)^2 yd$$

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$$= 2\left(\frac{1}{1.8\times 10^{11}}\right)\left(\frac{10^7}{5\times 10^{-2}}\right)^2 9\times 10^{-3}\times 1.8\times 10^{-2}$$

$$= \frac{2\times 9\times 10^2}{25} = \mathbf{72\ V.}$$

Example 9.5 A beam of positive ions moving along the *x*-axis enters a region of uniform electric field of intensity 3 KV/m along the *y*-axis and magnetic field of 1 Kilo Gauss along the *z*-axis. Calculate the speed of those ions which pass undeviated. What will happen to those ions which are moving (*i*) faster, (*ii*) slower than these ions ? (*Nagpur Uni.* 2008)

Solution. Given $E = 10^3$ V/m, $B = 10^3$ Gauss, 10^4 Gauss = 1 Wb/m^2

$$\therefore \qquad B = \frac{10^3}{10^4} = 0.1 \text{ Wb/m}^2 = 0.1 \text{ Tesla}$$

For positive ions which pass undeviated

$$\text{downward magnetic force} = \text{upward electric force}$$

$$qvB = qE$$

$$\therefore \qquad v = \frac{E}{B} = \frac{10^3}{0.1} = \mathbf{10^4\ m/s}$$

(*i*) The ions which move with speed *more* than this value will be deflected *downward*, and (*ii*) the ions which move with speed *less* than this value will be deflected *upward.*

Example 9.6 Protons are accelerated in a cyclotron in which the magnetic field strength is 1 Wb/m^2. What must be the frequency of the oscillator supplying power to the dees? (Given : mass of proton = 1.67 × 10^{-27} kg, electronic charge = 1.6 × 10^{-19} C)

Solution. $B = 1$ Wb/m^2, $m_p = 1.672 \times 10^{-27}$ kg

$$q = e = 1.6 \times 10^{-19} \text{ C}$$

$$T = \frac{2\pi m}{qB}$$

$$\therefore \qquad f = \frac{1}{T} = \frac{qB}{2\pi m_p}$$

$$= \frac{1.6\times 10^{-19}\times 1}{2\times 3.14\times 1.672\times 10^{-27}} = \frac{1.6\times 10^8}{2\times 3.14\times 1.672}$$

$$= 0.1524 \times 10^8 \text{ Hz}$$

$$= 15.24 \times 10^6 \text{ Hz} = \mathbf{15.24\ MHz}$$

Example 9.7 What would be the length of the last drift tube in a linear accelerator which produces 120 MeV C^{12} ions, using frequency of 70 MHz?

(1eV = 1.6 × 10^{-19} J, and 1 a.m.u. = 1.66 × 10^{-27} kg) (*Nagpur U.* 2009)

Solution. $E_n = 120 \text{ MeV} = 120 \times 1.6 \times 10^{-19} \times 10^6$ J

$= 1.2 \times 1.6 \times 10^{-11}$ J, $m = 12 \times 1.66 \times 10^{-27}$ kg

$f = 70 \text{ MHz} = 70 \times 10^6 \text{ Hz} = 7 \times 10^7$ Hz.

Let l_n be the length of the last tube.

We have,

$$\frac{1}{2} m v_n^2 = E_n$$

$$\therefore \quad v_n = \sqrt{\frac{2E_n}{m}}$$

According to the condition for synchronous acceleration

$$\frac{l_n}{v_n} = \frac{T}{2} = \frac{1}{2f}$$

$$\therefore \quad l_n = \frac{v_n}{2f} = \frac{1}{2f}\sqrt{\frac{2E_n}{m}}$$

$$= \frac{1}{2 \times 7 \times 10^7}\sqrt{\frac{2 \times 1.2 \times 1.6 \times 10^{-11}}{12 \times 1.66 \times 10^{-27}}}$$

$$= \frac{1}{14 \times 10^7}\sqrt{\frac{2 \times 1.2 \times 1.6 \times 10^{-11}}{1.2 \times 1.66 \times 10^{-26}}}$$

$$= \frac{1}{14 \times 10^7}\sqrt{\frac{2 \times 1.6 \times 10^{15}}{1.66}}$$

$$= \frac{10^7}{14 \times 10^7}\sqrt{\frac{2 \times 1.6 \times 10}{1.66}}$$

$$= \frac{1}{14} \times 4.391 = 0.3136 \text{ m}$$

$$= \mathbf{31.36 \text{ cm}}$$

Example 9.8 If deflection sensitivity of a CRO is 2mm/V, calculate the deflection of the spot when a voltage of 15 V applied. (*Nagpur Uni.* 2006)

Solution. For 1 volt, the deflection produced is 2mm.

$\therefore$ 15 V will produce deflection = 2 × 15 = 30 mm = **3 cm.**

Example 9.9 Determine the deflection sensitivity of a signal for a CRT in which l = 2 cm, L = 30 cm, d = 0.5 cm and V_a = 2000 V. (*Pbi. U.* 2001)

Solution. For a CRT, Electrostatic deflection sensitivity

$$S = \frac{lL}{2dV_a} \text{ m/volt}$$

Here l = 2 cm = 0.02 m; L = 30 cm = 0.3 m

d = 0.5 cm = 0.005 m ; V_a = 2000 V

$$\therefore \quad \text{Deflection sensitivity} = \frac{0.02 \times 0.3}{2 \times 0.005 \times 2000}$$

$$= 3 \times 10^{-4} \text{ m/V} = \mathbf{0.3\ mm/V}$$

Example 9.10 The length of deflecting plates in a *C.R.O* is 5 cm. They are separated by 4 mm. The distance of the fluorescent screen from the nearest edge of the deflecting plates is 15 cm. A d.c voltage of 25 V is applied to the deflecting plates. If the accelerating potential difference is 1000 volt find the displacement of the spot on the screen. (*Bang. U.* 2001)

Solution. For a *CRT*, total deflection due to electrostatic field is given by

$$\text{Total deflection} = \frac{lL}{2d}\frac{V_d}{v_a}$$

and the displacement of the spot on the screen is equal to total deflection. Referring to Fig. 9.10, we have

$$l = 5 \text{ cm} = 0.05 \text{ m};\ d = 4 \text{ mm} = 0.004 \text{ m};\ D = 15 \text{ cm}$$

$$L = \left[\frac{l}{2} + D\right] = 2.5 + 15 = 17.5 \text{ cm} = 0.175 \text{ m}$$

$$V_d = 25 \text{ V};\ V_a = 1000 \text{ V}.$$

$$\therefore \quad \text{Total deflection} = \frac{0.05 \times 0.175 \times 25}{2 \times 0.004 \times 1000}$$

$$= 27.34 \times 10^{-3} \text{ m} = \mathbf{27.34\ mm}$$

Example 9.11 A cyclotron with dees of radius 90 cm has a transverse magnetic field of 0.8 Tesla. Calculate the energies to which (*i*) a proton and (*ii*) deutron are accelerated.

Given Mass of the proton = 1.67×10^{-27} kg. Mass of the deutron = 3.34×10^{-27} kg.

Solution. (*i*) Proton.

$$\text{Mass of the proton } m = 1.67 \times 10^{-27} \text{ kg}$$

$$\text{Charge on the proton } e = 1.6 \times 10^{-19} \text{ C}.$$

$$\text{Energy of the emergent proton } E = \frac{1}{2}\frac{B^2 e^2 r^2}{m}$$

$$= \frac{0.8 \times 0.8 \times 1.6 \times 10^{-19} \times 1.6 \times 10^{-19} \times 0.9 \times 0.9}{2 \times 1.67 \times 10^{-27}}$$

$$= 0.3973 \times 10^{-11} \text{ J} = \frac{0.3973 \times 10^{-11}}{1.6 \times 10^{-13}} \text{ MeV}.$$

$$= 24.8 \text{ MeV}.$$

(*ii*) Deutron.

$$\text{Mass of the deutron } m = 3.34 \times 10^{-27} \text{ kg}$$

$$\text{Charge on the deutron } e = 1.6 \times 10^{-19} \text{ C}.$$

$\therefore$ Energy of the emergent deutron $= \dfrac{0.8 \times 0.8 \times 1.6 \times 10^{-19} \times 1.6 \times 10^{-19} \times 0.9 \times 0.9}{2 \times 3.34 \times 10^{-27}}$

$= 0.1986 \times 10^{-11}$ J

$=$ **12.4 MeV.**

Example 9.12 A cyclotron has a magnetic field of 10^4 Gauss and radius of 80 cm. Calculate the frequency of the alternating electric field that must be applied and to what energy deutrons can be accelerated ? Mass of deutron = 2 a.m.u. (*P.U.* 2000)

Solution. $B = 10^4$ Gauss = 1 T ; r = 80 cm = 0.8 m

m = 2 *a.m.u.* = $2 \times 1.66 \times 10^{-27}$ kg = 3.32×10^{-27} kg ;

$e = 1.6 \times 10^{-19}$ C.

$\therefore$ Frequency of alternating electric field

$$f = \frac{Be}{2\pi m} = \frac{1 \times 1.6 \times 10^{-19}}{2\pi \times 3.32 \times 10^{-27}}$$

$= 7.7 \times 10^6 = 7.7$ MHz

$$\text{Energy } E = \frac{1}{2}\frac{B^2e^2r^2}{m}$$

$$= \frac{1 \times 1 \times 1.6 \times 10^{-19} \times 1.6 \times 10^{-19} \times 0.8 \times 0.8}{2 \times 3.32 \times 10^{-27}} = 24.67 \times 10^{-13} \text{ J}$$

$=$ **15.4 MeV**

Example 9.13 Deutrons are to be accelerated with a cyclotron. If its magnets produce a flux density of 2.475 T, what must be the frequency of the oscillating potential applied to the Dee's ? Mass of $_1H^2$ = 2 a.m.u. (*P.U.* 2001; *Nagpur U.* 2009)

Solution. Mass of $_1H^2$ atom = 2 a.m.u. = $2 \times 1.66 \times 10^{-27}$ kg = 3.32×10^{-27} kg

$e = 1.6 \times 10^{-19}$ C, B = 2.475 T

Frequency of the applied field (or oscillating potential) $f = \dfrac{Be}{2\pi m}$

$$= \frac{2.475 \times 1.6 \times 10^{-19}}{2\pi \times 3.32 \times 10^{-27}} = 18.98 \times 10^6 \text{ Hz} = \mathbf{18.98 \text{ MHz}}$$

Example 9.14 Between the Dee's of a cyclotron 1.5 metre in diameter an alternating potential difference of 15 mega cycles is applied. Calculate the energy in MeV of the protons issuing out of the cyclotron. Mass of proton = 1.672×10^{-27} kg.

Solution. In terms of frequency of the applied field and radius of cyclotron Dee's the energy is given by

$$E = 2\pi^2 mf^2r_{max}^2$$

Mass of proton $= 1.672 \times 10^{-27}$ and $f = 15 \times 10^6$

$$\therefore \qquad E \text{ (in MeV)} = \frac{2\pi^2 \times 1.672 \times 10^{-27} \times 15 \times 10^6 \times 15 \times 10^6 \times .75 \times .75}{1.6 \times 10^{-13}}$$

$$= \mathbf{26.11\ MeV}$$

Example 9.15 A cyclotron with Dee's of diameter 1.8 m has a magnetic field of 0.8 tesla. Calculate the energy to which the doubly ionised helium ion He^{++} can be accelerated. Also calculate the number of revolutions the particle makes in attaining this energy. Mass of He^{++} = 6.68×10^{-27} kg.

Solution. Mass of the α-particle (He^{++})

$$m = 6.68 \times 10^{-27} \text{ kg}$$

Charge on He^{++} ion, $e = 2 \times 1.6 \times 10^{-19}$ C

Now $$E = \frac{B^2 e^2 r^2}{2m}$$

$$B = 0.8 \text{ Tesla; r} = 0.9 \text{ m}$$

$$\therefore \qquad E = \frac{0.8 \times 0.8 \times 2 \times 1.6 \times 10^{-19} \times 2 \times 1.6 \times 10^{-19} \times 0.9 \times 0.9}{2 \times 6.68 \times 10^{-27}}$$

$$= 0.397 \times 10^{-11} \text{ J} = \frac{0.397 \times 10^{-11}}{1.6 \times 10^{-13}} = \mathbf{24.8\ MeV.}$$

The frequency of the alternating electric field is given by

$$f = \frac{Be}{2\pi m} = \frac{0.8 \times 2 \times 1.6 \times 10^{-19}}{2\pi \times 6.68 \times 10^{-27}} = 0.061 \times 10^8 = 6.1 \times 10^6 \text{ s}^{-1}$$

This gives the number of times the He^{++} ion comes out of the gap between the dees each time undertaking a semi-circular path.

$\therefore$ Number of complete revolution made by He^{++} ion in attaining the above energy

$$= \frac{f}{2} = \frac{1}{2} \times 6.1 \times 10^6 = \mathbf{3.05 \times 10^6\ s^{-1}}$$

Example 9.16 Deutrons in a cyclotron describe a circle of radius 0.32 m just before emerging out of the Dee's. The frequency of the applied e.m.f. is 10 MHz. Find the flux density of the magnetic field and the velocity of the deutrons emerging out of the cyclotron. Mass of deutron is 3.32×10^{-27} kg and charge 1.6×10^{-19} C. (*Bang. U.* 2000)

Solution. The frequency of the applied electric field is given by

$$f = \frac{Be}{2\pi m} \qquad \therefore \quad B = \frac{2\pi m f}{e}$$

Here $m = 3.32 \times 10^{-27}$ kg; f = 10 MHz = 10×10^6 Hz;

$e = 1.6 \times 10^{-19}$ C

$$\therefore \qquad B = \frac{2\pi \times 3.32 \times 10^{-27} \times 10^7}{1.6 \times 10^{-19}} = 1.304 \text{ Tesla}$$

Radius of the circle just before deutrons emerge $r_{max} = 0.32$ m

Now $$Bev = \frac{mv^2}{r_{max}}$$

$\therefore$ $$v = \frac{Be\,r_{max}}{m} = \frac{1.304 \times 1.6 \times 10^{-19} \times 0.32}{3.32 \times 10^{-27}}$$

$$= \mathbf{2.01 \times 10^7\ ms^{-1}}$$

Example 9.17 A cyclotron of radius 0.462 m is used to accelerate deutrons. The oscilator frequency is 25 MHz. Find the magnetic flux density needed and also energy acquired by the deutron. Given m = 3.32 × 10^{-27} kg. (*Bang. U.* 2004)

Solution. $m = 3.32 \times 10^{-27}$ kg. f = 25 MHz = 25 × 10^6 Hz

$e = 1.6 \times 10^{-19}$ C

$$B = \frac{2\pi mf}{e} = \frac{2\pi \times 3.32 \times 10^{-27} \times 25 \times 10^6}{1.6 \times 10^{-19}} = 3.26 \text{ T}$$

$$E = 2\pi^2 mf^2 r_{max}^2$$

$$= 2\pi^2 \times 3.32 \times 10^{-27} \times 25 \times 10^6 \times 25 \times 10^6 \times 0.462 \times 0.462$$

$$= 87.4 \times 10^{-13} \text{ J} = \frac{87.4 \times 10^{-13}}{1.6 \times 10^{-13}} = \mathbf{54.6\ MeV.}$$

Example 9.18 A cyclotron oscillator frequency 1 MHz is used to accelerate protons. If the radius of the dees is 60 cm, find the magnetic field in Tesla.

(*Nagpur Uni.* 2008; *H.P.U.* 2002)

Solution. $m = 1.67 \times 10^{-27}$ kg; $f = 1$ MHz = 10^6 Hz

$e = 10^6 \times 10^{-19}$ C

$$B = \frac{2\pi mf}{e} = \frac{2 \times 3.142 \times 1.67 \times 10^{-27} \times 10^6}{1.6 \times 10^{-19}} = \mathbf{6.56\ T}$$

Example 9.19 A beam of charged particles (having same charge and momentum) enters a magnetic field and is deflected by 180° in a circular path of radius 20 cm. Find the width of the focal line if beam is in the form of a cone of 6°.

Solution. $R = 20$ cm = 0.2 m, $\phi = \left(\frac{6}{2}\right)^0 = 3° = \frac{3 \times \pi}{180}$ radian

Width of the focal line $= R\phi^2$

$$= 0.2 \times \left(\frac{3 \times \pi}{180}\right)^2$$

$$= \mathbf{5.48 \times 10^{-4}\ m}$$

Example 9.20 An electron moving in a horizontal direction with a speed of 5.0 × 10^7 m/sec enters a region where there is a uniform electric field of 2000 V/m directed upwards

in the plane of its motion. Find the electrons coordinates referred to the point of entry and the direction of its motion 4×10^{-8} second later.

Solution. $$z = \frac{1}{2}\left(\frac{qE}{m}\right)t^2 \quad \text{Here } q = e = 1.6 \times 10^{-19}\text{ C}$$

$$\therefore \quad y = \frac{1}{2}\left[\frac{2000 \times 1.6 \times 10^{-19}}{9.1 \times 10^{-31}}\right](4 \times 10^{-8})^2 = 0.28\text{ m}$$

Along x-axis, $$x = v \times t = 5 \times 10^7 \times 4 \times 10^{-8} = 2.0\text{ m}$$

$$\therefore \quad \tan\phi = \frac{y}{x} = \frac{0.28}{2.0} = 0.14$$

$$\therefore \quad \phi = \tan^{-1}(0.14) = 8° \text{ w.r.to horizontal axis.}$$

Example 9.21 In a Bainbridge mass spectrograph radius of curvature of the path of Ne^{20} ion is 0.26 m. Calculate that for Ne^{22} ions. Assume single ionization of both the ions.

(*Nagpur Uni.* 2007, 2008, 2005)

Solution. Given $m_1 \propto 20, \quad m_2 \propto 22, \quad R_1 = 0.26$ m

$$R_1 = km_1$$

$$R_2 = km_2$$

$$\therefore \quad \frac{R_1}{R_2} = \frac{m_1}{m_2}$$

or $$R_2 = \frac{m_2}{m_1} \times R_1 = \frac{22 \times 0.26}{20}$$

$$= \mathbf{0.286\text{ m}}$$

EXERCISE CH. 9

LONG QUESTIONS

1. (*a*) Describe the motion of an electron subjected to a uniform electric field acting normal to the electron velocity.

   (*b*) Show that an electron moving with uniform velocity follows a parabolic path in a transverse uniform electric field. [*Nagpur Uni.* 2009]
2. Show that the veloicty acquired by an electron in a uniform electrostatic field varies as the square root of potential difference through which it is accelerated. [*Nagpur Uni.* 2001]
3. Prove that an electron moves along a parabolic path, when it enters in a uniform electric field applied perpendicular to its motion. What will happen if the electric field is not uniform?
4. (*a*) Describe the path of electron of mass '*m*' and charge '*e*' moving with velocity *v* in (*i*) uniform magnetic field, and (*ii*) uniform electrostatic field. Both the fields are applied perpendicular to the direction of motion of the particle.

   (*b*) Does kinetic energy of a charged particle change when it enters a magnetic field ?
5. (*a*) Draw the block diagram of a cathode ray oscillograph. Describe its construction and explain its working. Give uses of *CRO*. [*Pbi. U.* 2004, 2002, 2000; *G.N.D.U.* 20004; *Pat. U.* 2004; *Nag. U.* 2002; *Gharwal U.* 2000]

(*b*) Why is a sweep circuit necessary in a *C.R.O*? What will happen if fly back time is comparable to sweep time in a sawtooth voltage? [*Nag. U.* 2000]

6. (*a*) Explain the necessity of time base circuit. [Nag. U. 2002]

(*b*) Write the applications and uses of C.R.O.

[*Pbi. U.* 2003; *M.D.U.* 2001; *Pat. U.* 2004; *Gharwal U.* 2000]

(*c*) Describe the working and function of electron gun in a CRO. Which type of emission is employed in it? [*M.D.U.* 2003]

(*d*) Describe the role of aquadag in cathod ray oscilloscope. [*P.U.* 2001]

7. (*a*) Explain the principle of electrostatic focussing in a CRO. [*Nagpur Uni.* 2007]

(*b*) What is the difference between a CRT and CRO?

(*c*) The wilemite is used on the screen of a CRO. We get (*i*) blue colour (*ii*) green colour (*iii*) red colour (*iv*) white colour.

8. (*a*) Describe dual beam cathode ray oscilloscope. [*G.N.D.U.* 2002, 2001]

(*b*) Why Willemite is most suitable coating material for screen of C.R.T? [*H.P.U.* 2003]

9. Explain carefuly the principle of Linear accelerator.

Deduce the expression for the energy of the particle and length of cylinders in terms of the constants of the apparatus. [*Nagpur Uni.* 2009, 2007, 2006; *G.N.D.U.* 2004, 2003; *H.P.U.* 2001; *Gauhati. U.* 2003; *P.U.* 2004; *K.U.* 2000]

10. (*a*) Describe the principle construction and working of a cyclotron. Derive expression for the maximum kinetic energy achieved by a particle of mass *m* in terms of the applied magnetic field and dee radius. Also state the relation in terms of the frequency of the applied electric field. Discuss its limitations. [*M.D.U. 2000, KU.* 2002, 2004; *Nag. Uni.* 2007, 2006; *G.N.D.U.* 2002; *Pbi. U.* 2003; *Bang. U.* 2001; *Kerala U.* 2001; *Cal. U.* 2003; *P.U.* 2002, 2000]

(*b*) Can we accelerate neutrons by a cyclotron?

11. (*a*) Show that the maximum radius of curvature of the path of a particle inside the dees of a cyclotron is proprotional to the square root of the number of times it crosses the gap between the dees.

(*b*) What are the primary functions of (*i*) electric field and (*ii*) magnetic field in a cyclotron?

12. (*a*) A Cyclotron is a called resonance device. Explain.

(*b*) Can a cyclotron be used to accelerate electrons? If not why?

[*K.U.* 2002; *Pbi. U.* 2001, 2000; *H.P.U.* 2002, 2001]

13. Give theory of Thomson's parabola method to determine the specific charge of positive ions. Give the necessary theory. [*Nagpur Uni.* 2007]

14. Describe the principle, construction and working of Bainbridge's mass spectrograph. How is the nuclear mass determined using the spectrograph? [*K.U.* 2002, 2000; *M.D.U.* 2002]

15. Draw figure showing the parabolic traces obtained on the photographic place in Thomson's method due to two isotopes of atomic masses m_1 and m_2 where $m_1 < m_2$. Explain how the traces are analysed to determine the atomic masses.

16. Draw a neat diagram of Thomson's apparatus for determining charge to mass ratio of positive ions.

Give a brief description of the apparatus. Describe Thomson's parabola method to determine e/m of the isotopes of an element in gaseous state.

17. Draw a neat diagram of cyclotrom and obtain resonance condition. [*Nagpur Uni.* 2002]

18. (*a*) Explain the principle and working of cyclotron. Show that the time spent by the charged particle in the Dee of the cyclotron is independent of its velocity.

(*b*) Mention the limitations on the energy achieved by a particle in the cyclotron?

[*Nagpur Uni.* 2007]

SHORT QUESTIONS

1. Show that a charged particle does not change its energy when fired in a magnetic field.
2. Magnetic field changes the velocity of a charged particle without changing its speed. Explain.

   [*Nagpur Uni.* 2001]
3. Show that the radius of orbit of a charged particle moving at right angles to magnetic field is proportional to its momentum. [*Nagpur Uni.* 2000]
4. Consider a particle of mass m and charge q moving with velocity v. The particle enters a region where a perpendicular uniform magnetic field B acts. Show that in the region the kinetic energy of the particle is proportional to the square of the radius of its orbit.
5. Explain the working of an electrostatic electron lens.
6. What is deflection sensitivity of a cathode ray tube? Obtain expression for (*i*) electrostatic deflection sensitivity and (*ii*) magnetic deflection sensitivity. Give their importance.

   [*Nagpur U.* 2009; *Bang. U.* 2004]
7. Explain how a CRO may be used to measure voltage. [Bang. U. 2000]
8. Discuss how a cathode ray osciloscope may be used to measure the phase difference between two a.c signals of the same frequency and amplitude. [*Kalkotta U.* 2001]
9. Explain how a C.R.O may be used to measure frequency.

   [*Nag. U.* 2002; *Kolkotta U.* 2001; *Bang. U.* 2000]
10. What is the difference between linear accelerator and circular acelerator? [*P.U.* 2001]
11. Which accelerator makes use of electromagnetic radiations for accelerating particles?

    [*G.N.D.U.* 2000]
12. What is cyclotron? Show that the maximum kinetic energy gained by a particle in a cyclotron is

    $$\text{K.E.} = 2\pi^2 m f^2 r_{max}^2$$

    symbols have their usual meaning.
13. What is velocity selector? Explain the function of velocity selector in Bainbridge mass spectrograph in determining the isotopes of various elements. [*Nagpur Uni.* 2008]
14. Explain with neat diagram the working of magnetic lens. Where it is used.
15. An electric field E is applied between the vertical deflecting plates of a CRO. Obtain an expression for the displacement of an electron entering the field. [*Nagpur Uni.* 2007]
16. Obtain an expression for the deflection produced on the screen at a distance L from the centre of the plates [*Nagpur Uni.* 2007]
17. Obtain expression for maximum K.E. of particles coming out of a cyclotron.

    [*Nagpur Uni.* 2007]
18. State any two limitations of a linear accelerator. [*Nagpur Uni.* 2007]
19. What is the path traced out by a beam of electrons when subjected to parallel E and B fields?

    [*Nagpur Uni.* 2007]

20. What is discharge tube? Describe its construction. [*Nagpur Uni.* 2008]

21. Explain the principle of 180° magnetic focusing. [*Agra U.* 2005; *Nagpur Uni.* 2008, 2006]

22. Describe the construction of an electron gun. [*Nagpur Uni.* 2009, 2008]

23. If a charged particle passes through, mutualy perpendicular electric and magnetic fields,then prove that its path will be cycloid. [*Agra Uni.* 2004]

24. Explain the principle of velocity selector. [*Agra Uni.* 2004]

25. Show that in Bainbridge mass spectrograph, the radii of curvature of path of positive ions is proportional to masses of ions. (*Nagpur U.* 2009)

NUMERICAL QUESTIONS

1. In Thomson's parabola method, parabolic traces were obtained on the photographic plate for Neon and hydrogen ions. The *X*-coordinates of the points of intersection of a straight line, paralel to the *x*-axis, with the parabolic traces were 4.77, 5, 22.4 mm. Find the atomic masses of the isotopes of neon. [Atomic mass of H = 1 amu]. [*Nagpur Uni.* 2007]

2. The pole pieces of a cyclotron are 1.2 m in diameter and provide a magnetic field of 1.6 Wb m^{-2}. What will be the energy of the alphas, deutrons and protons in such a machine? What should be the range of oscilator frequency to cover the acceleration of the above particles. Mass of the proton = 1.67×10^{-27} kg and charge 1.6×10^{-19} C.

[*Calicut. U.* 2003] (**Ans.** 12.20 to 24.40 MHz)

3. If the electric field in the velocity selector of a Bainbridge mass spectrograph is 10^4 V/m and magnetic flux density is 0.2 Wb/m^2; find the speed of an ion which will go undeviated through the crossed field. [*Nagpur U.* 2004] (**Ans.** 5×10^4 m/s)

4. In a cyclotron the magnetic flux density is 1.5 weber/m^2. A proton of mass 1.67×10^{-27} kg and charge 1.6×10^{-19} C is accelerated. The radius at which the proton leaves the system is 0.5 m. Find its maximum K.E. and the time of flight through one dee.

[*Nagpur Uni.* 2004] (**Ans.** 2.18×10^{-8} s)

5. The energy of a C^{12} ion when it emerges from the last drift tube of a linear accelerator is 120 MeV. What would be the energy of a O^{16} ion from the accelerator? (**Ans.** 160 MeV)

6. Two positive ray parabolas are obtained in a particular experiment for H_2 and a gas *X*. At a particular *y* value, the *x* values are in the ratio 4:1. Determine the isotopic mass of element *X* and identify it. Given m_H = 1amu [*Nagpur U.* 2009] (**Ans.** 16amu; $^{16}O_8$)

7. What is the energy in Joules of an electron accelerated through a potential of 1000 kV.

[*Nagpur U.* 2009] (**Ans.** 1.6×10^{-13}J)

8. A cyclotron oscillator's frequency is 10MHz. What would be the operating magnetic field for accelerating protons? If the radius of its dees is 60 cm, what is the K.E. of proton beam produced by the accelerator? Given e = 1.6×10^{-19}C and m = 1.67×10^{-27} Kg.

[*Nagpur U.* 2008] (**Ans.** 0.656 Wb/m^2, 7.4 MeV)

10 PROPERTIES OF MATTER

ELASTICITY

INTRODUCTION

During the study of mechanics in earlier chapters, we had started with the concept of a massive small particle and then the rigid body which we define as one, the distance between any two particles remains unaltered whatever the external forces applied to it. In other words, the body remains *un-deformated* in its shape, size and volume.

However, in practice, we never find such a body. Every material body gets demormated, to a smaller or larger extent, depending upon the way in which the forces act. Moreover, it has a tendency to recover its original shape and size on the removal of external applied forces, within some limit, called as *elastic limit*. This property of a body is known as *elasticity*.

Elasticity is the property by virtue of which material bodies regain their original shape and size after the external deforming forces are removed. When an external force acts on a body, there is a change in its length, shape and volume. The body is said to be strained. Such bodies are called *elastic bodies*. Some of the examples are steel, glass, ivory, quartz, rubber etc. The bodies who do not regain their original shape and size are called *plastic bodies*. Steel, glass are more elastic than rubber. Liquids and gases are highly elastic.

In this chapter, our discussion will be confined to those bodies which are *homogeneous,* means having a uniform composition and *isotropic* i.e. whose properties are the same in all directions. Metals, in the form of wires or rods behave as isotropic bodies for elastic properties. Solids, mostly crystals show different properties in different directions and are thus called as *non-isotropic* or *anisotropic*.

10.1 STRESS AND STRAIN

When an external force acts upon a body relative displacement of its various parts takes place. The displaced particles tend to come to their original position to restore the original length, volume or shape of the body and thus, exert a restoring force.

The restoring force per unit area comes into play inside the body is called the **stress.**

The reaction set up in the body is equal and opposite to the applied force, so long as there is no permanent change produced in the body. The restoring force, is therefore, equal to the applied force.

Hence, if the force F is applied normaly to the area of cross-section a of a wire, then

$$\text{Stress} = \frac{F}{a}$$

The external force acting on the body causes a relative displacement of its constituent particles. A change in the length, volume or shape takes place. The body is then said to be under a strain.

Strain *is defined as the ratio of the change in length, volume or shape to the original length, volume or shape. Strain is thus, a pure ratio and has no units.*

10.2 SMALL DEFORMATIONS AND HOOKE'S LAW

In 1676, Robert Hooke discovered a linear relationship between stress and strain for *small deformations i.e.* within elastic limit. This relation is known as *Hooke's Law.*

For stresses within the elastic limit of a material, the strain produced is directly proportional to the stress applied.

Thus, within elastic limit of a material

$$\text{Stress} \propto \text{Strain}$$

or
$$\frac{\text{stress}}{\text{strain}} = \text{constant}$$

The constant is called the modulus of elasticity of the material. The S.I. unit of modulus of elasticity is Newton per square metre (N/m^2 or Nm^{-2}) and its diamensions are as those of stress, *i.e.* [M° L^{-1} T^{-2}].

10.3 ELASTIC CONSTANTS FOR AN ISOTROPIC SOLIDS

According to the nature of strain, there are three moduli of elasticity of an isotropic material. These are

(*i*) Young's modulus, Y

(*ii*) Bulk modulus, K, and

(*iii*) Modulus of rigidity η.

For stresses within the elastic limit of a material, they are defined as follows:

(*i*) Young's modulus. *It is defined as the ratio of stress to longitudinal strain within elastic limits.*

When a change of length takes place the strain is known as **longitudinal strain**. *It is measured by the change in length per unit length.* If L is the length of a wire and an increase l in length is caused by a force, then

$$\text{Strain} = \frac{l}{L}$$

Hence, Young's modulus of elasticity $= \dfrac{\text{Stress}}{\text{Strain}}$

or
$$Y = \frac{F}{a} \div \frac{l}{L} = \frac{FL}{al} \qquad \text{... (1)}$$

(*ii*) Bulk modulus. *It is defined as the ratio of stress to volumetric strain.* When a force is applied normally to the surface of a body and a change in volume takes place, the strain is known as **volumetric strain.** *It is measured by the change in volume per unit volume* and is equal to v/V where v is the change in volume produced by the force F in the original volume V. Hence, bulk modulus K is given by

$$K = \frac{F}{a} \div \frac{v}{V} = \frac{FV}{av} = \frac{PV}{v} = \frac{P}{v/V} \qquad \text{... (2)}$$

where $\dfrac{F}{a} = P$ (applied pressure).

When a very small pressure dP is applied, the change in volume being very small is represented by dV, then

$$K = -\frac{dP}{dV/V} \qquad \text{... (3)}$$

The negative sign indicates that an increase in applied pressure causes a decrease in volume.

***(iii)* Modulus of rigidity.** *The coefficient or modulus or rigidity is defined as the ratio of the tangential stress to shearing strain.*

To find the value of the shearing strain, consider a solid cube *ABCDefgh*. The lower face *CDgh* is fixed and a *tangential force F* is applied over the face *ABef* so that it is displaced to the position of $A'B'e'f'$. Each horizontal layer of the cube is then displaced, the displacement being proportional to its distance from the fixed plane. Then

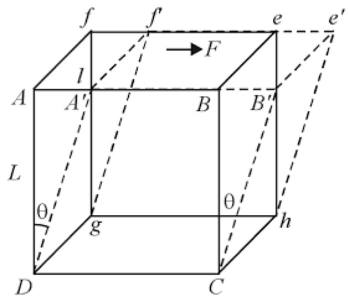

Fig. 10.1

$$\text{Shearing strain} = \frac{\text{Displacement of a plane}}{\text{Distance from the fixed plane}}$$

$$= \frac{AA'}{AD} = \frac{l}{L} = \tan\theta$$

When θ is small $\tan\theta = \theta$

∴ Shearing strain = θ

Hence the angle through which a line originally perpendicular to the fixed plane is turned is a measure of the shearing strain.

$$\text{Now modulus of rigidity} = \frac{\text{Tangential stress}}{\text{Shearing strain}}$$

$$\therefore \qquad \eta = \frac{F}{a} \div \theta = \frac{F}{a\theta} = \frac{T}{\theta} \qquad \text{... (4)}$$

where a is the area of the face *ABef* and T is the tangential force applied per unit area.

***(iv)* Poisson's ratio.** When a wire is stretched, its length increases but its diameter decreases. In general, when an elongation is produced by a *longitudinal stress* in a certain direction a contraction results in the *lateral dimensions* of the body under strain. The lateral strain is proportional to the longitudinal strain so long as it is small.

The ratio $\dfrac{\text{Lateral strain}}{\text{Longitudinal strain}}$ is called *Poisson's ratio* and is denoted by the letter σ.

Let the length of the wire = L, Diameter of the wire = D

Increase in length = l, Corresponding decrease in diameter = d

$$\text{The poisson's ratio } \sigma = \frac{d/D}{l/L} = \frac{\beta}{\alpha} \qquad \text{... (5)}$$

where β is the lateral strain and α the longitudinal strain.

Poisson's ratio is a dimensionless quantity.

10.4 GLASS IS MORE ELASTIC THAN RUBBER

Glass is more elastic than rubber. This is because for a given stress, the strain produced in glass is much less than that in rubber.

Suppose for rubber, Young's modulus Y_1 is

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$$Y_1 = \frac{F\,L}{a\,l_1} \qquad \text{... }(i)$$

For glass,
$$Y_2 = \frac{F\,L}{a\,l_2} \qquad \text{... }(ii)$$

For the two materials, F, L and a remaining the same, l_1 for rubber is more than l_2 for glass. Therefore, Y_2 is greater than Y_1.

10.5 LIMIT, YIELD POINT, ELASTIC FATIGUE

Elastic Limit. Elastic limit is the maximum stress within which the body exhibits the property of elasticity. Below the elastic limit, the body regains its original position or shape or size when the deforming force is removed. Beyond the elastic limit, the body does not regain completely its original position or shape or size even though the external force is withdrawn.

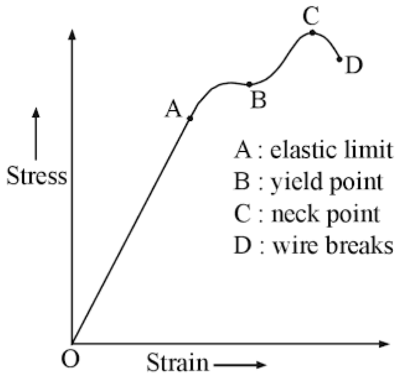

Fig. 10.2

Yield point. When a wire is loaded beyond the elastic limit, Hooke's law is no longer obeyed and the extension produced in the wire is not proportional to the stress. Fig. 10.2 shows stress-strain curve for a wire. The extension is more than the corresponding increase in stress. At this stage, the particles of the material go further apart. Now, if the load is withdrawn, the particles do not regain their original positions. Point B in Fig. 10.2 represents yield point.

Elastic fatigue. If a body is continuously subjected to stress and strain, after some period it gets fatigue. Consider two torsional pendulums A and B having similar wires. A is set into vibration. After A has come to rest, both the pendulums A and B are set into vibration simultaneously. It is found that due to elastic fatigue, A comes to rest earlier than B.

10.6 WORK DONE IN STRETCHING A WIRE

Whenever a body is deformed by aplication of external forces, the body gets strained. The work done is stored in the body in the form of energy, known as energy of strain. Let us calculate the work done in stretching a wire, producing longitudinal strain.

Consider a wire of length L, area of cross section a and Young's modulus of elasticity Y. Let l be the increase in length when a stretching force F is applied.

The Young's modulus of elasticity

$$Y = \frac{FL}{al} \quad \text{or} \quad F = \frac{Yal}{L}$$

$\therefore$ Work done in producing a stretching $dl = F.dl = \frac{Yal}{L} \times dl$

Hence, work done to produce a stretching of the wire from 0 to l

$$W = \int_0^l \frac{Yal}{L}\,dl = \frac{Ya}{L}\left[\frac{l^2}{2}\right]_0^l$$

$$= \frac{1}{2}\,\frac{Yal^2}{L} = \frac{1}{2}\,\frac{Yal}{L} \times l = \frac{1}{2}\,F \times l$$

$$= \frac{1}{2} \times \text{stretching force} \times \text{elongation produced} \qquad \text{... (6)}$$

10.7 RELATION BETWEEN Y, K AND σ

Consider a **unit** cube *ABCDefgh* which is homogeneous and isotropic and let forces T_x, T_y and T_z act perpendicular to the faces *BehC* and *AfgD*, *efgh* and *ABCD*, *ABef* and *CDgh* respectively as shown in Fig. 10.3.

If α is the increase in length per unit length per unit tension along the direction of the force, then the elongation produced in the edges *AB*, *Be* and *BC* will be $T_x\,\alpha$, $T_y\,\alpha$ and $T_z\,\alpha$ respectively.

If β is the contraction produced per unit length per unit tension in a direction perpendicular to the force, then the contraction produced perpendicular to the edges AB, Be and BC will be $T_x\,\beta$, $T_y\,\beta$ *and* $T_z\,\beta$ *respectively.*

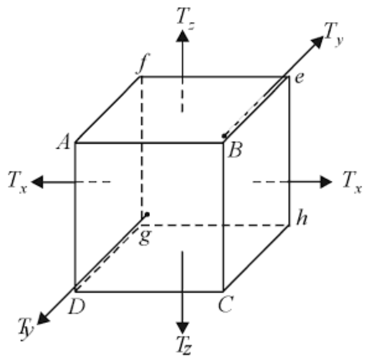

Fig. 10.3

Thus, the lengths of the edges are as follows:

$$AB = 1 + T_x\alpha - T_y\beta - T_z\beta$$
$$Be = 1 + T_y\alpha - T_x\beta - T_z\beta$$
$$BC = 1 + T_z\alpha - T_x\beta - T_y\beta$$

Hence, the new volume of the cube now becomes

$$AB \times Be \times BC = (1 + T_x\,\alpha - T_y\,\beta - T_z\beta) \times (1 + T_y\alpha - T_x\beta - T_z\beta) \times (1 + T_z\alpha - T_x\beta - T_y\beta)$$

Since α and β are very small quantities, terms containing their squares and higher powers can be neglected.

$$\therefore \text{ Volume} = 1 + \alpha\,(T_x + T_y + T_z) - 2\beta\,(T_x + T_y + T_z)$$
$$= 1 + (\alpha - 2\beta)\,(T_x + T_y + T_z)$$

If the deforming forces acting on the three faces are equal, then

$$T_x = T_y = T_z = T$$

$\therefore$ Volume $= 1 + 3T\,(\alpha - 2\beta)$

If instead of applying a stretching force T outwardly, a pressure P is applied on all the faces to compress the cube, the contraction in volume

$$= 3P\,(\alpha - 2\beta)$$

$\therefore$ Volumetric strain $= 3P\,(\alpha - 2\beta)/1$ [∵ It is a unit cube.]

Hence, Bulk modulus $K = \dfrac{\text{Stress}}{\text{Volumetric strain}} = \dfrac{P}{3P\,(\alpha - 2\beta)}$

$$= \frac{1}{3(\alpha - 2\beta)} \qquad ...(i)$$

Divide the numerator and the denominator by α, then

$$K = \frac{\frac{1}{\alpha}}{3\,(1 - 2\beta/\alpha)} = \frac{Y}{3\,(1 - 2\sigma)} \qquad ...(ii)$$

$$\left[\because \frac{1}{\alpha} = Y \text{ and } \frac{\beta}{\alpha} = \sigma \text{ (the Poisson's ratio)}\right]$$

or $$Y = 3K\,(1 - 2\sigma) \qquad ...(7)$$

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10.8 RELATION BETWEEN Y, η AND σ

When a tangential force F is applied to the upper face $ABef$ of a cube $ABCD$ ... then the face $ABCD$ is displaced to the position $A'B'CD$. The diagonal DB increases in length to DB' whereas the diagonal CA decreases CA'.

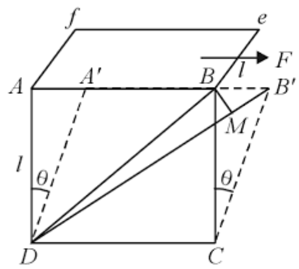

Fig. 10.4

Shearing stress $= \dfrac{F}{\text{area } ABef} = T$

Now, a shearing stress along AB is equivalent to *tensile stress* along DB and an equal *compression stress* along CA at right angles.

If α and β are the longitudinal and lateral strains per unit stress respectively, then

Extension along diagonal DB due to tensile stress

$$= DB.T.\ \alpha$$

and extension along diagonal DB due to compression stress along AC

$$= DB.T.\ \beta$$

$\therefore$ Total extension along $DB = DB.T.\ (\alpha + \beta) = \sqrt{2}.L.T.(\alpha + \beta)$

Draw a perpendicular BM on DB'. Then increase in the length of diagonal DB is practically equal to $B'M$.

Since θ is very small, therefore, angle $AB'C$ is nearly 90° and hence, $\angle BB'M = 45°$.

$$\therefore B'M = BB' \cos 45° = \frac{BB'}{\sqrt{2}} = \frac{l}{\sqrt{2}}$$

Hence,
$$T(\alpha + \beta)\, L\, \sqrt{2} = \frac{l}{\sqrt{2}}$$

$$T.\frac{L}{l} = \frac{1}{2(\alpha + \beta)} \qquad ...(i)$$

But
$$T.\frac{L}{l} = \frac{T}{l/L} = \frac{T}{\theta} = \eta \qquad ...(ii)$$

Hence, from (*i*) and (*ii*), we have, the coefficient of rigidity

$$\eta = \frac{1}{2(\alpha + \beta)} = \frac{1}{2\alpha(1 + \beta/\alpha)} \qquad ...(iii)$$

But Poisson's ratio
$$\sigma = \frac{\text{Lateral strain}}{\text{Longitudinal strain}} = \frac{\beta}{\alpha}$$

and Young's modulus
$$Y = \frac{\text{Stress}}{\text{Longitudinal strain}}$$

$$= \frac{1}{\text{Longitudinal strain per unit stress}} = \frac{1}{\alpha}$$

Substituting in (*iii*), we get

$$\eta = \frac{Y}{2\,(1 + \sigma)} \qquad ...\ (8)$$

or
$$Y = 2\eta\,(1 + \sigma) \qquad ...\ (8A)$$

10.9 RELATION BETWEEN Y, K AND η

The Relation between Y, η and σ (See article 10.9)

$$\eta = \frac{Y}{2(1+\sigma)} \qquad ...(i)$$

or $Y = 2\eta(1+\sigma)$

and, the relation between Y, K and σ (See article 10.8)

$$K = \frac{Y}{3(1-2\sigma)} \qquad ...(ii)$$

or $Y = 3K(1-2\sigma)$

From relation (i), we have $2 + 2\sigma = \frac{Y}{\eta}$

From relation (ii), we have $1 - 2\sigma = \frac{Y}{3K}$

Adding, we have

$$3 = \frac{Y}{3K} + \frac{Y}{\eta} = Y\left(\frac{1}{3K} + \frac{1}{\eta}\right) = Y\left(\frac{\eta + 3K}{3\eta K}\right)$$

Hence, $$Y = \frac{9\eta K}{\eta + 3K}$$

or $$\frac{9}{Y} = \frac{1}{K} + \frac{3}{\eta} \qquad ...(9)$$

10.10 RELATION BETWEEN K, η, σ

From relation (8) and (8 A), we have,

$$Y = 2\eta(1+\sigma)$$

and from relation (7), we have

$$Y = 3K(1-2\sigma)$$

$$\therefore \quad 2\eta(1+\sigma) = 3K(1-2\sigma)$$

or $$2\eta + 2\eta\sigma = 3K - 6K\sigma$$

or $$\sigma(2\eta + 6K) = 3K - 2\eta$$

$$\therefore \quad \sigma = \frac{3K - 2\eta}{6K + 2\eta} = \frac{3K - 2\eta}{2(3K + \eta)} \qquad ...(10)$$

10.11 LIMITING VALUE OF σ.

From relations (7) and (8 A), we have

$$3K(1-2\sigma) = 2\eta(1+\sigma) \qquad ...(i)$$

The bulk modulus K and coefficient of rigidity η are both positive quantities. If therefore, equation (i) is true and (i) *Poisson's ratio is to be positive*, the right hand expression and also the left hand expression must be positive. This is possible only if

$$2\sigma < 1 \text{ or } \sigma < 0.5$$

(ii) If σ *is a negative* quantity the left-hand expression is positive. The right hand expression also must then be positive. This is possible only if $1 + \sigma$ is positive or σ is not less than -1.

Thus for a homogeneous isotropic material the value of σ must lie between $+0.5$ and -1.

Theoretically it cannot be greater than $+0.5\left(i.e. +\frac{1}{2}\right)$ and less than – 1.

Practical limits of σ. In actual practice σ cannot be negative.

A negative value of σ would mean that on being extended a body would also expand laterally. Since, no substance behaves in this way *in actual practice σ lies between 0 and + 0.5.*

10.12 TORSION (TWISTING) OF A CYLINDER

Angle of twist. Consider a short cylinder of length l and radius a clamped at the upper end AB. Suppose a twisting couple is applied to the face $A'B'$ as shown by the arrow head, in a direction perpendicular to the length of the cylinder. The radius $O'P$ is twisted through an angle θ to the position $O'P'$ (fig. 10.5).

θ *is known as the* **angle of twist**.

This is an example of pure shear because the twist neither produces a change in length nor a change in the radius of the cylinder.

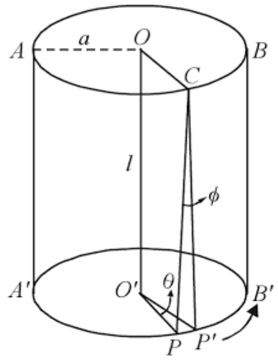

Fig. 10.5

Due to elasticity of the material a *restoring couple* is set up inside the cylinder which is equal and opposite to the *twisting couple*.

Angle of shear. Due to the application of the twisting couple a line such as CP on the rim of the cylinder parallel to OO' is displaced to the position CP' through an angle ϕ, due to the twisting couple. Here angle ϕ is the *angle of shear*. The displacement PP' is maximum for the points lying on the rim and goes on decreasing as we move towards O', the centre of the cylinder.

Couple required to twist a uniform solid cylinder. To find the value of the *twisting couple* imagine the solid cylinder to consist of a large number of *co-axial* cylindrical shells. Consider one such cylindrical shell of radius x and thickness dx. (Fig. 10.6)

Each radius of the lower end of the cylinder is turned through the same angle θ but the displacement is maximum at the rim and decreases as we move towards the centre O' were it is reduced to zero.

The angle of shear ϕ will have the maximum value when $x = a$ and least at O' where $x = 0$. This shows that the shearing strain is not constant throughout the cylinder. It is maximum on the rim and least for the innermost layer.

Thus angle of shear is the same for any one of the *hollow cylinders,* being greatest for the outermost and least for the innermost cylinder.

If the points Q and Q' are supposed to lie on the rim of the hollow cylinder of radius x and ϕ is the angle of shear, then from Fig. 10.5, we have $QQ' = l\phi$ (as ϕ is small)

And also from Fig. 10.6

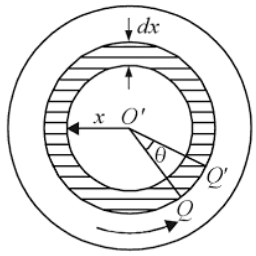

Fig. 10.6

$$QQ' = x\theta$$

$$\therefore \quad l\phi = x\theta$$

or $$\phi = \frac{x\theta}{l} \quad ...(i)$$

If η is the coefficient of rigidity, then

$$\eta = \frac{\text{Shearing stress}}{\text{Angle of shear}} = \frac{T}{\phi}$$

$$\therefore \quad T = \eta\phi = \frac{\eta x\theta}{l} \quad ...(ii)$$

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Now, the face area of the hollow cylinder = $2\pi x.dx$

$\therefore$ Total shearing force on this area $= 2\pi x dx.\ \dfrac{\eta x\theta}{l} = \dfrac{2\pi\eta\theta}{l}\ .\ x^2 dx$

Moment of the force about $OO' = \dfrac{2\pi\eta\theta}{l}\ .\ x^2 dx.x$

The twisting couple applied to the whole cylinder can be obtained by integrating this quantity for limits $x = 0$ and $x = a$.

$$\text{Hence, twisting couple } = \frac{2\pi\eta\theta}{l}\int_0^a x^3\, dx = \frac{2\pi\eta\theta}{l}\left[\frac{x^4}{4}\right]_0^a$$

$$= \frac{2\pi\eta\theta}{l}\,\frac{a^4}{4} = \frac{\pi\eta\theta a^4}{2l}$$

This relation is used to find the value of rigidity by the statical method.

If in the above relation $\theta = 1$ radian, then

$$\text{Twisting couple per unit angular twist } c = \frac{\pi\eta a^4}{2l} \qquad ...(11)$$

This twisting couple per unit angular twist of the wire or cylinder is called its **modulus of torsion** *or torsional rigidity.*

It is evident from relation (11) that the couple required is proportional to the *fourth* power of the radius.

Hollow Cylinder. For a hollow cylinder with inner radius r_1 and outer radius r_2, the twisting couple is obtained by integrating $\dfrac{2\pi\eta\theta}{l}\ x^3\ dx$ between the limits $x = r_1$ and $x = r_2$.

$$\text{Hence, twisting couple } = \frac{2\pi\eta\theta}{l}\int_{r_1}^{r_2} x^3\ dx = \frac{2\pi\eta\theta}{l}\left[\frac{x^4}{4}\right]_{r_1}^{r_2} = \frac{\pi\eta\theta}{2l}\,[r_2^4 - r_1^4]$$

$$\therefore \text{ Twisting couple per unit angular twist } c' = \frac{\pi\eta}{2l}\,[r_2^4 - r_1^4] \qquad ...(12)$$

Hollow cylinder is sronger than solid cylinder of same length, mass and material. Consider two cylinders of same length, mass and material. One is solid of radius r and the other is hollow of inner radius r_1 and outer radius r_2.

$$\text{Torsional couple for solid cylinder per unit angular twist } c = \frac{\eta\,\pi r^4}{2l}$$

$$\text{Torsional couple for hollow cylinder per unit angular twist } c' = \frac{\pi\eta\,(r_2^4 - r_1^4)}{2l}$$

$$\therefore \qquad \frac{c'}{c} = \frac{r_2^4 - r_1^4}{r^4} = \frac{(r_2^2 - r_1^2)\,(r_2^2 + r_1^2)}{r^4}$$

As the cylinders are of the same length and mass

$$\pi\,(r_2^2 - r_1^2)\,l\rho = \pi r^2 l\rho$$

$$\therefore \qquad r_2^2 - r_1^2 = r^2$$

$$\text{Hence,} \qquad \frac{c'}{c} = \frac{(r_2^2 - r_1^2)\,(r_2^2 + r_1^2)}{r^2 \times r^2} = \frac{r_2^2 + r_1^2}{r^2} \qquad ...(13)$$

Thus, $c' > c$ *i.e.*, torsional rigidity for hollow cylinder is greater than that for the solid cylinder of the same mass, length and material. Hence the hollow cylinder is stronger than solid cylinder.

This shows also that hollow shaft is stronger than the solid one.

10.13 WORK DONE IN TWISTING A WIRE

Let a wire of length l and radius r be fixed at the upper end. A couple is applied at the lower end of the wire so as to produce a twist of an angle θ at this end.

If c is the couple per unit angular twist of the wire, then the couple required to produce a twist θ in the wire = $c\theta$. Now, the work done in twisting the wire through a small angle $d\theta$ is given by

$$dW = c\theta \, d\theta$$

Hence, the total work done in twisting the wire through an angle θ is given by

$$W = \int_0^{\theta} c\theta \, d\theta = \frac{1}{2} c\theta^2$$

Now, $c = \eta \dfrac{\pi r^4}{2l}$, where, η is the coefficient of rigidity, r the radius and l the length of the wire

$$\therefore \qquad W = \frac{\eta \pi r^4}{4l} \theta^2 \qquad \text{... (14)}$$

The energy spent in doing the work is stored in the wire and is known as the **strain energy.**

10.14 BENDING OF BEAMS

A beam is defined as a structure of uniform cross-section, whose length is large as compared to its breadth and thicknes. For such a structure, the shearing stress for any given cross-section is negligible. Beams are used in the construction of bridges and infrastructure where heavy loads are to be supported. They are most commonly used in the structure of multistoried buildings.

Important Definitions

Neutral sufrace. When a metallic strip is fixed at one end and loaded at the other a bending is produced due to the moment of the load. The deformation produced by the load brings about restoring forces due to elasticity tending to bring the strip back to its original position. In equilibrium position

Restoring couple = Bending couple

These two couples act in the opposite directions.

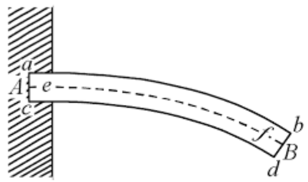

Fig. 10.7

Suppose a metallic strip consists of a large number of filaments of small thickness lying one above the other. When a load is applied at the end B, the end A being fixed, inner filaments like cd are shortened or compressed while the outer filaments like ab are elongated as shown in Fig. 10.7 and 10.8. Along the section lying in between these two portions a filament like ef is neither stretched nor compressed. Such a surface is called the *neutral surface.*

Plane of bending. *The plane in which bending takes place is known as plane of bending*. When the beam is placed horizontally the plane of bending is a vertical plane perpendicular to the beam.

Neutral axis. *The section of the neutral surface* (ef) *by the plane of bending which is perpendicular to it is called the neutral axis.*

The change in length of any filament is proportional to the distance of the filament from the neutral axis.

10.15 BENDING MOMENT

Consider a small part XY of the neutral axis of the strip bent into an arc of radius R subtending an angle θ at the centre of curvature O, as shown in Fig. 10.8.

Let $X'Y'$ be another filament at a distance x from the neutral surface, then

$$XY = R\theta$$

and $$X'Y' = (R + x)\,\theta$$

$\therefore$ Increase in length of the filament

$$= X'Y' - XY$$

$$= (R + x)\,\theta - R\theta = x\theta$$

$$\therefore \text{ Strain } = \frac{\text{change in length}}{\text{original length}} = \frac{x\theta}{R\theta} = \frac{x}{R}$$

Now Young's modulus $Y = \dfrac{\text{Stress}}{\text{Strain}}$

$$\therefore \qquad \text{Stress} = Y \times \text{Strain} = \frac{Yx}{R}$$

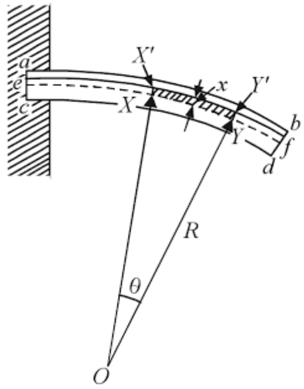

Fig. 10.8

Consider a section $ABCD$ (drawn rectangular for convenience) of the strip at right angles to its length and the plane of bending (Fig. 10.9). Then the forces acting on the strip are perpendicular to this section and the line EF lies on the neutral surface. *The forces producing elongations act in the upper half ABEF and those producing contraction act in the lower half CDEF in opposite directions perpendicular to the section ABCD and hence constitute a couple.*

To find the moment of this couple consider a small area δa lying at a distance x from the neutral axis EF, then

Force on area δa = stress × area $= \dfrac{Yx.\delta a}{R}$

Moment of the force about the axis EF

$$= \frac{Yx\delta a.x}{R} = \frac{Y.x^2\delta a}{R}$$

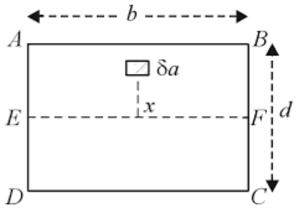

Fig. 10.9

Hence, moment of all the forces acting at various points of the whole face $ABCD$ are

$$= \frac{Y}{R}\,\Sigma x^2\,\delta a$$

To find the value of $\Sigma x^2\delta a$ let us suppose that we can divide the whole area into a number of such parts each of area δa and let the number of such parts be n, then

$$\Sigma x^2\delta a = x_1^2\delta a + x_2^2\delta a + \ldots\ldots.n \text{ times}$$

$$= n\delta a\,\frac{x_1^2 + x_2^2 + \ldots.. + x_n^2}{n} = ak^2$$

where $a = n\delta a$ is the area of the face $ABCD$ and k^2 is the square of radius of gyration k of $ABCD$ about the axis EF.

$$\therefore \qquad \frac{Y}{R}\,\Sigma\, x^2\delta a = \frac{Yak^2}{R} \qquad ...(i)$$

The quantity ak^2 = Moment of inertia of the beam if it has a unit mass per unit area and is called the **geometrical moment of inertia *I***.

Hence, moment of the restoring couple $= \dfrac{Yak^2}{R} = \dfrac{YI}{R}$ ...(*ii*)

In equilibrium

Restoring couple = bending couple (or bending moment).

Hence **Bending moment** *may be defined as the total moment of all the couples arising in a bent beam and trying to resist its deformation caused by an external couple.*

Flexural rigidity. In equation (*ii*) the quantity $YI = Yak^2$ *is called flexural rigidity.*

$$\text{Hence, bending moment} = \frac{YI}{R} = \frac{\text{Flexural rigidity}}{\text{Radius of curvature of neutral surface}}$$

(*i*) If the **cross-section of the beam is rectangular,** then

$$a = b \times d$$

where b is the breadth of the face $ABCD$ and d the thickness. The moment of inertia of the rectangle $ABCD$ about the axis EF parallel to the side AB

$$Mk^2 = M\,\frac{d^2}{12} \text{ or } k^2 = \frac{d^2}{12}$$

$\therefore$ Geometrical moment of Inertia $I = ak^2 = bd\,\dfrac{d^2}{12} = \dfrac{bd^3}{12}$

Substituting the value of I in (*ii*), we have

The moment of the restoring couple $= \dfrac{Y}{R}\,\dfrac{bd^3}{12} = \dfrac{Ybd^3}{12R}$

and in equilibrium this is equal to the **bending couple** (**or bending moment**).

(*ii*) If the **cross-section is circular** and has a radius r, then

$$a = \pi r^2$$

and moment of inertia $= \dfrac{Mr^2}{4}$ or $k^2 = r^2/4$

$\therefore$ Geometrical moment of inertia $I = ak^2 = \pi r^2\,\dfrac{r^2}{4} = \dfrac{\pi r^4}{4}$

Substituting this value of I in (*ii*), we have

The moment of the restoring couple $= \dfrac{Y}{R}\cdot\dfrac{\pi r^4}{4} = \dfrac{\pi Y r^4}{4R}$

and in equilibrium this is equal to the **bending couple** (or **bending moment**).

10.16 CANTILEVER

It is a beam fixed horizontally at one end and loaded at the other.

Depression at loaded end. Suppose, EF represents the *neutral axis* of the cantilever of length l fixed at the end E and loaded with a load W at F as shown in Fig. 10.10. Let the end F be depressed to the position F' under the action of the load.

Consider a section of the beam as at P at a distance x from the fixed end E, then the moment of the couple due to the load W

$$= W \times PF' = W(l - x)$$

As the point F' is very close to F, PF' is almost perpendicular to FW.

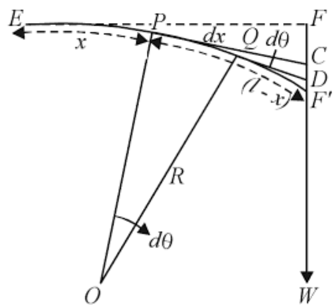

Fig. 10.10

Since, the beam is in equilibrium the bending couple is equal to the restoring couple $\frac{YI}{R}$ where R is the radius of curvature of the neutral axis at P, Y the Young's modulus and I the geometrical moment of inertia. Hence,

$$W(l - x) = \frac{YI}{R}$$

or $$\frac{1}{R} = \frac{W(l - x)}{YI}$$

As, we move towards the fixed end E the moment of the load increases. Hence the radius of curvature will be different at different points and decreases as we move towards E. Consider another point Q lying very close to P at distance dx. The radius of curvature at Q is practically the same as dx is very small.

$\therefore$ $$PQ = R\,.\,d\theta = dx$$

or $$d\theta = \frac{dx}{R} = \frac{W(l - x)\,dx}{YI}$$

Draw tangents at P and Q meeting the vertical line at C and D respectively. Then the depression of Q below P is evidently

$$CD = dy = (l - x)\,d\theta$$

where $d\theta$ is the angle between the two tangents.

$\therefore$ $$dy = \frac{(l - x)\,W\,.\,(l - x)\,dx}{YI}$$

$$= \frac{W}{YI}(l - x)^2\,dx$$

$\therefore$ Total depression $$FF' = y = \int_0^l \frac{W}{YI}(l - x)^2\,dx$$

$$= \frac{W}{YI}\int_0^l (l^2 + x^2 - 2lx)\,dx$$

$$= \frac{W}{YI}\left[l^2x + \frac{x^3}{3} - \frac{2lx^2}{2}\right]_0^l$$

$$= \frac{W}{YI}\left[l^3 + \frac{l^3}{3} - l^3\right]$$

$$= \frac{Wl^3}{3YI} \qquad \ldots (17)$$

Special Cases:

1. **Circular cross-section.** If r is the radius of circular cross-section, then

$$I = \frac{\pi r^4}{4}$$

$$\therefore \qquad y = \frac{4Wl^3}{3Y\,\pi r^4} \qquad \text{... (18)}$$

2. Rectangular Cross-section. If b is the breadth and d the depth of a rectangular rod, then

$$I = ak^2 = b.d \times \frac{d^2}{12} = \frac{bd^3}{12}$$

$$\therefore \qquad y = \frac{Wl^3}{3Y} \times \frac{12}{bd^3} = \frac{4Wl^3}{Ybd^3} \qquad \text{... (19)}$$

10.17 BAR SUPPORTED AT TWO ENDS AND LOADED IN THE MIDDLE

Suppose we take a rod of a certain material and support it at two knife edges A and B. If the rod is loaded at the centre C with a load W, then the reaction at each knife edge will be $W/2$ in the upward direction.

Since, the middle part of the rod is practically horizontal, it may be considered as equal to two inverted cantilevers fixed at C and being loaded at A and B with a load $W/2$ acting in the upward direction. If l is the length of the beam, then the depression y at C is given by

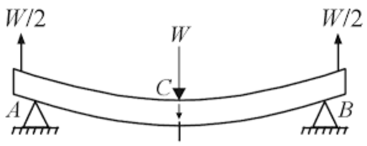

Fig. 10.11

$$y = \frac{W/2\,.\left(\frac{l}{2}\right)^3}{3YI} = \frac{Wl^3}{48YI}$$

(*i*) If the rod is *circular in cross-section* and has a radius r, then geometrical moment of inertia

$$I = ak^2 = \pi r^2 . \frac{r^2}{4} = \frac{\pi r^4}{4}$$

$$\therefore \qquad y = \frac{Wl^3}{48Y} . \frac{4}{\pi r^4} = \frac{Wl^3}{12Y\,\pi r^4}$$

$$\text{or} \qquad Y = \frac{Wl^3}{12\,y\,\pi\,r^4} \qquad \text{... (20)}$$

(*ii*) If the rod is *rectangular* and has a breadth b and depth d, then geometrical moment of inertia

$$I = ak^2$$

$$= b.d\,\frac{d^2}{12} = \frac{bd^3}{12}$$

$$\therefore \qquad y = \frac{Wl^3}{48Y} \times \frac{12}{bd^3} = \frac{Wl^3}{4Ybd^3}$$

$$\text{or} \qquad Y = \frac{Wl^3}{4bd^3 y} \qquad \text{...(21)}$$

Experimental determination. The depression y at the centre of the rod can be found out accurately with the help of a spherometer or a travelling microscope. By loading the rod at the centre with a load increasing in equal steps and then decreasing the load in the same equal steps, the mean depression y for a certain load can be found out. The breadth b and the depth d are also measured and thus, Young's modulus of elasticity is calculated from the relation given above.

10.18 I-SECTION GIRDERS

A steel girder undergoes bending under the action of a load. As such the filaments of the girder above the neutral surface are compressed and those below the neutral surface are extended. This compression or extension of a filament is proportional to its distance from the neutral surface and, therefore, the compression or extension increases as the distance of the filament from the neutral surface increases. Its value is zero at the neutral surface and maximum at the lower or upper faces. Therefore, outer layers undergo much greater strain than inner layers. Hence, to make the outer layers stronger than inner layers the girders are manufactured in I-shape. This considerably saves the material and reduces weight and cost without in any way sacrificing the strength of the girder.

SOLVED EXAMPLES

Example 10.1 A cube of aluminium of side 10 cm is subjected to a shearing force of 100 N. The top surface of the cube is displaced by 0.01 cm with respect to the bottom. Calculate the shearing stress, shearing strain and modulus of rigidity.

Solution. Each side of aluminium cube $L = 10$ cm $= 0.1$ m

Area of face $a = 0.1 \times 0.1 = 0.01$ m^2

Tangential force $F = 100$ N

Shearing stress $\quad T = \dfrac{F}{a} = \dfrac{100}{0.01} = 10^4 \text{ Nm}^{-2}$

Displacement $\quad l = 0.01$ cm $= 0.0001$ m

Thickness $\quad L = 10$ cm $= 0.1$ m

Shearing strain $\quad \theta = \dfrac{l}{L} = \dfrac{0.0001}{0.1} = 10^{-3}$

Modulus of rigidity $\quad \eta = \dfrac{T}{\theta} = \dfrac{10^4}{10^{-3}} = \mathbf{10^7\ Nm^{-2}}$

Example 10.2 What force is required to stretch a steel wire $\frac{1}{2}$ sq.cm. in cross-section to double its length? $Y = 2 \times 10^{11}$ Nm^{-2}.

Solution. Given $Y = 2 \times 10^{11}$ Nm^{-2}

Area of cross-section $a = \dfrac{1}{2}$ sq. cm $= 0.5 \times 10^{-4}$ m^2

When the length is doubled; increase in length l = original length L

Now Young's modulus $Y = \dfrac{FL}{al} \qquad \therefore\ F = \dfrac{Yal}{L} = Ya$

or Force required, $F = 2 \times 10^{11} \times 0.5 \times 10^{-4} = \mathbf{10^7}$ **Newton**

Example 10.3 A steel wire 1.5 mm in diameter is just stretched between two fixed points at a temperature 40°C. Determine the tension in the wire if the temperature falls to 30°C. Given that for steel $\alpha = 0.000012$/°C and Y for steel $= 20 \times 10^{10}$ Nm^{-2}.

Solution. Here $Y = 20 \times 10^{10}$ Nm^{-2} $\qquad \alpha = 0.00012$/°C

If L is the length of the wire at 40°C, then decrease in length at 30°C $= L \times \alpha \times (40 - 30)$

$$= L \times 0.000012 \times 10 = 0.00012\ L$$

$\therefore$ Longitudinal strain $= \dfrac{0.00012\,L}{L} = 0.00012$

Now, stress = $Y \times$ strain = $20 \times 10^{10} \times 0.00012 = 2.4 \times 10^7$ Nm^{-2}

Area of cross-section $a = \pi r^2 = \pi \times \left(\dfrac{1.5 \times 10^{-3}}{2}\right)^2 = 1.766 \times 10^{-6}\ \text{m}^2$

$$\text{Stress} = \frac{F}{a}$$

$\therefore$ Tension in the wire $F = a \times$ stress = $1.766 \times 10^{-6} \times 2.4 \times 10^7$ = **42.4 Newton**

Example 10.4 Calculate the length of the wire that will break under its own weight when suspended vertically. Given breaking stress = 9.8×10^8 Nm^{-2}, density of wire = 10^4 kg m^{-3} and $g = 9.8$ ms^{-2}.

Solution. When the wire hangs vertically its weight mg acts as a longitudinal force. Let L be the maximum length of the wire that can hang without breaking and a its area of cross-section, then

$$\begin{aligned}\text{Weight of wire } mg &= L \times a \times \rho \times g \text{ N}\\ &= L \times a \times 10^4 \times g \text{ N}\end{aligned}$$

Breaking load = Breaking stress $\times\, a = 9.8 \times 10^8 \times a$ N

$\therefore \qquad L \times a \times g \times 10^4 = 9.8 \times 10^8 \times a$

or $$L = \frac{9.8 \times 10^8}{9.8 \times 10^4} = 10^4\ \text{m} = \mathbf{10\ km}$$

The wire of length 10 km *will break* under its own weight if the applied stress is greater than the breaking stress. It will not break if the applied stress $\leq$ the breaking stress.

Example 10.5 A steel wire of length 2.00 m and cross-section 1×10^{-6} m^2 is held between rigid supports with a tension of 200 N. If the middle of the wire is pulled 5 mm sideways, calculate change in tension. Also calculate change in tension if temperature changes by 5°C. For steel $Y = 2.2 \times 10^{11}$ Nm^{-2} and $\alpha = 8 \times 10^{-6}$ deg^{-1}.

Solution. (*i*) Initial length of wire $L = 2$ m

Final length of wire when it is pulled sideways at the centre by 5 mm = 5×10^{-3} m

$$L' = 2[1^2 + (5 \times 10^{-3})^2]^{1/2} = 2\,[1 + 25 \times 10^{-6}]^{1/2}$$

$$= 2\left[1 + \frac{1}{2} \times 25 \times 10^{-6}\right]$$ by applying Binomial theorem and neglecting higher powers of 25×10^{-6} as it is a very small quantity.

$\therefore L' = 2 + 25 \times 10^{-6}$ m

Increase in length $l = L' - L = 25 \times 10^{-6}$ m

Area of cross-section of the wire $a = 1 \times 10^{-6}$ m^2

Let F be the additional tension in the wire due to increase in length, then

Young's modulus $$Y = \frac{\text{stress}}{\text{strain}} = \frac{F/a}{l/L} = \frac{FL}{al}$$

or $$F = \frac{Yal}{L} = \frac{2.2 \times 10^{11} \times 10^{-6} \times 25 \times 10^{-6}}{2} = \mathbf{2.75\ N}$$

(*ii*) When the temperature changes by $\theta = 5$°C, the change in length

$$l = L \times \alpha \times \theta = 2 \times 8 \times 10^{-6} \times 5 = 8 \times 10^{-5}\ \text{m}$$

$\therefore$ Change in tension $F = \dfrac{Yal}{L} = \dfrac{2.2 \times 10^{11} \times 1 \times 10^{-6} \times 8 \times 10^{-5}}{2}$ = **8.8 N**

Example 10.6 A wire 0.5 m long and 1 sq. mm in cross-section has Young's modulus 1.24 × 10^{11} N-m^{-2}. How much work is done in stretching it through 1 mm ? (*K.U.* 2002)

Solution. a = 1 sq. mm = 10^{-6} m^2, l = 1 mm = 10^{-3} m, L = 0.5 m

$Y = 1.24 \times 10^{11}$ N-m^2

Work done $W = \dfrac{1}{2}\dfrac{Yal}{L}l = \dfrac{1.24 \times 10^{11} \times 10^{-6} \times 10^{-3} \times 10^{-3}}{2 \times 0.5}$ = **0.124 J**

Example 10.7 A steel wire of length 2.0 m is stretched through 2.0 mm. The cross sectional area of wire is 40 mm^2. Calculate the elastic potential energy stored in the wire in the stretched condition. Young's modulus of steel = 2.0 × 10^{11} N/m^2.

Solution. L = 2.0 m, l = 2.0 mm = 2 × 10^{-3} m, a = 40 mm^2 = 40 × 10^{-6} m^2

$Y = 2 \times 10^{11}$ N/m^2

Elastic potential energy stored = Work done in stretching the wire

$$= \frac{1}{2} Y \frac{al^2}{L} = \frac{1}{2} \times \frac{2 \times 10^{11} \times 40 \times 10^{-6} \times 2 \times 10^{-3} \times 2 \times 10^{-3}}{2} = \mathbf{8\ J}$$

Example 10.8 If $\eta = 8 \times 10^{11}$ N/m^2 and $Y = 20 \times 10^{11}$ N/m^2 for iron, calculate Poisson's ratio.

Solution. From the relation $Y = 2\eta\,(1 + \sigma)$, we have,

$$\sigma = \frac{Y}{2\eta} - 1 = \frac{20 \times 10^{11}}{2 \times 8 \times 10^{11}} - 1 = \mathbf{0.25.}$$

Example 10.9 Calculate the Poisson's ratio for silver. Given Young's modulus for silver is 7.25 × 10^{10} N/m^2 and bulk modulus is 11 × 10^{10} N/m^2. (*Meerut U.*, 2001)

Solution. $Y = 7.25 \times 10^{10}$ N/m^2 $\quad K = 11 \times 10^{10}$ N/m^2

Now, $\quad Y = 3K\,(1 - 2\sigma) \qquad \therefore\ 1 - 2\sigma = \dfrac{Y}{3K}$ or $2\sigma = 1 - \dfrac{Y}{3K}$

$$\therefore\ \sigma = \frac{1}{2}\left[1 - \frac{Y}{3K}\right] = \frac{1}{2}\left[1 - \frac{7.25 \times 10^{10}}{3 \times 11 \times 10^{10}}\right] = \frac{1}{2}(1 - 0.22) = \mathbf{0.39}$$

Example 10.10 The volume of a solid does not vary with pressure. Find Poisson's ratio for the solid. (*Meerut U.*, 2003)

Solution. Young's modulus Y, bulk modulus K, and Poisson's ratio σ are connected by the relation

$$Y = 3K\,(1 - 2\sigma) \text{ or } 1 - 2\sigma = \frac{Y}{3K}$$

Now, bulk modulus $K = \dfrac{\text{Applied pressure}}{\text{Volumetric strain}} = \dfrac{P}{v/V}$

As the volume of the solid does not vary with pressure, the volumetric strain is zero or $v/V = 0$.

Therefore, $K = \infty$ (infinity) and $\dfrac{Y}{3K} = 0.$

Hence, $1 - 2\sigma = 0$ or $2\sigma = 1 \quad \therefore \sigma = \dfrac{1}{2}$ = **0.5**

Example 10.11 What couple must be applied to a wire one metre long, 1 mm in diameter in order to twist one end of it, through 90°, the other end remaining fixed. Rigidity of material of the wire is 2.8×10^{10} N-m^{-2}.

Solution. Here $l = 1$ m; radius $a = 0.5$ mm $= 0.5 \times 10^{-3}$ m

$\eta = 2.8 \times 10^{10}$ N-m^{-2}, $\theta = 90° = \pi/2$ radian

Couple per unit angular twist $c = \dfrac{\eta\pi a^4}{2l}$

$\therefore$ Couple for angular twist $\theta = c\theta = \eta\dfrac{\pi a^4}{2l} \times \dfrac{\pi}{2} = \dfrac{\eta\pi^2 a^4}{4l}$

$$= \frac{2.8 \times 10^{10} \times \pi^2 \times (0.5 \times 10^{-3})^4}{4 \times 1}$$

$$= \mathbf{43.19 \times 10^{-4}\ N\text{-}m}$$

Example 10.12 A circular bar one metre long and 8 mm diameter is rigidly clamped at one end in a vertical position. A couple of magnitude 2.5 Nm is applied at the other end. As a result a mirror fixed at this end deflects a spot of light by 0.15 m on the scale one metre away. Calculate the modulus of rigidity of the bar.

Solution. Here $l = 1$m, radius $a = 4$ mm $= 4 \times 10^{-3}$ m

Couple $c = 2.5$ N-m, $\eta = ?$

The spot of light is deflected through 0.15 m on a scale one metre away, the angle 2θ through which the reflected ray turns is given by

$$\tan 2\theta = 0.15 \text{ or } 2\theta = 8° 32'$$

$\therefore\ \theta = 4°16' = 0.0745$ rad

Twisting couple $c = \dfrac{\pi\eta a^4}{2l}.\theta$

or $$\eta = \frac{2cl}{\pi a^4\theta} = \frac{2 \times 2.5 \times 1}{\pi\,(4 \times 10^{-3})^4 \times 0.0745}$$

$$= \mathbf{8.344 \times 10^{10}\ Nm^{-2}}$$

Example 10.13 A cylindrical bar of length 1m and diameter 8 mm is fixed at one end and the other end is twisted through an angle of 5° by the application of a couple of 2.5 Nm. Calculate the modulus of rigidity of the material of the bar. (*Kerala U.* 2001)

Solution. $5° = \dfrac{5\pi}{180} = 0.0873$ rad. $\eta = \dfrac{2cl}{\pi a^4\theta} = \dfrac{2 \times 2.5 \times 1}{\pi\,(4 \times 10^{-3})^4 \times 0.0873}$

$$= \mathbf{7.12 \times 10^{10}\ Nm^{-2}}$$

Example 10.14 A power of 6 kilowatts is transmitted by a shaft of length 4 metres and radius 2.5 cm at a speed of 200 revolutions per minute. If the modulus of rigidity of the material is 9×10^{10} Nm^{-2}, calculate the relative twist between the ends of the shaft.

Solution. Here $l = 4$ m; $a = 2.5$ cm $= 2.5 \times 10^{-2}$ m

$\eta = 9 \times 10^{10}$ N–m^{-2}; Power = 6 K.watt = 6000 Js^{-1}

Time of one revolution $= \dfrac{60}{200} = \dfrac{3}{10}$ sec

Work done/rev $W = \dfrac{6000 \times 3}{10} = 1800$ J

If θ is the relative shift between the ends of the shaft and c the couple per unit angular twist, then

$$\frac{1}{2} c\theta^2 = W$$

or $$\frac{1}{2} \frac{\pi\eta a^4}{2l} \theta^2 = 1800$$

$\therefore$ $$\theta = \sqrt{\frac{1800 \times 4 \times 4}{\pi \times 9 \times 10^{10} \times (2.5 \times 10^{-2})^4}} = \mathbf{0.51\ radian}$$

Example 10.15 One end of a steel wire of length 0.2 m and radius 2×10^{-3} m is fixed. If the work done in twisting the free end of the wire is 3.85×10^{-2} J, calculate the angle through which the wire is twisted. Given rigidity modulus of steel = 8.075×10^{11} Nm^{-2}.

(*M.S.U. Tirunaveli* 2007; *Bang. U.*, 2000)

Solution. $$c = \frac{\eta\pi r^4}{2l} = \frac{8.075 \times 10^{11} \times 22 \times 2^4 \times 10^{-12}}{2 \times 0.2 \times 7} = 101.5$$

$$\frac{1}{2} c\theta^2 = W \quad \therefore \theta^2 = \frac{2W}{c} = \frac{2 \times 3.85 \times 10^{-2}}{101.5} = 0.07586 \times 10^{-2}$$

or $$\theta = 0.02754 \text{ rad} = \mathbf{1.58°}$$

Example 10.16 A sphere of mass 0.8 kg and radius 3 cm is suspended by a wire 1 m long of radius 0.5 m. If the time for one torsional vibration is 1.23 sec, determine the modulus of rigidity of the wire.

Solution. Here $l = 1$ m, $M = 0.8$ kg, $R = 3$ cm $= 3 \times 10^{-2}$ m

$a = 0.5$ mm $= 0.5 \times 10^{-3}$ m, $t = 1.23$ sec

M.I. of sphere $I = \dfrac{2}{5} MR^2 = \dfrac{2}{5} \times 0.8 \times (3 \times 10^{-2})^2$

$= 2.88 \times 10^{-4}$ kgm^2

Now $$t = 2\pi\sqrt{\frac{I}{c}} = 2\pi\sqrt{\frac{2lI}{\eta\pi a^4}}$$

$\therefore$ $$\eta = \frac{8\pi Il}{t^2 a^4} = \frac{8 \times \pi \times 2.88 \times 10^{-4} \times 1}{(1.23)^2 \times (0.5 \times 10^{-3})^4}$$

$$= \mathbf{7.654 \times 10^{10}\ N\text{-}m^{-2}}$$

Example 10.17 A cylindrical metal bar of length 0.24 m and diameter 4 cm is suspended by a wire 0.5 m long such that the axis of the bar is horizontal. The arrangement makes 100 torsional vibrations in 235.9 sec. Determine the co-efficient of rigidity of the material of wire. Given density of material of bar = 9×10^3 kgm^{-3} and radius of wire = 0.1 cm.

Solution. For cylindrical bar $L = 0.24$ m, $R = 2$ cm $= 2 \times 10^{-2}$ m

Density $\rho = 9 \times 10^3$ kgm^{-3}

Volume $V = \pi R^2 L = \pi \times 4 \times 10^{-4} \times 0.24 = 3.016 \times 10^{-4}$ m^3

$\therefore$ Mass $M = 3.016 \times 10^{-4} \times 9 \times 10^3 = 2.715$ kg.

M.I. of cylindrical bar about an axis passing through its centre and perpendicular to the length

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$$I = M\left[\frac{R^2}{4}+\frac{l^2}{12}\right] = 2.715\left[\frac{4\times10^{-4}}{4}+\frac{.24\times.24}{12}\right] = 133.04\times10^{-4}\ \text{kgm}^2$$

Time period $T = \dfrac{235.9}{100} = 2.359$ sec

Now $\quad T = 2\pi\sqrt{\dfrac{I}{c}}$ or $c = \dfrac{4\pi^2 I}{T^2}$ and $c = \dfrac{\pi\eta a^4}{2l}$

$$\therefore \quad \frac{\pi\eta a^4}{2l} = \frac{4\pi^2 I}{T^2}$$

or $$\eta = \frac{8\pi Il}{T^2a^4} = \frac{8\pi\times133.04\times10^{-4}\times0.5}{(2.359)^2\ (1\times10^{-3})^4} = \mathbf{3\times10^{10}\ Nm^{-2}}$$

Example 10.18 A rod of rectangular cross-section having breadth and thickness each 0.5 cm is bent in the form of an arc of radius of curvature 1000 cm. If Young's modulus of the material of the rod is 10^{12} dynes/cm^2 calculate (*i*) stress, strain at the curved sufrace and (*ii*) bending moment.

Solution. (*i*) When a bar is bent, the tensile strain on any filament at a distance x from the neutral surface is given by

$$\text{strain} = \frac{\text{change in length}}{\text{original length}} = \frac{x}{R}$$

where R is the radius of curvature of the neutral surface.

The neutral surface lies along the middle of the bar.

Thickness of the bar = 0.5 cm = 0.005 m

$\therefore$ For the convex surface of the bent bar $x = \dfrac{0.005}{2} = 0.0025$ m

Radius of curvature of the curved surface R = 1000 cm = 10 m

Hence tensile strain $= \dfrac{x}{R} = \dfrac{0.0025}{10} = 0.00025 = 25\times10^{-5}$

Young's modulus $Y = 10^{12}$ dynes/cm^2 = 10^{11} Nm^{-2}

$\therefore$ Tensile stress = Y × tensile strain = $10^{11}\times25\times10^{-5}$ = $\mathbf{25\times10^6\ Nm^{-2}}$

(*ii*) Geometrical moment of inertia of the rectangular bar

$$I = \frac{bd^3}{12}$$

Now $b = d = 0.005$ m

$$I = \frac{(0.005)^4}{12} = \frac{625\times10^{-12}}{12} = 52\times10^{-12}$$

$$\therefore \text{ Bending moment } = \frac{YI}{R} = \frac{10^{11}\times52\times10^{-12}}{10} = \mathbf{52\times10^{-2}\ Nm}$$

Example 10.19. A steel rod of length 50 cm, width 2 cm and thickness 1 cm is bent into the form of an arc of radius of curvature 2.0 m. Calculate the bending moment. Young's modulus of the material of the rod = 2×10^{11} N/m^2. (*Kerala U.*, 2001)

Solution. b = 2cm = 2×10^{-2} m; d = 1 cm = 10^{-2} m

$$\therefore \qquad I = \frac{bd^3}{12} = \frac{2 \times 10^{-2} \times 10^{-6}}{12} = \frac{1}{6} \times 10^{-8}$$

$$\therefore \text{ Bending moment } = \frac{YI}{R} = \frac{2 \times 10^{11}}{2} \times \frac{1}{6} \times 10^{-8} = \frac{10^3}{6} = \mathbf{166.67\ Nm}$$

Example 10.20 A brass bar 1 cm square in cross-section is supported on two knife edges one metre apart. A load of 1 kg at the centre of the bar depresses that point 2.51 mm. What is Young's modulus of brass? (*Purvanchal U.* 2005)

Solution. As the bar is 1 cm square

$\therefore \qquad b = d = 1 \text{ cm} = 10^{-2}$ m

Distance between knife edges $l = 1$ m

Depression $y = 2.51$ mm $= 2.51 \times 10^{-3}$ m

Load $W = 1$ kg wt. $= 9.8$ N

Now,

$$Y = \frac{Wl^3}{4bd^3 y}$$

$$= \frac{9.8 \times 1}{4 \times 10^{-2} \times 10^{-6} \times 2.51 \times 10^{-3}}$$

$$= \mathbf{9.761 \times 10^{10}\ N\text{-}m^{-2}}$$

Example 10.21 In an experiment the diameter of the rod was 1.26 cm and distance between the knife edges 0.7 m. On putting a load of 0.9 kg at the mid-point, the depression was 0.025 cm. Find Young's modulus of elasticity of the material of rod.

Solution. Here

$$l = 0.7 \text{ m},$$
$$W = 0.9 \text{ kg-wt} = 0.9 \times 9.8 \text{ N}$$
$$r = 0.63 \text{ cm} = 0.63 \times 10^{-2} \text{ m}$$
$$y = 0.025 \text{ cm} = 2.5 \times 10^{-4} \text{ m}$$

Now,

$$Y = \frac{Wl^3}{12\pi r^4 y}$$

$$= \frac{0.9 \times 9.8 \times 0.7 \times 0.7 \times 0.7}{12\pi \times (0.63 \times 10^{-2})^4 \times 2.5 \times 10^{-4}}$$

$$= \mathbf{20.374 \times 10^{10}\ Nm^2}$$

Example 10.22 Compare the loads required to produce equal depressions for two beams made of the same material and having the same length and weight with only difference that one has circular cross-section while the cross-section of the other is square.

Solution. If l is the length of each bar, ρ its density, r the radius of circular bar and a each side of the face of square bar, then

Mass of square bar $= a^2 l\rho$

Mass of circular bar $= \pi r^2 l\rho$

As, the mass of the bars are equal

$$\therefore \qquad \pi r^2 l\rho = a^2 l\rho$$

$$\text{or} \qquad \pi r^2 = a^2 \qquad \ldots(i)$$

If I_1 is the geometrical *M.I.* of the square bar and I_2 that of the circular bar, then

Depression for square bar $y = \frac{W_1}{YI_1} \cdot \frac{l^3}{3}$ ...(*ii*)

and Depression for circular bar $y = \frac{W_2}{YI_2} \cdot \frac{l^3}{3}$ ...(*iii*)

From (*ii*) and (*iii*), we get,

$$\frac{W_1}{I_1} = \frac{W_2}{I_2}$$

or
$$\frac{W_1}{W_2} = \frac{I_1}{I_2}$$

Now
$$I_1 = \frac{a^4}{12} \text{ and}$$

$$I_2 = \frac{\pi r^4}{4}$$

$\therefore$
$$\frac{W_1}{W_2} = \frac{a^4}{12} \times \frac{4}{\pi r^4}$$

$$= \frac{a^4}{3\pi r^4}$$

$$= \frac{\pi^2 r^4}{3\pi r^4} = \frac{\pi}{3} = \mathbf{1.05}$$

Example 10.23 A square bar of length 1m and cross-section 1cm² is clamped horizontally at one end and a weight of 1 kgm is applied at the other end. Neglecting weight of the bar calculate the depression of the loaded end. Given $Y = 9.78 \times 10^{10}$ N/m² and $g = 9.78$ m/sec².

(*Nag. U.* 2001; *M.D.U.* 2002)

Solution. Here $l = 1$m. As it is a square bar $d = b = 1$ cm $= 10^{-2}$ m.

$W = 1$ kgm $= 1 \times 9.78 = 9.78$ N; $Y = 9.78 \times 10^{10}$ Nm^{-2}

$\therefore$ Depression of the loaded end $y = \frac{4Wl^3}{Ybd^3}$ for a rectangular bar

$$= \frac{4Wl^3}{Yb^4} \text{ for a square bar}$$

$$= \frac{4 \times 9.78 \times 1}{9.78 \times 10^{10} \times 10^{-8}} = 4 \times 10^{-2} \text{ m} = \mathbf{4\ cm.}$$

EXERCISE CH. 10

LONG QUESTIONS

1. (*a*) What is the difference between angle of twist and angle of shear? Deduce an expression for the couple required to twist a uniform solid cylinder by an angle.

(*D.A.U. Agra* 2008; *Kerala U.* 2001; *Meerut U.* 2005, 2002, 2000; *M.D.U.* 2002; *Nagpur U.* 2005, 2003) (*b*) What is the value of the couple for a hollow cylinder of inner radius r_1 and outer radius r_2?

2. Define the term neutral surface, plane of bending, neutral axis and bending moment. Derive an expression for the couple required to bend a uniform straight metallic strip into an arc of a circle of small curvature. What is meant by flexural rigidity?

(*Kerala U.* 2001; *Luck. U.* 2001; *Nag. U.* 2001; *Gharwal U.* 2000; *Purvanchal U.* 2004; *M.S.U. Tiruneveli* 2007)

3. (*a*) What is a cantilever? Derive an expression for the depression of the loaded end of a cantilever of (*i*) circular cross-section and (*ii*) rectangular cross-section of negligible weight. (*Meerut U.* 2002; *K.U.* 2002, 2001; *M.D.U.* 2003; *Gauhati U.* 2007; *M.S.U. Tiruneveli* 2007)

(*b*) Hence derive an expression for the bending of a bar supported at the two ends and loaded in the middle. Describe an experiment to determine Y by bending.

(*Luck.U.* 2001; *K.U.* 2000)

4. Show that shear is equivalent to an elongation strain and compression strain at right angles to each other and each is half of shearing angle. (*Cal. U.* 2003) [**Ans.** 5×10^{-4} J]

5. Prove that a shearing stress is equivalent to a linear tensile stress and an equal compression stress mutually at right angles. (*Agra U.* 2005)

6. Show that the couple required per unit angular twist in the case of cylindrical wire

$$C = \frac{\pi \eta r^4}{2l}$$ (*Purvanchal U.* 2005, 2004; *M.S.U. Tiruneveli*, 2007)

7. Show that the bending moment for a thin uniform bar of rectangular cross-section is

$$\frac{\gamma b d^3}{12R}$$

8. Calculate the strain in a rod of isotropic material stretched in such a way that all lateral strains are prevented.

[**Hint:** $\beta = \frac{\alpha}{2}$ and $\sigma = 0.5$. It is an ideal case]

9. Define elastic constants. Establish relation between them. What are torsional oscillations? Derive an expression for the twisting couple per unit angular twist for a hollow cylinder.

10. Define a cantilever. Obtain an expression for the depression produced at its free end when the weight of the beam is negligible. (*M.S.U. Tiruneveli*, 2007)

SHORT QUESTIONS

1. Explain the terms stress and strain. Define Young's modulus Y, the bulk modulus K, rigidity modulus η and poisson's ratio σ. Write dimensions of σ.

(*M.D.U.* 2003; *Meerut U.* 2003; *Nagpur U.* 2001; *K.U.* 2000; *M.S.U. Tirumeveli*, 2007; *Agra U.* 2006; *Madurai U.* 2003)

2. Prove that glass is more elastic than rubber.

3. Explain the terms:

(*i*) Elastic limit,

(*ii*) Yield point,

(*iii*) Elastic fatigue.

4. Estimate the work done in stretching a wire. (*Kerala U.* 2001)

5. If Y, K and σ represent Young's modulus, Bulk modulus and Poisson's ratio respectively, then prove that $K = \dfrac{Y}{3(1-2\sigma)}$. (*Purvanchal U.*, 2006, 2004; *Agra U.*, 2005, 2006; *M.D.U.* 2001; *M.S.U. Tiruneveli*, 2007)

6. Show that for a homogeneous isotropic medium $Y = 2\eta\ (1 + \sigma)$ where letters have their usual meaning. (*Meerut U.* 2003; *Cal. U.* 2003; *K.U.* 2002; *M.D.U.* 2003)

7. If Y, K, η and σ represent the Young's modulus, bulk modulus, coefficient of rigidity and Poisson's ratio, then derive various relations connecting each other and prove that σ is less than 0.5 and cannot be less than – 1. What are the practical limits for Poisson's ratio (σ) ? (*Madurai U.* 2003; *Kerala U.* 2001; *Nagpur U.* 2001; *Indore U.* 2001; *M.D.U.* 2001; *K.U.* 2001, 2000; *Bang. U.* 2000; *Guwahati U.* 2000; *Gharwal U.* 2000)

8. Explain why a hollow cylinder is stronger than a solid cylinder of the same length, mass and material. (*Gharwal U.* 2001, 2000; *K.U.* 2001, *Bang. U.* 2000)

9. Deduce an expression to calculate the work done in twisting a cylindrical wire by an angle θ. Prove that a hollow cylinder is stronger than a solid cylinder of same mass and material. (*M.S.U. Tiruneveli*, 2007; *D.A.U. Agra U.* 2007, 2004)

10. In case of bending of a rod, Young's modulus only comes into play and not modulus of rigidity even though there is a change in shape. Explain.

11. Explain the term 'Neutral surface' and 'Bending moment' of a beam. (*Nagpur U.* 2005)

12. Prove that

$$\frac{3}{Y} = \frac{1}{3K} + \frac{1}{\eta}$$

(*Purvanchal U.* 2006; *Agra*, 2006)

13. Show that

$$\sigma = \left(\frac{Y}{2\eta}\right) - 1$$

(*Purvanchal U.* 2006; *Agra* 2006)

14. Show that the work done per unit volume in straining a body is equal to $\frac{1}{2}$(stress × strain).

15. Obtain a relation between elastic coefficients Y, η and σ. (*Agra U.* 2007)

16. Two cylinders have the same length and mass and are made of same material; one is solid while the other which is hollow has an external radius twice the internal radius. Compare the torsional rigidities of the two cylinders. Show that second cylinder is stronger.

17. Explain why steel girders have I section?

(*Madurai U.* 2003, *M.D.U.* 2002, *Bang. U.* 2000)

18. Show that the potential energy per unit volume of a strained wire is $\frac{1}{2}$ × stress × strain.

19. If Y, η and K represent Youngs modulus, coefficient of rigidity and bulk modulus respectively, then prove that

$$\frac{9}{Y} = \frac{3}{\eta} + \frac{1}{K}$$

(*Purvanchal U.* 2007; *Agra*, 2006)

20. A circular hollow rod and a circular solid rod of same length and same mass are to be twisted by same amount. Which one will be more difficult? (*Purvanchal U.* 2004)

[**Ans.** Hollow cylinder]

21. Two spheres, one hollow and one solid, are of same mass, same external radius and same external appearance. Which one will roll down faster on an inclined plane?

(*Purvanchal U.* 2004) [**Ans.** Solid sphere rolls down faster than hollow sphere.]

22. Show that shearing strain θ is equivalent to the extensional strain or compressional strain and it is equal to the half of the shearing strain (θ). (*Purvanchal U.* 2005)

24. Find the Poisson's ratio of a material whose volume is not changeable at any pressure. (*Purvanchal U.* 2005)

25. Show that the torsional rigidity is greater for a hollow cylinder than for a solid one of the same material, mass, length and cross-section area. [**Hint:** Since M.E. greater] (*D.A.U.*, 2008; *Agra*, 2006)

26. Obtain in terms of bending moment M, an expression for the longitudinal stress in a beam at *a* distance z from the neutral axis. (*Agra U.* 2006)

27. Obtain an expression for the moment of inertia of a diatomic molecule. (*Agra U.* 2006)

28. What are theorical limits of Poisson's ratio for an isotropic homogeneous body? (*Agra U.* 2005)

29. Explain bending moment. Derive the expression for bending moment. (*M.S.U. Tiruneveli*, 2007)

30. State and prove the relation between Y, η and σ for a homogeneous, isotropic body where the symbols have their usual meaning. (*Nagpur Uni.* 2009)

31. Define neutral axis of a beam. (*Nagpur Uni.* 2009)

NUMERICAL QUESTIONS

1. Calculate the value of Young's modulus given $\eta = 2 \times 10^{10}$ N/m^2 and $\sigma = 0.25$. (*Nagpur U.*, 2001) [**Ans.** 5×10^{10} Nm^{-2}]

2. Calculate Poisson's ratio σ for Brass.
Given: $Y = 10 \times 10^{10}$ N/m^2 and $K = 10 \times 10^{10}$ N/m^2
[**Hint:** $Y = 3K(1 - 2\sigma)$] (*Nagpur U.* 2005) [**Ans.** 0.33]

3. The end of a rectangular cantilever depresses 10 mm under a certain load. Calculate the depression under the same load, for another cantilever of same material two times in length, two times in width and three times in thickness. (*Nagpur U.* 2005) [**Ans.** 1.48 mm]

4. A solid cylinder of radius 5 cm is converted into a hollow cylinder of same mass and length and external radius 7 cm. If the restoring couple per unit radian twist in original cylinder is C, deduce the same for the new hollow cylinder. [**Ans.** $C' = 2.92\ C$]

5. Find the work done in stretching a wire of 1 sq. mm cross-section. Young's modulus 2×10^{11} N/m^2 and 2 m long through 0.1 mm. [**Ans.** 5×10^{-4}J]

6. A gold wire 0.32 mm in diameter elongates by 1 mm, when stretched by a force of 0.33 kg wt. and twists through 1 radian when equal and opposite torques of 145 dyne-cm are applied at its ends. Find the Poisson's ratio for gold. (given $g = 9.8$ ms^{-2}.) [**Ans.** 0.427]

7. A uniform rod of length 1 m is clamped horizontally at one end. A weight of 0.1 kg is attached at the free end. Calculate the depression of the free end of the rod. The diameter of the rod is 0.02 m.
Young's modulus of the material of the rod = 1×10^{10} Nm^{-2}. [**Ans.** 4.1 mm]

8. Poisson's ratio for a material is 0.379 and rigidity is 2.87×10^{-2} Nm^{-2}, find Young's modulus. [**Ans.** 7.196×10^{-2} Nm^{-2}]

9. A cylinder of diameter 4 cm and length 5 cm is suspended horizontally by a steel wire of length 100 cm and radius 0.02 cm. Calculate the time of one vibration. The coefficient of rigidity of steel is 8×10^{11} dynes/cm^2 and the density of lead is 11.4 gms/c.c. [**Ans.** 6.586 seconds]

Hint. $M = \pi R^2 l\rho$ and $I = M\left[\frac{R^2}{4} + \frac{L^2}{12}\right]$

$$t = 2\pi\sqrt{\frac{I}{c}} = 2\pi\sqrt{\frac{I.2l}{\eta\pi r^4}}$$

10. A metallic strip of width 2 cm and thickness 3 mm supported horizontally on knife edges 80 cm apart is loaded with 50 gm at its middle. Find by how much the centre of the metallic strip is depressed. Young's modulus for the material is 2×10^{11} Nm^{-2} .

[**Ans.** 0.581×10^{-2} m]

11. A steel rod of circular cross-section of radius 1 cm is rigidly fixed at one end, the other end which is at a distance of 1 metre from the fixed end, is loaded with 8 kg. Calculate the deflection of the rod. Given Young's modulus of elasticity = 2×10^{11} Nm^{-2}.

(*M.S.U. Tiruneveli*, 2007) [**Ans.** 1.663×10^{-2} m]

12. A steel wire of 1.00 mm radius is bent in the form of a circular arc of radius 50 cm. Calculate (*i*) The bending moment (*ii*) maximum stress. Given $Y = 2 \times 10^{11}$ Nm^{-2}.

(*Agra U.*, 2003) [**Ans.** (*i*) 0.314 Newton metre (*ii*) 4×10^8 Nm^{-2}]

11

FLUID MECHANICS: VISCOSITY

INTRODUCTION

When a fluid (liquid or gas) flows, different layers move with different velocities, giving rise to velocity gradiant. Faster moving layer tries to accelerate the slower moving layer, at the same time the slower moving layer tries to retard the faster moving layer. This result in a frictional force *i.e.,* viscous force between any two adjacent layers in relative motion. This property of the fluid by the virtue of which it opposes the relative motion between the adjacent layers is known as viscosity. Knowledge of viscosity of liquid is useful in many fields, viz. oils are used in machines as lubricant, aeroplane or rocket moving through air or submarines moving inside water. The lubricant should be such that its viscosity almost remains constant even if its temperature changes.

11.1 VISCOSITY

When the motion of a liquid over a horizontal solid surface is slow and steady, its layer in contact with the solid surface is stationary. In other words, its velocity along the sufrace is zero. The velocity of any other layer is proportional to its distance from the stationary layer and is maximum for the topmost layer. If we consider any particular layer of the liquid, we find that the layer *immediately above it is moving faster* than the layer *immediately below it.* Hence the upper layer tends to increase the velocity of the lower layer whereas the lower layer tends to decrease the velocity of the upper layer. The two layers together tend to destroy their relative motion as if there is some backward dragging force acting tangentially on the layers. Consequently, if a relative velocity is to be maintained between the two layers of a liquid, an external force is required to ovecome this backward drag.

This property by virtue of which a liquid opposes relative motion between its different layers is called **viscosity**.

Coefficient of Viscosity. Let a layer RS of a liquid move with velocity v relative to a parallel layer PQ which is at a distance x from it. Let the force on an area A required to produce this motion be F. This force must act on RS in the direction of motion. An equal force will, therefore, act on it in the oppostie direction due to viscosity. This backward dragging force F depends upon the following factors:

(*i*) $F \propto -v$

i.e., F is directly proportional to the relative velocity v and acts in a direction opposite to the direction of motion of the liquid.

(*ii*) $F \propto A$ and

(*iii*) $F \propto \frac{1}{y}$

Combining all these factors, we have

$$F \propto -A\frac{v}{y}$$

or $$F = -\eta A\frac{v}{y}$$

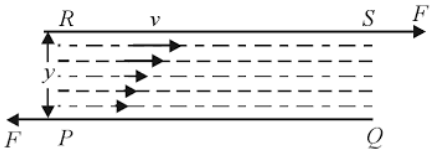

Fig. 11.1

where η is a constant depending upon the nature of the liquid and is called the **coefficient of viscosity.** If the two layers are very close together, then denoting the distance between them by dy and relative velocity by dv, we get

$$F = -\eta A\frac{dv}{dy} \qquad \text{... } (i)$$

$\frac{dv}{dy}$ is called the **velocity gradient** or the rate of change of velocity with distance.

If $$A = 1 \text{ and } \frac{dv}{dy} = 1, \text{ then}$$

$$F = \eta \text{ (numerically).}$$

The coefficient of viscosity is thus defined as the tangential force per unit area required to maintain a unit velocity gradient, i.e., a unit relative velocity between two layers a unit distance apart.

Unit. A fluid has a viscosity of one **poise** if a tangential force of 1 dyne per square cm is required to maintain a relative velocity of 1 cm per second between two layers 1 cm apart. The S.I. unit of viscosity is a **Decapoise.** A fluid has a viscosity of one **decapoise** if, a tangential force of 1 Newton per square metre is required to maintain a relative velocity of 1 metre per second between two layers one metre apart.

Now, $$\eta = \frac{Fy}{Av}$$

$\therefore$ $$\text{Poise} = \frac{\text{dyne} \times \text{cm}}{\text{cm}^2 \times \text{cm per second}}$$

$$\text{Deca poise} = \frac{\text{Newton} \times \text{metre}}{\text{metre}^2 \times \text{metre per second}} = \frac{10^5 \text{ dyne} \times 100 \text{ cm}}{10^4 \text{ cm}^2 \times 100 \text{ cm per second}}$$

$$= \frac{10 \text{ dyne} \times \text{cm}}{\text{cm}^2 \times \text{cm per second}} = 10 \text{ poise}$$

$\therefore$ 1 Deca poise = 10 poise

Dimensions. The dimensions of viscosity are $[M^1L^{-1}T^{-1}]$ as proved below.

$$\text{Viscosity} = \frac{Fy}{Av} = \frac{M^1 L^1 T^{-2} L^1}{L^2 L^1 T^{-1}} = M^1 L^{-1} T^{-1}$$

11.2 STREAMLINE AND TURBULENT FLOW

Consider a liquid flowing in a pipe and let the velocity of flow be v_1 at A, v_2 at B and so on as shown in Fig. 11.2. *If with time the velocity at every point in the liquid remains constant in magnitude as well as direction, then the flow is said to be* **steady**. In other words, in a steady flow each particle follows exactly the same path and has the same velocity as its predecessor. In such a case the liquid is said to have an orderly or a **streamline flow**. *The line along which the liquid moves when the flow is steady is known as* **streamline.** The tangent at any point on this line gives the direction of the velocity of flow at that point. A streamline, therefore, represents a fixed path curved or straight followed by an orderly procession of particles. In tubes of constant area of cross-section all the streamlines are parallel to the axis of the tube. The flow is streamline only as long as the velocity of the liquid does not exceed a particular value called the **critical velocity**. When the velocity is greater than the critical value the flow of the liquid does not remain steady but becomes **turbulent.** When the flow is unsteady or turbulent there are eddies and whirlpools in the motion and the paths as well as the velocities of the particles are continuously changing.

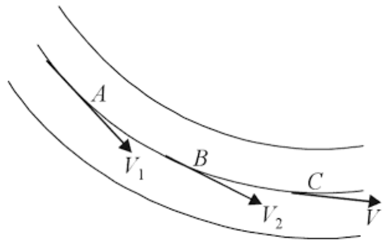

Fig. 11.2

11.3 CRITICAL VELOCITY

To show the existence of critical velocity experimentally, consider a tube AB about 2 centimetres in diameter in which the liquid (water) enters at C and leaves at D. At the end A of the tube AB there is a rubber cork through which passes an inlet tube I, drawn out into a capillary about 10 cm. long and 0.5 mm. in diameter. The tube I is connected to a reservoir containing ink, the flow of ink through it

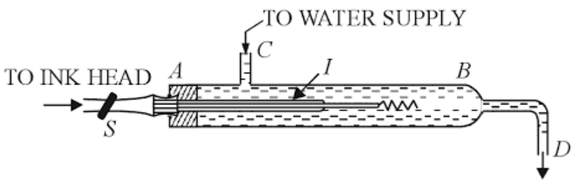

Fig. 11.3

is controlled by a screw type pinch-cock S. At C is connected a constant level tank, the position of which is so adjusted that water flows very slowly through the tube AB. In this position, it is seen that long column of ink escapes from the jet along the central axis of the stream. Under these conditions the liquid is having a streamline motion and all filaments move parallel to the axis of the tube. The head of water is slowly increased so that the velocity of flow also increases. It is seen that beyond a certain velocity the coloured band of ink is broken up by eddies and mixes with water. This state shows the turbulent flow.

Hence the **critical velocity** *of a liquid is that velocity of flow above which the flow ceases to be streamline.*

Critical Velocity of liquid by the method of dimensions:

The expression for critical velocity v_c can easily be deduced by the method of dimensions. The value of v_c is found to depend upon:

(*i*) η the coefficient of viscosity

(*ii*) ρ the density of the liquid, and

(*iii*) r the radius of the tube.

Hence, the $v_c = K\,\eta^{\alpha}\,\rho^b\,r^c$ where K is a constant.

Substituting the dimensions of the various quantities, we have

$$[LT^{-1}] = [ML^{-1}T^{-1}]^a\ [ML^{-3}]^b\ [L]^c$$

$$[LT^{-1}] = [M^{a+b}L^{-a-3b+c}\,T^{-a}]$$

According to the principle of homogeneity of dimensions

$$a + b = 0;\ -a - 3b + c = +1 \text{ and } -a = -1$$

$$\therefore \quad a = 1 \quad b = -1 \quad \text{and} \quad c = -1$$

Hence, $v_c = K\dfrac{\eta}{\rho r}$ or $\dfrac{v_c \rho r}{\eta} = K$

This constant K is known as Reynold's number. It is a pure numerical and is independent of the system of units used. Its value is about 2000.

Significance of Reynold's number. The significance of the equation

$$v_c = K\frac{\eta}{\rho r} \quad \text{... (2)}$$

lies in the fact that a non-dimensional quantity of the form $\dfrac{v_c \rho r}{\eta}$ determines the process that takes place during the motion of a liquid through a tube and when this quantity attains a certain value the flow changes from a steady streamline flow to a turbulent one in which eddies are formed. Thus the critical velocity of a liquid is

(*i*) directly proportional to its viscosity,

(*ii*) inversely proportional to its density and

(*iii*) inversely proportional to the radius of the tube.

It, therefore, follows that narrow tubes, low density and high viscosity help in producing orderly motion and wide tubes, high density and low viscosity tend to produce turbulent motion.

11.4 POISEUILLE'S FORMULA FOR STEADY FLOW OF A LIQUID THROUGH A NARROW TUBE

About 1940 French physicist Poiseuille derived the formula for the volume of liquid flowing per second through a cylindrical tube of circular cross-section. The derivation is based on the following assumptions:

Assumptions. (*i*) The flow is steady and stream line. The stream lines are everywhere parallel to the axis of the tube.

(*ii*) The pressure over any section normal to the axis of the tube is constant, so that there is no radial flow.

(*iii*) The liquid layer in contact with the walls of the tube is at rest and the velocity of any layer of the liquid is only a function of distance of the layer from the axis of the tube.

Poiseuille determined the viscosity of a liquid by measuring the volume of the liquid flowing through a capillary tube.

Theory. Consider a capillary tube of length l and radius a as shown in Fig. 11.4. If the flow of the liquid through it is steady, the velocity of the liquid along the walls is zero and is maximum along the axis of the tube.

Consider a cylindrical layer of the liquid co-axial with the tube of inner radius r and outer radius $r + dr$, shown separately in Fig. 11.5. The velocity of the liquid at a

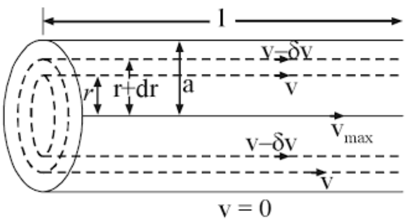

Fig. 11.4

distance r from the axis of the tube is v and at a distance $r + dr$ is $v - dv$, so that $\frac{dv}{dr}$ is the *velocity gradient.*

The surface area of the cylinder = $2\pi rl$.

The liquid on the inner side of this cylindrical layer is moving faster while that on the outer side is moving slower.

$\therefore$ The tangential force exerted by the outer layer on the inner layer *opposite* to the direction of motion is given by

$$F = -\eta\ 2\pi rl\ \frac{dv}{dr}$$

The forward push due to the difference of pressure P on the two sides of the cylinder of radius r

$$= P \times \pi r^2$$

When the motion is steady, there is no acceleration of the liquid and thus, we have

$$-\eta\, 2\pi rl\ \frac{dv}{dt} = P\pi r^2$$

$$dv = \frac{-P}{2\eta l}\, rdr$$

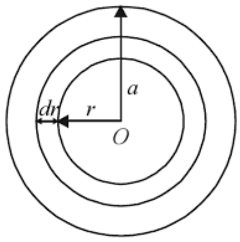

Fig. 11.5

Integrating both sides, we have

$$v = -\frac{P}{2\eta l} \times \frac{r^2}{2} + C$$

where C is the constant of integration.

But as the velocity of the liquid along the sides of the tube is zero, therefore if $r = a$.

$$v = 0$$

Substituting these values, we get

$$0 = -\frac{P}{2\eta l} \times \frac{a^2}{2} + C$$

or
$$C = \frac{P}{2\eta l} \times \frac{a^2}{2}$$

Hence,
$$v = \frac{P}{4\eta l}\,[a^2 - r^2] \qquad ...(3)$$

This gives the velocity of flow at a distance r from the axis of the tube through the cylindrical layer.

The area of cross-section of the cylindrical layer of radius r and thickness $dr = 2\pi rdr$

$\therefore$ Volume of liquid passing per second through this area $dV = v.2\pi rdr$

Hence the volume of the liquid passing through the whole tube per second is given by

$$\int dV = \int_0^a 2\pi vrdr$$

or
$$V = \int_0^a \frac{P}{4\eta l}\,(a^2 - r^2)\ 2\pi rdr$$

$$= \frac{\pi P}{2\eta l}\int_0^a (a^2 r - r^3)\ dr = \frac{\pi P}{2\eta l}\left[\frac{a^2 r^2}{2} - \frac{r^4}{4}\right]_0^a \qquad ...\ (4)$$

$$= \frac{\pi P}{2\eta l}\left[\frac{a^4}{2} - \frac{a^4}{4}\right] = \frac{\pi P a^4}{8\eta l}$$

$$\therefore \qquad \eta = \frac{\pi P a^4}{8lV} \qquad \text{... (5)}$$

Thus if we know P, a, l and V the coefficient of viscosity can be determined. This is the Poiseuille's formula for the flow of liquid through the capillary tube. It is true only for streamline flow.

The flow is streamline, as we know, when the average velocity of flow is less than critical velocity. But the critical velocity is inversely proportional to the radius of the tube. Hence, the above formula fails in case of tube of wide bore. Moreover, the pressure difference between the ends of the tube should be small in order to have streamline flow.

11.5 POISEUILLE'S FLOW METHOD FOR DETERMINING THE COEFFICIENT OF VISCOSITY OF LIQUID

Formula: Poiseuile's formula for the volume of liquid flowing per second through a cylindrical tube of circular cross-section is

$$V = \frac{\pi P a^4}{8\eta l}$$

$$\therefore \qquad \eta = \frac{\pi a^4}{8l}\left(\frac{P}{V}\right) = \left(\frac{\pi \rho g a^4}{8l}\right)\left(\frac{h}{V}\right). \qquad \text{... (6)}$$

Experimental Arrangement: A long and narrow capilary tube C of uniform bore is placed horizontaly having its ends inserted into two unions U and V. The experimental liquid (say water) is allowed to enter the capillary at union U at constant pressure by making use of constant level apparatus A. The liquid will leave the capillary through the other union V from where it can be collected in a measuring jar. A pinch cock K regulates the flow. The unions are connected to manometer M by means of two rubber tubes. The difference h between the levels of liquid in two limbs of manometer M gives the pressure difference between the ends of the capilary tube (See Fig. 11.6).

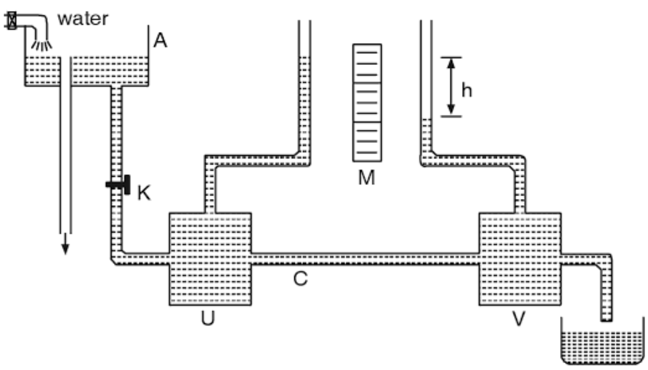

Fig. 11.6

Procedure: With the help of the pinch cock K the flow is adjusted so that the liquid leaves the capillary tube drop by drop. When the flow becomes steady, the emerging liquid is collected in a graduated jar for a certain interval of time. Thus, we find volume V flowing per second for a constant difference of levels h in the manometer. The experiment is repeated a number of times for different values of h. A graph of V against h is plotted and the average value of h/V is determined. The length and radius of the capillary tube are determined in usual manner.

Then, the value of η at room temperature is calculated using the above formula.

11.6 MAIN SOURCES OF ERROR AND CORRECTIONS APPLIED TO POISEUILLE'S FORMULA.

In deducing Poiseuille's equation the following two factors have not been considered and therefore, some corrections are necessary.

(*i*) **Length.** When the liquid enters the tube its motion is accelerated at the inlet end and as a result of this the velocity distribution becomes uniform and stream line only when the liquid has travelled a short length of the tube. This necessitates a correction for the effective length of the tube from l to $(l + 1.64\,a)$.

(*ii*) **Pressure.** When the liquid leaves the tube it has some velocity. Hence, the entire pressure difference $P = h\rho g$ is not used up in overcoming the viscous resistance. But a part of it is spent in imparting kinetic energy to the liquid. The effective pressure is less than the actual pressure by an amount $\dfrac{V^2\rho}{\pi^2 a^4}$.

Hence, effective pressure $P_1 = h\rho g - \dfrac{V^2\rho}{\pi^2 a^4}$

Substituting the value of P_1 instead of P in Poiseuille's equation we get,

$$\eta = \frac{\pi a^4\ h\rho g}{8V\,(l + 1.64\,a)} - \frac{V\rho}{8\pi\ (l + 1.64\,a)} \qquad \text{... (7)}$$

This is the corrected version.

11.7 PARABOLIC NATURE OF VELOCITY PROFILE

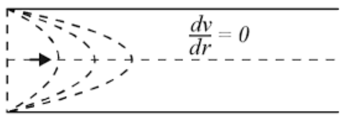

Fig. 11.7

The velocity of a liquid flowing through a uniform capillary tube

$$v = \frac{P}{4\eta l}\,(a^2 - r^2) \qquad (\because \text{Eq. 3})$$

shows that velocity of flow at a distance r from the axis of the tube through the layer is a parabola as shown in Fig. 11.7.. Hence, the velocity profile is parabolic in nature.

11.8 VARIATION OF VISCOSITY OF LIQUIDS/GASES WITH TEMPERATURE

The viscosity of liquids decreases with temperature, *e.g.*, the viscosity of water decreases from 0.0101 poise at 20°C to 0.0047 at 60°C and that of castor oil from 24.18 poise at 10°C to 9.86 at 20°C. No definite relation has been found to exist between viscosity and temperature, but various empirical formula have been suggested from time to time which are all approximate. One of these relations has the form

$$\log \eta = a + \frac{b}{T}$$

where a and b are constants and T is the absolute temperature.

In the case of gases, the viscosity increases with temperature. According to the kinetic theory of gases, the variation of viscosity with temperature is given by

$$\eta = a\eta_0\ T^{1/2}$$

where a is a constant, η_0 the viscosity at 0°C and T the absolute temperature at which the viscosity is η. Sutherland taking into consideration the small force of attraction between the neighbouring molecules modified the formula to

$$\eta = \eta_0 \frac{aT^{1/2}}{1 + \frac{S}{T}} \qquad \text{... (8)}$$

where S is known as Sutherland's constant. For small ranges of temperatures this formula holds true for many gases.

Variation of viscosity with pressure. The viscosity of liquids, in general, increases with increase of pressure. For example, the viscosity of ether at 20°C is raised by about 60% when pressure is increased to 500 atmospheres.

In case of water at normal temperature, there is decrease in viscosity for the first few hundred atmospheres. For more viscous liquids the increase in viscosity with pressure is much more than in case of fairly mobile liquids.

For moderate pressures, the viscosity of a gas is found to be independent of pressure. But at very high pressures, the viscosity increases with increasing pressure.

11.9 EQUATION OF CONTINUITY

Consider a horizontal pipe (fig. 11.8). A liquid is flowing through the pipe. The flow of liquid is assumed to be streamline.

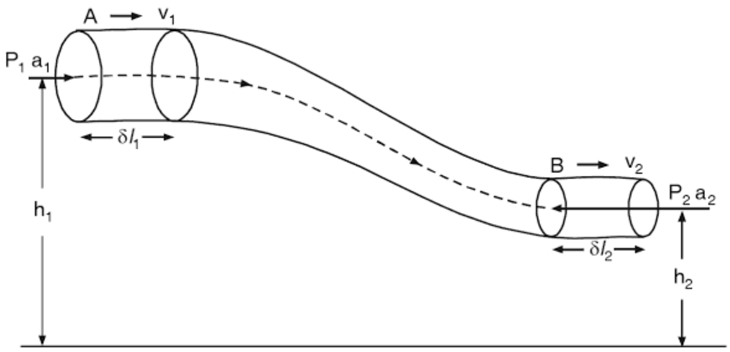

Fig. 11.8

Let a_1 be the area of cross-section of tube AB at A

P_1 be the pressure at A,

v_1 be the velocity at A, h_1 be the height of A and m be the mass of the liquid flowing through every cross-section in time δt sec.

Let a_2, P_2, v_2 and h_2 be the corresponding quantities at B.

The liquid at A in time δt moves through a distance $\delta l_1 = v_1 \cdot \delta t$. In the same time, the liquid at B moves through

$$\delta l_2 = v_2 \cdot \delta t.$$

In the steady flow, the same mass m of the liquid passes every cross-section in time δt.

$$\therefore \qquad m = a_1 \delta l_1 \rho = a_2 \delta l_2 \rho$$

$$\text{or} \qquad = a_1 v_1 \delta t \rho = a_2 v_2 \delta t \rho$$

$$\text{or} \qquad a_1 v_1 = a_2 v_2 = \frac{m}{\rho \delta t}$$

$\frac{m}{\rho \delta t}$ is the volume of liquid flowing per sec., which is constant.

$$\therefore \quad a_1 v_1 = a_2 v_2 = \text{constant} \quad \text{... (9)}$$

This equation is known as *equation of continuity*.

11.10 BERNOULLI'S THEOREM

Statement: *When a non-viscous and incompressible liquid flows steadily in stream-lines the sum of pressure energy, kinetic energy, and potential energy per unit mass at any point in a streamline remains constant.*

In symbols, the theorem is stated as follows:

$$\frac{P}{\rho} + hg + \frac{1}{2}v^2 = \text{constant} \quad \text{... (10)}$$

where P/ρ is the pressure head, or the potential energy per unit mass due to pressure, hg is the elevation head or the potential energy per unit mass due to gravity, and $\frac{v^2}{2}$ is the kinetic energy per unit mass.

Proof. Consider a streamline liquid flow through pipe *AB* as shown in fig. 11.8.

The energy of a liquid in motion at any point consists of the following three forms:

(*i*) Potential energy (*ii*) Kinetic energy (*iii*) Pressure energy.

(*i*) **Potential energy.** The potential energy of a liquid of mass *m* at a height *h* above the ground level = *mgh*

$\therefore$ P.E. per unit mass $= gh$

and P.E. per unit volume $= \rho gh$

where ρ is the density or mass per unit volume.

(*ii*) **Kinetic energy.** The kinetic energy of a liquid of mass *m* moving with a velocity *v* is given by

$$\text{K.E.} = \frac{1}{2}mv^2$$

$$\therefore \quad \text{K.E. per unit mass} = \frac{1}{2}v^2$$

$$\text{and} \quad \text{K.E. per unit volume} = \frac{1}{2}\rho v^2$$

(*iii*) **Pressure energy.** When a liquid flows through a pipe *AB* as shown, the volume *V* of the liquid entering *A* in a time *t* is equal to the volume of the liquid leaving *B* in the same time.

Let a_1 be the area of cross-section of the tube, v_1 the velocity of the liquid and P_1 the pressure exerted by the liquid at *A* and a_2, v_2 and P_2 the corresponding values at *B*, then

Force exerted by the liquid at $A = P_1 a_1$

Distance travelled by the liquid in time $t = v_1 t$

$\therefore$ Work done $= P_1 a_1 v_1 t$

But $a_1 v_1 t = V$

where *V* is the volume of the liquid entering *A*.

$\therefore$ Work done $= P_1 V$

This work is stored in the liquid and is known as its pressure energy.

Hence, pressure energy at $A = P_1 V$

$\therefore$ Pressure energy per unit volume at $A = P_1$

and pressure energy per unit mass at $A = \frac{P_1}{\rho}$

If m is the mass of the liquid entering A in a given time, then

Pressure energy of the liquid at $A = m\frac{P_1}{\rho}$

Potential energy of the liquid at $A = mgh_1$

where h_1 is the height of the centre of the tube at A from the ground level.

Kinetic energy of the liquid at $A = \frac{1}{2}mv_1^2$

$\therefore$ Total energy at $A = m\frac{P_1}{\rho} + mgh_1 + \frac{1}{2}mv_1^2$

Similarly total energy at $B = m\frac{P_2}{\rho} + mgh_2 + \frac{1}{2}mv_2^2$

Since no accumulation of liquid any where in the tube and applying the principle of conservation of energy

$$m\frac{P_1}{\rho} + mgh_1 + \frac{1}{2}mv_1^2 = m\frac{P_2}{\rho} + mgh_2 + \frac{1}{2}mv_2^2$$

or $$\frac{P_1}{\rho} + gh_1 + \frac{1}{2}v_1^2 = \frac{P_2}{\rho} + gh_2 + \frac{1}{2}v_2^2 = \text{constant.} \quad ...(i)$$

or $$\frac{P}{\rho} + gh + \frac{1}{2}v^2 = \text{constant} \quad ... (ii)$$

This is **Bernoulli's theorem.**

Dividing (i) by g, we get

$$\frac{P_1}{\rho g} + h_1 + \frac{v_1^2}{2g} = \frac{P_2}{\rho g} + h_2 + \frac{v_2^2}{2g} = \text{constant} \quad ...(iii)$$

or $$\frac{P}{\rho g} + h + \frac{v^2}{2g} = \text{constant.} \quad ... (11)$$

$\frac{P}{\rho g}$ is called the *pressure head, h the elevation head* and $\frac{v^2}{2g}$ the *velocity head* and the Bernoulli's theorem may be put in the following form:

In streamline motion of a liquid the sum of the pressure head, the elevation head and the velocity head is constant at all points.

11.11 EULER'S EQUATION

For a non-steady flow of a non-viscous and incompressible liquid through a tube, the Euler's equation is

$$\rho\left(V\frac{\partial v}{\partial l} + \frac{\partial v}{\partial t}\right) = -\frac{\partial P}{\partial l} - \rho g\frac{\partial h}{\partial l} \quad ... (i)$$

Dividing by ρ and re-arranging the terms, we get

$$\frac{1}{\rho}\frac{\partial P}{\partial l} + v\frac{\partial v}{\partial l} + g\frac{\partial h}{\partial l} + \frac{\partial v}{\partial t} = 0 \quad ... (ii)$$

where v is the velocity of the liquid at a cross-section at distance l from same reference point, ρ is the density of the liquid, $\partial P/\partial l$ is the rate of change of pressure with distance l, $\partial v/\partial l$ is the rate of change of velocity with distance; $\partial h/\partial l$ is the rate of change of vertical height of cross-section, and $\partial v/\partial t$ is the acceleration of the liquid at the cross-section.

This equation (*ii*) is Euler's equation and is applicable to both steady and non-steady flow of liquids.

Derivation of Bernoulli's Theorem:

For steady or streamline flow, $\dfrac{\partial v}{\partial t} = 0$

Substituting in Eq. (*ii*), we get

$$\frac{1}{\rho}\frac{\partial P}{\partial l} + v\frac{\partial v}{\partial l} + g\frac{\partial h}{\partial l} = 0$$

Integrating w.r.to l along stream line flow, we get

$$\frac{1}{\rho}\int\frac{dP}{dl}\,dl + \int v\frac{dv}{dl}\,dl + g\int\frac{dh}{dl}\,dl = \text{constant}$$

or
$$\frac{1}{\rho}\int dP + \int v\,dv + g\int dh = \text{constant}$$

or
$$\frac{P}{\rho} + \frac{1}{2}v^2 + gh = \text{constant}$$

This is the Bernoulli's equation for steady flow of unit mass.

11.12 APPLICATIONS OF BERNOULLI'S THEOREM

Followings are some of the applications of Bernoulli's theorem.

11.12.1. Venturimeter

Measurement of rate of discharge. When a liquid flows through a horizontal pipe, with a constriction, as shown in Fig. 11.9, the speed of the liquid increases as it enters the constriction whereas the pressure decreases.

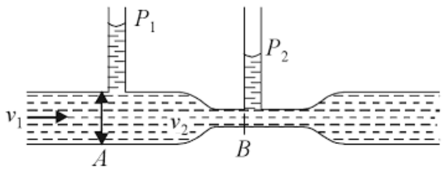

Fig. 11.9

Suppose water flows from a wide tube of area of cross-section a_1 at A into a narrow tube of area of cross-section a_2 at B, then the pressure of water at A, *i.e.*, P_1 is greater than the pressure of water at B, *i.e.*, P_2 whereas the velocity of water at B, *i.e.*, v_2 is greater than the velocity of water at A *i.e.*, v_1.

According to Bernoulli's theorem

$$\frac{P_1}{\rho g} + h_1 + \frac{v_1^2}{2g} = \frac{P_2}{\rho g} + h_2 + \frac{v_2^2}{2g} \quad ...(i)$$

When the tube is horizontal $h_1 = h_2$

$$\therefore \quad \frac{P_1}{\rho} + \frac{v_1^2}{2} = \frac{P_2}{\rho} + \frac{v_2^2}{2} \quad ...(ii)$$

or
$$P_1 - P_2 = \frac{\rho}{2}(v_2^2 - v_1^2) = \frac{\rho v_1^2}{2}\left(\frac{v_2^2}{v_1^2} - 1\right) \quad ...(iii)$$

Now the volume of water entering per second into the pipe at A = volume of water leaving the pipe at B per second.

$\therefore \qquad a_1 v_1 = a_2 v_2$

or $$\frac{v_2}{v_1} = \frac{a_1}{a_2}$$

Substituting in (*iii*), we have

$$P_1 - P_2 = \frac{\rho v_1^2}{2}\left(\frac{a_1^2}{a_2^2} - 1\right) = \frac{\rho v_1^2}{2}\left(\frac{a_1^2 - a_2^2}{a_2^2}\right)$$

or $$v_1^2 = \frac{2(P_1 - P_2)\, a_2^2}{\rho(a_1^2 - a_2^2)}$$

$\therefore$ $$v_1 = a_2\sqrt{\frac{2\,(P_1 - P_2)}{\rho(a_1^2 - a_2^2)}} = a_2\sqrt{\frac{2\,(P_1 - P_2)}{(a_1^2 - a_2^2)}}$$

since the density of water is 1 gm. per c.c. in C.G.S. system.

$\therefore$ Volume of water entering at A per second or the rate of flow of water through the pipe

$$= a_1 v_1 = a_1 a_2\sqrt{\frac{2\,(P_1 - P_2)}{(a_1^2 - a_2^2)}} \qquad ...(12)$$

Thie equation given above is used to determine the rate of flow of water through a pipe or the rate of discharge through city water mains. For this purpose a tube similar to that shown in Fig. 11.9, is used having a constriction between the inlet and the outlet sections of large diameters. Such a tube is called a **venturi tube.** A meter used to measure the rate of flow of water, which utilises a venturi tube and is *calibrated* with the help of the above equation is called a **venturi flow meter.**

11.12.2 Pitot's tube

It is a device used for the measurement of the velocity of liquids or gases through pipes. The working of Pitot's tube is based upon Bernoulli's principle. It consists of a U-tube having small aperture at the ends X and Y. The plane of the aperture at X is parallel to the direction of flow of the liquid and that of the aperture at Y is perpendicular to the direction of flow. If the pipe is horizontal, then the plane of the aperture at X is horizontal and that at Y is vertical (fig. 11.10).

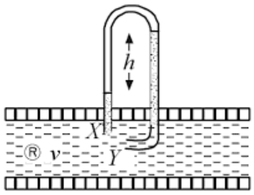

Fig. 11.10

Suppose the velocity of flow of the liquid in the pipe is v and pressure head (or static pressure) is P_x. The velocity of the liquid falls rapidly to zero as it enters the aperture at Y and hence, the velocity head falls to zero so that the pressure head is very much increased, say to P_y.

Applying Bernoulli's theorem for horizontal flow to points X and Y, we have

$$\frac{P_x}{\rho g} + h + \frac{v^2}{2g} = \frac{P_y}{\rho g} + h + 0$$

or $$P_x + \frac{1}{2}\rho v^2 = P_y$$

$\therefore$ $$v = \sqrt{\frac{2(P_y - P_x)}{\rho}}$$

Now $P_y - P_x = h\rho g$ where h is the difference in heights of the liquid column in the two limbs of Pitot's tube.

Hence, $$v = \sqrt{2gh}$$

The device is sometimes calibrated to read the value of velocity v directly and is then known as *speed indicator.*

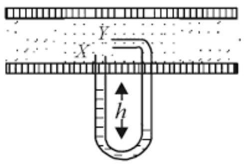

Fig. 11.11

The arrangement for finding the velocity of flow of gases is shown in Fig. 11.11. Here again

$$v = \sqrt{\frac{2(P_y - P_x)}{\rho}}$$

where ρ is the density of the gas. If d is the density of the liquid (usually mercury) in Pitot's tube, then

$$P_y - P_x = hdg$$

and

$$v = \sqrt{\frac{2hdg}{\rho}} \qquad \text{... (13)}$$

11.12.3 Lift of an Aeroplane

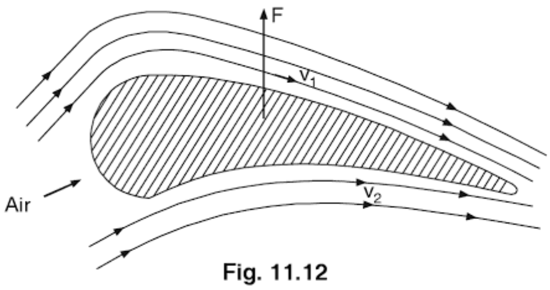

Fig. 11.12

The wings of the aeroplane are made tapering as shown in fig. 11.12. The upper surface is made convex and the lower surface is made concave. Due to this shape of the wing, the air currents at the top have a large velocity than at the bottom ($v_1 > v_2$) consequently, according to Bernoulli's equation, the pressure above the surface of the wing is less as compared to the lower surface of the wing. The difference of pressure creates an upward force on the wing, normal to the flow of air. This force provides upward lift for the aeroplane.

11.12.4 Bunsen Burner

In bunsen burner, the gas enters the base and comes out of the nozle N. As the cross-section of nozzle is very small, the velocity of the gas emerging out is very large. Consequently, pressure just above the nozzle decreases. Air from the atmosphere rushes into the burner. The mixtur of gas and air moves up in the burner and burns at the top.

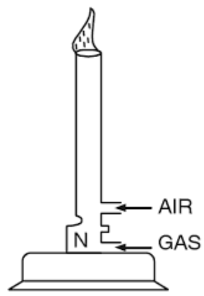

Fig. 11.13

11.12.5 Atomiser or Sprayer

An atomiser consists of an air pump T and a bottle containg liquid to be sprayed (fig. 11.14). Air is blown through the tube T using by compressing a rubber bulb fitted at its neck. As air rushes out through the jet (orifice O) with high velocity. The pressure at the orifice O of tube A, according to Bernoulli's theorem, decreases and thereby the liquid rises up the tube A. The drops of the liquid coming out of this tube are divided into in very fine enormous droplets by high velocity jet.

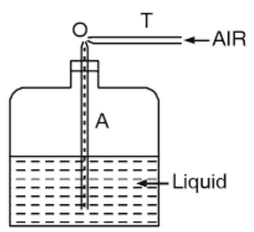

Fig. 11.14

11.12.6 Carburettor

Carburettor is a chamber to mix air and petrol vapour in an internal combustion engine. Air is allowed to enter through a nozzle with large velocity. The pressure is lowered and the petrol is sucked up into the chamber. The petrol vaporizes quickly. The mixed vapours of petrol and air are fed into the carburettor chamber where it burns at the moment spark is produced by spark plug.

11.12.7 Blowing of Roofs

Due to wind, storm or cyclone, the roofs are blown off. When a high velocity wind blows over the roof, a considerably lower pressure at the top of the roof is created. As the pressure on the lower side of roof is higher roofs are easily blown off, without damaging the walls of the building.

11.13 STOKES' LAW

When a body falls through a liquid (or gas) it carries along with it the layer of the fluid in contact and thus tends to produce a realtive motion between the layers of the fluid. The relative motion is opposed by forces due to viscosity. The *opposing force* increases with the velocity of the body and if the body is small it soon becomes equal to the *driving force* producing motion. The body then moves with a constant velocity, known as **terminal velocity.**

The viscous force acting on the body is given by Stokes' law which is stated as below:

When a small spherical body falls vertically under gravity in a viscous liquid with the terminal velocity, the viscous force F acting upwards on the body is given by

$$F = 6\pi\eta rv \qquad \text{... (14)}$$

where h is the coefficient of viscosity of the liquid, r is the radius of the spherical body and v is the terminal velocity.

11.14 DERIVATION OF STOKES' FORMULA BY METHOD OF DIMENSIONS

For a small sphere falling through a viscous medium the opposing force is directly *proportional* to the velocity v and also depends upon:

(*i*) radius r of the sphere

(*ii*) coefficient of viscosity η of the medium and

(*iii*) density of the medium d.

Combining all these factors, we have $F = Kvr^a\eta^b d^c$

where K is a constant and a, b and c are the dimensional coefficients. Now, putting the dimensions of various quantities on both sides of the equation, we get

$$M^1L^1T^{-2} = (L^1T^{-1})\,(L^a)\,(M^bL^{-b}T^{-b})(M^cL^{-3c})$$
$$= M^{b+c}L^{1+a-b-3c}\,T^{-1-b}$$

Comparing the coefficients of similar terms, we get

$$b + c = 1 \qquad \text{...}(i)$$
$$1 + a - b - 3c = 1 \text{ or } a - b - 3c = 0 \qquad \text{...}(ii)$$
$$-1 - b = -2 \text{ or } b = 1 \qquad \text{...}(iii)$$

Substituting $b = 1$ in (*i*), we have $c = 0$

and substituting $b = 1$ and $c = 0$ in (*ii*), we have $a = 1$

$\therefore \quad F = kvr\eta$

Stokes' calculated the value of K to be 6π.

Hence, $\quad F = 6\pi\eta rv$

If the density of the material of the body is ρ, then downward force due to gravity on the small sphere $= \frac{4}{3}\pi r^3 \rho g$ and upward thrust due to buoyancy = weight of the fluid displaced.

$$= \frac{4}{3}\pi r^3 dg$$

$\therefore$ Resultant downward force $F = \frac{4}{3}\pi r^3 (\rho - d)\, g$

Hence, $\frac{4}{3}\pi r^3 (\rho - d)\, g = 6\,\pi\eta r v$ or $v = \frac{2r^2 (\rho - d)\, g}{9\eta}$ ... (15)

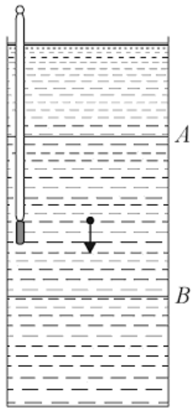

Fig. 11.15

11.15 DETERMINATION OF VISCOSITY OF A LIQUID

The viscosity of a viscous liquid like castor oil is determined by finding the teminal velocity of a small sphere of a suitable size falling through the liquid.

A small sphere of about one mm. radius is placed on the surface of the liquid almost at the centre. It begins to fall down and acquires a constant velocity after falling through a distance of about 10 to 15 cm. The time t taken by the sphere to fall through the distance AB is noted.

$\therefore$ Terminal velocity $v = \frac{AB}{t}$ ... (16)

The radius of the sphere is measured by a travelling microscope. The coefficient of viscosity is then determined by the formula given above.

Application. Stokes' formula is used to find the rate of fall of an ion in an electric field and thus the charge on the ion is calculated as in Millikan's method.

11.16 TORRICELLI'S THEOREM

Velocity of efflux. Consider an ideal liquid of density ρ contained in a tank provided with a narrow orifice O. Let H be the total depth of the tank, h the height of the surface of the liquid above O and h' the height of the orifice above the bottom of the tank.

Suppose v is the velocity of discharge. As the liquid flows out of the orifice it falls through a height h (from the top of the liquid surface upto the orifice)

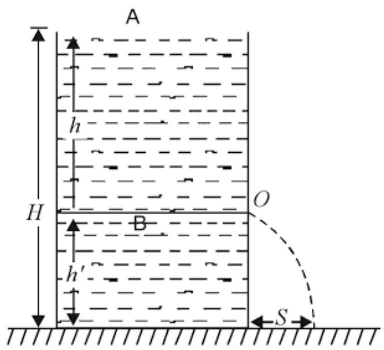

Fig. 11.16

According to Bernoulli's theorem,

$$\frac{v^2}{2} + gh + \frac{P}{\rho} = \text{constant}$$

As P is same at the free surface and at orifice,

$$\frac{v^2}{2} + gh = \text{constant.}$$

At A, height = h and $v = 0$

At B, height = 0 and velocity is v

$$\therefore \qquad gh = \frac{v^2}{2} \quad \text{or} \quad v = \sqrt{2gh}$$

This is known as velocity of efflux. This result was first derived by Torricelli and hence known as Torricelli's theorem.

Statement: According to Torricelli's theorem, the velocity of efflux of a liquid through an orifice is equal to that which a body acquires in falling freely under gravity from the surface of the liquid to the orifice. This result is true only for ideal liquid free from viscosity.

In practice, a liquid will have some value for viscosity and hence the velocity of efflux is never reached to a value equal to $\sqrt{2gh}$.

Here the velocity of efflux is an application of Bernoulli's theorem.

Range of flow. As the liquid escapes through the orifice O, it has zero initial velocity in the vertical direction. If flows out in the form of a parabolic jet and strikes the ground at a distance S from the tank after a time t. The vertical distance covered by the jet in time $t = h'$

$$\therefore \qquad h' = \frac{1}{2} gt^2 \qquad \text{or} \qquad t = \sqrt{\frac{2h'}{g}}$$

Hence, the horizontal distance covered by the jet or range of liquid flow

$$S = vt = \sqrt{2gh}\sqrt{\frac{2h'}{g}} = 2\sqrt{hh'}$$

SOLVED EXAMPLES

Example 11.1 Calculate the speed at which the velocity head of a stream of water is equal to 0.50 m of Hg.

Solution. Velocity head = $\dfrac{v^2}{2g}$ metres of Hg

Given: Velocity head = 0.50 m of Hg

= 0.50 × 13.6 m of water

$$\therefore \qquad \frac{v^2}{2g} = 0.5 \times 13.6$$

$$v^2 = 2 \times 9.8 \times 0.5 \times 13.6 = 9.8 \times 13.6$$

$$v = \sqrt{9.8 \times 13.6} = \mathbf{11.54\ m/s}$$

Example 11.2 A railway engine is fitted with a tube whose one end is inside a reservoir of water is between the rails. The other end of the tube is 4m above the surface of water in the reservoir. Calculate the speed of with which the water rushes out of the upper end, of the engine is moving with a speed of 108 km/hr.

Solution. Applying Bernoulli's theorem,

$$mgh_1 + \frac{1}{2} mv_1^2 = mgh_2 + \frac{1}{2} mv_2^2$$

$$gh_1 + \frac{1}{2} v_1^2 = gh_2 + \frac{1}{2} v_2^2$$

$$\frac{1}{2} v_1^2 = g(h_2 - h_1) + \frac{1}{2} v_2^2$$

$$v_1 = \sqrt{2g(h_2 - h_1) + v_2^2}$$

Here $(h_1 - h_2) = 4$ m; $g = 9.8$ m/s^2; $v_2 = 108$ km/hr = 30 m/s

$$\therefore \qquad v_1 = \sqrt{2 \times 9.8 \times (-4) \times (30)^2}$$

$$v_1 = \sqrt{900 - 78.4}$$

$$\mathbf{v_1 = 28.66\ m/s}$$

Example 11.3 Water flows through a horizontal pipe line of varying cross-section. At a point where the pressure of water is 0.05m of mercury the velocity of flow is 0.25m/s. Calculate the pressure at another point where velocity of flow is 0.4 m/s. Density of water = 10^3 kg/m^3. (*Nag. U.*, 2001)

Solution. Here, $P_1 = 0.05$ m of Hg $= 0.05 \times 13.6 \times 10^3 \times 9.8$

$= 6.664 \times 10^3\ \text{Nm}^{-2}$

$v_1 = 0.25\ \text{ms}^{-1}$, $v = 0.4\ \text{ms}^{-1}$ $P_2 = ?$

As the pipe is horizontal, according to Bernoulli's theorem

$$\frac{P_1}{\rho} + \frac{v_1{}^2}{2} = \frac{P_2}{\rho} + \frac{v_2{}^2}{2}$$

or
$$\frac{6.664 \times 10^3}{10^3} + \frac{0.25^2}{2} = \frac{P_2}{10^3} + \frac{0.4^2}{2}$$

or
$$\frac{P_2}{10^3} = 6.664 + \frac{0.25^2}{2} - \frac{0.4^2}{2}$$

or
$$P = (6.664 + 0.03125 - 0.08)\ 10^3 = 6.61525 \times 10^3\ \text{Nm}^{-2}$$

$$= \frac{6.61525 \times 10^3}{13.6 \times 10^3 \times 9.8} = \mathbf{0.0496\ m\ of\ mercury}$$

Example 11.4 A pipe is running full of water. At a certain point *A* it tapers from 0.6 m diameter to 0.2 m diameter at *B*. The pressure difference between *A* and *B* is 1 m of water column. Find the rate of flow of water through the pipe.

Solution. Let p_1, v_1 be the pressure and velocity at A and p_2 and v_2 the corresponding values at B, then

$$p_1 - p_2 = 1 \times 10^3 \times 9.8\ \text{Nm}^{-2}$$

Rate of flow at $A = v_1 \times$ area $= \pi \times 0.3^2 \times v_1 = 0.09\ \pi\ v_1$

Rate of flow at $B = v_2 \times$ area $= \pi \times 0.1^2 \times v_2 = 0.01\ \pi\ v_2$

For a steady flow $0.01\ \pi v_2 = 0.09\ \pi v_1$

or $v_2 = 9v_1$

As the height remains the same, according to Bernoulli's theorem, we have

$$\frac{p_1}{\rho} + \frac{v_1^2}{2} = \frac{p_2}{\rho} + \frac{v_2^2}{2}$$

or
$$\frac{p_1 - p_2}{\rho} = \frac{1}{2}(v_2^2 - v_1^2)$$

or
$$\frac{10^3 \times 9.8}{10^3} = \frac{1}{2}(81\ v_1^2 - v_1^2)$$

$\therefore$
$$v_1 = \sqrt{\frac{9.8}{40}} = 0.495\ \text{ms}^{-1}$$

$\therefore$ Rate of flow $= \pi \times 0.09 \times 0.495 = \mathbf{0.14\ m^3\ s^{-1}}$

Example 11.5 Water issues into the air from a horizontal nozzle whose area of cross-section is 0.125×10^{-4} m^2. Its speed is such that 1.875 kg emerge in one minute. The water strikes a fixed wall which is at right angles to the nozzle and 0.5 m from it and then falls in a vertical plane. Calculate the vertical distance below the nozzle of the point where the jet strikes the wall and the force which the water exerts on the wall.

Solution. Volume of water flowing out per second $V = \dfrac{1.875}{60 \times 10^3} = 31.25 \times 10^{-6}\ \text{m}^3$

Area of nozzle $a = 0.125 \times 10^{-4}$ m^2

If v is the velocity with which water issues, then $V = av$

or $$v = \frac{V}{a} = \frac{31.25 \times 10^{-6}}{0.125 \times 10^{-4}} = 250 \times 10^{-2}\ \text{ms}^{-1}$$

Distance of wall from the nozzle = 0.5 m

$\therefore$ Time taken by water to reach the wall $= \dfrac{0.5}{250 \times 10^{-2}} = 0.2$ sec.

Vertical distance through which water falls in 0.2 sec.

$$= \frac{1}{2} gt^2 = \frac{1}{2} \times 9.8 \times (.2)^2 = 0.196 \text{ m}$$

Hence, the water jet will strike the wall at a point 0.196 m below the nozzle.

Force exerted on wall = Momentum imparted by water in one second

$$= \frac{1.875}{60} \times 250 \times 10^{-2} = \mathbf{7.813 \times 10^{-2}\ N}$$

Example 11.6 A tank contains water to a height H. Calculate the range of flow of water from an orifice at depth $\dfrac{H}{4}, \dfrac{H}{2}$ and $\dfrac{3H}{4}$ from the surface of water. (*Indore U.*, 2001)

Solution. (*i*) *When the orifice is at a depth* $\dfrac{H}{4}$

$$h = \frac{H}{4},\ h' = H - \frac{H}{4} = \frac{3H}{4}$$

$$\text{Range} \quad S = 2\sqrt{hh'} = 2\sqrt{\frac{H}{4} \times \frac{3H}{4}} = \frac{\sqrt{3}}{2} H$$

(*ii*) *When the orifice is at a depth* $\dfrac{H}{2}$

$$h = \frac{H}{2},\ h' = H - \frac{H}{2} = \frac{H}{2}$$

$$\text{Range} \quad S = 2\sqrt{hh'} = 2\sqrt{\frac{H}{2} \times \frac{H}{2}} = H$$

(*iii*) *When the orifice is at a depth* $\dfrac{3H}{4}$

$$h = \frac{3H}{4},\ h' = H - \frac{3H}{4} = \frac{H}{4}$$

$$\text{Range} \quad S = 2\sqrt{hh'} = 2\sqrt{\frac{3H}{4} \times \frac{H}{4}} = \mathbf{\frac{\sqrt{3}}{2} H.}$$

Example 11.7 A plate of metal 100 sq. cm. in area rests on a layer of castor oil 2 mm thick whose co-efficient of viscosity is 15.5 poise. Calculate the horizontal force required to move the plate with a speed of 0.03 ms^{-1}.

Solution. Area A = 100 sq. cm. = 10^{-2} m^2, v = 0.03 ms^{-1}, r = 2 mm = 0.2×10^{-2} m, η = 15.5 poise = 1.55 deca-poise

Horizontal viscous force $F = -\eta A \dfrac{v}{r} = \dfrac{-1.55 \times 10^{-2} \times 0.03}{0.2 \times 10^{-2}} = -\ 0.2325$ N

$\therefore$ External force required = 0.2325 N

Example 11.8 A horizontal tube of 1 mm bore is joined to another horizontal tube of 0.5 mm bore. Water enters at the free end of the first tube at a presure equal to 0.5 m of water above the atmospheric pressure and leaves at the free end of the second tube at the atmospheric pressure. Calculate the pressure at the junction of the tubes if the lengths of the tubes are equal.

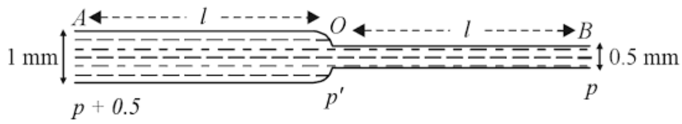

Fig. 11.17

Solution. According to Poiseuille's equation $V = \dfrac{\pi P a^4}{8\eta l}$

If p' is the pressure at the junction O of the two tubes each of length l, then difference of pressure between A and $O = p + 0.5 - p'$

$\therefore$ Volume of water flowing through AO per second

$$= \frac{\pi\ (p + 0.5 - p')\ (0.5 \times 10^{-3})^4}{8\eta l}$$

Difference of pressure between O and $B = p' - p$

$\therefore$ Volume of water flowing per second through $OB = \dfrac{\pi\ (p' - p)\ (0.25 \times 10^{-3})^4}{8\eta l}$

As the two tubes are joined end to end, the volume of water flowing per second through them is the same.

$$\therefore \qquad \frac{\pi (p + 0.5 - p')\ (0.5 \times 10^{-3})^4}{8\eta l} = \frac{\pi (p' - p)\ (0.25 \times 10^{-3})^4}{8\eta l}$$

or $$p + 0.5 - p' = \frac{(p' - p)}{16}$$

$$17\ (p' - p) = 8$$

$$\therefore \qquad p' - p = \frac{8}{17} = \textbf{0.47 m of water column.}$$

Hence, pressure at O is 0.47 m of water column.

Example 11.9 A capillary tube of radius *a* and length *l* is fitted horizontally at the bottom of a cylindrical flask of cross-section area *A*. Initially there is water in the flask up to a height *h*. What time would be required for half the liquid to flow out, if the coefficient of viscosity of the liquid is η?

Solution. According to Poiseuille's equation the volume V of the water flowing through a tube of length l and radius a is given by

$$V = \frac{\pi P a^4}{8\eta l} \qquad ...(i)$$

As A is the area of cross-section of the vessel and h the height of water above the capillary tube.

$\therefore$ Pressure head $P = hg$ [$\because$ $\rho = 1$ for water in C.G.S. system]

Suppose in a small time dt the level of water in the vessel falls through a height dh, then

Volume of water flowing in time $dt = A.dh$

$\therefore$ Rate of flow $V = -A\dfrac{dh}{dt}$

The negative sign shows that the height decreases with time.

Substituting the value of P and V in (i), we have

$$-A\frac{dh}{dt} = \frac{\pi hga^4}{8\eta l} \quad \text{or} \quad dt = -\frac{8\eta lA}{\pi ga^4}\cdot\frac{dh}{h} \qquad ...(ii)$$

Let t be the time in which the initial height h is reduced to $h/2$, then

$$\int_0^t dt = -\frac{8\eta lA}{\pi ga^4}\int_h^{h/2}\frac{dh}{h}$$

$$\therefore \qquad t = -\frac{8\eta lA}{\pi ga^4}\left[\log_e h\right]_h^{h/2} = \frac{8\eta lA}{\pi ga^4}\left[\log_e h - \log_e \frac{h}{2}\right]$$

$$= \frac{8\eta lA}{\pi ga^4}\log_e 2 = \mathbf{2.3026 \times A\,\frac{8\eta l}{\pi ga^4}\log_{10} 2}$$

Example 11.10 Calculate the mass of water flowing in 10 minute through a tube 0.1 cm in diameter, 40 cm long if there is a constant pressure head of 20 cm of water. The co-efficient of viscosity of water is 0.0089 C.G.S. units.

Solution. Here $l = 40$ cm $= 0.4$ m, $\eta = 0.0089$ C.G.S. units $= 0.00089$ deca-poise

$P = 20$ cm of water column $= 0.2 \times 9.8 \times 10^3 = 1.96 \times 10^3$ Nm^{-2}

Diameter $= 0.1$ cm or Radius $a = 0.05 \times 10^{-2}$ m

Rate of flow $V = \dfrac{P\pi a^4}{8\eta l} = \dfrac{1.96\times10^3\times\pi\times(0.05\times10^{-2})^4}{8\times0.00089\times0.4} = 135.13\times10^{-9}\ \text{m}^3$

Mass flowing/sec $= 135.13 \times 10^{-9} \times 10^3 = 135.13 \times 10^{-6}$ kg

Mass flowing in 10 min $= 135.13 \times 10^{-6} \times 600 = \mathbf{81.078 \times 10^{-3}}$ **kg**

Example 11.11 In the Poiseuille experiment the following observations were made. Volume of water collected in 5 minutes = 40 c.c.; Head of water = 0.4 m; length of capillary tube = 0.602 m and radius of capillary tube = 0.52 × 10^{-3} m. Calculate the co-efficient of viscosity of water.

Solution. Volume of water collected per second

$$V = \frac{40}{5\times60}\ \text{cm}^3 = \frac{40}{5\times60}\times10^{-6}\ \text{m}^3 = \frac{2}{15}\times10^{-6}\ \text{m}^3$$

Head of water $h = 0.4$ m

$\therefore$ Difference of pressure $P = h\rho g = 0.4 \times 10^3 \times 9.8$ Nm^{-2} $= 3.92 \times 10^3$ Nm^{-2}

Length of capillary tube $l = 0.602$ m

Radius of the capillary tube $a = 0.52 \times 10^{-3}$ m

Now, co-efficient of viscosity $\eta = \dfrac{\pi P a^4}{8lV}$

$$= \frac{3.142 \times 3.92 \times 10^3 \times (0.52)^4 \times 10^{-12} \times 15}{8 \times 0.602 \times 2 \times 10^{-6}} = \mathbf{1.4 \times 10^{-3}\ Nm^{-2}\ (or\ deca\text{-}poise)}$$

Example 11.12 If two capillaries of radii r_1 and r_2 and length l_1 and l_2 are joined in series, derive an expression for the rate of flow of the liquid through the arrangement using Poiseuille's formula.

Solution. According to Poiseuille's formula, the rate of flow V of a liquid through a capillary tube of length l and radius r is given by

$$V = \frac{\pi p r^4}{8\eta l}$$

where p is the pressure difference across the ends of the tube and η the co-efficient of viscosity.

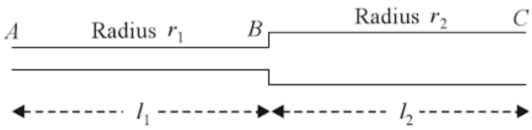

Fig. 11.18

Consider two capillaries of lengths l_1 and l_2 having radii r_1 and r_2 respectively connected in series. If p_1 is the pressure difference between the ends of capillary AB and p_2 that between the ends of the capillary BC, then as the same volume of liquid is flowing through each of the capillaries

$$V = \frac{\pi p_1 r_1^4}{8\eta l_1} = \frac{\pi p_2 r_2^4}{8\eta l_2}$$

So that $$p_1 = \frac{8\eta l_1}{\pi r_1^4} V \text{ and } p_2 = \frac{8\eta l_2}{\pi r_2^4} V$$

If p is the effective pressure across the ends A and C, then

$$p = p_1 + p_2 = \left(\frac{8\eta l_1}{\pi r_1^4} + \frac{8\eta l_2}{\pi r_2^4}\right) V$$

$\therefore$ Rate of flow $$V = \frac{\pi p}{8\eta\left(\dfrac{l_1}{r_1^4} + \dfrac{l_2}{r_2^4}\right)}$$

Example 11.13 Two drops of water of the same size are falling through air with terminal velocity 1 ms^{-1}. If the two drops combine to form a single drop, calculate the terminal velocity.

Solution. Let r be the radius of each drop and r_1 that of the combined drop, then

$$4/3\ \pi\ r_1^3 = 2 \times 4/3\ \pi\ r^3$$

$\therefore$ $$r_1 = 2^{1/3} r$$

If v is the terminal velocity of each drop and v_1 that of the combined drop then according to Stoke's law

$$v = \frac{2r^2 (\rho - d)\, g}{9\eta}$$

and $$v_1 = \frac{2r_1^2 (\rho - d)\, g}{9\eta}$$

$$\therefore \qquad \frac{v_1}{v} = \frac{r_1^2}{r^2}$$

$$= \frac{2^{2/3} r^2}{r^2} = 2^{2/3}$$

$$= \mathbf{1.588\ ms^{-1}} \qquad [\because v = 1\ \text{ms}^{-1}]$$

Example 11.14 Eight drops of water of the same size are falling through air with terminal velocity of 10 m/sec. If the eight drops combine to form a single drop what will be the new terminal velocity?

Solution. $$\frac{4}{3}\pi r_1^3 = 8 \times \frac{4}{3}\pi r^3 \quad \text{or} \quad r_1 = 2r$$

$$\frac{v_1}{v} = \frac{r_1^2}{r^2} = \frac{2^2\, r^2}{r^2} = 4 \quad \text{or} \quad \mathbf{v_1 = 4 \times 10 = 40\ m/s}$$

Example 11.15 A steel ball of radius 2×10^{-3} m falls in a vertical column of castor oil. The co-efficient of viscosity of castor oil is 0.7 Nm^{-2} and its density 0.98×10^3 kg m^{-3}. The density of steel is 7.8×10^3 kg m^{-3} and g = 9.8 ms^{-2}. Find its terminal velocity.

Solution. According to Stokes' formula, the terminal velocity is given by

$$v = \frac{2r^2(\rho - d)\, g}{9\eta}$$

Now, radius of the ball $r = 2 \times 10^{-3}$ m

Density of steel ball $\rho = 7.8 \times 10^3$ kg m^{-3}

Density of castor oil $d = 0.98 \times 10^3$ kg m^{-3} ; $g = 9.8$ ms^{-2}

Viscosity of castor oil $\eta = 0.7$ Nm^{-2}

$$\therefore \qquad v = \frac{2 \times (2 \times 10^{-3})^2\ (7.8 \times 10^3 - 0.98 \times 10^3)\, 9.8}{9 \times 0.7}$$

$$= \frac{2 \times 4 \times 10^{-3} \times 6.82 \times 9.8}{9 \times 0.7}$$

$$= \mathbf{84.87 \times 10^{-3}\ ms^{-1}}$$

EXERCISES CH.11

LONG QUESTIONS

1. Define coefficient of viscosity of a liquid. Describe the Poiseuille's method for measuring the coefficient of viscosity of a liquid. Derive the formula used with its two correction terms. (*Purvanchal U.* 2004; *M.S.U. Tiruneveli* 2007; *Meerut* 2004; *D.A.U. Agra* 2008)
2. (*a*) Give with necessary theory Poiseuille's method of determining the coefficient of viscosity of a liquid. State clearly the assumptions made.

   (*b*) Why is correction of Poiseuille's equation necessary? Obtain the corrected version.
   (*Bhopal U.* 2004; *Utkal U.* 2003; *Nag. U.* 2003, 2001; *Meerut U.* 2003, 2002, 2001, 2000; *Indore U.* 2001; *Guwahati U.* 2000; *Gharwal U.* 2000; *Purvanchal U.* 2006, 2004; *Agra U.* 2005, 2003)
3. Deduce an expression for the distribution of velocity of a liquid flowing through a uniform capilary tube of circular cross-section. What is the nature of velocity profile?

4. Derive Stokes' formula for the velocity of a small sphere falling through a viscous liquid using the method of dimensions. Explain how this is utilised to determine the viscosity of a liquid like castor oil. Mention one more application of Stokes' law. (*Guwahati U.* 2000)
5. State and prove Bernoulli's theorem for a liquid along a stream line. (*Osm. U.*, 2004; *Nag. U.*, 2003, 2001; *Indore U.*, 2001)
6. Show that for a liquid in stream line motion

$$\frac{P}{\rho} + gh + \frac{v^2}{2} = \text{constant}$$

7. Give Euler's equation for a non-steady flow or non-viscous and incompressible liquid through a tube.

   Hence, derive Bernoulli's theorem.
8. A small spherical drop of radius a, of a material of density r falls from rest in still air of viscosity η. Now,

   (*i*) Write down the equation of motion of the drop,

   (*ii*) Evaluate the terminal velocity,

   (*iii*) Evaluate the time and distance traversed when the drop has acquired 90% of the terminal velocity.
9. Obtain Stokes' law for the motion of a body in a viscous medium from dimensional considerations.

SHORT QUESTIONS

1. Distinguish between streamline flow and turbulent flow of a liquid and explain the significance of Reynold's number. (*Nagpur U.* 2009; *Agra U.* 2006)
2. Define viscosity and coefficient of viscosity and give its units.
3. What is critical velocity? How it makes a difference of streamline and turbulent flow.
4. Derive critical velocity by using method of dimensions.
5. Explain the term viscosity and coefficient of viscosity. Give the dimensions of coefficient of viscosity. (*Bhopal U.*, 2004; *Nag. U.*, 2009, 2003, 2001; *Gharwal U.*, 2000)
6. Distinguish between streamline and turbulent flow of a liquid. Discuss briefly the idea or critical velocity and explain the significance of the Reynold's number. (*Meerut U.* 2003; *Osm. U.* 2004; *Nag. U.* 2003, 2001; *Bang. U.* 2000; *D.A.U. Agra* 2008,2007, 2006)
7. State and explain rate of flow of liquid. What is the equation of continuity? (*D.A.U. Agra* 2008)
8. Show how Bernoulli's theorem is applied to measure the rate of discharge through city water mains. Explain the principle and application of venturimeter. (*Osm. U.* 2004)
9. Describe the working of a Pitot's tube.
10. Show how the coefficient of viscosity of two liquids may be compared.
11. How does the viscosity vary with temperature and pressure? (*Meerut U.* 2003, *Gharwal U.* 2000)
12. State and prove Bernoulli's theorem. Explain the lifting action of an aeroplane. (*Nag. U.* 2005)

13. State Stokes' law of viscous force. (*Nag. U.* 2005)

14. Explain the following applications using Bernoulli's theorem:
(*i*) Atomiser
(*ii*) Venturimeter
(*iii*) Bunsen burner
(*iv*) Carburettor

15. What do you mean by equation continuity? Derive it for steady current. (*Agra U.* 2004, 2003; *S.M.U. Tiruneveli*, 2007)

16. Explain the parabolic nature of velocity profile of a streamline flow.

17. Discuss main sources of errors and the corrections applied in Poiseuille's formula.

18. Calculate the velocity of efflux and range of flow of a liquid through an orifice in a reservoir.

19. Exaplain the terms in detail:
(*i*) Coefficient of viscosity
(*ii*) Velocity of efflux
(*iii*) Velocity gradient (*Purvanchal U.* 2004)

20. What do you mean by Reynold's number? Give its significance. (*Agra U.* 2007)

21. Derive Poiseuille's formula for the rate of steady flow of liquid through a capilary tube of circular cross-section. Why does it fails in the case of a gas? (*Agra U.* 2006)

22. Discuss the rate of flow of liquid in a capillary tube. Explain the analogy between liquid flow and electric current. (*M.S.U. Tiruneveli*, 2007)

23. State Bernoulli's theorem and apply it to obtain an expression for the reduction of pressure when water flows through a constriction in a pipe. (*M.S.U. Tiruneveli*, 2007)

24. Define critical velocity and Reynold's number. (*Nagpur Uni.* 2009)

25. Describe any two applications of Bernoulli's theorem. (*Nagpur Uni.* 2009)

NUMERICAL QUESTIONS

1. A Pitot's tube is fixed to a water pipe of diamete 10 cms and the difference of pressure indicated by the gauge is 4 cm. of water column. Find the volume of water flowing per second through the pipe.

2. Calculate the speed at which the velocity head of a streamline water is equal to 2 m of Hg. [**Ans.** 23.08 m/s]

3. Air is streaming past a horizontal airplane wing such that its speed is 120 ms^{-1} over the upper surface and 190 ms^{-1} at the lower surface. If the density of air is 1.3 kg/m^{-3}; Find the difference in pressure between the top and the bottom of the wing. If the wing is 10 m long and has an average width of 2 m, calculate the gross lift of the wing. (*Roorkee Engineering Entrance Exam.* 2000)[**Ans.** 81900 N]

4. A tank containing water has an orifice on one verticle side. If the centre of the orifice is 4.9 m below the surface level in the tank, find the velocity of discharge, assuming that there is no wastage of energy. [**Ans.** 9.8 m/s]

Hint: $\frac{1}{2}mv^2 = mgh \qquad \therefore \quad v = \sqrt{2gh}$.

5. Calculate the mass of water flowing in 10 minutes through a tube 0.1 cm in diameter, 40 cm long if, there is a constant pressure head of 20 cm of water. The coefficient of viscosity of water is 0.0089 C.G.S. Units. [**Ans.** 81.078 × 10^{-3} kg]

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Hint: $V = \dfrac{\pi P a^4}{8\eta l}$

6. In a Poiseuille's experiment the following observations were made. Volume of water collected in 5 minutes = 40 c.c.; Head of water = 0.4 m; length of capillary tube = 0.602 m and radius of capillary tube = 0.52×10^{-3} m. Calculate the coefficient of viscosity of water. [**Ans.** 1.4×10^{-3} Nm^{-2} (or deca-poise)]

7. If two capillaries of radii r_1 and r_2 and length l_1 and l_2 are joined in series; derive an expression for the rate of flow liquid through the arrangement using Poiseuille's formula.

[**Ans.** $V = \dfrac{\pi P}{8\eta}\left(\dfrac{l_1}{r_1^4} + \dfrac{l_2}{r_2^4}\right)$]

8. In an experiment with Poiseuille's apparatus the following figures were obtained: Volume of water issuing out/mt. = 7.08 c.c.; Head of water = 34.1 cm.; Length of the tube = 56.45 cm.; Radius of the tube = 0.0514 cm. Find the coefficient of viscosity.

[**Ans.** 0.01377 poise]

9. A capillary tube of bore 1 mm. and length 20 cm. is fitted horizontally to a sufficiently big vessel kept full of alcohol of density 0.8 gm./c.c. The depth of the centre of the capillary tube below the surface of alcohol is 30 cms. If the viscosity of alcohol is 0.012 poise find the amount that will flow in 5 minutes. [**Ans.** 72.19 c.c.]

10. A water drop is observed to fall through a gas of density 0.001 gm/c.c. with a constant velocity of 980 cm/sec. What is the radius of the drop? The co-efficient of viscosity of the gas is 2×10^{-4} poise. [**Ans.** 3×10^{-2} cm]

11. Three capillaries of lengths 8 l, 0.2 l, 2 l and radii r, 0.2 r and 0.5 r respectively are connected in series. If the total pressure across the system is p, deduce the pressure across the shortest capillary. (*Meerut U.* 2004)

Hint: $p = p_1 + p_2 + p_3 = \left(\dfrac{8\eta l_1}{\pi r_1^4} + \dfrac{8\eta l_2}{\pi r_2^4} + \dfrac{8\eta l_3}{\pi r_3^4}\right) = 0.7575\, p$

12. What is Stokes' law? A horizontal tube has different cross-sectional areas at points A and B. The diameters of the tube at A and B are 4 cm and 2 cm respectively. The pressure difference between the two points is 8 cm when a liquid of density 8 gm/cm^3 flows through this. Calculate the rate of flow of liquid in the tube (g = 980 cm/sec^2)

Hint: $p = p_1 + p_2 = \dfrac{8\eta l_1}{\pi r_1^4} + \dfrac{8\eta l_2}{\pi r_2^4}$ (*Purvanchal U.* 2007)

13. A venturimeter is connected to a horizontal main of radius 20 cm. If the radius of the throat of the venturimeter be 15 cm and difference of water level in the piezometer tubes be 10 cm. Calculate the rate of flow of water per hour through the main.

(*M.S.U. Tiruneveli*, 2007) [**Ans.** 43.11×10^4 litre/hour]

14. Water flows through a horizontal capillary tube of 1 mm internal diameter of length 70 cm under pressure of a column of water 30 cm in height. Find the rate of flow of water through the capillary tube. $\eta = 10^{-3}$ N-s/m^2. (*Nagpur Uni.* 2008)

[**Hint.** Rate of flow = $\dfrac{P\pi a^4}{8\eta l} = \dfrac{h\rho g \pi a^4}{8\eta l}$

Here $h = 30 \times 10^{-2}$ m, $\rho = 10^3$ kg/m^3, g = 9.8 m/s^2, $a = 0.5 \times 10^{-3}$ m, l = 0.7 m]

12

SURFACE TENSION

INTRODUCTION

It's our common experience that a liquid drop always assumes spherical shape. For example, if a small quantity of mercurry is put on a table, it splits up into a number of spheres of different sizes. Similarly, rain drops and dew drops are observed to be spherical in shape. This is because, for a given volume, mathematically, sphere has minimum surface area. This property of the liquid to reduce its surface area is called surface tension. Due to surface tension, the liquid surface behaves like a stretched elastic membrane having a natural tendency to contract as if it is under tension. This property helps small insects to walk freely on the surface of water. A gently placed metallic needle on water surface floats, although density of needle is more than that of water. Water easily rises up in huge trees against gravity due to capillary action, all these are the effect of surface tension. In this chapter, we will discuss its origin in terms of molecular theory, pressure over curved surfaces etc. in detail.

12.1 SURFACE TENSION

A stretched body is in a state of tension and has a natural tendency to contract. For example, when we stretch a rubber tube it has a tendency to shorten its length and if a rubber sheet is stretched it has a tendency to reduce its area. On account of the *cohesive forces* between the molecules of a liquid, the free surface of a liquid always behaves like *stretched membrane or sheet and tends to contract to the smallest possible area*. Since, for a given volume, a sphere has the least surface area, the liquid assumes a spherical shape. This is why the rain drops and mercury globules are spherical. Ordinarily the effect is not so marked as the liquids tend to spread due to the force of gravity. If the force of gravity is eliminated the liquid will assume a perfectly spherical shape.

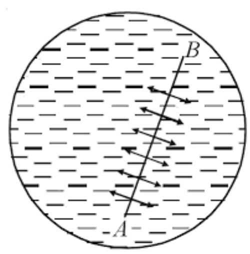


Fig. 12.1

Suppose a line AB is drawn in the free surface of a liquid. The molecules lying just on its one side try to pull away from the molecules lying just on the other side in order to decrease the surface area.

Hence surface tension is defined as the force per unit length acting on either side of a line drawn in the liquid surface in equilibrium, the direction of the force being tangential to the surface and perpendicular to the line.

It is measured in Newton per meter in S.I. units.

Laplace explained the phenomenon of surface tension on the basis of the molecular theory, discussed below:

12.2 MOLECULAR THEORY OF SURFACE TENSION

The molecules of a liquid attract each other. This force of attraction between them is called the force of *cohesion.* The force of cohesion varies inversely as some high power of the distance. It, therefore, becomes negligible at an appreciable distance from a molecule.

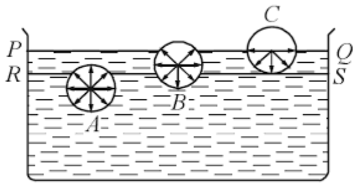


Fig. 12.2

The maximum distance up to which the force of cohesion between two molecules can act is called their molecular range. It is of the order of 10^{-9} m. being different for different substances.

The sphere drawn with the molecule as centre having a radius (r) *equal to the molecular range is called the sphere of influence.*

A molecule attracts and is in turn attracted by the molecules lying within its *sphere of influence.* Let us consider three molecules of a liquid; A well inside it, B just below the surface and C on the surface, with their sphere of influence drawn around them.

The sphere of influence of A lies wholly in the liquid. The molecule is attracted equally in all directions as shown. *Hence there is no resultant cohesive force acting on it.*

The sphere of influence of B lies partly outside the liquid. The number of molecules in the upper half, attracting it upwards, is less than the number of molecules in the lower half attracting it downwards. *Hence, there is a resultant downward force acting on it.*

The sphere of influence of C is exactly half outside the liquid and half inside it. *Hence, it is attracted downwards with the maximum force, perpendicular to the surface.*

If a plane RS is drawn parallel to the free surface PQ of the liquid at a distance equal to the molecular range (distance r), then the layer of the liquid between the planes PQ and RS is called the *surface film. Hence, all the molecules in the surface film are pulled downward due to the cohesive force between molecules.* The downward pull, however, increases as we go up from the plane RS towards the free surface (PQ) of the liquid.

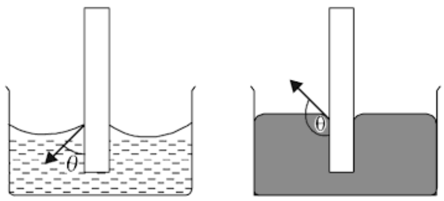
If a molecule is brought from within the liquid to the surface film, work has to be done against the downward cohesive force and its potential energy increases. *Hence, the potential energy of the molecules lying within the surface film is greater than the potential energy of the molecules lying below.* But, a system in equilibrium always tries to have the *lowest potential* energy. Thus, in order to decrease the potential energy of the molecules in the surface film the area of the film must be least. *This is why the free surface of a liquid always tends to have the minimum surface area.*

12.3 ANGLE OF CONTACT

If a plate of glass is dipped in water with its sides vertical, it will be observed that water is drawn up along the plate and assumes a curved shape as shown in Fig. 12.3 (*i*).

If, on the other hand, the plate is dipped in mercury the surface is again curved but is depressed below as shown in Fig. 12.3 (*ii*).

The angle θ between the tangent to the liquid surface at the point of contact and the solid surface inside the liquid is known as **angle of contact** *between the solid and the liquid.*



(*i*) (*ii*)

Fig. 12.3

The angle of contact for *pure water* and *clean* glass is zero, for ordinary water and glass it is 8° and for silver and water is 90°. For mercury and clean glass the angle of contact is *obtuse* being about 140°.

The angle of contact depends upon

(*i*) *the nature of the liquid and the solid.*

(*ii*) *the material which exists above the free surface of the liquid.* The angle of contact between mercury and glass, when air is above mercury, is different from the angle of contact when there is a layer of water above mercury.

(*iii*) It is independent of the angle of inclination of the solid to the liquid surface.

To find whether the angle of contact between a liquid and a solid is obtuse or acute consider the case of a solid, liquid and air in contact. Let θ be the angle of contact of the liquid with the solid and T_1 the surface tension for air liquid, T_2 for air solid and T_3 for liquid-solid surface respectively as shown in Fig. 12.4 (*a*).

For equilibrium

$$T_3 + T_1 \cos\theta = T_2$$

or
$$\cos\theta = \frac{T_2 - T_3}{T_1}$$

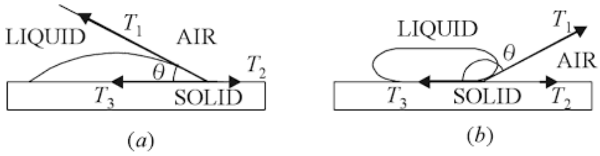


Fig. 12.4

Thus, if T_2 is greater than T_3, cos θ is positive and θ is less than 90° (acute) as shown in Fig. 12.4 (*a*), but if T_2 is less than T_3, cos θ is negative and θ is obtuse as shown in Fig. 12.4 (*b*). If, however, T_2 is greater than $(T_1 + T_3)$, there will be no equilibrium and the liquid will spread over the solid.

12.4 CURVATURE OF LIQUID SURFACE

Dip a glass capillary tube vertically in a liquid. Consider a molecule A on the surface touching the wall of the tube. Force between two same molecules (liquid-liquid) is called force of cohesion and the force between two different molecules (liquid-solid) is called force of adhesion. In addition to its weight which may be neglected, it is acted upon by the following forces:

(*i*) The outward pull due to the molecules of the solid wall called the *force of adhesion.* This force acts along AB at right angles to the wall of the tube as shown in Fig. 12.5 (*i*). Let it be denoted by F_1.

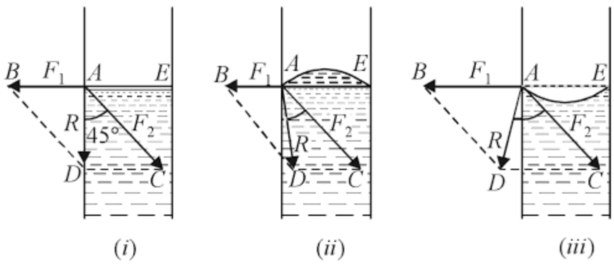


Fig. 12.5

(*ii*) The pull due to other molecules of the liquid, called the *force of cohesion.* The molecule A is pulled along AE by this force by the molecules in the surface, along AD [Fig. 12.5 (*i*)] by the molecules vertically below it with the same force and along directions lying between AD and AE by other molecules. The resultant of all these forces acts along AC and is inclined to the vertical at an angle of 45°. Let this be denoted by F_2. The two forces F_1 and F_2 are thus inclined at an angle of 135°.

The direction of resultant R of the forces F_1 and F_2 depends upon the magnitude of F_1 and F_2.

(1) If $F_1 = F_2 \sin 45° = \frac{F_2}{\sqrt{2}}$, the resultant acts vertically downwards as in Fig. 12.5 (*i*) and the liquid surface is plane.

(2) If F_1 is less than $\frac{F_2}{\sqrt{2}}$ as shown in Fig. 12.5 (*ii*), the resultant R *lies inwards* and the liquid surface will become *convex.*

(3) If F_1 is greater than $\frac{F_2}{\sqrt{2}}$, as shown in Fig. 12.5 (*iii*), the resultant R *lies outwards* and the liquid surface will become *concave.*

This is because a liquid cannot withstand any shearing stress, its surface at every point sets itself at right angle to the resultant force.

In the case of water, the force of *adhesion* F_1 is greater than $\frac{1}{\sqrt{2}}$ times the force of *cohesion* F_2 and the meniscus is *concave*. This is also the case with all liquids which wet the walls of the tube.

In the case of mercury the force of adhesion F_1 is less than $\frac{1}{\sqrt{2}}$ times the force of cohesion F_2 and the meniscus is *convex.* This is also the case with all liquids which do not wet the walls of the tube.

12.5 SURFACE ENERGY

The surface of a liquid is in a state of tension. The force of surface tension tends to decrease the surface area. Hence when the area of a liquid surface is increased work is done against surface tension. This work is stored in the surface as surface energy.

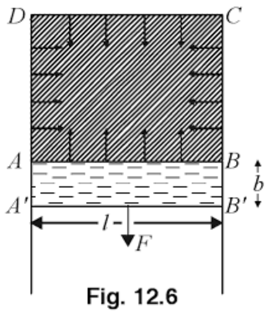


Fig. 12.6

Consider a wire frame $ABCD$ in which the wire AB is movable. Form a soap film over it. The wire AB is pulled inwards due to surface tension by a force $2T \times l$, where T is the surface tension and l is the length AB. The factor 2 appears because there are two surfaces of the soap film. If the film is pulled by a small distance b to the position $A'B'$ keeping the temperature constant, then

The work done = $2T \times l \times b$

Increase in area = $2 \times l \times b$

$$\therefore \text{ Energy spent per unit area } = \frac{2Tlb}{2lb} = T \qquad \text{... (1)}$$

This energy is stored in the surface.

Hence the surface energy per unit area of a surface is numerically equal to the surface tension.

12.6 PRESSURE ON CURVED LIQUID SURFACE

All the molecules lying within the surface film of a liquid are pulled downwards due to the cohesive force between the molecules. *This downward force exerted per unit area of a liquid surface is called cohesion pressure.*

If the free surface of a liquid is *plane*, the resultant force on a molecule due to surface tension is *zero* as shown in Fig. 12.7 (*i*) and the cohesion pressure is negligible.

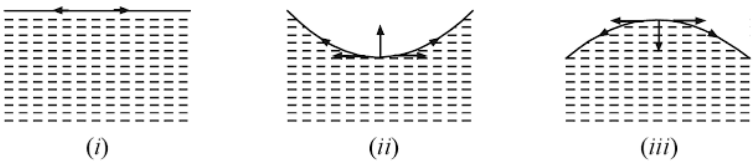


Fig. 12.7

If the free surface of a liquid is *concave*, the resultant force on it acts *outwards* (away from the liquid) as shown in Fig. 12.7 (*ii*) and the cohesion pressure is *decreased.*

If the free surface of a liquid is *convex*, the resultant force on it acts *inwards* (into the liquid) as shown in Fig. 12.7 (*iii*) and the cohesion pressure is *increased.*

12.7 EXCESS OF PRESSURE INSIDE A LIQUID DROP

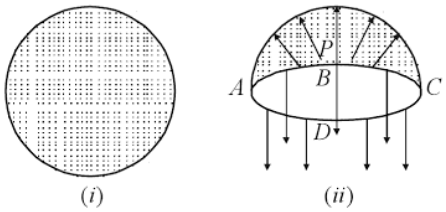


Fig. 12.8

A spherical liquid drop has a convex surface as in Fig. 12.8 (*i*). The molecules on the surface, therefore, experience a resultant force acting inwards due to surface tension. Hence, the pressure within the drop is greater than the pressure outside it by a certain amount, say p. Now consider the equilibrium of the upper half of the drop as shown in Fig. 12.8 (*ii*). There are two forces acting on it.

(*i*) *The upward force* on the plane face *ABCD* due to the excess pressure in the other half. If r is the radius of the drop, then this upward force

$$= p \times \pi r^2$$

(*ii*) The *downward force* due to surface tension acting round the edge of the circle *ABCD*. If T is the surface tension, then this force

$$= T \times 2\pi r$$

If we neglect the weight of the drop, the hemisphere is in equilibrium under the action of these two forces.

$\therefore$
$$p \times \pi r^2 = T \times 2\pi r$$

or
$$p = \frac{2T}{r} \qquad \text{... (2)}$$

The treatment would be similar if we consider an air bubble surrrounded on all sides by a liquid.

In other words, the pressure inside a spherical *drop* exceeds that outside it by $\frac{2T}{R}$.

12.8 EXCESS OF PRESSURE INSIDE A BUBBLE IN AIR

The case of a soap bubble is different from a spherical drop. It has air inside as well as outside it and, therefore, has *two surfaces* like a spherical shell.

The total force due to surface tension will be $2 \times T \times 2\pi r$.

Hence for equilibrium

$$p \times \pi r^2 = 2 \times T \times 2\pi r$$

or
$$p = \frac{4T}{r} \qquad \text{... (3)}$$

In other words, the pressure inside a spherical *bubble* exceeds that outside it by $\frac{4T}{R}$.

Thus the excess pressure inside a spherical drop or a bubble is inversely proportional to its radius *i.e.*,

$$p \propto \frac{1}{r}$$

12.9 EXCESS OF PRESSURE ACTING ON ONE SIDE OF CURVED LIQUID FILM OVER THE OTHER SIDE

If there is a curved liquid film having two free surfaces. Suppose the film has two curvatures. The inward pressure on it due to surface tension is balanced by an equal pressure acting outwards. To find the value of excess pressure consider a *small* curvilinear rectangular element $ABCD$ of a liquid surface. Its side AB is of length x and radius of curvature r_1 with centre at O_1 and the side BC is of length y and radius of curvature r_2 with centre at O_2 as shown in Fig. 12.9.

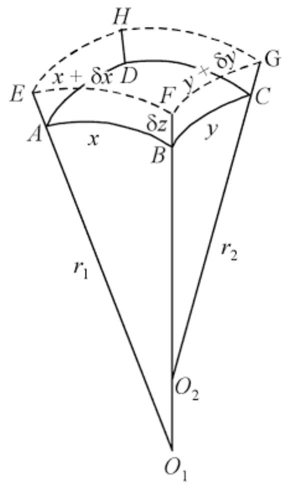


Fig. 12.9

$\therefore$ Area of the element $ABCD = xy$

If the excess pressure on the concave side is p when the liquid surface is at rest, then the outward thrust on the element $ABCD = pxy$.

If the liquid surface is moved outward parallel to itself through a very small distance δz, such that the curvature in the two planes remains the same, then

$$\text{Work done} = pxy.\delta z$$

The side AB increases in length from x to $x + \delta x$ and the side BC from y to $y + \delta y$.

$\therefore$ New area of the element $EFGH$

$$= (x + \delta x)(y + \delta y)$$
$$= xy + x\delta y + y\delta x + \delta x\, \delta y$$

But $\delta x \delta y$ being the product of two very small quantities can be neglected.

$\therefore$ Increase in area of the surface $= x\delta y + y\delta x$

Hence increase in surface energy $= T(x\delta y + y\delta x)$

Now the work done in moving the surface outward is equal to the increase in surface energy.

$\therefore$
$$pxy.\delta z = T(x\delta y + y\delta x)$$

or
$$p = T\left(\frac{1}{y}.\frac{\delta y}{\delta z} + \frac{1}{x}.\frac{\delta x}{\delta z}\right) \quad ...(i)$$

As the side EF is parallel to AB and AB is very small, the figures ABO_1 and EFO_1 can be taken as *similar triangles*.

$\therefore$
$$\frac{EF}{AB} = \frac{FO_1}{BO_1}$$

or
$$\frac{x + \delta x}{x} = \frac{r_1 + \delta z}{r_1}$$

Subtracting 1 from both sides, we have

$$\frac{\delta x}{x} = \frac{\delta z}{r_1}$$

or
$$\frac{1}{x}\frac{\delta x}{\delta z} = \frac{1}{r_1}$$

Similarly
$$\frac{1}{y}\frac{\delta y}{\delta z} = \frac{1}{r_2}$$

Substituting these values in equation (i), we have

$$p = T\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

Now in the case of an air or a soap bubble there are two surfaces, an inner and an outer one, and the increase in area is twice as much as given above. Hence

$$p = 2T\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \qquad \text{... (4)}$$

When the centres of curvature of AB and AC lie on opposite sides, one of the surfaces is convex and the other is a concave. The radii r_1 and r_2 of the two surfaces in such a case will have opposite signs and hence

$$p = 2T\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \qquad \text{... (5)}$$

if r_2 is greater than r_1.

12.10 CAPILLARY ACTION

A capillary is a tube of very fine bore. When a capillary tube, open at both the ends, is dipped into a liquid vertically, the liquid either rises or falls inside it due to the phenomenon of surface tension. This phenomenon is known as *capillary action*. It is observed that the liquids (like water) that wet the glass, rise inside the capillary tube while those which do not wet (like mercury) the glass, show a depression inside the capillary tube. This peculiar behaviour is due to the cohessive and adhesive force between the liquid and the material of the capillary tube.

12.11 RISE OF LIQUID IN A CAPILLARY

When a capillary tube, open at both ends, is dipped vertically in a liquid, the surface of the liquid inside the tube is generally curved. If the liquid wets the tube as in the case of water the surface is *concave upwards*, and the pressure in the liquid just below the meniscus is less than atmospheric pressure above it by an amount $\frac{2T}{r}$, where T is the surface tension of the liquid and r the radius of curvature of the meniscus. Hence, the liquid rises in the capillary tube and the weight of the liquid in it balances this difference of pressure.

Let h be the height of the liquid in the tube from the horizontal surface in the vessel to the tangent plane at the bottom B of meniscus, r the radius of the tube and ρ the density of the liquid.

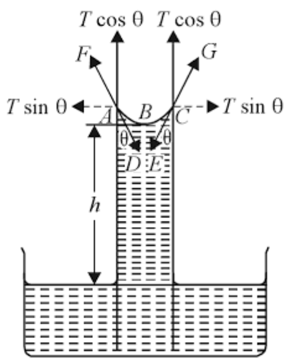


Fig. 12.10

The volume of the liquid between the liquid surface in the vessel and the tangent plane at $B = \pi r^2 h$.

If the capillary is *fine*, the meniscus is hemispherical and its radius of curvature is equal to the radius of the tube.

$\therefore$ Volume of the liquid in the meniscus = Volume of the cylinder of height r–volume of the hemisphere.

$$= \pi r^2 \times r - \frac{2}{3}\pi r^3 = \frac{1}{3}\pi r^3$$

$\therefore$ Total volume of the liquid = $\pi r^2 h + \frac{1}{3}\pi r^3 = \pi r^2\left(h + \frac{r}{3}\right)$

Hence weight of the liquid $= \pi r^2\left(h + \frac{r}{3}\right)\rho g$

The weight of the liquid is supported by forces due to surface tension. To explain this :

Draw a tangent plane AD to the liquid surface at any point A and let it make an angle θ with the vertical wall of the tube. Then θ is the *angle of contact* and the surface tension T at A acts *inwards* along AD. It, therefore, exerts a pull on the glass. According to Newton's third law of motion, '*action and reaction are equal and opposite*'. Hence the glass pulls the liquid at A *outwards along* AF with the same force. The outward reaction due to surface tension can be resolved into two components.

(*i*) $T \cos \theta$ acting vertically upwards

and (*ii*) $T \sin \theta$ acting in the horizontal direction.

Considering the meniscus which touches the inner surface of the tube all round the circumference $2\pi r$, the horizontal components acting in one half of the circumference are equal and opposite to those acting in the other half and hence cancel each other.

$\therefore$ Total force acting vertically upward = $2\pi r . T \cos \theta$

This vertically upward force supports the weight of the liquid acting in the downward direction.

$$\therefore \quad 2\pi r\ T\ \cos\theta = \pi r^2\ (h + r/3)\ \rho g$$

or

$$T = \frac{r\,(h + r/3)\ \rho g}{2\cos\theta}$$

If the capillary is very fine r is very-very small and $r/3$ can be neglected as compared to h. In such a case

$$T = \frac{r\ h\ \rho g}{2\cos\theta} \qquad \text{... (6)}$$

$$\therefore \text{ Liquid rises through a height } h = \frac{2T\ \cos\theta}{r\rho g} \qquad \text{... (6 A)}$$

$$\text{For a liquid for which } \theta = 0;\ \cos\theta = 1;\ h = \frac{2T}{r\rho g} \qquad \text{... (6 B)}$$

From where the Energy comes

When a liquid rises against gravity in a capillary tube, its potential energy increases. Question arises, "From where does this energy come?"

Now, there are three media; glass, liquid and air and they have three surfaces of separation, *i.e.*, (*i*) glass-air surface, (*ii*) liquid-air surface and (*iii*) glass-liquid surface.

Each surface has its own surface tension which is different in different cases. *The surface energy per unit area of surface is always equal to its surface tension.*

As the liquid rises in the capillary tube the *glass-liquid* surface *increases* and the *glass-air* surface *decreases* by the same amount. Moreover, the surface of the liquid in the capillary is *concave* instead of plane and hence there is an *increase* in the liquid-air surface, therefore, *the surface energy of glass-air surface decreases whereas the surface energy of glass-liquid surface and liquid air surface increases.*

But, on the whole, there is a decrease in the total energy of the system and this energy is responsible for raising the liquid against gravity.

If the Length of the Tube is Smaller than *h*

The liquid in a capillary rises to a *maximum* height h, given by

$$T = \frac{r\,(h + r/3)\ \rho g}{2\cos\theta}$$

Neglecting the factor $r/3$, we get

$$T = \frac{rh\ \rho g}{2\cos\theta}$$

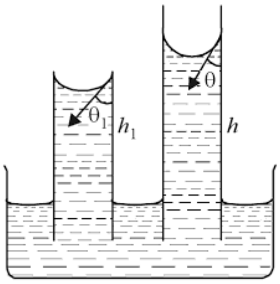


Fig. 12.11

or $$\frac{2T}{r\rho g} = \frac{h}{\cos\theta} = \text{a constant}$$

as T, r, ρ and g are constants at a place for a given liquid and the capillary tube.

When the length of the capillary tube is less than the maximum height to which the liquid can rise, the liquid will rise to the top of the capillary as shown in Fig. 12.11 (smaller tube) and in its attempt to rise further the radius of curvature of the liquid surface will increase. This will also cause an increase in the value of the angle of contact θ and the liquid will have no tendency to further increase its curvature when the new angle of contact θ_1 is such that

$$\frac{h}{\cos\theta} = \frac{h_1}{\cos\theta_1}$$

12.12 ROLE OF ANGLE θ IN ELEVATION AND DEPRESSION

The elevation or depression of a liquid in a capillary tube is given by

$$T = \frac{r(h + r/3)\rho g}{2\cos\theta}$$

The factor $r/3$ is negligibly small. Hence neglecting $r/3$, we have

$$T = \frac{rh\rho g}{2\cos\theta} \qquad ...(i)$$

where T is the surface tension of the liquid, h the elevation or depression of the liquid column in the capillary tube with respect to the liquid level in the vessel outside, ρ the density of the liquid, θ the angle of contact between the liquid and the solid material of the capillary and r the radius of the capillary tube.

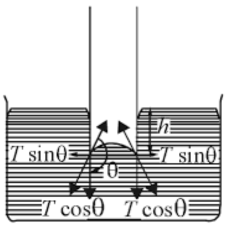


Fig. 12.12

When the liquid wets the sides of the capillary, it rises in the capillary tube as shown in Fig. 12.10. The weight of the liquid column is supported by the upward component $T\cos\theta$ of the reaction due to surface tension.

When the liquid does not wet the sides of the capillary, it is depressed below the outside level, because the vertical component of the reaction of the force of surface tension acts in the downward direction as shown in Fig. 12.12. It is because the angle of contact θ is obtuse ($> \pi/2$). The liquid column moves downwards till the hydrostatic pressure over the area of cross-section of the capillary tube due to the level of the liquid outside balances the downward components of the reaction of the surface tension. In such a case again

$$2\pi r T\cos\theta = \pi r^2 (h + r/3)\rho g.$$

Neglecting $\frac{r}{3}$ we get the relation

$$T = \frac{rh\rho g}{2\cos\theta}$$

which is the same as relation (i).

12.13 SATURATION VAPOUR PRESSURE

Molecules evaporate from the free surface of a liquid and form a pressure. This vapour is called vapour pressure and depends on the number of molecules per unit volume in the space above the liquid surface. At the same time, some molecules from vapour return to the liquid surface and condense on it. The vapour pressure attains a maximum steady value at a given temperature when the number of molecules returning to the liquid surface per second is equal to the number leaving it per second. The liquid, then said to be in equilibrium with its vapour and the vapour pressure, is called *saturated vapour pressure* at that temperature.

The saturated vapour pressure depends upon the nature of curvature of the liquid surface.

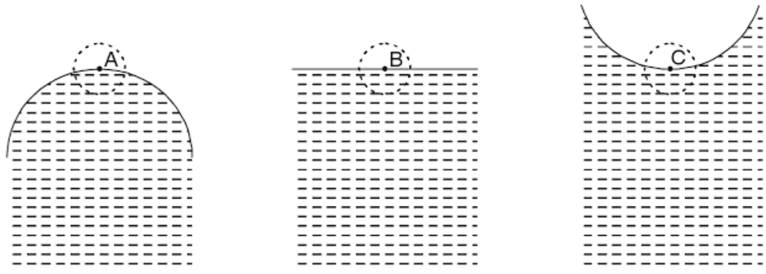


Fig. 12.13

A molecule leaves the surface only when it possesses sufficient kinetic energy to do the necessary work against the attraction of its neighbouring molecules. Suppose, we draw the sphere of molecular attraction round a particular molecule A in a convex surface, B molecule in a plane surface and molecule C in a concave surface as shown in Fig. 12.13 (*i*), (*ii*) and (*iii*), respectively. We can see from figures that the attracting liquid molecules in the sphere of molecular attraction is less for convex surface than compared to plane surface. Hence, the molecule A can be more easily evaporated than the molecule B. Thus, the number of molecules leaving the convex surface per second is greater than the plane surface. Therefore, the saturated vapour pressure over a convex surface is greater than that over a plane surface.

Similarly, if we consider the molecule C over a concave surface [Fig. 12.13 (*iii*)], then in this case the number of attracting molecules be more than plane surface and consequently, the saturated vapour pressure over a concave surface is less than that over a plane surface.

12.14 RELATION FOR VAPOUR PRESSURE OVER A CURVED SURFACE

Dip a capillary tube vertically in a liquid which wets the tube. The liquid rises to a height h in it. Enclose the whole arrangement in a bell-jar and exhaust it of the air it contains. The liquid evaporates and equilibrium is reached when the space within the bell-jar is saturated with vapour.

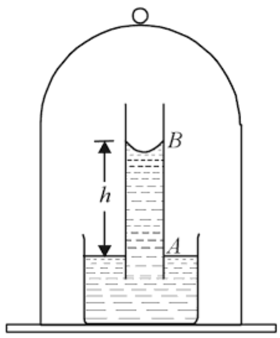


Fig. 12.14

Let P be the vapour pressure over the horizontal surface A and σ the density of the vapour, then

Vapour pressure *just above* the concave surface $B = P - h\sigma g$

If ρ is the density of the liquid, then

Pressure *just below* the concave surface $B = P - h\rho g$

$\therefore$ Pressure *just above* the concave surface B is greater than the pressure *just below* it and excess of pressure

$$= (P - h\sigma g) - (P - h\rho g) = hg\,(\rho - \sigma)$$

As the tube is narrow the meniscus is *nearly hemispherical* and has a radius r equal to that of the tube.

$\therefore$ Excess of pressure just above the meniscus over that *just below it* $= \dfrac{2T}{r}$

Hence $$\frac{2T}{r} = hg\,(\rho - \sigma)$$

or $$h = \frac{2T}{rg\,(\rho - \sigma)}$$

Now, vapour pressure *just above* the concave surface B

$$= P - h\sigma g$$

$$= P - \frac{2T}{r}\,\frac{\sigma}{(\rho - \sigma)}$$

Hence, vapour pressure over the concave surface is less than the vapour pressure over the plane surface by an amount p given by

$$p = \frac{2T}{r}\,\frac{\sigma}{(\rho - \sigma)}$$

It can be proved in a similar way that the vapour pressure over a convex surface is greater than the vapour pressure over the plane surface by the same amount.

12.15 FORMATION OF RAINDROPS (Role of dust nuclei)

It has been proved above that the vapour pressure over a concave liquid surface is less and over a convex liquid surface is greater than the vapour pressure over a plane surface. If we have a space *saturated* with vapour, a liquid drop placed in it will evaporate. The reason is that surface of a drop is convex and the maximum vapour pressure over a convex surface must be more than that on a plane surface. The drop evaporates to increase the pressure and as it evaporates its radius decreases and the value of saturation vapour pressure over it must rise further. The drop, therefore, evaporates more and more rapidly to increase the vapour pressure. The space then contains more vapour than that required for saturation at that temperature without condensation taking place and the vapour is said to be *supersaturated.*

If the atmosphere is absolutely free from dust or charged particles, a saturated vapour or even supersaturated vapour does not condense into drops, because as soon as a tiny drop is formed, the saturation vapour pressure required over it must be greater than the existing pressure. The drop again evaporates and no condensation takes place. When dust particles or ions are present the vapour condenses on them and the size of the drop formed even in the beginning is sufficiently large. The surface of a large drop is almost flat and it has no tendency to evaporate. As more and more of the vapours condense over it, its size increases and its tendency to evaporate is further reduced. Thus, dust particles or charged ions play an important role in condensation of vapour and formation of raindrops.

SOLVED EXAMPLES

Example 12.1. Calculate the work done in spraying a drop of mercury of 1 mm radius into one million identical drops all of the same size. Surface tension of mercury is 550 × 10^{-3} Nm^{-1}.

(*Cal. U.* 2003; *Bhopal U.* 2004; *Nagpur U.* 2007)

Solution. Radius of big drop = 1 mm = 10^{-3} m   Surface tension $T = 550 \times 10^{-3}$ Nm^{-1}

Number of droplets formed = 10^6

Let r be the radius of each small droplets, then   Volume of big drop = Volume of all small drops

$\therefore$ $$\frac{4}{3}\pi R^3 = 10^6 \times \frac{4}{3}\pi r^3$$

or $$R = 100\,r$$

$\therefore \qquad r = \dfrac{10^{-3}}{100} = 10^{-5}$ m

Surface area of big drop = $4\pi \times (10^{-3})^2 = 4\pi \times 10^{-6}$ m^2

Surface area of all small drops = $4\pi \times (10^{-5})^2 \times 10^6 = 4\pi \times 10^{-4}$ m^2

Increase in surface area $= 4\pi\,(10^{-4} - 10^{-6}) = 4\pi \times 10^{-6}\,(100 - 1) = 4\pi \times 10^{-6} \times 99$ m^2

Energy expended $= 4\pi \times 10^{-6} \times 99 \times 550 \times 10^{-3}$

$= 6843.3 \times 10^{-7}$ J = $\mathbf{6.843 \times 10^{-4}}$ **J**

Example 12.2. What is the difference of pressure between the inside and outside of a spherical drop of water of radius 1 mm? Surface tension of water = 73 × 10^{-3} Nm^{-1}.

Solution. $r = 1.0$ mm $= 10^{-3}$ m; $T = 73 \times 10^{-3}$ Nm^{-1}

If p is the difference of pressure between the inside and outside of a liquid drop, then

$$p = \frac{2T}{r} = \frac{2 \times 73 \times 10^{-3}}{10^{-3}} = \mathbf{146\ Nm^{-2}}$$

Example 12.3. A soap bubble is slowly enlarged from a radius of 0.01 m to 0.1 m. Calculate the work done in the process. Surface tension of soap solution is 26 × 10^{-3} Nm^{-1}.

(*Nagpur U.*, 2008)

Solution. As the radius of soap bubble is increased its surface area increases. Hence work done in the process = Surface tension × increase in surface area

$= 26 \times 10^{-3} \times 4\pi\,[(0.1)^2 - (0.01)^2]$

$= 26 \times 10^{-3} \times 4\pi \times 99 \times 10^{-4} = \mathbf{3.235 \times 10^{-3}}$ **J**

Example 12.4. What amount of energy will be liberated if 1000 droplets of water each of 10^{-8} m in diameter coalesce to form one large spherical drop? Surface tension of water = 75 × 10^{-3} Nm^{-1}.

Solution. Radius of each droplet = 0.5×10^{-8} m

Volume of 1000 droplets $= \dfrac{4}{3}\pi\,(0.5 \times 10^{-8})^3 \times 1000$

If R is the radius of bigger drop formed, then

$$\frac{4}{3}\pi R^3 = \frac{4}{3}\pi\,(0.5 \times 10^{-8})^3 \times 1000$$

$\therefore \quad R = 10 \times 0.5 \times 10^{-8} = 5 \times 10^{-8}$ m

Surface area of big drop = $4\pi\,(5 \times 10^{-8})^2$ m^2

Surface area of 1000 droplets = $1000 \times 4\pi\,(0.5 \times 10^{-8})^2$ m^2 = $10 \times 4\pi\,(5 \times 10^{-8})^2$ m^2

Decrease in area = $10 \times 4\pi\,(5 \times 10^{-8})^2 - 4\pi\,(5 \times 10^{-8})^2$

$= 4\pi\,(5 \times 10^{-8})^2\,(10 - 1) = 4\pi \times 9 \times (5 \times 10^{-8})^2$ m^2

$\therefore$ Energy liberated = $4\pi \times 9 \times (5 \times 10^{-8})^2 \times 75 \times 10^{-3} = \mathbf{2.12 \times 10^{-14}}$ **J.**

Example 12.5. A small hollow sphere which has a small hole in it is immersed in water to a depth of 0.4 m before any water penetrates into it. If the surface tension of water is 73 × 10^{-3} Nm^{-1}, find the radius of hole.

Solution. Depth of water $h = 0.4$ m, Density of water = 10^3 kg m^{-3}

Excess pressure of water at a depth h metre

$= hdg = 0.4 \times 10^3 \times 9.8 = 3.92 \times 10^3$ Nm^{-2}

If r is the radius of the hole, then before water enters into the hollow sphere, an air bubble of radius r will escape. Thus, excess pressure inside the bubble

$$p = \frac{2T}{r} = \frac{2 \times 73 \times 10^{-3}}{r} \text{ Nm}^{-2}$$

Hence, in equilibrium

$$hdg = \frac{2T}{r}$$

or $$r = \frac{2T}{hdg} = \frac{2 \times 73 \times 10^{-3}}{3.92 \times 10^{3}} = \mathbf{37.25 \times 10^{-6} \text{ m}}$$

Example 12.6. Calculate the excess pressure inside a soap bubble of radius 3 × 10⁻³ m. Surface tension of soap solution is 20 × 10⁻³ Nm⁻¹. Also calculate surface energy.

Solution. Radius of soap bubble = 3×10^{-3} m; Surface Tension $T = 20 \times 10^{-3}$ Nm^{-1}

Excess pressure $p = \frac{4T}{r} = \frac{4 \times 20 \times 10^{-3}}{3 \times 10^{-3}} = 26.67 \text{ Nm}^{-2}$

Surface energy = Surface tension × area of surface

$$= 4\pi r^2 T = 4\pi \times 9 \times 10^{-6} \times 20 \times 10^{-3}$$

$$= \mathbf{22.62 \times 10^{-7} \text{ Joule.}}$$

Example 12.7. Calculate the work done in blowing a soap bubble of radius 10 cm and surface tension 30 dynes per cm.

Solution. Work done = Surface tension × area of surface.

A soap bubble has two surfaces.

∴ Work done = 2 × surface tension × area of surface

$$= 2 \times T \times 4\pi r^2 = 2 \times 30 \times 4 \times \frac{22}{7} \times 10 \times 10$$

$$= 7.54 \times 10^4 \text{erg} = \mathbf{7.54 \times 10^{-3} \text{ J}}$$

Example 12.8. A spherical bubble is rising slowly through a column of mercury in a long vertical tube. If the radius of bubble at a depth of 1.24 m is 0.1 mm, calculate the depth at which the radius will be 0.12 mm. Surface tension of mercury is 560 × 10⁻³ Nm⁻¹, atmospheric pressure = 1 Standard atmosphere and density of mercury = 13.6 × 10³ Kg m⁻³.

Solution. Total pressure of mercury at a depth of 1.24 m

$$= (1.24 + 0.76)\ 13.6 \times 10^3 \times 9.8 = 266.56 \times 10^3 \text{ N-m}^{-2}$$

Radius of air bubble $r_1 = 0.1$ mm $= 0.1 \times 10^{-3}$ m

Surface tension of mercury $T = 560 \times 10^{-3}$ Nm^{-1}

Excess pressure inside the bubble $= \frac{2T}{r_1} = \frac{2 \times 560 \times 10^{-3}}{0.1 \times 10^{-3}} = 11.2 \times 10^3 \text{ Nm}^{-2}$

Total pressure of air in the bubble,

$$P_1 = 266.56 \times 10^3 + 11.2 \times 10^3 = 277.76 \times 10^3 \text{ Nm}^{-2}$$

Volume of air bubble $V_1 = \frac{4}{3} \pi\ (0.1 \times 10^{-3})^3 \text{ m}^3$

Let h be the depth at which radius $r_2 = 0.12 \times 10^{-3}$ m^3

Pressure at depth $h = (0.76 + h)\ 13.6 \times 10^3 \times 9.8$ Nm^{-2}

Excess pressure inside the bubble $= \frac{2T}{r_2} = \frac{2 \times 560 \times 10^{-3}}{0.12 \times 10^{-3}} = 9333.3 \text{ Nm}^{-2} = 9.333 \times 10^3 \text{ Nm}^{-2}$

Volume of bubble $V_2 = \frac{4}{3}\pi\,(0.12 \times 10^{-3})^3\ m^3$

Total pressure $P_2 = (0.76 + h)\ 13.6 \times 10^3 \times 9.8 + 9.333 \times 10^3$

According to Boyles law $P_2V_2 = P_1V_1$

$$\therefore\quad 10^3\,[(0.76 + h)\,133.28 + 9.333] \times \frac{4}{3}\pi\,(0.12 \times 10^{-3})^3$$

$$= 277.76 \times 10^3 \times \frac{4}{3}\pi\,(0.1 \times 10^{-3})^3$$

or $$133.28h + 101.293 + 9.333 = \frac{277.76\,(0.1)^3}{(0.12)^3}$$

$$\therefore\quad 133.28h = 277.76 \times 0.57847 - 110.626$$

or $$h = \frac{50.05}{133.28} = \mathbf{0.3755\ m}$$

Example 12.9. A wire ring 0.03 m radius is rested flat on the surface of a liquid and is then raised. The pull required is 3.03 gm wt more before the film breaks than it is after. Calculate the surface tension of the liquid.

Solution. When a ring is gradually raised out of the liquid, in addition to the weight an extra downward pull due to surface tension also acts. As the liquid touches the ring on the outside as well inside.

$\therefore$ Extra downward pull = $2 \times 2\pi rT$

where r is the radius of the ring. This is on the supposition that the wire forming the ring is very thin so that the inside and outside radii are the same. Now

$$r = 0.03$$

$\therefore$ Extra downward pull = 3.03 gm wt = $3.03 \times 10^{-3} \times 9.8$ N

$$\therefore\quad T = \frac{3.03 \times 10^{-3} \times 9.8}{4\pi \times 0.03} = \mathbf{78.76 \times 10^{-3}\ Nm^{-1}}$$

Example 12.10. The two arms of a U-tube have diameters 2 mm and 1 mm. The tube is partly filled with water and is held with its arms vertical. Find the difference in the levels of water in the two limbs if the surface tension of water is 70×10^{-3} Nm^{-1}.

Solution. The rise of liquid in a capillary tube is given by the relation

$$\frac{2T\cos\theta}{r} = h\rho g$$

where T is the surface tension of the liquid, θ the angle of contact, r the radius of the tube, h the height of the liquid column, ρ the density of the liquid and g the acceleration due to gravity.

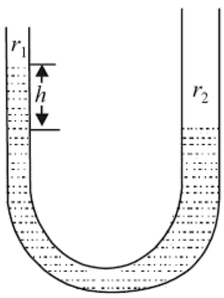


Fig. 12.15

If we have a U-tube, the two limbs of which have different radii say r_1 and r_2, then

Rise of liquid in the limb of radius $r_1 = \frac{2T\cos\theta}{r_1\rho g}$

Rise of liquid in the limb of radius $r_2 = \frac{2T\cos\theta}{r_2\rho g}$

$\therefore$ Difference in the height of liquid column in the two limbs of the U-tube

$$h = \frac{2T\cos\theta}{\rho g}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

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Now $r_1 = 1\text{ mm} = 1 \times 10^{-3}\text{ m}$; $r_2 = 2\text{ mm} = 2 \times 10^{-3}\text{ m}$

$$T = 70 \times 10^{-3}\ \text{Nm}^{-1};\ \theta = 0;\ \rho = 10^3\ \text{Kgm}^{-3};\ g = 9.8\ \text{ms}^{-2}$$

$$\therefore \quad h = \frac{2 \times 70 \times 10^{-3}}{10^3 \times 9.8}\left[\frac{1}{1 \times 10^{-3}} - \frac{1}{2 \times 10^{-3}}\right] = \frac{2 \times 70 \times 1}{10^3 \times 9.8 \times 2}$$

$$= 7.1 \times 10^{-3}\ \text{m} = \mathbf{7.1\ mm.}$$

Example 12.11. By how much will the surface of liquid be depressed in a glass tube of radius 0.02 cm if the angle of contact of the liquid is 135° and surface tension 547×10^{-3} Nm^{-1}? Assume the density to be 13.5×10^3 kg^{-3} and $g = 9.8$ m^{-2}.

Solution. $T = 547 \times 10^{-3}\ \text{Nm}^{-1},\ r = 0.02\ \text{cm} = 2 \times 10^{-4}\ \text{m}$

$$\rho = 13.5 \times 10^3\ \text{kgm}^{-3},\ g = 9.8\ \text{ms}^{-2}$$

$$\theta = 135° \quad \therefore \cos 135 = -\cos 45 = -0.7071$$

Now
$$h = \frac{2T \cos\theta}{\rho g r} = \frac{-2 \times 547 \times 10^{-3} \times 0.7071}{13.5 \times 10^3 \times 9.8 \times 2 \times 10^{-4}}$$

$$= \mathbf{-2.924 \times 10^{-2}\ m}$$

Negative sign indicates that liquid is depressed.

Example 12.12. Calculate the height to which a liquid will rise in a capillary tube of radius 0.2 mm when surface tension of liquid is 26×10^{-3} Nm^{-1} and density 800 kg m^{-3}. Take angle of contact 0.

Solution. Here $r = 0.2\ \text{mm} = 2 \times 10^{-4}\ \text{m},\ T = 26 \times 10^{-3}\ \text{Nm}^{-1}$

$$\theta = 0,\ \rho = 800\ \text{kgm}^{-3},\ g = 9.8\ \text{ms}^{-2}$$

Now
$$T = \frac{r(h + r/3)\rho g}{2\cos\theta} = \frac{\rho g h r}{2}$$

$$\therefore \quad h = \frac{2T}{\rho g r} = \frac{2 \times 26 \times 10^{-3}}{800 \times 9.8 \times 2 \times 10^{-4}} = \mathbf{0.033\ m}$$

EXERCISE CH. 12

LONG QUESTIONS

1. (*a*) Deduce an expression for the difference of pressure on the two sides of a spherical drop.

   (*b*) How will the problem be altered in dealing with a spherical soap bubble? Hence show that the pressure inside a spherical bubble of radius r exceeds that outside it by $\frac{4T}{R}$, T being the surface tension. (*Nag. U.* 2001; *Gauhati U.* 2000; *Cal. U.* 2003)

2. Explain the term surface energy. Derive the relation between surface tension and surface energy. (*Indore U.* 2001)

3. Show that the excess pressure acting on the curved surface of a curved membrane is given by $p = 2T\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$, where r_1 and r_2 are the radii of curvature and T the surface tension of the membrane.

4. (***a***) Derive an expression for the height h through which a liquid of surface tension T will rise in a capillary tube of radius r. Explain clearly from where the enrgy comes when the liquid rises against gravity in the capillary tube.

(***b***) What will happen if the length of the capillary tube is smaller than h?

5. (***a***) Obtain the relation between the vapour pressure over a curved surface and that over a flat surface.

(***b***) What is the function of dust nuclei in the formation of clouds?

6. Define angle of contact for a liquid in contact with solid. Show that the height to which a liquid rises in a capillary tube of radius r is given by

$$h = \left(\frac{2T}{\rho r g}\right) - \frac{r}{3}.$$

SHORT QUESTIONS

1. Explain what do you understand by surface tension of a liquid and state its unit.

(*Bhopal U.*, 2004; *Nag. U.* 2009, 2003)

2. Give an account of molecular theory of surface tension.

(*Cal. U.*, 2003; *Bhopal U.*, 2004)

3. Explain what do you understand by the angle of contact in the case of a liquid.

4. Account for the curvature of the surface of a liquid in the neighbourhood of a solid surface.

5. Explain why a liquid is either raised or depressed in a capillary tube.

NUMERICAL QUESTIONS

1. Air bubble of radius 0.2 mm is situated just below the surface of water. Calculate the gauge pressure (excess of pressure) inside the air bubble. Surface tension of water = 7.2 $\times 10^{-2}$ N/m. [**Ans.** 720 N/m^2]

2. A sphere of water of radius 1 mm is sprayed into a million drops all of the same size. Find the energy expended in doing so. Surface tension of water = 72 $\times$ 10^{-3} N/m.

[**Ans.** 8.96 $\times$ 10^{-5} *J*]

3. A U-tube made up of two capillary tubes of diameter 0.5 mm and 1.0 mm respectively contains water (S.T. 72 $\times$ 10^{-3} N/m). What would be the difference of levels between the two arms? [**Ans.** 1.47 $\times$ 10^{-2} m]

4. Calculate the excess pressure inside a spherical air bubble in water.

Given: Diameter of bubble = 0.5 mm, Surface tension of water = 75 $\times$ 10^{-3} N/m.

(*Nag. U.* 2007) [**Ans.** 600 Nm^{-2}]

5. Calculate the excess of pressure between inside and outside of a soap bubble of radius 1 cm. Surface tension of soap solution is 3.2 $\times$ 10^{-2} N/m. [**Ans.** 12.8 N/m^2]

6. In a capilary tube water rises to a height 0.1 m. In the same capillary tube mercury is dipressed by 3.42 $\times$ 10^{-2} m.

Angle of contact for water = 0°

Angle of contact for mercury = 135°

Calculate the surface tension of mercury given that the surface tension of water as 72 $\times$ 10^{-3} N/m. Density of mercury = 13.6 $\times$ 10^2 kg/m^3. [**Ans.** 0.4728 N/m]

SECTION II

1. MATHEMATICAL BACKGROUND
2. ELECTROSTATICS
3. ELECTRIC CURRENTS
   (Steady and Alternating)
4. MAGNETOSTATICS
5. TIME VARYING FIELDS
6. ELECTROMAGNETIC WAVES

13 MATHEMATICAL BACKGROUND-I

VECTOR ANALYSIS

INTRODUCTION

The language in which physics speaks becomes much simpler when it is expressed in terms of vector analysis. Vector analysis provides an effective tool for writing physics and at the same time makes it possible to visualise the physical meaning of the equations distinctly and exactly, particularly in electromagnetic theory. Maxwell had shown the inbuilt potential of 'field equations' and the associated physical meaning with them. A very powerful mathematical technique has been devised called *vector analysis* for writing physical laws, and is discussed in this chapter.

13.1 SCALAR AND VECTOR QUANTITIES

All physical quantities, in general, can be divided into two classes *i.e.*, (*i*) scalar quantities and (*ii*) vector quantities.

1. Scalar Quantities or Scalars. *Physical quantities which have only a magnitude but no direction* are called **scalar quantities** or simply **scalars.** Examples of these are mass, volume, density, temperature, speed, work, heat etc. These quantities can be added according to ordinary laws of algebra.

2. Vector Quantities or Vectors. *Physical quantities which are completely known by their magnitude as well as direction* are called **vector quantities** or simply **vectors.** Examples of these are displacement, velocity, acceleration, force, momentum, electric field intensity etc.

13.2 GRAPHICAL REPRESENTATION

In a diagram, a vector is represented by a straight line with an arrow head whose direction gives the direction of the vector and whose length gives the magnitude of the vector on some suitable scale. *The magnitude of the vector in itself is a scalar quantity.*

A vector is represented by a letter with an arrow head on its top. Thus, vector A is represented as $\vec{A}$ (Fig. 13.1).

The absolute value of the vector is indicated by A. The magnitude of the vector is always taken as *positive*. A negative sign before the vector indicates that a vector has merely changed the sense of direction *i.e.,* it interchanges the *arrow head and tail* without changing the length.

13.3 SOME LAWS OF VECTOR ALGEBRA

(*i*) *Equal vectors.* Two vectors are considered equal if their magnitude and directions are the same.

As shown in Fig. 13.1 vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ are equal. A given vector can be moved around at will provided its length and direction are not changed. Thus $\vec{A} = \vec{B} = \vec{C}$

This equation means that the vectors have the same magnitude and same direction.

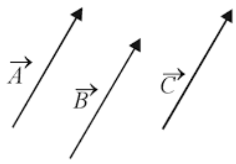

Fig. 13.1

(*ii*) *Negative vector.* A vector having the same magnitude as that of the given vector $\vec{A}$, but directed in the opposite sense is called a negative of the given vector. Symbolically, the negative of vector $\vec{A}$ is represented by $-\vec{A}$ (Fig. 13.2).

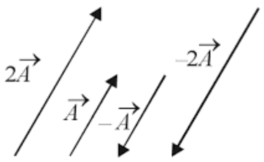

Fig. 13.2

(*iii*) *Scalar multiple of a vector.* The product of a vector $\vec{A}$ and a real number m is a vector $m\vec{A}$. This means that length is m times the magnitude of $\vec{A}$. The direction of the vector is the same or opposite, according as m is positive or negative. Thus

$$m\,(-\vec{A}) = -m\,\vec{A} \text{ and } -m\,(-\vec{A}) = m\,\vec{A}$$

(*iv*) *Position vector.* The position of a point with respect to an arbitrary origin can be represented by a vector called the position vector. If O is an arbitrary origin and P a point in space, then the position vector of P with respect to O is represented by $\vec{OP}$ (Fig 13.3).

Fig. 13.3

(*v*) *Co-initial vectors.* Vectors are said to be co-initial if they have a common initial point.

(*vi*) *Collinear or parallel vectors.* Vectors acting along the same line or along parallel lines are called collinear vectors.

If two vectors $\vec{A}$ and $\vec{B}$ are parallel and act in the same direction as shown in Fig. 13.4 (*a*), the resultant vector $\vec{R}$ is given by $\vec{R} = \vec{A} + \vec{B}$.

When the two vectors have opposite directions as shown in Fig. 13.4 (*b*), the resultant vector $\vec{R}$ is given by, $\vec{R} = \vec{A} - \vec{B}$

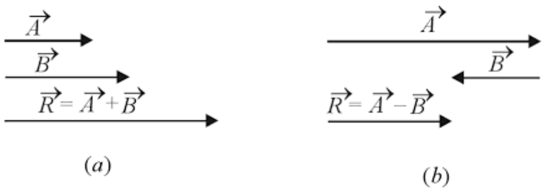

Fig. 13.4

(*vii*) *Coplanar vectors.* Vectors which are confined to the same plane are called coplanar vectors.

(*viii*) *Null vector.* If the initial and terminal points of a vector are coincident, the vector is called a null vector and has a *zero magnitude* and no direction. It is also called as zero vector.

(*ix*) *Unit vector.* A vector of unit magnitude is called a unit vector. A unit vector in the direction of $\vec{A}$ is written as $\hat{A}$, read as A hat or A caret or a A cap. Thus vector $\vec{A}$ is written as

$$\vec{A} = \hat{A}\,A$$

13.4 COMPOSITION OF VECTORS

The vector quantities in mechanics can be compounded together by **parallelogram and triangle** *law of additions.*

The parallelogram law states that the sum of the two vectors $\vec{A}$ and $\vec{B}$ is the vector $\vec{R}$ determined by the diagonal *oc* of the parallelogram of which the adjacent sides *oa* and *ab* represent the two vectors.

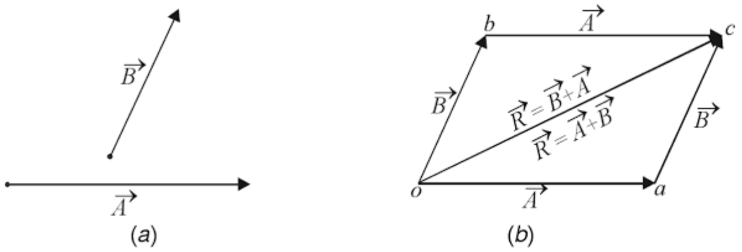

Fig. 13.5

Consider two vectors $\vec{A}$ and $\vec{B}$ acting in different directions as shown in Fig. 13.5 (*a*). On a diagram to a scale layout draw *oa* representing the vector $\vec{A}$ and *ac* representing the vector $\vec{B}$ such that its tail is at the head of vector $\vec{A}$. Draw a line *oc* from the tail of $\vec{A}$ to the head of $\vec{B}$ to get the vector sum $\vec{R}$. This is a vector equivalent in length and direction to the successive vectors $\vec{A}$ and $\vec{B}$. Thus

$$\vec{R} = \vec{A} + \vec{B}$$

If we represent vector $\vec{B}$ along *ob* and draw the vector $\vec{A}$ along *bc* such that its tail is at the head of vector $\vec{B}$, the line *oc* again represents the vector sum $\vec{R}$. Then $\vec{R} = \vec{B} + \vec{A}$

Hence the two vector sums are mathematically equal *i.e.,* $\vec{A} + \vec{B} = \vec{B} + \vec{A}$

This shows that the vector sum is **commutative.**

***(ii)* Associative law.** Suppose the point *O* is acted upon by a number of vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ and $\vec{D}$ as shown in Fig. 13.6 (*a*). From a point *O*, draw a line *oa* to a scale layout representing the vector $\vec{A}$ in magnitude and direction. Draw a line *ab* on the same scale representing the vector $\vec{B}$ such that the tail of $\vec{B}$ coincides with the head of $\vec{A}$, then the line *ob* represents the vector sum $\vec{E}$. Hence $\vec{E} = \vec{A} + \vec{B}$. Similarly, draw *bc* representing the vector $\vec{C}$ and *cd* representing the vector $\vec{D}$ and join *od* representing the vector $\vec{R}$.

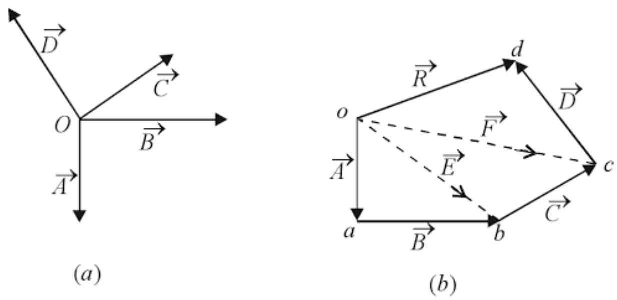

Fig. 13.6

Now $\vec{F}$ is the vector sum of $\vec{E}$ and $\vec{C}$ $\quad\therefore\ \vec{F} = \vec{C} + \vec{E} = \vec{A} + \vec{B} + \vec{C}$

Again $\vec{R}$ is the vector sum of $\vec{F}$ and $\vec{D}$ $\quad\therefore\ \vec{R} = \vec{F} + \vec{D} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$

This law holds good for any number of vectors. Hence the vector addition is **associative.**

(*iii*) Vector subtraction. The negative of a vector is a vector of the same magnitude but opposite in direction. Hence subtraction of vector $\vec{B}$ from vector $\vec{A}$ is the same as addition of vector $-\vec{B}$ to vector $\vec{A}$. Thus $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

13.5 RECTANGULAR OR ORTHOGONAL UNIT VECTORS

The vectors can be resolved into component vectors along the three orthogonal axes of the cartesian co-ordinate system in which the three axes *OX, OY* and *OZ* are mutually perpendicular to each other. The unit vectors along the *X, Y* and *Z* axes are represented by $\hat{i}$, $\hat{j}$, and $\hat{k}$ respectively. The rectangular system such as shown in Fig. 13.7 is called the right-handed co-ordinate system. The word right-handed has been derived from the fact that a right-handed screw when rotated from *X* to *Y*-axis through a small angle will move the screw in the positive direction of *Z*-axis.

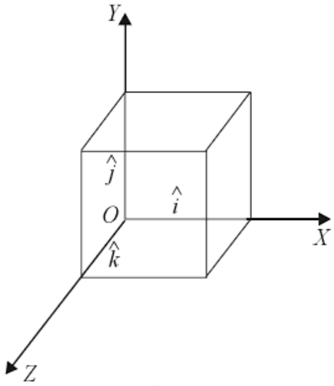

Fig. 13.7

A vector $\vec{A}$ taken along *X*- axis can be written as $\vec{A} = \hat{i}\,A$ where *A* is the magnitude of $\vec{A}$.

13.6 ORTHOGONAL RESOLUTION OF VECTORS AND DIRECTION COSINES

Let the vector $\vec{A} = \vec{OP}$ in a rectangular co-ordinate system be resolved into its component vectors in the directions of *X, Y* and *Z* axes. Let the initial point of the vector *A* be at the origin *O* of the co-ordinate system. Draw a rectangular parallelopiped with its three edges at *O* and lying along the three cartesian axes such that the vector $\vec{A}$ becomes the diagonal through the solid figure.

Let $\vec{A_x}$, $\vec{A_y}$ and $\vec{A_z}$ be the vector intercepts along *X, Y* and *Z* axes respectively and A_x, A_y and A_z be their magnitudes respectively, then $\vec{A_x} = A_x\,\hat{i}$, $\vec{A_y} = A_y\,\hat{j}$ and $\vec{A_z} = A_z\,\hat{k}$.

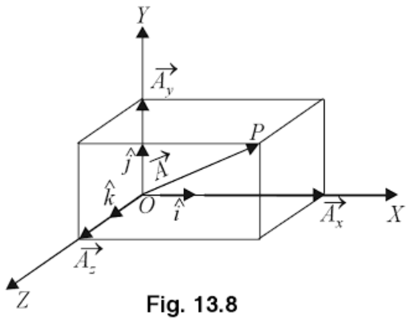

Fig. 13.8

$$\therefore\ \vec{A} = \vec{A_x} + \vec{A_y} + \vec{A_z} = A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}$$

By Pythagoras theorem in three dimensions we have,

$$A^2 = A_x^2 + A_y^2 + A_z^2$$

or $$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

The unit vector $\hat{A}$ in the direction of $\vec{A}$ can be found out in terms of the unit rectangular vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ by dividing the vector $\vec{A}$ by its modulus *A*.

$$\therefore \qquad \hat{A} = \frac{\vec{A}}{A} = \frac{A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

Direction cosines. The cosine of the angles which the vector $\vec{A}$ makes with the *X*- axis, *Y*-axis and *Z*-axis are known as *X*- direction cosine, *Y*-direction cosine and *Z*-direction cosine respectively. These are denoted by *l, m* and *n* respectively.

Suppose the vector $\vec{A}$ makes angle α, β and γ with X, Y and Z axes respectively , then

$$l = \cos\alpha = \frac{A_x}{A}\,;\ m = \cos\beta = \frac{A_y}{A}\,;\ n = \cos\gamma = \frac{A_z}{A}$$

where A_x, A_y and A_z are the intercepts of the vector $\vec{A}$ along the X, Y and Z axes respectively and A is the magnitude (modulus) of the vector $\vec{A}$. Squaring and adding the cosines, we get

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{A_x}{A}\right)^2 + \left(\frac{A_y}{A}\right)^2 + \left(\frac{A_z}{A}\right)^2 = \frac{A_x^2 + A_y^2 + A_z^2}{A^2}$$

or $$l^2 + m^2 + n^2 = \frac{A_x^2 + A_y^2 + A_z^2}{A^2} = \frac{A^2}{A^2} = 1$$

Let $\vec{A_1}$ and $\vec{A_2}$ be the two vectors, whose magnitudes are represented by A_1 and A_2 respectively, then

$$\vec{A_1} = A_{1x}\,\hat{i} + A_{1y}\,\hat{j} + A_{1z}\,\hat{k}$$

or $$\frac{\vec{A_1}}{A_1} = \frac{A_{1x}}{A_1}\,\hat{i} + \frac{A_{1y}}{A_1}\,\hat{j} + \frac{A_{1z}}{A_1}\,\hat{k}$$

But $$\frac{A_{1x}}{A_1} = l_1\,;\ \frac{A_{1y}}{A_1} = m_1\,;\ \frac{A_{1z}}{A_1} = n_1$$

$\therefore$ $$\vec{A_1} = A_1\,(l_1\,\hat{i} + m_1\hat{j} + n_1\,\hat{k})$$

Similarly $$\vec{A_2} = A_2\,(l_2\,\hat{i} + m_2\hat{j} + n_2\,\hat{k})$$

Now $\vec{A_1}\cdot\vec{A_2} = A_1\,A_2\,\cos\theta$ where θ is the angle between the vectors $\vec{A_1}$ and $\vec{A_2}$ [For proof see **article 13.6**]

$\therefore$ $$\cos\theta = \frac{\vec{A_1}\cdot\vec{A_2}}{A_1\,A_2}$$

$$= \frac{A_1\,(l_1\,\hat{i} + m_1\,\hat{j} + n_1\,\hat{k})\cdot A_2\,(l_2\,\hat{i} + m_2\,\hat{j} + n_2\,\hat{k})}{A_1\,A_2}$$

$$= l_1\,l_2 + m_1\,m_2 + n_1\,n_2$$

as $$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\,.\,\hat{k} = 1 \text{ and } \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\,.\,\hat{i} = 0$$

[For proof see **Art. 13.7**]

Example 13.1. If vector $\vec{A} = \hat{i} + 2\hat{j} - 2\hat{k}$, $\vec{B} = 2\hat{i} + \hat{j} + \hat{k}$ and $\vec{C} = \hat{i} - 3\hat{j} - 2\hat{k}$, find the magnitude and direction cosines of the vector ($\vec{A} + \vec{B} + \vec{C}$).

Solution. Vector sum $$\vec{R} = \vec{A} + \vec{B} + \vec{C}$$

$$= \hat{i} + 2\hat{j} - 2\hat{k} + 2\hat{i} + \hat{j} + \hat{k} + \hat{i} - 3\hat{j} - 2\hat{k} = 4\hat{i} - 3\hat{k}$$

$\therefore$ $R_x = 4\hat{i}$, $R_y = 0$ and $R_z = -3k$

Hence magnitude of $$\vec{R} = \left|\vec{R}\right| = \sqrt{4^2 + 0^2 + 3^2} = 5$$

Direction cosines of $\vec{R}$

$$l = \frac{R_x}{R} = \frac{4}{5}\ ;\ m = \frac{R_y}{R} = \frac{0}{5} = 0\ ;\ n = \frac{R_z}{R} = \frac{-3}{5}$$

13.7 POSITION VECTOR AND DISPLACEMENT VECTOR

If the co-ordinates of a point are (x, y, z) then the vector $\vec{r}$ joining the origin to this point is expressed as $(x\hat{i} + y\hat{j} + z\hat{k})$.

This vector is known as the **position vector** of this point.

If a particle P having co-ordinates (x_1, y_1, z_1) is represented by position vector $\vec{r_1}$, then in Fig. 13.9,

$$\vec{r_1} = x_1\,\hat{i} + y_1\,\hat{j} + z_1\,\hat{k}$$

If this particle moves to another point Q which has a position vector $\vec{r_2}$ given by

$$\vec{r_2} = x_2\,\hat{i} + y_2\,\hat{j} + z_2\,\hat{k}$$

where (x_2, y_2, z_2) are the co-ordinates of Q, then the displacement from P to Q is represented by vector $\vec{r}$ given by

$$\vec{r} = \vec{r_2} - \vec{r_1} \qquad \because\ \vec{r_2} = \vec{r_1} + \vec{r}$$

or $$\vec{r} = (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j} + (z_2 - z_1)\,\hat{k}$$

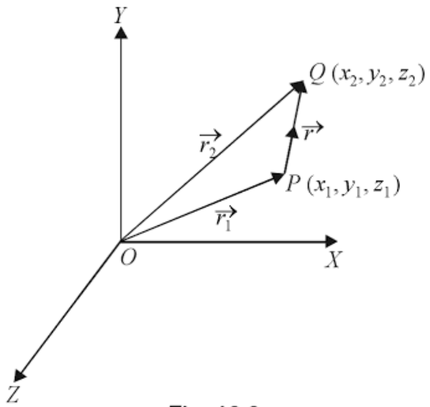

Fig. 13.9

If PQ represents a small vectorial change $d\vec{r} = \vec{r_2} - \vec{r_1}$ and $x_2 - x_1 = dx$, $y_2 - y_1 = dy$ and $z_2 - z_1 = dz$, then $d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$\vec{r_2} - \vec{r_1} = \vec{r}$ represents a **line element** and $d\vec{r}$ represents an infinitesimally small line element.

13.8 PRODUCT OF VECTORS

There are two ways in which we define the product of two vectors $\vec{A}$ and $\vec{B}$ as,

***(i)* Scalar product or dot product.** Scalar product of two vectors $\vec{A}$ and $\vec{B}$ is $\vec{A}\,.\,\vec{B}$ (read as $\vec{A}$ dot $\vec{B}$)

***(ii)* Vector product or Cross product.** $\vec{A} \times \vec{B}$ is vector product (read as $\vec{A}$ cross $\vec{B}$)

The dot product $\vec{A}\,.\,\vec{B}$ gives a scalar result, while the cross product $\vec{A} \times \vec{B}$ gives a vector result.

13.9 SCALAR OR DOT PRODUCT OF TWO VECTORS

The scalar product of two vectors $\vec{A}$ *and* $\vec{B}$ *is defined as a scalar quantity which is equal to the product of the magnitudes of the given two vectors and the cosine of the angle between their directions.* Thus, if θ is the given angle between the directions of the two vectors $\vec{A}$ and $\vec{B}$ (Fig. 13.10), then

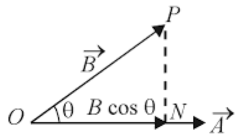

Fig. 13.10

$$\vec{A} \cdot \vec{B} = AB \cos\theta$$
$$= A\,(B\cos\theta) = A\,(ON)$$

It is thus clear that the scalar product of two vectors is equivalent to the product of the *magnitude* (modulus) of one vector and the component of the other vector in the direction of the former.

Properties of scalar product of two vectors

(1) Condition for two perpendicular vectors. When the two vectors are mutually perpendicular, then $\theta = 90°$ and $\cos\theta = 0$

$$\therefore \qquad \vec{A} \cdot \vec{B} = AB\cos\theta = 0$$

Thus if $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along the three co-ordinate axes mutually perpendicular to each other, then

$$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$$

Hence the two non-zero vectors are orthogonal vectors if and only if

$$\vec{A} \cdot \vec{B} = 0$$

(2) Condition for two collinear vectors. The two vectors are collinear if $\theta = 0$ or π.

(*i*) If $\theta = 0 \quad \cos\theta = 1$

$$\therefore \qquad \vec{A} \cdot \vec{B} = AB\cos\theta = AB$$

In other words, scalar product of two like vectors is positive and equal to the product of their moduli.

(*ii*) If the two vectors are antiparallel, then

$$\theta = \pi \text{ and } \cos\theta = -1 \qquad \therefore\ \vec{A} \cdot \vec{B} = AB\ \cos\pi = -AB$$

This shows that the scalar product of two antiparallel vectors is negative and is equal to the product of their moduli.

(3) Scalar product of two equal vectors. If $\vec{A} = \vec{B}$ and $\theta = 0$, then $\vec{A} \cdot \vec{A} = AA\cos\theta = A^2$

Thus, the square of any vector is equal to the square of its magnitude (modulus).

If $\hat{i}$, $\hat{j}$, and $\hat{k}$ are unit vectors along three axes, then

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$$

(4) Scalar product is distributive i.e,

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$$

Proof.

Consider three vectors A, B and C as shown in Fig. 13.11. Let the angle between the vectors $\vec{A}$ and $\vec{B}$ be α, and between $\vec{A}$ and $\vec{C}$ be β. Let $\vec{R}$ be the resultant of $\vec{B}$ and $\vec{C}$ making an angle θ with $\vec{A}$.

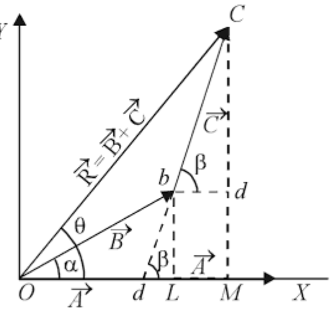

Fig. 13.11

Now $\qquad \vec{R} = \vec{B} + \vec{C}$

and $\qquad \vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{R}$

$$= AR \cos\theta = A\,(OM)$$

where $\qquad OM = R\cos\theta$ is the component of $\vec{R}$ along the vector $\vec{A}$.

Again $\qquad \vec{A} \cdot \vec{B} = AB\cos\alpha = A(OL)$

Similarly $\qquad \vec{A} \cdot \vec{C} = AC\cos\beta = A\,(LM)$

Now $\qquad A\,(OM) = A\,(OL + LM) = A\,(OL) + A\,(LM)$

$\therefore \qquad \vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$

(5) Scalar product in terms of rectangular components. The vectors $\vec{A}$ and $\vec{B}$ in terms of their components can be written as

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \text{ and } \vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

$\therefore$
$$\vec{A} \cdot \vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \cdot (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$
$$= A_x B_x\,(\hat{i} \cdot \hat{i}) + A_x B_y\,(\hat{i} \cdot \hat{j}) + A_x B_z\,(\hat{i} \cdot \hat{k}) + A_y B_x\,(\hat{j} \cdot \hat{i})$$
$$+ A_y B_y\,(\hat{j} \cdot \hat{j}) + A_y B_z\,(\hat{j} \cdot \hat{k}) + A_z B_x\,(\hat{k} \cdot \hat{i}) + A_z B_y\,(\hat{k} \cdot \hat{j}) + A_z B_z\,(\hat{k} \cdot \hat{k})$$
$$= A_x B_x + A_y B_y + A_z B_z$$

$\because \qquad \hat{i} \cdot \hat{j} = \hat{i} \cdot \hat{k} = \hat{j} \cdot \hat{i} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = \hat{k} \cdot \hat{j} = 0$

and $\qquad \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$

(6) Angle between two vectors. We know that

$$\vec{A} \cdot \vec{B} = AB\cos\theta$$

or
$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{AB} = \frac{A_x B_x + A_y B_y + A_z B_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\,\sqrt{B_x^2 + B_y^2 + B_z^2}}$$

Example 13.2. If $\vec{A} = \lambda\vec{B}$ prove that $\dfrac{A_x}{B_x} = \dfrac{A_y}{B_y} = \dfrac{A_z}{B_z}$.

Solution. $\qquad \vec{A} = \lambda\vec{B}$

or $\qquad (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) = \lambda(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$

or $\qquad (A_x - \lambda B_x)\hat{i} + (A_y - \lambda B_y)\hat{j} + (A_z - \lambda B_z)\hat{k} = 0$

As $\hat{i}, \hat{j}, \hat{k}$ are non-co-planar vectors, the equation is satisfied only if $A_x - \lambda B_x = 0$; $A_y - \lambda B_y = 0$; $A_z - \lambda B_z = 0$.

$\therefore$
$$A_x = \lambda B_x \text{ or } \frac{A_x}{B_x} = \lambda$$
$$A_y = \lambda B_y \text{ or } \frac{A_y}{B_y} = \lambda$$

and $$A_z = \lambda B_z \text{ or } \frac{A_z}{B_z} = \lambda$$

Thus $$\frac{A_x}{B_x} = \frac{A_y}{B_y} = \frac{A_z}{B_z}$$

Example 13.3. If $|\vec{A}| = |\vec{B}|$ prove that $(\vec{A} + \vec{B})$ is perpendicular to $(\vec{A} - \vec{B})$.

Solution. $$|\vec{A}| = |\vec{B}|$$

$$|\vec{A}|^2 = |\vec{B}|^2 \text{ or } \vec{A}\,.\,\vec{A} = \vec{B}\,.\,\vec{B}$$

Now $$(\vec{A} + \vec{B})\,.\,(\vec{A} - \vec{B}) = \vec{A}\,.\,\vec{A} - \vec{A}\,.\,\vec{B} + \vec{B}\,.\,\vec{A} - \vec{B}\,.\,\vec{B} = 0$$

As $$\vec{A}\,.\,\vec{B} = \vec{B}\,.\,\vec{A}$$

As the scalar product of the vectors $(\vec{A} + \vec{B})$ and $(\vec{A} - \vec{B})$ is zero, the vectors $(\vec{A} + \vec{B})$ is perpendicular to $(\vec{A} - \vec{B})$.

Example 13.4. Two sides of a triangle are formed by the vectors $\vec{A} = 3\hat{i} + 6\hat{j} - 2\hat{k}$ and $\vec{B} = 4\hat{i} - \hat{j} + 3\hat{k}$. Determine all the angles of the triangle.

Solution. Let $\vec{C}$ be the third side of the triangle, then

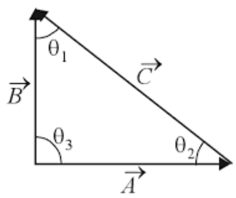

Fig. 13.12

$$\vec{A} + \vec{C} = \vec{B}$$

$$\vec{C} = \vec{B} - \vec{A}$$

$$= (4\hat{i} - \hat{j} + 3\hat{k}) - (3\hat{i} + 6\hat{j} - 2\hat{k})$$

$\therefore$ $$\vec{C} = \hat{i} - 7\hat{j} + 5\hat{k}$$

Angle between the vectors $\vec{A}$ and $\vec{B}$ is given by

$$\cos\theta_3 = \frac{\vec{A}\,.\,\vec{B}}{AB} = \frac{A_xB_x + A_yB_y + A_zB_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\,\sqrt{B_x^2 + B_y^2 + B_z^2}}$$

$$= \frac{12 - 6 - 6}{\sqrt{A_x^2 + A_y^2 + A_z^2}\,\sqrt{B_x^2 + B_y^2 + B_z^2}} = 0$$

$\therefore$ $$\theta_3 = 90°$$

Angle between vectors $\vec{A}$ and $\vec{C}$ is given by

$$\cos(\pi - \theta_2) = \frac{\vec{A}\,.\,\vec{C}}{AC} = \frac{A_xC_x + A_yC_y + A_zC_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\,\sqrt{C_y^2 + C_y^2 + C_z^2}} = \frac{-49}{\sqrt{49}\,\sqrt{75}}$$

$\therefore$ $$\cos\theta_2 = \frac{7}{5\sqrt{3}} = 0.8083$$

or $$\theta_2 = 36°4' \qquad \text{Hence } \theta_1 = 53°56'$$

Example 13.5 Show that the vectors $\vec{A} = 3\hat{i} - 2\hat{j} + \hat{k}$, $\vec{B} = \hat{i} - 3\hat{j} + 5\hat{k}$ and $\vec{C} = 2\hat{i} + \hat{j} - 4\hat{k}$ form a right angled triangle.

Solution. The vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ will form a triangle if one of the vectors is equal to the vector sum of the remaining two vectors.

Now $$\vec{B}+\vec{C}=\hat{i}-3\hat{j}+5\hat{k}+2\hat{i}+\hat{j}-4\hat{k}=3\hat{i}-2\hat{j}+\hat{k}=\vec{A}$$

Hence the three vectors form a triangle.

Again
$$\vec{A}\,.\,\vec{B}=(3\hat{i}-2\hat{j}+\hat{k})\,.\,(\hat{i}-3\hat{j}+5\hat{k})=14$$
$$\vec{A}\,.\,\vec{C}=(3\hat{i}-2\hat{j}+\hat{k})\,.\,(2\hat{i}+\hat{j}-4\hat{k})=0$$
$$\vec{B}\,.\,\vec{C}=(\hat{i}-3\hat{j}+5\hat{k})\,.\,(2\hat{i}+\hat{j}-4\hat{k})=-21$$

As $\vec{A}\,.\,\vec{C}=0$, the vectors $\vec{A}$ and $\vec{C}$ are orthogonal and triangle formed is a right angled triangle.

Example 13.6. A particle moves from a point (3, –4, –2) m to a point (–2, 3, 5) m under the influence of a force $\vec{F}=(-2\hat{i}+3\hat{j}+4\hat{k})$ Newton. Calculate the work done by the force.

Solution. Displacement of the particle

$$\vec{r}=(-2-3)\hat{i}+(3+4)\hat{j}+(5+2)\hat{k}$$
$$=(5\hat{i}+7\hat{j}+7\hat{k})\text{ m}$$
$$\vec{F}=(-2\hat{i}+3\hat{j}+4\hat{k})\text{ N}$$

$\therefore$ Work done $W=\vec{F}\,.\,\vec{r}=(-2\hat{i}+3\hat{j}+4\hat{k})\,.\,(-5\hat{i}+7\hat{j}+7\hat{k})$

$$=(10+21+28)\text{ Joules}=59\text{ J}$$

Example 13.7. If $\vec{A}=3\hat{i}-\hat{j}+2\hat{k}$ and $\vec{B}=\hat{i}-2\hat{j}+2\hat{k}$, find scalar product of two vectors. (*Nagpur U., s* 2006)

Solution.
$$\begin{aligned}\vec{A}\cdot\vec{B} &= \left(3\hat{i}-\hat{j}+2\hat{k}\right)\cdot\left(\hat{i}-2\hat{j}+2\hat{k}\right)\\ &= 3+2+4\\ &= 9\end{aligned}$$

Example 13.8. Find the angle between the vectors $\left(2\hat{i}+2\hat{j}+3\hat{k}\right)$ and $\left(6\hat{i}-3\hat{j}+2\hat{k}\right)$.

(*Nagpur U., s* 2007)

Solution. Let $\vec{A}=2\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{B}=6\hat{i}-3\hat{j}+2\hat{k}$ and θ be the angle between them, then

$$\vec{A}\cdot\vec{B} = \text{AB}\cos\theta$$

$$\therefore\quad \cos\theta = \frac{\vec{A}\cdot\vec{B}}{AB}=\frac{A_xB_x+A_yB_y+A_zB_z}{\sqrt{A_x^2+A_y^2+A_z^2}\,\sqrt{B_x^2+B_y^2+B_z^2}}$$
$$=\frac{12-6+6}{\sqrt{4+4+9}\,\sqrt{36+9+4}}=\frac{12}{\sqrt{17}\,\sqrt{49}}$$
$$=\frac{12}{7\sqrt{17}}=\frac{12}{7\times 4.123}=0.4158$$

$$\therefore\quad \theta = \cos^{-1}(0.4158)=65.45°$$

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13.10 VECTOR PRODUCT OR CROSS PRODUCT

The cross product of two vectors $\vec{A}$ and $\vec{B}$ denoted as $\vec{A} \times \vec{B}$ and read as ($\vec{A}$ cross $\vec{B}$) whose directions are inclined at an angle θ is the vector whose magnitude is $AB \sin\theta$ and direction is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$. It is **positive** if the rotation from $\vec{A}$ to $\vec{B}$ is anti-clockwise [Fig. 13.13 (*a*)] and **negative** if the rotation from $\vec{A}$ to $\vec{B}$ is clockwise [Fig 13.13 (*b*)].

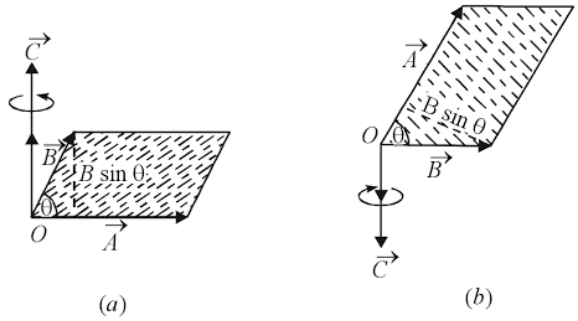

Fig. 13.13

Thus $$\vec{A} \times \vec{B} = AB \sin\theta\ \hat{n} = \vec{C}$$

where $\hat{n}$ is a unit vector perpendicular to the vectors $\vec{A}$ and $\vec{B}$.

The unit vector $\hat{n}$ is called *unit normal.*

13.11 PROPERTIES OF VECTOR PRODUCT OF TWO VECTORS

(1) Commutative law is not obeyed. From the definition of the vector product it is evident that the change in the order of vectors in a cross product reverses the direction of the product as $\sin(-\theta) = -\sin\theta$

$$\vec{B} \times \vec{A} = -AB \sin\theta\ \hat{n} = -\vec{A} \times \vec{B}$$

Hence, the vector product of the two vectors is not commutative and the order of the terms should be strictly maintained.

(2) However **distributive** law holds good *i.e.,*

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

(3) Area of a parallelogram. The magnitude of the vector product $\vec{A} \times \vec{B}$ gives the area of the parallelogram formed by the two vectors as adjacent sides. Referring to Fig. 13.13, we find that

Height of parallelogram $h = B \sin\theta$

Now $$\vec{A} \times \vec{B} = AB \sin\theta = Ah = \text{Base} \times \text{height of parallelogram}$$
$$= \text{Area of the parallelogram}$$

Since $\vec{A} \times \vec{B}$ is a vector whose direction is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$, the area is a vector quantity. In the case of an **open** surface, the direction of the area is taken along $\hat{n}$ which is normal to the plane containing the parallelogram. The area is taken as **positive** if the direction of $\hat{n}$ coincides with the direction of advance of a right handed screw when rotated so as to describe the boundary of the (open) surface in a positive sense *i.e.,* **anticlockwise** direction.

In the case of a **closed** surface the direction of the area vector is normal to the surface pointing **outwards** *i.e.,* away from the closed surface of which the area forms a part. The advantage of representing an area by a vector is that the cross product of the two vectors $\vec{A}$ and $\vec{B}$ representing the sides being equal to $AB \sin\theta$ where θ is the angle contained between the two vectors automatically gives the magnitude of the area and the direction of the resultant vector gives the direction of the area as a vector.

(4) Collinear vectors. If two vectors are collinear, then $\theta = 0$ or π, *i.e.*, $\sin\theta = 0$

$$\therefore \qquad \vec{A} \times \vec{B} = AB \sin\theta \ \hat{n} = 0$$

Thus, two vectors are collinear (parallel, or antiparallel) if the vector product $\vec{A} \times \vec{B} = 0$.

(5) Equal vectors. If two vectors are equal then $\theta = 0$, and vector product

$$\vec{A} \times \vec{A} = AA \sin\theta \ \hat{n} = 0$$

Hence, vector product of two equal vectors is always zero. In a particular case if $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along X, Y and Z axes, then $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$.

(6) Vector product of two unit vectors. We know

$$\vec{A} \times \vec{B} = AB \sin\theta \ \hat{n} \qquad \text{or} \qquad \frac{\vec{A}}{A} \times \frac{\vec{B}}{B} = \sin\theta \ \hat{n}$$

$$\text{or} \qquad \hat{A} \times \hat{B} = \sin\theta \ \hat{n}$$

Hence, vector product of two unit vectors is a vector whose magnitude is equal to the sine of the angle between their directions and whose direction is perpendicular to the plane containing the given vectors. The direction is determined by the right hand rule.

(7) Vector product of two perpendicular vectors. In this case $\theta = \pi/2$ and $\sin\theta = 1$

$$\vec{A} \times \vec{B} = AB \sin\frac{\pi}{2} \ \hat{n} = AB\,\hat{n}$$

Hence, vector product of two perpendicular vectors is a vector whose magnitude is equal to the product of the magnitudes of the given vectors and whose direction is along $\hat{n}$ such $\vec{A} \times \vec{B}$ and $\hat{n}$ form a right handed system.

If $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the unit vectors along the three axes X, Y and Z respectively, then

$$\hat{i} \times \hat{j} = \hat{k}\,, \ \hat{j} \times \hat{k} = \hat{i} \ \text{ and } \ \hat{k} \times \hat{i} = \hat{j}$$

$$\text{Also} \qquad \hat{j} \times \hat{i} = -\hat{k}, \ \hat{k} \times \hat{j} = -\hat{i} \ \text{ and } \ \hat{i} \times \hat{k} = -\hat{j}$$

(8) Vector product in terms of components. Let (A_x, A_y, A_z) and (B_x, B_y, B_z) be the orthogonal projections of the vectors $\vec{A}$ and $\vec{B}$ along X, Y and Z axes respectively, then

$$A = \hat{i}\,A_x + \hat{j}\,A_y + \hat{k}A_z \ \text{ and } \ \vec{B} = \hat{i}\,B_x + \hat{j}\,B_y + \hat{k}B_z$$

$$\begin{aligned}
\therefore \qquad \vec{A} \times \vec{B} &= (\hat{i}A_x + \hat{j}A_y + \hat{k}A_z) \times (\hat{i}\,B_x + \hat{j}\,B_y + \hat{k}\,B_z) \\
&= A_x B_y \hat{k} - A_x B_z \hat{j} - A_y B_x \hat{k} + A_y B_z \hat{i} + A_z B_x \hat{j} - A_z B_y \hat{i} \\
&= \hat{i}\,(A_y B_z - A_z B_y) + \hat{j}\,(A_z B_x - A_x B_z) + \hat{k}\,(A_x B_y - A_y B_x) \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}
\end{aligned}$$

(9) Determination of Angle from vector product.

We know $\vec{A} \times \vec{B} = AB \sin\theta \; \hat{n}$

Also $$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}$$

$$= AB \sin\theta \; \hat{n}$$

Squaring both sides and taking self-product on either side, we have

$$A^2 B^2 \sin^2\theta = (A_y B_z - A_z B_y)^2 + (A_z B_x - A_x B_z)^2 + (A_x B_y - A_y B_x)^2$$

$\because$ $$\hat{n} \cdot \hat{n} = 1 \text{ and } \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$$

As $$A^2 = A_x^2 + A_y^2 + A_z^2 \text{ and } B^2 = B_x^2 + B_y^2 + B_z^2$$

$\therefore$ $$\sin\theta = \left\{ \frac{(A_y B_z - A_z B_y)^2 + (A_z B_x - A_x B_z)^2 + (A_x B_y - A_y B_z)^2}{(A_x^2 + A_y^2 + A_z^2)(B_x^2 + B_y^2 + B_z^2)} \right\}^{\frac{1}{2}}$$

Example 13.9. The position and velocity vectors of two particles at any instant are $\vec{r_1}$, $\vec{r_2}$ and $\vec{v_1}, \vec{v_2}$ respectively. Prove that they will collide if

$$(\vec{r_1} - \vec{r_2}) \times (\vec{v_1} - \vec{v_2}) = 0 \cdot$$

Solution. Let the two particles P and Q have position vectors $\vec{r_1}$ and $\vec{r_2}$ as shown in Fig. 13.14.

The velocity vectors of P and Q are $\vec{v_1}$ and $\vec{v_2}$ at that instant as shown. Let the two particles collide at a point C after a time interval t, then

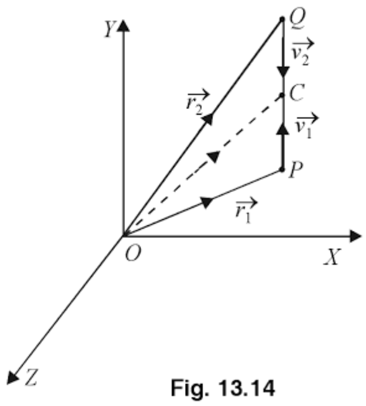

Fig. 13.14

Position vector of particle P at C

$$\overrightarrow{OC} = \overrightarrow{OP} + \overrightarrow{PC}$$
$$= \vec{r_1} + \vec{v_1}\, t$$

as $\vec{v_1}\, t$ is the distance travelled by the particle in time t.

Position vector of particle Q at C

$$\overrightarrow{OC} = \overrightarrow{OQ} + \overrightarrow{QC} = \vec{r_2} + \vec{v_2}\, t$$

The particles P and Q will collide if

$$\vec{r_1} + \vec{v_1}\, t = \vec{r_2} + \vec{v_2}\, t$$

or $$(\vec{r_1} - \vec{r_2}) = -(\vec{v_1} - \vec{v_2})\, t$$

For this condition to be satisfied

$$(\vec{r_1} - \vec{r_2}) \times (\vec{v_1} - \vec{v_2}) = -t\,[(\vec{v_1} - \vec{v_2}) \times (\vec{v_1} - \vec{v_2})]$$
$$= 0$$

As vector product of two equal vectors is zero.

Example 13.10. Find a unit vector that is perpendicular to $\vec{A} = (\hat{i} + 2\hat{j} + 3\hat{k})$ and $\vec{B} = (3\hat{i} + 2\hat{j} + \hat{k})$. Calculate the sine of the angle between them. (*Nagpur U. s* 2008)

Solution. The vector $\vec{C}$ perpendicular to both $\vec{A}$ and $\vec{B}$ is given by the vector product of the two vectors.

$\therefore$ $$\vec{C} = \vec{A} \times \vec{B} = (\hat{i} + 2\hat{j} + 3\hat{k}) \times (3\hat{i} + 2\hat{j} + k)$$

$$= \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix} = -4\hat{i} + 8\hat{j} - 4\hat{k}$$

The magnitude of vector $\vec{C} = -4\hat{i} + 8\hat{j} - 4\hat{k}$ is

$$|\vec{C}| = \sqrt{4^2 + 8^2 + 4^2} = \sqrt{96}$$

Unit vector in the direction of $\vec{C}$ is given by

$$\hat{C} = \frac{\vec{C}}{|\vec{C}|} = \frac{-4\hat{i} + 8\hat{j} - 4\hat{k}}{\sqrt{96}}$$

Again magnitude of $\vec{A} = |\vec{A}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

and magnitude of $\vec{B} = |\vec{B}| = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$

If θ is the angle between the two vectors $\vec{A}$ and $\vec{B}$, then

$$|\vec{C}| = |\vec{A}||\vec{B}| \sin\theta$$

or

$$\sin\theta = \frac{|\vec{C}|}{|\vec{A}||\vec{B}|} = \frac{\sqrt{96}}{\sqrt{14}\sqrt{14}} = \sqrt{\frac{96}{196}} = \sqrt{0.4898} = 0.6999$$

Example 13.11. Find the Cartesian components of a vector $\vec{C}$ which is perpendicular to the vectors $\vec{A} = 2\hat{i} - \hat{j} - 4\hat{k}$ and $\vec{B} = 3\hat{i} - \hat{j} - \hat{k}$. (*Gauhati U.* 2000; *Luck. U.* 2001)

Solution.

$$\vec{C} = \vec{A} \times \vec{B} = (2\hat{i} - \hat{j} - 4\hat{k}) \times (3\hat{i} - \hat{j} - \hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & -4 \\ 3 & -1 & -1 \end{vmatrix} = -(3\hat{i} + 10\hat{j} - \hat{k})$$

Example 13.12. The diagonals of a parallelogram are given by vectors $(3\hat{i} + \hat{j} + 2\hat{k})$ and $(\hat{i} - 3\hat{j} + 4\hat{k})$. Find the area of the parallelogram.

Solution. Consider a parallelogram *PQRS* such that the diagonal *PR* is represented by the vector $\vec{A} = 3\hat{i} + \hat{j} + 2\hat{k}$ and the diagonal *SQ* by the vector $\vec{B} = \hat{i} - 3\hat{j} + 4\hat{k}$.

Now the diagonal *PR* and *SQ* bisect each other at *O*. Hence

$$\overrightarrow{PO} = \frac{1}{2}\vec{A} = \frac{1}{2}(3\hat{i} + \hat{j} + \hat{k})$$

$$\text{and } \overrightarrow{OQ} = \frac{1}{2}\vec{B} = \frac{1}{2}(\hat{i} - 3\hat{j} + 4\hat{k})$$

Area of parallelogram *PQRS* = 4 × area of Δ *POQ*

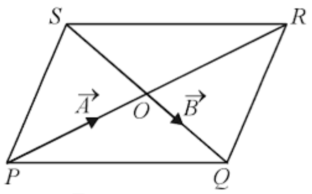

Fig. 13.15

$$= 4 \times \frac{1}{2}\ \vec{PO} \times \vec{OQ}$$

$$= 2\left[\frac{1}{2}(3\hat{i} + \hat{j} + 2\hat{k}) \times \frac{1}{2}(\hat{i} - 3\hat{j} + 4\hat{k})\right]$$

$$= \frac{1}{2}(3\hat{i} + \hat{j} + 2\hat{k}) \times (\hat{i} - 3\hat{j} + 4\hat{k})$$

$$= \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 1 & -3 & 4 \end{vmatrix} = \frac{1}{2}(10\hat{i} - 10\hat{j} - 10\hat{k})$$

$$= \frac{1}{2}(10\hat{i} - 10\hat{j} - 10\hat{k}) = (5\hat{i} - 5\hat{j} - 5\hat{k})$$

$$= \sqrt{5^2 + 5^2 + 5^2} = \sqrt{75} = 8.66 \text{ sq. units.}$$

Example 13.13. If $\vec{A} + \vec{B} + \vec{C} = 0$, prove that $\vec{A} \times \vec{B} = \vec{B} \times \vec{C} = \vec{C} \times \vec{A}$.

Solution. In rectangular component form

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

and
$$\vec{C} = C_x\hat{i} + C_y\hat{j} + C_z\hat{k}$$

Given $\quad \vec{A} + \vec{B} + \vec{C} = 0$

$\therefore \quad \vec{A} = -(\vec{B} + \vec{C})$

or $\quad A_x\hat{i} + A_y\hat{j} + A_z\hat{k} = -(B_x + C_x)\hat{i} - (B_y + C_y)\hat{j} - (B_z + C_z)\hat{k}$

Hence $\quad A_x = -(B_x + C_x);\ A_y = -(B_y + C_y);\ A_z = -(B_z + C_z)$

Now $\quad \vec{A} \times \vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \times (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$$

Substituting the values of A_x, A_y and A_z in terms of B_x, B_y, B_z and C_x, C_y, C_z, we have

$$\vec{A} \times \vec{B} = \{-(B_y + C_y)B_z - [-(B_z + C_z)B_y]\}\hat{i}$$
$$+ \{-(B_z + C_z)B_x - [-(B_x + C_x)B_z]\}\hat{j}$$
$$+ \{-(B_x + C_x)B_y - [-(B_y + C_y)B_x]\}\hat{k}$$

Now $-(B_y + C_y)B_z - [-(B_z + C_z)B_y]$

$$= -B_yB_z - C_yB_z + B_zB_y + C_zB_y = B_yC_z - B_zC_y$$

and $-(B_z + C_z)B_x - [-(B_x + C_x)B_z]$

$$= -B_zB_x - C_zB_x + B_xB_z + C_xB_z = B_zC_x - C_zB_x$$

Also $-(B_x + C_x)B_y - [-(B_y + C_y)B_x]$

$$= -B_xB_y - C_xB_y + B_yB_x + C_yB_x = B_xC_y - B_yC_x$$

$\therefore$ $$\vec{A} \times \vec{B} = (B_y C_z - B_z C_y)\hat{i} + (B_z C_x - B_x C_z)\hat{j} + (B_x C_y - B_y C_x)\hat{k}$$
$$= \vec{B} \times \vec{C}$$

Similarly $\vec{A} \times \vec{B} = \vec{C} \times \vec{A}$

Hence $\vec{A} \times \vec{B} = \vec{B} \times \vec{C} = \vec{C} \times \vec{A}$

Example 13.14. If $\vec{A} = 3\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{B} = \hat{i} - 2\hat{j} + 2\hat{k}$, find the vector product of two vectors. (*Nagpur U. s* 2005, 2006)

Solution. $$\vec{C} = \vec{A} \times \vec{B} = (3\hat{i} - \hat{j} + 2\hat{k}) \times (\hat{i} - 2\hat{j} + 2\hat{k})$$
$$= \begin{vmatrix} i & j & k \\ 3 & -1 & 2 \\ 1 & -2 & 2 \end{vmatrix} = 2\hat{i} - 4\hat{j} - 5\hat{k}$$

13.12 TRIPLE PRODUCT OF VECTORS

The vector product of two vectors $\vec{B}$ and $\vec{C}$ is a vector which can give both scalar and vector product with a third vector $\vec{A}$. There are, therefore, two triple products

$$\vec{A} \,.\, (\vec{B} \times \vec{C})$$

This is known as **scalar triple product** and

$$\vec{A} \times (\vec{B} \times \vec{C})$$

This is known as **vector triple product.**

13.13 SCALAR TRIPLE PRODUCT

$\vec{A} \,.\, (\vec{B} \times \vec{C})$ is a scalar tripple product. We have proved that $\vec{B} \times \vec{C}$ is a vector normal to the plane of $\vec{B}$ and $\vec{C}$ and its magnitude is equal to the area of a parallelogram having the sides represented by $\vec{B}$ and $\vec{C}$ shown shaded in Fig. 13.16.

The scalar product of $\vec{A}$ and $(\vec{B} \times \vec{C})$ is a product of this area and the projection of $\vec{A}$ along the vector $(\vec{B} \times \vec{C})$ which is equal to $h = A \cos\theta$.

$\therefore$ $$\vec{A} \,.\, (\vec{B} \times \vec{C}) = \text{Area of base} \times \text{height } h$$

This is the volume of the parallelopiped enclosed by the vectors $\vec{A}, \vec{B}$ and $\vec{C}$ as its edges.

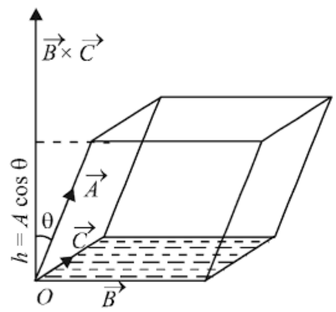

Fig. 13.16

As any face of the parallelogram enclosed can be taken as its base, three equivalent expressions for volume are

Volume $$V = \vec{A} \,.\, (\vec{B} \times \vec{C}) = \vec{B} \,.\, (\vec{C} \times \vec{A}) = \vec{C} \,.\, (\vec{A} \times \vec{B})$$

The scalar triple product is **positive** if the angle θ as shown in Fig. 13.16 is **acute** *i.e.,* if $\vec{A}, \vec{B}$ and $\vec{C}$ are **right handed** system of vectors and **negative** if θ is obtuse *i.e.,* if $\vec{A}, \vec{B}$ and $\vec{C}$ form a **left handed** system of vectors. Also in the scalar triple product, the final result is the same even if the position of dot and cross may be interchanged. The above expression, therefore, can be written as

Volume $$V = (\vec{A} \times \vec{B}) \,.\, \vec{C} = (\vec{B} \times \vec{C}) \,.\, \vec{A} = (\vec{C} \times \vec{A}) \,.\, \vec{B}$$

We can also write the scalar triple product of $\vec{A}, \vec{B}, \vec{C}$ as $[\vec{A}\vec{B}\vec{C}]$.

Thus in scalar triple product the position of dot and cross may be interchanged without changing the value of the product provided the **cyclic order** of vectors is maintained.

Again $\qquad \vec{C} \times \vec{B} = -\vec{B} \times \vec{C}$

$\therefore \qquad \vec{A} \,.\, (\vec{C} \times \vec{B}) = -\vec{A} \,.\, (\vec{B} \times \vec{C})$

Scalar Triple Product in Terms of Rectangular Components. The scalar triple product in terms of rectangular components of vectors may be written in the form of a determinant. Now,

$$\vec{A} \,.\, (\vec{B} \times \vec{C}) = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \,.\, \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}).\,[B_y C_z - B_z C_y)\hat{i} + (B_z C_x - B_x C_z)\hat{j} + (B_x C_y - B_y C_x)\hat{k}]$$

$$= A_x\,(B_y C_z - B_z C_y) + A_y\,(B_z C_x - B_x C_z) + A_z\,(B_x C_y - B_y C_x)$$

$$= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

The scalar triple product of three unit vectors $\hat{i}, \hat{j}, \hat{k}$

$$[\hat{i}\,\hat{j}\,\hat{k}] = [\hat{j}\,\hat{k}\,\hat{i}] = [\hat{k}\,\hat{i}\,\hat{j}] = 1$$

Condition for co-planar vectors. The condition that the three vectors may be co-planar is that their scalar triple product should vanish. In other words,

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = 0$$

In such a case, the volume of the parallelopiped formed by the three vectors = 0. Hence, these lie in one plane. The above condition is satisfied if two of the vectors are **parallel** or **equal.**

13.14 PHYSICAL SIGNIFICANCE AND IMPORTANT FEATURES OF SCALAR TRIPLE PRODUCT

(*i*) The scalar triple product $\vec{A} \cdot (\vec{B} \times \vec{C})$ represents the volume of the parallelopiped enclosed by the vectors $\vec{A}, \vec{B}$ and $\vec{C}$ as its edges.

(*ii*) As any face of the parallelogram enclosed by two of the three vectors can be taken as the base, there are three equivalent expressions for volume as

$$\text{Volume } V = \vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

(*iii*) In scalar triple product, the position of 'dot' and 'cross' may be interchanged without changing the value of the product, provided the *cyclic order* is maintained.

(*iv*) If $\vec{A} \cdot (\vec{B} \times \vec{C}) = 0$, the volume of the parallelopiped formed by the vectors = 0, Hence, the three vectors are co-planar.

Example 13.15. Prove that the vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 4\hat{j} + 5\hat{k}$ are co-planar. (*Meerut U.,* 2001)

Solution. Let $\vec{A} = 2\hat{i} - \hat{j} + \hat{k}$; $\vec{B} = \hat{i} + 2\hat{j} - 3\hat{k}$; $\vec{C} = 3\hat{i} - 4\hat{j} + 5\hat{k}$. The three vectors $\vec{A}, \vec{B}$ and $\vec{C}$ are co-planar if $\vec{A} \,.\, (\vec{B} \times \vec{C}) = 0$

Now $$\vec{A}\,.\,(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \begin{vmatrix} +2 & -1 & +1 \\ +1 & +2 & -3 \\ +3 & -4 & +5 \end{vmatrix}$$

$$= 2\,(10 - 12) - 1\,(-9 - 5) + 1\,(-4 - 6)$$
$$= -4 + 14 - 10 = 0$$

As $\vec{A}\,.\,(\vec{B}\times\vec{C}) = 0$, the three vectors are co-planar.

13.15 VECTOR TRIPLE PRODUCT

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}\,(\vec{A}\,.\,\vec{C}) - \vec{C}\,(\vec{A}\,.\,\vec{B})$$

To evaluate this expression we shall first find the value of $\vec{B}\times\vec{C}$ which in terms of rectangular components is given by

$$\vec{B}\times\vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= (B_y C_z - B_z C_y)\hat{i} + (B_z C_x - B_x C_z)\hat{j} + (B_x C_y - B_y C_x)\hat{k}$$

Now $$\vec{A}\times(\vec{B}\times\vec{C}) = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\times[(B_y C_z - B_z C_y)\hat{i}$$
$$+ (B_z C_x - B_x C_z)\hat{j} + (B_x C_y - B_y C_x)\hat{k}]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ (B_y C_z - B_z C_y) & (B_z C_x - B_x C_z) & (B_x C_y - B_y C_x) \end{vmatrix}$$

The first term of the above determinant

$$= [A_y\,(B_x C_y - B_y C_x) - A_z\,(B_z C_x - B_x C_z)]\,\hat{i}$$
$$= [A_y B_x C_y - A_y B_y C_x - A_z B_z C_x + A_z B_x C_z]\,\hat{i}$$

Add and subtract $A_x B_x C_x$, collect positive terms and negative terms and arrange, then we get

First term $= [(A_y B_x C_y + A_z B_x C_z + A_x B_x C_x) - (A_y B_y C_x + A_z B_z C_x + A_x B_x C_x]\,\hat{i}$

$$= [\,B_x(A_x C_x + A_y C_y + A_z C_z) - C_x\,(A_x B_x + A_y B_y + A_z B_z)\,]\ \hat{i}$$
$$= B_x\,(\vec{A}\,.\,\vec{C})\,\hat{i} - C_x\,(\vec{A}\,.\,\vec{B})\,\hat{i}$$

Similarly, the second and third terms of the determinant are

2nd term $= B_y\,(\vec{A}\,.\,\vec{C})\,\hat{j} - C_y\,(\vec{A}\,.\,\vec{B})\,\hat{j}$

3rd term $= B_z\,(\vec{A}\,.\,\vec{C})\,\hat{k} - C_z\,(\vec{A}\,.\,\vec{B})\,\hat{k}$

Adding all these terms, we get

$$\vec{A}\times(\vec{B}\times\vec{C}) = (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})\,(\vec{A}\,.\,\vec{C}) - (C_x\hat{i} + C_y\hat{j} + C_z\hat{k})\,(\vec{A}\,.\,\vec{B})$$
$$= \vec{B}\,(\vec{A}\,.\,\vec{C}) - \vec{C}\,(\vec{A}\,.\,\vec{B})$$

As the vector $\vec{B} \times \vec{C}$ is perpendicular to the plain containing $\vec{B}$ and $\vec{C}$, the vector $\vec{A} \times (\vec{B} \times \vec{C})$ will be perpendicular to the plane containing $\vec{A}$ and $(\vec{B} \times \vec{C})$ *i.e.*, it will be in the plane of $\vec{B}$ and $\vec{C}$, the vectors inside the brackets. This is also clear from the result $(\vec{A} . \vec{C})$ and $(\vec{A} . \vec{B})$ being scalar quantities the resultant lies in the plane of $\vec{B}$ and $\vec{C}$.

Example 13.16. Prove that $\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = 0$.

(*Nagpur U. s*/2009, *s*/2008, *s*/2005; *Gauhati U.* 2000; *Meerut U.* 2002)

Solution. According to triple vector product

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}\,(\vec{A} . \vec{C}) - \vec{C}\,(\vec{A} . \vec{B})$$
$$\vec{B} \times (\vec{C} \times \vec{A}) = \vec{C}\,(\vec{B} . \vec{A}) - \vec{A}\,(\vec{B} . \vec{C})$$
$$\vec{C} \times (\vec{A} \times \vec{B}) = \vec{A}\,(\vec{C} . \vec{B}) - \vec{B}\,(\vec{C} . \vec{A})$$

Now, $\vec{A} . \vec{C} = \vec{C} . \vec{A}, \quad \vec{A} . \vec{B} = \vec{B} . \vec{A}, \vec{B} . \vec{C} = \vec{C} . \vec{B}$

Adding, we get $\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = 0$

Example 13.17. Prove that $(\vec{A} + \vec{B}) . [(\vec{B} + \vec{C}) \times (\vec{C} + \vec{A})] = 2\vec{A} . (\vec{B} \times \vec{C})$

Solution. $(\vec{A} + \vec{B}) . [(\vec{B} + \vec{C}) \times (\vec{C} + \vec{A})]$

$$= (\vec{A} + \vec{B}) . [(\vec{B} \times \vec{C}) + (\vec{B} \times \vec{A}) + (\vec{C} \times \vec{C}) + (\vec{C} \times \vec{A})]$$
$$= \vec{A} . (\vec{B} \times \vec{C}) + \vec{A} . (\vec{B} \times \vec{A}) + \vec{A} . (\vec{C} \times \vec{C}) + \vec{A} . (\vec{C} \times \vec{A})$$
$$+ \vec{B} . (\vec{B} \times \vec{C}) + \vec{B} . (\vec{B} \times \vec{A}) + \vec{B} . (\vec{C} \times \vec{C}) + \vec{B} . (\vec{C} \times \vec{A})$$

Now $\quad \vec{A} . (\vec{B} \times \vec{A}) = 0; \vec{A} . (\vec{C} \times \vec{C}) = 0; \vec{A} . (\vec{C} \times \vec{A}) = 0$

$$\vec{B} . (\vec{B} \times \vec{C}) = 0; \vec{B} . (\vec{B} \times \vec{A}) = 0; \vec{B} . (\vec{C} \times \vec{C}) = 0$$

$\therefore \quad (\vec{A} + \vec{B}) . [(\vec{B} + \vec{C}) \times (\vec{C} + \vec{A})] = \vec{A} . (\vec{B} \times \vec{C}) + \vec{B} . (\vec{C} \times \vec{A}) = 2\vec{A} . (\vec{B} \times \vec{C})$

Example 13.18. If $\vec{A}$ and $\vec{B}$ are two vectors, show that the component of $\vec{A}$ perpendicular to $\vec{B}$ is given by $\dfrac{\vec{B} \times (\vec{A} \times \vec{B})}{B^2}$.

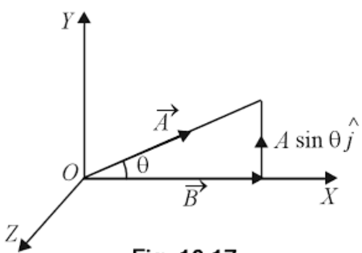

Fig. 13.17

Solution. Suppose the vectors $\vec{A}$ and $\vec{B}$ lie in the X-Y plane with the vector $\vec{B}$ coinciding with the X-axis, then

$$\vec{A} \times \vec{B} = AB \sin\theta\,\hat{n}$$

where $\hat{n}$ is a unit vector perpendicular to the plane containing the vectors $\vec{A}$ and $\vec{B}$ *i.e.*, in the direction of –Z-axis.

or $$\hat{n} = -\hat{k}$$

$\therefore$ $$\vec{A} \times \vec{B} = -AB \sin\theta\,\hat{k}$$

Hence $$\vec{B} \times (\vec{A} \times \vec{B}) = -AB \sin\theta\,\vec{B} \times \hat{k}$$
$$= -AB \sin\theta\, B\,(\hat{i} \times \hat{k}) \qquad [\because \vec{B} = B\hat{i}\,]$$

$$= -AB^2 \sin\theta\,(\hat{i} \times \hat{k})$$

$$= AB^2 \sin\theta\,\hat{j} \qquad [\because\ \hat{i} \times \hat{k} = -\hat{j}\,]$$

or
$$A \sin\theta\ \hat{j} = \frac{\vec{B} \times (\vec{A} \times \vec{B})}{B^2}$$

where $A \sin\theta\ \hat{j}$ is the component of $\vec{A}$ perpendicular to $\vec{B}$ and lies along the Y-axis as shown in Fig. 13.17.

Example 13.19. Prove that $\hat{i} \times (\vec{A} \times \hat{i}) + \hat{j} \times (\vec{A} \times \hat{j}) + \hat{k} \times (\vec{A} \times \hat{k}) = 2\vec{A}$.

Solution. Applying the relation $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}\,(\vec{A}\,.\,\vec{C}) - \vec{C}\,(\vec{A}\,.\,\vec{B})$ we have

$$\hat{i} \times (\vec{A} \times \hat{i}) = \vec{A}(\hat{i}\,.\,\hat{i}) - \hat{i}\,(\hat{i}\,.\,\vec{A})$$

$$= \vec{A} - \hat{i}\,(\hat{i}\,.\,\vec{A})$$

$$\hat{j} \times (\vec{A} \times \hat{j}) = \vec{A} - \hat{j}\,(\hat{j}\,.\,\vec{A})$$

$$\hat{k} \times (\vec{A} \times \hat{k}) = \vec{A} - \hat{k}\,(\hat{k}\,.\,\vec{A})$$

$\therefore$
$$\hat{i} \times (\vec{A} \times \hat{i}) + \hat{j} \times (\vec{A} \times \hat{j}) + \hat{k} \times (\vec{A} \times \hat{k})$$

$$= 3\vec{A} - \hat{i}\,(\hat{i}\,.\,\vec{A}) - \hat{j}\,(\hat{j}\,.\,\vec{A}) - \hat{k}\,(\hat{k}\,.\,\vec{A})$$

Now
$$\hat{i}\,.\,\vec{A} = \hat{i}\,.\,(A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) = A_x$$

Similarly
$$\hat{j}\,.\,\vec{A} = A_y \text{ and } \hat{k}\,.\,\vec{A} = A_z$$

$\therefore$
$$3\vec{A} - \hat{i}\,(\hat{i}\,.\,\vec{A}) - \hat{j}\,(\hat{j}\,.\,\vec{A}) - \hat{k}\,(\hat{k}\,.\,\vec{A})$$

$$= 3\vec{A} - [A_x\hat{i} + A_y\hat{j} + A_z\hat{k}]$$

$$= 3\vec{A} - \vec{A} = 2\vec{A}$$

Example 13.20. What should be the orientation between unit vectors $\hat{A}$ and $\hat{B}$ such that their sum has unit magnitude? (G.N.D.U. 2003)

Solution. If $\hat{A}$ and $\hat{B}$ are two vectors inclined at an angle θ, then magnitude $|\vec{R}|$ of the resultant is given by

$$|\vec{R}| = |\vec{A} + \vec{B}| = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

where A and B are the magnitudes of $\hat{A}$ and $\hat{B}$ respectively.

If $\hat{A}$ and $\hat{B}$ are unit vectors, then magnitude of each = 1. When their sum has unit magnitude $|\vec{R}| = 1$

$\therefore$
$$1 = \sqrt{1 + 1 + 2\cos\theta} \quad \text{or} \quad 1 = 2\,(1 + \cos\theta)$$

or
$$1 + \cos\theta = \frac{1}{2} \quad \text{or} \quad \cos\theta = -\frac{1}{2} \quad \therefore \quad \theta = 120°$$

Hence the orientation between the two vectors should be 120°.

Example 13.21. If unit vectors $\hat{A}$ and $\hat{B}$ are inclined at an angle θ then prove that $|\hat{A} - \hat{B}| = 2 \sin \theta/2$. (*M.D.V.* 2003)

Solution.
$$|\vec{A} - \vec{B}| = \sqrt{A^2 + B^2 - 2AB\cos\theta}$$

For unit vectors
$$\left|\hat{A}-\hat{B}\right| = \sqrt{1+1-2\cos\theta} = \sqrt{2(1-\cos\theta)}$$
$$\left|\hat{A}-\hat{B}\right|^2 = 2\,(1-\cos\theta) = 2\times 2\sin^2\theta/2$$
$$\left|\hat{A}-\hat{B}\right| = 2\sin\theta/2$$

Example 13.22. If $\vec{A}+\vec{B}+\vec{C} = 0$ prove that $\vec{A}\cdot(\vec{B}\times\vec{C}) = 0$ (*Meerut. U.* 2003)

Solution. If $\vec{A}+\vec{B}+\vec{C} = 0$, the three vectors $\vec{A}, \vec{B}$ and $\vec{C}$ are co-planar. As $\vec{A}\cdot(\vec{B}\times\vec{C})$ represents the volume of the paralelopiped enclosed by the vectors $\vec{A}\cdot\vec{B}$ and $\vec{C}$ as its edges and the three vectors are co-planar, the volume of the parallelopiped is zero.

$$\therefore \quad \vec{A}\cdot(\vec{B}\times\vec{C}) = 0$$

Example 13.23. Prove that $(\vec{A}+\vec{B})\cdot[(\vec{B}+\vec{C})\times(\vec{C}+\vec{A})] = 2\vec{A}\cdot(\vec{B}\times\vec{C})$ (*Nag. U. s*/2004)

Solution. $(\vec{A}+\vec{B})\cdot[(\vec{B}+\vec{C})\times(\vec{C}+\vec{A})]$
$$= (\vec{A}+\vec{B})\cdot[(\vec{B}\times\vec{C})+(\vec{B}\times\vec{A})+(\vec{C}\times\vec{C})+(\vec{C}\times\vec{A})]$$
$$= \vec{A}\cdot(\vec{B}\times\vec{C})+A\cdot(\vec{B}\times\vec{A})+\vec{A}\cdot(\vec{C}\times\vec{C})+\vec{A}\cdot(\vec{C}\times\vec{A})$$
$$+\vec{B}\cdot(\vec{B}\times\vec{C})+\vec{B}\cdot(\vec{B}\times\vec{A})+\vec{B}\cdot(\vec{C}\times\vec{C})+\vec{B}\cdot(\vec{C}\times\vec{A})$$

Now
$$\vec{A}\cdot(\vec{B}\times\vec{A}) = 0;\ \vec{A}\cdot(\vec{C}\times\vec{C}) = 0;\ \vec{A}\cdot(\vec{C}\times\vec{A}) = 0$$
$$\vec{B}\cdot(\vec{B}\times\vec{C}) = 0;\ \vec{B}\cdot(\vec{B}\times\vec{A}) = 0;\ \vec{B}\cdot(\vec{C}\times\vec{C}) = 0$$
$$\therefore (\vec{A}+\vec{B})\cdot[(\vec{B}+\vec{C})\times(\vec{C}+\vec{A})] = \vec{A}\cdot(\vec{B}\times\vec{C})+\vec{B}\cdot(\vec{C}\times\vec{A})] = 2\vec{A}\cdot(\vec{B}\times\vec{C})$$

13.16 SOME IMPORTANT FORMULAE IN VECTOR CALCULUS:

Some results are given below for ready reference:

(*i*) $\dfrac{d\vec{A}}{dt} = \lim\limits_{\delta t\to 0}\dfrac{\vec{A}(t+\delta t)-\vec{A}(t)}{\delta t}$

(*ii*) $\dfrac{d}{dt}(\vec{A}+\vec{B}) = \dfrac{d\vec{A}}{dt}+\dfrac{d\vec{B}}{dt}$

(*iii*) $\dfrac{d\vec{A}}{dt} = \dfrac{d\vec{A}}{dS}\dfrac{dS}{dt}$

(*iv*) $\dfrac{d(u\vec{A})}{dt} = \dfrac{du}{dt}\vec{A}+u\dfrac{d\vec{A}}{dt}$

(*v*) $\dfrac{d\vec{A}}{dt} = \dfrac{\partial A_x}{\partial t}\hat{i}+\dfrac{\partial A_y}{\partial t}\hat{j}+\dfrac{\partial A_z}{\partial t}\hat{k}$

(*vi*) $\dfrac{d(\vec{A}\cdot\vec{B})}{dt} = \dfrac{d\vec{A}}{dt}\cdot\vec{B}+\vec{A}\cdot\dfrac{d\vec{B}}{dt}$

(*vii*) $\dfrac{d(\vec{A}\times\vec{B})}{dt} = \dfrac{d\vec{A}}{dt}\times\vec{B}+\vec{A}\times\dfrac{d\vec{B}}{dt}$

(*viii*) $\dfrac{d}{dt}\left(\vec{A}\times\dfrac{d\vec{A}}{dt}\right) = \vec{A}\times\dfrac{d^2\vec{A}}{dt^2}$

(*ix*) The derivative of a vector of constant direction is parallel to that vector.

(*x*) The derivative of a vector of constant magnitude is perpendicular to the vector itself.

(*xi*) $\dfrac{d}{dt}(\vec{A}\cdot\vec{B}\times\vec{C}) = \dfrac{d\vec{A}}{dt}\cdot\vec{B}\times\vec{C}+\vec{A}\cdot\dfrac{d\vec{B}}{dt}\times\vec{C}+\vec{A}\cdot\vec{B}\times\dfrac{d\vec{C}}{dt}$

(*xii*) $\dfrac{d}{dt}(\vec{A}\times\vec{B}\times\vec{C}) = \dfrac{d\vec{A}}{dt}\times(\vec{B}\times\vec{C})+\vec{A}\times\left(\dfrac{d\vec{B}}{dt}\times\vec{C}\right)+\vec{A}\times\left(\vec{B}\times\dfrac{d\vec{C}}{dt}\right)$

Note. In the derivation of the product of vector, the order of placement of vectors should not be ordinarily changed.

13.17 SCALAR AND VECTOR FIELD

A field is a region in space in which a function u is defined at all points. Thus in Cartesian co-ordinates $u = f(x, y, z)$ specifies a field.

Scalar field. *A region in space in which a scalar quantity is continuous and is defined by a single value at every point of the position variable is caled a scalar field.*

If ϕ is a scalar function of position variable $\vec{r}$ with a set of co-ordinates (x, y, z) then we denote the scalar field as $\phi = (\vec{r}) = \phi(x, y, z)$.

Examples. (*i*) Variation of temperature at various points along a metal rod one end of which is heated while the other end is kept cold is an example of *scalar temperature field.*

(*ii*) Variation of electric potential at various points surrounding a charged body is an example of *scalar potential field*.

It should be noted that a surface passing through all such points which have the same value of scalar field is called *level surface*, as for example *equipotiential surface*.

Vector field. *A region in which a vector quantity is continuous and is defined by a single value (in magnitude and direction) at every point of the position variable is called a vector field.*

If V is vector function of position variable r with a set of co-ordinates (x, y, z) then we denote the vector field as

$$V(r) = \vec{V}(x, y, z)$$

Examples. (*i*) An *electric field* is an example of a vector field. The force experienced by a unit positive charge placed at any point in the field gives the *magnitude* of the vector field and the direction in which the unit positive charge moves, if free to do so gives the *direction* of the vector field.

A vector field is represented by *flux lines* or *lines of flow.*

(*ii*) The *gravitational field* acting on a body is another example of vector field.

However, a quantity which possesses both magnitude and direction is *not necessarily a vector*. A vector besides possessing magnitude and direction must obey rules of addition and multiplication according to vector algebra. For example, it should add according to parallelogram law of vectors.

As an example, '*finite rotation*' has a magnitude and direction but is not a vector quantity. Similarly, *electric current* possesses both magnitude and direction but is not a vector.

13.18 GRADIENT OF A SCALAR FIELD IN CARTESIAN CO-ORDINATES

Consider a point P in the scalar field having position vector. $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and let the value of the scalar field at this point be $\phi(\vec{r}) = \phi(x, y, z)$ where ϕ (xyz) is continuous differentiable function of three independent space co-ordinates x, y, z.

The partial derivative of ϕ with respect to x (keeping y and z constant) $= \dfrac{\partial \phi}{\partial x}$. It measures the rate of change of ϕ at the point P along the x-direction. Similarly, $\dfrac{\partial \phi}{\partial y}$ = rate of change of ϕ at the point P along the Y-direction and $\dfrac{\partial \phi}{\partial z}$ = rate of change of ϕ at the point P along the Z-direction. The function ϕ, therefore, has different rates of variation along different directions. With the partial derivatives $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}$ we can construct at every point in space a vector whose components along the X, Y and Z directions are equal to the respective partial derivatives of the scalar function ϕ. This vector is known as the *gradient of* ϕ *or grad* ϕ.

$$\therefore \qquad grad\ \phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} \qquad \ldots (i)$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi = \vec{\nabla}\phi \qquad \ldots (ii)$$

The symbol $\vec{\nabla}$ (read as Del) is called vector differential operator or Del operator. It should be clearly noted that $\vec{\nabla}$ is not a vector but an operator which obeys laws of vectors

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

13.19 PHYSICAL SIGNIFICANCE OF GRAD ϕ

Consider a point P having position co-ordinates x, y, z, and point Q very close to it having position co-ordinates $x + dx, y + dy, z + dz$ as shown in Fig. 13.18. the vector distance between the two points P and Q is

$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

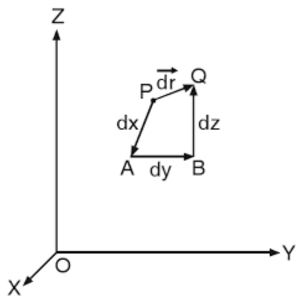

Fig. 13.18

Let ϕ be the value of a scalar function at P and $\phi + d\phi$ its value at Q, then the change in value of $\phi = d\phi$ is known as the *total differential of* ϕ.

Now rate of chnage of ϕ along X-axis = $\frac{\partial \phi}{\partial x}$

$$\therefore \qquad \text{Value of } \phi \text{ at } A = \phi + \frac{\partial \phi}{\partial x}dx$$

The rate of change of ϕ along Y-axis = $\frac{\partial \phi}{\partial y}$

$$\therefore \qquad \text{Value of } \phi \text{ at } B = \phi + \frac{\partial \phi}{\partial x}dx + \frac{\partial}{\partial y}\left(\phi + \frac{\partial \phi}{\partial x}dx\right)dy$$

$$= \phi + \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial^2 \phi}{\partial x\,\partial y}dx\,dy$$

$$= \phi + \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$$

because the terms $\frac{\partial^2 \phi}{\partial x\,\partial y}dx\,dy$ contain the product of two very small quantities and can be neglected.

The rate of change of ϕ along Z – axis = $\frac{\partial \phi}{\partial z}$

$$\therefore \qquad \text{Value of } \phi \text{ at } Q = \phi + \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial}{\partial z}\left(\phi + \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy\right)dz$$

$$= \phi + \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz$$

Neglecting the terms $\frac{\partial^2\phi}{\partial x \partial z} dx\, dz$ and $\frac{\partial^2\phi}{\partial y \partial z} dy\, dz$

$$\therefore \qquad d\phi = \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz$$

$$= \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$\text{or} \qquad d\phi = \vec{\nabla}\phi \cdot d\vec{r} = \vec{\nabla}\phi \cdot \hat{r}\, dr$$

$$\therefore \qquad \frac{d\phi}{dr} = \vec{\nabla}\phi \cdot \hat{r}$$

where $\hat{r}$ is a unit vector along PQ the direction of displacement. Thus $\frac{d\phi}{dr}$ is the directional derivative of ϕ. The rate of change is maximum if $\hat{r}$ is along $\vec{\nabla}\phi$ *i.e.*, the angle between $\vec{\nabla}\phi$ and $\hat{r}$ is zero.

Hence gradient of a scalar field ϕ defines a vector field the magnitude of which is equal to the maximum rate of change of ϕ and the direction of which is the same as the direction of displacement along which the rate of change is maximum.

Gradient of scalar field is a vector. From relation (*i*) of article 13.18 we have

$$\text{grad } \phi = \vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k},$$

This shows that grad ϕ is a vector whose x, y, z components are

$$\frac{\partial\phi}{\partial x}\hat{i}, \frac{\partial\phi}{\partial y}\hat{j} \text{ and } \frac{\partial\phi}{\partial z}\hat{k}$$

Hence, the gradient of scalar field is a vector.

For example, electric potential V is a scalar function but $-\vec{\nabla}V = \vec{E}$ the electric field intensity is a vector which shows how V varies in the neighbourhood of a point in space.

Vector from a scalar field. The gradient of a scalar field (function) grad $\phi = \vec{\nabla}\phi$ is a vector. Thus, by considering gradient of a scalar field we get a vector field starting from a scalar field.

13.20 GEOMETRICAL INTERPRETATION OF GRAD ϕ

The value of $\frac{d\phi}{dr} = 0$ when we move in a direction perpendicular to the direction $\vec{\nabla}\phi$ *i.e.*, when the angle between $\hat{r}$ in the direction of displacement and $\vec{\nabla}\phi$ = 90° as shown in Fig. 13.19. In such a case because $d\phi$ = 0, ϕ = a constant. This defines a three-dimensional surface and $\vec{\nabla}\phi$ a vector normal to the surface.

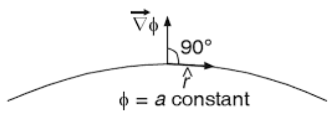

Fig. 13.19

A unit vector normal to the surface $\phi(x, y, z) =$ a constant is given by

$$\frac{\vec{\nabla}\phi}{\left|\vec{\nabla}\phi\right|}$$

In other words, *the gradient of a scalar function at any point is directed normally to the surface in the scalar field over which the value of scalar function is constant.*

Example 13.24. Calculate grad r where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Solution.

$$\text{grad } r = \vec{\nabla} r = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{\frac{1}{2}}$$

because

$$r = \left|\vec{r}\right| = (x^2 + y^2 + z^2)^{\frac{1}{2}}$$

Now

$$i\frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{\frac{1}{2}} = \hat{i}\left[\frac{1}{2}(x^2 + y^2 + z^2)^{1/2} 2x\right] = i\,[x\,(x^2 + y^2 + z^2)^{-1/2}]$$

$$= \hat{i}\,\frac{x}{(x^2 + y^2 + z^2)^{1/2}} = \hat{i}\,\frac{x}{r}$$

Similarly,

$$\hat{j}\frac{\partial}{\partial y}(x^2 + y^2 + z^2)^{1/2} = \hat{j}\,\frac{y}{r}$$

and

$$\hat{k}\frac{\partial}{\partial z}(x^2 + y^2 + z^2)^{1/2} = \hat{k}\,\frac{z}{r}$$

$\therefore$

$$\vec{\nabla} r = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}$$

$$= \frac{\vec{r}}{r} = \frac{\vec{r}}{\left|\vec{r}\right|} = \hat{r}$$

Example 13.25. Find the value of $\vec{\nabla} r^n$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Solution.

$$\vec{\nabla} r^n = \left(i\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{\frac{n}{2}}$$

$$= \left[\frac{n}{2}(x^2 + y^2 + z^2)^{\frac{n}{2}-1} 2x\right]\hat{i} + \left[\frac{n}{2}(x^2 + y^2 + z^2)^{\frac{n}{2}-1} 2y\right]\hat{j}$$

$$+ \left[\frac{n}{2}(x^2 + y^2 + z^2)^{\frac{n}{2}-1} 2z\right]\hat{k}$$

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$$= n(x^2+y^2+z^2)^{\frac{n}{2}-1}(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= nr^{n-2}\vec{r}$$

Example 13.26. Prove that $\vec{\nabla}\left(\frac{1}{r^n}\right) = \frac{-n}{r^{n+2}}\vec{r}$ **where** $r = \sqrt{x^2+y^2+z^2}$ (*P.U.* 2003)

Solution. $$\vec{\nabla}\left(\frac{1}{r^n}\right) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{-\frac{n}{2}}$$

$$= \left[-\frac{n}{2}(x^2+y^2+z^2)^{-\frac{n}{2}-1}2x\right]\hat{i}+\left[-\frac{n}{2}(x^2+y^2+z^2)^{-\frac{n}{2}-1}2y\right]\hat{j}$$

$$+\left[-\frac{n}{2}(x^2+y^2+z^2)^{-\frac{n}{2}-1}2z\right]\hat{k}$$

$$= -n(x^2+y^2+z^2)^{\frac{(n+2)}{2}}(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= -\frac{n}{r^{n+2}}\vec{r}$$

Example 13.27. Given $\phi = x^4+y^4+z^4$ **determine** $\vec{\nabla}\phi$

Solution. $$\vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)(x^4+y^4+z^4)$$

$$= \left(\frac{\partial}{\partial x}x^4\right)\hat{i}+\left(\frac{\partial}{\partial y}y^4\right)\hat{j}+\left(\frac{\partial}{\partial z}z^4\right)\hat{k}$$

$$= 4x^3\hat{i}+4y^3\hat{j}+4z^3\hat{k}$$

$$= 4(x^3\hat{i}+y^3\hat{j}+z^3\hat{k})$$

Ex. *If* $\phi = x^{3/2}+y^{3/2}+z^{3/2}$ *find* $\vec{\nabla}\phi$ (*Pbi. U.* 2002, 2001)

Hint. $\vec{\nabla}\phi = \frac{3}{2}[x^{1/2}\hat{i}+y^{1/2}\hat{j}+z^{1/2}\hat{k}]$

Example 13.28. If $\phi(x, y, z) = 3x^2y - y^3x^2$ **be any scalar function of** x, y, z**, find out grad** ϕ **at 1, 2, 2.**

Solution. $$grad\ \phi = \vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}$$

Now $$\phi = 3x^2y - y^3x^2$$

$$\therefore \quad \frac{\partial \phi}{\partial x} = 6xy - 2xy^3; \ \frac{\partial \phi}{\partial y} = 3x^2 - 3y^2x^2; \ \frac{\partial \phi}{\partial z} = 0$$

$$\therefore \quad grad \ \ \phi = (6xy - 2xy^3)\,\hat{i} + (3x^2 - 3y^2x^2)\,\hat{j}$$

At the point 1, 2, 2

$$6xy - 2xy^3 = 12 - 16 = -4; \ 3x^2 - 3y^2x^2 = 3 - 12 = -9$$

$$\therefore \quad grad \ \ \phi = -4\hat{i} - 9\hat{j}$$

Ex. *If* $\phi\,(x, y, z) = 3x^2y - yz^2$ *find grad* ϕ *at the point* $(1, 2, -1)$ (*M.D.U.* 2001)

Hint. $$\frac{\partial \phi}{\partial x} = 6xy; \ \frac{\partial \phi}{\partial y} = 3x^2 - z^2; \ \frac{\partial \phi}{\partial z} = -2yz$$

$$\therefore \quad \vec{\nabla}\phi = 6xy\hat{i} + (3x^2 - z^2)\hat{j} - 2yz\hat{k}$$

At point $(1, 2, -1)$, *grad* $\phi = 12\hat{i} + 2\hat{j} + 4\hat{k}$

Example 13.29. If $\phi\,(x, y, z) = 3x^3y - y^2z^2$, find $\vec{\nabla}\phi$ at 1, – 1, 1. (*Nagpur U.* w/2007)

Solution. Now $\quad \phi = 3x^3y - y^2z^2$

$$\frac{\partial \phi}{\partial x} = 9x^2y; \quad \frac{\partial \phi}{\partial y} = 3x^3 - 2yz^2; \quad \frac{\partial \phi}{\partial z} = -2y^2z$$

$$\therefore \quad \vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$= 9x^2y\,\hat{i} + (3x^3 - 2yz^2)\,\hat{j} - 2y^2z\,\hat{k}$$

At point $(1, -1, 1)$, $\vec{\nabla}\phi = -9\hat{i} + 5\hat{j} - 2\hat{k}$

Example 13.30. Find the gradient of scalar function $\phi\,(x, y) = x^2 - y^2$. (*Nagpur U. s*/2007)

Solution. $$grad \ \ \phi = \vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j}$$

Now $\quad \phi\,(x, y) = x^2 - y^2$

$$\frac{\partial \phi}{\partial x} = 2x; \quad \frac{\partial \phi}{\partial y} = -2y$$

$$\therefore \quad grad \ \ \phi = \vec{\nabla}\phi = 2x\hat{i} - 2y\hat{j} = 2\,(x\hat{i} - y\hat{j})$$

Example 13.31. Find a unit normal to the surface $x^2y + 2xz = 4$ at point $(2, -2, 3)$.

Solution. The point $(2, -2, 3)$ satisfies the equation $x^2y + 2xz = 4$

Therefore, the point $(2, -2, 3)$ lies on the surface represented by the equation $x^2y + 2xz = 4$

A unit normal to the surface of $\phi\,(x, y, z) =$ a constant is given by $\dfrac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$

Here ϕ is $\quad x^2y + 2xz = 4$

$$\therefore \qquad \vec{\nabla}\phi = \vec{\nabla}(x^2y + 2xz)$$

$$= \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

$$\frac{\partial \phi}{\partial x} = 2xy + 2z; \quad \frac{\partial \phi}{\partial y} = x^2; \quad \frac{\partial \phi}{\partial z} = 2x$$

At the point (2, – 2, 3) $\quad \frac{\partial \phi}{\partial x} = -2; \quad \frac{\partial \phi}{\partial y} = 4; \quad \frac{\partial \phi}{\partial z} = 4$

$$\therefore \qquad \vec{\nabla}\phi = -2\hat{i} + 4\hat{j} + 4\hat{k}$$

Also $\quad \left|\vec{\nabla}\phi\right| = \sqrt{(-2)^2 + (4)^2 + (4)^2} = \sqrt{36} = 6$

$\therefore$ Unit normal to the surface $x^2y + 2xz = 4$ at the point (2, – 2, 3)

$$\frac{-2\hat{i} + 4\hat{j} + 4\hat{k}}{6} = -\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

13.21 LINE INTEGRAL OF A VECTOR FUNCTION

Any integral along a curve is called a line integral.

Let $\vec{F}(x, y, z)$ be a vector function and AB be a curve (Fig. 13.20)

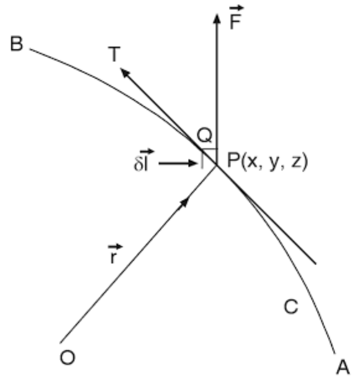

Fig. 13.20

Line integral of a vector function $\vec{F}$ along the curve AB is the integral of the component of $\vec{F}$ along the tangent to the curve at a point P.

The curve AB is divided into a large number of small line elements.

Let dl be the length of one such line element at P, θ the angle between $\vec{F}$ and the tangent PT at P.

The component of $\vec{F}$ along the tangent at $P = F\cos\theta$

$\therefore$ the product of $F\cos\theta$ and dl at P

$$= (F \cos \theta)\ (dl) = F\ dl \cos \theta = \vec{F} \cdot d\vec{l}$$

The limit of the sum of these products as the number of the line elements tends to infinity and the length of each line element tends to zero is called the line integral of $\vec{F}$ from A to B and is denoted by

$$\int_A^B \vec{F} \cdot d\vec{l}, \quad \text{or} \quad \int_C \vec{F} \cdot d\vec{l}$$

Thus $$\int_C \vec{F} \cdot d\vec{l} = \lim_{n \to \infty} \sum_{i=1}^{n} \vec{F}_i \cdot d\vec{l}_i$$

If $\vec{F}$ is a variable force acting on a particle along the curve AB, then the line integral $\int_A^B \vec{F} \cdot d\vec{l}$ represents the work done by $\vec{F}$ in moving the particle from A to B. If C is a simple closed curve, *i.e.* a curve which does not intersect itself anywhere, the line integral around the curve is denoted by $\oint \vec{F} \cdot d\vec{l}$.

13.22 SURFACE INTEGRAL OF A VECTOR FUNCTION

Let $\vec{F}(x, y, z)$ be a vector function of position and S be the area of a given surface (Fig. 13.21).

Surface integral of a vector function $\vec{F}$ over the surface S is the integral of the component of $\vec{F}$ along the normal to the surface.

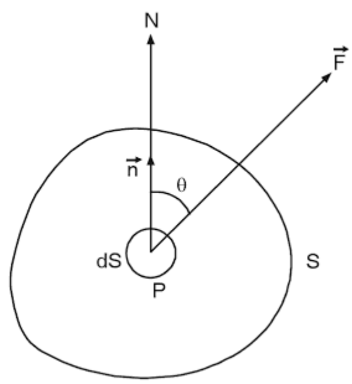

Fig. 13.21

The surface S is divided into a large number of small surface elements.

Let dS be the area of one such surface element at P, θ the angle between $\vec{F}$ and the normal PN to dS, $\vec{n}$ the unit vector in the direction of PN.

The component of $\vec{F}$ along the normal PN to $dS = F \cos \theta$

The product of $F \cos \theta$ and dS at P

$$= (F \cos \theta)\ (dS) = FdS \cos \theta$$

$$= \vec{F} \cdot \vec{n}\ dS = \vec{F} \cdot d\vec{S}$$

The limit of the sum of these products as the number of the surface elements tends to infinity and the area of each surface element tends to zero is called the surface integral of $\vec{F}$ over the surface S and is denoted by

$$\iint_S \vec{F} \cdot d\vec{S}, \qquad \text{or} \qquad \iint_S \vec{F} \cdot \vec{n}\, dS$$

Thus

$$\iint_S \vec{F} \cdot d\vec{S} = \lim_{n \to \infty} \sum_{i=1}^{n} \vec{F}_i \cdot d\vec{S}$$

This surface integral is called flux of vector $\vec{F}$ over area S.

Other surface integrals are.

$$\iint_S \phi\ dS, \qquad \iint_S \phi\ \vec{n}\ dS, \qquad \iint_S \vec{F} \times d\vec{S}$$

where ϕ is a scalar point function.

The surface integral over a closed surface is sometimes denoted by $\oiint$

13.23 FLUX OF A VECTOR

If a small plane area of magnitude dS is drawn around a point in the field of a vector, then the product of the component of the vector along the normal to the area at that point, and the magnitude of the area is called the flux of the vector through that area.

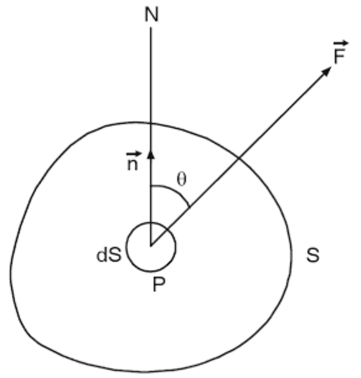

Fig. 13.22

In Fig. 13.22, the flux $d\phi$ of the vector $\vec{F}$ at point P through the surface area of magnitude dS around P is given by

$$d\phi = F \cos \theta\ dS$$

$$= \vec{F} \cdot \vec{n}\ dS = \vec{F} \cdot d\vec{S}$$

where θ is the angle between the direction of $\vec{F}$ at P and the normal PN, and $\vec{n}$ is the unit vector in the direction of the normal PN to dS.

The total flux of $\vec{F}$ through the surface of area S is

$$\phi = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_S \vec{F} \cdot d\vec{S}$$

where $d\vec{S}$ is a vector area of magnitude dS whose direction is that of the outward drawn unit normal $\vec{n}$ (or $\hat{n}$).

13.24 VOLUME INTEGRAL

The triple integral of a function $f(x, y, z)$ over a volume V enclosed by a closed surface is called volume integral of a function over the volume V, and is denoted by

$$\iiint_V f(x, y, z)\, dV .$$

Volume integrals usually occur as integrals of scalar quantities.

13.25 LAMINAR (LAMELLAR) VECTOR FIELD

A vector field which can be expressed as the gradient of a scalar field is known as a lamellar (or laminar) vector field.

Electric field is an example of a lamellar field since

$$\vec{E} = -\vec{\nabla} V$$

where V is the electrical potential which is a scalar function.

The word lamellar (or laminar) means that the field can be divided into laminas or layers over which the value of the scalar function whose gradient gives the vector field $\vec{E}$ remains constant.

Equipotential surfaces in the electric field are an example of such laminae.

13.26 DIVERGENCE OF A VECTOR FIELD

Just as we can get a vector field starting from a scalar field, we can get a scalar field starting from a vector field.

Consider a vector field $\vec{A}$, the magnitude and direction of which is a function of the position co-ordinaes at a point, then the scalar product of the differential operator $\vec{\nabla}$ and the vector $\vec{A}$ is a *scalar* function of position co-ordinates x, y, z and is known as the *divergence of the vector* $\vec{A}$.

$$\therefore \qquad \text{div. } \vec{A} = \vec{\nabla} \cdot \vec{A}$$

where

$$\vec{\nabla} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right)$$

and

$$\vec{A} = (\hat{i} A_x + \hat{j} A_y + \hat{k} A_z)$$

$$\therefore \qquad \vec{\nabla} \cdot \vec{A} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (\hat{i} A_x + \hat{j} A_y + \hat{k} A_z)$$

$\vec{\nabla} \cdot \vec{E} \neq 0$, means the divergence of electric field is not equal to zero *i.e.,* the *electric field is not solenoidal*.

Example 13.32. Calculate the divergence of the vector function

$$\vec{A} = xy\,\hat{i} + yz\,\hat{j} + zx\,\hat{k}. \qquad \text{(Nagpur Uni. s/2007)}$$

Solution.

$$\vec{\nabla} \cdot \vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (xy\,\hat{i} + yz\,\hat{j} + zx\,\hat{k})$$

$$= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(zx)$$

$$= y + z + x$$

Example 13.33. If $\vec{A} = x^3 z\,\hat{i} + 3y^2 z^2\,\hat{j} - 4\,xyz^2\,\hat{k}$, find div $\vec{A}$ at the point (2, – 1, 1).

(*Nagpur Uni. s*/2009, *s*/2006)

Solution.

$$\text{div } \vec{A} = \vec{\nabla} \cdot \vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (x^3 z\,\hat{i} + 3y^2 z^2\,\hat{j} - 4xyz^2\,\hat{k})$$

$$= \frac{\partial}{\partial x}(x^3 z) + \frac{\partial}{\partial y}(3y^2 z^2) + \frac{\partial}{\partial z}(-4xyz^2)$$

$$= 3x^2 z + 6yz^2 - 8xyz$$

At point (2, – 1, 1), $\quad \text{div } \vec{A} = \vec{\nabla} \cdot \vec{A} = (12 - 6 + 16) = 22$

Example 13.34. If ϕ is a scalar field and $\vec{A}$ a vector field, find the value of div $\phi(\vec{A})$

(*Meerut U.* 2003)

Solution. If ϕ is a scalar field and $\vec{A}$ a vector field, then div. $(\phi\vec{A}) = \vec{\nabla} \cdot (\phi\vec{A})$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (\phi\,A_x\,\hat{i} + \phi\,A_y\,\hat{j} + \phi\,A_z\,\hat{k})$$

$$= \frac{\partial}{\partial x}(\phi\,A_x) + \frac{\partial}{\partial y}(\phi\,A_y) + \frac{\partial}{\partial z}(\phi\,A_z)$$

$$= \frac{\partial\phi}{\partial x}A_x + \phi\,\frac{\partial A_x}{\partial x} + \frac{\partial\phi}{\partial y}A_y + \phi\,\frac{\partial A_y}{\partial y} + \frac{\partial\phi}{\partial z}dz + \phi\,\frac{\partial A_z}{\partial z}$$

$$= \left(\frac{\partial\phi}{\partial x}A_x + \frac{\partial\phi}{\partial y}A_y + \frac{\partial\phi}{\partial z}A_z\right) + \phi\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$$

$$= \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right) \cdot (A_x\,\hat{i}\;A_y\,\hat{j} + A_z\,\hat{k}) + \phi\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$$

$$= \vec{\nabla}\phi \cdot \vec{A} + \phi\,(\vec{\nabla} \cdot \vec{A})$$

or $\quad div\,(\phi\,\vec{A}) = grad\,\phi \cdot \vec{A} + \phi\;div\,(\vec{A})$ or $\vec{\nabla} \cdot (\phi\vec{A}) = \vec{\nabla}\phi \cdot \vec{A} + \phi\vec{\nabla} \cdot \vec{A}$

Example 13.38. Find the constant '*a*' so that the vector

$$\vec{A} = (x + 3y)\,\hat{i} + (2y + 3z)\,\hat{j} + (x + az)\,\hat{k}$$

is a solenoidal vector. (*G.N.D.U.* 2001)

Solution. $$\vec{\nabla} \cdot \vec{A} = \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(2y + 3z) + \frac{\partial}{\partial z}(x + az) = 1 + 2 + a = 0$$

$$\therefore \quad a = -3$$

Example 13.39. If the vector $\vec{A} = 3xyz\,\hat{i} + 2xy^2\,\hat{j} - x^2yz\,\hat{k}$ and scalar function $\phi = 3x^2 - yz$, evaluate *div* $(\phi\vec{A})$ at position (1, – 1, 1).

Solution. $$div\,(\phi\vec{A}) = \vec{\nabla}\phi \cdot \vec{A} + \phi\, div\,(\vec{A})$$

Here $$\phi = 3x^2 - yz \quad \text{and} \quad \vec{A} = 3xyz\,\hat{i} + 2xy^2\,\hat{j} - x^2yz\,\hat{k}$$

$$\therefore \quad grad\,\phi = \vec{\nabla}\phi = \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right) = 6x\hat{i} - z\,\hat{j} - y\,\hat{k}$$

$$\therefore \quad grad\,\phi \cdot \vec{A} = \vec{\nabla}\phi \cdot \vec{A} = (6x\hat{i} - z\,\hat{j} - y\,\hat{k}) \cdot (3xyz\,\hat{i} + 2xy^2\,\hat{j} - x^2yz\,\hat{k})$$
$$= 18x^2yz - 2xy^2z + x^2y^2z$$
$$= xyz\,(18x - 2y + xy)$$

$$div\,\vec{A} = \vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 3yz + 4xy - x^2y$$

$$\therefore \quad \phi\,(div\,\vec{A}) = \phi\,(\vec{\nabla} \cdot \vec{A}) = (3x^2 - yz)\,(3yz + 4xy - x^2y)$$

At the point 1, – 1, 1

$$grad\,\phi \cdot \vec{A} = \vec{\nabla}\phi \cdot \vec{A} = -1(18 + 2 - 1) = -19$$

$$\phi\,(div\,\vec{A}) = (4)\,(-6) = -24$$

$$\therefore \quad (div\,\phi\vec{A}) = \phi\,(\vec{\nabla} \cdot \vec{A}) = grad\,\phi \cdot \vec{A} + \phi\,(div\,\vec{A})$$

or $$\vec{\nabla} \cdot (\phi\vec{A}) = \vec{\nabla}\phi \cdot \vec{A} + \phi\,(\vec{\nabla} \cdot \vec{A})$$
$$= -19 - 24 = -43$$

Example 13.40 If $\phi = x^2 - y^2 + 2z$, find *div. grad* ϕ.

Solution. $$div.\ grad\,\phi = \vec{\nabla} \cdot \vec{\nabla}\phi$$

Now $$\vec{\nabla}\phi = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)(x^2 - y^2 + 2z)$$

$$= 2x\hat{i} - 2y\,\hat{j} + 2\,\hat{k}$$

$$\therefore \quad \vec{\nabla} \cdot \vec{\nabla}\phi = \vec{\nabla} \cdot (2x\hat{i} - 2y\,\hat{j} + 2\,\hat{k})$$

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$$\sum (div \vec{A})\, \Delta v_i = \iiint_v (div \vec{A})\, \Delta v$$

$$\therefore \quad \sum \iint_{\Delta S_i} \vec{A} \cdot d\vec{s} = \iiint_v (div \vec{A})\, \Delta v \qquad \dots (ii)$$

Comparing (i) and (ii), we get

$$\iint_S \vec{A} \cdot d\vec{s} = \iiint_v (div \vec{A})\, dv = \iiint_v (\vec{\nabla} \cdot \vec{A})\, dv$$

The importance of this theorem lies in the fact that *it helps us to convert a surface integral into a volume integral and vice versa.*

Example 13.41 Using Gauss's divergence theorem evaluate

$$\iint_S \boldsymbol{x\, dy\, dz + y\, dz\, dx + z\, dx\, dy}$$

where S is the sphere $x^2 + y^2 + z^2 = 1$.

Solution. The sphere $x^2 + y^2 + z^2 = 1$ has a radius unity and its centre is at the origin. As a general case, consider a sphere of radius r, with centre at the origin, then Gauss's divergence theorem can be put in the form

$$\oiint \vec{r} \cdot d\vec{s} = \iiint_v (\vec{\nabla} \cdot \vec{r})\, dv$$

where $\vec{r}$ is the radius vector, $d\vec{s}$ a small vector area element and dv a small volume element.

Now for the given sphere

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

and $$d\vec{s} = dy\,dx\,\hat{i} + dx\,dz\,\hat{j} + dx\,dy\,\hat{k}$$

$$\therefore \quad \vec{r} \cdot d\vec{s} = (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (dy\,dz\,\hat{i} + dx\,dz\,\hat{j} + dx\,dy\,\hat{k})$$

$$= x\, dy\, dz + y\, dx\, dz + z\, dx\, dy$$

$$\therefore \quad \iint_{S_2} \vec{A} \cdot d\vec{s} = \iint_{S_2} (\hat{i} + y^2\hat{j} + z^2\hat{k}) \cdot ds\hat{i} = \iint_{S_2} ds = S_2$$

The area of the surface S_2 ($y = 1$, $z = 1$) = 1

Hence $$\iint_{S_1} \vec{A} \cdot d\vec{s} + \iint_{S_2} \vec{A} \cdot d\vec{s} = 1$$

Similarly $$\iint_{S_3} \vec{A} \cdot d\vec{s} + \iint_{S_4} \vec{A} \cdot d\vec{s} = 1$$

and $$\iint_{S_5} \vec{A} \cdot d\vec{s} + \iint_{S_6} \vec{A} \cdot d\vec{s} = 1$$

$$\therefore \qquad \iint_S \vec{A}\cdot d\vec{s} = \iint_{S_1} \vec{A}\cdot d\vec{s} + \iint_{S_2} \vec{A}\cdot d\vec{s} + \iint_{S_3} \vec{A}\cdot d\vec{s} + \iint_{S_4} \vec{A}\cdot d\vec{s} + \iint_{S_5} \vec{A}\cdot d\vec{s}$$

$$+ \iint_{S_6} \vec{A}\cdot d\vec{s} = 1+1+1 = 3$$

Example 13.42. A vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Prove that $\iint_S \vec{r}\cdot d\vec{s} = 3V$ where V is the volume enclosed by the surface S.

Solution. According to Gauss's divergence theorem

$$\oiint_S \vec{A}\cdot d\vec{s} = \iiint_V (\vec{\nabla}\cdot\vec{A})\, dv$$

Here $$\vec{A} = \vec{r}$$

$$\therefore \qquad \oiint_S \vec{r}\cdot d\vec{s} = \iiint_V (\vec{\nabla}\cdot\vec{r})\, dv$$

Now $$\vec{\nabla}\cdot\vec{r} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= 1 + 1 + 1 = 3$$

$$\therefore \qquad \iint_S \vec{r}\cdot d\vec{s} = \iiint_V 3\,dv = 3\iiint_V dv = 3V$$

13.31 CURL OF A VECTOR FIELD

Consider a vector field $\vec{A}$ the magnitude and direction of which is a function of position co-ordinates at a point, then the vector (cross) product of the differential operator $\vec{\nabla}$ and the vector $\vec{A}$ is a *vector* function of position co-ordinates x, y, z and is known as the *curl* of the vector $\vec{A}$.

$$\therefore \qquad Curl\ \vec{A} = \vec{\nabla}\times\vec{A}$$

Now $$\vec{\nabla} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$$

and $$\vec{A} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})$$

$$\therefore \qquad \vec{\nabla}\times\vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\times(A_x\hat{i} + A_y\hat{j} + A_z\hat{k})$$

$$= \frac{\partial A_y}{\partial x}\hat{i}\times\hat{j} + \frac{\partial A_z}{\partial x}\hat{i}\times\hat{k} + \frac{\partial A_x}{\partial y}\hat{j}\times\hat{i} + \frac{\partial A_z}{\partial y}\hat{j}\times\hat{k} + \frac{\partial A_x}{\partial z}\hat{k}\times\hat{i} + \frac{\partial A_y}{\partial z}\hat{k}\times\hat{j}$$

$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}$$

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The above expression can be put in determinant form as under

$$Curl\ \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Physical significance. Consider a vanishingly small area $\Delta\vec{S}$ enclosed by a closed path which forms its boundary. Let $\vec{A}$ be a vector field defined everywhere in the region and $d\vec{l}$ a vector representing a small element of the path, its direction being that of the tangent to the path at the element $d\vec{l}$. If θ is the angle between the vector $\vec{A}$ at any point on the curve and the direction of the element, then

$$\vec{A}\cdot d\vec{l} = A\ dl \cos\theta$$

As $A\cos\theta$ is the component of the vector $\vec{A}$ along the element $d\vec{l}$, the line integral of the vector field $\vec{A}$ for the closed path

$$= \oint A\ dl\cos\theta = \oint \vec{A}\cdot d\vec{l}$$

where the symbol $\oint$ represents the integration over the entire closed path.

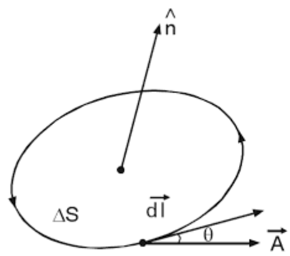

Fig. 13.26

It is a scalar quantity. The path encloses an area $\Delta\vec{S}$.

$\therefore$ Line integral per unit area $= \dfrac{1}{\Delta S}\oint \vec{A}\cdot d\vec{l}$

Now, consider a unit vector $\hat{n}$ in a direction perpendicular to the area S, the positive direction of $\hat{n}$ being related to the direction of the line integral by the right handed screw rule. *The curl of the vector field $\vec{A}$ (curl $\vec{A}$) is defined as the vector, the magnitude of whose component in the direction of the unit vector $\hat{n}$ is given by the line integral of the vector $\vec{A}$ for the closed path per unit area enclosed by it when the area becomes vanishingly small.* As the magnitude of the component of a vector in the direction of the unit vector is given by the scalar product of the vector and the unit vector $\hat{n}$.

$$\therefore \qquad Curl\ \vec{A}\cdot\hat{n} = \lim_{\Delta S\to 0}\frac{1}{\Delta S}\oint \vec{A}\cdot d\vec{l}$$

As $\Delta S \to 0$ let it be denoted by dS, then

$$Curl\ \vec{A}\cdot\hat{n}\,dS = \oint \vec{A}\cdot d\vec{l}$$

$$\therefore \qquad Curl\ \vec{A}\cdot d\vec{S} = \oint \vec{A}\cdot d\vec{l}$$

or $$Curl\ \vec{A} = \frac{1}{d\vec{S}} \oint \vec{A} \cdot d\vec{l} \qquad \ldots (i)$$

***Curl* of a vector field is a vector.** *Curl* of a vector field $\vec{A}$ is given by the cross (vector) product of the operator $\vec{\nabla}$ which behaves as a vector and $\vec{A}$ *i.e.*,

$$Curl\ \vec{A} = \vec{\nabla} \times \vec{A}$$

$\therefore$ $Curl\ \vec{A}$ is a *vector.*

Relation (*i*) also shows that $Curl\ \vec{A}$ is a vector.

13.32 CURL OF A VECTOR IN CARTESIAN CO-ORDINATES

To calculate the value of the curl of a vector we use the definition

$$Curl\ \vec{A} = \underset{\Delta S \to 0}{\text{Lim}} \frac{1}{\Delta \vec{S}} \oint \vec{A} \cdot d\vec{l}$$

where $\oint \vec{A} \cdot d\vec{l}$ is the line integral of the vector field $\vec{A}$ for the closed path enclosing a vanishingly small area $\Delta\vec{S}$. To find the value of $Curl\ \vec{A}$ in Cartesian co-ordinates we shall take up one component at a time. Consider a point P having co-ordinates x, y, z. For simplicity, with P at its centre draw a rectangular path $ABCD$ of surface parallel to XY plane having its surface AB of length Δx parallel to X-axis and surface BC of length Δy parallel to Y-axis.

In such a case the unit vector $\hat{n}$ = unit vector $\hat{k}$ parallel to Z axis and we shall get the Z-component of the $Curl\ \vec{A}$ denoted as $curl_z\ \vec{A}$.

According to right hand screw rule, the direction of line integral around the closed rectangular path must be clockwise as seen by someone looking up in the direction $\hat{n}$. Let A_x, A_y, A_z be the components of the vector $\vec{A}$ in the directionn of X, Y, Z, axes respectively at the point x, y, z.

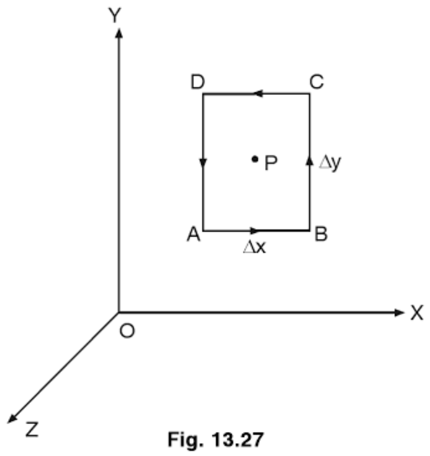

Fig. 13.27

The co-ordinates of a point at the centre of AB are $x, \left(y - \frac{\Delta y}{2}\right), z$. If the rate of change of A_x along the Y-axis is $\frac{\partial A_x}{\partial y}$ then the value of A_x at the centre of AB

$$= A_x - \frac{\partial A_x}{\partial y} \frac{\Delta y}{2}$$

Similarly, the value of A_x at the centre of CD

$$= A_x + \frac{\partial A_x}{\partial y} \frac{\Delta y}{2}$$

The co-ordinates of a point at the centre of BC are $\left(x + \frac{\Delta x}{2}\right), y, z$. If the rate of change of A_y along the X-axis is $\left(\frac{\partial A_y}{\partial x}\right)$ then the value of A_y at the centre of BC

$$= A_y + \frac{\partial A_y}{\partial x}\frac{\Delta x}{2}$$

Similarly, the value of A_y at the centre of DA

$$= A_y - \frac{\partial A_y}{\partial x}\frac{\Delta x}{2}$$

Hence
$$\oint \vec{A}\cdot d\vec{l} = \left(A_x - \frac{\partial A_x}{\partial y}\frac{\Delta y}{2}\right)\Delta x + \left(A_y + \frac{\partial A_y}{\partial x}\frac{\Delta x}{2}\right)\Delta y$$
$$- \left(A_x + \frac{\partial A_x}{\partial y}\frac{\Delta y}{2}\right)\Delta x - \left(A_y - \frac{\partial A_y}{\partial x}\frac{\Delta x}{2}\right)\Delta y$$

$$= \frac{\partial A_y}{\partial x}\Delta x\,\Delta y - \frac{\partial A_x}{\partial y}\Delta x\,\Delta y$$

$$= \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\Delta x\,\Delta y$$

$$Curl_z A = \lim_{\Delta S \to 0} \frac{1}{\Delta \vec{S}} \oint \vec{l}\cdot d\vec{l}$$

$$= \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \qquad \ldots [\because\ \Delta S = \Delta x\,\Delta y]$$

Similarly, x and y components of the curl $\vec{A}$ are

$$Curl_x A = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

$$Curl_y A = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}$$

$$\therefore \qquad Curl\ \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}$$

$$= \vec{\nabla}\times\vec{A}$$

Irrotational field. *A vector field whose curl vanishes* (= 0) *is called irrotational field.* If $\vec{A}$ represents an irrotational field, then *curl* $\vec{A} = \vec{\nabla} \times \vec{A} = 0$

Example. Electric field $\vec{E}$ is an example of an irrotational field since

$$\vec{\nabla} \times \vec{E} = 0$$

[**Note.** A vector field whose curl is not equal to zero is called a *rotational vector field.* Magnetic field $\vec{B}$ is an example of rotational vector field since $\vec{\nabla} \times \vec{B} \neq 0$]

Example 13.43. If $\vec{E} = (x + y)\,\hat{i} + (y - 2x)\,\hat{j} - 2z\,\hat{k}$, prove that Curl $\vec{E} = -3\hat{k}$ and div $\vec{E} = 0$. (*Pbi. U.* 2000)

Solution. Given $\vec{E} = (x+y)\,\hat{i} + (y-2x)\,\hat{j} - 2z\,\hat{k}$

(*i*) $$Curl\ \vec{E} = \vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (x+y) & (y-2x) & (-2z) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-2z) - \frac{\partial}{\partial z}(y-2x)\right] + \hat{j}\left[\frac{\partial}{\partial z}(x+y) - \frac{\partial}{\partial x}(-2z)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y-2x) - \frac{\partial}{\partial y}(x+y)\right]$$

$$= \hat{k}(-2-1) = -3\hat{k}$$

(*ii*) $$div\ \vec{E} = \vec{\nabla} \cdot \vec{E} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left[(x+y)\,\hat{i} + (y-2x)\,\hat{j} + (-2z)\,\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(x+y) + \frac{\partial}{\partial y}(y-2x) + \frac{\partial}{\partial z}(-2z)$$

$$= 1 + 1 - 2 = 0$$

Example 13.44. For a vector $\vec{A} = 2x^2 y\,\hat{i} + 3yz\,\hat{j} + x^2 y^2 z^2\,\hat{k}$, find curl $\vec{A}$ at the point (1, – 2, 0). (*Nagpur Uni. s*/2007)

Solution.

$$Curl\ \vec{A} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x^2 y & 3yz & x^2 y^2 z^2 \end{vmatrix}$$

As the divergence of the vector field $\vec{A} = 0$, *the field is solenoidal.*

(*ii*)
$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}xy - \frac{\partial}{\partial z}xz\right] + \hat{j}\left[\frac{\partial}{\partial z}yz - \frac{\partial}{\partial x}xy\right] + \hat{k}\left[\frac{\partial}{\partial x}xz - \frac{\partial}{\partial y}yz\right]$$

$$= \hat{i}\,[x - x] + \hat{j}\,[y - y] + \hat{k}\,[z - z] = 0$$

Since curl of the vector field $\vec{A}$ vanishes (= 0) *the field is irrotational*.

Example 13.47. If $\vec{A} = 2x^2z^2\hat{i} - 2xy^2\,\hat{j} + 2x^2y^2\,\hat{k}$, find the value of curl $\vec{A}$ at the point (1, 1, 1).

Solution.
$$Curl\,\vec{A} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x^2z^2 & -2xy^2 & 2x^2y^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2x^2y^2) - \frac{\partial}{\partial z}(-2xy^2)\right] + \hat{j}\left[\frac{\partial}{\partial z}(2x^2z^2) - \frac{\partial}{\partial x}(2x^2y^2)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(-2xy^2) - \frac{\partial}{\partial y}(2x^2z^2)\right]$$

$$= \hat{i}\,[4x^2y] + \hat{j}\,[4x^2z - 4xy^2] + \hat{k}\,[-2y^2]$$

Putting $x = 1$, $y = 1$, and $z = 1$, we get

$$\vec{\nabla} \times \vec{A} = 4\,\hat{i} + 0\,\hat{j} - 2\,\hat{k} = 4\,\hat{i} - 2\,\hat{k}$$

13.33 STOKE'S THEOREM

Stoke's theorem states that the surface integral of the curl of a vector $\vec{A}$ over a surface $\vec{S}$ of any shape is equal to the line integral of the vector field $\vec{A}$ over the boundary of that surface.

or
$$\iint_S (curl\,\vec{A}) \cdot d\,\vec{S} = \oint \vec{A} \cdot d\,\vec{l}$$

where $\oint$ represents the line integral over the closed path enclosing the surface $\vec{S}$.

The sense of integration is related with the positive normal to the surface S according to the right hand screw rule.

Stoke's theorem may also be stated in the form '*Stoke's theorem states that the line integral of a vector field $\vec{A}$ around any closed curve C is equal to the surface integral of the curl of $\vec{A}$ over an open surface S bounded by the curve C*'. Mathematically.

$$\oint_C \vec{A} \cdot d\vec{l} = \iint_S (Curl\ A) \cdot d\vec{s} = \iint_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{s}$$

we shall prove the theorem given in the form

$$\iint_S (Curl\ A) \cdot d\vec{s} = \oint \vec{A} \cdot d\vec{l}$$

Suppose a smooth closed curve l encloses vector area $\vec{S}$ in a vector field $\vec{A}$. Let the area $\vec{S}$ be divided into a large number of small areas $\Delta S_1, \Delta S_2, \ldots \Delta S_i, \ldots$ etc. having perimeters Δl_1, Δl_2, ... Δl_i etc. respectively, The line integrals of the vector $\vec{A}$ around each of the small paths Δl_1, Δl_2 ... Δl_i etc. will be in the same sense. Therefore, the line integral along the common boundary of two smal areas like ΔS_1 and ΔS_2 will cancel each other being in the opposite direction. Hence, the sum of all these line integrals will be equal to the line integral around l the boundary enclosing the whole area $\vec{S}$ because in traversing the small areas all parts of the line integral will cancel out except for those parts which are along the outer boundary l. Thus, the line integral around the closed curve l is equal to the sum of the line integrals around the paths Δl_1, Δl_2, ... Δl_i ... etc.

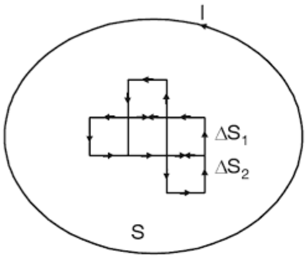

Fig. 13.28

or
$$\oint_l \vec{A} \cdot d\vec{l} = \sum \oint_{\Delta l_i} \vec{A} \cdot d\vec{l} \quad \ldots (i)$$

Any one of the small areas ΔS_i wil have a *curl* of the vector field of which the normal component

$$Curl_n\ \vec{A} = \lim_{\Delta s_i \to 0} \frac{1}{\Delta S_i} \oint_{\Delta l_i} \vec{A} \cdot d\vec{l}$$

or
$$\oint_{\Delta l_i} \vec{A} \cdot d\vec{l} = (Curl_n\ \vec{A})\ \Delta S_i$$

$\therefore$
$$\sum_{\Delta l_i} \oint \vec{A} \cdot d\vec{l} = \sum (Curl_n\ \vec{A})\ \Delta S_i$$

When $\Delta S_i = ds$ *i.e.*, an infinite number of small volume elements are involved and $\Delta S_i \to 0$, then

$$\sum (Curl_n\ \vec{A})\ \Delta S = \iint_S (Curl_n\ \vec{A}) \cdot d\vec{s} = \iint_S Curl\ \vec{A} \cdot d\vec{s}$$

$\therefore$
$$\sum_{\Delta l_i} \oint \vec{A} \cdot d\vec{l} = \iint_S Curl\ \vec{A} \cdot d\vec{s} \quad \ldots (ii)$$

From (*i*) and (*ii*), we get

$$\iint_S Curl\,\vec{A}\cdot d\,\vec{s} = \oint_l \vec{A}\cdot d\,\vec{l}$$

or
$$\iint_S (\vec{\nabla}\times\vec{A})\cdot d\,\vec{s} = \oint_l \vec{A}\cdot d\,\vec{l}$$

Importance. The importance of Stoke's theorem lies in the fact that it helps us to convert the line integral of a vector into the surface integral of the curl of that vector and vice-versa.

e.g. to prove Ampere's law in differential form $\vec{\nabla}\times\vec{B} = \mu_0\,\vec{J}$ we convert $\oint \vec{B}\cdot d\,\vec{l}$ into $\iint_S (\vec{\nabla}\times\vec{B})\cdot d\,\vec{s}$ using Stoke's theorem and equate it to $\mu_0 \iint_S \vec{J}\cdot d\,\vec{s}$, so that $\vec{\nabla}\times\vec{B} = \mu_0\,\vec{J}$.

Example 13.48. If the line integral of a vector $\vec{A}$ around a closed curve is equal to the surface integral of the vector $\vec{B}$ taken over the surface bounded by the given closed curve show that $\vec{B} = curl\,\vec{A}$.

Solution. Let a closed curve be bounded by the surface *S*, then

$$\oint \vec{A}\cdot d\,\vec{l} = \oiint_S \vec{B}\cdot d\,\vec{s} \qquad \text{... }(i)$$

According to Stoke's theorem,

$$\oint \vec{A}\cdot d\,\vec{l} = \oiint_S Curl\,\vec{A}\cdot d\,\vec{s} \qquad \text{... }(ii)$$

∴ From (*i*) and (*ii*), we have

$$\oiint \vec{B}\cdot d\,\vec{s} = \oiint Curl\,\vec{A}\cdot d\,\vec{s}$$

Hence
$$\vec{B} = Curl\,\vec{A}$$

Example 13.49. A vector $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$. Show that $\oint \vec{r}\cdot d\,\vec{r} = 0$.

Solution. According to Stoke's theorem

$$\oint \vec{A}\cdot d\,\vec{l} = \iint_S (\vec{\nabla}\times\vec{A})\cdot d\,\vec{s}$$

Here $\vec{A} = \vec{r}$ and $d\,\vec{l}$ = a line element = $d\,\vec{r}$

∴
$$\oint \vec{A}\cdot d\,\vec{l} = \oint \vec{r}\cdot d\,\vec{r}$$

∴
$$\oint \vec{r}\cdot d\,\vec{r} = \iint_S (\vec{\nabla}\times\vec{r})\cdot d\,\vec{r}$$

But
$$\vec{\nabla}\times\vec{r} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\times(x\,\hat{i} + y\,\hat{j} + z\,\hat{k})$$

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$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$$

$$\therefore \qquad \oint \vec{r} \cdot d\vec{r} = 0$$

Example 13.50. If the vector $\vec{A} = K(-y\hat{i} + x\hat{j})$, calculate (*i*) curl $\vec{A}$ (*ii*) $\oint \vec{A} \cdot d\vec{l}$ for closed curve $x^2 + y^2 = r^2$, $z = 0$.

Solution. (*i*) Vector $\vec{A}$

$$= K(-y\hat{i} + x\hat{j})$$

$$\therefore \qquad Curl\ \vec{A} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -Ky & +Kx & 0 \end{vmatrix}$$

$$= [+K - (-K)]\hat{k} = +2K\hat{k}$$

(*ii*) $\oint \vec{A} \cdot d\vec{l}$ for the closed curve $x^2 + y^2 = r^2$, $z = 0$

$$\vec{A} \cdot d\vec{l} = K(-y\hat{i} + x\hat{j}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= K(-y\,dx + x\,dy) \qquad \ldots (i)$$

The closed curve $x^2 + y^2 = r^2$ is satisfied if

we put $x = r\cos\theta$ and $y = r\sin\theta$

because $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

Hence $dx = -r\sin\theta\, d\theta$ and $dy = r\cos\theta\, d\theta$

Substituting in (*i*), we have

$$\vec{A} \cdot d\vec{l} = K[r^2\sin^2\theta\, d\theta + r^2\cos^2\theta\, d\theta] = Kr^2\, d\theta$$

$$\therefore \qquad \oint \vec{A} \cdot d\vec{l} = \oint Kr^2 d\theta = Kr^2 \oint d\theta = 2\pi Kr^2 \qquad \ldots (ii)$$

because a closed curve extends an angle 2π at its centre.

13.34 GREEN'S THEOREM IN A PLANE

Green's theorem in a plane is a special case of Stoke's theorem. It states:

'*If S is a closed region in xy plane bounded by a simple closed curve C and ϕ and ψ are continuous functions of x and y having continuous derivatives, then*

$$\oint_C \phi\, dx + \psi\, dy = \iint_S \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx\, dy$$

where the curve C is traversed in the anticlockwise direction.

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$$= \iint_S \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx\, dy$$

This equation is known as Green's theorem in a plane.

Expression of Green's Theorem in Vector Form

Let $\phi\,(x, y) = F_1$ and $\psi\,(x, y) = F_2$

Then, $$\phi\, dx + \psi\, dy = F_1\, dx + F_2\, dy = (F_1 \hat{i} + F_2 \hat{j}) \cdot (dx \hat{i} + dy \hat{j}) = \vec{F} \cdot d\vec{l}$$

and $$\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \qquad \dots (i)$$

$$= x - \text{component of } curl\, \vec{F}.$$

$$\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} = curl\, \vec{F} \cdot \hat{k} = (\nabla \times \vec{F}) \cdot \hat{k} \qquad \dots (ii)$$

Substituting these expressions in the mathematical statement of Green's theorem, we get

$$\oint_C \vec{F} \cdot d\vec{l} = \iint_S (\nabla \times \vec{F}) \cdot \hat{k}\, dR \qquad \dots (iii)$$

where $\hat{k}$ is unit vector in the direction of z-axis and $dR = dx\, dy$.

The above expression (*iii*) is Green's theorem in vector form. The equation shows that this is Stoke's theorem in the XY plane. Thus, Green's theorem in the plane is a special case of Stoke's theorem.

Example 13.51. Find the work done by a force of magnitude 15 units in displacing a particle at a point (1, 1, 1) to the point (2, 1, 3). Given that the force acts along the direction of the vector $(\hat{i} + 2\hat{j} + 2\hat{k})$.

Solution. Given : $|\vec{F}| = 15$ units; $A\,(1, 1, 1)$, $B\,(2, 1, 3)$,

$$\text{direction of force} = \hat{i} + 2\hat{j} + 2\hat{k}$$

$\therefore$ Unit vector in the direction $\vec{F}$ is given by

$$\hat{n} = \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$$

$\therefore$ The force in magnitude and direction is given by

$$\vec{F} = 15\hat{n} = \frac{15}{3}(\hat{i} + 2\hat{j} + 2\hat{k}) = 5\,(\hat{i} + 2\hat{j} + 2\hat{k})$$

Displacement $\vec{AB}$: Displacement from point A (1, 1, 1) to point B (2, 1, 3) is given by

$$\vec{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$
$$= (2 - 1)\hat{i} + (1 - 1)\hat{j} + (3 - 1)\hat{k}$$

$$= \hat{i} + 0\hat{j} + 2\hat{k}$$

$$\text{Work done} = \vec{F} \cdot \vec{AB} = 5(\hat{i} + 2\hat{j} + 2\hat{k}) \cdot (\hat{i} + 0\hat{j} + 2\hat{k})$$

$$= 5[(1)(1) + (2)(0) + (2)(2)]$$

$$= 5 \times 5 = 25 \text{ units.}$$

EXERCISE CH. 13

LONG QUESTIONS

1. (*a*) Deduce the expression for the scalar product of two vectors in terms of their rectangular components.

(*b*) Hence deduce an expression for the angle between the two vectors.

2. (*a*) Deduce the expression for vector product of two vectors in terms of their components.

(*b*) Hence derive an expression for the angle between two vectors.

3. (*a*) What are scalar and vector triple products? (*Meerut U.* 2002, 2001)

(*b*) Show that $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \times (\vec{C} \cdot \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$. (*Gauhati U.* 2007)

4. (*a*) Describe gradient of a scalar field in Cartesian co-ordinates. Explain its physical significance. Show that the gradient of a scalar function at any point is directed normally to the surface in the scalar field over which the value of scalar function is constant. (*Purvanchal U.* 2004; *D.A.U. Agra.* 2007)

(*b*) The gradient of a scalar field is a vector. Hence explain how you can produce a vector from a scalar field. (*Meerut. U.* 2003; *M.D.U.* 2003; *H.P.U.* 2003, 2000; *K.U.* 2001; *P.U.* 2000; *Nagpur Uni. s*/2008, *s*/2006, *Pbi. U.* 2000)

5. (*a*) Define divergence of a vector field. What is its physical meaning? Give two examples. (*Gauhati U.* 2007)

(*b*) Divergence of a vector field is a scalar quantity. Hence explain how you can produce a scalar field from a vector field. (*Pbi. U.* 2003, 2001, 2000; *H.P.U.* 2001; *G.N.D.U.* 2001; *K.U.* 2000; *Nagpur Uni. s*/2007; *Magadh U.* 2001, 2003)

6. (*a*) Derive an expression for divergence of a vector field in Cartesian co-ordinates from first principles. (*Purvanchal U.* 2004; *D.A.U. Agra* 2006)

(*b*) What do you mean by a solenoidal vector field? Give one example. What is the meaning of $\vec{\nabla} \cdot \vec{E} \neq 0$. (*Pbi. U.* 2001; *H.P.U.* 2001; *Purvanchal U.* 2004)

7. State and prove Gauss's divergence theorem. (*P.U.* 2001, 2000; *Nagpur U.* 2009; *K.U.* 2003; *H.P.U.* 2003, 2002, 2000; *M.D.U.* 2001; *Gharwal. U.* 2000; *Meerut* 2004; *D.A.U. Agra* 2007, *Gauhati U.* 2007; *Magadh U.* 2003)

8. Define curl of a vector field and give its physical significance. Show that curl of a vector field is a vector quantity. (*P.U.* 2000; *G.N.D.U.* 2004, 2000; *Pbi. U.* 2003; *Gauhati U.* 2007; *H.P.U.* 2003; *Nagpur Uni.* 2009, *s*/2007, *w*/2007; *Meerut* 2005)

9. State and prove Stoke's theorem. Give its importance. (*Pbi. U.* 2001, 2000; *P.U.* 2001; *M.D.U.* 2003, 2000; *G.N.D.U.* 2000; *Nagpur U. s*/2008; *Magadh U.* 2002; *H.P.U.* 2003; *Meerut U.* 2003; *Purvanchal U.* 2004; *D.A.U. Agra* 2006)

10. (*a*) Show by actual computation that curl gradient of a scalar function is always zero or curl grad $\phi = 0$ (*P.U.* 2001; *H.P.U.* 2001; *M.D.U.* 2001; *Meerut* 2005)

(*b*) Show that the curl of a uniform electric field is zero. (*H.P.U.* 2003; *P.U.* 2003)

11. Show that a vector field whose curl is everywhere zero can be expressed as the gradient of another suitable scalar field. What is this type of field called? (*G.N.D.U.* 2004)

12. Prove that div curl $\vec{A} = 0$ or $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$

OR

If a vector $\vec{B}$ is curl of another vector $\vec{A}$, then the divergence of such a vector is zero.

(*Pbi. U.* 2003; *Meerut U.* 2003, 2002; *P.U.* 2001, 2000; *Agra U.* 2006; *H.P.U.* 2002, 2000; *Gauhati U.* 2007)

13. State and prove Green's theorem in a plane. (*Gauhati U.* 2000; *Magadh U.* 2002; *D.A.U. Agra* 2007)

14. (*i*) Define vector field.

(*ii*) State any two properties of a vector product.

(*iii*) State and prove Stoke's theorem.

(*iv*) What do you understand by $curl\ \vec{F} = 0$, where $\vec{F}$ is the vector field.

(*Nagpur Uni.* 2008)

15. (*i*) Explain physical meaning of the divergence of a vector field.

(*ii*) Calculate the divergence of the vector function

$$\vec{A} = xy\,\hat{i} + yz\,\hat{j} + zx\,\hat{k}$$

(*iii*) State and prove Green's theorem. (*Nagpur Uni.* 2007)

SHORT QUESTIONS

1. Define scalar (dot) product of two vectors. Give its important properties.

2. Define direction cosines of a vector. If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two vectors, show that the angle θ between them is given by

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2.$$

3. Define cross-product of two vectors.

4. Express the scalar triple product in terms of rectangular components.

5. Prove that if the scalar triple product vanishes the vectors are co-planar.

6. Define vector field. State any two properties of vector product.

(*Nagpur Uni. s*/2008, *w*/2007)

7. What is a field? What are scalar and vector fields? Explain. Give one example of each.

(*Pbi. U.* 2003, 2002; *M.D.U.* 2002; *H.P.U.* 2001; *Purvanchal U.* 2004; *Agra U.* 2003)

8. A quantity possesses both magnitude and direction. Is it necessarily a vector? Explain.

(*H.P.U.* 2001)

9. Explain 'Physical significance, and 'Geometrical interpretation' of grad ϕ.

10. What is lamelar vector field?

11. Show that the surface integral of a vector over a surface is equal to the volume integral of the divergence of the vector. (*Nagpur Uni. s*/2007)

12. Calculate the value of the curl of a vector in terms of Cartesian co-ordinates.

(*Magadh U.* 2003)

13. What is an irrotational field? Give one example. (*Pbi. U.* 2001; *H.P.U. G.N.D.U.* 2003)

14. What do you understand by $curl\ \vec{F} = 0$, where $\vec{F}$ is the vector field? (*Nag. U. s*/2008)

15. What do you understand by irrotational vector? Show that the motion is irrotational if the body is moving with a velocity $\vec{V} = (2x\hat{i} + 2y\hat{j})\ \text{ms}^{-1}$. (*Meerut Uni.* 2002)

16. $Curl\ \vec{F} = 0$ and $div\ \vec{F} = 0$ what do you conclude from this. (Pbi. Uni. 2003)

17. Calculate the value of the divergence of a vector product of two vectors or prove that

$$div\ (\vec{A} \times \vec{B}) = \vec{B} \cdot curl\ \vec{A} - \vec{A} \cdot curl\ \vec{B}$$

(*Indore U.* 2001; *Gharwal U.* 2000; *Meerut U.* 2002; *M.D.U.* 2002; *M.S.U. Tiruneveli* 2007)

18. If $\vec{A}$ and $\vec{B}$ are two differentiable vectors, then show that

$$\vec{\nabla} \times (\vec{A} + \vec{B}) = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{B}$$ (*H.P.U.* 2003)

19. Explain the term flux of vector field. State and prove Green's theorem. (*Nagpur U. s*/2009, *s*/2006, *s*/2007, *w*/2007)

20. Define line and surface integrals of a vector. (*Gauhati U.* 2007; *Nagpur Uni. s*/2005, *w*/2007; *Magadh U.* 2002)

21. What is meant by volume integral? (*Nagpur Uni. s*/2008, *Magadh U.* 2002)

22. What do you understand by curl and divergence of a vector field V? (*Magadh* 2003)

NUMERICALS

1. Find the scalar product of two vectors $\vec{A}$ and $\vec{B}$ where

$\vec{A} = \hat{i} + 2\hat{j} + \hat{k}$ and $\vec{B} = 2\hat{i} + \hat{j} - 3\hat{k}$. (*Nagpur Uni. s*/2005) **[Ans.** 1]

2. What is the unit vector perpendicular to both $\vec{A}$ and $\vec{B}$ where $\vec{A} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{B} = (3\hat{i} + 4\hat{j} - \hat{k})$? Calculate the sine of the angle between them.

(*D.A.U. Agra* 2006) **[Ans.** $(-3\hat{i} + 5\hat{j} + 11\hat{k}) / \sqrt{155}$; $\sin\theta = 0.997$)]

3. Prove that the vectors $2\hat{i} - \hat{j} + \hat{k}, \hat{i} + 2\hat{j} + 3\hat{k}, 3\hat{i} - 4\hat{j} + 5\hat{k}$ are co-planar. (*Meerut U.* 2001)

4. Show that for any vector $\vec{A}$

$\vec{A} \cdot \vec{A} = A^2$ and $\vec{A} \times \vec{A} = 0$ (*Nagpur Uni. s*/2006, *s*/2007)

5. If $\vec{A} \times \vec{B} = 0$ and $\vec{A} \neq 0$, $\vec{B} \neq 0$, find the angle between $\vec{A}$ and $\vec{B}$. (*Nag. U. w*/2007)

6. Show that the vector triple product

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}\,(\vec{A} \cdot \vec{C}) - \vec{C}\,(\vec{A} \cdot \vec{B}).$$

(*Meerut U.* 2003, 2002; *D.A.U. Agra* 2003)

7. Prove that $(\vec{B} \times \vec{C}) \cdot \{\vec{A} \times (\vec{B} \times \vec{C})\} = 0$

[**Hint:** Find $(\vec{B} \times \vec{C}) \cdot \{\vec{A} \times (\vec{B} \times \vec{C})\}$ using formula in Q. 13 and show that $(\vec{B} \times \vec{C}) \cdot \vec{B} = (\vec{B} \times \vec{C}) \cdot \vec{C} = 0$]

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8. Show that the vector field $\vec{A} = \frac{-2z^2y}{x^3}\hat{i} + \frac{z^2}{x^2}\hat{j} + \frac{2yz}{x^2}\hat{k}$ is irrotational. (*G.N.D.U.* 2002)

9. Show that the field $\vec{E} = 6xy\,\hat{i} + (3x^2 - 3y^2)\,\hat{j}$ is irrotational. (*H.P.U.* 2002, 2000)

10. Prove that $curl\,(\phi\,\vec{A}) = \phi\, curl\,\vec{A} + grad\,\phi \times \vec{A}$ (*Indore U.* 2001)

11. Prove that

$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{A}) = (\vec{A}\cdot\vec{C})\,(\vec{B}\cdot\vec{A}) - (\vec{B}\cdot\vec{C})\,(\vec{A}\cdot\vec{A})$$ (*Magadh U.* 2003)

12. If $\vec{r}$ is the position vector of a point, then show that:

$$div\ grad\ r^n = n\,(n+1)\,r^{n-2}.$$ (*Meerut* 2005, 2004)

13. Prove that :

(*i*) $grad\ r^m = mr^{m-2}\vec{r}$

(*ii*) $\nabla^2\left(\frac{1}{r}\right) = 0$

(*iii*) $curl\ \vec{r} = 0$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ (*Purvanchal U.* 2006)

14. Prove that:

(*i*) $Curl\,(\phi\,\vec{A}) = \phi\, Curl\,\vec{A} - \vec{A}\times grad\,\phi$

(*ii*) $div\left(\frac{\vec{r}}{r^3}\right) = 0$

(*iii*) $div\ \vec{r} = 3$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ (*Purvanchal U.* 2006)

15. Verify Green's theorem in plane for $\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ where C is the boundary of the region defined by

(*a*) $y = x^{\frac{1}{2}};\ y = x^2$

(*b*) $y = 0,\ x = 0,\ y + x = 1$ (*D.A.U. Agra* 2005)

16. If a is a constant vector, then show that

(*i*) $grad\,(\vec{a}\cdot\vec{r}) = \vec{a}$

(*ii*) $curl\,(\vec{a}\times\vec{r}) = 2\,\vec{a}$ (*Gauhati U.* 2007)

17. Prove that the three vectors

$$\vec{A} = 3\hat{i} + \hat{j} + 2\hat{k};\quad \vec{B} = \hat{i} - \hat{j} - \hat{k};\quad \vec{C} = \hat{i} + 5\hat{j} - 4\hat{k}$$

are at right angles to each other. (*Gauhati U.* 2007)

18. Prove that the vector $\vec{A} = 3y^4z^2\,\hat{i} + 4x^3z^2\,\hat{j} - 3x^2y^2\,\hat{k}$ is a solenoidal.

(*M.S.U. Tirunelveli* 2007)

14 MATHEMATICAL BACKGROUND-II

PARTIAL DERIVATIVES

INTRODUCTION

In the previous chapter, we have realised that vector analysis is a very powerful tool for visualising the physical meaning of the equations distinctly and exactly. Particularly, in electromagnetic theory change in field and potential can be understood clearly in terms of vector differentiation and integration. Partial differentiation and total differentiation is discussed in early articles of this chapter.

14.1 FUNCTIONS OF TWO AND THREE VARIABLES

In partial differentiation of a function of two, or more variables, only one variable varies. Area of a rectangle depends upon its length and breadth. Hence, we can say that area of a triangle is a function of two variables *i.e.* its lengths and breadth. Mathematically, z is called a function of two variables x and y, if z has one definite value for every pair of values of x and y. Symbolically, it is written as

$$z = f(x, y)$$

The variables x and y are called independent variables while z is called the dependent variable.

On the same line, volume is a quantity and has three variables *viz.* volume of a rectangular block has 3 variables *i.e.* length, breadth and height. Therefore, volume is a function of three independent variables.

Thus, we can define z as a function of two or more than two variables.

14.2 PARTIAL DERIVATIVES

Let $z = f(x, y)$ be a function of two variables x and y. If we keep y as constant and vary x alone, then z is a function of x only. The derivative or the differential coefficient of z with respect to x, keeping y as constant is called the partial derivative of z with respect to x and is denoted by one of the symbols:

$$\frac{\partial z}{\partial x}, \quad \frac{\partial f}{\partial x}, \quad f_x(x, y), \quad D_x f$$

By definition,

$$\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$$

Similarly, the partial derivative of z w.r.to y keeping x as constant is denoted by

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$$\frac{\partial z}{\partial y}, \quad \frac{\partial f}{\partial y}, \quad f_y(x, y), \quad D_y f$$

By definition,

$$\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}$$

Example 14.1. If $z(x + y) = x^2 + y^2$, show that

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{dz}{dx} - \frac{dz}{dy}\right)$$

Solution. $z(x + y) = x^2 + y^2$, $\quad z = \dfrac{x^2 + y^2}{x + y}$

$$\frac{\partial z}{\partial x} = \frac{(x + y)\,2x - (x^2 + y^2)\cdot 1}{(x + y)^2} = \frac{x^2 + 2xy - y^2}{(x + y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(x + y)\,2y - (x^2 + y^2)\cdot 1}{(x + y)^2} = \frac{-x^2 + 2xy + y^2}{(x + y)^2}$$

Now,

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = \frac{x^2 + 2xy - y^2}{(x + y)^2} - \frac{-x^2 + 2xy + y^2}{(x + y)^2}$$

$$= \frac{2x^2 - 2y^2}{(x + y)^2} = \frac{2(x - y)}{x + y}$$

$$\therefore \quad \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \frac{4(x - y)^2}{(x + y)^2} \qquad \ldots (i)$$

and

$$\text{R.H.S.} = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

$$= 4\left[1 - \frac{x^2 + 2xy - y^2}{(x + y)^2} - \frac{-x^2 + 2xy + y^2}{(x + y)^2}\right]$$

$$= 4 \cdot \frac{x^2 + 2xy + y^2 - x^2 - 2xy + y^2 + x^2 - 2xy - y^2}{(x + y)^2}$$

$$= 4 \cdot \frac{x^2 - 2xy + y^2}{(x + y)^2} = 4 \cdot \frac{(x - y)^2}{(x + y)^2} \qquad \ldots (ii)$$

From Eq. (*i*) and (*ii*), we have

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

Hence Proved

Example 14.2. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ for $f(x, \underline{y}) = x^2 - xy + y^2$ (*Nagpur U. s*/2008)

Solution. $$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 - xy + y^2) = 2x - y \cdot 1 + 0 = 2x - y$$

and $$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 - xy + y^2) = 0 - x \cdot 1 + 2y = -x + 2y$$

Example 14.3. If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, then find the value of $x\frac{du}{dx} + y\frac{du}{dy}$.

Solution. $$u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{1}{y} + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) = \frac{1}{\sqrt{y^2 - x^2}} - \frac{y}{x^2 + y^2}$$

$\therefore$ $$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2 - x^2}} - \frac{xy}{x^2 + y^2} \qquad \text{... }(i)$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\left(-\frac{x}{y^2}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = -\frac{x}{y\sqrt{y^2 - x^2}} + \frac{x}{x^2 + y^2}$$

$\therefore$ $$y\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{y^2 - x^2}} + \frac{xy}{x^2 + y^2} \qquad \text{... }(ii)$$

Adding Eq. (*i*) and (*ii*), we get

$$x\frac{\partial y}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Example 14.4. If $u = (1 - 2xy + y^2)^{-1/2}$, prove that $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = y^2u^3$.

Solution. $$u = (1 - 2xy + y^2)^{-1/2} \qquad \text{... }(i)$$

Differentiating Eq. (*i*) partially w.r.to x, we get

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(1 - 2xy + y^2)^{-3/2} \cdot (-2y)$$

$$x\frac{\partial u}{\partial x} = xy\,(1 - 2xy + y^2)^{-3/2} \qquad \text{... }(ii)$$

Differentiating Eq. (*i*) partially w.r.to y, we get

$$\frac{\partial u}{\partial y} = -\frac{1}{2}(1 - 2xy + y^2)^{-3/2} \cdot (-2x + 2y)$$

$$y\frac{\partial u}{\partial y} = (xy - y^2)\,(1 - 2xy + y^2)^{-3/2} \qquad \text{... }(iii)$$

Subtracting Eq. (*iii*) from (*ii*), we get

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = xy\,(1 - 2xy + y^2)^{-3/2} - (xy - y^2)\,(1 - 2xy + y^2)^{-3/2}$$

$$= y^2\,(1 - 2xy + y^2)^{-3/2}$$

$$= y^2\,u^3$$

Hence Proved.

Example 14.5. If $z = e^{ax+by} \cdot f(ax - by)$, prove that

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$$

Solution.

$$z = e^{ax+by} \cdot f(ax - by)$$

$$\frac{\partial z}{\partial x} = a\,e^{ax+by} \cdot f(ax - by) + e^{ax+by} \cdot a\,f'(ax - by)$$

$$b\frac{\partial z}{\partial x} = ab\,e^{ax+by} \cdot f(ax - by) + ab\,e^{ax+by} \cdot f'(ax - by) \qquad \ldots (i)$$

Similarly,

$$\frac{\partial z}{\partial y} = b\,e^{ax+by} \cdot f(ax - by) + e^{ax+by} \cdot (-b)\,f'(ax - by)$$

$$a\frac{\partial z}{\partial y} = ab\,e^{ax+by} \cdot f(ax - by) - ab\,e^{ax+by} \cdot f'(ax - by) \qquad \ldots (ii)$$

Adding Eq. (*i*), and (*ii*), we get

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2ab\,e^{ax+by} \cdot f(ax - by)$$

$$= 2abz$$

Hence Proved.

14.3 GEOMETRIC INTERPRETATION OF PARTIAL DERIVATIVES

Let us discuss the geometrical interpretation of the partial derivatives $\frac{dz}{dx}$ and $\frac{dz}{dy}$ of a function of two variables x and y *i.e.* $Z = f(x, y)$.

Let $Z = f(x, y)$ be a surface S. Suppose a plane $y = k$, (where k is constant) parallel to XZ plane intersects the surface $Z = f(x, y)$ along the curve APB, as shown in fig. 14.1. The section APB is a plane curve whose equations are

$$Z = f(x, y) \qquad \ldots (i)$$

and

$$y = k \qquad \ldots (ii)$$

Let $P(x, k, y)$ be a point on the curve. Then, the slope of the tangent to the curve at point P is $\frac{dz}{dy}$.

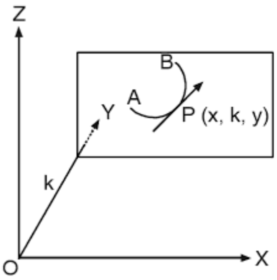

Fig. 14.1

Similarly, $\frac{dz}{dy}$ is the slope of the tangent to the curve of intersection of the surface $z = f(x, y)$ with a plane $y = c$ parallel to YZ plane (not shown in diagram).

14.4 TOTAL DIFFERENTIAL OF A FUNCTION

As we know, in partial differentiation of a function of two or more variables, only one variable varies, while remaining remain constant. However, in total differentiation, increaments are provided in all the variables viz. as x changes from x to $x + \delta x$, y also changes from $y + \delta y$, z from z to $z + \delta z$ and so on.

Verification:

$$u = x^3 + y^3 = a^3 \cos^3 t + b^3 \sin^3 t$$

$$\frac{du}{dt} = -3\, a^3 \cos^2 t \sin t + 3\, b^3 \sin^2 t \cos t$$

Example 14.7. Find total differential of the following function:

$$\boldsymbol{z = f(x, y) = x^2 y + xy^2}$$ (*Nagpur U. w*/2007)

Solution. $z = f(x, y) = x^2 y + xy^2$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \quad \dots (i)$$

$$\frac{\partial f}{\partial x} = 2xy + y^2$$

and $$\frac{\partial f}{\partial y} = x^2 + 2xy$$

Substituting in Eq. (i), we have

$$dz = (2xy + y^2)\, dx + (x^2 + 2xy)\, dy$$

Example 14.8. If $z = \sqrt{x^2 + y^2}$ then show that, $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z$. (*Nagpur U. s*/2005)

Solution. $z = \sqrt{x^2 + y^2}$

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\therefore \quad x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{x^2}{\sqrt{x^2 + y^2}} + \frac{y^2}{\sqrt{x^2 + y^2}} = \frac{x^2 + y^2}{\sqrt{x^2 + y^2}} = \sqrt{x^2 + y^2} = z$$ **Hence proved.**

14.4.2 Total differential of a function of three variables

Let $u = f(x, y, z)$ ... (i)

be a function of three variables x, y, z. If δx, δy and δz are the small increments in x, y and z respectively, then the corresponding increment in u will be $(u + \delta u)$ and is given by

$$u + \delta u = f(x + \delta x, y + \delta y, z + \delta z) \quad \dots (ii)$$

Proceeding in the similar way that of article 14.4.1, we get

$$du = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \quad \dots (iii)$$

Example 14.9. If $u = f(y - z, z - x, x - y)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

(*A.M.I.E. w*/2001, *A.M.I.E. T.E. s*/2002)

Solution. Let $r = y - z, \quad s = z - x, \quad t = x - y$

So that $u = f(r, s, t)$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial f}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial f}{\partial t} \cdot \frac{\partial t}{\partial x}$$

$$= \frac{\partial f}{\partial r}(0) + \frac{\partial f}{\partial s}(-1) + \frac{\partial f}{\partial t}(1)$$

$$= -\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \qquad \text{... } (i)$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{\partial f}{\partial r}(1) + \frac{\partial f}{\partial s}(0) + \frac{\partial f}{\partial t}(-1)$$

$$= \frac{\partial f}{\partial r} - \frac{\partial f}{\partial t} \qquad \text{... } (ii)$$

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= \frac{\partial f}{\partial r}(-1) + \frac{\partial f}{\partial s}(1) + \frac{\partial f}{\partial t}(0)$$

$$= -\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s} \qquad \text{... } (iii)$$

Adding Eq. (*i*), (*ii*) and (*iii*), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$ **Hence Proved.**

Example 14.10. If $u = x^2 + y^2 + z^2 - 2xyz = 1$, show that

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0$$ (*A.M.I.E.* 2000)

Solution. $$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

Given: $$u = x^2 + y^2 + z^2 - 2xyz = 1 \qquad \text{... } (i)$$

$$\frac{\partial u}{\partial x} = 2x - 2yz = 2\,(x - yz)$$

$$\frac{\partial u}{\partial y} = 2y - 2xz = 2\,(y - xz)$$

$$\frac{\partial u}{\partial z} = 2z - 2xy = 2\,(z - xy)$$

$\therefore$ $$du = 2\,(x - yz)\,dx + 2\,(y - xz)\,dy + 2\,(z - xy)\,dz = 0$$

or $$(x - yz)\,dx + (y - xz)\,dy + (z - xy)\,dz = 0 \qquad \text{... } (ii)$$

Rearranging Eq. (*i*), we get

$$x^2 - 2xyz = 1 - y^2 - z^2$$

or $$x^2 - 2xyz + y^2z^2 = 1 - y^2 - z^2 + y^2z^2 = (1 - y^2) - z^2\,(1 - y^2) = (1 - y^2)\,(1 - z^2)$$

or $$(x - yz)^2 = (1 - y^2)\,(1 - z^2)$$

or $$(x - yz) = \sqrt{(1 - y^2)\,(1 - z^2)}$$

Similarly, $$(y - xz) = \sqrt{(1-x^2)(1-z^2)}$$

and $$(z - xy) = \sqrt{(1-x^2)(1-y^2)}$$

Substituting in Eq. (*ii*), we have

$$\sqrt{(1-y^2)(1-z^2)}\,dx + \sqrt{(1-x^2)(1-z^2)}\,dy + \sqrt{(1-x^2)(1-y^2)}\,dz = 0$$

Dividing throughout by $\sqrt{(1-x^2)(1-y^2)(1-z^2)}$, we have

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0$$

Hence Proved.

Example 14.11. If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{zx}\right)$ show that

$$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$$

Solution. $$u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right) = f(r, s) \qquad \text{... (i)}$$

where $$r = \frac{y-x}{xy} \quad \text{and} \quad s = \frac{z-x}{zx}$$

$$r = \frac{1}{x} - \frac{1}{y} \qquad s = \frac{1}{x} - \frac{1}{z}$$

$$\frac{\partial r}{\partial x} = -\frac{1}{x^2} \qquad \frac{\partial s}{\partial x} = -\frac{1}{x^2}$$

$$\frac{\partial r}{\partial y} = \frac{1}{y^2} \qquad \frac{\partial s}{\partial y} = 0$$

$$\frac{\partial r}{\partial z} = 0 \qquad \frac{\partial s}{\partial z} = \frac{1}{z^2}$$

From Eq. (*i*),

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z} \cdot dz$$

Since u is a function of r, s *i.e.* $f(r, s)$, we write

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial f}{\partial s} \cdot \frac{\partial s}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial r} \cdot \left(-\frac{1}{x^2}\right) + \frac{\partial f}{\partial s}\left(-\frac{1}{x^2}\right) = -\frac{1}{x^2}\frac{\partial f}{\partial r} - \frac{1}{x^2}\frac{\partial f}{\partial s}$$

or $$x^2 \frac{\partial u}{\partial x} = -\frac{\partial f}{\partial r} - \frac{\partial f}{\partial s} \qquad \text{... }(ii)$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial f}{\partial s} \cdot \frac{\partial s}{\partial y}$$

$$= \frac{\partial f}{\partial r}\left(\frac{1}{y^2}\right) + \frac{\partial f}{\partial s}(0) = \frac{1}{y^2}\frac{\partial f}{\partial r}$$

or
$$y^2 \frac{\partial u}{\partial y} = \frac{\partial f}{\partial r} \qquad \ldots (iii)$$

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial z}$$

$$= \frac{\partial f}{\partial r}(0) + \frac{\partial f}{\partial s}\left(\frac{1}{z^2}\right) = \frac{1}{z^2}\frac{\partial f}{\partial s}$$

or
$$z^2 \frac{\partial u}{\partial z} = \frac{\partial f}{\partial s} \qquad \ldots (iv)$$

Adding Eq. (*ii*), (*iii*) and (*iv*), we have

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = -\frac{\partial f}{\partial r} - \frac{\partial f}{\partial s} + \frac{\partial f}{\partial r} + \frac{\partial f}{\partial s} = 0$$

Hence Proved.

14.5 PARTIAL DERIVATIVES OF HIGHER ORDERS

Let $z = f(x, y)$, then $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ can be further differentiated w.r.to x and y as they are functions of x and y. For example,

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \quad \text{or} \quad \frac{\partial^2 f}{\partial x^2} \quad \text{or} \quad f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\,\partial x} \quad \text{or} \quad \frac{\partial^2 f}{\partial y\,\partial x} \quad \text{or} \quad f_{yx}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x\,\partial y} \quad \text{or} \quad \frac{\partial^2 f}{\partial x\,\partial y} \quad \text{or} \quad f_{xy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \quad \text{or} \quad \frac{\partial^2 f}{\partial y^2} \quad \text{or} \quad f_{yy}$$

These are called the second order derivatives using partial differentiation. Here, it should be noted that

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial x\,\partial y}$$

Moreover, higher order partial differentiation can be obtained.

14.6 IMPORTANT DEDUCTIONS

Let
$$z = f(x, y), \text{ then}$$

$$dz = \frac{\partial f}{\partial x}\cdot dx + \frac{\partial f}{\partial y}\cdot dy$$

If
$$z = 0, \qquad dz = 0$$

then
$$0 = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\frac{\partial f}{\partial x}dx = -\frac{\partial f}{\partial y}dy$$

or, $$\frac{\partial f}{\partial y}\cdot\frac{dy}{dx} = -\frac{\partial f}{\partial x}$$

or, $$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}} \qquad \textbf{(Remember)} \quad \ldots (i)$$

We can find $\dfrac{d^2y}{dx^2}$ by differentiating Eq. (i), we have

Let $\dfrac{\partial f}{\partial x} = p$, $\dfrac{\partial f}{\partial y} = q$, $\dfrac{\partial^2 f}{\partial x^2} = r$, $\dfrac{\partial^2 f}{\partial x\,\partial y} = s$, $\dfrac{\partial^2 f}{\partial y^2} = t$

From Eq. (i), $$\frac{dy}{dx} = -\frac{p}{q}$$

Differentiating again, we have

$$\frac{d^2y}{dx^2} = \frac{q\dfrac{dp}{dx} - p\dfrac{dq}{dx}}{q^2} \qquad \ldots (ii)$$

But
$$\frac{dp}{dx} = \frac{\partial p}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial p}{\partial y}\cdot\frac{dy}{dx} = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y}\cdot\frac{dy}{dx}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{dy}{dx}$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y\,\partial x}\cdot\frac{dy}{dx} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y\,\partial x}\left(\frac{-\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}\right)$$

$$= r - s\frac{p}{q} = \frac{qr - ps}{q}$$

$$\frac{dq}{dx} = \frac{\partial q}{\partial x} + \frac{\partial q}{\partial y}\cdot\frac{dy}{dx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\left(-\frac{p}{q}\right)$$

$$= \frac{\partial^2 f}{\partial x\,\partial y} - \frac{\partial^2 f}{\partial y^2}\cdot\frac{p}{q} = s - \frac{tp}{q} = \frac{qs - tp}{q}$$

Substituting in Eq. (ii), we have

$$\frac{d^2y}{dx^2} = -\frac{q\dfrac{qr - ps}{q} - p\dfrac{qs - tp}{q}}{q^2} = -\frac{q^2r - pqs - pqs + p^2t}{q^3}$$

$$= -\frac{q^2r - 2pqs + p^2t}{q^3}$$

or $$\frac{\partial^2 z}{\partial y \partial x} = \frac{2x^2 - x^2 - y^2}{x^2 + y^2} = \frac{x^2 - y^2}{x^2 + y^2}$$ **Hence Proved.**

Example 14.13. If $v = (x^2 + y^2 + z^2)^{m/2}$, then find the value of m ($m \neq 0$) which will make

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$$

Solution. $v = (x^2 + y^2 + z^2)^{m/2}$

$$\frac{\partial v}{\partial x} = \frac{m}{2}(x^2 + y^2 + z^2)^{\frac{m}{2}-1} \cdot (2x)$$

$$= mx\,(x^2 + y^2 + z^2)^{\frac{m}{2}-1}$$

$$\frac{\partial^2 v}{\partial x^2} = mx\left(\frac{m}{2} - 1\right)(x^2 + y^2 + z^2)^{\frac{m}{2}-2} \cdot (2x) + m\,(x^2 + y^2 + z^2)^{\frac{m}{2}-1}$$

$$= m\,(m-2)\,x^2\,(x^2 + y^2 + z^2)^{\frac{m}{2}-2} + m\,(x^2 + y^2 + z^2)^{\frac{m}{2}-1}$$

$$= m\,(x^2 + y^2 + z^2)^{\frac{m}{2}-2} \cdot [(m-2)\,x^2 + x^2 + y^2 + z^2] \qquad \ldots (i)$$

Similarlay, $$\frac{\partial^2 v}{\partial y^2} = m\,(x^2 + y^2 + z^2)^{\frac{m}{2}-2} \cdot [(m-2)\,y^2 + x^2 + y^2 + z^2] \qquad \ldots (ii)$$

$$\frac{\partial^2 v}{\partial z^2} = m\,(x^2 + y^2 + z^2)^{\frac{m}{2}-2}\,[(m-2)\,z^2 + x^2 + y^2 + z^2] \qquad \ldots (iii)$$

Adding Eq. (*i*), (*ii*) and (*iii*), we have

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = m\,(x^2 + y^2 + z^2)^{\frac{m}{2}-2}[(m-2)(x^2 + y^2 + z^2) + 3\,(x^2 + y^2 + z^2)]$$

$$= m\,(x^2 + y^2 + z^2)^{\frac{m}{2}-1}[m - 2 + 3]$$

$$0 = m\,(x^2 + y^2 + z^2)^{\frac{m}{2}-1} \cdot (m+1)$$

$$0 = m\,(m+1)\,(x^2 + y^2 + z^2)^{\frac{m}{2}-1}$$

or $m\,(m + 1) = 0$ or $m = 0, -1$

$\therefore$ $m = -1$ $(\because\ m \neq 0)$

Example 14.14. If $u = e^{xyz}$, find the value of $\frac{\partial^3 u}{\partial x \partial y \partial z}$. (*A.M.I.E.* 2000)

Solution. $u = e^{xyz}$

$$\frac{\partial u}{\partial z} = e^{xyz}\,(xy)$$

$$\frac{\partial^2 u}{\partial y \partial z} = e^{xyz}\,(x) + e^{xyz}\,(xz)\,(xy)$$

$$= e^{xyz}\,(x + x^2yz)$$

Example 14.16. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}.$$

Solution. $$u = \log(x^3 + y^3 + z^3 + 3xyz) \qquad \dots (i)$$

Differenting partially Eq. (*i*), w.r.to x, we have

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz} \qquad \dots (ii)$$

Similarly, $$\frac{\partial u}{\partial y} = \frac{3y^2 - 3zx}{x^3 + y^3 + z^3 - 3xyz} \qquad \dots (iii)$$

and $$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz} \qquad \dots (iv)$$

Adding Eq. (*ii*), (*iii*) and (*iv*), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{(x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)} = \frac{3}{x+y+z}$$

or $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u = \frac{3}{x+y+z}$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= -3(x+y+z)^{-2} - 3(x+y+z)^{-2} - 3(x+y+z)^{-2}$$

$$= \frac{-9}{(x+y+z)^2}$$

Hence Proved.

Example 14.17. If $u = f(r)$ and $x = r\cos\theta$, $y = r\sin\theta$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

Solution. $$u = f(r), \qquad x = r\cos\theta, \qquad y = r\sin\theta \qquad \dots (i)$$

$$r^2 = x^2 + y^2 \qquad \text{so that} \qquad 2r\frac{\partial r}{\partial x} = 2x$$

$$\therefore \qquad \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} = \frac{\partial f}{\partial r}\cdot\frac{x}{r}$$

Differentiating partially again w.r. to x, we get

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$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial^2 f}{\partial r^2}\cdot\frac{\partial r}{\partial x}\right)\cdot\frac{x}{r}+\frac{\partial f}{\partial r}\cdot\left[\frac{r\cdot 1 - x\dfrac{\partial r}{\partial x}}{r^2}\right]$$

$$= \left(\frac{\partial^2 f}{\partial r^2}\cdot\frac{x}{r}\right)\frac{x}{r}+\frac{\partial f}{\partial r}\left[\frac{r - x\cdot\dfrac{x}{r}}{r^2}\right]$$

$$= \frac{\partial^2 f}{\partial r^2}\cdot\frac{x^2}{r^2}+\frac{\partial f}{\partial r}\cdot\frac{r^2-x^2}{r^3}$$

$$= \frac{\partial^2 f}{\partial r^2}\cdot\frac{x^2}{r^2}+\frac{\partial f}{\partial r}\cdot\frac{y^2}{r^3} \quad \text{... }(ii)$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 f}{\partial r^2}\cdot\frac{y^2}{r^2}+\frac{\partial f}{\partial r}\cdot\frac{x^2}{r^3} \quad \text{... }(iii)$$

Adding Eq. (*ii*) and (*iii*), we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 f}{\partial r^2}\cdot\frac{x^2}{r^2}+\frac{\partial f}{\partial r}\cdot\frac{y^2}{r^3}+\frac{\partial^2 f}{\partial r^2}\cdot\frac{y^2}{r^2}+\frac{\partial f}{\partial r}\cdot\frac{x^2}{r^3}$$

$$= \frac{\partial^2 f}{\partial r^2}\left(\frac{x^2}{r^2}+\frac{y^2}{r^2}\right)+\frac{\partial f}{\partial r}\left(\frac{y^2}{r^3}+\frac{x^2}{r^3}\right)$$

$$= \frac{\partial^2 f}{\partial r^2}\left(\frac{x^2+y^2}{r^2}\right)+\frac{\partial f}{\partial r}\left(\frac{x^2+y^2}{r^3}\right)$$

$$= \frac{\partial^2 f}{\partial r^2}\cdot 1+\frac{\partial f}{\partial r}\cdot\frac{1}{r} = \frac{\partial^2 f}{\partial r^2}+\frac{\partial f}{\partial r}\cdot\frac{1}{r}$$

$$= f''(r)+\frac{1}{r}f'(r)$$

Hence Proved.

Example 14.18. If $x^3 + y^3 = 3axy$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$

Solution. $x^3 + y^3 = 3axy$ or $x^3 + y^3 - 3axy = 0$

$$f(x, y) = x^3 + y^3 - 3axy = 0$$

Let $p = \dfrac{\partial f}{\partial x} = 3x^2 - 3ay$, $\quad q = \dfrac{\partial f}{\partial y} = 3y^2 - 3ax$

$r = \dfrac{\partial^2 f}{\partial x^2} = 6x$, $\quad s = \dfrac{\partial^2 f}{\partial y\,\partial x} = -3a$, $t = \dfrac{\partial^2 f}{\partial y^2} = 6y$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{3x^2-3ay}{3y^2-3ax} = \frac{ay-x^2}{y^2-ax} \quad \text{... }(i)$$

$$\frac{d^2y}{dx^2} = -\frac{q^2r - 2pqs + p^2t}{q^3} \qquad \textbf{(Remember)}$$

$$= -\frac{(3y^2 - 3ax)^2\, 6x - 2\,(3x^2 - 3ay)\,(3y^2 - 3ax)\,(-3a) + (3x^2 - 3ay)^2 6y}{(3y^2 - 3ax)^3}$$

$$= -\frac{2x\,(y^2 - ax)^2 + 2a\,(x^2 - ay)\,(y^2 - ax) + 2y\,(x^2 - ay)^2}{(y^2 - ax)^3}$$

$$= \frac{2a^3xy}{(ax - y^2)^3}$$

Example 14.19. If $u = x \log xy$ where $x^3 + y^3 + 3xy = 1$, find $\frac{du}{dx}$.

Solution. $u = x \log xy$

$$\frac{\partial u}{\partial x} = x\left(\frac{1}{xy}\cdot y\right) + 1 \log xy = 1 + \log xy \qquad \ldots (i)$$

$$\frac{\partial u}{\partial y} = x \cdot \frac{1}{xy} \cdot x = \frac{x}{y} \qquad \ldots (ii)$$

$$x^3 + y^3 + 3xy = 1$$

Differentiating, we get

$$3x^2 + 3y^2\frac{dy}{dx} + 3x\frac{dy}{dx} + 3y = 0$$

or
$$\frac{dy}{dx} = -\frac{x^2 + y}{x + y^2} \qquad \ldots (iii)$$

We know that

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$= \frac{\partial u}{\partial x}\cdot 1 + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$= (1 + \log xy) + \frac{x}{y}\left(-\frac{x^2 + y}{x + y^2}\right) \qquad [\because \text{Eq. } (i), (ii), (iii)]$$

$$= 1 + \log xy - \frac{x}{y}\cdot\frac{x^2 + y}{x + y^2}$$

Example 14. 20. If $x^2 - y^2 + u^2 + 2v^2 = 1$, $x^2 + y^2 - u^2 - v^2 = 2$, find $\frac{du}{dx}$ and $\frac{dv}{dx}$.

Solution. $x^2 - y^2 + u^2 + 2v^2 = 1 \qquad \ldots (i)$

$$x^2 + y^2 - u^2 - v^2 = 2 \qquad \ldots (ii)$$

Differentiating Eq. (i) partially w.r. to x, we get

$$2x + 2u\frac{\partial u}{\partial x} + 4v\frac{\partial v}{\partial x} = 0$$

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or $$u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x} = -x \qquad \text{... }(iii)$$

Differentiating Eq. (*ii*) partially w.r. to x, we get

$$2x-2u\frac{\partial u}{\partial x}-2v\frac{\partial v}{\partial x} = 0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x} = x \qquad \text{... }(iv)$$

Subtracting Eq. (*iv*) from (*iii*), we have

$$v\frac{\partial v}{\partial x} = -2x \quad \text{or} \quad \frac{\partial v}{\partial x} = -\frac{2x}{v}$$

Similarly, we get $\dfrac{\partial u}{\partial x} = \dfrac{3x}{u}$.

14.7 APPLICATIONS OF PARTIAL DIFFERENTIATION

In physics, there are huge number of applications of partial differentiation. In practice, we come across systems with two or more variables, particularly in heat and thermodynamics. In order to study the system distinctly, we study the system with one parameter varying, keeping remaining parameters constant. This is what, we do in partial differentiation. Determination of error, Jacobians, Taylor's series, maxima and minima are the main applications of partial differentiation. Some of the applications are discussed below:

14.7.1 Error Determination

The most powerful tool of partical differentiation is error determination of any physical quantity measured.

In differential calculus, we know that

$$\lim_{\partial x\to 0}\frac{\partial y}{\partial x} = \frac{dy}{dx}$$

$$\frac{\delta y}{\delta x} = \frac{dy}{dx} \quad \text{approximately}$$

or $$\delta y = \left(\frac{dy}{dx}\right)\delta x \quad \text{(approximately)}$$

Then,

(*i*) δx is known as **'absolute error'** in x.

(*ii*) $\dfrac{\delta x}{x}$ is known as **'relative error'** in x.

(*iii*) $\left(\dfrac{\delta x}{x}\right)\times 100$ is known as **'percentage error'** in x.

Example 14.21. The power dissipated in a resistor is given by $P=\dfrac{E^2}{R}$. Using calculus, find the approximate percentage change in P when E is increased by 3% and R is decreased by 2% (*Nagpur U. s*/2005)

Solution. $P=\dfrac{E^2}{R}$ or $\log P = 2\log E-\log R$

On differentiating, we get

$$\frac{\delta P}{P} = 2\frac{\delta E}{E} - \frac{\delta R}{R}$$

or, $$\frac{100 \times \delta P}{P} = 2 \times \frac{100 \times \delta E}{E} - \frac{100 \times \delta R}{R}$$

or, $$\frac{100 \times \delta P}{P} = 2 \times (3) - (-2) = 8$$

∴ Percentage change (*i.e.* error) in $P = 8\%$

Example 14.22. The deflection at the centre of a rod of a length *l* and diameter *d* supported at its ends and loaded at the centre with a weight *w* varies as $wl^3 d^{-4}$. What is the percentage increase in the deflection corresponding to the percentage increase in *w*, *l* and d of 3, 2 and 1 respectively?

Solution. Let the deflection of the centre of the rod be

$$D = k\frac{w\,l^3}{d^4}$$

Taking logarithmic, we get

$$\log D = \log k + \log w + 3 \log l - 4 \log d$$

On differentiating, we get

$$\frac{\delta D}{D} = \frac{\delta w}{w} + 3\frac{\delta l}{l} - 4\frac{\delta d}{d}$$

$$\begin{aligned} 100 \times \frac{\delta D}{D} &= 100 \times \frac{\delta w}{w} + 3 \times 100 \times \frac{\delta l}{l} - 4 \times 100 \times \frac{\delta d}{d} \\ &= 3 + 3 \times 2 - 4 \times 1 \\ &= 3 + 6 - 4 \\ &= 5\ \% \end{aligned}$$

Example 14.23. The period *T* of a simple pendulum is

$$\boldsymbol{T = 2\pi\sqrt{\frac{l}{g}}}$$

Find the maximum error in *T* due to possible errors upto 1 % in *l* and 2.5 % in *g*.

Solution. $$T = 2\pi\sqrt{\frac{l}{g}}$$

or $$\log T = \log 2\pi + \frac{1}{2}\log l - \frac{1}{2}\log g$$

Differentiating,

$$\frac{\delta T}{T} = 0 + \frac{1}{2}\frac{\delta l}{l} - \frac{1}{2}\frac{\delta g}{g}$$

or $$\left(\frac{\delta T}{T}\right) \times 100 = \frac{1}{2}\left[\left(\frac{\delta l}{l}\right) \times 100 - \left(\frac{\delta g}{g}\right) \times 100\right]$$

But $\frac{\delta l}{l} \times 100 = 1$, $\frac{\delta g}{g} \times 100 = 2.5$

Substituting, we get

$$\left(\frac{\delta T}{T}\right) \times 100 = \frac{1}{2}[1 \pm 2.5]$$

Maximum error in $T = 1.75$ %

Example 14.24. The power consumed in an electrical resistor is given by $P = E^2/R$ watts. If E = 200 volts and R = 8 ohm, by how much does the power change if E is decreaed by 5 volts and R is decreased by 0.2 ohm? (*Nagpur U. s*/2006)

Solution.
$$P = \frac{E^2}{R}$$
$$\log P = 2 \log E - \log R$$

Differentiating,

$$\frac{\delta P}{P} = 2 \times \frac{\delta E}{E} - \frac{\delta R}{R} \quad \ldots (i)$$

Here power is $P = E^2/R = \dfrac{(200)^2}{8} = \dfrac{40000}{8} = 5000$ Watt.

Substituting, $\delta E = -5$ volt, $\delta R = -0.2$ ohm. in Eq. (i),

$$\frac{\delta P}{P} = 2 \times \frac{(-5)}{200} - \frac{(-0.2)}{8}$$
$$= -\frac{1}{20} + \frac{1}{40} = -\frac{1}{40}$$
$$\delta P = \left(-\frac{1}{40}\right) \times 5000 = -125 \text{ Watt.}$$

The power consumption is decreased by 125 Watts.

Example 14.25. Find the possible percentage error in computing the parallel resistance r of three resistances r_1, r_2, r_3 from the formula $\dfrac{1}{r} = \dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3}$ if r_1, r_2, r_3 are each in error by plus 1.2%.

Solution. Here $\dfrac{1}{r} = \dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3}$, being parallel combination ... (i)

Differentiating, we get

$$-\frac{1}{r^2} dr = -\frac{1}{r_1^2} dr_1 - \frac{1}{r_2^2} dr_2 - \frac{1}{r_3^2} dr_3$$

Rearranging,

$$\frac{1}{r}\left(\frac{100\, dr}{r}\right) = \frac{1}{r_1}\left(\frac{100\, dr_1}{r_1}\right) + \frac{1}{r_2}\left(\frac{100\, dr_2}{r_2}\right) + \frac{1}{r_3}\left(\frac{100\, dr_3}{r_3}\right)$$
$$= \frac{1}{r_1}(1.2) + \frac{1}{r_2}(1.2) + \frac{1}{r_3}(1.2)$$
$$= 1.2\left[\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right]$$

$$= 1.2\left(\frac{1}{r}\right) \qquad [\because \text{Eq. } (i)]$$

$$\therefore \quad \frac{100\,dr}{r} = 1.2\,\%, \quad \text{the error in } r.$$

14.7.2 Jacobians

If u and v are functions of the two independent variables x and y, then the determinant

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

is called the **Jacobian** of u, v with respect to x, y and is written as

$$\frac{\partial(u, v)}{\partial(x, y)} \text{ or } J\left(\frac{u, v}{x, y}\right)$$

Similarly, the Jacobian of u, v, w with respect to x, y, z is

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$$

Example 14.26. If $x = r\cos\theta$, $y = r\sin\theta$; evaluate $\dfrac{\partial(x, y)}{\partial(r, \theta)}$, and $\dfrac{\partial(r, \theta)}{\partial(x, y)}$.

Solution.

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

Now,

$$r^2 = x^2 + y^2 \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial \theta}{\partial x} = \frac{-y}{x^2 + y^2} = -\frac{y}{r^2}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} \qquad \frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2} = \frac{x}{r^2}$$

$$\frac{\partial (u, v, w)}{\partial (x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} y+z & z+x & x+y \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} y+z & z+x & x+y \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} x+y+z & x+y+z & x+y+z \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} \quad R_1 + R_2$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 0 \qquad (\because R_1 = R_3)$$

Hence, there exists a functional relationship between u, v, w.

Now, $w^2 = (x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$

$$w^2 = v + 2u$$

$$w^2 - v - 2u = 0,$$

This is the required relationship.

Example 14.31. Verify whether the following funcitons are functionally dependent, and if so, find the relation between them.

$$u = \frac{x+y}{1-xy}, \qquad v = \tan^{-1} x + \tan^{-1} y$$

Solution.

$$\frac{\partial (u, v)}{\partial (x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

Hence, u and v are functionally related.

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$v = \tan^{-1} u$$

or

$$u = \tan^{-1} v$$

14.8 JACOBIANS OF IMPLICIT FUNCITONS

The variables x, y, u, v are connected by implicit functions

$$f_1(x, y, u, v) = 0 \qquad \text{... } (i)$$

$$f_2(x, y, u, v) = 0 \qquad \text{... } (ii)$$

where x, y are implicit functions of x, y.

Differentiating Eq. (*i*) and (*ii*) w.r.t. to x, y, we get

$$\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f_1}{\partial v}\cdot\frac{\partial v}{\partial x} = 0 \quad \ldots (iii)$$

$$\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f_1}{\partial v}\cdot\frac{\partial v}{\partial y} = 0 \quad \ldots (iv)$$

$$\frac{\partial f_2}{\partial x} + \frac{\partial f_2}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f_2}{\partial v}\cdot\frac{\partial v}{\partial x} = 0 \quad \ldots (v)$$

$$\frac{\partial f_2}{\partial y} + \frac{\partial f_2}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f_2}{\partial v}\cdot\frac{\partial v}{\partial y} = 0 \quad \ldots (vi)$$

Now we have,

$$\frac{\partial(f_1, f_2)}{\partial(u, v)} \times \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix} \times \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial u}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial f_1}{\partial u} \times \dfrac{\partial u}{\partial x} + \dfrac{\partial f_1}{\partial u} \times \dfrac{\partial v}{\partial x} & \dfrac{\partial f_1}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial f_1}{\partial v}\dfrac{\partial v}{\partial y} \\ \dfrac{\partial f_2}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial f_2}{\partial v}\dfrac{\partial v}{\partial x} & \dfrac{\partial f_2}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial f_2}{\partial v}\dfrac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} -\dfrac{\partial f_1}{\partial x} & -\dfrac{\partial f_1}{\partial y} \\ -\dfrac{\partial f_2}{\partial x} & -\dfrac{\partial f_2}{\partial y} \end{vmatrix} \qquad [\because \text{ Eq. } (iii), (iv), (v) \text{ and } (vi)]$$

$$= (-1)^2 \frac{\partial(f_1, f_2)}{\partial(x, y)}$$

$$\therefore \qquad \frac{\partial(u, v)}{\partial(x, y)} = (-1)^2 \frac{\partial(f_1, f_2)/\partial(x, y)}{\partial(f_1, f_2)/\partial(u, v)}$$

Example 14.32. If $x^2 + y^2 + u^2 - v^2 = 0$ and $uv + xy = 0$ prove that

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{x^2 - y^2}{u^2 + v^2}$$

Solution. Let $f_1 = x^2 + y^2 + u^2 - v^2$ and $f_2 = uv + xy$

$$\frac{\partial(f_1, f_2)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ y & x \end{vmatrix} = 2(x^2 - y^2)$$

$$\frac{\partial(f_1, f_2)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix} = 2(u^2 + v^2)$$

Using the relation

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SHORT QUESTIONS

1. What is the Jacobian of u, v, w with respect to x, y, z ? (*Nagpur U. s*/2005)
2. Define Jacobian in three diamensions and state any two properties. (*Nagpur U. s*/2007)
3. If $f(x, y) = x^3 y + e^{xy^2}$, find f_x and f_y.
4. Verify that $f_{xy} = f_{yx}$ for the functions:

   (*a*) $\dfrac{2x - y}{x + y}$ (*b*) $x \tan xy$ (*c*) $\cos h\,(y + \cos x)$ (*d*) x^y

5. Show that $z = \log\{(x - a)^2 + (y - b)^2\}$, satisfies $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$ except at (a, b).
6. If $f(x, y, z) - (x^2 + y^2 + z^2)^{-1/2}$, show that
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$
7. If $z = x^3 - xy + y^3$, $x = r\cos\theta$, $y = r\sin\theta$, find $\dfrac{\partial z}{\partial r}, \dfrac{\partial z}{\partial \theta}$.

   [**Hint:** $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cdot\dfrac{\partial x}{\partial r} + \dfrac{\partial z}{\partial y}\cdot\dfrac{\partial y}{\partial x} = (3x^2 - y)\cos\theta + (3y^2 - x)\sin\theta$

   $\dfrac{\partial z}{\partial \theta} = \dfrac{\partial z}{\partial x}\cdot\dfrac{\partial x}{\partial \theta} + \dfrac{\partial z}{\partial y}\cdot\dfrac{\partial y}{\partial \theta} = (3x^2 - y)(-r\sin\theta) + (3y^2 - x)\,r\cos\theta$]

8. If $z = e^{xy^2}$, $x = t\cos t$, $y = t\sin t$, compute $\dfrac{dz}{dt}$ at $t = \dfrac{\pi}{2}$. [**Ans.** $-\dfrac{\pi^3}{8}$]
9. Given that $u = 1/\sqrt{x^2 + y^2 + z^2}$, verify that (*D.A.V. Agra* 2008)
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad ; \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -u$$
10. If $z = a\tan^{-1}(y/x)$, show that

    (*i*) $(1 + q^2)\,r - 2pqs + (1 + p^2)\,t = 0$

    (*ii*) $(rt - s^2)/(1 + p^2 + q^2)^2 = -a^2/(x^2 + y^2 + a^2)^2$

    where $\dfrac{\partial z}{\partial x} = p$, $\dfrac{\partial z}{\partial y} = q$, $\dfrac{\partial^2 z}{\partial x^2} = r$, $\dfrac{\partial^2 z}{\partial x\,\partial y} = s$, $\dfrac{\partial^2 z}{\partial y^2} = t$

11. If $u = f(x + 2y) + g(x - 2y)$, show that
$$4\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$$
12. If $u = \phi(x + at) + \psi(x - at)$, show that
$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$$

    Prove that, if $y = x + at$, $z = x - at$, the equation becomes $\dfrac{\partial^2 u}{\partial y\,\partial z} = 0$

13. If $u = \dfrac{x^2+y^2+z^2}{x}$, $v = \dfrac{x^2+y^2+z^2}{y}$ and $w = \dfrac{x^2+y^2+z^2}{z}$, find $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$.

14. If $u = \cos x$, $v = \sin x \cos y$, $w = \sin x \sin y \cos z$, then show that

$$\frac{\partial(u,v,w)}{\partial(x, y, z)} = (-1)^2 \sin^3 x \sin^2 y \sin z.$$

15. Find $\dfrac{\partial u}{\partial r}$ and $\dfrac{\partial u}{\partial \theta}$ if $u = e^{r\cos\theta} \cdot \cos(r \sin\theta)$.

16. If $u = e^{xyz}$, find the value of $\dfrac{\partial^3 u}{\partial x\,\partial y\,\partial z}$

17. If $u = (1 - 2xy + y^2)^{-1/2}$, prove that

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[y^2 \frac{\partial u}{\partial y}\right] = 0$$

18. If $u = x^y$, show that $\dfrac{\partial^3 u}{\partial x^2 \partial y} = \dfrac{\partial^3 u}{\partial x\,\partial y\,\partial z}$.

19. If $z = \log(e^x + e^y)$, show that $rt - s^2 = 0$

20. If $z = y\,f(x^2 - y^2)$ show that $y\dfrac{\partial z}{\partial x} + x\dfrac{\partial z}{\partial y} = \dfrac{xz}{y}$

21. Show that $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x\,\partial y} + \dfrac{\partial^2 z}{\partial y^2} = 0$ where $z = x \cdot f(x+y) + y \cdot f(x+y)$

22. If $x^x y^y z^z = c$, show that at $x = y = z$, $\dfrac{\partial^2 z}{\partial x\,\partial y} = -(x \log e^x)^{-1}$.

23. If $x = r\cos\theta$, $y = r\sin\theta$, prove that

(*a*) $\dfrac{\partial r}{\partial x} = \dfrac{\partial x}{\partial r}$, $\quad r\dfrac{\partial \theta}{\partial x} = \dfrac{1}{r}\cdot\dfrac{\partial x}{\partial \theta}$ $\qquad$ (*b*) $\dfrac{\partial^2 \theta}{\partial x^2} + \dfrac{\partial^2 \theta}{\partial y^2} = 0$

24. If $v = (x^2 + y^2)\cdot f(x, y)$, prove that $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = (x^4 - y^4)\, f''(x, y)$.

25. If $u = e^{xyz} f\left(\dfrac{xy}{z}\right)$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2xyzu \;;\quad y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2xyzu$$

26. If $u = \dfrac{1}{x}[(x-y) + \psi(x+y)]$, then show that $\dfrac{\partial}{\partial x}\left(\dfrac{x^2\,\partial u}{\partial x}\right) = x^2\dfrac{\partial^2 u}{\partial y^2}$.

27. If $x = \dfrac{r}{2}(e^{\theta} + e^{-\theta})$, $y = \dfrac{r}{2}(e^{\theta} - e^{-\theta})$, prove that $\dfrac{\partial x}{\partial r} = \dfrac{\partial r}{\partial x}$.

28. If $u = \log \dfrac{x^3 + y^3}{x^2 + y^2}$, prove that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = 1$.

29. If $u = f\left(\dfrac{y}{x}\right) + \sqrt{x^2 + y^2}$ find the value of

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$$ [**Ans.** 0]

30. If $z = xy / (x + y)$, find the value of

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2}$$ [**Ans.** 0]

31. If $u = \dfrac{x^2 y^2 z^2}{x^2 + y^2 + z^2} + \cos \dfrac{xy + yz}{x^2 + y^2 + z^2}$ show that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = \frac{4x^2 y^2 z^2}{x^2 + y^2 + z^2}$$

32. Prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 u}{\partial s^2} + \dfrac{\partial^2 s}{\partial t^2}$

where $x = s \cos \alpha - t \sin \alpha$ and $y = s \sin \alpha + t \cos \alpha$.

33. If $z = z(u, v)$, $u = x^2 - 2xy - y^2$ and $v = y$. Show that

$$(x + y) \frac{\partial z}{\partial x} + (x - y) \frac{\partial z}{\partial y} = 0 \text{ is equivalent to } \frac{\partial z}{\partial v} = 0$$

34. If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that

$$(y^2 - zx) \frac{\partial u}{\partial x} + (x^2 - yz) \frac{\partial u}{\partial y} + (z^2 - xy) \frac{\partial u}{\partial z} = 0$$

35. If $z = f(x, y)$, where $x = uv$, $y = \dfrac{u + v}{u - v}$ show that $2x \dfrac{\partial z}{\partial x} = u \dfrac{\partial z}{\partial u} + v \dfrac{\partial z}{\partial v}$.

36. If $u = x^2 + y^2$ where $x = s + 3t$, $y = 2s - t$ prove that

$$\frac{\partial^2 u}{\partial t^2} = 2 \frac{\partial^2 u}{\partial s^2}$$

37. $z = f(u, v)$ where $u = x \cos \theta - y \sin \theta$, $v = x \sin \theta + y \cos \theta$, show that

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = u \frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v}, \ \theta \text{ being constant.}$$

38. Given $u = F(x, y)$, $x = e^s \cos t$, $y = e^s \sin t$, show that

$$\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 s}{\partial t^2} = e^{2s} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

39. If v be a potential function such that $v = v(r)$ and $r^2 = x^2 + y^2 + z^2$; show that

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = \frac{\partial^2 v}{\partial r^2} + \frac{2}{r} \frac{\partial v}{\partial r}$$

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40. Given that $w = x + 2y + z^2$, $x = r/s$, $y = r^2 + e^s$ and $z = 2r$ show that

$$r\frac{\partial w}{\partial r} + s\frac{\partial w}{\partial s} = 12r^2 + 2s\,e^s$$

41. Find $\dfrac{d^2y}{dx^2}$ if $x^5 + y^5 = 5a^3xy$ $\left[\textbf{Ans. } \dfrac{6\,a^3xy\,(2a^6 + x^3y^3)}{(a^3x - y^4)^3}\right]$

42. Find $\dfrac{d^2y}{dx^2}$ if $ax^2 + 2hxy + by^2 = 1$. $\left[\textbf{Ans. } \dfrac{h^2 - ab}{(hx + by)^3}\right]$

43. If $x = uv$, $y = \dfrac{u+v}{u-v}$, find $\dfrac{\partial(u, v)}{\partial(x, y)}$ $\left[\textbf{Ans. } \dfrac{(u-v)^2}{4uv}\right]$

44. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, find $\dfrac{\partial(r, \theta, \phi)}{\partial(x, y, z)}$ $\left[\textbf{Ans. } \dfrac{1}{r^2\sin\theta}\right]$

45. $u = x + y + z$, $u^2v = y + z$, $u^3w = z$, show that $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = u^{-5}$

46. $u = x + y + z$, $uv = y + z$, $uvw = z$, Evaluate $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$. [**Ans.** $u^2 v$]

47. If $u^3 + v^3 = x + y$, $u^2 + v^2 = x^3 + y^3$, then prove that

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{1}{2}\,\frac{y^2 - x^2}{2uv\,(u - v)}.$$

48. If $u = \dfrac{x}{\sqrt{1-r^2}}$, $v = \dfrac{y}{\sqrt{1-r^2}}$, $w = \dfrac{z}{\sqrt{1-r^2}}$ show that $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = \dfrac{1}{(1-r^2)^{5/2}}$ where $r^2 = x^2 + y^2 + z^2$.

49. If $u_1 = x_1 + x_2 + x_3 + x_4$, $u_1u_2 = x_2 + x_3 + x_4$, $u_1u_2u_3 = x_3 + x_4$, $u_1u_2u_3u_4 = x_4$

Show that $\dfrac{\partial(x_1, x_2, x_3, x_4)}{\partial(u_1, u_2, u_3, u_4)} = u_1^3 \cdot u_2^2 \cdot u_3$

50. Given $u = e^x\cos y + e^y\sin z$, find all first partial derivatives and verify that

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}\,;\quad \frac{\partial^2 u}{\partial x\,\partial z} = \frac{\partial^2 u}{\partial z\,\partial x}\,;\quad \frac{\partial^2 u}{\partial y\,\partial z} = \frac{\partial^2 u}{\partial z\,\partial y}$$

15

REPEATED INTEGRALS

15.1 DOUBLE INTEGRATION

We know, from our previous study that, the line integral of a function $y = f(x)$ between limits a and b is, given by

$$\int_a^b f(x)\,dx = \lim_{\substack{n \to \infty \\ \delta x \to 0}} \left[f(x_1)\,\delta x_1 + f(x_2)\,\delta x_2 + f(x_3)\,\delta x_3 + \ldots\ldots + f(x_n)\,\delta x_n \right]$$

Let us consider a function $f(x, y)$ of two independent variables x, y having definite value at each point in the finite region R of xy plane. The region R is divided into n elementary areas $\delta A_1, \delta A_2, \delta A_3, \ldots\ldots \delta A_n$. Let $(x_1, y_1), (x_2, y_2), \ldots\ldots$ be the points within these areas respectively, then the limit of the sum of $f(x_1, y_1)\,\delta A_1; f(x_2, y_2)\,\delta A_2; \ldots\ldots$ as the number of sub-divisions increases indefinitely and the area of each sub-division decreases to zero is defined as the *double integral* of (x, y) over the region R and is expressed as

$$\iint_R f(x, y)\,dA$$

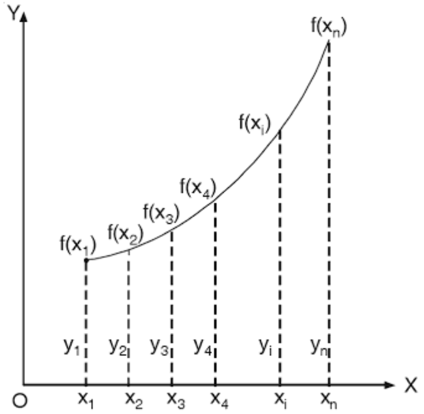

Fig. 15.1

Thus,

$$\iint_R f(x, y)\,dA = \lim_{\substack{n \to \infty \\ \delta A \to 0}} \left[f(x_1, y_1)\,\delta A_1 + f(x_2, y_2)\,\delta A_2 + \ldots\ldots\ldots + f(x_n, y_n)\,\delta A_n \right]$$

15.2 EVALUATION OF DOUBLE INTEGRAL

For evaluation of a double integral, it is expressed as the repeated integral $\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y)\,dx\,dy$, then its value is evaluated as under:

First Method.

When y_1, y_2 are functions of x, and x_1, x_2 are constants, $f(x, y)$ is first integrated w.r. to y keeping x constant between the limits y_1, y_2. The result is then integrated w.r. to x between the limits x_1, x_2, *i.e.*,

$$I_1 = \iint_R f(x, y)\, dx\, dy$$

$$= \int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} f(x, y)\, dy}\, dx \quad \ldots (i)$$

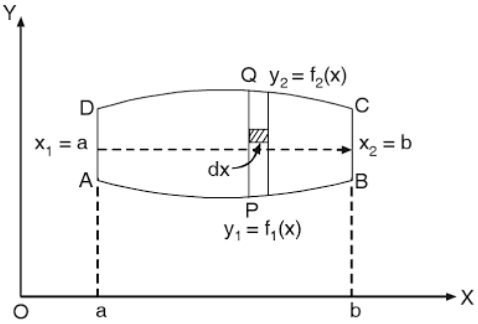

Fig. 15.2

Fig. 15.2 shows the process of integration. The integral inside the rectangle in Eq. (*i*) means that the integration is along one edge of the strip *PQ* between the limits y_1 and y_2 (x remaining constant). The outer integral of rectangle in Eq. (*i*) between the limits $x_1 = a$ and $x_2 = b$ means the sliding of PQ edge from $x_1 = a$ to $x_2 = b$, so that the whole region *ABCD* is covered.

Second Method

When x_1 and x_2 are functions of y and y_1, y_2 are constants, $f(x, y)$ is first integrated w.r. to x keeping y constant between the limits x_1, x_2 and then resulting expression is integrated w.r. to y between the limits c and d. *i.e.*

$$I_2 = \int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x, y)\, dx}\, dy \quad \ldots (ii)$$

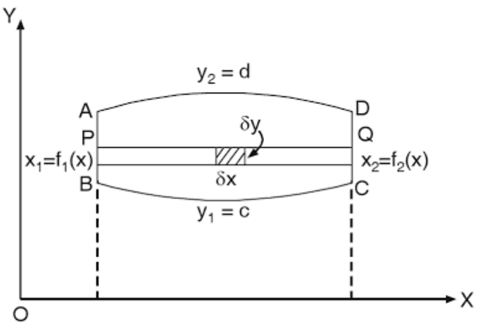

Fig. 15.3

The process of integration is shown in fig. 15.3. The integral inside the rectangle in Eq. (*ii*) means that the integration is along one edge of the strip *PQ* between the limits x_1 and x_2 (y remaining constant). The outer integral of rectangle in Eq. (*ii*) between the limits $y_1 = c$ and $y_2 = d$ means the sliding of *PQ* from $y_1 = c$ to $y_2 = d$ so that the region *ABCD* is covered.

Note: For constant limits it does not matter whether we first integrate w.r. to x and then w.r. to y or vice versa.

Example 15.1. Evaluate $\displaystyle\int_0^1 dx \int_0^x e^{y/x} dy$ (*Nagpur U. s*/2005)

Solution. Using $\int e^x dx = e^x$, we get

$$\int_0^1 dx \int_0^x e^{y/x} dy = \int_0^1 dx \left[x \cdot e^{y/x} \right]_0^x$$

$$= \int_0^1 (x\,e - x)\, dx$$

$$= (e - 1) \int_0^1 x\, dx$$

$$= (e - 1) \left[\frac{x^2}{2} \right]_0^1$$

$$= \frac{1}{2}(e-1)$$

Example 15. 2. Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{dy\,dx}{1+x^2+y^2}$

Solution.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{dy\,dx}{1+x^2+y^2} = \int_0^1 dx \int_0^{\sqrt{1-x^2}} \frac{dy}{1+x^2+y^2}$$

$$= \int_0^1 dx \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1} \frac{y}{\sqrt{1-x^2}}\right]_0^{\sqrt{1-x^2}}$$

$$= \int_0^1 dx \frac{1}{\sqrt{1+x^2}} \cdot \frac{\pi}{4} = \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}}\, dx$$

$$= \frac{\pi}{4}\left[\log\left(x+\sqrt{x^2+1}\right)\right]_0^1$$

$$= \frac{\pi}{4} \log(\sqrt{2}+1).$$

Example 15.3. Find the area of the first quadrant of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

From this find the total area enclosed by the ellipse.

Solution. The area is obtained by the double integral:

$$\iint dx\, dy$$

For the first quadrant, the limits for y are $y_1 = 0$ and y_2 is evaluated as under:

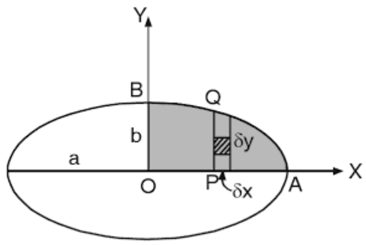

Fig. 15.4

$$\frac{x^2}{y^2} + \frac{y^2}{b^2} = 1$$

$$\therefore \qquad y_2^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$= \frac{b^2}{a^2}(a^2 - x^2)$$

or

$$y_2 = \frac{b}{a}\sqrt{a^2 - x^2}.$$

And the limits for x are $x_1 = 0$ and $x_2 = a$.

$$\therefore \qquad \iint dx\, dy = \int_0^a dx \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dy$$

$$= \int_0^a dx\,[y]_0^{\frac{b}{a}\sqrt{a^2-x^2}} = \int_0^a dx\left[\frac{b}{a}\sqrt{a^2-x^2}\right]$$

$$= \frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$= \frac{b}{a}\left[\frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{b}{a}\left[0+\frac{1}{2}a^2\sin^{-1}(1)\right]$$

$$= \frac{b}{a}\times\frac{1}{2}a^2\times\frac{\pi}{2}$$

$$= \frac{\pi ab}{4} \qquad \ldots (iii)$$

This is the area of one quadrant of the ellipse.

$\therefore$ The total area enclosed by the ellipse

$$= 4\times\left(\frac{\pi ab}{4}\right)$$

$$= \pi ab \qquad \ldots (iv)$$

Example 15.4. Find the area of the surface enclosed by the circle

$$x^2 + y^2 = a^2$$

Solution. Take $b = a$ in Eq. (iv) of Example. 15.3

Area of the surface enclosed by the circle

$$= \pi\, a \times a \qquad (\because\ b = a)$$

$$= \pi\, a^2$$

Example 15.5. Evaluate $\iint_R xy\,dx\,dy$ where R is the quadrant of the circle $x^2 + y^2 = a^2$ where $x \geq 0$ and $y \geq 0$.

Solution. Let the region of integration to the first quadrant of the circle OAB.

$$\iint_R xy\,dx\,dy$$

where $x^2 + y^2 = a^2$ or, $y = \sqrt{a^2-x^2}$ First we integrate w.r. to y and then w.r. to x. The limits for y are 0 to $\sqrt{a^2-x^2}$ and that for $x = 0$ to a.

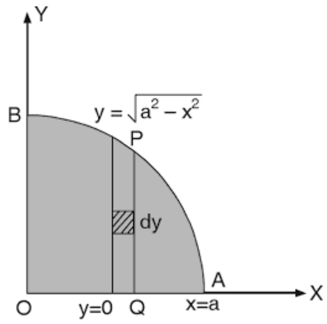

Fig. 15.5

$$= \int_0^a x\,dx \int_0^{\sqrt{a^2-x^2}} y\,dy$$

$$= \int_0^a x\,dx\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}}$$

$$= \frac{1}{2}\int_0^a x\,(a^2 - x^2)\,dx$$

$$= \frac{1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{a^4}{8}$$

Example 15.6. Evaluate $\iint_A xy\,dx\,dy$ where A is the domain bounded by x – axis, ordinate $x = 2a$ and the curve $x^2 = 4\,ay$.

Solution. Given $\iint_A xy\,dx\,dy$

The area of integration is OAB. Hence we integrate first w.r. to y and then w.r. to x.

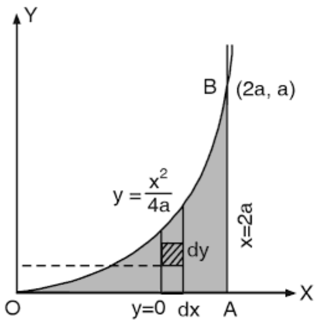

Fig. 15.6

The limits of integration are 0 to $2a$ for x and 0 to $\frac{x^2}{4a}$ for y.

Thus,

$$\iint_A xy\,dx\,dy = \int_0^{2a}\int_0^{x^2/4a} xy\,dx\,dy$$

$$= \int_0^{2a} x\,dx \int_0^{x^2/4a} y\,dy = \int_0^{2a} x\,dx\left[\frac{y^2}{2}\right]_0^{x^2/4a} = \int_0^{2a} x\cdot\frac{x^4}{32\,a^2}\,dx$$

$$= \frac{1}{32\,a^2}\int_0^{2a} x^5 dx = \frac{1}{32\,a^2}\left[\frac{x^6}{6}\right]_0^{2a}$$

$$= a^{4/3}.$$

Example 15.7. Evaluate $\int_0^2 \int_0^{\sqrt{2x - x^2}} (x^2 + y^2)\,dy\,dx$

Solution. $\int_0^2 \int_0^{\sqrt{2x - x^2}} (x^2 + y^2)\,dy\,dx$

The limits of $y = \sqrt{2x - x^2}$ or $y^2 = 2x - x^2$

or $x^2 + y^2 - 2x = 0$ ... (1)

represents a circle whose centre is (1, 0) and radius = 1. Lower limit of y is 0 *i.e.* x – axis.

Region of integration is upper half circle. Let us convert Eq. (1) into polar coordinates by putting

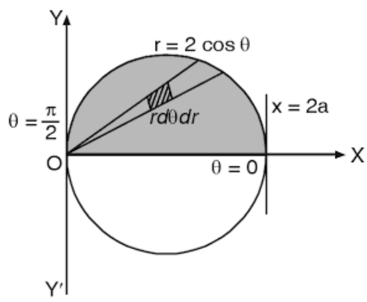

Fig. 15.7

$x = r\cos\theta,\quad y = r\sin\theta$

$r^2 - 2r\cos\theta = 0\quad$ or $\quad r = 2\cos\theta$

Limits of r are 0 to $2\cos\theta$.

Limits of θ are 0 to $\pi/2$.

$$\therefore\quad \int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)\,dy\,dx = \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\,(r\,d\theta)\,dr$$

$$= \int_0^{\pi/2} d\theta \int_0^{2\cos\theta} r^3\,dr = \int_0^{\pi/2} d\theta \left[\frac{r^4}{4}\right]_0^{2\cos\theta}$$

$$= 4\int_0^{\pi/2} \cos^4\theta\,d\theta = 4\times\frac{3\times 1\times\pi}{4\times 2\times 2}$$

$$= \frac{3\pi}{4}$$

Example 15.8. Evaluate $\iint_s \sqrt{xy - y^2}\,dy\,dx$, where s is a triangle with vertices (0, 0), (10, 1) and (1, 1).

Solution. Let the vertices of a triangle OBA be (0, 0), (10, 1) and (1, 1).

Equation of OA is $x = y$

Equation of OB is $x = 10\,y$

The region of ΔOBA, given by the limits $y \le x \le 10\,y$ and $0 \le y \le 1$.

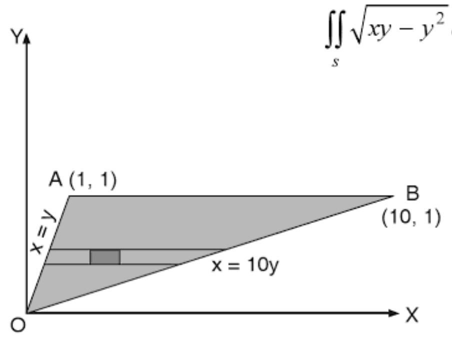

Fig. 15.8

$$\iint_s \sqrt{xy-y^2}\,dy\,dx = \int_0^1 dy \int_y^{10y} (xy - y^2)$$

$$= \int_0^1 dy \left[\frac{2}{3}\,\frac{1}{y}\,(xy-y^2)^{3/2}\right]_y^{10y}$$

$$= \int_0^1 \frac{2}{3}\cdot\frac{1}{y}\,(9y^2)^{3/2}\,dy = 18\int_0^1 y^2\,dy$$

$$= 18\left[\frac{y^3}{3}\right]_0^1 = \frac{18}{3} = 6.$$

15.3 CHANGE OF VARIABLES OF INTEGRATION

Change of variables of integration means the change of order of integration. Some of the problems connected with double integrals, which seems to be complicated, can be made simple and easy to handle by a change in the order of integration.

It should be noted that on changing the order of integration, the limits of integration do change. To find the new limits, we draw the rough sketch of the region of integration. Let us take some examples:

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Example 15.9. Change the order of integration in

$$\mathbf{I} = \int_0^1 \int_{x^2}^{2-x} xy\, dx\, dy \text{ and hence evaluate the same.}$$

(*Nagpur U. s*/2001)

Solution.

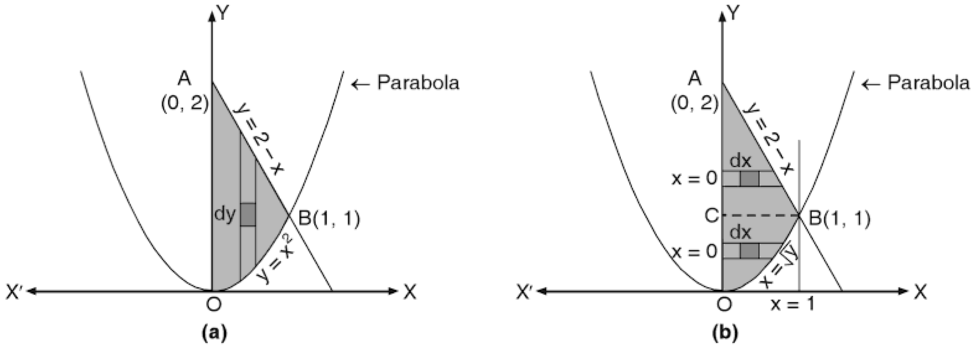

Fig. 15.9

$$\mathrm{I} = \int_0^1 \int_{x^2}^{2-x} xy\, dx\, dy$$

The region of integration is the figure bounded by parabola $y = x^2$ and the line $y = 2 - x$ as shown by shaded portion in fig. 15.9. The point of intersection of the parabola and the line is $B(1, 1)$.

In the figure 15.9 (*a*), we have taken a elementary strip parallel to y axis and the order of integration is

$$\int_0^1 x\, dx \int_{x^2}^{2-x} y\, dy$$

In the figure 15.9 (*b*), we have taken one elementary strip parallel to x axis in the area OBC and another strip in the area ABC. The limits of x in the area OBC are 0 and $\sqrt{y}$ and the limits of x in the area ABC are 0 and $2 - y$. So, the integral using fig. 15.9 (*b*) will be

$$= \int_0^1 y\, dy \int_0^{\sqrt{y}} x\, dx + \int_1^2 y\, dy \int_0^{2-y} x\, dx = \int_0^1 y\, dy \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} + \int_1^2 y\, dy \left[\frac{x^2}{2}\right]_0^{2-y}$$

$$= \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 y(2-y)^2 dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\int_1^2 (4y - 4y^2 + y^3)\, dy$$

$$= \frac{1}{6} + \frac{1}{2}\left[2y^2 - \frac{4}{3}y^3 + \frac{y^4}{4}\right]_0^2$$

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$$\int_0^{2a} dx \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} V\, dy = \int_0^{2a} dy \int_{y^2/2a}^{2a} V\, dx + \int_0^{a} dy \int_{y^2/2a}^{a+\sqrt{a^2-y^2}} V\, dx + \int_0^{a} dy \int_{a+\sqrt{a^2-y^2}}^{2a} V\, dx$$

15.4 APPLICATIONS OF DOUBLE INTEGRALS

(1) Area in Cartezian Co-ordinates $\int_a^b \int_{y_1}^{y_2} dx\, dy$ = Area.

Let the curves AB and CD be $y_1 = f_1(x)$ and $y_2 = f_2(x)$ respectively.

Let the ordinates AD and BC be $x = a$ and $x = b$ respectively.

So the area enclosed by the two curves $y_1 = f_1(x)$ and $y_2 = f_2(x)$ and $x = a$ and $x = b$ is $ABCD$.

Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be the two neighbouring points, then the area of the elemental rectangle $PQ = \delta x \cdot \delta y$

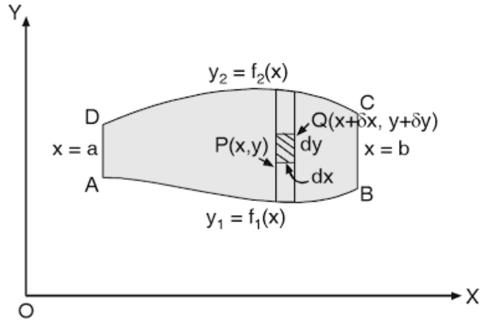

Fig. 15.12

$$\text{Area of the vertical strip} = \lim_{\delta y \to 0} \sum_{y_1}^{y_2} \delta x \cdot \delta y = \delta x \int_{y_1}^{y_2} dy$$

Here δx the width of the strip constant throughout.

If we add all the strips from $x = a$ to $x = b$, we get

$$\text{Area of } ABCD = \lim_{\delta x \to 0} \sum_a^b \delta x \int_{y_1}^{y_2} dy = \int_a^b dx \int_{y_1}^{y_2} dy = \int_a^b \int_{y_1}^{y_2} dx \cdot dy$$

(2) Area in Polar coordinates

$$\text{Area} = \iint r\, d\theta\, dr$$

Let us consider the area enclosed by the curve $r = f(\theta)$.

Let $P(r, \theta)$, $Q(r + \delta r, \theta + \delta\theta)$ be two neighbouring points.

Draw arcs PL and QM, radii r and $r + \delta r$.

$PL = r\delta\theta$, $PM = \delta r$

$$\begin{aligned}\text{Area of rectangle } PLQM &= PL \times PM \\ &= (r\,\delta\theta)\,(\delta r) = r\,\delta\theta \cdot \delta r\end{aligned}$$

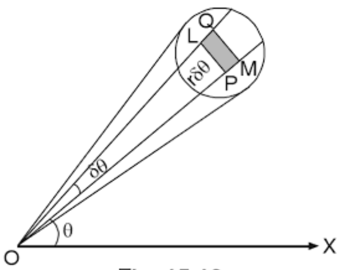

Fig. 15.13

The whole area A is composed of such small rectangles.

Hence,
$$A = \lim_{\substack{\delta r \to 0 \\ \delta\theta \to 0}} \sum\sum r\,\delta\theta \cdot \delta r = \iint r\, d\theta \cdot dr$$

Example 15.12. Find the area between the parabola $y^2 = 4ax$ and $x^2 = 4ay$.

Solution.

$$y^2 = 4ax \qquad \ldots (i)$$
$$x^2 = 4ay \qquad \ldots (ii)$$

$$= \left[2x - \frac{x^2}{2} + \frac{1}{3}(4-2x)^{3/2}\right]_0^2$$

$$= \left[4 - 2 - \frac{4}{3}\right] = \frac{2}{3}$$

15.5 TRIPLE INTEGRAL

Let us consider a function $f(x, y, z)$ of independent variables x, y, z having definite value at each point in three dimensional finite volume V. Imagine that the volume V is divided into n elementary volumes $\delta V_1, \delta V_2, \delta V_3, ... \delta V_n$. Let $(x_1, y_1, z_1), (x_2, y_2, z_2)$ .... be the points within these volumes.

Then the limit of the sum of $f(x_1, y_1, z_1)\, \delta V_1\, f(x_2, y_2, z_2)\, \delta V_2$, ... as the number of the sub-divisions increases indefinitely and the volume of each sub-division decreases to zero is defined as the *triple integral* of $f(x, y, z)$ over the volume V and is written as

$$\iiint_V f(x, y, z)\, dV$$

Thus,

$$\iiint_V f(x, y, z)\, dV = \lim_{\substack{n\to\infty \\ \delta V \to 0}} [f(x_1, y_1, z_1)\, \delta V_1 + f(x_2, y_2, z_2)\, \delta V_2 + ...$$

$$....... + f(x_n, y_n, z_n)\, \delta V_n]$$

15.6 EVALUATION OF TRIPLE INTEGRAL.

For evaluation of a triple integral, it can be expressed as the repeated integral:

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x, y, z)\, dx\, dy\, dz \qquad ... \text{(i)}$$

If x_1, x_2 are constants; y_1, y_2 are either constants or functions of x and z. Similarly z_1, z_2 are either constants or functions of x and y, then the integral is evaluated as follows:

First $f(x, y, z)$ is integrated w.r. to z between the limits z_1 and z_2, keeping x and y constant; then the expression is integrated w.r. to y between the limits y_1 and y_2, keeping x constant. The expression thus obtained is finally integrated w.r. to x between the limits x_1 and x_2.

Thus,

$$I = \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} \boxed{\int_{z_1}^{z_2} f(x, y, z)\, dz}\, dy}\, dx} \qquad ... (ii)$$

The rectangles on R.H.S. of Eq. (*ii*) show the order of integration.

Example 15.14. Using triple integral find the volume of a sphere of radius *a*.

Solution. We use spherical polar coordinates to find the volume of a sphere of radius a. An element of the volume of the sphere is shown in fig. 15.16.

In the spherical polar coordinates r, θ, ϕ the lengths of the edges of the volume elements are dr, $rd\theta$ and $r \sin\theta\, d\phi$.

Example 15.15. Evaluate $\iiint\limits_R (x + y + z)\, dx\, dy\, dz$

where $R : 0 \le x \le 1, \quad 1 \le y \le 2, \quad 2 \le z \le 3$.

Solution.

$$\int_0^1 dx \int_1^2 dy \int_2^3 (x + y + z)\, dz = \int_0^1 dx \int_1^2 dy \left[\frac{(x + y + z)^2}{2} \right]_2^3$$

$$= \frac{1}{2} \int_0^1 dx \int_1^2 dy\, [(x + y + z)^2 - (x + y + z)^2]$$

$$= \frac{1}{2} \int_0^1 dx \int_1^2 (2x + 2y + 5) \cdot 1\, dy$$

$$= \frac{1}{2} \int_0^1 dx \left[\frac{(2x + 2y + 5)^2}{4} \right]_1^2 = \frac{1}{8} \int_0^1 dx\, [(2x + 4 + 5)^2 - (2x + 2 + 5)^2]$$

$$= \frac{1}{8} \int_0^1 (4x + 16) \cdot 2\, dx = \int_0^1 (x + 4)\, dx$$

$$= \left[\frac{x^2}{2} + 4x \right]_0^1 = \frac{1}{2} + 4$$

$$= \frac{9}{2}$$

Example 15.16. Compute $\iiint \dfrac{dx\, dy\, dz}{(x + y + z + 1)^3}$ if the region of integration is bounded by the coordinate planes and the plane $x + y + z = 1$.

Solution. Let the given region be R, then R is expressed as

$0 \le z \le 1 - x - y, \quad 0 \le y \le 1 - x, \quad 0 \le x \le 1$

$$\iiint\limits_R \frac{dx\, dy\, dz}{(x + y + z + 1)^3} = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} \frac{dz}{(x + y + z + 1)^3}$$

$$= \int_0^1 dx \int_0^{1-x} dy \left[\frac{1}{-2(x + y + z + 1)^2} \right]_0^{1-x-y}$$

$$= -\frac{1}{2} \int_0^1 dx \int_0^{1-x} dy \left[\frac{1}{(x + y + 1 - x - y + 1)^2} - \frac{1}{(x + y + 1)^2} \right]$$

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$$= -\frac{1}{2}\int_0^1 dx \int_0^{1-x}\left[\frac{1}{4}-\frac{1}{(x+y+1)^2}\right]dy = -\frac{1}{2}\int_0^1 dx\left[\frac{y}{4}+\frac{1}{x+y+1}\right]_0^{1-x}$$

$$= -\frac{1}{2}\int_0^1 dx\left[\frac{1-x}{4}+\frac{1}{x+1+1-x}-\frac{1}{x+1}\right] = -\frac{1}{2}\int_0^1\left(\frac{1-x}{4}+\frac{1}{2}-\frac{1}{x+1}\right)dx$$

$$= -\frac{1}{2}\left[-\frac{(1-x)^2}{8}+\frac{x}{2}-\log(x+1)\right]_0^1$$

$$= -\frac{1}{2}\left[\frac{1}{2}-\log 2+\frac{1}{8}\right] = -\frac{1}{2}\left[\frac{5}{8}-\log 2\right] = \frac{1}{2}\log 2-\frac{5}{16}.$$

Example 15.17. Evaluate $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{a} r^2 \sin\theta\, dr\, d\theta\, d\phi$ **.**

Solution. $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{a} r^2\sin\theta\, dr\, d\theta\, d\phi$

$$= \int_0^{2\pi} d\phi\int_0^{\pi/4}\sin\theta\, d\theta\int_0^a r^2\, dr$$

$$= \int_0^{2\pi} d\phi\int_0^{\pi/4}\sin\theta\, d\theta\left[\frac{r^3}{3}\right]_0^a$$

$$= \frac{a^3}{3}\int_0^{2\pi} d\phi\int_0^{\pi/4}\sin\theta\, d\theta = \frac{a^3}{3}\int_0^{2\pi} d\phi\,[-\cos\theta]_0^{\pi/4}$$

$$= \frac{a^3}{3}\int_0^{2\pi} d\phi\left(-\cos\frac{\pi}{4}+\cos\theta\right) = \frac{a^3}{3}\left(1-\frac{1}{\sqrt{2}}\right)\int_0^{2\pi} d\phi$$

$$= \frac{a^3}{3}\left(1-\frac{1}{\sqrt{2}}\right)(\phi)_0^{2\pi}$$

$$= \frac{a^3}{3}\left(1-\frac{1}{\sqrt{2}}\right)\cdot 2\pi$$

$$= \frac{2\pi\, a^3}{3\sqrt{2}}\left(\sqrt{2}-1\right)$$

15.7 CHANGE TO SPHERICAL COORDINATES

The relation between the cartesian and spherical polar coordinates of a point are given by the relations

$$x = r\sin\theta\cos\phi$$

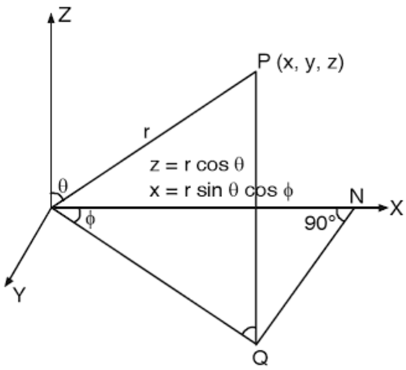

Fig. 15.17

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$$\delta V = \delta x\, \delta y\, \delta z$$

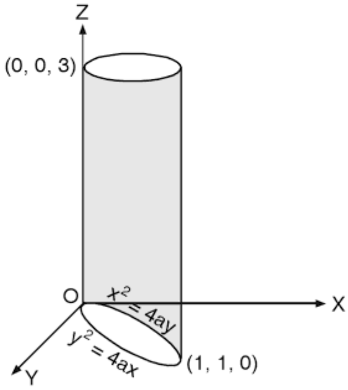
$$\therefore \qquad V = \iiint dx\, dy\, dz$$

The mass of the rectangular object having density ρ will be

$$\text{Mass} = \text{volume} \times \text{density} = \iiint \rho\, dx\, dy\, dz\,.$$

Example 15.19. Find the volume of the region bounded by the surface $y = x^2$, $x = y^2$ and the planes $z = 0$; $z = 3$.

Solution. Given : $y = x^2, \quad x = y^2, \quad z = 0, \quad z = 3$



$$\text{Required volume} = \iiint dx\, dy\, dz$$

$$= \int_0^1 dx \int_{x^2}^{\sqrt{x}} dy \int_0^3 dz$$

$$= \int_0^1 dx \int_{x^2}^{\sqrt{x}} dy\, [z]_0^3$$

$$= 3\int_0^1 dx\, [y]_{x^2}^{\sqrt{x}}$$

$$= 3\int_0^1 (\sqrt{x} - x^2)\, dx$$

$$= 3\left[\frac{2\,x^{3/2}}{3} - \frac{x^3}{3}\right]_0^1$$

$$= 3\left[\frac{2}{3} - \frac{1}{3}\right] = 1$$

EXERCISE CH. 15

1. Define double and triple integrals. (*Nagpur U. s*/2007)

2. Define double and triple integrals. Explain the method of evaluation of double integral as repeated integral over a region A. (*Nagpur U. s*/2006)

3. What is meant by triple integral. (*Nagpur U. w*/2007)

4. What is meant by volume integral? (*Nagpur U. s*/2008)

5. Evaluate $\iint_A (x+y)^2\, dx\, dy$, where A is the area bounded by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

[**Ans.** $\dfrac{1}{4}\pi a b\, (a^2 + b^2)$]

6. Evaluate $\iint_R x^2\, dx\, dy$, where R is the two dimensional region bounded by the curves $y = x$ and $y = x^2$. [**Ans.** $\dfrac{1}{20}$]

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7. Evaluate $\int_0^1 \int_{y^2}^{y} (1 + xy^2)\, dx\, dy$ **[Ans.** $\frac{41}{210}$]

8. Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} dx\, dy$ **[Ans.** $\frac{\pi a^2}{4}$]

9. Evaluate $\iint (x^2 + y^2)\, x\, dx\, dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$.

[**Hint:** Convert into polar coordinates]. **[Ans.** $\frac{a^5}{5}$]

10. Evaluate $\iint_A r^2 \sin\theta\, d\theta$ where A is the region of the circle $r = 2a \cos \theta$ lying above initial line. **[Ans.** $\frac{2a^3}{3}$]

11. Evaluate on changing the order of integration of $\int_0^1 \int_{x^2}^{x} (x^2 + y^2)^{-\frac{1}{2}} dy\, dx$

[Ans. $\int_0^1 dy \int_y^{\sqrt{y}} (x^2 + y^2)^{-\frac{1}{2}} dx$]

12. Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2 \sqrt{x^2 + y^2}\, dx\, dy$ by changing into polar coordinates. **[Ans.** $\frac{\pi a^5}{20}$]

13. Evaluate the integral : $\int_0^{\log 2} \int_0^{x} \int_0^{x+\log y} e^{x+y+z} dz\, dy\, dx$ **[Ans.** $\frac{8}{3} \log 2 - \frac{19}{9}$]

14. Evaluate $\int_0^1 dx \int_0^2 dy \int_1^2 x^2 yz\, dz$ **[Ans.** 1]

15. Evaluate $\int_1^2 \int_0^1 \int_{-1}^1 (x^2 + y^2 + z^2)\, dx\, dy\, dz$ **[Ans.** 6]

16. Evaluae $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} dz\, dy\, dx$ **[Ans.** $\frac{e^{4a}}{8} - \frac{3}{4} e^{2a} + e^{a} - \frac{3}{8}$]

17. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\, dz\, dy\, dx$ **[Ans.** $\frac{1}{48}$]

18. Calculate the volume of the solid bounded by the surface $x = 0$, $y = 0$, $x + y + z = 1$ and $z = 0$ **[Ans.** $\frac{1}{6}$]

19. Calculate the volume of the solid bounded by the following surfaces : $z = 0$, $x^2 + y^2 = 1$, $x + y + z = 3$. **[Ans.** 3π]

20. Find by triple integration the volume of a solid bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the paraboloid $x^2 + y^2 = 3z$. [**Ans.** $\frac{19\pi}{6}$]

21. Find the volume cut from the paraboloid $4z = x^2 + y^2$ by plane $z = 4$. [**Ans.** $32\ \pi$]

22. Find the volume of the elipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$ [**Ans.** $\frac{4\pi\, abc}{3}$]

23. Find the volume of a cone of height h and base radius r. [**Ans.** $\frac{1}{3}\pi r^2 h$]

16 ELECTROSTATICS

ELECTRIC FIELD

INTRODUCTION

We are familiar with the basic fact that there exists fundamentally two types of electric charges, positive and negative in nature. Benjamin Franklin, a pioneer of electrostatics gave the name of 'positive charge' to the charge on a glass rod rubbed with silk, and a 'negative charge' to that on an ebonite rod rubbed with fur. From our previous experimental knowledge, we also know that two similar charges repel each other and unlike charges attract.

In this chapter, we study the physics of stationary electric charges i.e. charges at rest called electrostatics. The force of attraction or repulsion between static charges is given by Coulomb's law of electrostatic forces.

16.1 COULOMB'S LAW OF ELECTROSTATIC FORCE

For two electric charges at rest, like charges repel each other and unlike charges attract each other. Coulomb's law states that

"*The magnitude of the force between two point charges is directly proportional to the product of the magnitude of the charges and inversely proportional to the square of the distance between them.*"

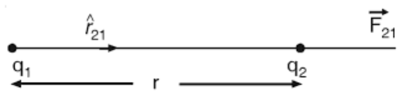

Fig. 16.1

If q_1 and q_2 are the magnitude of two charges and r the distance between them, then the force

$$F = K\frac{q_1 q_2}{r^2}$$

where K is the constant of proportionality, the value of which depends upon the nature of the medium separating the two charges and the system of units in which various quantities are expressed. This is inverse square law.

In the **vector** form, Coulomb's law can be stated as

$$\vec{F}_{21} = K\frac{q_1 q_2}{|\vec{r}_{21}|^2}\hat{r}_{21} \qquad \ldots(i)$$

where $\vec{F}_{21}$ is the force on charge q_2 due to charge q_1, $\vec{r}_{21}$ the vector distance of q_2 from q_1 and $\hat{r}_{21}$ a unit vector in the direction of $\vec{r}_{21}$ *i.e.*, to charge q_2 from charge q_1.

The unit vector $\hat{r}_{21} = \dfrac{\vec{r}_{21}}{\left|\vec{r}_{21}\right|}$

$$\therefore \qquad \vec{F}_{21} = K\frac{q_1 q_2 \vec{r}_{21}}{\left|\vec{r}_{21}\right|^3} \qquad ...(ii)$$

16.2 COULOMB'S LAW IN TERMS OF POSITION VECTORS

Let $\vec{r}_1$ and $\vec{r}_2$ represent the position vectors of two point charges q_1 and q_2 respectively [Fig. 16.2], then

$$\vec{r}_2 = \vec{r}_1 + \vec{r}_{21}$$

or $$\vec{r}_{21} = \vec{r}_2 - \vec{r}_1$$

and $$\left|\vec{r}_{21}\right| = \left|\vec{r}_2 - \vec{r}_1\right|$$

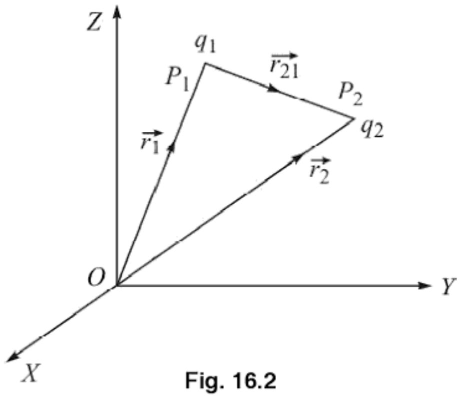

Fig. 16.2

Substituting in Eq. (*ii*) of art. (16.1), we get

Force on charge q_2 due to charge q_1

$$\vec{F}_{21} = K\frac{q_1 q_2}{\left|\vec{r}_2 - \vec{r}_1\right|^3}(\vec{r}_2 - \vec{r}_1) \qquad ...(i)$$

1. In C.G.S. System

In C.G.S. system, the constant $K = \dfrac{1}{k}$ and for any medium

$$\vec{F}_{21}\text{ (medium)} = \frac{1}{k}\frac{q_1 q_2}{\left|\vec{r}_{21}\right|^2}\hat{r}_{21} = \frac{1}{k}\frac{q_1 q_2}{\left|\vec{r}_{21}\right|^3}\hat{r}_{21}$$

$$= \frac{1}{k}\frac{q_1 q_2 (\vec{r}_2 - \vec{r}_1)}{\left|\vec{r}_{21} - \vec{r}_1\right|^3}$$

k is known as *specific inductive capacity or dielectric constant* of the medium and its value for vaccum or free space is taken to be **unity.** For air at N.T.P. it is also very nearly equal to 1(1.0059).

$\therefore$ For vacuum or free space

$$\vec{F}_{21}\text{ (vacuum)} = \frac{q_1 q_2}{\left|\vec{r}_{21}\right|^2}\hat{r}_{21} = \frac{q_1 q_2 \vec{r}_{21}}{\left|\vec{r}_{21}\right|^3}$$

$$= \frac{q_1 q_2 (\vec{r}_2 - \vec{r}_1)}{\left|\vec{r}_2 - \vec{r}_1\right|^3}$$

In the C.G.S. system, the force is measured in dynes, distance in c.m. and charge in *e.s.u.* or stat-coulomb.

2. In S.I. System

In S.I. units $K = \dfrac{1}{4\pi\varepsilon}$ where ε is the permittivity of the medium.

$$\therefore \quad \text{For any medium } \vec{F}_{21} \text{ (medium)} = \frac{1}{4\pi\varepsilon} \frac{q_1 q_2 (\hat{r}_{21})}{\left|\vec{r}_{21}\right|^2} = \frac{1}{4\pi\varepsilon} \frac{q_1 q_2 \vec{r}_{21}}{\left|\vec{r}_{21}\right|^3}$$

$$= \frac{1}{4\pi\varepsilon} \frac{q_1 q_2 (\vec{r}_2 - \vec{r}_1)}{\left|\vec{r}_2 - \vec{r}_1\right|^3} \qquad \ldots (i)$$

For vacuum or free space

$$\vec{F}_{21} \text{ (vacuum)} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{\left|\vec{r}_{21}\right|^2} \hat{r}_{21} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2 \vec{r}_{21}}{\left|\vec{r}_{21}\right|^3}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2 (\vec{r}_2 - \vec{r}_1)}{\left|\vec{r}_2 - \vec{r}_1\right|^3} \qquad \ldots (ii)$$

where ε_0 is the permittivity of free space. The value of $\dfrac{1}{4\pi\varepsilon_0} = 9 \times 10^9$ newton-m^2/coulomb^2 [Nm^{2}C^{-2}] and $\varepsilon_0 = 8.85 \times 10^{-12}$ coulomb^2/Newton m^2 [C^{2}N^{-1}m^{-2}].

16.3 RELATION BETWEEN COULOMB AND STAT-COULOMB

Coulomb. The S.I. unit of charge is a *coulomb*. It is defined from the basic unit of current in S.I. *i.e.*, an *ampere*.

A coulomb is the quantity of charge that passes through any cross-section of a conductor when a current of one ampere flows through it for one second.

Stat-coulomb. The C.G.S. unit of charge is a *stat-coulomb*.

The charge is said to be one stat-coulomb if it experiences a force of repulsion of one dyne when placed in vacuum at a distance of one cm from an equal and similar charge.

The magnitude of electrostatic force, between two charges of one coulomb each placed one metre apart in vacuum is given by

$$F = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2}$$

$$= 9 \times 10^9 \text{ N m}^2 \text{ C}^{-2} \times \text{C}^2 \text{ m}^{-2}$$

$$= 9 \times 10^9 \text{ N} = 9 \times 10^{14} \text{ dynes}$$

Now let 1 coulomb = n stat-coulomb

$$\therefore \text{ In C.G.S. units} \quad F = \frac{q_1 q_2}{r^2} = \frac{n \times n}{100^2} = \frac{n^2}{10^4} \text{ dynes}$$

or $$\frac{n^2}{10^4} = 9 \times 10^{14}$$

$$\therefore \quad n^2 = 9 \times 10^{18} \text{ or } n = 3 \times 10^9$$

Hence 1 coulomb = 3×10^9 stat-coulomb

16.4 DIELECTRIC CONSTANT

The dielectric constant of a medium is defined as the ratio of the magnitude of the force between two charges placed a certain distance apart in vacuum to the force between the same two charges placed the same distance apart in the medium.

The force between two charges q_1 and q_2 placed at a distance r in a medium in C.G.S. units, taking magnitudes only is given by

$$F_m = \frac{1}{k}\frac{q_1 q_2}{r^2}$$

and in free space $$F_0 = \frac{q_1 q_2}{r^2}$$

$$\therefore \qquad \frac{F_0}{F_m} = k \qquad \dots (i)$$

where k is the *dielectric constant* of medium.

Similarly in S.I. units, taking magnitudes only

$$F_m = \frac{1}{4\pi\varepsilon}\frac{q_1 q_2}{r^2}$$

and $$F_0 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$$

$$\therefore \qquad \frac{F_0}{F_m} = \frac{\varepsilon}{\varepsilon_0} = \varepsilon_r \qquad \dots (ii)$$

ε_r is known as *relative permittivity and is the ratio of the permittivity of the medium to the permittivity of free space (or vacuum).*

From (*i*) and (*ii*), we have

$$k = \varepsilon_r$$

Thus the dielectric constant of a medium is the same as its relative permittivity, both being dimensionless constants.

16.5 COULOMB'S LAW IN ACCORDANCE WITH NEWTON'S THIRD LAW OF MOTION

Let $\vec{r_1}$ and $\vec{r_2}$ represent the position vectors of two point charges q_1 and q_2 respectively, then the force on charge q_2 due to charge q_1 (in S.I. units) in free space is given by

$$\vec{F}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{\left|\vec{r_2} - \vec{r_1}\right|^3}(\vec{r_2} - \vec{r_1}) \qquad \dots (i)$$

Now $$(\vec{r_1} - \vec{r_2}) = -(\vec{r_2} - \vec{r_1})$$

and $$|\vec{r_1} - \vec{r_2}| = |\vec{r_2} - \vec{r_1}|$$

$$\therefore \qquad \vec{F}_{12} = -\frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{|\vec{r_2} - \vec{r_1}|^3}(\vec{r_2} - \vec{r_1}) \qquad \dots (ii)$$

Comparing (*i*) and (*ii*), we have

$$\vec{F}_{12} = -\vec{F}_{21}$$

This relation shows that *the force* $\vec{F}_{21}$ *on the charge* q_2 *due to the charge* q_1 *is equal and opposite to the force* $\vec{F}_{12}$ *on the charge* q_1 *due to the charge* q_2. Hence Coulomb's law is in accordance with Newton's third law of motion. In other words, the electric force between two static charges is Newtonian in character.

16.6 LIMITATIONS OF COULOMB'S LAW

(*i*) Coulomb's law is valid only for point charges, (*ii*) Coulomb's law is strictly applicable to charges at rest. When a charge is in motion forces other than electrostatic forces come into play and Coulomb's law cannot be used to measure the charges in motion.

(*iii*) Coulomb's law fails to explain the stability of nucleus, since in nucleus, there are large number of protons, all having positive charge. According to Coulomb's law, they should repel each other. However, nucleus has stable identity. Hence, Coulomb's law fails to explain the stability of the nucleus.

16.7 ELECTRICAL FORCES AND NUCLEAR FORCES

In nucleus, there are large number of protons closely packed, all having positive charge. According to Coulomb's law, they should strongly repel each other due to the small value of r ($\because F \propto \frac{1}{r^2}$). Inspite of this strong repulsion, the nucleus has a stable nature. This failure of coulomb's law, is because, in nuclei, some *non-electrical* and *non-gravitational* forces, in addition to electric forces, called the nuclear forces exist. These nuclear forces hold protons together, inspite of the electrical repulsion between them. Hence, the nuclear forces are known as short range forces.

16.8 COULOMB'S LAW AND NUCLEAR FISSION

It should be noted that Coulomb's law is valid over a wide range from 10^{-13} cm to distances of several kilometers. Therefore, coulomb's force is a large range force. However, the nuclear forces are of short range. Nuclear forces fall off much more rapidly than $1/r^2$ and this has important consequences. If the nucleus has too many protons (uranium atomic number Z = 92); the electrical force of repulsion becomes predominant over the nuclear force. The nuclear forces act mainly between each proton or neutron and its nearest neighbour, while the electrical forces act over larger distances, giving a repulsion between each proton and all of the others in the nucleus. The greater is the number of protons in the nucleus, stronger is the electrical repulsion and so it is easier to break such nucleus and results into nuclear fission.

16.9 INFLUENCE OF OTHER CHARGES

Coulomb's force between two charges is a two body interaction and is independent of the presence of other charges. Hence the Coulomb's force that one charge exerts on another charge remains unchanged if other charges are brought nearby.

16.10 PRINCIPLE OF SUPERPOSITION OF CHARGES

(Net force due to number of discrete charges)

To find the net force due to a number of discrete charges q_1, q_2, q_3 ..., q_n located at position vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3,......,\vec{r}_n$ on a test charge q_0 located at position vector $\vec{r}_0$, we calculate the force due to each one of the charges on the test charge and add all the forces vectorially. Now force on q_0 due to charge q_1

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$$\vec{E} = \frac{\vec{F}}{q_0}$$

where $\vec{F}$ is the total experienced by the charge q_0.

The value of q_0 should be so small that it should not disturb the electric field. Therefore, to be more exact, the ratio $\frac{\vec{F}}{q_0}$ as the magnitude of the test charge q_0 is made smaller and smaller so that it finally tends to zero *i.e.*, the limiting value of

$$\lim_{q_0 \to 0} \frac{\vec{F}}{q_0}$$

defines a vector known as electric field at the point.

The S.I. unit of electric field intensity is Newton/coulomb (NC^{-1}).

16.15 ELECTRIC FIELD DUE TO A POINT CHARGE

Let a point charge q known as the *source charge* be placed at the point B having a position vector $\vec{R}$ and a test charge q_0 at the point A having the position vector $\vec{r}$, then the force on the charge q_0 due to the charge q is given by

$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{qq_0}{|\vec{r} - \vec{R}|^3} (\vec{r} - \vec{R})$$

The electric field at A known as the *observation point* is given by

$$\vec{E} = \frac{\vec{F}}{q_0} = \frac{1}{4\pi\varepsilon_0} \frac{q}{|\vec{r} - \vec{R}|^3} (\vec{r} - \vec{R}) \quad \text{... } (i)$$

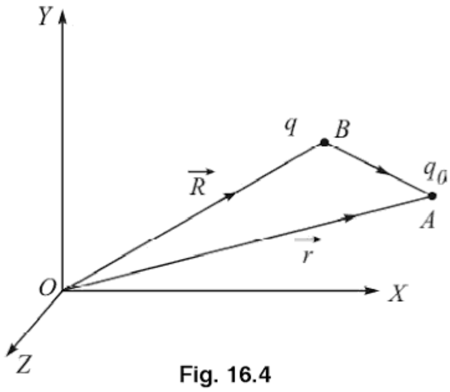

Fig. 16.4

Note. *Note that in the relation (i) in the expression* $(\vec{r} - \vec{R})$, $\vec{r}$ *the position vector of the observation point (test charge) precedes the position vector of the source charge.*

The direction of electric intensity is the same as that of $(\vec{r} - \vec{R})$ and is *from the source charge* to the *observation point* (or test charge).

When the source charge lies at the origin O. Then $R = 0$

In such a case
$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^3} \vec{r} \quad \text{... } (ii)$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \frac{\vec{r}}{r}$$

or
$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r} \quad \text{... } (iii)$$

where $\hat{r}$ is a unit vector in the direction of $\vec{r}$ and $\vec{r}$ is the position vector of the observation point.

The magnitude of electric intensity $E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$

The vector quantity $\vec{E}$ depends only on the charge q and on the position of the point where test charge q_0 was placed; and is called *intensity of electric field*, or field strength at that point. Thus, the intensity of electric field due to a charge q is defined as the force experienced by a unit positive (for convenience) charge placed at the point at which the field is to be determined. However, it should be remembered that the test charge q_0 should be as small as possible, so that its pressure would not disturb the primary charge q, which is responsible for the field $\vec{E}$.

Unit of $\vec{E}$

From the definition of $\vec{E}$ given by Eq. (*iii*), its SI unit is Newton/Coulomb. However, after the discussion of electric potential, its equivalent unit is volt/metre.

16.16 FIELD LINES

Recalling Eq. (*iii*) of article 16.12, the force $\vec{F}$ acting on a test charge q_0 placed in electric field due to charge q will be repulsive *i.e.* from q to q_0 and hence direction of $\vec{E}$ is radially outward from q as shown in Fig. 16.5(*a*).

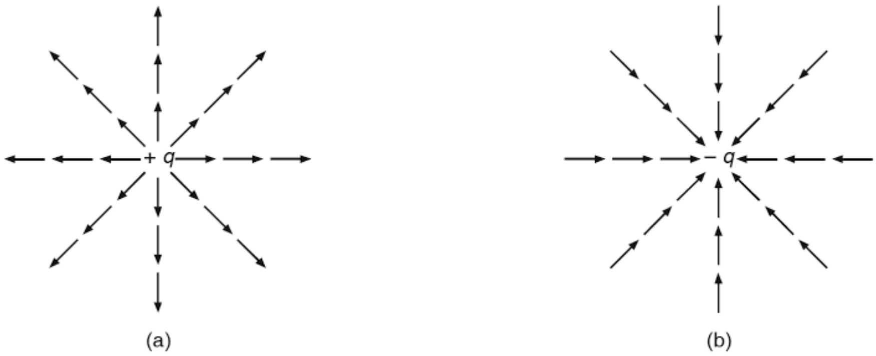

Fig. 16.5

On the otherhand, if the charge under consideration is ($-q$), the situation is just opposite. In this case, $\vec{E}$ is directed inward or toward ($-q$). This is shown in Fig. 16.5(b). The magnitude of $\vec{E}$ increases as the test charge q_0 moves towards $-q$. The lines pointed away or towards the charge q are known as *field lines*, and the density of lines of force gives the magnitude of $\vec{E}$.

Thus, $\vec{E}$ is strong where the lines are close and $\vec{E}$ is weak where the lines are far apart. Let us study electric lines of force and line charge density is detail.

16.17 ELECTRICAL LINES OF FORCE

If a small test charge q_0 ($q_0 \to 0$) is placed in an electric field $\vec{E}$, then it will move along a curve such that the tangent to the curve at every point gives the direction of electric field $\vec{E}$ at that point. Such a line (or curve) is known as *line of electric force*.

(*i*) It is assumed that when a point charge q is placed in a medium of permittivity ε, the number of lines of electric force originating from the point $= \frac{q}{\varepsilon}$.

Suppose we have a point charge q and we draw a sphere of radius r around the point, then magnitude of electric field strength at any point on the spherical surface

$$|\vec{E}| = E = \frac{1}{4\pi\varepsilon}\frac{q}{r^2} = \frac{q/\varepsilon}{4\pi r^2}$$

$$= \frac{\text{Number of lines of electric force}}{\text{Area of the sphere through which the lines cross}}$$

$=$ Number of lines of electric force crossing per unit area.

The strength of the electric field at a point is, therefore, taken to be proportional to the number of lines crossing a unit area around that point.

(ii) The lines of force originate from a positive charge and terminate on a negative charge as shown.

The lines of electric force do not cross each other, because if they cross each other it would mean that there are at least two directions of the electric field $\vec{E}$ at that point *i.e.* $\vec{E}$ will not be unique which is against the concept of vector field and is an erroneous idea.

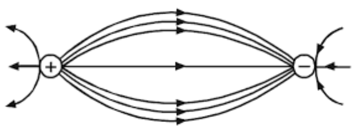

Fig. 16.6

16.18 LINE CHARGE DENSITY

Consider a charged thin wire AB . If Δq is the charge on a small length Δl of the thin lines charge, surrounding the point C, then

Average charge per unit length $= \frac{\Delta q}{\Delta l}$

The limiting value of $\frac{\Delta q}{\Delta l}$ as $\Delta l \to 0$ is know as the line charge density λ at the point C on the infinitesimally small element of length Δl.

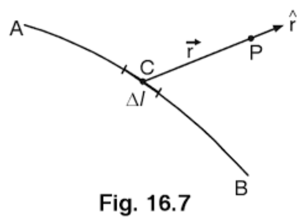

Fig. 16.7

$$\therefore \quad \lambda = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl} \quad \text{... } (i)$$

Hence the line charge density may be defined as the charge per unit length on a given line curved or straight.

From (*i*) we have $dq = \lambda\, dl$

$\therefore$ Total charge on a length l of the line is given by

$$q = \int_l dq = \int_l \lambda\, dl$$

If we have a uniform charge distribution and q is the charge on a length l, then

$$q = \lambda\, l \text{ or } \lambda = \frac{q}{l}$$

16.19 ELECTRIC FIELD $\vec{E}$ DUE TO A LINE CHARGE

Let AB be a line element of charge having a uniform line charge density λ. Consider a small length dl of the line element containing the point C having a position vector $\vec{R}$, then

Charge on the element of length $dl = dq = \lambda\, dl$

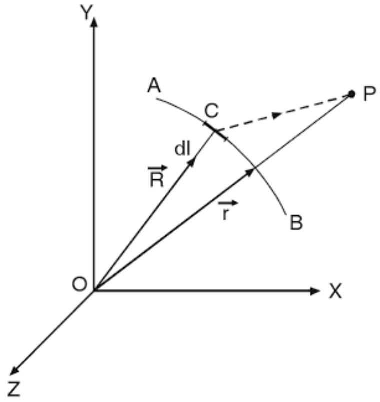

Fig. 16.8

The electric field intensity $\vec{dE}$ at an observation point P having position vector $\vec{r}$ due to the charge dq on the small line element of length dl in free space is given by

$$\vec{dE} = \frac{1}{4\pi\varepsilon_0}\frac{dq\,(\vec{r}-\vec{R})}{|\vec{r}-\vec{R}|^3} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dl\,(\vec{r}-\vec{R})}{|\vec{r}-\vec{R}|^3}$$

$\therefore$ Electric intensity at P due to the whole line charge AB,

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_A^B \frac{\lambda\, dl\,(\vec{r}-\vec{R})}{|\vec{r}-\vec{R}|^3}$$

If the point C on the small line element dl lies at the origin then $\vec{R} = 0$. In such a case

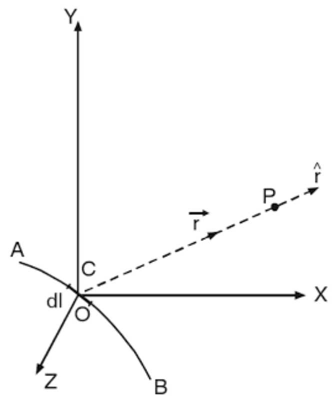

Fig. 16.9

$$\vec{dE} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{|\vec{r}|^3}\vec{r} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dl\, \vec{r}}{|\vec{r}|^3}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\lambda dl}{r^2}\frac{\vec{r}}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dl}{r^2}\hat{r}.$$

where $\hat{r}$ is a unit vector drawn from the charge dq at C to the point P i.e., in the direction of $\vec{r}$.

For the whole line charge AB the electric field intensity

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_A^B \frac{\lambda\, dl\, \vec{r}}{|\vec{r}|^3} = \frac{1}{4\pi\varepsilon_0}\int_A^B \frac{\lambda\, dl\, \hat{r}}{r^2}$$

16.20 ELECTRIC FIELD $\vec{E}$ DUE TO INFINITELY LONG CHARGED CONDUCTOR

Consider a long straight line charge of infinite length in the form of a uniformly charged straight wire placed the X-axis and extending from $x = -\infty$ to $x = +\infty$ in the free space. Let P be the point where the electric intensity is to be calculated. Draw PO perpendicular to the line charge and let O represent the origin and OPY the Y-axis of the co-ordinate system.

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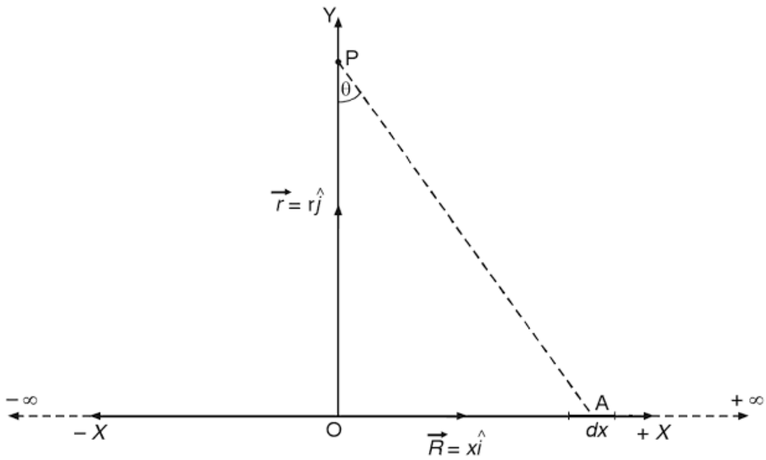

Fig. 16.10

Let $\qquad \vec{OP} = \vec{r} = r\hat{j}$

Consider a small element of length dx of the line charge at a distance x from O.

Let $\qquad \vec{OA} = \vec{R} = x\hat{i}$

If λ is the line charge density, then charge on the length $dx = dq = \lambda dx$

Electric field at P due to this charge element at A in S.I. units in free space

$$\vec{dE} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx(\vec{r}-\vec{R})}{|\vec{r}-\vec{R}|^3} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx\,(r\hat{j}-x\hat{i})}{|r\hat{j}-x\hat{i}|^3}$$

Now $\qquad |r\hat{j}-x\hat{i}| = \sqrt{r^2+(-x)^2} = \sqrt{r^2+x^2}$

$$\vec{dE} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{(r^2+x^2)^{3/2}}(r\hat{j}-x\hat{i}) \qquad \text{... }(ii)$$

Hence total electric field at P due to whole line charge element extending from $-\infty$ to $+\infty$.

$$\vec{E} = \int_{-\infty}^{+\infty}\vec{dE} = \frac{\lambda}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty}\frac{rdx}{(r^2+x^2)^{3/2}}\hat{j} - \frac{\lambda}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty}\frac{xdx}{(r^2+x^2)^{3/2}}\hat{i}$$

... (iii)

To find the value of the integral,

put $\qquad x = r\tan\theta \quad \therefore\ dx = r\sec^2\theta\, d\theta$

and $\qquad (r^2+x^2)^{3/2} = (r^2+r^2\tan^2\theta)^{3/2} = r^3\,(1+\tan^2\theta)^{3/2} = r^3\sec^3\theta$

Also when $x = +\infty;\ \theta = +\dfrac{\pi}{2}$ and when $x = -\infty;\ \theta = -\dfrac{\pi}{2}$

Substituting in (*iii*), we have

$$\vec{E} = \frac{\lambda}{4\pi\varepsilon_0}\int_{-\pi/2}^{+\pi/2}\frac{r^2\sec^2\theta\,d\theta}{r^3\sec^3\theta}\hat{j} - \frac{\lambda}{4\pi\varepsilon_0}\int_{-\pi/2}^{+\pi/2}\frac{r^2\tan\theta\sec^2\theta\,d\theta}{r^3\sec^3\theta}\hat{i}$$

$$= \left[\frac{\lambda}{4\pi\varepsilon_0 r}\int_{-\pi/2}^{+\pi/2}\cos\theta\,d\theta\right]\hat{j} - \left[\frac{\lambda}{4\pi\varepsilon_0 r}\int_{-\pi/2}^{+\pi/2}\sin\theta\,d\theta\right]\hat{i}$$

Now $$\int_{-\pi/2}^{+\pi/2}\cos\theta\,d\theta = [\sin\theta]_{-\pi/2}^{+\pi/2} = 2$$

and $$\int_{-\pi/2}^{+\pi/2}\sin\theta\,d\theta = [-\cos\theta]_{-\pi/2}^{+\pi/2} = 0$$

$\therefore$ $$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{r}\hat{j} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{j} \qquad \text{... }(iv)$$

The magnitude of $|\vec{E}| = \dfrac{\lambda}{2\pi\varepsilon_0 r}$ Coulomb/m^2. The direction of $\vec{E}$ is along + Y direction.

Equation (iv) shows that the electric field due to an infinitely long uniformly charged wire is inversely proportional to the perpendicular distance of the observation point from the wire.

16.21 ELECTRIC FORCE BETWEEN TWO PARALLEL INFINITE LONG CHARGED CONDUCTORS

Let A and B be two parallel infinite wires placed to Y-axis as shown in Fig. 16.11. Let λ_1 be the uniform line charge density of wire A and λ_2 that of wire B, then,

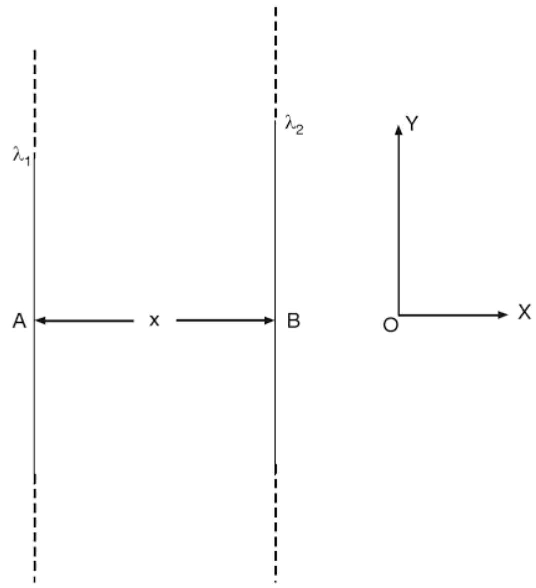

Fig. 16.11

Electric intensity due to the wire A at a distance x from it

$$\vec{E} = \frac{1}{2\pi\varepsilon_0}\frac{\lambda_1}{x}$$

The direction of $\vec{E}$ is perpendicular to the length of the wire A *i.e.* along + X axis.

A small length dl of the wire B carries a charged $dq = \lambda_2\, dl$

$\therefore$ Force acting on the length dl of the wire B due to the charge on the wire A

$$= \frac{1}{2\pi\varepsilon_0}\frac{\lambda_1\lambda_2\, dl}{x}$$

$\therefore$ Electric force per unit length $= \dfrac{1}{2\pi\varepsilon_0}\dfrac{\lambda_1\lambda_2}{x}$

The force acts in a direction perpendicular to the length of the wire B *i.e.*, along + X-axis. Similarly the force acting on the wire A due to the charge on the wire $B = \dfrac{1}{2\pi\varepsilon_0}\dfrac{\lambda_1\lambda_2}{x}$ in a direction perpendicular to the length of the wire A *i.e.*, along – X-axis.

16.22 ELECTRIC FIELD $\vec{E}$ DUE TO CHARGED RING ON ITS AXIS

Consider a circular line charge (or uniformly charged ring) of radius a, having total charge q, then line charge density $\lambda = \dfrac{1}{2\pi a}$.

Let the centre of the line charge (or ring) be at the origin O of the co-ordinate system and the circular line charge (or ring) lie in the XY plane so that the axis of the circular line charge or ring is represented by the Z-axis,

Let P be the point where electric field is to be calculated so that

$$\vec{OP} = \vec{r} = z\hat{k}$$

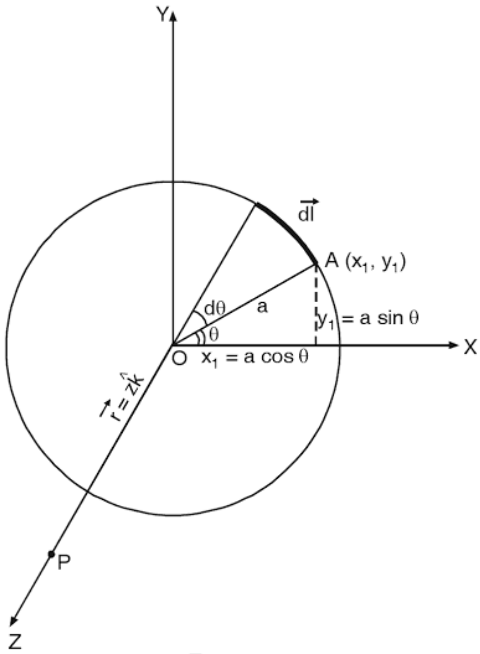

Fig. 16.12

Consider a small element of length $\vec{dl}$ at A making angles θ and $\theta + d\theta$ with the X-axis, then

$$dl = a\, d\theta$$

and charge on the element

$$dq = \lambda\, dl = \frac{q}{2\pi a}\, a\, d\theta = \frac{q}{2\pi}\, d\theta$$

Let the position vector of the charge element dl be $\vec{r_1}$ with co-ordinate (x_1, y_1) as it lies in the X-Y plane.

[**Note.** In this question we have taken the position vector of the source charge as $\vec{r_1}$ instead of $\vec{R}$ as in earlier question because we have taken the co-ordinate of the point A as x_1, y_1].

$$\therefore \quad \vec{r_1} = x_1\hat{i} + y_1\hat{j} = a\cos\theta\,\hat{i} + a\sin\theta\,\hat{j}$$

Electric field at P due to this charge element at A

$$\vec{dE} = \frac{dq}{4\pi\varepsilon_0}\frac{(\vec{r}-\vec{r_1})}{|\vec{r}-\vec{r_1}|^3} = \frac{q\,d\theta}{2\pi}\frac{1}{4\pi\varepsilon_0}\frac{(\vec{r}-\vec{r_1})}{|\vec{r}-\vec{r_1}|^3} \qquad \text{... } (i)$$

Substituting the value of $\vec{r} = z\hat{k}$ and $\vec{r_1} = x\hat{i} + y\hat{j}$, we have

$$(\vec{r}-\vec{r_1}) = z\hat{k} - x_1\hat{i} - y_1\hat{j} = z\hat{k} - a\cos\theta\,\hat{i} - a\sin\theta\,\hat{j}$$

and

$$|\vec{r}-\vec{r_1}| = (z^2 + a^2\cos^2\theta + a^2\sin^2\theta)^{1/2} = (z^2 + a^2)^{1/2}$$

in Eq. (i), we have

$$\vec{dE} = \frac{q}{2\pi}\frac{1}{4\pi\varepsilon_0}\frac{z\hat{k} - a\cos\theta\,\hat{i} - \sin\theta\,\hat{j}}{(z^2+a^2)^{3/2}}\,d\theta$$

The total electric field due to whole circular line charge (or ring) is obtained by integrating the above expression between the limits $\theta = 0$ to $\theta = 2\pi$

$$\therefore \qquad \vec{E} = \int_0^{2\pi}\vec{dE} = \frac{qz\hat{k}}{2\pi(z^2+a^2)^{3/2}}\frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}d\theta$$

$$-\frac{qa}{2\pi(z^2+a^2)^{3/2}}\frac{1}{4\pi\varepsilon_0}\left[\hat{i}\int_0^{2\pi}\cos\theta\,d\theta + \hat{j}\int_0^{2\pi}\sin\theta\,d\theta\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{qz\hat{k}\,2\pi}{2\pi(z^2+a^2)^{3/2}} = \frac{1}{4\pi\varepsilon_0}\frac{qz}{(z^2+a^2)^{3/2}}\hat{k} \qquad \text{... } (ii)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{2\pi a\lambda z}{(z^2+a^2)^{3/2}}\hat{k} \qquad [\because\ q = 2\pi\, a\lambda]$$

$$= \frac{a\lambda}{2\varepsilon_0}\frac{z}{(z^2+a^2)^{3/2}}\hat{k} \qquad \text{... } (iii)$$

Since $\int_0^{2\pi}\cos\theta\,d\theta = 0$ and $\int_0^{2\pi}\sin\theta\,d\theta = 0$

For positive values of z ($z > 0$) $\vec{E}$ is along the + Z axis and for negative values of z ($z < 0$) $\vec{E}$ is along-Z axis.

16.23 ELECTRIC DIPOLE

A pair of equal and opposite point charges separated by a small distance is called an electric dipole.

A molecule made up a positive and a negative ion is an example of electric dipole in nature.

Electric dipole moment *is defined as the product of one of the charges and the vector distance separating the two charges.*

Thus two equal and opposite charges + q and – q separated by a small distance $2l$ constitute a dipole. The electric dipole moment is given by

$$\vec{p} = q2\vec{l} = 2q\vec{l}$$

The vector $\vec{l}$ is drawn from the negative to the positive charge and is along the axis of the dipole. The electric dipole moment is a *vector* quantity and its direction is also from negative to the positive charge.

16.24 ELECTRIC FIELD DUE TO A DIPOLE

Consider an electric dipole lying along the X-axis with its mid point at the origin O of the co-ordinate system. Let the magnitude of each charge be q and $2\vec{l}$ the vector distance between the charges.

Dipole moment of the electric dipole

$$\vec{p} = q2\vec{l} = 2ql\,\vec{i}$$

(i) Point on the axial line. Consider a point P on the axial line of the dipole at a vector distance $\vec{r} = r\hat{i}$ from the origin. Let $\vec{r_+}$ and $\vec{r_-}$ be the position vectors of charges $+q$ and $-q$ respectively then $\vec{r_+} = \vec{i} = l\hat{i}$ and $\vec{r_-} = -\vec{l} = -l\hat{i}.$

Electric field at P due to the charge $+q$

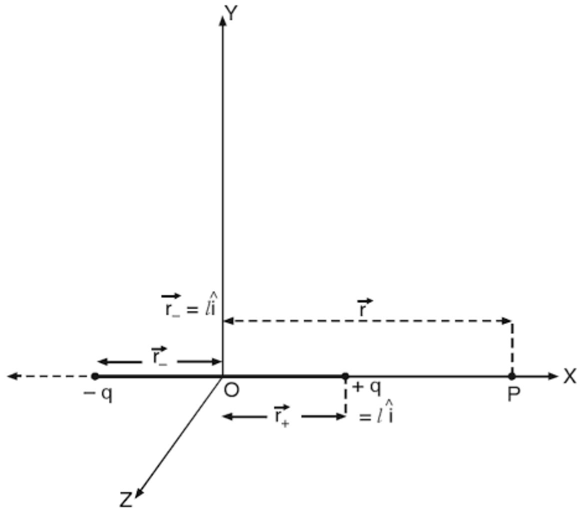

Fig. 16.13

$$\vec{E_+} = \frac{q}{4\pi\varepsilon_0}\frac{(\vec{r}-\vec{r_+})}{|\vec{r}-\vec{r_+}|^3} = \frac{q}{4\pi\varepsilon_0}\frac{(r\hat{i}-l\hat{i})}{|r\hat{i}-l\hat{i}|^3}$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{(r-l)\hat{i}}{(r-l)^3} = \frac{q}{4\pi\varepsilon_0(r-l)^2}\hat{i}$$

Similarly electric field at P due to the charge $-q$

$$\vec{E}_- = \frac{-q}{4\pi\varepsilon_0}\frac{r\hat{i}+l\hat{i}}{(r\hat{i}+l\hat{i})^3} = -\frac{q}{4\pi\varepsilon_0(r+l)^2}\hat{i}$$

The resultant electric field at P

$$\vec{E} = \vec{E}_+ + \vec{E}_- = \frac{q\hat{i}}{4\pi\varepsilon_0}\left[\frac{1}{(r-1)^2} - \frac{1}{(r+1)^2}\right]$$

$$= \frac{q\hat{i}}{4\pi\varepsilon_0}\left[\frac{4rl}{(r^2-l^2)^2}\right]$$

Now $\quad 2lq\hat{i} = \vec{p}$

$\therefore$
$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{2r\vec{p}}{(r^2-l^2)^2} \qquad \text{... } (i)$$

Equation (i) shows that the electric field $\vec{E}$ acts along $+X$ direction.

When *P* lies a far off point. When the point P lies at a vary large distance as compared to the length of the dipole *i.e.* $r >> l$, then l^2 be can be neglected as compared to r^2 and we get

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{2\vec{p}}{r^3} \qquad \text{... } (ii)$$

This shows that the electric field due to a dipole at a far off point varies inversely as the cube of the distance from the dipole.

The direction of electric field is along $+X$-axis as the direction $\vec{p}$ is from $-q$ to $+q$ *i.e.*, along $+X$-axis

(*ii*) Point of the equatorial line. As the origin of the co-ordinate system lies at the mid point of the dipole, the equatorial line co-incides with the Y-axis.

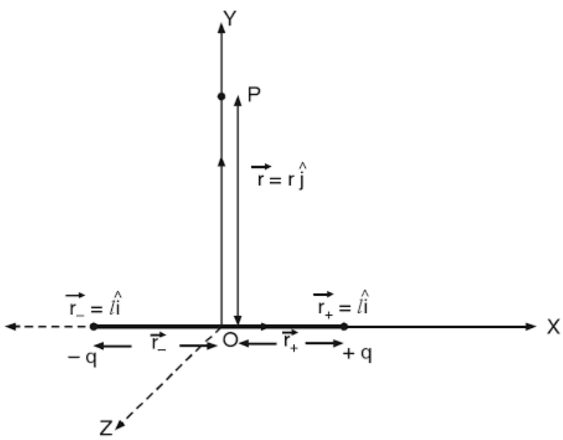

Fig. 16.14

Consider a point P on the Y-axis (Equatorial line) at a vector distance $\vec{r} = r\hat{j}$ from the origin. Then electric field at P due the charge $+q$

$$\vec{E}_+ = \frac{q}{4\pi\varepsilon_0}\frac{(\vec{r}-\vec{r}_+)}{|\vec{r}-\vec{r}_+|^3} = \frac{q}{4\pi\varepsilon_0}\frac{r\hat{j}-l\hat{i}}{|r\hat{j}-l\hat{i}|^3} = \frac{q}{4\pi\varepsilon_0}\frac{r\hat{j}-l\hat{i}}{(r^2+l^2)^{3/2}}$$

because $|r\hat{j}-l\hat{i}| = (r^2 + l^2)^{1/2}$

Similarly electric field at P due to the charge $-q$

$$\vec{E}_- = \frac{-q}{4\pi\varepsilon_0}\frac{r\hat{j}+l\hat{i}}{(r^2+l^2)^{3/2}}$$

The resulant electric field at P

$$\vec{E} = \vec{E}_+ + \vec{E}_-$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{r\hat{j}-l\hat{i}}{(r^2+l^2)^{3/2}} - \frac{r\hat{j}+l\hat{i}}{(r^2+l^2)^{3/2}}\right]$$

$$= \frac{2l\hat{i}}{(r^2+l^2)^{3/2}} = \frac{-\vec{p}}{4\pi\varepsilon_0(r^2+l^2)^{3/2}} \qquad \ldots (iii)$$

Eq. (iii) shows that the electric field $\vec{E}$ acts along the $-X$ direction.

When P lies at a far off point. For a very far off point l^2 can be neglected as compared to r^2 and we get

$$\vec{E} = -\frac{\vec{p}}{4\pi\varepsilon_0 r^3} \qquad \ldots (iv)$$

This relation show that the electric field $\vec{E}$ points in a direction opposite to dipole moment vector $\vec{p}$. As the direction of $\vec{p}$ is from $-q$ to $+q$ *i.e.*, along $+X$ axis, the electric field points along $-X$-axis. This relation also shows that the electric field due to a short dipole varies inversely as the cube of the distance of he point.

Comparing relation (ii) and (iv) we find that

(1) The magnitude of the electric field due to a small dipole at a far off point on the axial line is *twice* the value of the electric field at a far off point on the equatorial line.

(2) The electric field on the axial line is directed along $+X$ axis and on the equatorial line along $-X$ axis.

16.25 ELECTRIC FIELD DUE TO A SHORT DIPOLE

Consider an electric dipole lying along the X-axis with its mid point O at the origin of the co-ordinate system. Let the magnitude of each charge be q and $2\vec{l}$ the vector distance between the two charges then

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because when $r >> l$, l^2 can be neglected as compared to r^2 and $\frac{l^2}{r^2}$ tends to zero.

$$\therefore \qquad |\vec{r}-\vec{l}| = r\left[1-\frac{2\vec{r}\cdot\vec{l}}{r^2}\right]^{1/2}$$

and
$$\frac{1}{|\vec{r}-\vec{l}|^3} = r^{-3}\left(1-\frac{2\vec{r}.\vec{l}}{r^2}\right)^{-3/2}$$

$$= \frac{1}{r^3}\left(1-\frac{3\vec{r}\cdot\vec{l}}{r^2}+\ldots \text{ terms containing higher powers}\right)$$

As the terms containing higher powers can be neglected,

$$\frac{1}{|\vec{r}-\vec{l}|^3} = \frac{1}{r^3}+\frac{3\vec{r}\cdot\vec{l}}{r^5}$$

Similarly,

$$\frac{1}{|\vec{r}+\vec{l}|^3} = \frac{1}{r^3}-\frac{3\vec{r}\cdot\vec{l}}{r^5}$$

$$\therefore \qquad \frac{1}{|\vec{r}-\vec{l}|^3}-\frac{1}{|\vec{r}+\vec{l}|^3} = \frac{6\vec{r}.\vec{l}}{r^5}$$

and
$$\frac{1}{|\vec{r}-\vec{l}|^3}+\frac{1}{|\vec{r}+\vec{l}|^3} = \frac{2}{r^3}$$

Substituting these values in Eq. (i), we have

$$\vec{E} = \frac{q\vec{r}}{4\pi\varepsilon_0}\frac{6\vec{r}\cdot\vec{l}}{r^5}-\frac{q\vec{l}}{4\pi\varepsilon_0}\frac{2}{r^3}$$

$$= \frac{3\vec{r}(\vec{r}\cdot 2q\vec{l})}{4\pi\varepsilon_0 r^5}-\frac{2q\vec{l}}{4\pi\varepsilon_0 r^3}$$

or
$$\vec{E} = \frac{3\vec{r}(\vec{r}\cdot\vec{p})}{4\pi\varepsilon_0 r^5}-\frac{\vec{p}}{4\pi\varepsilon_0 r^3} \qquad \ldots (ii)$$

Relation (ii) gives the electric field due to a small dipole located at the origin at any point away from it.

(*i*) when *P* lies on the axial line. When the point P lies on the axial line

$$\vec{r} = r\hat{i} \text{ and } \vec{l} = l\hat{i}$$

$$\therefore \qquad \vec{r}\,.\,\vec{p} = \vec{r}\cdot 2q\,\vec{l} = r\hat{i}\cdot 2ql\hat{i} = r2ql = rp$$

$$\therefore \qquad \vec{E} = \frac{3r\hat{i}\,r\,p}{4\pi\varepsilon_0 r^5} - \frac{p\hat{i}}{4\pi\varepsilon_0 r^3} = \frac{2p}{4\pi\varepsilon_0 r^3}\hat{i} = \frac{2\vec{p}}{4\pi\varepsilon_0 r^3}$$

Compare the above result with relation (*ii*), article 16.24.

(*ii*) When *P* lies on the equatorial line. When the point *P* lies on the equatorial line

$$\vec{r} = r\hat{j} \text{ and } \vec{l} = l\hat{i}$$

$$\therefore \qquad \vec{r}\cdot\vec{p} = r\hat{j}\cdot 2ql\hat{i} = 2qrl(\hat{j}\cdot\hat{i}) = 0$$

$$\therefore \qquad \vec{E} = -\frac{p}{4\pi\varepsilon_0 r^3}$$

Compare the above result with relation (*iv*) of article 16.24.

16.26 ELECTRIC MULTIPOLES

When large number of dipoles are collectively arranged in a small space (or region), such a system of charges in known as electric multipoles. This increases the complexity. We use dipole (two-poles), quadrupole (four-poles), octupole (eight-poles) etc. or in general, 2^n poles, where n is called the order of multipole. For monopole $n = 0$, for dipole $n = 1$, for quadrupole $n = 2$ and so on.

16.27 ELECTRIC QUADRUPOLE

A quadrupole is an arrangement of two parallel dipoles of equal but opposite polarity separated by small distance.

Opposite polarity means their dipole moments are oppositely directed. Fig 16.16 shows some quadrupole arrangements.

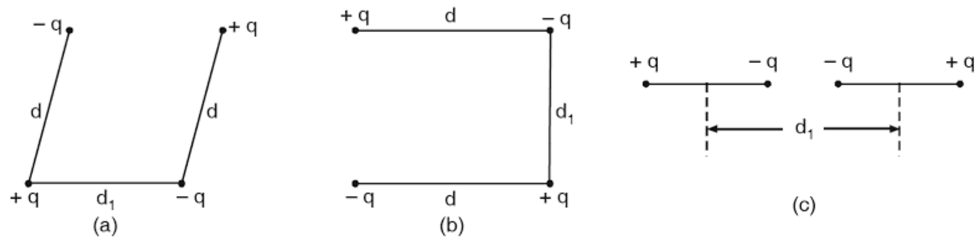

Fig. 16.16

When a multipole consists of point charges lying on one single axis, the multipole is called an axial multipole or linear dipole [Fig. 16.16 (*c*)].

SOLVED EXAMPLES

Example16.1 Two charges +5 coulomb and +15 coulomb are located at points (2, –4, 3) and (–3, 2, 1) metre. Calculate the force on +15 coulomb.

Solution. Here $\qquad q_1 = +5\text{C}\ q_2 = +15\text{C}$

$$\vec{r}_1 = 2\hat{i} - 4\hat{j} + 3\hat{k} \text{ and } \vec{r}_2 = -3\hat{i} + 2\hat{j} + \hat{k}$$

$$\therefore \quad \vec{r}_2 - \vec{r}_1 = -5\hat{i} + 6\hat{j} - 2\hat{k}$$

Force on the charge q_2 (+15C)

$$\vec{F}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{|\vec{r}_2 - \vec{r}_1|^3}(\vec{r}_2 - \vec{r}_1)$$

Now $$|\vec{r}_2 - \vec{r}_1| = \sqrt{(-5)^2 + (6)^2 + (-2)^2} = \sqrt{65}$$

$$\therefore \quad \vec{F}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{5\times 15}{65^{3/2}}(5\hat{i} + 6\hat{j} - 2\hat{k})$$

$$= \frac{9\times 10^9 \times 5 \times 15}{65^{3/2}}(5\hat{i} + 6\hat{j} - 2\hat{k})$$

$$= \mathbf{1.288 \times 10^9\ (-5\hat{i} + 6\hat{j} - 2\hat{k})\ Newton}$$

Example 16.2 How far should be the two protons if electric force between them is equal to the weight of a proton? (*Nagpur University w*/2004)

Solution. Rest mass of the proton $m = 1.67 \times 10^{-27}$ kg

Charge on the proton $e = 1.6 \times 10^{-19}$ C

Let r be the distance between the two protons when electric force between them is equal to the weight of a proton $= mg = 1.67 \times 10^{-27} \times 9.8$ Newton

Electrical force between two protons $= \dfrac{1}{4\pi\varepsilon_0}\dfrac{e\times e}{r^2} = \dfrac{9\times 10^9 \times 1.6\times 10^{-19}\times 1.6\times 10^{-19}}{r^2}$

$$\therefore \quad \frac{9\times 10^9 \times 1.6\times 10^{-19}\times 1.6\times 10^{-19}}{r^2} = 1.67 \times 10^{-27} \times 9.8$$

$$\therefore \quad r^2 = \frac{9\times 10^9 \times 1.6\times 10^{-19}\times 1.6\times 10^{-19}}{1.67\times 10^{-27}\times 9.8} = 1.408\times 10^{-2}$$

$$\therefore \quad r = 1.186 \times 10^{-1}\text{ m} = \mathbf{0.1186\ m}$$

Example 16.3 Using x, y and z co-ordinates and measuring lengths in meters charges of + 20 coulomb and +30 coulomb are placed at positions (8, 0, 0) and (0, 8, 0) respectively. Evaluate the force experienced by a charge of + 10 coulomb at the position (4, 4, 0). Use vectors to get your answer.

Solution. Here

$$q_1 = +20\text{ C} \qquad \vec{r}_1 = 8\hat{i}$$

$$q_2 = +30\text{ C} \qquad \vec{r}_2 = 8\hat{j}$$

$$q_0 = +10\text{ C} \qquad \vec{r}_0 = 4\hat{i} + 4\hat{j}$$

Example 16.7 A charge of $10\sqrt{2}$ Coulomb is located at $(3\hat{i}+4\hat{j}+5\hat{k})$ m. Calculate the electric field intensity at a point having position vector $(5\hat{i}+4\hat{j}+3\hat{k})$ m.

Solution. Given : $q = 10\sqrt{2}$ C; $\vec{R} = 3\hat{i}+4\hat{j}+5\hat{k}$, $\vec{r} = 5\hat{i}+4\hat{j}+3\hat{k}$

$$(\vec{r}-\vec{R}) = (5\hat{i}+4\hat{j}+3\hat{k})-(3\hat{i}+4\hat{j}+5\hat{k}) = 2\hat{i}-2\hat{k}$$

$$\therefore \quad |\vec{r}-\vec{R}| = \sqrt{2^2+(-2)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$$\therefore \quad \vec{E} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{|\vec{r}-\vec{R}|^3}\cdot(\vec{r}-\vec{R})$$

$$= 9\times 10^9 \times \frac{10\sqrt{2}}{(2\sqrt{2})^3}\times 2(\hat{i}-\hat{k}) = \frac{9\times10^9\times10(\hat{i}-\hat{k})}{8}$$

$$= 1.125\times10^{10}(\hat{i}-\hat{k}) = \mathbf{11.25\times10^9(\hat{i}-\hat{k})\ NC^{-1}}$$

Example 16.8 A negative point charge 10^{-8} coul is situated in air at the origin of a rectangular coordinate system. What is the electric field intensity at a point on the positive x-direction 3 meters from the origin?

Solution. Field Intensity

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r^2}\hat{r}$$

Along x-direction,

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r^2}\hat{i} = -\frac{1}{4\pi\varepsilon_0}\times\frac{10^{-8}\,\text{coul}}{3^2\,\text{m}^2}\hat{i}$$

$$= -\frac{9\times10^9\ \text{Nm}^2/\text{coul}^2\times10^{-8}\ \text{coul}}{3^2\,\text{m}^2}\hat{i}$$

$$= \mathbf{-10\hat{i}\ N/coul}$$

i.e., the electric field $\vec{E}$ is directed in the negative x-axis and its magnitude is 10 N per coul, (or volt per metre).

Example 16.9 A positive point charge of 10^{-9} coul is situated in air at the origin ($x = 0$, $y = 0$), and a negative point charge of -2×10^{-9} coul is situated on the y-axis 1 metre from the origin ($x = 0$, $y = 1$). Find the total electric field intensity at P on the x-axis 2 metres from the origin ($x = 2$, $y = 0$).

Solution. The electric field $\vec{E}_1$ at P due to the charge at (0, 0) is

$$E_1 = \frac{1}{4\pi\varepsilon_0}\times\frac{q_1}{r^2}\hat{r} \quad \text{or} \quad \vec{E}_1 = \frac{\hat{i}\,9\times10^9\ \text{Nm}^2/\text{coul}^2\times10^{-9}\ \text{coul}}{2^2\text{m}^2}$$

$$= \hat{i}\,(2.25)\ \text{N/coul} \qquad \ldots (i)$$

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$$\therefore \qquad |\vec{r}| = \sqrt{3^2+4^2} = 5 \text{ m}$$

Hence electric field at O

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}|^3}\vec{r}$$

$$= 9\times10^9\times\frac{100}{5^3}(-3\hat{i}-4\hat{j})$$

$$= 9\times10^9\left(-\frac{12}{5}\hat{i}-\frac{16}{5}\hat{j}\right)$$

$$= 9\times10^9-(2.4\hat{i}-3.2\hat{j})$$

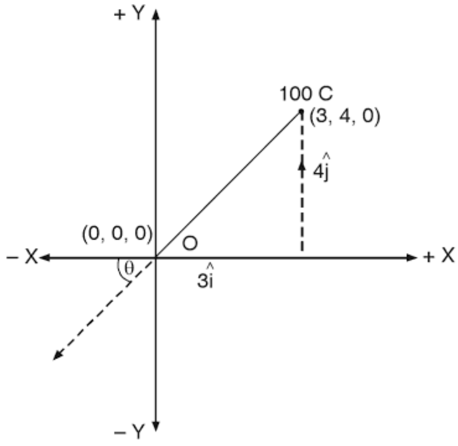

Fig. 16.20

$$\therefore \qquad |\vec{E}| = 9\times10^9\sqrt{(2.4)^2+(3.2)^2} = \mathbf{36\times10^9 NC^{-1}}$$

Example 16.17 Calculate the net electrical force on a unit positive charge placed at the centre of a square of side 'b' which has charges q, $2q$, $-4q$ and $2q$ placed in order at 4 corners.

Solution. Let the square $ABCD$ of each side 'b' have charges q, $+2q$, $-4q$ and $2q$ at the corners A, B, C and D Electrical intensity at 0 due to $2q$ and B and $2q$ and D cancel each other. Therefore electric intensity at O

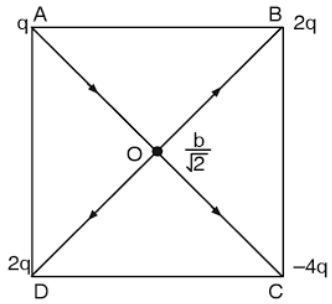

Fig. 16.21

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{b^2/2}+\frac{4q}{b^2/2}\right] = \left[\frac{\mathbf{1}}{\mathbf{4\pi\varepsilon_0}}\frac{\mathbf{10q}}{\mathbf{b^2}}\right]$$

EXERCISE CH. 16

LONG QUESTIONS

1. (*a*) State Coulomb's law in vector notation.

(*Nagpur U. s*/2009, *s*/2003, *w*/2004, *s*/2004; *M.S.U. Tiruneveli* 2007; *G.N.D.U.* 2004; *Meerut U.* 2002)

(*b*) Write the law for the force between two point charges when their position vectors are given.

(*c*) State the law in C.G.S. and S.I. units.

2. What is line charge density? Derive an expression for the electric field $\vec{E}$ due to an infinitely long uniformly charged straight wire using Coulomb's law.

(*D.A.U. Agra* 2006)

3. Two parallel infinite wires have uniform line charge densities λ_1 and λ_2 separated by a distance x. Calculate electric force per unit length on one wire as a result of the other.

4. Derive an expression for the electric field due to a circular line charge at a point on its axis.

5. Derive an expression for electric field at a point situated on the axis of a uniformly charged ring. (*Pbi.U.*, 2001; *H.P.U.*, 2000)

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6. (*a*) What is an electric dipole? What is dipole moment? Calculate the electric field due to a point (*i*) on the axial line and (*ii*) on the equatorial line.

   (*b*) Show that the value of electric field due to a small dipole at a far off point on the axial line is twice that on the equational line. (*Agra U.*, 2005; *G.N.D.U.*, 2004)

7. Calculate the electric field at a far off arbitrary point due to a small point dipole. Hence find the value of the field at a point (*i*) on the axial line and (*ii*) on the equatorial line. (*H.P.U.*, 2002,)

8. Show that the electric field due to electric dipole at a point on its axis is twice as strong as that at a point at the same distance along the perpendicular bisector. (*Nagpur Uni. w*/2004 *s*/2004)

9. Obtain an expression for electric field strength due to group of point charges. (*Nagpur Uni., s*/2003, *w*/2007)

SHORT QUESTIONS

1. Define a coulomb and a state-coulomb. Derive a relation between the two (*Pbi.U.*, 2002)
2. Define dielectric constant of a medium. Show that the dielectric constant is equal to relative permittivity.
3. Prove that Coulomb's law is in accordance with Newton's third law of motion. (*Nagpur Uni*., *s*/2008)

OR

4. Show the electric force between static charges is Newtonian.
5. What are limitations of Coulomb's law? (*Pbi. U.*, 2003; *H.P.U.*, 2002)
6. Is the Coulomb force that one charge exerts on another charge changed if other charges are brought nearby? Hence find the net force due to a number of discrete charges. (*G.N.D.U.*, 2000)
7. State Coulomb law. Is it universal law? (*Nagpur Uni. w*/2007)
8. What is dielectric? Give some examples. (*Nagpur Uni. w*/2007)
9. Write briefly what you know about quantisation of charge. Why it is not possible for a body to have charge of 1.5 *e*? (*H.P.U.*, 2000)
10. Define electric field intensity. State its value for a point charge and give its units. (*Nagpur Uni., w*/2007, *w*/2004, *s*/2004; *Pbi. U.*, 2002)
11. What are electric lines of force?
12. Why two electric lines of force do not cross each other. (*G.N.D.U.*, 2001)
13. What is an electric dipole? (*Nagpur Uni. s*/2006. *w*/2007, *s*/2008, *s*/2007)
14. State the principle of superposition of charges in Electrostatics. Explain with suitable example. (*M.S.U. Tiruneveli*, 2007)

NUMERICALS

1. A charge of $10\sqrt{2}$ Coulomb is located at $(3\hat{i}+4\hat{j}+5\hat{k})m$. Calculate the electric field intensity a point having position vector $(5\hat{i}+4\hat{j}+3\hat{k})m$. (*Pbi.U.*, 2000; *H.P.U.*, 2000)

17

ELECTRIC POTENTIAL

INTRODUCTION

There are two main properties of a vector field, namely *flux* and *rotation* which manifest into *flow* and *rotation* respectively. The electric vector $\vec{E}$, has a divergence nature, as explained in terms of flux flow. In addition to this important property, an electric field is a non-curl or curl free field i.e. irrotational property. In vector notation, it is expressed as $\vec{\nabla}\times\vec{E}=0$. According to the theorem of vector analysis, curl of gradient of a scalar is zero. In other words, $\vec{E}$ whose curl is zero, can be expressed as a gradient of a scalar function, the electric potential (V). Thus, the electric field is described not only in terms of (vector) the intensity of electric field $\vec{E}$ but also by a scalar quantity the electric potential (V). Thus, $\vec{E}$ and V are related to each other. It's a matter of convenience to take either vector or scalar form, depending upon the problem under consideration.

17.1 CONSERVATIVE FIELD

A vector field for which the line integral depends only on the end points but is independent of the actual path, is known as a conservative field.

In other words, the integral of a conservative field for a closed path is zero.

Thus if $\oint \vec{A}.\vec{dl}=0, \vec{A}$ represent a conservative field.

Conservative field as gradient of scalar field. A conservative field can always be expressed as the gradient of a scalar field. For example, if a vector field $\vec{A} = grad\,\phi = \vec{\nabla}\phi$ where ϕ is a scalar field, then the vector field $\vec{A}$ is a conservative field.

To prove, consider two points A and B and let $\vec{A}$ be a vector field, then the line integral $\int \vec{A}.\vec{dr}$ where $\vec{dr}$ is an infinitesimally small displacement gives a scalar function, say ϕ

$$\therefore \qquad \int_A^B \vec{A}.\vec{dr} = \phi_B - \phi_A$$

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and
$$\underset{\Delta x \to 0}{\mathrm{Lt}} \frac{1}{\Delta x} \int_A^B \vec{A}.\vec{dr} = \frac{\phi(x+\Delta x, y, z) - \phi(x, y, z)}{\Delta x} = \frac{\partial \phi}{\partial x} \quad ...(i)$$

Now $\int_A^B \vec{A}.\vec{dr}$ depends only on the initial and final values of ϕ at points A and B and is therefore independent of the path.

If we choose the *straight line* path from A to B, then $\vec{dr} = dx\hat{i}$ for this path since it is parallel to x-axis.

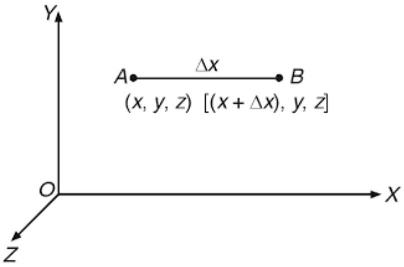

Fig. 17.1

$$\therefore \quad \underset{\Delta x \to 0}{\mathrm{Lt}} \frac{1}{\Delta x} \int_A^B \vec{A}.\vec{dr} = \underset{\Delta x \to 0}{\mathrm{Lt}} \frac{1}{\Delta x} \int_A^B \vec{A}.dx\hat{i}$$

$$= \underset{\Delta x \to 0}{\mathrm{Lt}} \frac{1}{\Delta x} \int_A^B A_x.dx$$

As the length of the path approaches zero in the limit A_x = constant

$$\therefore \quad \underset{\Delta x \to 0}{\mathrm{Lt}} \frac{1}{\Delta x} \int_A^B A_x.dx = \frac{A_x}{\Delta x} \int_A^B dx = \frac{A_x}{\Delta x} \Delta x = A_x \quad ...(ii)$$

From (i) and (ii) we have

$$A_x = \frac{\partial \phi}{\partial x}$$

similarly we can prove that

$$A_y = \frac{\partial \phi}{\partial y} \text{ and } A_z = \frac{\partial \phi}{\partial z}$$

Hence
$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

$$= \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} = \vec{\nabla}\phi$$

Thus for a *conservative field*

$$\vec{A} = \vec{\nabla}\phi = grad\, \phi$$

i.e., a conservative field is gradient of a scalar field.

Curl of conservative field is zero. A conservative field is always given by the gradient of scalar field (or function). If $\vec{A}$ is a conservative field and ϕ a scalar function. then

$$\vec{A} = grad\, \phi = \vec{\nabla}\phi$$

and
$$Curl\, \vec{A} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{\nabla}\phi = 0$$

because curl of gradient of a scalar function is zero.

Non-conservative field. A vector field for which the line integral depends upon the path and for which the integral for a closed path is not equal to zero is called a non-conservative field. For a non-conservative field

$$Curl\ \vec{A} \neq 0 \ \text{ or } \ \vec{\nabla} \times \vec{A} \neq 0$$

Examples. Electric field due to a stationary charge is conservative. Magnetic field is non-conservative.

17.2 ELECTRIC FIELD IS CONSERVATIVE

An electric field is the example of a conservative field because, an electric field is the negative gradient of scalar function (ϕ).

or
$$\vec{E} = -grad\,\phi = -\vec{\nabla}\phi$$

The curl of an electric field is given by

$$Curl\ \vec{E} = -curl\ grad\,\phi = -\vec{\nabla} \times \vec{\nabla}\phi = 0$$

We summarise below various conditions for a vector field to be *conservative*.

(*i*) *A vector field for which the line integral depends only on the end points and is independent of the actual path taken is conservative field.*

(*ii*) *The line integral of a conservative field over a closed path is zero.* If $\vec{A}$ is a conservative field

$$\oint \vec{A}.\vec{dl} = 0$$

(*iii*) *A conservative field can always be represented as the gradient of a scalar field.* If $\vec{A}$ is a conservative field and ϕ the corresponding scalar field, then

$$\vec{A} = \vec{\nabla}\phi = grad\,\phi$$

(*iv*) The *curl* of a conservative field is zero. If $\vec{A}$ is a conservative field

$$Curl\,\vec{A} = \vec{\nabla} \times \vec{A} = 0$$

17.3 LINE INTEGRAL OF A VECTOR FIELD

Consider a path *ACB* between two points *A* and *B* in a vector field *F*. Divide the path *ACB* into a very large number of small elements dl_1, dl_2, dl_3, ... dl_i ...etc. such that the corresponding values of the vector field are $\vec{F_1}, \vec{F_2}, \vec{F_3}, \ldots \vec{F_i}$ etc. respectively.

Each element $\vec{dl}$ is also a vector element as it has a magnitude *dl* and the direction at the point say *P* is parallel (tangent) to the path while going from *A* to *B*.

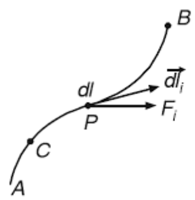

Fig. 17.2

The summation of the scalar products.

$$\vec{F_1}.\vec{dl_1} + \vec{F_2}.\vec{dl_2} + \vec{F_3}.\vec{dl_3} + ..\vec{F_i} \cdot \vec{dl_i} + \ldots = \sum \vec{F_i}.\vec{dl_i}$$

for all elements in the limit $\vec{dl}_i \to 0$ is known as the *line integral* of the vector field $\vec{F}$.

$\therefore$ Line integral of a vector field = $\int_{A \text{ along } ACB}^{B} \vec{F}.\vec{dl}$

Line integral of an electrostatic field. As an example, let us now suppose that the vector field is an electric field $\vec{E}$ then a *small* test charge q_0 placed at P experience a force

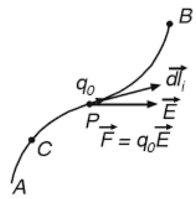

Fig. 17.3

$$\vec{F} = q_0 \vec{E}$$

$\therefore$ Work done by the field when the test charge moves through a small vector distance $\vec{dl}$

$$dW = \vec{F} \cdot dl = q_0 \vec{E} \cdot \vec{d} l$$

Total work done when the test charge moves from A to B

$$W = \int_A^B dW = \int_A^B \vec{F}.\vec{dl} = \int_A^B q_0 \vec{E}.\vec{dl} = q_0 \int_A^B \vec{E}.\vec{dl}$$

$\therefore$ Work done by the field per unit charge

$$= \frac{W}{q_0} = \int_A^B \vec{E}.\vec{dl}$$

In other words *line integral of an electrostatic field from a point A to B gives the work done by the field when a unit positive charge moves from A to B under the effect of the field.*

17.4 LINE INTEGRAL OF A VECTOR FIELD OVER A CLOSED PATH

A conservative vector field can be expressed as the gradient of a scalar field.

For example, if a vector field $\vec{A} = grad\,\phi = \vec{\nabla}\phi$ where ϕ is a scalar field, then the vector field $\vec{A}$ is a conservative field.

The line integral of the vector field $\vec{A}$ between two points A and B is given by $\int_A^B \vec{A}.\vec{dr}$ where $\vec{dr}$ is a small vector element of distance.

$$\therefore \qquad \int_A^B \vec{A}.\vec{dr} = \int_A^B \vec{\nabla}\phi.\vec{dr} = \int_A^B d\phi$$

because
$$\vec{\nabla}\phi.\vec{dr} = \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right).(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi$$

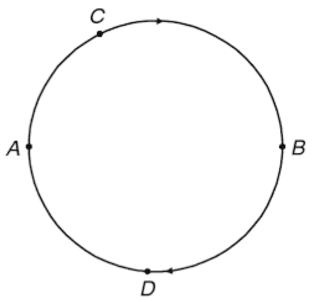

Fig. 17.4

$$\therefore \qquad \int_A^B \vec{A}.\vec{dr} = \int_A^B d\phi = \phi_B - \phi_A \qquad \ldots (i)$$

where ϕ_B is the value of scalar function ϕ at B and ϕ_A its value at A.

According to relation (i) *the line integral of the conservative vector field $\vec{A}$ depends only on the values of scalar field at the initial and final position A and B and is independent of the path join joining A and B.*

Now consider a closed path $ACBDA$. The line integral along the closed path $ACBDA$ is the sum of the line integrals from A to B along the path ACB and from B to A along the path BDA. (fig 17.4).

$$\therefore \qquad \oint_{\text{ACBDA}} \vec{A}.\vec{dr} = \int_{\substack{A \\ along\ ACB}}^{B} \vec{A}.\vec{dr} + \int_{\substack{B \\ along\ BDA}}^{A} \vec{A}.\vec{dr}$$

To evaluate $\int_{\substack{A \\ along\ ACB}}^{B} \vec{A}.\vec{dr}$ all vector segments are taken pointing in the direction A to B via C and to evaluate $\int_{\substack{B \\ along\ BDA}}^{A} \vec{A}.\vec{dr}$ all the vector segments are taken pointing in the opposite direction from B to A via D.

$$\therefore \qquad \int_{\substack{A \\ along\ ACB}}^{B} \vec{A}.\vec{dr} = -\int_{\substack{B \\ along\ BDA}}^{A} \vec{A}.\vec{dr}$$

$$\therefore \qquad \oint_{\text{ACBDA}} \vec{A}.\vec{dr} = \int_{\substack{A \\ along\ ACB}}^{B} \vec{A}.\vec{dr} - \int_{\substack{A \\ along\ ACB}}^{B} \vec{A}.\vec{dr} = 0$$

Hence the line integral of a conservative field (OR a vector field which is gradient of a scalar field) over a closed path is zero.

17.5 LINE INTEGRAL OF ELECTRIC FIELD IS PATH INDEPENDENT

The electric field is a *conservative field*. In other words, when a charge moves from one point to another in an electric field, the work done is independent of the path taken by it.

To prove this, consider the electric field due to a point Charge Q, lying at O, the origin of the co-ordinate system and let A and B be two points in the field having vector distances $\vec{r_1}$ and $\vec{r_2}$ respectively from the origin. Join A to B by any path say ACB. Let P be a point on the path at a vector distance $\vec{r}$ and having co-ordinate x, y, and z then

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

and $$|r^2| = x^2 + y^2 + z^2$$

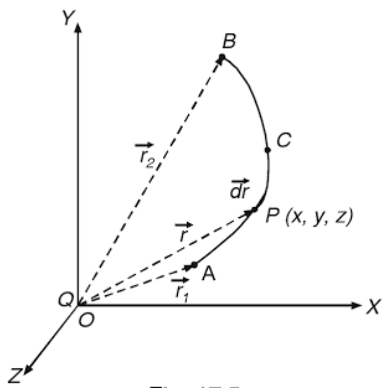

Fig. 17.5

17.6 WORK DONE IN ELECTRIC FIELD IS INDEPENDENT OF PATH

Suppose a small test charge q_0 moves from A to B in an electric field along the path ACB in a *quasi-static manner* and W_1 is the work done *by* the field. Let W_2 be the work done *by* the field when the same charge moves along the path ADB between the same two points. This also means that when the same charge is moved back from B to A by the path BCA an amount of work W_1 will have to be done *against* the electric field. Similarly if the test charge is move back along the path BDA an amount of work W_2 will have to be done *against* the field. If W_1 and W_2 are not equal, let W_1 be greater than W_2. Let the charge q_0 go along the path ACB and return by the path BDA. In this way a net *positive* amount of work $(W_1 - W_2)$ is done by the electric field during the process which leaves the test charge q_0 in its original position at A. According to the principle of conservation of energy this is not possible unless an amount of energy $(W_1 - W_2)$ is made to enter the system from outside. This do not happen in the case of an electric field because magnitude and positions of all the charges remains unchanged during the process. Thus no work can be obtained by moving a charge away from the point A and back to it again.

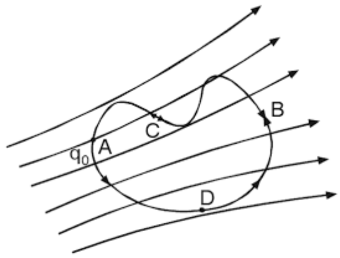

Fig. 17.6

Hence $$W_1 - W_2 = 0 \quad \text{or} \quad W_1 = W_2$$

Thus equal and same amount of work is done on a charge along all possible paths joining A and B. In other words, *the work done when a charge moves from one point to another in an electric field, is independent of the path.*

17.7 LINE INTEGRAL OF AN ELECTRIC FIELD OVER A CLOSED PATH IS ZERO

The work done in taking a unit positive charge from A to B in an electric field $\vec{E}$ is given by

$\int_A^B \vec{E}.d\vec{r}$ where $d\vec{r}$ is a small element of the path.

$\therefore$ Work done in moving a unit positive charge from A to B via the path ACB *by* the electric field

$$= W_1 = \int_A^B \vec{E}.d\vec{r}$$

and work done in moving a unit positive charge from B to A via the path BDA *against* the field

$$= W_2 = \int_B^A \vec{E}.d\vec{r} = -\int_A^B \vec{E}.d\vec{r}$$

$\therefore$ Total work done in taking a unit positive charge from A to B and back again to A over the closed path $ACBDA$ is given by

$$\oint \vec{E}.d\vec{l} = \int_A^B \vec{E}.d\vec{r} - \int_A^B \vec{E}.d\vec{r} = 0 \qquad \ldots (i)$$

Hence the line integral of an electric field over a closed path is zero.

In other words, since the electrostatic force is a conservative force, the net work done during a closed path is zero.

Using Stoke's theorem Eq. (i) becomes

(*i*) When charge lies at the origin. The electric potential at a point in an electric field is defined as the work done in taking a unit positive charge from infinity to that point against the electrical forces.

It is given by $$V = -\int_{\infty}^{r} \vec{E}.d\vec{r}$$

where $\vec{r}$ is the position vector of the point and $\vec{E}$ the electric field.

Consider a charge q lying at the origin O and a point B at a position vector $\vec{r}$, then

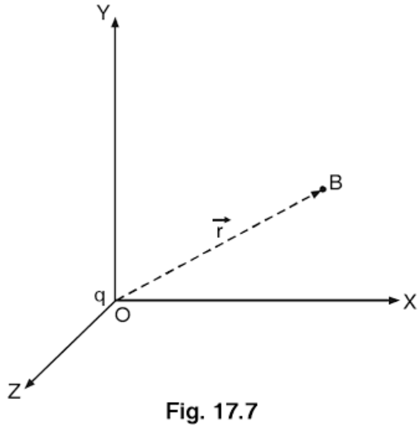

Fig. 17.7

Electric potential at B, $$V_B = -\int_{\infty}^{r} \vec{E}.d\vec{r} \quad \ldots (i)$$

The electric field at B due to the charge q at the origin

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}|^3}\vec{r}$$

Substituting in Eq (*i*), we have

$$V_B = -\frac{1}{4\pi\varepsilon_0}\int_{\infty}^{r}\frac{\vec{r}.\vec{dr}}{|\vec{r}|^3} = \frac{1}{4\pi\varepsilon_0}\int_{\infty}^{r}\frac{rdr}{r^3} = -\frac{1}{4\pi\varepsilon_0}\int_{\infty}^{r}\frac{dr}{r^2}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\right]_{\infty}^{r} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

Hence is general for a point lying at a distance r from a charge q, electric potential $V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$.

(*ii*) When a charge is located at an arbitrary point. Suppose the charge q, lies at A at the position vector $\vec{r'}$.

Let the position vector of the observation point B be $\vec{r}$, then

$$\vec{r} = \vec{r'} - \vec{r_0}$$

where $\vec{r_0}$ is the vector distance of B from the point charge q.

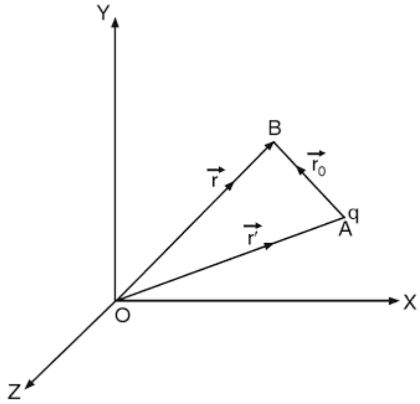

Fig. 17.8

$$\therefore \quad \vec{r_0} = \vec{r} + \vec{r'}$$

and $$|\vec{r}_0| = |\vec{r} - \vec{r}\,'|$$

Hence electric potential at B due to the point charge q

$$= V_B = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_0} = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r} - \vec{r}\,'|}$$

Note that $\vec{r}$ is the position vector of the observation point and $\vec{r}\,'$ the position vector of the source charge q.

17.11 ELECTRIC POTENTIAL DUE TO CONTINUOUS CHARGE DISTRIBUTION (CCD)

Potential is a scalar quantity. Therefore, for a number of point charges $q_1, q_2, q_3, \ldots q_n$ located at position vector $\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots \vec{r}_n$ etc. respectively, the net potential at the observation point at the position $\vec{r}$ is given by

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{|\vec{r} - \vec{r}_1|} + \frac{1}{4\pi\varepsilon_0}\frac{q_2}{|\vec{r} - \vec{r}_2|} + \ldots + \frac{1}{4\pi\varepsilon_0}\frac{q_n}{|\vec{r} - \vec{r}_n|}$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_{i=i}^{n}\frac{q_i}{|\vec{r} - \vec{r}_i|}$$

Now $\frac{1}{4\pi\varepsilon_0}\frac{q_1}{|\vec{r} - \vec{r}_1|} = V_1$ potential at the observation due to charge q_1

Similarly $\frac{1}{4\pi\varepsilon_0}\frac{q_2}{|\vec{r} - \vec{r}_2|} = V_2$ and so on

$$\therefore \quad V = V_1 + V_2 + V_3 + \ldots + V_n = \sum_{i=i}^{n} V_i$$

Continuous Charge Distribution. In the case of continuous charge distribution q is replaced by elementary charge dq, its vector distance from the origin by $\vec{r}_i$ the summation sign being replaced by the integral.

$$\therefore \quad V = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{|\vec{r} - \vec{r}_i|}$$

For a line charge distribution

$$V = \frac{1}{4\pi\varepsilon_0}\int\frac{\lambda\, dl}{|\vec{r} - \vec{r}_i|}$$

where λ is charge per unit length.

For surface charge distribution

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma\, ds}{|\vec{r} - \vec{r_i}|}$$

where σ is charge per unit surface area.

For volume charge distribution

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho\, dv}{|\vec{r} - \vec{r_i}|}$$

where ρ is charge per unit volume.

17.12 ELECTRIC POTENTIAL DIFFERENCE BETWEEN TWO POINTS

The difference of electric potential between two point A and B in an electric field is given by

$$V_B - V_A = -\int_A^B \vec{E}.d\vec{r} \qquad \ldots (i)$$

where $\int_A^B \vec{E}.d\vec{r}$ is the line integral of the electric field between the two points.

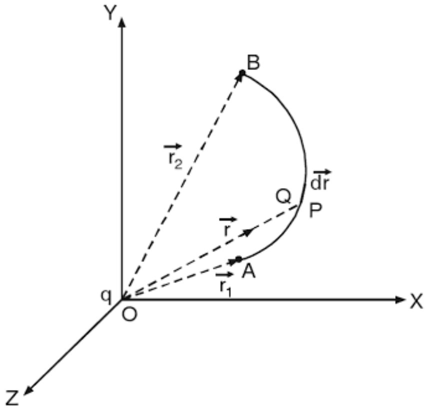

Fig. 17.10

If a charge q lies at the origin of co-ordinate system and the radial distances of the points A and B are $\vec{r_1}$ and $\vec{r_2}$ respectively, as shown in Fig. 17.10, then, the electric field at a point P at a radial distance $\vec{r}$ due to a point charge q lying at the origin is given by

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}|^3}\vec{r}$$

Substituting this value $\vec{E}$ in relation (i), we get

$$V_B - V_A = -\int_A^B \vec{E}.d\vec{r} = -\frac{1}{4\pi\varepsilon_0}\int_{r_1}^{r_2}\frac{q}{|\vec{r}|^3}\vec{r}.d\vec{r}$$

Now $$\vec{r}.d\vec{r} = \frac{1}{2}d(r^2) = \frac{1}{2}2r\,dr = rdr$$

$\therefore$ $$V_B - V_A = -\frac{1}{4\pi\varepsilon_0}\int_{r_1}^{r_2}\frac{q}{r^2}dr$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r}\right]_{r_1}^{r_2} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_2} - \frac{1}{r_1}\right]$$

17.13 POTENTIAL GRADIENT

The potential rise between two points along an electric field is called the gradient of the potential gradient.

Thus, gradient of $$V = \lim_{\Delta L \to 0} \frac{\Delta V}{\Delta L} = \frac{dV}{dl}$$

where ΔV is charge in potential for ΔL charge in length.

If the element of the length dl is at an angle θ with $\vec{E}$, then the work done i.e., potential change

$$dV = \vec{E}.d\vec{l}$$

$$= -\left|\vec{E}\right|\left|d\vec{l}\right|\cos\theta$$

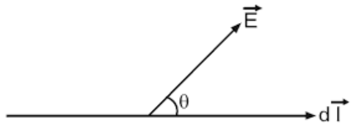

Fig. 17.10

$\therefore$ $$\text{grad } V = -\frac{dV}{dl} = -\left|\vec{E}\right|\cos\theta$$

Since, grad $V = \vec{\nabla}V$ in vector notation, $\vec{\nabla}V = -\left|\vec{E}\right|\cos\theta$

17.14 EQUIPOTENTIAL SURFACE (OR CONTOURS)

The locus of all the points which have the same potential is called an equipotential surface.

For a point charge, the equipotential surfaces are a family of concentric spheres. Similarly, for a spherical charge the equipotential surfaces are also concentric spheres due to symmetry considerations. For a uniform electric field, the equipotential surfaces are planes perpendicular to the field.

Two equipotential surfaces cannot intersect. Two equipotential surfaces cannot intersect. If they intersect the point of intersection wil have two different values of potential which is not possible.

Line integral of electric field on equipotential surface is zero. The potential difference between two points A and B in an electric field is given by

$$V_B - V_A = -\int_A^B \vec{E}.\vec{dr}$$

For an equipotential surface $V_B = V_A$ or $V_B - V_A = 0$

$\therefore$ $$\int_A^B \vec{E}.\vec{dr} = 0$$

Hence the line integral of electric field between any two points or an equipotential surface is zero.

Electric field $\vec{E}$ is always perpendicular to surface of charged conductor. The surface of a charged conductor is an equipotential surface. The equipotential surfaces are always at right angles to the lines of electric force and to the direction of electric field $\vec{E}$. If electric field $\vec{E}$ were not at right angles to the equipotential surface it would have a component lying in that surface. Then work would have to be done in moving a test charge from one point to the other on the equipotential surface. But no work is done to move a test charge between any two points on an equipotential surface because there is no difference of potential.

Therefore, $\vec{E}$ is always at right angles to an equipotential surface and no work is done in moving a charge from one point to other on an equipotential surface.

Proof : Consider an equipotential surface S as shown in Fig 17.11. Let A and B be two *very close* points on the surface S.

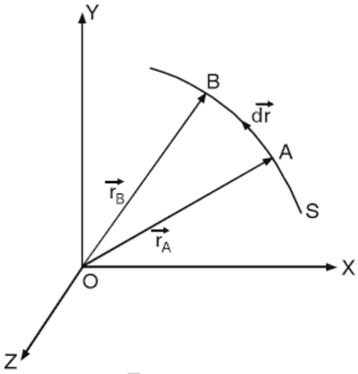

Fig. 17.11

Position vector of point $A = \vec{r}_A$

Position vector of point $B = \vec{r}_B$

$$\therefore \quad \vec{AB} = \vec{OB} - \vec{OA}$$

or
$$d\vec{r} = \vec{r}_B - \vec{r}_A \qquad \text{... }(i)$$

The potential difference between the points A and B,

$$V_B - V_A = -\int_A^B \vec{E}.\vec{dr} \qquad \text{... }(ii)$$

where $\vec{E}$ is the electric field at any point between A and B. Since the point A and B are very close, the electric field $\vec{E}$ has approximately the same value everywhere along the displacement from A to B. Taking $\vec{E}$ to be a constant Eq. (ii) becomes

$$V_B - V_A = -\vec{E}.\int_A^B d\vec{r} = -\vec{E}.[\vec{r}]_A^B = -\vec{E}.\left(\vec{r}_B - \vec{r}_A\right)$$

or
$$V_B - V_A = \vec{E}.\vec{AB} \qquad [\because \vec{r}_B - \vec{r}_A = \vec{AB} \text{ Eq. } (i)]$$

But the points A and B lie on the equipotential surface S i.e.,$V_A - V_B$ or $V_B - V_A = 0$

$$\therefore \quad -\vec{E}.\vec{AB} = 0 \qquad \text{...}(iii)$$

Eq. (iii) shows that electric field $\vec{E}$ is perpendicular to $\vec{AB}$ as the scalar product of the vectors $\vec{E}$ and $\vec{AB}$ is zero.

Thus we find that electric field is always perpendicular (or normal) to the equipotential surface.

17.15 WORK DONE ON EQUIPOTENTIAL SURFACE IS ZERO

The difference of electric potential between two points A and B in an electric field is defined as the work done in taking a unit positive charge from A to B against the electrical forces. The potential at every point on an equipotential surface is the same. Therefore if the two points A and B lie on an equipotential surface the work done in taking a unit positive charge from A to B on the equipotential surface is zero.

Hence the work done in moving a charge (+ ve or – ve) on the equipotential surface is zero.

17.15 IMPORTANCE FEATURES OF ELECTRIC FIELD STRENGTH $(\vec{E})$ AND ELECTRIC POTENTIAL (V)

The potential at a point can be zero if the field strength there is non-zero. This is because potential is a scalar quantity whereas field strength is a vector.

The converse is also true *i.e.* the field strength at a point can be zero, even if the potential there is non-zero Examples:

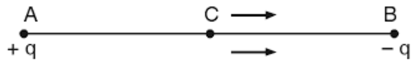

Fig. 17.12 (*a*)

(*i*) Consider two charges $+q$ and $-q$ lying at the points A and B respectively, then for a point C midway between them [Fig. 17.12 (*a*)].

The electric potential is given by

$$+\frac{1}{4\pi\varepsilon_0}\frac{q}{AC}-\frac{1}{4\pi\varepsilon_0}\frac{q}{BC} = 0 \qquad [\because AC = BC]$$

But the electric field at C to the charge $+q$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{AC^2} \text{ along } AC \text{ produced.}$$

and electric field at C due to the charge $-q$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{BC^2} \text{ along } AC \text{ produced.}$$

$$\therefore \quad \text{Total electric field} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{AC^2}+\frac{q}{BC^2}\right] \neq 0$$

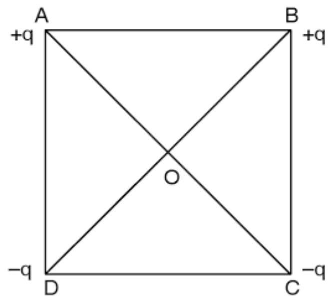

Fig. 17.12 (*b*)

even when $AC = BC$. The field acts along AC produced. Similarly if charge $+q$ each are placed at the corners A and B of a square and charges $-q$ each at the corners C and D, then potential at O, the point of intersection of two diagonals is zero but the field is non-zero [Fig. 17.12 (*b*)].

(*ii*) If, on the other hand in the first case both the charges are positive, the electric field at C the mid-point of AB is zero since the field at C due to the charge of $+q$ at A and a charge $+q$ at B are equal and opposite and hence net fields is zero. But the net electric potential is not zero because potential due to the two charges $+q$ at A and $+q$ and B are both positive and add up.

Similarly in the second case if all the four charges are positive, the net electric field at O is zero but the net electric potential is non-zero.

(*iii*) Moreover, it is possible for a body to have a charge and still be at zero potential. As an example consider a charged parallel plate capacitor having two plates A and B as shown in Fig. 17.13. The plate B is earthed. When the plate A is given a positive charge it induces an equivalent (bound) negative charge on the plate B and the (free) positive charge flows to the earth. Thus the plate B has a charge but no potential hence zero potential.

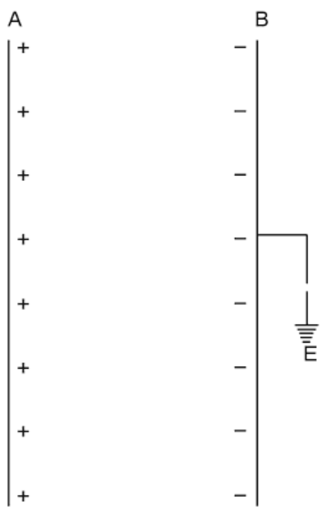

Fig. 17.13

17.16 ELECTRIC FIELD AS A GRADIENT OF POTENTIAL *i.e.* $\vec{E} = -\vec{\nabla} V = -\text{grad}\, V$

The expression $\vec{E} = -\vec{\nabla} V$ means that the electric field $\vec{E}$ at a point is the negative gradient of potential (V) at that point.

The electric potential at a point P in an electric field $\vec{E}$ is defined as the work done in bringing a unit positive charge from infinity to that point *against* the electrical forces. In other words, the electric potential is the negative line integral of the electric field.

$$\therefore \qquad V = -\int_{\infty}^{P} \vec{E}.d\vec{r}$$

or
$$dV = -\vec{E}.d\vec{r} \qquad \ldots (i)$$

If the co-ordinates of the point P are $x, y, z,$ then $V(x, y, z)$ is a function of these co-ordinates. The partial derivatives

$$\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y} \text{ and } \frac{\partial V}{\partial z}$$

represent the rate of change of V with $x, y,$ and z respectively.

$$\therefore \qquad dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

$$= \left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right).(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

But $dx\hat{i} + dy\hat{j} + dz\hat{k} = d\vec{r}$ where $dx,\ dy$ and dz are components of the displacement vector $d\vec{r}$ along $x, y,$ and z directions respectively. Also

$$\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)V = \vec{\nabla}V$$

$$= \text{grad}\, V$$

$$\therefore \qquad dV = \vec{\nabla}V.d\vec{r} \qquad \ldots (ii)$$

From (i) and (ii) we have

$$-\vec{E}.d\vec{r} = \vec{\nabla}V.d\vec{r}$$

or
$$\vec{E} = -\vec{\nabla}V = -\text{grad}\, V \qquad \ldots (iii)$$

Relation (iii) may be written in the form

$$\vec{E}(x, y, z) = \frac{\partial V(xyz)}{\partial x}\hat{i} + \frac{\partial V(xyz)}{\partial y}\hat{j} + \frac{\partial V(xyz)}{\partial z}\hat{k}$$

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Hence the electric field intensity has a magnitude which is equal to the maximum rate of change of electric potential and its direction is the same as the direction along which rate of change of potential is maximum.

Significance of negative sign. The negative sign implies that the electric intensity $\vec{E}$ points in the direction of decreasing V.

Note: If in a certain region of space the electric field $\vec{E}$ is zero. The electric potential V need not be necessarily zero, as explained below:

The electric field at a point is the negative of the gradient of electric potential at the point *i.e.*,

$\vec{E} = -\vec{\nabla}V$. If in a certain region of space the electric field $\vec{E} = 0$ it means $-\vec{\nabla}V = 0$ or V = a constant.

Thus the electric potential has a constant value not necessarily zero.

For example, in the case of a uniformly charged spherical shell the electric field is zero inside but of the electric potential has a constant value everywhere in the shell.

17.17 POTENTIAL DUE TO A LINE CHARGE

Consider a straight line charge placed along the X-axis and extending from $X = a$ to $x = b$.

Let P be a point on the Y-axis at a distance $\vec{r}$ from the origin O, then position vector of the point P,

$$\vec{r} = r\hat{j}$$

Take a small element of length dx at a distance x from O, then position vector of the charge element,

$$\vec{r}_1 = x\hat{i}$$

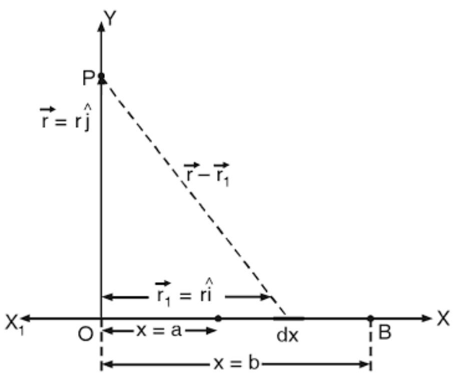

Fig. 17.14

If λ is the line charge density, then the charge on the element $dx = dq = \lambda dx$.

The potential at P due to this charge element dq is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{|\vec{r}-\vec{r}_1|}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx}{|r\hat{j}-r\hat{i}|} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx}{\sqrt{r^2+x^2}} \quad \ldots (i)$$

because $$|r\hat{j}-r\hat{i}| = \sqrt{r^2+x^2}$$

$\therefore$ Potential at P due to the whole line charge

$$V = \int_a^b dV = \frac{\lambda}{4\pi\varepsilon_0}\int_a^b \frac{dx}{\sqrt{r^2+x^2}}$$

Now $$\int \frac{dx}{\sqrt{r^2+x^2}} = \log_e (x+\sqrt{r^2+x^2})$$

Hence
$$V = \frac{\lambda}{4\pi\varepsilon_0}\left[\log_e (x+\sqrt{r^2+x^2})\right]_a^b$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left[\log_e (b+\sqrt{r^2+b^2}) - \log_e (a+\sqrt{r^2+a^2})\right]$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\log_e \frac{b+\sqrt{r^2+b^2}}{a+\sqrt{r^2+a^2}} \qquad \ldots (ii)$$

If the line charge extends on both sides of the origin say from $x = -a$ to $x = +b$, then the potential at P = Potential of the portion from $-a$ to the origin + potential of the portion from the origin to $+b$ as potential is a scalar quantity.

$$\therefore \qquad V = \frac{\lambda}{4\pi\varepsilon_0}\left[\int_{-a}^{0}\frac{dx}{\sqrt{r^2+x^2}} + \int_{0}^{b}\frac{dx}{\sqrt{r^2+x^2}}\right]$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left[\int_{-a}^{0}\frac{dx}{\sqrt{r^2+x^2}} + \int_{0}^{b}\frac{dx}{\sqrt{r^2+x^2}}\right]$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left\{\left[\log_e (x+\sqrt{r^2+x^2})\right]_0^a + \left[\log_e (x+\sqrt{r^2+x^2})\right]_0^b\right\}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left\{\left[\log_e (a+\sqrt{r^2+a^2}) - \log_e r\right] + \left[\log_e (b+\sqrt{r^2+b^2}) - \log_e r\right]\right\}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left[\log_e \frac{a+\sqrt{r^2+a^2})}{r} + \log_e \frac{b+\sqrt{r^2+b^2})}{r}\right]$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left\{\log_e\left[\frac{a}{r}+\frac{\sqrt{r^2+a^2}}{r}\right] + \log_e\left[\frac{b}{r}+\frac{\sqrt{r^2+b^2}}{r}\right]\right\}$$

If $a = b$ *i.e.*, the line charge extends from $-b$ to $+b$, we have

$$V = \frac{2\lambda}{4\pi\varepsilon_0}\log_e\left[\frac{b}{r}+\frac{\sqrt{r^2+a^2}}{r}\right] = \frac{\lambda}{2\pi\varepsilon_0}\log_e\left[\frac{b}{r}+\frac{\sqrt{r^2+b^2}}{r}\right] \qquad \ldots (iii)$$

Potential due to an infinitely long line charge. If $b = \infty$, $V = \frac{\lambda}{2\pi\varepsilon_0}\log_e \infty = \infty$. This shows that electric potential due to infinite charged wire is infinite. Hence, if the line charge extends from $-\infty$ to $+\infty$ the result becomes, indeterminate. However, if b is large as compared to r so that r^2 can be neglected as compared to b^2, then

$$V = \frac{\lambda}{2\pi\varepsilon_0}\log_e \frac{2b}{r} \qquad \ldots (iv)$$

17.18 ELECTRICAL POTENTIAL (V) DUE TO ELECTRIC DIPOLE

Consider a dipole consisting of charges + q and – q separated by a *small* distance $2\vec{l}$, lying along the X-axis with its centre at the region, then

Dipole moment of the electric dipole $\vec{p} = 2q\vec{l}$

Consider a point P having a position vector $\vec{r}$, then

Potential at P due to the charge + q

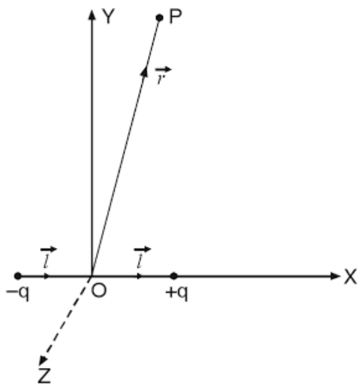

Fig. 17.16

$$V_+ = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}-\vec{l}|}$$

and potential at P due to the charge – q

$$V_- = -\frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}+\vec{l}|}$$

∴ Net potential at P due to the electric dipole

$$V = V_+ + V_-$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{r}-\vec{l}|}-\frac{1}{|\vec{r}+\vec{l}|}\right]$$

To find value of $|\vec{r}-\vec{l}|$, consider the term

$$|\vec{r}-\vec{l}|^2 = (\vec{r}-\vec{l}).(\vec{r}-\vec{l})$$

$$= r^2+l^2-2\vec{r}.\vec{l}$$

$$= r^2\left[1+\frac{l^2}{r^2}-2\frac{\vec{r}.\vec{l}}{r^2}\right]$$

$$\therefore \qquad |\vec{r}-\vec{l}| = r\left[1+\frac{l^2}{r^2}-\frac{2\vec{r}.\vec{l}}{r^2}\right]^{1/2}$$

or

$$\frac{1}{|\vec{r}-\vec{l}|} = \frac{1}{r}\left[1+\frac{l^2}{r^2}-\frac{2\vec{r}.\vec{l}}{r^2}\right]^{-1/2}$$

When P is at a *far off* point from O or the dipole is a *point dipole or short dipole i.e., r > 1,*

$\frac{l^2}{r^2}$ can be neglected as compared to 1. Hence

$$\frac{1}{|\vec{r}-\vec{l}|} = \frac{1}{r}\left[1-\frac{2\,\vec{r}\,.\,\vec{l}}{r^2}\right]^{-1/2}$$

Expanding the terms on the right hand side and neglecting terms containing $\frac{l^2}{r^2}$ and higher powers we get

$$\frac{1}{|\vec{r}-\vec{l}|} = \frac{1}{r}\left[1+\frac{\vec{r}\,.\,\vec{l}}{r^2}\right] = \frac{1}{r}+\frac{\vec{r}\,.\,\vec{l}}{r^3}$$

Similarly

$$\frac{1}{|\vec{r}+\vec{l}|} = \frac{1}{r}-\frac{\vec{r}\cdot\vec{l}}{r^3}$$

Substituting these values of $\frac{1}{|\vec{r}-\vec{l}|}$ and $\frac{1}{|\vec{r}+\vec{l}|}$ in relation (i) we get

$$V = \frac{2q}{4\pi\varepsilon_0}\frac{\vec{r}\,.\,\vec{l}}{r^3} = \frac{\vec{r}\,.2q\,\vec{l}}{4\pi\varepsilon_0 r^3} = \frac{\vec{r}\,.\,\vec{p}}{4\pi\varepsilon_0 r^3} = \frac{\vec{p}\,.\,\vec{r}}{4\pi\varepsilon_0 r^3} = \frac{1}{4\pi\varepsilon_0}\vec{p}\,.\left[\frac{\vec{r}}{r^3}\right] = -\frac{1}{4\pi\varepsilon_0}\vec{p}\,.\vec{\nabla}\left(\frac{1}{r}\right)$$

as $$\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3} \quad \text{[See Example 13.27]}$$

Also $$\frac{\vec{p}\,.\,\vec{r}}{4\pi\varepsilon_0 r^3} = \frac{\vec{p}\,.\,\hat{r}}{4\pi\varepsilon_0 r^2} \qquad \text{... }(ii)$$

where $\hat{r}$ is a unit vector in the direction of $\vec{r}$. If θ is the angle between $\vec{p}$ and $\vec{r}$, then

$$V = \frac{pr\cos\theta}{4\pi\varepsilon_0 r^3} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} \qquad \text{... }(iii)$$

Thus we find that the *potential at a point due to a point dipole varies inversely as the square of the distance.*

Case (*i*) Potential on the axial line. When the point P lies on the axis of the dipole (or axial line), the vector $\vec{p}$ and $\vec{r}$ are in the same direction (or $\theta = 0$ and $\cos\theta = 1$) *i.e.,* along the X-axis

$$\therefore \qquad \vec{p}\,.\,\vec{r} = pr$$

Substituting in (*iii*), we have

$$V = \frac{pr}{4\pi\varepsilon_0 r^3} = \frac{p}{4\pi\varepsilon_0 r^2}$$

Note. When the point P lies on the axial line of the dipole in a direction opposite to that of $\vec{p}$; $\theta = \pi$ and $\cos\theta = -1$.

$$\therefore \qquad V = -\frac{p}{4\pi\varepsilon_0 r^2}$$

Case (*ii*) Potential on the normal to the axis (Equatorial line). When the point P lies on the normal to the axis, $\vec{P}$ and $\vec{r}$ are at right angles to each other ($\vec{P}$ along X-axis and $\vec{r}$ along Y-axis) or $\theta = \pi/2$ and $\cos\theta = 0$

$$\therefore \qquad \vec{p}\cdot\vec{r} = 0$$

Substituting in (*iii*), we have

$$V = 0$$

Thus the electric potential at a point on the equatorial line of a dipole is zero.

17.19 ELECTRIC FIELD $(\vec{E})$ DUE TO ELECTRIC DIPOLE

The electric field at any point having polar coordinates (r, θ) is obtained by $\vec{E} = -\vec{\nabla}V$ and in polar coordinates

$$\vec{E}_r + \vec{E}_\theta = \hat{r}\,|\vec{E}_r| + \hat{\theta}\,|\vec{E}_\theta|$$

$$= -\hat{r}\frac{\delta v}{\delta r} - \hat{\theta}\frac{1}{r}\frac{\delta v}{\delta\theta} \qquad \ldots (i)$$

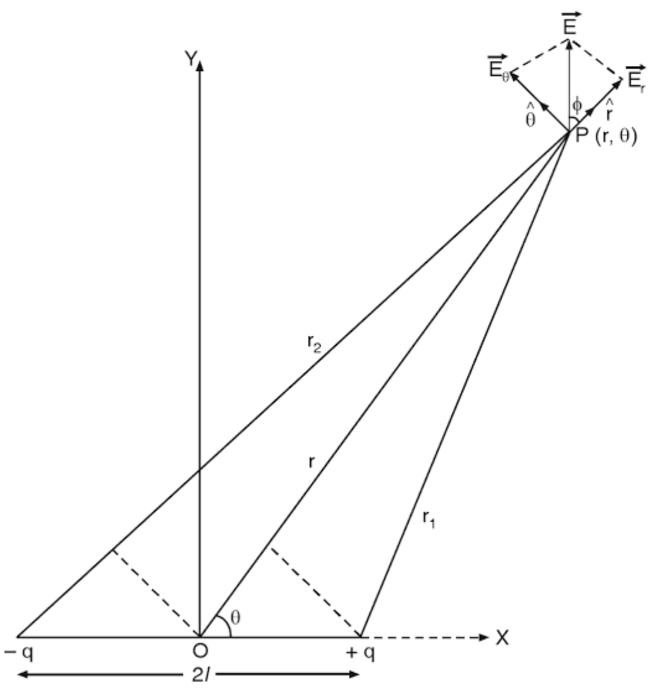

Fig. 17.17

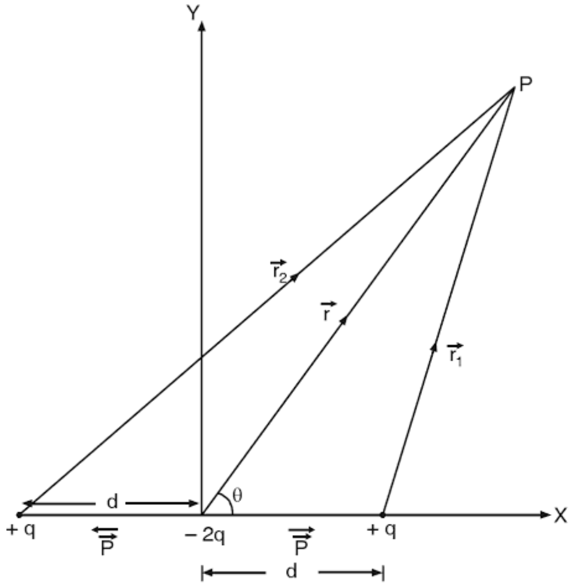

Fig. 17.19

Here $$\frac{1}{r_1} = \frac{1}{(r^2+d^2-2\,\mathrm{rd}\cos\theta)^{1/2}} = \frac{1}{r\left[1+\frac{d^2}{r^2}-\frac{2d}{r}\cos\theta\right]^{1/2}}$$

$$\frac{1}{r_1} = \frac{1}{r}\left[1+\left(\frac{d^2}{r^2}-\frac{2d}{r}\cos\theta\right)\right]^{-1/2}$$

since r >> d, we expand using Binomial theorem as

$$\left[(1-x)^{-1/2}\right] = \left[1-\frac{x}{2}+\frac{3x^2}{8}\right]$$

Hence,

$$\frac{1}{r_1} = \frac{1}{r}\left[1-\frac{1}{2}\left(\frac{d^2}{r^2}-\frac{2d}{r}\cos\theta\right)+\frac{3}{8}\left(0+\frac{4d^2}{r^2}-\cos^2\theta\right)+...+\text{ higher}\right.$$

order terms. Since r >> d, higher order terms are neglected.

$$\frac{1}{r_1} = \frac{1}{r}\left[1-\frac{d^2}{2r^2}+\frac{d}{r}\cos\theta+\frac{3}{2}\frac{d^2}{r^2}\cos^2\theta\right]$$

$$\therefore \qquad \frac{1}{r_1} = \frac{1}{r}\left[1+\frac{d^2}{2r^2}(3\cos^2\theta-1)+\frac{d}{r}\cos\theta\right]$$

Similarly,

$$\frac{1}{r_2} = \frac{1}{r}\left[1+\frac{d^2}{2r^2}(3\cos^2\theta-1)-\frac{d}{r}\cos\theta\right]$$

After substitution in Eq. (i) for $V_{(r)}$, we get,

$$V_{(r)} = \frac{q}{4\pi\in_0}.\frac{1}{r}\left[1+\frac{d^2}{2r^2}(3\cos^2\theta-1)\frac{d}{r}\cos\theta+1+\frac{d^2}{2r^2}(3\cos^2\theta-1)-\frac{d}{r}\cos\theta-2\right]$$

$$= \frac{q}{4\pi\in_0}\frac{1}{r}\left[\frac{d^2}{r^2}(3\cos^2\theta-1)\right]$$

$$V_{(r)} = \frac{qd^2}{4\pi\in_0 r^3}(3\cos^2\theta-1) \qquad \ldots (ii)$$

The product $2qd^2$ is called the electric quadrupole moment denoted by Q_d, then

$$V_{(r)} = \frac{Q_d(3\cos^2\theta-1)}{8\pi\in_0 r^3} \qquad \ldots (iii)$$

Thus, the potential due to a linear quadrupole varies inversely as the cube of the distance.

17.21 ELECTRIC FIELD $\vec{E}$ DUE TO QUADRUPOLE

The electric potential due to quadrupole at any point situated far off is given by

$$V_{(r)} = \frac{qd^2}{4\pi\in_0 r^3}(3\cos^2\theta-1) \qquad (\because \text{ Eq } (ii) \text{ of article 17.20})$$

By making use of relation, $\vec{E} = -\vec{\nabla}V$, we can calculate the electric field.

The X-component in cartesian co-ordinate system is

$$\left|\vec{E}_x\right| = -\frac{\partial V}{\partial x} = -\frac{qd^2}{4\pi\in_0}.\frac{\partial}{\partial x}\left(\frac{3\cos^2\theta-1}{r^3}\right)$$

$$= -\frac{qd^2}{4\pi\in_0}.\frac{\partial}{\partial x}\left(\frac{3x^2}{r^5}-\frac{1}{r^3}\right) \qquad (\text{Since } x = r\cos\theta)$$

$$= -\frac{qd^2}{4\pi\in_0}\left[\frac{6x}{r^5}+\frac{\partial}{\partial x}(r^{-5}).3x^2-\frac{\partial}{\partial x}(r^{-3})\right]$$

$$= -\frac{qd^2}{4\pi\in_0}\left[\frac{6x}{r^5}+\frac{\partial}{\partial x}(r^{-5})\frac{\partial}{\partial x}3x^2-\frac{\partial}{\partial_r}(r^{-3})\frac{\partial_r}{\partial x}\right]$$

But $\qquad r^2 = x^2+y^2+z^2$

$$\therefore \qquad 2r\frac{\partial_r}{\partial x} = 2x \text{ OR } \frac{\partial_r}{\partial x} = \frac{x}{r}$$

$$\therefore \qquad \left|\vec{E}_x\right| = -\frac{qd^2}{4\pi \in_0}\left[\frac{6x}{r^5} + 3x^2(-5r^{-6})\frac{x}{r} + 3r^{-4}\frac{x}{r}\right]$$

$$= -\frac{qd^2}{4\pi \in_0}\left[\frac{6x}{r^5} - \frac{15x^3}{r^7} + \frac{3x}{r^5}\right]$$

$$= \frac{3qd^2}{4\pi \in_0 r^2}\left(\frac{5x^3}{r^2} - 3x\right)$$

$$= \frac{3qd^2}{4\pi \in_0 r^5}\left(\frac{5r^3\cos^3\theta}{r^2} - 3r\cos\theta\right) \qquad \text{(since } x = r\cos\theta\text{)}$$

$$= \frac{3qd^2}{4\pi \in_0 r^5}(5r\cos^3\theta - 3r\cos\theta)$$

$$= \frac{3qd^2}{4\pi \in_0 r^4}(5\cos^3\theta - 3\cos\theta)$$

Similarly Y-Component of the field is obtained as

$$\left|\vec{E}_Y\right| = -\frac{\partial V}{\partial y} = -\frac{qd^2}{4\pi \in_0}\frac{\partial}{\partial y}\left(\frac{3\cos^2\theta - 1}{r^3}\right)$$

proceeding in the same way, we get,

$$\left|\vec{E}_Y\right| = \frac{3qd^2}{4\pi \in_0 r^5}(5\cos^2\theta - 1)\,y$$

and Z-component of the field as,

$$\left|\vec{E}_Z\right| = \frac{3qd^2}{4\pi \in_0 r^5}(5\cos^2\theta - 1)\,z$$

Therefore, the magnitude of the intensity,

$$\left|\vec{E}\right| = \sqrt{\left|\vec{E}_x\right|^2 + \left|\vec{E}_y\right|^2 + \left|\vec{E}_z\right|^2}$$

$$= \frac{3qd^2}{4\pi \in_0}\left[\frac{1}{r^8}(5\cos^3\theta - 3\cos\theta)^2 + \frac{1}{r^{10}}(5\cos^2\theta - 1)^2 y^2 + \frac{1}{r^{10}}(5\cos^2\theta - 1)^2 z^2\right]^{1/2}$$

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$$= \frac{3qd^2}{4\pi \epsilon_0 r^4}\left[(5\cos^3\theta - 3\cos\theta)^2 + \frac{1}{r^2}(5\cos^2\theta - 1)^2(y^2 + z^2)\right]^{1/2}$$

$$= \frac{3qd^2}{4\pi \epsilon_0 r^4}\left[(5\cos^3\theta - 3\cos\theta)^2 + (5\cos^2\theta - 1)(1 - \cos^2\theta\right]^{1/2}$$

Since $y^2 + z^2 = r^2 - x^2 = r^2 - r^2 \cos^2\theta$ $(\because x = r\cos\theta)$

$\therefore$ $y^2 + z^2 = r^2 (1 - \cos^2\theta)$

Solving squares and product, we get

$$|\vec{E}| = \frac{3qd^2}{4\pi \epsilon_0 r^4}\left[5\cos^4\theta - 2\cos^2\theta + 1\right]^{1/2} \quad \ldots (i)$$

Thus, the electric field of a linear quadrupole varies inversely as the fourth power of the distance.

The following important cases arise:

Case 1 : Field on the axis of quadrupole.

i.e. if $\theta = 0$, the field is maximum

$$|\vec{E}|_{\max} = \frac{6qd^2}{4\pi \epsilon_0 r^4} = \frac{3Q_d}{4\pi \epsilon_0 r^4}$$

(as $2qd^2 = Q_d$ = the quardupole moment) ... (*ii*)

Case 2 : Field along the equatorial line.

i.e. if $\theta = 90°$, the field is minimum

$$|\vec{E}|_{\min} = \frac{3qd^2}{4\pi \epsilon_0 r^4} = \frac{3Q_d}{8\pi \epsilon_0 r^4} \quad \ldots (iii)$$

Thus, the field on the axis of the quadrupole is double to that of the field at the same distance on equatorial line.

Case 3 : Field on the – ve X-axis:

i.e. $\theta = 180°$, then, the field is again maximum, but in a direction of negative X-axis.

17.22 FORCE AND TORQUE ON A DIPOLE IN UNIFORM ELECTRIC FIELD

Note. *In this article we shall denote the length of the dipole by 2dl instead of 2l to indicate that the length of the dipole is infinitessimally small.*

Consider an electric dipole consisting of charge +q and –q separated by a distance $2d\vec{l}$ apart, then dipole moment $\vec{p} = 2q\, d\vec{l}$

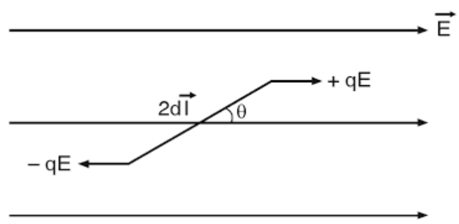

Fig. 17.20

Force. When such a dipole is placed in a uniform electric field $\vec{E}$ with its axis making angle θ with the direction of $\vec{E}$ as shown, then

Force on the charge $+q = +qE$

i.e., in the direction of the field

Force on the charge $-q = -qE$

i.e., this force is equal and opposite to the force on the charge $+q$.

$\therefore$ Net force = 0

Hence the net translatory force on a dipole in a uniform electric field is zero.

Torque. It has been shown that the net translatory force on a dipole in a uniform electric field is zero. The two forces do not act along the same straight line and constitute a couple, the moment of which is given by

$$qE\,2dl \sin\theta = pE \sin\theta$$

Since $\vec{E}$ and $\vec{p}$ are vector quantities.

$$pE \sin\theta = \vec{p} \times \vec{E}$$

$\therefore$ Torque on the dipole

$$\tau = \vec{p} \times \vec{E}$$

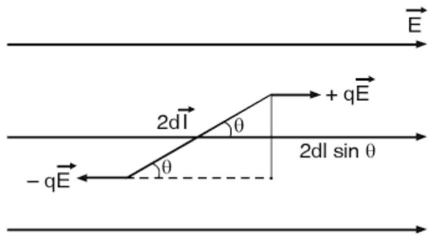

Fig. 17.21

The torque = 0 when $\vec{p}$ is parallel to $\vec{E}$.

17.23 ENERGY OF A DIPOLE

The potential energy of a dipole in an electric field is defined as the amount of work done against the electric field in bringing the dipole from infinity and placing it in the desired orientation in the field.

Let the dipole make an angle θ with the field $\vec{E}$, then the magnitude of the couple (torque) acting on it = $pE \sin\theta$. Work done in turning the dipole from the position making an angle θ with the field to the position making an angle $\theta + d\theta$ is given by

$$dW = pE \sin\theta\, d\theta$$

$\therefore$ Work done in rotating the dipole through an angle θ in uniform electric field

$$W = \int dW = \int_0^\theta pE \sin\theta\, d\theta = [-pE\cos\theta]_0^\theta = [+pE\cos\theta]_\theta^0$$

$$= pE(\cos 0 - \cos\theta) = pE(1 - \cos\theta) \qquad \ldots (ii)$$

In general, the work done in rotating the dipole from an initial position θ_1 to the final position θ_2 is

$$W = \int dW = pE\int_{\theta_1}^{\theta_2} \sin\theta\, d\theta$$

$$= -pE(\cos\theta_2 - \cos\theta_1) \qquad \ldots (iii)$$

Clearly, this work done is stored up in the form of potential energy of the dipole in the final position.

When the dipole is placed perpendicular to the direction of the electric field the work done = 0. The equipotential lines are always at right angles to the lines of electric field, therefore, the work done in bringing the charge $+q$ and the charge $-q$ to the same potential being equal and opposite

$$\therefore \qquad \phi_E = \iint_s \vec{E}.\vec{dS}$$

where $\iint_s$ represents the surface integral over the area $\vec{S}$.

From relation (*ii*), we find that' *'The electric flux through a vector area* $\vec{dS}$ *is given by the dot product of electric field vector* $\vec{E}$ *and area vector* $\vec{dS}$.

Dimension and units of electric flux.The electric intensity $\vec{E} = \frac{\vec{F}}{q_0}$. The units of $\vec{E}$ therefore, are Newton/coulomb [NC^{-1}].

Now electric flux $\phi_E = \vec{E}.\vec{S}$. The units of electric flux therefore, are Newton metre^2 per coulomb, [$Nm^2\ C^{-1}$].

Electric flux density. *Electric flux is defined as electric flux per unit area.* If ϕ_E is the electric flux through an areas A in a uniform electric field, then,

Electric flux density $E = \frac{\phi_E}{A}$

Unit. The S.I. unit of electric flux density is [NC^{-1}] *Newton per Coulomb.*

For a vector field through a plane surface *positive flux* implies lines of electric force of the field *emerging out* of the surface, *negative flux* means lines of force of the field are *directed* into the surface. The flux is zero if as many lines of force are directed into the surface as emerge out of it.

17.25 ELECTRIC FLUX OVER A SURFACE DUE TO A POINT CHARGE

Consider a point charge q and a surface S in free space. Let r be the distance from the charge to a point P on the surface, $\hat{n}$ the outwardly directed unit normal to the surface at the point P contained by a small element of surface area vectorially represented by $\vec{dS}$. The electric field $\vec{E}$ at the point P on the surface due to the charge q is directed along the line from the charge q to the point P. Let the direction of $\vec{E}$ make an angle θ with the unit $\hat{n}$ then the electric flux over the area $\vec{dS}$ is given by

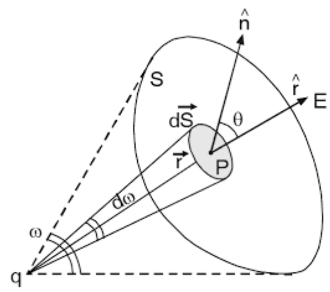

Fig. 17.23

$$d\phi_E = \vec{E}.\vec{dS} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}\cdot\hat{n}\,dS$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dS\cos\theta \qquad (\because\ \hat{r}\cdot\hat{n} = \cos\theta\,)$$

Now $\frac{dS\cos\theta}{r^2} = d\omega$ the small solid angle subtended by the surface dS at the position of the point charge q

$$\therefore \qquad d\phi_E = \frac{1}{4\pi\varepsilon_0} q \, d\omega$$

Hence electric flux over the whole of the surface S due to the point charge q

$$\phi_E = \frac{1}{4\pi\varepsilon_0} \iint_s q \, d\omega = \frac{1}{4\pi\varepsilon_0} q \, \omega$$

if q has a constant value and ω is the solid angle subtended by the whole surface S at the point charge q.

17.26 GAUSSIAN SURFACE

A hypothetical closed surface of any shape drawn in an electric field for the purpose of solving problems concerning electric flux is called *Gaussian surface.* The shape of the Gaussian surface is decided on the basis of the symmetry of the problem so that the value of electric flux $\phi_E = \oiint \vec{E} \cdot \vec{dS}$ (for free space) can be calculated.

17.27 GAUSS'S THEOREM

Gauss's theorem states that *'The total electric flux in free space (or vacuum) through a closed surface is equal to* $\frac{1}{\varepsilon_0}$ *times the charge enclosed by the surface'.*

or

$$\phi_E = \oiint_s \vec{E} . \vec{dS} = \frac{q}{\varepsilon_0}$$

Consider a charge q situated at O, the origin of the rectangular co-ordinate system. Let S be a Gaussian surface around this. Now consider a small elementary area $\vec{dS}$ at a vector distance $\vec{r}$ from the charge q.

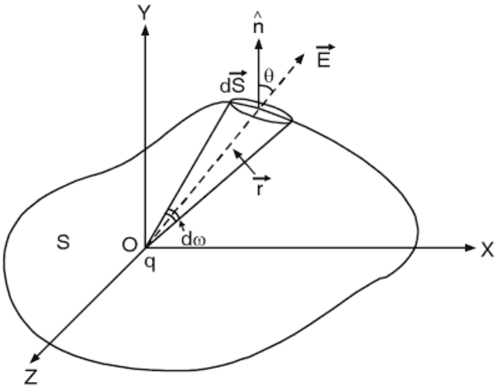

Fig. 17.24

The electric flux through the area $\vec{dS}$ is given by

$$d\phi_E = \vec{E} . \vec{dS}$$

where $\vec{E}$ is the electric field vector at $\vec{r}$.

The total electric flux over the (closed) Gaussian surface due to the charge q inside it, in free space is given by

$$\phi_E = \int d\phi_E = \oiint_s \vec{E} . \vec{dS}$$

Now, the electric intensity $\vec{E}$ at point on the elementary surface $\vec{dS}$ at the position vector $\vec{r}$ due to the charge q at the origin is given by

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$$

where $\hat{r}$ is a unit vector in the direction of electric intensity $\vec{E}$.

Also $\vec{dS} = ds \, \hat{n}$ where $\hat{n}$ is a unit vector in the direction of positive (outward draw) normal to the surface $\vec{dS}$.

$$\therefore \qquad \phi_E = \frac{1}{4\pi\varepsilon_0}\oiint \frac{q}{r^2}\hat{r}.\hat{n}\,dS$$

$$= \frac{1}{4\pi\varepsilon_0}\oiint q\frac{dS\cos\theta}{r^2} \qquad [\because\ \hat{r}.\hat{n} = \cos\theta\,]$$

Now $\dfrac{dS\cos\theta}{r^2} = d\omega$ the small solid angle subtended by the elementary are $\vec{dS}$ at q.

$$\therefore \qquad \phi_E = \frac{1}{4\pi\varepsilon_0}\oiint q\,d\omega = \frac{q}{4\pi\varepsilon_0}\oiint d\omega = \frac{q}{\varepsilon_0}$$

as $\oiint d\omega = 4\pi$ *i.e.*, the solid angle subtended by a closed surface at a point inside it is 4π.

Hence $\phi_E = \dfrac{q}{\varepsilon_0}$ where $\phi_E = \oiint \vec{E}.\vec{dS}$

The statement $\oiint \vec{E}.\vec{dS} = \dfrac{q}{\varepsilon_0}$ is known as the **integral form of Gauss's law.**

If there are several charges inside, the positive charges give the values $\dfrac{q_1}{\varepsilon_0} + \dfrac{q_2}{\varepsilon_0} + \ldots$ and the negative charges give the values $\dfrac{-q_1'}{\varepsilon_0} - \dfrac{q_2'}{\varepsilon_0} - \ldots$ and the total electric flux due to all the charges $= \dfrac{1}{\varepsilon_0}(q_1 + q_2 + \ldots - q_1' - q_2' \ldots)$. Thus the total charge inside refers to the algebraic sum of the charges.

Hence for a number of point charges within the closed surface, Gauss's theorem can be sated as

$$\phi_E = \oiint \vec{E}.\vec{dS} = \frac{1}{\varepsilon_0}\sum q_i$$

When the charge lies outside the surface. If the charge lies outside a closed surface the electric flux inwards is equal to the electric flux outwards and the resultant flux for the whole surface is zero. For a re-entrant surface (a surface with convulsions) as shown in Fig. 17.25 the theorem is equally applicable. When the charge q lies inside as at A a small cone cuts the surface an **odd** number of times so that electric flux for the cone at A

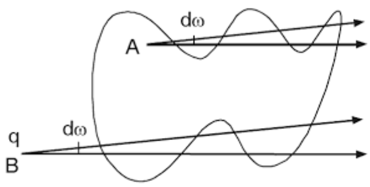

Fig. 17.25

$$= +q\,d\omega - q\,d\omega + q\,d\omega - q\,d\omega + q\,d\omega = q\,d\omega$$

If the charge lies outside as at B, the cone cuts the surface an even number of times and the contribution of any cone

$$= q\,d\omega - q\,d\omega + q\,d\omega - q\,d\omega = 0.$$

Thus, the total flux over a surface due to a charge lying outside is zero.

Gaussian surface. The symmetry is the most crucial condition, which is satisfied in the following three cases:

(*i*) spherical symmetry: construct a Gaussian surface as concentric sphere.

(*ii*) Cylindrical symmetry: construct a Gaussian surface as coaxial cylinder.

(*iii*) Plane symmetry: construct "Pill box" which straddles the surface.

17.29.1 Electric Field Due to a Charged Straight Wire

Consider an infinitely long straight wire having a constant line charge density λ. Let P be a point at a distance a from the straight line charge. The direction of the electric field due to the electric charge will be *radially outward* due to *symmetry*. Draw a cylinder of height h and radius a coaxial with the line charge and closed at both ends by plain caps A and B normal to the axis to represent the Gaussian surface. The closed surface consists of three parts A, B and C. Now if we consider a very small area element $\vec{dS}$ on the surface A, then area vector $\vec{dS}$ acts along the *outward drawn normal* and is at right angles to the electric field vector $\vec{E}$.

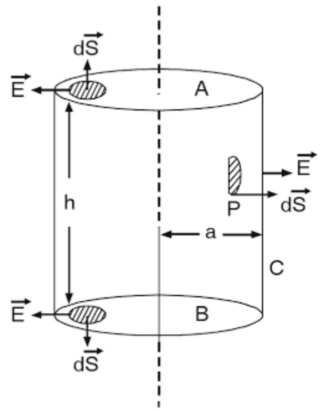

Fig. 17.26

$\therefore$ For the surface A; $\iint\limits_A \vec{E}.\vec{dS} = 0$

Similarly for the surface B; $\iint\limits_B \vec{E}.\vec{dS} = 0$

For the surface C the area vector $\vec{dS}$ and the electric field vector $\vec{E}$ are parallel and act in the same direction

$$\therefore \qquad \iint\limits_C \vec{E}.\vec{dS} = \iint\limits_C E\,dS$$

where E and dS are the magnitudes of vectors $\vec{E}$ and $\vec{dS}$ respectively.

As all points on the curved surface of the cylinder C are equidistant from the axis, the electric field is the same at every point. Hence $E = a$ constat

$$\therefore \qquad \iint\limits_C E\,dS = E\iint\limits_C dS = 2\pi\,ahE \qquad \text{... }(i)$$

where 2π ah is the area of the curved surface of the cylinder of height h and radius a.

According to Gauss's theorem, the electric flux over whole of the closed surface

$$\phi_E = \oiint\limits_C \vec{E}.\vec{ds} = \iint\limits_A \vec{E}.\vec{dS} + \iint\limits_B \vec{E}.\vec{dS} + \iint\limits_C \vec{E}.\vec{dS}$$

and given by
$$\phi_E = \frac{q}{\varepsilon_0}$$

Now the total charge within the Gaussian surface = charge on the wire of height $h = \lambda h$

$$\therefore \qquad E = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r} \qquad \text{... }(iii)$$

If the point P is infinitely close to the charged surface, then $r = a$ and hence

$$E = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{a} \qquad \text{... }(iv)$$

If σ is the surface charge density then charge per unit length

$$\lambda = 2\pi\, a\, \sigma$$

Hence
$$E = \frac{1}{2\pi\varepsilon_0}\frac{2\pi a\sigma}{a} = \frac{\sigma}{\varepsilon_0}$$

For a point inside the charged cylinder *i.e.* for $r < a$, no charge is enclosed within the Gaussian surface.

$$\therefore \qquad E2\pi\, rh = 0$$

or
$$E = 0$$

The results are equally applicable to a hollow conducting cylinder. The surface of the conducting cylinder is an *equipotential surface* and the electric field intensity is everywhere normal to the surface and its direction is, therefore, along the outward drawn normal at every point.

17.29.3 Two Co-Axial Cylinders

Suppose we have two co-axial cylinders of radii a and b where $a > b$ having charge per unit length $+\lambda$ on the inner cylinder and $-\lambda$ on the outer cylinder. In actual practice we come across such a case when we have two co-axial conducting, hollow cylinders, the inner charged and the outer earthed.

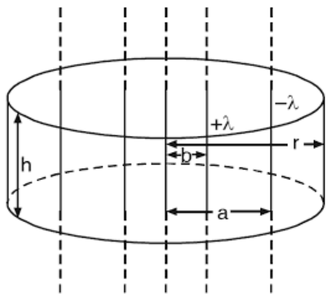

Fig. 17.28

(i) when the point P lies outsider the cylinder at a distance r such that $r > a$, the charge enclosed within the Gaussian cylinder of length $h = (+\lambda h - \lambda h) = 0$ due to the inner as well as the outer charged cylinders.

$\therefore$ For $\qquad r > a, E = 0$

(ii) When the point P lies in between the two cylinder $a > r > b$ then the charge $+\lambda h$ is only enclosed within the Gaussian cylinder and the charge $-\lambda h$ is outside it.

$$\therefore \qquad E = +\frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r} \qquad \text{[See Eq. }(iii)\text{ of article 17.29.2]}$$

(iii) When the point P lies inside the two cylinder $r < b$, no charge is enclosed within the Gaussian surface

$$\therefore \qquad E = 0$$

17.29.4 Electric Field Near a Charged Infinite Plane

Let AB be a part of a uniformly charged infinite plane of a non-conducting material having a *uniform* surface charge density σ. From symmetry considerations the electric field is in a direction perpendicular to the sheet and hence everywhere normal to it. Consider a right cylinder $a\ b\ c\ d$

$$\therefore \qquad 2A\,|\vec{E}| = \frac{\sigma A}{\varepsilon_0} \text{ where } \sigma \text{ is surface charge density.}$$

$$\therefore \qquad \vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n} \text{ where } \hat{n} \text{ is unit vector to the surface.}$$

Note: The field due to an infinite plane is independent of the distance of how far away the point where the field is required.

17.29.6 Electric Field at Any Point Due to Two Parallel Sheets of Charge

Suppose there are two parallel plane sheets A and B having uniform surface charge densities $+\sigma_1$ and $+\sigma_2$ respectively.

Let the sheet A be to the left of sheet B.

There are three regions :

Region 1– To the *left* of sheet A.

Region 2 – *Between* the two sheets A and B.

Reigion 3 – To the *right* of sheet B.

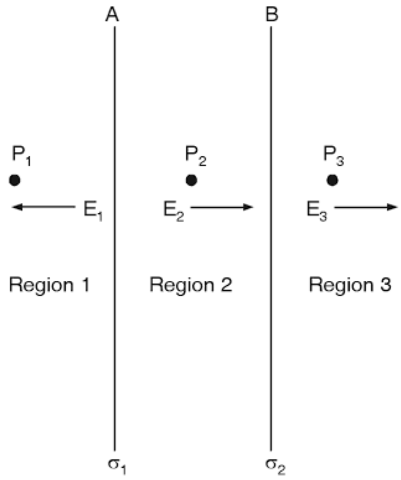

Fig. 17.31

Region 1. Consider a point P_1 in the region 1, then Electric field at P_1 due to the surface charge density $+\sigma_1$ on A

$$= \frac{\sigma_1}{2\varepsilon_0} \text{ along } -X \text{ direction.}$$

Electric field at P_1 due to the surface charge density $+\sigma_2$ on B

$$= \frac{\sigma_2}{2\varepsilon_0} \text{ along } -X \text{ direction}$$

$$\therefore \quad \text{Net electric field at} \quad P_1 = E_1 = \frac{1}{2\varepsilon_0}(\sigma_1+\sigma_2) \text{ along } -X \text{ direction}$$

or

$$E_1 = -\frac{1}{2\varepsilon_0}(\sigma_1+\sigma_2) \qquad \ldots (i)$$

The *negative* sign indicates that the field acts in the $-X$ direction *i.e.*, from *right to left*.

Region 2. Now consider a point P_2 in region 2, then

Electric field at P_2 due to the surface charge density $+\sigma_1$ on A

$$= +\frac{\sigma_1}{2\varepsilon_0} \text{ along } +X \text{ direction and}$$

Electric field at P_2 due to the surface charge density $+\sigma_2$ on B

$$= -\frac{\sigma_2}{2\varepsilon_0} \text{ along } -X \text{ direction}$$

$$\therefore \quad \text{Net electric field at} \quad P_2 = E_2 = \frac{1}{2\varepsilon_0}(\sigma_1-\sigma_2)$$

Total electric flux through the Gaussian surface $\oint\!\!\oint \vec{E}.\vec{dS} = E \oint\!\!\oint dS = E\,4\pi\,r^2$

According to Gauss's theorem

$$\oint\!\!\oint \vec{E}.\vec{dS} = \frac{q}{\varepsilon_0}\frac{r^3}{a^3}$$

or $$E4\pi\,r^2 = \frac{q}{\varepsilon_0}\frac{r^3}{a^3}$$

or $$E = \frac{1}{4\pi\varepsilon_0}\frac{qr}{a^3}$$

The field acts radially outward. The above relation shows that the *electric field at any point inside a uniformly charged solid sphere is directly proportional to its distance form the centre of the sphere.*

17.30 COULOMB'S LAW AS A SPECIAL CASE OF GAUSS'S LAW (COULOMB'S LAW FROM GAUSS'S LAW)

Consider an isolated point charge q located at O, the origin of the co-ordinate system. Draw a sphere of radius r with q as centre to represent a *spherical Gaussian surface*. From considerations of *spherical symmetry* the electric field intensity $\vec{E}$ at any point on the surface must have the same magnitude and must be directed along the outward drawn radius *i.e.* $\vec{E}$ must be normal to the surface.

Let P be a point on the surface of the sphere having a position vector $\vec{r}$, then

Electric intensity at P, $\vec{E} = E\hat{r}$ where $\hat{r}$ is a unit vector in the direction of $\vec{r}$.

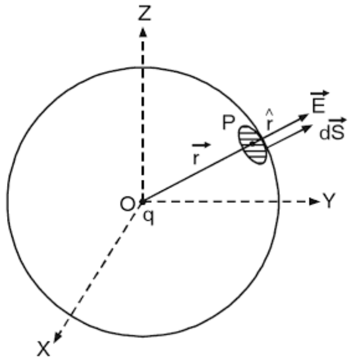

Fig. 17.35

Now consider a small area element $\vec{dS}$ about the point P.

For a spherical surface the area vector $\vec{dS}$ at any point is also directed along the outward drawn normal to the surface.

Therefore, the electric field vector $\vec{E}$ and area vector $\vec{dS}$ at any point on the surface of the sphere are parallel to each other.

Hence $$\vec{E}.\vec{dS} = E\,dS$$

where E and dS are the magnitudes of vectors $\vec{E}$ and $\vec{dS}$ respectively.

According to Gauss's theorem

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$$\oint\!\!\oint_{C.S} \vec{E}.\vec{ds} = \frac{q}{\varepsilon_0} \qquad \ldots (i)$$

where $\oint\!\!\oint_{C.S}$ represent integration over a closed surface.

But $$\oint\!\!\oint \vec{E}.\vec{dS} = \oint\!\!\oint E\,dS = E \oint\!\!\oint dS = E\,4\pi\,r^2 \qquad \ldots (ii)$$

From (*i*) and (*ii*), we have

$$E 4\pi\, r^2 = \frac{q}{\varepsilon_0}$$

or $$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \qquad \ldots (iii)$$

The above relation gives the magnitudes of the electric field intensity at any point at a distance r from the isolated point charge q and from considerations of symmetry its direction is along the line joining the charge to the point.

Hence $$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

If a second charge q' is placed at the point P where the value of $\vec{E}$ has been calculate, then the magnitude of the force that acts on it is given by

$$F = Eq' = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{r^2}$$

Taking the direction of the force also into consideration

$$\vec{F} = \frac{1}{\varepsilon_0}\frac{qq'}{r^2}\hat{r}$$

which is *Coulomb's law*.

Hence Coulomb's law can be deduced form Gauss's law and considerations of symmetry.

In other words, *Coulomb's law is a special case of Gauss's law.*

17.31 DIFFERENTIAL FORM OF GAUSS'S LAW (OR COULOMB'S LAW)

To prove $$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$$

Gauss's theorem or the *integral form* of Coulomb's law states that

$$\oint\!\!\oint \vec{E}.\vec{dS} = \frac{q}{\varepsilon_0}$$

To convert it to *differential form* of Coulomb's law or Gauss's law we make use of Gauss's divergence theorem of vector calculus, which states

$$\oint\!\!\oint \vec{A}.\vec{dS} = \iiint_V (\vec{\nabla}.\vec{A})\,dv$$

For an electrostatic field $\vec{E}$, we have

$$\oiint \vec{E}.\vec{dS} = \iiint_V (\vec{\nabla}.\vec{E})dv$$

$\therefore$ Gauss's theorem can be written as

$$\oiint \vec{E}.\vec{dS} = \iiint_V (\vec{\nabla}.\vec{E})\,dv = \frac{q}{\varepsilon_0} \qquad \text{... }(i)$$

If ρ is the charge density at a point within the volume V, then

$$q = \iiint_V \rho\,dv \text{ or } \frac{q}{\varepsilon_0} = \frac{1}{\varepsilon_0}\iiint_V \rho\,dv = \iiint_V \frac{\rho}{\varepsilon_0}dv \qquad \text{... }(ii)$$

Comparing (i) and (ii), we have

$$\therefore \qquad \vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \text{... }(iii)$$

This is the differential form of Gauss's law (or Coulomb's law)

17.32 ELECTRIC FIELD ON THE SURFACE OF A CONDUCTOR

Let C be a point very close to a charged conducting surface AB having a local surface charge density σ. Draw through C a small Gaussian cylinder CD whose sides are perpendicular to the charged surface and the ends C and D are parallel to the surface such that the end C lies just outside the charged surface and the end D just inside it. If ΔS is the area of the charged surface enclosed within the cylinder, then charge within the cylinder

$$= \sigma\Delta S$$

$\therefore$ According to Gauss's theorem total electrical flux over the surface of the closed cylinder

$$= \sigma\Delta S/\varepsilon_0 \qquad \text{... }(i)$$

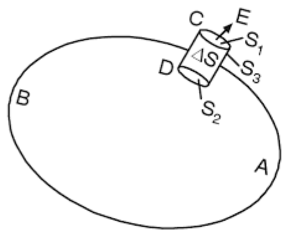

Fig. 17.36

The surface of the conductor AB is an equipotential surface. The direction of the electric field intensity at every point is, therefore, along the outward drawn normal. The closed cylinder has tree surface S_1, S_2, and S_3.

$\therefore$ Electric flux over the surface S_1 at $C = \iint_{S_1} \vec{E}.\vec{dS}$

where $\vec{E}$ is the electric field just outside the cylinder.

The electric field over the area S_2 at D is zero as *the electric field intensity inside a charged conductor is zero.* The electric flux over the curved surface S_3 of the cylinder is also zero as the direction of electric field intensity is parallel to the surface.

$\therefore$ Total electric flux over the closed surface of the cylinder CD

$$= \oiint \vec{E}.\vec{dS} = \iint_{S_1} \vec{E}.\vec{dS} + \iint_{S_2} \vec{E}.\vec{dS} + \iint_{S_3} \vec{E}.\vec{dS}$$

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$$= \iint_{S_1} \vec{E}.\vec{dS} = \iint_{S_1} E\, dS = E\iint_{S_1} dS = E\,\Delta S \qquad \text{... } (ii)$$

From (*i*) and (*ii*) we have

$$E\Delta S = \sigma\Delta S/\varepsilon_0$$

or

$$E = \frac{\sigma}{\varepsilon_0} \qquad \text{... } (iii)$$

Thus the *local field* at a point on the surface of a conductor is equal to $\frac{1}{\varepsilon_0}$ times the *surface charge density* σ at that point.

Electric field near a non-conducting sheet

The electric field near a plane non-conducting thin sheet of charge is $\frac{\sigma}{2\varepsilon_0}$ ... (*iv*)

For proof See article 17.29.4 eq. (*iii*)

Comparing equations (*iii*) and (*iv*), *we find that the electric field near a charged conducting sheet is twice as great as near a non-conducting sheet with the same surface density of charge.*

Excess charge lies on the surface of a conductor. Consider a conductor carrying a charge q. The electric field intensity $\vec{E}$ at any point inside it is zero.

Now consider a Gaussian surface just inside the conductor infinitely close to the actual surface as shown by dotted line in Fig. 17.37.

The total electric flux through this surface

$$\oiint \vec{E}.\vec{dS} = 0 \text{ since } \vec{E} = 0$$

According to Gauss's theorem $\oiint \vec{E}.\vec{dS} = \frac{q}{\varepsilon_0} = 0$ or $q = 0$

Thus there is no charge anywhere within the surface of the conductor *i.e. the excess charge lies on the surface of the conductor.*

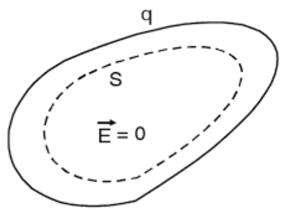

Fig. 17.37

17.33 SCREENING OF FIELD $\vec{E}$ BY A CONDUCTOR

Imagine that there is a cavity in a conductor. Since there is no charge in the cavity, the field within the cavity is always zero. In other words, if we place a charge or an equipment inside the cavity, it will remain shielded from the external field Fig. 17.38 (*a*).

If there is a charge placed in the cavity , $\vec{E} \neq 0$ as shown in Fig. 17.38 (*b*). It should be remembered that the cavity and contents are isolated from the outside. No external fields penetrate the conductor, these are cancelled by the induced charges on the outer surface. Also the field due to the charges within the cavity is nullified by the induced charges on the inner surface for all external points.

Application: This is reason that we are most safe inside a metal car during the thunderstorm If lightning strikes, then the person inside the car is shielded by the metal body. This is also the

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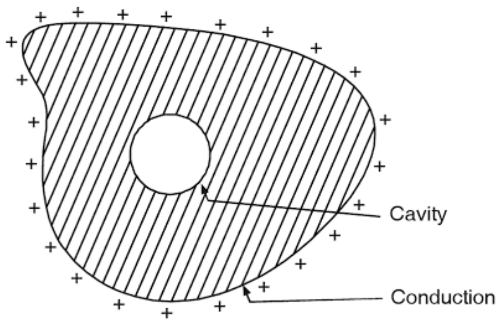

Fig. 17.38 (*a*)

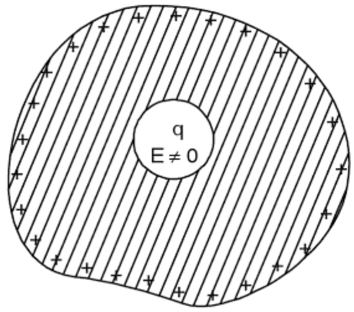

Fig. 17.38 (*b*)

principle of the famous *Faradays's cage*. The encloser need not be a solid conductor. Even a wire mesh serves to isolate the charges and equipment and apparatus inside the cage are well shielded from external strong electric fields.

17.34 CAPACITANCE OF A PARALLEL PLATE CAPACITOR

Consider two metal plates each of area A held at a distance d apart as shown in Fig. 17.39 and be given $+Q$ and $-Q$ charges respectively.

$\therefore$ The surface charges density

$$\sigma = \frac{Q}{A}$$

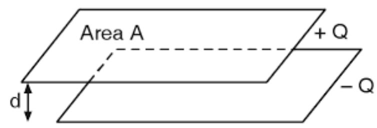

Fig. 17.39

The electric field $\vec{E}$ inside it is

$$E = \frac{\sigma}{\varepsilon_0} = \frac{1}{\varepsilon_0}\frac{Q}{A}$$

where ε_0 is the permittivity of free space between the two parallel plates.

The potential difference between the plates is V given by

$$V = ED \qquad (\because E = V/d)$$

or
$$V = \frac{Qd}{A\varepsilon_0}$$

or
$$\frac{Q}{V} = \frac{\varepsilon_0 A}{d} = \text{constant} = C$$

$$\therefore \qquad \text{Capacity } C = \frac{\varepsilon_0 A}{d} \text{ farad.}$$

Units: Unit of capacitance is coul/volt and is called Farad.

$$\therefore \qquad 1 \text{ Farad} = \frac{1\,\text{coulomb}}{1\,\text{volt}}$$

However, farad (F) is a very large unit. Therefore, in practice we use smaller units as micro-farad and pico-farad (*i.e.* micro-micro Farad) denoted by μF and pF respectively.

$$1\mu\text{F} = 10^{-6}\text{ F and } 1\mu\mu\text{ F} = 1\text{pF} = 10^{-12}\text{ F.}$$

Depending upon the size and geometry (shape) of the conducting plates; we can design the capacitors, generally of the type:

(*i*) a parallel plate capacitor

(*ii*) a cylindrical capacitor

(*iii*) a spherical capacitor

17.35 MECHANICAL FORCE PER UNIT AREA ON THE SURFACE OF A CHARGED CONDUCTOR

The charge on any small area of a conductor experiences an outward mechanical force due to the repulsion of the charge on the rest of the area. Let *AB* represent a charged conducting surface. Consider a small element of area *dS* and let *P* be a point just outside it. If σ is the surface density of charge, then electric field at *P*, $E = \frac{\sigma}{\varepsilon_0}$.

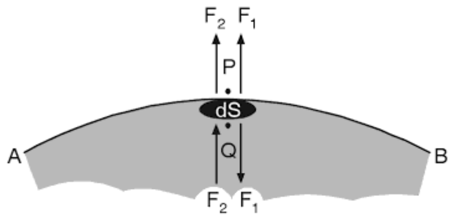

Fig. 17.40

The electric field at *P* may be regarded as made up of two parts:

(*i*) *An outward* F_1 due to the charge on the small area *dS* very close to *P* and

(*ii*) *An outward force* F_2 due to the charge on the rest of the surface.

$$\therefore \quad F_1 + F_2 = E = \sigma/\varepsilon_0$$

Now consider a point *Q just inside* the charged surface.

Since *Q* lies inside the conductor.

$\therefore$ Electric field at Q = 0.

The field at *Q* may again be regarded as made up of two parts. As the point *Q* lies on the opposite side of *P*, but still very close to it, the force F_1 due to the charge on the small area *dS* will be equal in magnitude but opposite in direction.

Since *Q* lies very near *P* the force *Q* due to the rest of the charged surface will be the same and in the same direction as at *P*

$$\therefore \quad -F_1 + F_2 = 0 \qquad \text{... (ii)}$$

Comparing (*i*) and (*ii*), we have $F_1 + F_2 = \sigma/2\varepsilon_0$

The charge on area *dS* is $\sigma\, dS$ and a unit charge on it experience an outward force $F = \frac{\sigma}{2\varepsilon_0}$ due to the charge on the rest of the surface.

$$\therefore \quad \text{Outward force on the area } dS = \sigma\, dS \frac{\sigma}{2\varepsilon_0} = \frac{\sigma^2 dS}{2\varepsilon_0}$$

$$\therefore \quad \text{Outward force per unit area (or outward pressure)} = \frac{\sigma^2}{2\varepsilon_0} \qquad \text{... (iii)}$$

$$\text{Now} \qquad E = \frac{\sigma}{\varepsilon_0} \text{ or } \sigma = \varepsilon_0 E$$

$\therefore$ Outward force per unit area or electrostatic pressure acting along the outward drawn normal to the surface

17.38 FORCE ACTING ON CONDUCTING SPHERE IN ELECTRIC FIELD

Consider a conducting uncharged sphere A of volume V is placed at P where an electric field $\vec{E}$ due to charge $+q$ present at O. Let $OP = r$. The dotted lines in Fig. 17.41 show the direction of lines

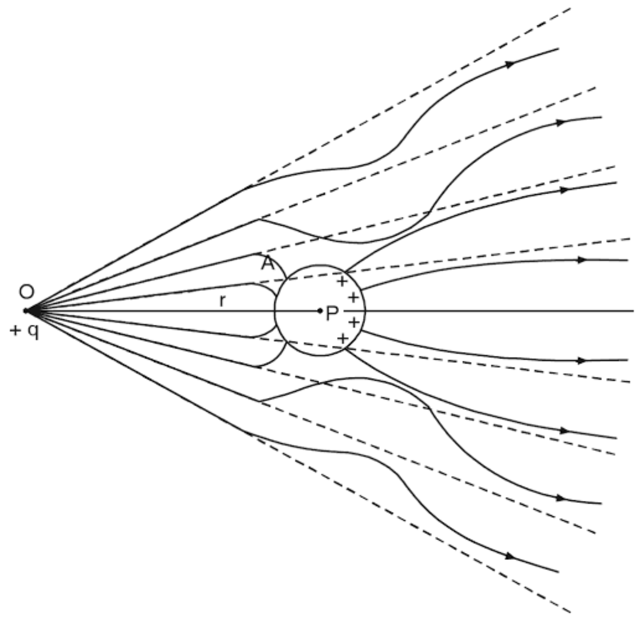

Fig. 17.41

of force when A is placed in the field. But we assume that the presence of the uncharged spherical conductor in the field does not produce any distortion in the lines of force *i.e.* the average intensity through the region now occupied by the sphere is the same as the electric intensity at P before the introduction of the sphere.

The electric intensity at P before the introduction of the sphere is $E = \dfrac{q}{4\pi\epsilon_0 kr^2}$, where k is dielectric constant of the medium between O and A. Hence, the potential energy which disappears is

$$W = \frac{\epsilon_0 kE^2}{2}.V = \frac{\epsilon_0 k}{2}\frac{q^2}{16\pi^2\epsilon_0^2 k^2 r^4}.V = \frac{q^2.V}{32\pi^2\epsilon_0 kr^4} \text{ Joules}$$

Let the sphere be moved now through a small distance dr towards O, the work done is stored in the form of potential energy.

$$\therefore \qquad dW = -F.dr$$

$$\therefore \qquad F = -\frac{dW}{dr} = -\frac{1}{dr}\left(\frac{q^2 v}{32\pi^2\epsilon_0 kr^4}\right) = +\frac{q^2 v}{8\pi^2\epsilon_0 kr^5}$$

$\therefore$ Force acting on the sphere $F = \dfrac{q^2.v}{8\pi^2\epsilon_0 kr^5}$

17.39 ELECTRICAL IMAGES

There are various methods, as discussed earlier, for calculation of $\vec{E}$ and electric potential V at a point. But, in some complicated problems, its not easy to find $\vec{E}$ and V. Lord Kelvin suggested

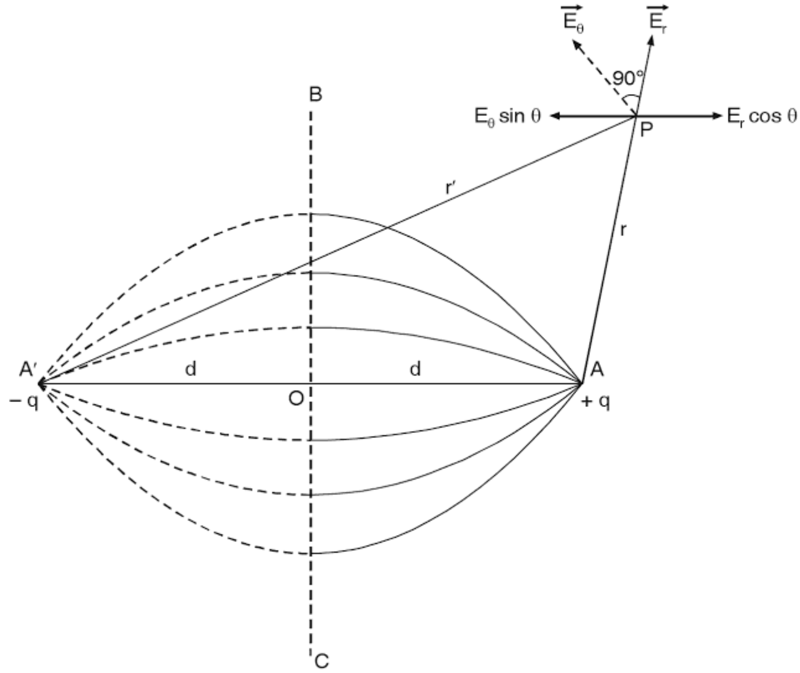

Fig. 17.42

a very powerful tool to resolve such problems on the basis of "distribution of charges". This method is known as the method of 'electrical images' due to optical analogy *i.e.* mirror image.

Let us consider a charge $+q$ placed infront of a conducting plane BC. Due to induction negative charge will appear over the surface of BC; but we do not know the surface charge density over BC, hence it is very difficult to calculate the potential V and electric vector $\vec{E}$ at any point P.

But the method of images makes it possible because the whole of this induced charge can be replaced by a single point charge $(-q)$ at A'; *i.e. the* electrical image of $+q$ at A shown in Fig. 17.42.

Boundary Condition

Let two point charges $+q$ and $-q$ be a distance $2d$.

Then potential at P due to these charges is given by

$$V_P = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} - \frac{q}{r'}\right)$$

in which r and r' are the distances of P from $+q$ and $-q$ respectively. It point lies on the midplane BC, then $r = r'$ and potential there is zero.

So the boundary condition is that the potential is zero all over the infinite plane BC. The condition is satisfied by earthing the plane BC. Hence the potential is zero.

Definition. An electrical image is defined as a point charge on one side of a surface, which will produce on the other side of the surface, the same electric field as it produced by the actual electrification of the surface. The actual electrification is then ignored and the image or images are used instead of plane to find the field .

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17.40 POINT CHARGE IN FRONT OF A GROUND INFINITE CONDUCTOR

Consider a point charge $+q$ placed infront of an infinitely long conducting plane (or wire or conductor) BC, situated at point A of distance d. In order to satisfy the boudary condition imagine an induced charge $-q$ on the other side (electrical image of $+q$) of BC at the same distance d from BC along the line AO produced and the conductor BC is earthed (to make potential zero on BC) as shown in Fig. 17.43. This satisfies the required boundary condition.

Consider any point P at a distance r from A and r' from $-q$ respectively.

From $\Delta\, APA'$,

$$r' = \left(r^2 + 4d^2 + 4rd\cos\theta\right)^{\frac{1}{2}}$$

Hence, the potential at P,

$$V_P = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{1}{r'}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{1}{\left(r^2 + 4d^2 + 4rd\cos\theta\right)^{\frac{1}{2}}}\right] \quad \ldots (i)$$

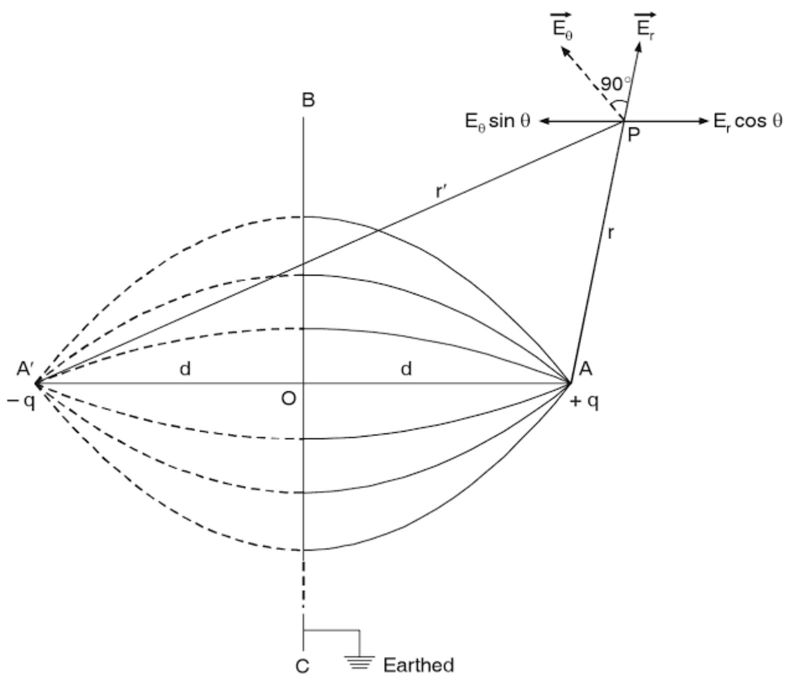

Fig. 17.43

The intensity of electric field at P in the direction of AP *i.e.* along r

$$E_r = -\frac{\partial V_P}{\partial r}$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r^2} - \frac{r + 2d\cos\theta}{(r^2 + 4d^2 + 4rd\cos\theta)^{3/2}}\right] \quad \ldots (ii)$$

And the intensity at P in a direction perpendicular to r is

Example 17.4 The intensity of the electric field at a point is given by $\vec{E} = 6xy\hat{i} + (3x^2 - 3y^2)\ \hat{j} + 4z\hat{k}$ Newton/Coulomb. Calculate the amount of work done by the field in taking a unit positive charge from the origin to the point (x_1, y_1, z_1).

If $x_1, y_1, z_1 = 1, 1, 1$ find the work done for a charge of 1 coulomb.

Solution. The work done by the field in taking a unit positive charge from the origin (0, 0, 0) to the point (x_1, y_1, z_1) is given by

$$\int_{000}^{x_1y_1z_1} \vec{E}.d\vec{r}$$

Now $$\vec{E} = 6xy\hat{i} + (3x^2 - 3y^2)\hat{j} + 4z\hat{k}$$

and $$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$\therefore$ $$\vec{E}.d\vec{r} = [6xy\hat{i} + (3x^2 - 3y^2)\hat{j} + 4z\hat{k}].[dx\hat{i} + dy\hat{j} + dz\hat{k}]$$
$$= 6xydx + (3x^2 - 3y^2)\ dy + 4zdz$$

Hence $$\int_{000}^{x_1y_1z_1} \vec{E}.d\vec{r} = \int_{000}^{x_1y_1z_1} [6xydx + (3x^2 - 3y^2)dy + 4zdz]$$

$$= [3x^2y + 3x^2y - y^3 + 2z^2]_{000}^{x_1y_1z_1} = 6x_1^2y_1 - y_1^3 + 2z_1^2$$

Substituting $x = 1$, $y = 1$ and $z = 1$ in the above relation, work done for a charge of 1 coulomb $= 6 - 1 + 2 = 7$ Joules.

Example 17.5 Eight charges of value 3μ*C*, 5μ*C*, 7μ*C*, 10μ*C*, 15μ*C*, – 1μ*C*, – 5μ*C* and – 7μ*C* are placed symmetrically on a circle of radius 0.4 m in air. Calculate the potential at the centre of the circle. (*Nagpur Uni.* 2005)

Solution. Potential $$V = \frac{1}{4\pi\epsilon_0}\frac{\sum q_i}{r}$$

$$= 9 \times 10^9 \times \frac{27 \times 10^{-6}}{0.4}$$

$$= \mathbf{6.075 \times 10^5\ Volt.}$$

Example 17.6 Determine the energy gained by α-particle when it is accelerated through a potential difference of 1000 volts.

Solution. The difference of potential between two points A and B is given by

$$V_B - V_A = \frac{U_B - U_A}{q}$$

where $U_B - U_A$ is the energy gained by the charge q in moving through a potential difference $V_B - V_A$ $(V_B > V_A)$.

If $V_B - V_A = V$ and $U_B - U_A = U$, then

$$U = qV$$

i.e., the energy gained by a particles of charge q coulomb when accelerated through a potential difference of V volts = qV Joules

$$\text{Charge on the } \alpha\text{-paticle } q = 2 \times 1.6 \times 10^{-19}\text{ C}$$

$$\text{Potential difference } V = 1000\text{ V}$$

$$\therefore \quad \text{Energy gained} = 2 \times 1.6 \times 10^{-19} \times 1000 = \mathbf{3.2 \times 10^{-16}\ J}$$

Example 17.7 In an electron gun electrons of charge *e* and *m* are accelerated through a potential difference *V*. Find the maximum speed attained by electrons.

Solution. Let v be the maximum velocity gained by the electron, then

Kinetic energy gained by the electron = $\frac{1}{2}mv^2$

Work done on the electron when it is accelerated through a potential difference $V = eV$

$$\therefore \quad \frac{1}{2}mv^2 = eV$$

$$\text{or} \quad V = \sqrt{\frac{\mathbf{2eV}}{\mathbf{m}}}$$

Example 17.8 A point charge 5×10^{-9} *C* is situated at the origin of the co-ordinates. Find the potential difference between the points *A* (1, 2, 2) and $B(\sqrt{5},2,4)$. (*M.D.U.* 2001)

Solution. Co-ordinate of points A; $x = 1$, $y = 2$, $z = 2$

$$\therefore \quad \text{Radial distance of } A, \quad r_1 = \sqrt{x^2+y^2+z^2} = \sqrt{1+4+4} = 3$$

$$\text{Co-ordinate of point } B; \quad x = \sqrt{5}, y = 2, z = 4$$

$$\therefore \quad \text{Radial distance of } B, \quad r_2 = \sqrt{5+4+16} = 5$$

$$\text{Now} \quad V_B - V_A = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_2} - \frac{1}{r_1}\right] = 5 \times 10^9 \times 9 \times 10^9 \times \left[\frac{1}{5} - \frac{1}{3}\right]$$

$$= -6 \text{ volt}$$

$$\text{or} \quad V_A - V_B = \mathbf{6\ volt.}$$

Example 17.9 The electric field in the *xy* plane is given by $\vec{E} = (y\hat{j} + x\hat{j})$, find the potential difference between two points *A* to *B* having co-ordinates (0, 0) and (2, 2) respectively.

$$\textbf{Solution.} \quad V_B - V_A = -\int_A^B \vec{E}.d\vec{r}$$

in the x-y plane $\quad d\vec{r} = dx\hat{i} + dy\hat{j}$ and $\vec{E} = y\hat{i} + x\hat{j}$

$$\therefore \quad \vec{E}.\vec{dr} = (y\hat{i} + x\hat{j}).(dx\hat{i} + dy\hat{j}) = ydx + xdy$$

$$\therefore \quad V_B - V_A = -\int_A^B y\,dx + xdy = [2xy]_{0,0}^{2,2} = 8 \text{ units.}$$

$$\frac{\partial V}{\partial x} = 12x - 5y \; ; \; \frac{\partial V}{\partial y} = -5x + 8y \; ; \; \frac{\partial V}{\partial z} = 6z$$

$\therefore$
$$\vec{E} = -[(12x - 5y)\hat{i} + (-5x + 8y)\hat{j} + (6z)\hat{k}]$$

At the point 2, 0, –3,

$$\vec{E} = -[24\hat{i} - 10\hat{j} - 18\hat{k}] = -24\hat{i} + 10\hat{j} + 18\hat{k}$$

Force on a charge $2 \times 10^{-10}\, C$ at (2, 0, –3)

$$= 2 \times 10^{-10}(-24\hat{i} + 10\hat{j} + 18\hat{k})$$

Example 17.16 Electric potential in a region is given by $V(x, y, z) = 4x^2 - 3y^2 - 9z^2$. Find electric field at P (3, 4, 5). (*Pbi. U.*, 2001)

Solution.
$$\vec{E} = -\vec{\nabla}V = -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right)$$

$$\vec{E} = -8x\hat{i} + 6y\hat{j} + 18z\hat{k}$$

At P (3, 4, 5),
$$\vec{E} = -24\hat{i} + 24\hat{j} + 90\hat{k}$$

Example 17.17 The potential function at a point is given by $V(x, y, z) = 10(x^2 + y^2 + z^2)^{-1/2}$. Find electric field intensity at point (2, 4, 4).

Solution.
$$(x, y, z) = 10(x^2 + y^2 + z^2)^{-1/2}$$

Now electric field
$$\vec{E} = -\vec{\nabla}V$$

$$= -\left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\left[10(x^2 + y^2 + z^2)^{-1/2}\right]$$

$$= \left[\frac{1}{2} \times 10(x^2 + y^2 + z^2)^{-3/2}\, 2x\right]\hat{i} + \left[\frac{1}{2} \times 10(x^2 + y^2 + z^2)^{-3/2}\, 2y\right]\hat{j}$$

$$+\left[\frac{1}{2} \times 10(x^2 + y^2 + z^2)^{-3/2}\, 2z\right]\hat{k}$$

$\therefore$
$$\vec{E} = \frac{10x}{(x^2 + y^2 + z^2)^{3/2}}\hat{i} + \frac{10y}{(x^2 + y^2 + z^2)^{3/2}}\hat{j} + \frac{10z}{(x^2 + y^2 + z^2)^{3/2}}\hat{k}$$

At the point 2, 4, 4,

$$(x^2 + y^2 + z^2)^{3/2} = (2^2 + 4^2 + 4^2)^{3/2} = 216$$

$$\vec{E} = \frac{20}{216}\hat{i} + \frac{40}{216}\hat{j} + \frac{40}{216}\hat{k}$$

$$= \frac{5}{54}\hat{i} + \frac{5}{27}\hat{j} + \frac{5}{27}\hat{k}$$

Example 17.18 Show that the electric field satisfies the equation

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}, \vec{\nabla}.\vec{E} = 0$$

Solution. For proof of $\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$ See article 17.31 relation (*iii*)

Now $\vec{E} = -\vec{\nabla}.\vec{V} = -grad\ V$ where V is the electric potential [For proof See article 17.16 Eq. (*iii*)]

$$\therefore \qquad \vec{\nabla} \times \vec{E} = -\ \vec{\nabla} \times \vec{\nabla} V = 0$$

since $\vec{\nabla} \times \vec{\nabla} = 0$ being the vector product of two equal vectors.

Example 17.19 Electric potential in a region of space is given by

$$V(x, y, z) = 50x^2 - 75y \ (V \text{ in volts})\ x, y, z \text{ in meters. Find}$$

(*i*) Magnitude of electric field at point (1, 1, 0) (*ii*) Charge density (*iii*) Show the electric field is not uniform. (*G.N.D.U.*, 2002; *Pbi.U.*, 2000)

Solution. Given $V = 50x^2 - 75y$

Electric field $\vec{E} = -\nabla \vec{V}$

$$= -\left[\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right](50x^2 - 75y)$$

$$= -\ [100\,x\hat{i} + 75\,\hat{j}\,]$$

or

$$\vec{E} = -100\,x\hat{i} + 75\,\hat{j}$$

Magnitude of $\vec{E}$ at 1, 1, 0 $\vec{E} = -100\,x\hat{i} + 75\,\hat{j}$

$$\therefore \qquad |\vec{E}| = \sqrt{100^2 + 75^2} = 125\ \text{Vm}^{-1}$$

(*ii*) Now $\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$

$$\vec{\nabla}.\vec{E} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(-100\,x\hat{i} + 75\,\hat{j}\,) = -100$$

$$\therefore \qquad \rho = -100\varepsilon_0 = -\ 100 \times 8.85 \times 10^{-12} = -\ 8.85 \times 10^{-10}\ \text{Cm}^{-3}$$

(*iii*) Electric field $\vec{E} = 100\,x\hat{i} + 75\,\hat{j}$

The y-component of $\vec{E}$ is constant but the x-component of $\vec{E}$ varies with x. Hence the resultant electric field $\vec{E}$ is not uniform.

Example 17.20 The electric potential in space is given by 3*x* + 4*y* – *z*. Show that the electric field intensity is uniform everywhere. If electric potential is constant in a certain region of space, what inference can be drawn about the electric field?

Solution. Given $\phi = 3x + 4y - z$

Now electric field intensity $\vec{E} = -\vec{\nabla}\phi$

Earth being a conductor, the surface charge density of earth's surface σ is given by $E = \dfrac{\sigma}{\varepsilon_0}$

or
$$\sigma = \varepsilon_0 E = 4\pi\varepsilon_0 \frac{E}{4\pi}$$

But
$$\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9 \text{ NC}^{-2} \text{ m}^2$$

$\therefore$
$$\sigma = \frac{1}{9\times10^9}\frac{300}{4\pi} = \frac{1}{12\pi}\times10^{-7}\,\text{Cm}^{-2}$$

Example 17.31 A soap bubble of radius 5 mm is formed at the end of an open glass tube. What charge should be given to the bubbles so that it is in equilibrium? Surface tension = 0.033 Nm^{-1} (*P.U.* 2000)

Solution. The soap bubble will be in equilibrium when

$$\frac{4T}{r} = \frac{q^2}{32\varepsilon_0\pi^2 r^4}$$

$\therefore$
$$q^2 = 4T \times 32\ \varepsilon_0\ \pi^2\ r^3 = 4\pi\ \varepsilon_0 \times 32\ \pi\ T\ r^3$$

Now
$$T = 0.033 \text{ Nm}^{-1} = 33\times10^{-3} \text{ Nm}^{-1},\ r = 5 \text{ mm} = 5 \times 10^{-3} \text{ m}$$

$$\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9 \text{ Nm}^2 \text{ C}^{-2}$$

$\therefore$
$$q^2 = \frac{1}{9\times10^9}\times32\times\frac{22}{7}\times33\times10^{-3}\times(5\times10^{-3})^3 = 46.095\times10^{-18}$$

or
$$q = 6.79 \times 10^{-9} \text{ C.}$$

Example 17.32 Find the surface density of charge of a horizontal charged plate so that the gold foil weighing 40 mg/cm^2 when placed on the plate will rise. (*Kerala. U.* 2001)

Solution. Suppose the surface density of charge required to raise the gold foil = σ

$\therefore$ Upward pressure on the gold foil = $\dfrac{\sigma^2}{2\varepsilon_0}$ Newton m^{-2}

Downward weight of the foil per sq metre = $40 \times 10^{-6} \times 10^4 \times 9.8$ Newton m^{-2}

For the foil to rise
$$\frac{\sigma^2}{2\varepsilon_0} = 40 \times 10^{-6} \times 10^4 \times 9.8$$

or
$$\sigma^2 = 2\varepsilon_0 \times 40 \times 10^{-6} \times 10^4 \times 9.8$$
$$= 2 \times 8.85 \times 10^{-12} \times 40 \times 10^{-6} \times 10^4 \times 9.8$$
$$\sigma = 8.33 \times 10^{-6} \text{ coulomb m}^{-2}$$

Example 17.33 Three charges + 1.5 *q*, + 1.5 *q* and – 3 *q* are placed at the vertices of an equilateral triangle of side *b*. Find the dipole moment of the charge distribution.

Solution. Let the charges + 1.5 q, +1.5 q and –3 q be placed at the corners of an equilateral triangle ABC as shown in Fig. 17.44. From A draw AO perpendicular to BC and let O be the origin of the co-ordinate system so that OC represents the + X axis, and OB the – X axis and OA the +Y axis.

Since each side of the triangle is b, $OC = \frac{b}{2}, OB = \frac{b}{2}$ and $OA = b \sin 60° = \frac{b}{2}\sqrt{3}$

The co-ordinate of the points A, B, C, therefore are $\left(0, \frac{\sqrt{3}}{2}b\right)\left(-\frac{b}{2}, 0\right)$ and $\left(+\frac{b}{2}, 0\right)$ respectively.

As the algebraic sum of all the charges is zero, the monopole moment = 0.

To find the dipole moment $\vec{p}$ we shall calculate the x and y components $\vec{p_x}$ and $\vec{p_y}$ separately.

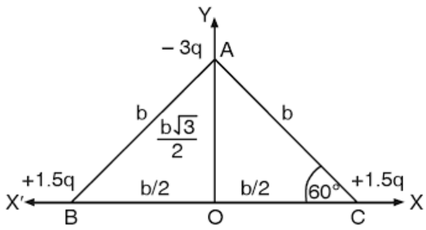

Fig. 17.44

$$\therefore \quad \vec{p_x} = \left[+1.5q\left(\frac{b}{2}\right) + 1.5q\left(-\frac{b}{2}\right) - 3q\,(0)\right]\hat{i} = 0$$

$$\text{and} \quad \vec{p_y} = \left[+1.5q\,(0) + 1.5q(0) - 3q\left(\frac{\sqrt{3}}{2}b\right)\right]\hat{j}$$

$$= -3\frac{\sqrt{3}}{2}qb\hat{j}$$

$$\therefore \quad \vec{p} = \vec{p_y} = -\frac{3\sqrt{3}}{2}qb\hat{j}$$

i.e., the dipole moment acts away from the point A along the perpendicular drawn from the charge $-3q$ on the line joining the charges $+ 1.5\,q$ and $+ 1.5\,q$.

Example 17.34 Four charges + *q*, – *q* + 2*q* and –2*q* are located at the corners of a square of side *a*. Find (*i*) Monopole moment (*ii*) Electric dipole moment of charge distribution

(*P.U.*, 2002)

Solution. Let the charges $+ q$, $- q$, $+2q$ and $-2q$ be placed at the corners A, B, C and D of the square.

Let A be the origin of the co-ordinate system, AB, the X-axis and AD the Y-axis. As the algebraic sum of all the charges is zero, the monopole moment = 0.

To find the dipole moment $\vec{p}$ we shall calculate the x and y components $\vec{p_x}$ and $\vec{p_y}$ separately, The co-ordinates of the points A, B, C, D are [0, 0], [a, 0], [a, a] and [0, a] respectively.

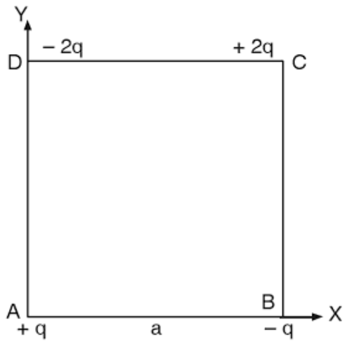

Fig. 17.45

$$\therefore \quad \vec{p_x} = + q \times 0 + (-q \times a) + 2q \times a + (-2q \times 0)$$

$$= q\,a$$

(***b***) What is the value of potential at a point (*i*) on the axis of the dipole (*ii*) on the normal to the axis?

12. (***i***) What is an electric dipole?

(***ii***) Derive an expression for electric potential at any point at a distance *r* from an electric dipole.

Hence obtain an expression for electric field and show that electric field along the axis of dipole is double than that of equatorial line. (*Nagpur Uni.* 2008, *Meerut* 2004)

13. (***i***) What is an electric dipole?

(***ii***) Define electric dipole moment. State its direction and its S.I. unit.

(*Nagpur Uni. s*/2009)

(***iii***) Assume the expression for electrical potential at any point due to an elctric dipole and hence derive expression for electric field intensity at any point on the axis of dipole.

(***iv***) Show that electrical field intensity on axial point is double that of equatorial point

(*Nagpur Uni. w*/2007)

14. Deduce the relation for an elctric field due to a quadrupole (in polar form). Hence find the field at a point (*i*) on the axial line and (*ii*) on the eqatorial line. (*Agra* 2004)

15. (***a***) Calculate the force and torque on a dipole in an external field.

(***b***) Derive an expression for the potential energy of an elctric dipole placed in a uniform electric field. When is the energy of dipole minimum? (*Nagpur Uni.* 2007)

16. Calculate the elctric field at a far off point due to a small electric dipole. Hence find the field at a point (*i*) on the axial line and (*ii*) on equatorial line.

(*Nagpur Uni.* 2003, 2007, *w/*2007, 2008)

17. (***a***) Explain the meaning of the term electric flux. What are its dimensions and S.I. units

(*G.N.D.U.* 2003, 2002, 2000, *Nagpur U.* 2008)

(***b***) Define electric flux density write its S.I. units. (*M.D.U.* 2000)

Under what conditions is the flux of a vector field through a plane surface (*i*) positive (***ii***) negative (*iii*) zero?

18. (***a***) State and prove Gauss's theorem in electrostatics. Prove that total flux over a surface due to a charge lying outside is zero.

(*b*) Write the law for a volume distribution of charge.

(*Meerut. U.* 2003; *P.U.* 2001, 2000; *Gharwal.U.* 2000; *Indore.U.* 2001; *Gauhati.U.* 2000; *M.D.U.* 2001; *H.P.U.* 2002, 2001; *G.N.D.U.* 2001; *K.U.* 2000; *Nagpur.U.*2006; *D.A.U Agra* 2007, 2005)

19. (***a***) Apply Gauss's theorem to calculate the electric field due to an infinitely long, uniformly charged, hollow cylinder. Is the result applicable to a hollow conducting cylinder? (*D.A.U. Agra* 2007)

(***b***) Find the electric due to two co-axial cylinders of radii a and b ($a > b$) the inner cylinder having a charge $+\lambda$ and the outer cylinder $-\lambda$ per unit length when the point lies at a distance r

(***i***) outside the cylinder $r > a$

(***ii***) between the cylinders $a > r > b$

(***iii***) inside the cylinder $r < b$ (*Nagpur Uni.* 2003)

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2. Under what conditions is the flux of vector field through a plane surface positive, negative and zero. (*Nagpur Uni.* 2007)

3. An electric dipole of length 10^{-2} m and a dipole moment 4×10^{-6} cm is enclosed in a cubical box 1 m × 1 m × 1 m . Calculate the electric flux over its surface.

(*Pbi. U.* 2001, 2000)

[**Ans.** $\phi_E = 0$]

4. An electric dipole of moment 5×10^{-6} cm is enclosed in a card board spherical shell of radius 1m. Find the electric flux over the whole surface. (*Pbi. U.* 2002, *H.P.U.* 2001)

[**Ans.** $\phi_E = 0$]

5. If 2000 flux through lines enter a given volume of space and 4000 lines diverge form it, find the total charge within the volume. (*M.D.U* 2001)

[**Ans.** 17.70×10^{-9} C]

6. Calculate the capacitance of a parallel plate capacitor having 20 cm × 20 cm square plates separated by a distance of 1.00 mm in vacuum

(**Hint:** $C = \dfrac{\varepsilon_0 A}{d} = \dfrac{8.85 \times 10^{-12} \times (0.2)^2}{10^{-3}\,\text{m}} = 354\ pF$) (*Nagpur Uni.,* 2006)

7. An insulated soap bubble 10 cm. radius is charged with 20 e.s.u. or ($20 \times 3.33 \times 10^{-10}$ Coulomb). Find the increase in radius due to the charge. Assume the atmospheric pressure $= 10^5$ N/m^2.

(**Hint:** $p \propto \dfrac{1}{v}$.; Boyle's law, $\therefore\ p \propto \dfrac{1}{r^3}$ or $p = \dfrac{k}{r^3}$ $\therefore$ $\dfrac{dp}{dr} = \dfrac{-3p}{r} dr = \dfrac{-r}{3p} dp$)

18 ELECTRIC FIELDS IN DIELECTRICS

INTRODUCTION

Substances like glass, ebonite, paraffine wax and mica are typical examples of insulators, used as dielectrics. A capacitor (or condenser) is a device for storing charge. If a dielectric material is introduced between the two plates forming a parallel plate capacitor, its capacity to store the charge is enhanced. This is true for other shaped viz. cylindrical, sphericals capacitors too. Therefore, dielectrics play a crucial role in the field of electricity and electronics. In this chapter, we will study the nature of electric fields developed in dielectrics.

18.1 CLASSIFICATION OF SUBSTANCES

Matter is composed of molecules which in turn may consist of one or more atoms. An atom has a positively charged nucleus and revolving round it are one or more electrons so that the atom, on the whole, is neutral. These electrons revolve in more or less circular orbits having radii of the order of 10^{-10} meter and are, therefore, confined within a very small region. These are known as *bound electrons*.

Electrons revolving in orbits close to the nucleus are more strongly bound to it than the electrons moving in the outer orbits. The atoms in a solid are very closely packed, the inter atomic distance being of the order of 5×10^{-10} meter. The neighbouring atoms exert *inter-atomic* forces due to the interaction of charge distribution on them. In some of the atoms these forces are sufficiently strong and are able to detach the outermost electrons which is most loosely bound from the parent atom. These electrons move about freely within the boundaries of the material just like the molecules of a gas in a vessel and collide with each other as well as with the positively charged ions formed due to the detachment of the electron (negatively charged) from the neutral atom. These electrons are known as *free electrons*. All substances can be classified into three types as per their behaviour.

(i) Conductors

In a *good conductor* each atom on an average gives rise to one free electron. In a solid the number of atoms per c.c. is of the order of 10^{22}. We therefore, suppose that the number of free electrons is also the same *i.e.*, 10^{22} electrons/c.c. Metals like Ag, Cu, Al are typical examples of good conductors.

(ii) Insulators

In an *insulator* or *bad conductor* the atoms do not lose outermost electron due to inter atomic forces and there are practically no free electrons. The conductivity of such materials is hardly 10^{-20}

times the conductivity of a good conductor. Substances like glass, ebonite and mica are typical examples of insulators. Such substance are also known as *dielectrics*.

The interatomic forces in liquids are much weaker than those in solids because the interatomic distances are about 10 times larger. Hence liquids are mainly insulators. Mercury is an exception. The conductivity of electrolytes like $CuSO_4$ solution is due to the dissociation of the molecule into positively charged Cu^{++} and negatively charged SO_4^{--} ions which move about freely in the liquid like free electrons and when a potential difference is applied the positive ions move from positive to negative and negative ions from negative to positive electrode thereby giving rise to a conduction current.

In gases the inter atomic distances are again 10 times greater than in the case of liquids. The interatomic forces are even weaker and there are practically no free electrons. Gases are, therefore, mostly *insulators*.

(iii) Semiconductors

Semiconductors are such materials which have conductivity about half way between good conductors and insulators i.e., their conductivity is about 10^{-10} times the conductivity of good conductors. These substance form a very useful class and find a number of applications in solid state electronic devices.

18.2 DIELECTRICS

A dielectric substance is a material which does not conduct electricity i.e., dielectrics are insulators. In such substance the electrons are firmly bound to the atoms of the material. These electrons can, therefore, move through very small distances of the order of atomic dimensions. There are practically no free electrons in dielectrics. Familiar examples are glass, mica, ebonite, paraffine wax and air.

Importance. The importance of dielectric, lies in the fact that dielectric materials are very good insulators. Electric current flows through high tension wires because air (as dielectric) is very good insulator. High voltage transformers are immersed in oil, a high insulator dielectric, to avoid leakage. Porcelain clamps are used at the top of electric poles to insulate the poles from earth. Dielectrics like rubber, plastic etc. are used as cover (sheaths) for underground and overhead cables. Dielectric like mica and paper are used in high quality capacitors employed in radio and television transmission and reception circuits, and other electronic devices.

18.3 NON-POLAR AND POLAR MOLECULES

According to the quantum theory *neutral* atoms in their *ground* state consist of a central *positively* charged nucleus surrounded by a *spherically symmetric cloud of equal negative charge* of smoothly varying charge density. The radius of this electron cloud constitutes the atomic radius and is of the order of 10^{-10} m. Thus for an atom, in its ground state, the centre of gravity of its negative charge lies exactly at its nucleus which is taken to be a *point* positive charge. The dipole moment of an atom is, therefore, zero.

For a molecule the positive charge is supposed to be concentrated, at the nuclear points and the negative charge forming a cloud of smoothly varying density around the constituent nuclei. Depending upon the shape of this cloud and variation of charge density inside it, the molecule of dielectric can be classified into two types (*i*) *non-polar molecules* and (*ii*) *polar molecules.*

(*i*) Non-polar molecules. *The molecules in which the centre of gravity of positive charges exactly coincides with that negative charges and the net dipole moment is zero are called non-polar molecules.*

Consider an atom having atomic number Z. The positive charge on the atom is $+ Ze$. This charge is concentrated within the nucleus and can be supposed to act at the centre of the atom. The negative charge on the atom is also $-Ze$ as the atom is on the whole, neutral. In a spherically symmetric atom this charge is distributed uniformly over the volume of the atom *i.e.*, the electrons revolving round the nucleus behave as spherically symmetric cloud of negative charge with the centre of gravity of the cloud at the centre of the atom. Thus the centre of gravity of the positive as well as the negative charge will coincide.

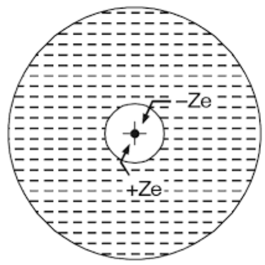

Fig. 18.1

The atom will , therefore, have zero dipole moment and hence will be *non-polar*.

These molecules do not have any permanent dipole moment because the dipole length is zero. Familiar example are H_2, O_2, N_2, CO_2, CCl_4 and CH_4.

(*ii*) Polar molecules. *The molecule in which the centre of gravity of positive charges does not coincide with that of the negative charges are known as polar molecules.* Such molecules, therefore, constitute a permanent dipole and have a permanent dipole moment. Familiar examples are H_2O, HCl and NH_3. These molecules consist of dissimilar atoms and their dipole moments are of the order of 10^{-29} coulomb meter which means a separation of 1 Å (10^{-10} m) between the centres of positive and negative charges of magnitude 1 electronic unit (1.6×10^{-19} C).

Non-Polar Molecule (*a*) *Polar Molecule* (*b*)

Fig. 18.2

Polar molecules in the absence of electric field. In the absence of any external electric field, in any bulk material, the molecular dipoles are randomly oriented and their electric dipole moments points in all possible directions cancelling out the effect of each other as shown in Fig 18.3. Hence, if we take a *macroscopic* view the material will not show any dipole moment and the *net resultant dipole moment per unit volume is zero.* It should be noted that though the individual molecules have dipole moments, the dipole moment of the bulk dielectric sample is zero.

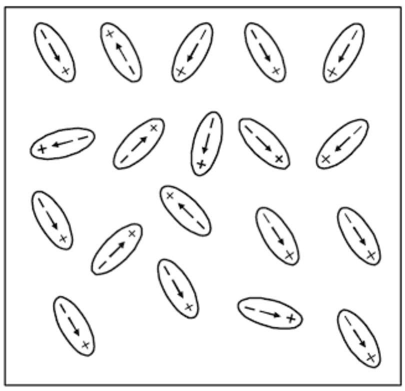

Fig. 18.3

Polar molecule in the presence of electric field. When an electric field is applied the electric dipole moments tend to align themselves in the direction of external field.

If $\vec{E}_0$ is the applied external field and $\vec{p}$ the molecular dipole moment, then

Torque experienced by each molecule $\vec{\tau} = \vec{p} \times \vec{E}_0$.

This torque tends to align the molecules in the direction of the applied electric field. This tendency of alignment is opposed by thermal agitation of the molecules. Thus the degree of alignment of the molecules depends upon

(*a*) The intensity of the applied electric field and

(*b*) The temperature of the dipole.

Greater the intensity of the electric field greater will be the separation between the centres of gravity of the positive and negative charges. The increase in dipole length will increase the dipole moment and thus the dipole moment per unit volume will increase. The net effect is a greater torque tending to align the molecules in the direction of the applied electric field.

Lower the temperature lesser will be the thermal vibrations of the molecules and hence it will be easier for the molecular dipoles to align in the direction of the field.

18.4 PHYSICAL MEANING OF POLARISATION OF A DIELECTRIC

As a result of alignment, the dielectric material when placed in an electric field acquires a net dipole moment. This dipole moment being due to the applied electric field is known as *induced dipole moment.*

When the molecules of a dielectric are aligned completely or partially in the direction of the electric field, the dielectric is said to be polarised. A polarised dielectric is shown in Fig. 18.4.

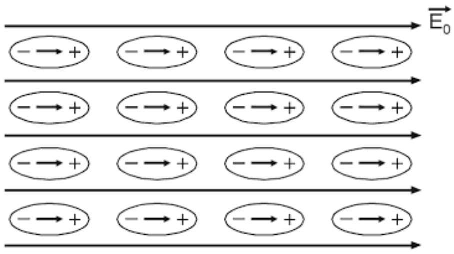

Fig. 18.4

18.5 EFFECT OF ELECTRIC FIELD ON NON-POLAR DIELECTRIC

Fig. 18.5 (*a*) shows a non-polar molecule. When such a non-polar molecule is placed in an electric field, the centre of the positive charge moves in the direction of the field and the centre of the negative charge in an opposite direction Fig. 18.5 (*b*). This separation of positive and negative charges continues till the force on either of them due to the external field is completely balanced

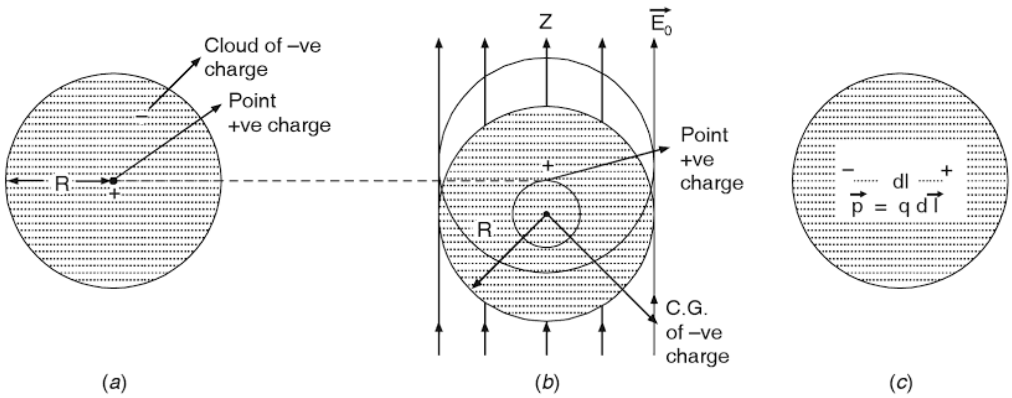

Fig. 18.5 : Induced dipole moment

by the internal force arising due to their relative displacements. The molecule develops a dipole moment known as *induced dipole* moment. Such a molecule is said to be *polarised* as shown in Fig. 18.5 (c).

The induced dipole moment *i.e.* polarisation in non-polar molecules lasts only for the time the electric field is applied. It disappears as soon as the electric field is removed.

Atomic dipole moment

If dl is the separation between the centres of positive and negative charges and q is the positive or negative charge on the atom, then

$$\text{Atomic dipole moment} \quad \vec{p} = q\, d\vec{l} \qquad \dots (i)$$

Thus atomic dipole moment is the product of the positive or negative charge on the atom and the distance between the centres of the positive and the negative charges in the direction of the applied electric field.

Atomic polarisability

The net induced electric dipole moment $\vec{P}$ for an atom is proportional to the strength of the applied electric field $\vec{E}_0$ and its direction is also parallel to that of $\vec{E}_0$. Thus, $\vec{P} \propto \vec{E}_0$

$$\therefore \qquad \vec{p} = \alpha \vec{E}_0$$

$$\text{or} \qquad \alpha = \frac{\vec{p}}{\vec{E}_0} \qquad \dots (ii)$$

α is the constant of proportionality and is known as the *atomic polarisability.*

Hence atomic polarisability is defined as the electric dipole moment induced in the atom by an electric field of unit strength.

Induced dipole moment $(\vec{p})$:

To calculate the value of induced atomic dipole moment, consider an atom of atomic number Z, then positive charge on the nucleus = + Ze.

As the atom is on whole neutral, the charge carried by all the electrons = – Ze.

It is assumed that the positive charge lies exactly at the centre of sphere of radius R and the negative charge is distributed uniformly in the form of spherical cloud so the centre of gravity of the negative charge also lies exactly at the centre of the sphere and coicides with the centre of gravity of the positive charge as shown in Fig. 18.5 (*a*). In such a case the atom has zero dipole moment.

An electric field $\vec{E}_0$ is now applied in the Z-direction. The positive charge being firmly fixed in the nucleus is supposed to continue in its original position. On the other hand it is assumed that the negatively charged spherical cloud is not distorted in shape by the external electric field but is only bodily displaced in the negative Z-direction as shown in Fig. 18.5 (*b*).

In the equilibrium position, let r be the separation between the positively charged nucleus and the centre of negative charge distribution, then

The magnitude of induced dipole moment of the atom $p = Ze\, r$ ... (*iii*)

Assuming that the volume charge density ρ in the cloud of negative charge has a uniform value and R is the radius of the atom

$$\rho = \frac{-Ze}{\frac{4}{3}\pi R^3}$$

and charge within a radius of sphere r

$$= \frac{4}{3}\pi r^3 \rho = \frac{-Ze}{\frac{4}{3}\pi R^3}\frac{4}{3}\pi r^3 = -Ze\frac{r^3}{R^3}$$

The field at the nucleus due to negative charge within the sphere of radius r

$$\vec{E} = -Ze\frac{r^3}{R^3}\cdot\frac{1}{4\pi\varepsilon_0 r^2}\hat{k} = -\frac{Zer}{4\pi\varepsilon_0 R^3}\hat{k}$$

i.e., the field acts in the negative Z-direction.

The total force on the nucleus due to negatively charged sphere

$$\vec{F} = Ze\vec{E} = -\frac{Zer}{4\pi\varepsilon_0 R^3}(Ze)\hat{k}$$

The charge contained in the sphere of radius r on the boudary of which the positive charge lies is the only charge exerting a net force on the positive charge. The charge contained in the rest of the sphere of the negative charge does not contribute to the force as the positively charged nucleus lies within the negative charge distribution.

The force on the nucleus due to the external electric field $\vec{E}_0$

$$\vec{F'} = ZeE_0\hat{k}$$

along positive Z-axis. As the positive charge is in equilibrium the net force on it is zero.

i.e., $$\vec{F} + \vec{F'} = 0$$

This is possible when

$$ZeE_0\hat{k} - \frac{Zer(Ze)}{4\pi\varepsilon_0 R^3}\hat{k} = 0.$$

or $$Zer = 4\pi\varepsilon_0 R^3 E_0 \qquad \ldots (iv)$$

Comparing (*iii*) and (*iv*), we have

Magnitude of induced atomic dipole moment $p = Zer = 4\pi\varepsilon_0 R^3 E_0$ ... (*v*)

Putting $4\pi\varepsilon_0 R^3 = \alpha$, the induced atomic polarisability.

Thus, *induced atomic polarisability* (α) *is defined as atomic dipole moment per unit polarising electric field.*

The direction of the dipole moment p is from the negative charge to positive charge *i.e.*, along the direction of the electric field.

Thus, in vector form, Eq. (*v*) is written as

$$\vec{E} = \vec{E}_0 + \vec{E}_p = \vec{E}_0 - \frac{\vec{P}}{\varepsilon_0} \quad \ldots (iii)$$

Hence *net electric field* $\vec{E}$ *within the dielectric is defined as the vector sum of the electric field* $\vec{E}_0$ *due to free charge density* σ_{free} *and polarisation field* $\vec{E}_p$ *due to bound charge density* σ_p.

Why electric field inside a dielectric decrease?

It is clear from equation (*iii*) that the electric field in the dielectric is reduced due to charges induced on its surface by the applied electric field $\vec{E}_0$ as the induced electric field due to dielectric polarisation $\vec{E}_p = -\frac{\vec{P}}{\varepsilon_0}$ acts in a direction opposite to that of $\vec{E}_0$.

Electric displacement vector *D*

Relation (*iii*) can be put as

$$\vec{E}_0 = \vec{E} + \frac{\vec{P}}{\varepsilon_0}$$

or

$$\varepsilon_0 \vec{E}_0 = \varepsilon_0 \vec{E} + \vec{P}$$

The quantity $\varepsilon_0 \vec{E}_0 = \varepsilon_0 \vec{E} + \vec{P}$ within the dielectric is given a special name, the *electric displacement vector* and is denoted by $\vec{D}$.

$$\therefore \quad \vec{D} = \varepsilon_0 \vec{E}_0 = \varepsilon_0 \vec{E} + \vec{P} \quad \ldots (iv)$$

Relation between three electric vector $\vec{E}, \vec{P}$ and $\vec{D}$

Equation (*iv*) gives the relation between $\vec{E}, \vec{P}$ and $\vec{D}$, the three electric vectors as

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

In free space or vacuum, there is no polarisation $(\vec{P} = 0)$

$$\therefore \quad \vec{D} = \varepsilon_0 \vec{E}$$

As all the vectors in relation (*iv*) act in the same direction we consider their magnitudes and get

$$D = \varepsilon_0 E + P$$

Units. The units of D are the same as those of P or $\varepsilon_0 E$ *i.e.*, Cm^{-2}.

18.12 CAPACITY OF PARALLEL PLATE CAPACITOR WITH DIELECTRIC

A parallel plate capacitor consists of two plates placed parallel to one another and separated by a small distance containing porcelain slab or some other dielectric medium. One of the plates is earthed and a charge is given to other plate as shown in Fig. 18.10

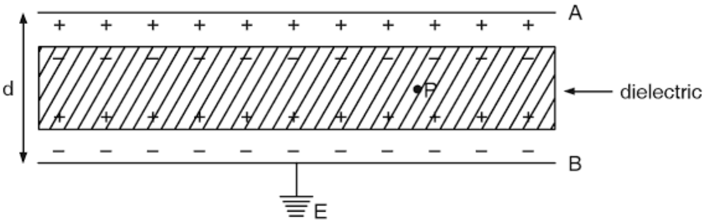

Fig. 18.10

Let A be the surface area of each plate, $+\sigma$ be the surface charge density on the plate A and $-\sigma$ that induced on the plate B, d be the distance between the two plates, then the electric field near a charged conducting plate in a dielectric

$$= \frac{\sigma}{2\varepsilon_0\,\varepsilon_r} = \frac{\sigma}{2k\,\varepsilon_0} \qquad (\because\ k = \varepsilon_r)$$

Therefore, force experienced by a unit positive charge at a point

(*i*) due to the upper plate $A = \dfrac{\sigma}{2k\,\varepsilon_0}$ (repulsion), and

(*ii*) due to the lower plate $B = \dfrac{\sigma}{2k\,\varepsilon_0}$ (attraction).

The two forces act in the same direction *i.e.* downward, the resultant force on a unit positive charge at P

$$F = \frac{\sigma}{2k\,\varepsilon_0} + \frac{\sigma}{2k\,\varepsilon_0}$$

$$= \frac{\sigma}{k\varepsilon_0} \qquad \text{... }(i)$$

Hence, the work done in moving a unit +ve charge from plate B to plate A against the electric force

$$= \frac{\sigma}{k\,\varepsilon_0}.d$$

assuming the field to be uniform.

$\therefore$ Potential difference between the two plates

$$V_A - V_B = \frac{\sigma}{k\,\varepsilon_0}d \qquad \text{... }(ii)$$

Charge on the plate A is $Q = A\sigma$

$\therefore$ Potential difference $V_A - V_B = \dfrac{\sigma}{\varepsilon_0}\left(d - t + \dfrac{t}{\varepsilon_r}\right)$

Hence capacitance

$$C = \frac{A\sigma}{\dfrac{\sigma}{\varepsilon_0}\left(d - t + \dfrac{t}{\varepsilon_r}\right)} = \frac{\varepsilon_0 A}{d - t + \dfrac{t}{\varepsilon_r}}$$

$$= \frac{\varepsilon_r\,\varepsilon_0\,A}{\varepsilon_r\,d - t\,(\varepsilon_r - 1)} \qquad \text{... } (i)$$

In terms of k,

$$C = \frac{\varepsilon_0 A}{d - t + \dfrac{t}{k}} = \frac{k\,\varepsilon_0\,A}{kd - t\,(k-1)} \qquad \text{... } (ii)$$

18.14 SIGNIFICANCE OF DIELECTRIC IN A PARALLEL PLATE CAPACITOR

On the introduction of dielectric slab of dielectric constant k between the plates of a capacitor, the electric field intensity at a point between the two plates falls from $\dfrac{\sigma}{\varepsilon_0}$ to $\dfrac{\sigma}{k\,\varepsilon_0}$ where σ is the charge per unit area on the charged plate. Thus the potential difference between the two plates decreases from $\dfrac{\sigma d}{\varepsilon_0}$ to $\dfrac{\sigma d}{k\,\varepsilon_0}$ where d is the distance between the two plates. As the capacitance $C = \dfrac{Q}{V}$, the capacitance increases from $\dfrac{A\sigma}{\dfrac{\sigma}{\varepsilon_0} d} = \dfrac{\varepsilon_0 A}{d}$ to $\dfrac{A\sigma}{\dfrac{\sigma}{k\,\varepsilon_0} d} = \dfrac{k\,\varepsilon_0\,A}{d}$ *i.e*, it becomes k times the capacitance before introduction of the dielectric slab. This is why presence of the dielectric in the capacitor enhances its capacity.

$$\therefore \qquad C_{\text{d}} = \varepsilon_r \,.\, C_a, \text{ also } Q_d = \varepsilon_r\, Q_a$$

where C_d is capacity of capacitor with dielectric and C_a is a capacity of capacitor with air.

As $\varepsilon_r > 1$, the capacitor will store more charge than it does without it.

However, there is no change in capacity is observed on the introduction of a thin metallic sheet between the plates of a parallel plate capacitor, as explained below:

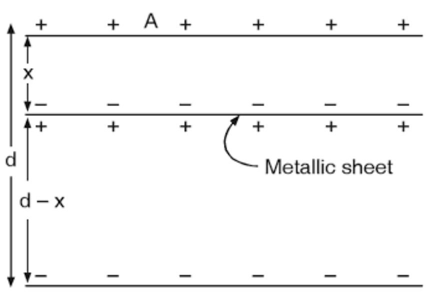

Fig. 18.12

Fig. 18.12 shows a very thin metallic sheet of area A introduced between the two plates of a parallel plate capacitor each of area A separated by a distance d. Let the distance of a thin metallic sheet from the upper plate be x and from the lower plate $(d - x)$.

A negative charge is induced on the upper face of the metal sheet and positive charge on the lower face thereby converting the single capacitor into two capacitors connected in series as the negative of the upper capacitor is connected to the positive of the lower capacitor.

The capacity of the upper capacitor $C_1 = \dfrac{\varepsilon_0 A}{x}$

The capacity of the lower capacitor $C_2 = \dfrac{\varepsilon_0 A}{d-x}$

If C is the combined capacity of C_1 and C_2 connected in series,

$$\frac{1}{C} = \frac{1}{C_1}+\frac{1}{C_2} = \frac{x}{\varepsilon_0 A}+\frac{d-x}{\varepsilon_0 A} = \frac{d}{\varepsilon_0 A}$$

or

$$C = \frac{\varepsilon_0 A}{d}$$

Before introducing the metal sheet, the capacity of the capacitor is also

$$C = \frac{\varepsilon_0 A}{d}$$

Therefore, no change in capacity on the introduction of a thin metallic sheet between the plates of parallel plate capacitor.

An *e.m.f.* source is connected to a capacitor and then disconnected. Let us examine what happens to the charge, the potential difference and the capacitance, if a dielectric slab is introduced between the plates of a parallel plate capacitor.

Charge. The charge on the plate of the capacitor *remains unchanged.*

Potential difference. On introducing the dielectric the net electric field between the plates of the capacitor decreases from E_0 of E where $E = E_0/\varepsilon r$, ε_r being the relative permittivity (or dielectric constant k) of the medium. The potential difference between the plates, therefore, decreases.

Capacitance. The capacitance of a parallel plate capacitor with air as dielectric is $C_a = \dfrac{\varepsilon_0 A}{d}$

and with the given dielectric is $C_d = \dfrac{\varepsilon_0 \varepsilon_r A}{d}$ *i.e.*, with the introduction of the dielectric slab the capacitance increase ε_r times.

18.15 CAPACITOR WITH TWO DIFFERENT DIELECTRICS FILLED

Surface density of charge on the plate P

$$= \sigma = +\frac{Q}{A}$$

Surface density of charge of the plate Q

$$= -\sigma = -\frac{Q}{A}$$

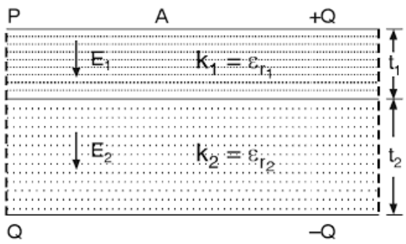

Fig. 18.13

Electric field in the slab of dielectric constant

$$k_1 = \text{relative permittivity } \varepsilon_{r_1}$$

$$E_1 = \frac{\sigma}{\varepsilon_0 \varepsilon_{r_1}} \text{ from } P \text{ to } Q$$

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Electric field in the slab of dielectric constant k_2 = relative permittivity ε_{r_2}

$$E_2 = \frac{\sigma}{\varepsilon_0 \varepsilon_{r_2}} \text{ from } P \text{ to } Q$$

(*ii*) Potential difference across the plates

Work done in taking a unit positive charge from plate Q to P

$$= \frac{\sigma}{\varepsilon_0 \varepsilon_{r_1}} t_1 + \frac{\sigma}{\varepsilon_0 \varepsilon_{r_2}} t_2$$

$\therefore$ Potential difference $V = V_P - V_Q = \dfrac{\sigma}{\varepsilon_0}\left[\dfrac{t_1}{\varepsilon_{r_1}} + \dfrac{t_2}{\varepsilon_{r_2}}\right]$

(*iii*) Capacitance of the capacitor

Capacitance of the capacitor

$$C = \frac{A\sigma}{\dfrac{\sigma}{\varepsilon_0}\left[\dfrac{t_1}{\varepsilon_{r_1}} + \dfrac{t_2}{\varepsilon_{r_2}}\right]} = \frac{\varepsilon_0 A}{\dfrac{t_1}{\varepsilon_{r_1}} + \dfrac{t_2}{\varepsilon_{r_2}}}$$

18.16 ENERGY STORED IN A CAPACITOR

Consider a capacitor of capacitance C having a potential difference V between the plates.

$\therefore$ Charge on the plate $Q = CV$

The charge on the positive plate = + Q and that on the negative plate = – Q. If we take an infinitesimal charge + dQ from the negative plate to the positive plate the work done against the potential difference

$$dW = V\,dQ = \frac{Q}{C} dQ$$

$\therefore$ Work done to charge the capacitor to a charge Q from the uncharged state

$$= W = \int dW = \int_O^Q \frac{Q}{C} = \frac{1}{2}\frac{Q^2}{C}$$

The energy $U = \dfrac{1}{2}\dfrac{Q^2}{C}$ is stored in the capacitor as potential energy of charge. This may also be put in the form

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} QV = \frac{1}{2} CV^2$$

This energy is stored in the capacitance in the form of electric field energy of the capacitor. It resides in the dielectric medium between the capacitor plates and is given by $\dfrac{1}{2}\varepsilon_0 \varepsilon_r E^2$ per unit volume and is discussed below.

$$\therefore \qquad \text{Energy density} = \frac{1}{2}\varepsilon_0\,\varepsilon_r\,E^2 = \frac{1}{2}\varepsilon_0\,\varepsilon_r\,\vec{E}.\vec{E} = \frac{1}{2}\vec{D}.\vec{E} \qquad ...(iii)$$

where $$\vec{D} = \varepsilon_r\,\varepsilon_0\,\vec{E} = \varepsilon\vec{E}$$

Thus when the value of the potential difference V between the plates of a capacitor (or E) is kept constant, the energy increase k (or ε_r) times when a medium of dielectric constant k (or relative permittivity ε_r) is introduce between the planes. This energy is drawn from the source (or battery) which keeps V constant.

(*ii*) Keeping charge Q constant

The energy stored in a parallel plate capacitor is given by

$$U = \frac{1}{2}\frac{Q^2}{C}$$

where Q is the charge on the capacitor and C its capacity.

If we have a parallel plate capacitor of capacitance C_a with air in between the plates each of area A separated by a distance d apart and introduce into it a dielectric of the same thickness d and dielectric constant k (or relative permittivity ε_r) then the new capacitance

$$C_d = k\,C_a = \varepsilon_r\,C_a$$

The energy stored in the capacitor with air in the gap between the capacitor plates

$$U_a = \frac{1}{2}\frac{Q^2}{C_a}$$

If the charge Q on the capacitor is kept constant and a plate having dielectric constant k (or relative permittivity ε_r) is introduced, the energy

$$U_d = \frac{1}{2}\frac{Q^2}{C_d} = \frac{1}{2}\frac{Q^2}{\varepsilon_r\,C_a} = \frac{1}{\varepsilon_r}U_a$$

i.e. the energy is reduced to $\frac{1}{\varepsilon_r}$, $\left(\text{or}\,\frac{1}{k}\right)$ of its original value.

The decrease in energy equal to $U_a\left(1-\frac{1}{k}\right)$ is due to the fact the a part of the energy is now used up in inducing the polarisation charge in the dielectric which in turn reduces the value of E or V.

18.18 CAPACITANCE OF A SPHERICAL CAPACITOR

Let A and B be the two concentric spheres of radii a and b respectively separated by air. Let the outer sphere be earthed and a positive charge q be given to the inner sphere. Consider a concentric shell of radius x and thickness dx as shown in Fig. 18.14.

Electric field at a point distant x from the centre O is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{x^2}$$

(*ii*) between the outer surface of B and the earth.

The capacity of this capacitor = The capacity of the spherical conductor of radius $b = 4\,\pi\,\varepsilon_0\,b$

$$\therefore \quad \text{Total capacity} \qquad C = 4\pi\varepsilon_0 \frac{ab}{b-a} + 4\pi\varepsilon_0\, b$$

$$= 4\pi\varepsilon_0 \left[\frac{ab}{b-a} + b\right] = 4\pi\varepsilon_0 \frac{b^2}{b-a} \qquad \ldots (iii)$$

18.19 CAPACITANCE OF A CYLINDRICAL CAPACITOR

A cylindrical capacitor consists of two coaxial cylinders, the space between the cylinders contains air or some other dielectric. Let A and B represent the sections of two coaxial cylinders of radius a and b respectively. Let the cylinder B be earthed and a charge λ per unit length be given to the cylinder A.

Consider a Point P within the two cylinders at a distance r from the axis, then magnitude of electric field intensity

$$E = \frac{1}{2\pi\varepsilon_0} \frac{\lambda}{r}$$

The field at every point is directed radially outward.

$\therefore$ The difference of potential between two cylinders

$$V_a - V_b = \int_b^a -\vec{E}.d\vec{r}$$

As the vector $\vec{E}$ and $d\vec{r}$ are both in the same direction.

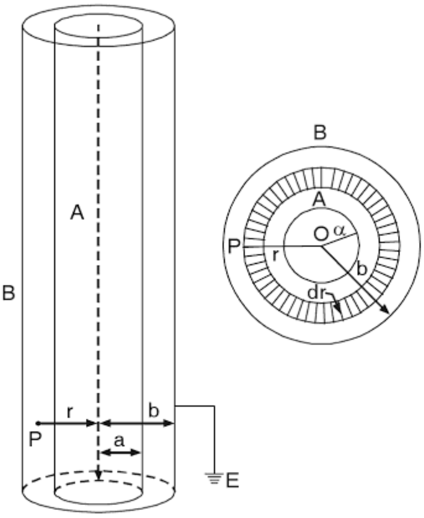

Fig. 18.16

$$V_a - V_b = -\int_b^a E\,dr = -\int_b^a \frac{\lambda}{2\pi\varepsilon_0} \frac{dr}{r}$$

$$= \frac{\lambda}{2\pi\varepsilon_0} [-\log_e r]_b^a = \frac{\lambda}{2\pi\varepsilon_0} \log_e \frac{b}{a}$$

Hence capacitance per unit length of the capacitor $C = \dfrac{\lambda}{\dfrac{\lambda}{2\pi\varepsilon_0} \log_e \dfrac{b}{a}} = \dfrac{2\pi\varepsilon_0}{\log_e \dfrac{b}{a}}$

The capacitance of a length l of the cylindrical capacitor $C_1 = \dfrac{2\pi\varepsilon_0\, l}{\log_e \dfrac{b}{a}}$

When the space between the two cylinders is filled with a medium of dielectric constant k or relative permittivity ε_r

$$E = \frac{1}{2\pi\varepsilon_0 \varepsilon_r}\frac{\lambda}{r}$$

and $$C_d = \frac{2\pi\varepsilon_0 \varepsilon_r}{\log_e \frac{b}{a}} \text{ Also } C_1 \text{ (with dielectric)} = \frac{2\pi\varepsilon_0 \varepsilon_r l}{\log_e \frac{b}{a}}$$

In terms of k, $$C_d = \frac{2\pi\varepsilon_0 k}{\log_e \frac{b}{d}} \text{ and } C_1 \text{ (with dielectric)} = \frac{2\pi\varepsilon_0 kl}{\log_e \frac{b}{a}}$$

18.20 MECHANISM OF POLARISATION

Since the polarisability α is defined in terms of dipole moment p, its magnitude is a measure of the extent to which electric dipoles are formed by ions, atoms and molecules. The electric dipoles are formed through a *variety of mechanism*, each one contributing to the value of α, hence α is regarded as total polarisability obtained as the sum of individual polarisabilities, each arising from a particular mechanism. The different polarisabilities are:

1. Electronic Polarisability

Centre of gravity of the electrons and protons do not coincide, resulting into induced dipole moment, *i.e.*,

$$p_e = \alpha_e E_{\text{int}}$$

P_e posseses all the characteristics of an assembly of dipoles produced by the displacement of electrons which have natural frequency equal to, or higher than, that of visible light. Therefore, α_e is called the optical or electronic polarisability.

2. Molecular Polarisability

Molecules having permanent dipole moment *i.e.*, polar molecules are affected in two possible ways:

(*a*) Atomic Polarisability : Firstly, the field may cause the atoms to be displaced altering the distance between them. The change in dipole moment of the molecule takes place. This is known as atomic polarisability, represented by α_a.

(*b*) Orientational or Dipolar Polarisability: Secondly, the molecule as a whole may rotate about its axis of symmetry, so that the dipoles align with the field. This is referred as orientational polarisation or dipole polarisation, contributing to orientinal or dipole polarisability, denoted by α_d.

3. Interfacial Polarisability

This mainly attributes to large number of defects in structure of crystals, *viz.* latice vacancies, impurity centres dislocations etc. and the polarisability is denoted by α_i.

Thus, the total polarisability α will be

$$\alpha = \alpha_e + \alpha_a + \alpha_d + \alpha_i \quad \text{... }(i)$$

18.21 PHYSICAL SIGNIFICANCE OF ELECTRIC DISPLACEMENT VECTOR

The electric field vector $\vec{E}$ depends not only on the magnitude and position of charge but also upon the permittivity of the medium. However electric displacement $\vec{D}$ is an electrical quantity which is independent of the medium. This concept can be understood by considering the Faraday's experiment. A sphere carrying a charge Q was placed inside without touching another hollow sphere. The outer sphere was earthed, for a moment and the inner charged sphere was removed. The charge on outer sphere was found to be equal but opposite to that of the inner sphere. The same result is obtained if spheres of various sizes are chosen or if the space between spheres is filled with different dielectric media. This shows that there was some kind of electric displacement which only depends upon the magnitude of charge Q on inner sphere and is does not depend upon the dielectric media. The S.I. unit of this displacement ψ is equal to the charge. Thus

$$\psi = Q$$

The displacement density D at a point is the displacement per unit area at the point. Thus,

$$D = \frac{\psi}{4\pi r^2}$$

$4\pi r^2$ being the surface area of the sphere of radius r, with the charge at its centre. Now since $\psi = Q$,

$$D = \frac{Q}{4\pi r^2}$$

Multiplying numerator and Denominator by $\in$, we get

$$D = \frac{\in Q}{4\pi \in r^2} = \in E$$

$$\therefore \qquad D = \in E$$

In vector from, $$\vec{D} = \in \vec{E}$$

18.22 CLAUSSIUS-MOSSOTTI EQUATION

The electric field inside a dielectric is somewhat less than the externally applied field. It is, therefore necessary to determine the local field or internal field at a point inside the dielectric after it gets polarized. The point where electric field is to be calculated is considered to be enclosed within an imaginary spherical cavity as shown in figure 18.17. The radius of spherical cavity is large as compared to the interatomic spacing.

The net local electric field at the point will be given by

$$E_{\text{loc}} = E_0 + E' + E_1 + E_2 \qquad \dots (i)$$

where,

$E_0 \rightarrow$ The field due to externally applied charges.

$E' \rightarrow$ Field due to the induced charges developed on the faces of dielectric.

$E_1 \rightarrow$ Field due to opposite charges on the inner surface of spherical cavity, and

$E_2 \rightarrow$ The field at the centre of cavity due to atomic dipoles inside the cavity.

The effect of E_0 and E' is to produce to net field E, so that

$$E_{\text{loc}} = E + E_1 + E_2$$

The field E_2 at the centre of spherical cavity can be shown to be zero for cubical or spherical symmetry. Hence

$$E_{\text{loc}} = E + E_1 \qquad \dots (ii)$$

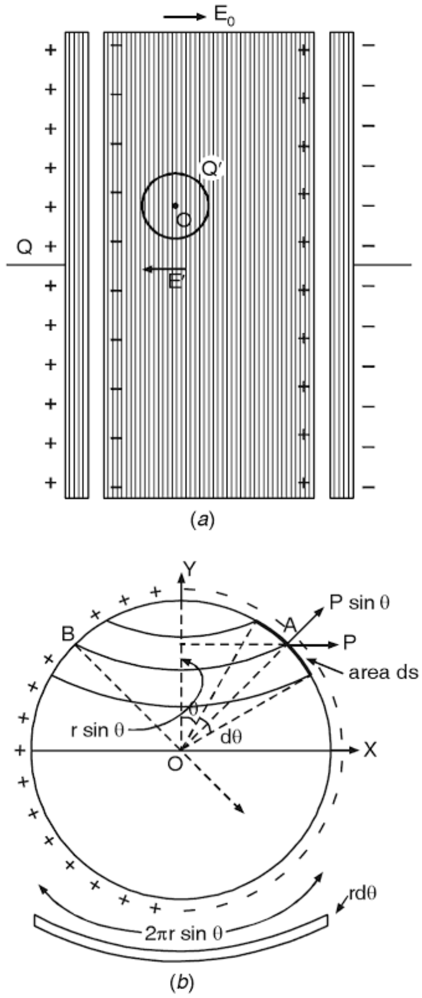

Fig. 18.17

The contribution E_1 due to charges on inner surface of the spherical cavity can be calculated as under:-

Consider an elementary area ds at the position A on the inner surface of spherical cavity as shown in figure 18.17 (b) Let θ be the angle between OA and the Y-axis.

The polarization vector $\vec{P}$ has the direction from negative to positive induced charge. The component of P normal to the area ds is $P \sin\theta$. By definition, the polarization is the charge per unit area so that $P \sin\theta$ is charge per unit area near ds. Thus, the total charge on area ds is $P \sin\theta\, ds$. The force on unit positive charge placed at the centre of cavity due to charge to charge on ds is (by Coulomb's law)

$$= \frac{P \sin\theta\, ds}{4\pi\epsilon_0 r^2} \text{ (along } OA)$$

Component of this force along the direction parallel to external field is

$$\frac{P \sin\theta\, ds}{4\pi \epsilon_0 r^2} . \sin\theta$$

For every area ds on the right side of the cavity there is another area ds carrying equal positive charge (at B is figure) on the left side of the cavity. The vertical components of forces at O due to charges on area ds at A and B are equal and opposite and therefore cancel each other. Their horizontal components, however, add together. The net force on unit positive charge at O due to the charge on a strip of with $rd\theta$ and length $2\pi r$ $\sin\theta$ (circumference of the strip) is

$$\Delta F = \frac{P\sin\theta}{4\pi \epsilon_0 r^2} . \sin\theta \times (\text{area of the strip})$$

$$= \frac{P\sin^2\theta}{4\pi \epsilon_0 r^2} \times (2\pi r \sin\theta \times r d\theta)$$

$$= \frac{P}{2\epsilon_0} \sin^3\theta\, d\theta$$

The net force due to the complete spherical cavity is

$$E = \int_0^{\pi} \frac{P}{2\epsilon_0} - \sin^3\theta\, d\theta$$

Let $\quad \cos\theta = x$, hence $\sin\theta\, d\theta = -\,dx$

when $\quad \theta = 0, x = +1$ and $\theta = \pi, x = -1$

$$\therefore \quad E_1 = -\frac{P}{2\epsilon_0}\int_{+1}^{-1} x^2 dx = \frac{P}{2\epsilon_0}\int_{-1}^{+1} x^2 dx = \frac{P}{2\epsilon_0}\left[\frac{x^3}{3}\right]_{-1}^{+1}$$

$$\therefore \quad E_1 = \frac{P}{3\epsilon_0} \qquad \text{... }(iii)$$

From equation (ii), the local field at O is given by

$$E_{loc} = E + E_1 = E + \frac{P}{3\epsilon_0} \qquad \text{... }(iv)$$

The field E_{loc} is called **Lorentz field**. Every dipole has a dipole moment $P = \alpha_e E_{loc}$. where α_e is electronic polarizability. If there and n dipoles per unit volume, the dipole moment per unit volume or polarization is

$$P = np$$

$$= n\,\alpha_e E_{loc} \qquad \text{... }(v)$$

$$= n\alpha_e\left[E + \frac{P}{3\epsilon_0}\right] \qquad \text{... }(vi)$$

But $\quad D = \epsilon_0 E + P$

$$\epsilon E = \epsilon_0 E + P$$

$$\varepsilon_0 \cdot \varepsilon_r E = \epsilon_0 E + P \qquad (\because D = \epsilon E)$$

$$\therefore \quad P = \epsilon_0 \epsilon_r E - \epsilon_0 E \qquad (\because \epsilon = \epsilon_0 . \epsilon_r)$$

From equation (*i*) and (*ii*), we get,

$$\frac{4}{3}\pi a^3 n = \frac{\epsilon_r - 1}{\epsilon_r + 2} \qquad ...(iii)$$

where $\frac{4}{3}\pi a^3$ is the volume of each dipole (molecule). Thus, equation (*iii*) can be used to calculate molecular volume if dielectric constant ϵ_r and n are known, which agrees well with the experimental value.

18.24 LIMITATION OF CLAUSSIUS-MOSSOTTI EQUATION

The Claussius–Mossotti equation

$$\frac{n\alpha_e}{3\epsilon_0} = \left[\frac{\epsilon_r - 1}{\epsilon_r + 2}\right]$$

is derived by taking $E_2 = 0$ (see eq. (*ii*) of article 18.22), in accordance with the following assumptions:

(1) Polarisation is considered as proportional to the field, it means the polarisation of the molecules by elastic displacement only.

(2) Absence of short-range interaction.

(3) Isotropy of the polarisability of the molecules.

All these conditions are satisfied with neutral molecules having no constant dipoles i.e. non-polar (molecules of symmetrical structure). Thus, the equation is applicable to neutral liquids and specially to gases in which the molecules are far apart.

18.25 BOUNDARY CONDITIONS FOR DIELECTRIC MATERIALS

1. Interface between two homogeneous dielectrics

Consider the interface between two homogeneous dielectric media having primittivities ϵ_1 and ϵ_2 in region 1 and 2 as shown in Fig. 18.18. Let us examine, how the electric field changes at the boundary between two different media. The electric field change that occurs in going one medium to another is determined by applying basic concepts in electrostatics, namely:

(*i*) the first is Gauss's law *i.e.*, $\oiint \vec{D}.d\vec{s} = Q$, and

(*ii*) the second is that an electrostatic field is a conservative field *i.e.*, no work is done in transporting a charge around a closed path in an electrostatic field, *i.e.*,

$$\oint \vec{E}.d\vec{l} = 0$$

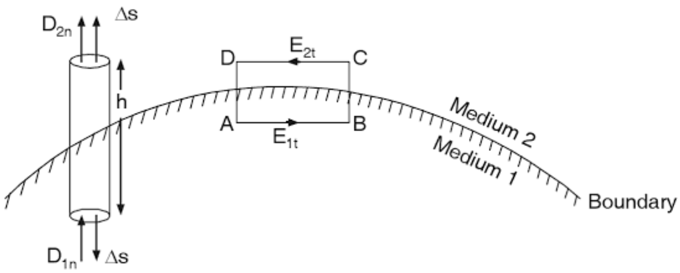

Fig. 18.18

1. Variable air condensers : This is popularity known as gang condensers used in radio and T.V. receivers for tuning.

2. Condensers with solid dielectric : The solid dielectrics commonly used are:

(*a*) Mica – these are known as mica condensers.

(*b*) Paper – two thin strips of aluminium or tin foils insulated by paper and rolling it into a bundle, safely used up to 500 volts.

(*c*) Ceramic – for large capacitance using ceramic as dielectric is used.

3. Electrolytic condenser : It is made by putting electrolyte between two aluminum plates.

SOLVED EXAMPLES

Example 18.1 Calculate the induced dipole moment per unit volume of He gas if placed in a field of 6000 volts/cm. The atomic polarisability of He = 0.18 × 10^{-40} farad m^2 and density of He is 2.6 × 10^{25} atoms per m^3. Also calculate the separation between the centres of positive and negative charges.

Solution. Electric field $\vec{E}$ = 600 volts/cm = 6 × 10^5 volts/m

Atomic polarisability of *He*, α = 0.18 × 10^{-40} Farad m^2

Number of atoms per m^3 N = 2.6 × 10^{25}

Dipole moment of *He* atom $\vec{p} = \alpha\vec{E} = 0.18 \times 10^{-40} \times 6 \times 10^5 = 1.08 \times 10^{-35}$ C- m

Induced dipole moment per unit volume $\vec{P} = N\vec{p} = 2.6 \times 10^{25} \times 1.08 \times 10^{-35}$

$= 2.81 \times 10^{-10}$ C/m^2

Charge on *He* nucleus $q = 2e = 2 \times 1.6 \times 10^{-19}$ C

$\therefore \qquad \vec{p} = qd\vec{l}$

Separation between the positive and negative charges of the atom

$$d\vec{l} = \frac{\vec{p}}{q} = \frac{1.08 \times 10^{-40}}{2 \times 1.6 \times 10^{-19}} = \mathbf{3.37 \times 10^{-17}\ m}$$

Example 18.2 A condenser with two horizontal metal plates separated by a distance of 4 mm is given potential of 9.8 V.A particle of mass 0.01 g and charge (– *q*) is rest at a point between the plates. Find the value of charge *q*. (*Nag. U.* 2001)

Solution. As the charged particles is at rest between the horizontal condenser plates, its downward weight is balanced by the upward electric force

or $$mg = qE$$

Now $$E = \frac{V}{d} = \frac{9.8}{4 \times 10^{-3}} = 2.45 \times 10^3\ \text{Vm}^{-1}(\text{NC}^{-1})$$

and $$mg = 0.01 \times 10^{-3} \times 9.8 = 9.8 \times 10^{-5}\ \text{N}$$

$\therefore$ $$q \times 2.45 \times 10^3 = 9.8 \times 10^{-5}$$

or $$q = \frac{9.8 \times 10^{-5}}{2.45 \times 10^3} = 4 \times 10^{-8} C = 0.04 \times 10^{-6} C$$

$$= \mathbf{0.04\ \mu C.}$$

Example 18.3 Dielectric constant of gas at N.T.P is 1.00074. Calculate dipole moment of each atom of the gas when it is held in an external field of 3×10^4 Vm^{-1}.

Solution. Electric field $E = 3 \times 10^4$ volt/m

Dielectric constant = relative permittivity

$\varepsilon_r = 1.000074$

Now $\varepsilon_r = 1 + \chi_e$ where χ_e is the electric susceptibility.

$\therefore$ $\chi_e = \varepsilon_r - 1 = 1.000074 - 1 = 0.000074$

If P is the induced dipole moment per unit volume, then $\chi_e = \dfrac{P}{\varepsilon_0 E}$

or $P = \chi_e \varepsilon_0 E$

where $\varepsilon_0 = 8.85 \times 10^{-12} \times C^2 N^{-1} m^{-2}$

$\therefore$ $P = 0.000074 \times 8.85 \times 10^{-12} \times 3 \times 10^4$

$= 19.65 \times 10^{-12}$ Cm

Number of gas atoms per gm atom $= 6.06 \times 10^{23}$

Volume per gm atom = 22.4 litre = 22.4×10^{-3} cubic metre.

$\therefore$ Number of gas atoms per cubic metre

$$N = \frac{6.06\times10^{23}}{22.4\times10^{-3}} = 2.7\times10^{25}$$

$\therefore$ Dipole moment per atom $p = \dfrac{P}{N}$

or $$p = \frac{19.65\times10^{-12}}{2.7\times10^{25}} = \mathbf{7.28\times10^{-37}\ \text{coulomb metre}}$$

Example 18.4 A potential difference of 200 volts is applied across the two plates of a parallel plate capacitor. The area of each plate is 100 π cm^2 and separation between the plates is 1 mm. The space between the plates is filled with a mica sheet having $k = 6$. Calculate (*i*) Charge on each plate and (*ii*) Electric field intensity within the sheets.

Solution. Area of capacitor plates $A = 100\,\pi$ cm^2

$= 100\,\pi \times 10^{-4} = \pi \times 10^{-2}$ m^2

Separation between the plates $d = 1$ mm $= 10^{-3}$ m

Dielectric constant of the medium k = relative permittivity $\varepsilon_r = 6$

Capacitance $$C = \frac{\varepsilon_0 \varepsilon_r A}{d} = \frac{\varepsilon_0\, 6\pi\times10^{-2}}{10^{-3}} = 60\,\pi\,\varepsilon_0 \text{ farad}$$

Now $$\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9$$

$\therefore$ $$\pi\varepsilon_0 = \frac{1}{4\times9\times10^9}$$

Hence capacitance $$C = 60\,\pi\varepsilon_0 = \frac{60}{4\times9\times10^9} = \frac{1}{6}\times10^{-8} = 1.67\times10^{-9} \text{ farad}$$

Potential difference between the plates = 200 volts

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Example 18.7 Two parallel plates having equal and opposite charges are separted by a slab 2 cm thick and having dielectric constant 3. If the electric field strength inside is 10^6 Vm^{-1} calculate the polarisation and displacement of vector. (*K.U.* 2000)

Solution. Electric field strength between the plates $E = 10^6$ Vm^{-1} = 10^6 NC^{-1}

$$\text{Dielectric constant } k = 3$$

$$\text{Permittivity of free space } \varepsilon_0 = 8.85 \times 10^{-12}\ C^2\ \text{N}^{-1}\ \text{m}^{-2}$$

$\therefore$ Displacement vector $D = k\varepsilon_0 E = 3 \times 8.85 \times 10^{-12} \times 10^6 = 26.55 \times 10^{-6}$ Cm^{-2}

Again $D = \varepsilon_0 E + P$

or $P = \text{D} - \varepsilon_0 E = 3\,\varepsilon_0 E - \varepsilon_0 E = 2\,\varepsilon_0 E = 2 \times 8.85 \times 10^{-12} \times 10^6$

$= \mathbf{17.70 \times 10^{-6}\ Cm^{-2}}$

Example 18.8 Three charges $q_1 = 3\mu C$, $q_2 = -6\mu C$, $q_3 = 7\mu C$ are located at points (–1, 1,0) (1, 2, 0) and (3, –1, 0) respectively. If the distances are measured in metres find the total configurational energy of the system.

Solution. Distance between q_1 and q_2

$$= \sqrt{(-1-1)^2 + (1-2)^2}$$

$$\therefore\quad |\vec{r_1} - \vec{r_2}| = \sqrt{5}$$

Distance between q_2 and q_3

$$= \sqrt{\{(1-3)^2 + (2-(-1))^2\}}$$

$$\therefore\quad |\vec{r_2} - \vec{r_3}| = \sqrt{13}$$

Distance between q_3 and q_1

$$= \sqrt{(3+1)^2 + (-1-1)^2}$$

$$\therefore\quad |\vec{r_3} - \vec{r_1}| = \sqrt{20}$$

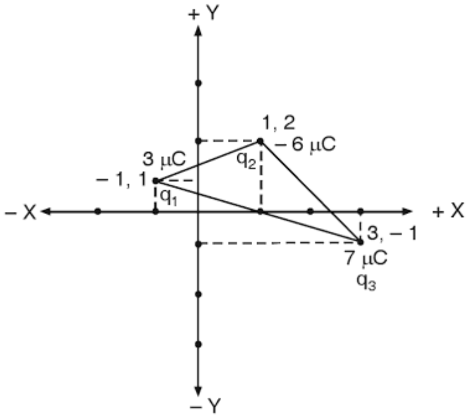


Fig. 18.19

Total energy of the configuration

$$U = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_2}{|\vec{r_1} - \vec{r_2}|} + \frac{q_2 q_3}{|\vec{r_2} - \vec{r_3}|} + \frac{q_3 q_1}{|\vec{r_3} - \vec{r_1}|}\right]$$

$$= 9 \times 10^9 \times 10^{-12}\left[\frac{-18}{\sqrt{5}} - \frac{42}{\sqrt{13}} + \frac{21}{\sqrt{20}}\right]$$

$$= \mathbf{-0.135\ J}$$

Example 18.9 Two protons in thorium nucleus are 3 × 10^{-15} m apart. What is their mutual potential energy?

Solution. The mutual potential energy of a system of two charges q_1 and q_2 a distance r apart is given by

$$U_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q}{r}$$

Now $q_1 = q_2 = 1.6 \times 10^{-19}$ C; $\frac{1}{4\pi\varepsilon_0} = 9\times10^9$; $r = 3\times10^{-15}$ m

$$\therefore \quad \text{Potential energy} = \frac{9\times10^9 \times 1.6\times10^{-19} \times 1.6\times10^{-19}}{3\times10^{-5}}$$

$$= \mathbf{7.68 \times 10^{-14}\ J.}$$

Example 18.10 A capacitor consists of two metallic discs each 1 metre in diameter placed parallel to each other at a distance of 4 mm. The potential between the plates is 10, 000 volts. Calculate the energy stored by the capacitor.

$$\boldsymbol{\varepsilon_0 = 8.85 \times 10^{-12}\ C^2/N\text{–}m^2}$$

Solution. Area of each disc of the capacitor $A = \frac{\pi}{4}D^2 = \frac{\pi}{4}\times1^2 = \frac{\pi}{4}\text{m}^2$

Distance between the disc $d = 4$ mm $= 4 \times 10^{-3}$ m

Capacitance of the capacitor $C = \frac{\varepsilon_0 A}{d} = \frac{8.85\times10^{-12}}{4\times10^{-3}}\times\frac{\pi}{4} = \frac{8.85\,\pi}{16}\times10^{-9}$ Farad

Voltage applied $V = 10,\,000\ V = 10^4\ V$

Energy stored in the capacitor $= \frac{1}{2}CV^2 = \frac{1}{2}\times\frac{8.85\pi}{16}\times10^{-9}\times10^8 = \mathbf{0.87J}$

Example 18.11 A 10 μF capacitor to 100 *V* is connected in parallel to an uncharged capacitor. After making the connection the common voltage on the capacitor is found to be 30 V. What is the capacity of the second capacitor? Also calculate net loss of energy during the process of connection.

Solution. Capacity of first capacitor $C_1 = 10\ \mu\text{F} = 10 \times 10^{-6}$ F

Potential $V_1 = 100$ V

Let capacity of second capacitor $= C_2$

Common potential after connection $V_2 = 30$ V

Now, as the total charge is conserved

$$C_1V_1 = (C_1 + C_2)\,V_2$$

or

$$C_2 = \frac{C_1V_1}{V_2} - C_1 = \frac{10\times10^{-6}\times100}{30} - 10\times10^{-6}$$

$$= \frac{70}{3}\times10^{-6}\ \text{F} = 23.3\ \mu\text{F}$$

Energy before connection $= \frac{1}{2}C_1V_1^2 = \frac{1}{2}\times10\times10^{-6}\times100\times100$

$= 50 \times 10^{-3}$ J

Energy after connection $= \frac{1}{2}C_1V_2^2 + \frac{1}{2}C_2V_2^2$

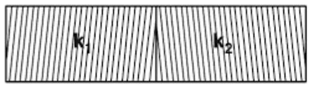
$$= 402 \times 10^{-12} \text{ F}$$
$$= \mathbf{402\ \mu\mu\ F}$$

Example 18.14. A parallel plate capacitor is filled with two dielectric of same dimensions but different constant k_1 and k_2 respectively. Calculate its capacitance.

(*Nagpur Uni*. *s*/2009, *s*/2007)

Solution. A parallel plate capacitor is filled with two dielectrics as shown in Fig. 18.22 (*a*). Here, two capacitors are connected parallel, each of area (surface) $\frac{A}{2}$ and thickness same *i.e.* d.

$\therefore$ Total capacitance, $\quad C_p = C_1 + C_2$



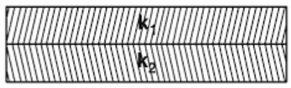
(*a*)

$$= \frac{k_1 \epsilon_0 \frac{A}{2}}{d} + \frac{k_2\ \epsilon_0\ \frac{A}{2}}{d}$$

$$= \frac{\epsilon_0\ A}{d}\left(\frac{k_1 + k_2}{2}\right) \qquad \text{... } (i)$$

The arrangement of Fig. 18.22 (*b*) can be regarded as two capacitors in series. Each capacitor will be of same area A but width $\frac{d}{2}$. Therefore, the total capacitance will be

$$\frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2}$$



(*b*)

Fig. 18.22

$$= \frac{\frac{d}{2}}{k_1\ \epsilon_0\ A} + \frac{\frac{d}{2}}{k_2\ \epsilon_0\ A} = \frac{d}{2}\left[\frac{1}{k_1\ \epsilon_0\ A} + \frac{1}{k_2\ \epsilon_0\ A}\right]$$

$$\therefore \qquad C_s = \frac{2 \epsilon_0\ A}{d}\left(\frac{k_1\, k_2}{k_1 + k_2}\right) \qquad \text{... } (ii)$$

Example 18.15. Three parallel capacitors and three series capacitor are connected in parallel. If the capacity of each capacitor is '*C*', find the capacitance of their combination.

(*Nagpur Uni*. 2006)

Solution. The equivalent capacitance of three parallel capacitors is

$$C_p = C + C + C = 3C \qquad \text{... } (i)$$

The equivalent capacitance of three series capacitors is

$$\frac{1}{C_s} = \frac{1}{C} + \frac{1}{C} + \frac{1}{C} = \frac{3}{C}$$

or

$$C'_s = \frac{C}{3} \qquad \text{... } (ii)$$

Now Eq (*i*) and Eq. (*ii*) are connected in parallel combination.

Therefore, the equivalent capacitance will be

$$C'_p = 3C + \frac{3}{C} = 3\left(C + \frac{1}{C}\right).$$

Example 18.16. The distance between the plates of parallel plate capacitor of capacitance 'C' is 'd'. A slab of dielectric constant 'k' and thickness $\frac{'3d'}{4}$ is inserted between the plates. What is the capacitance of the system.

Solution. Here, the two capacitors are of same surface area A but different with *i.e.* $\frac{d}{4}$ and $\frac{3d}{4}$, filled with air and dielectric with dielectric constant K respectively. Therefore, their capacities will be :

(*i*) Filled with air

$$C_1 = \frac{\varepsilon_0 A}{\frac{d}{4}} = \frac{4\varepsilon_0 A}{d}$$

(*ii*) Filled with dielectric

$$C_2 = \frac{k\varepsilon_0 A}{3\frac{d}{4}} = \frac{4k\varepsilon_0 A}{3d}$$

Both C_1 and C_2 are connected in series,

$$\therefore \qquad \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

or

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{\frac{4\varepsilon_0 A}{d} \times \frac{4K\varepsilon_0 A}{3d}}{\frac{4\varepsilon_0 A}{d} + \frac{4K\varepsilon_0 A}{3d}}$$

or

$$C = \frac{4K\varepsilon_0 A}{(3+k)d}.$$

EXERCISE CH. 18

LONG QUESTIONS

1. (***a***) Explain the terms conductors and insulators.
   (***b***) What is dielectric substance? Give examples. Discuss the importance of dielectrics. (*Nagpur Uni.* 2008; *G.N.D.U.*, 2000)
2. What are polar and non-polar molecules? Discuss the effect of electric fields on polar dielectrics. What is meant by polarisation of a dielectric? (G.N.D.U. 2004; *K.U.* 2001, 2000; *Kerala U.* 2001; *Nagpur U.* 2001, 2007, 2008; *P.U.* 2001; *Gauhati U.* 2000; *Meerut. U.* 2002; *M.D.U.* 20002; *Purvanchal U.* 2006)
3. (***a***) What happens when a non-polar molecule is placed in an electric field? Define atomic dipole moment and atomic polarisability. What are its dimensions? Give its S.I. Units.
   (***b***) How long does polarisation of non-polar molecules last?
   (*P.U.* 2001, 2000; *H.P.U.* 2001; *G.N.D.U.* 2001; *Nagpur U.* 2008, 2007; *Kerala U.* 2001)
4. What is atomic polarisability? Find a relation between dipole moment and atomic polarisability.

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9. Explain dielectric polarisation. Define polarisation vector $\vec{P}$ with its appropriate expressions. (*Purvanchal U.* 2005, *Nagpur Uni.* 2004)

10. Explain the electric displacement vector $\vec{D}$. (*Gauhati U.* 2007)

11. Write Claussius-Mossotti equation and give its molecular interpretation. (*Nagpur Uni.* 2004)

12. Define polarisation $\vec{P}$ and dielctric susceptibility χ_e and establish the relation $\vec{P} = \varepsilon_0 \chi_e \vec{E} = \varepsilon_0 (k-1)\vec{E}$. (*Purvanchal U.* 2007, 2005)

13. Find an expression for the energy stored in a capacitor. In what form the energy is stored. (*Nagpur U.* 2009; *Pbi. U.* 2000)

14. Obtain the boundary condition satisfied by E and D at the interface between two homogeneous dielectrics. (*Nagpur Uni.* 2008, 2006, *D.A.U. Agra* 2003)

15. What is the effective capacitance, when capacitor C_1 and C_2 are connected in (*i*) series and (*ii*) in parallel? (*Nagpur uni.* 2004, 2006)

Hints. (*i*) In series, $\frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2}$ (*ii*) In parallel, $C_p = C_1 + C_2$

16. Show that the electrostatic energy per unit volume in a dielectric is $\frac{1}{2}\vec{D}.\vec{E}$ where symbols have their usual meanings. (*H.P.U.* 2003, 2002, *P.U.* 2002)

17. An isolated air gap parallel plate capacitor C_a has a charge Q. A dielectric having constant K is inserted between the plates in such a way that charge on capacitor is not changed. Find the change in energy stored in the capacitor. (*G.N.D.U.* 2003, *P.U.* 2003).

18. Discuss different types of condensers.

19. What is dielectric strength of a material? In what units it is measured?

20. What is dielectric breakdown? Give its mechanism.

21. Show that the Lorentz field in a dielectric materials in equal to $\left[E + \frac{p}{3\varepsilon_0}\right]$. (*Nagpur U.* 2008)

22. What do you mean by molecular polarisability? Deduce an expression for the electric field on a molecule within a dielectric .Hence obtain Claussius-Mossotti relation. How can we find diameter of an atom with its help (*Meerut U.* 2005, 2003)

23. What do you mean by electronic polarisability? Derive the relation for electronic polarisability. (*Purvanchal U.* 2005)

24. Distinguish between electronic, orientational and ionic polarisability. (*Agra U.* 2004)

25. What is parallel plate capacitor? Calculate its capacitance when the space between the two plates is filled up with air. How is this expression modified when the space is partially filled with a slab of thickness t and dielectric constant k? (*Gauhati U.* 2007)

26. Explain the terms:

(i) Electric polarisation

(ii) Polarisability

(iii) Polarisation vector. (*Gauhati U.* 2007)

19 ELECTRIC CURRENTS

STEADY CURRENT

INTRODUCTION

When the charges are static i.e. stationary, we get an electrostatic field in their vicinity and a static potential is developed. However, if there is a potential difference, the charges start to move. This constitutes an electric current, the magnitude of which depends upon the resistance (R) offered to its flow, given by Ohm's law. In this chapter, we will study the difference between electric current (I) and current density ($\vec{J}$) and the equation of continuity. Applications of Kirchhoff's laws helps us to analyse the electrical network, whatsoever complicated it is. At the end of the chapter, we will study the growth and decay of current in an electrical circuit containing the electrical elements i.e. resistance (R), inductance (L) and capacitance (C).

19.1 DRIFT VELOCITY

Consider a conductor AB the two ends of which are connected to a battery. A *steady* electric field $\vec{E}$ is thus established in the conductor in the direction A to B. Before the field is applied the free electrons in the conductor move in all random directions like the molecules of a gas confined to a vessel, but as soon as the electric field is established the free electrons at the end B experience a force $-\vec{E}e$ from B to A in a direction opposite to that of the field $\vec{E}$. The electrons are, therefore, accelerated in this direction. In the process, the electrons collide with each other and with the positive ions in the conductor. At each collision the momentum gained in the direction of the force acting on the charge carrier due to the electric field is lost and the electron is accelerated afresh after each collision. Thus, due to collisions, a *backward* force acts on the electrons. The force is known as *collision drag*. The overall effect of these collisions is that the electrons *slowly drift* with a *constant average drift velocity* in the direction of $-\vec{E}$. The organised transport of charge by the electrons over distances very large as compared to atomic distances constitutes an electric current.

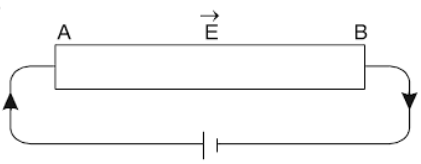

Fig. 19.1

The *vector average velocity with which the charge carriers move under the effect of the electric field is known as drift velocity, the average being macroscopic i.e., taken over a volume large as compared to molecular volume.*

Consider the case of free electrons as charge carriers in a conductor and let a number n_1 have a velocity v_1, n_2 a velocity v_2 ..., n_i a velocity v_i and so on, then

Average velocity $$v = \frac{1}{N}\sum_{i=1}^{i=N} n_i v_i$$

where $N = n_1 + n_2 + \ldots\ n_i + \ldots =$ total number of electrons.

19.2 CURRENT AND CURRENT DENSITY

Current. *The conventional current is defined as the rate of flow of positive charge through any cross-sectional area of a conductor.*

If a net charge q passes through any cross-section of a conductor in a time t, then

Current $$\text{I} = \frac{q}{t}$$

When q is in *coulombs* and t in *seconds*, the current I is is measured in *amperes.*

If the rate of flow of charge is not constant, the current varies with time and the instantaneous value represented as i is given by

$$i = \frac{dq}{dt}$$

The current I is characteristic of a particular conductor and is a *macroscopic quantity.*

The direction of conventional current is taken as the direction of flow of positive charge. Current is a *scalar* quantity.

Units. The S.I. unit of current is an *ampere.*

A current of one ampere is said to flow through a conductor if one coulomb of charge passes through any cross-sectional area of the conductor in 1 *second.*

The C.G.S. unit of current is a stat-ampere.

As $$1 \text{ Coulomb} = 3 \times 10^9 \text{ stat coulomb}$$

$$1 \text{ Ampere} = 3 \times 10^9 \frac{\text{stat coulomb}}{\text{second}} = 3 \times 10^9 \text{ stat-ampere.}$$

Current density. *The current density at a point in a conductor carrying current is defined as the current per unit area of cross-section of the conductor the area being taken in a direction normal to the current.* It may also be defined as the quantity of charge passing per second through a unit area of cross-section of the conductor taken normal to the direction of flow of charge.

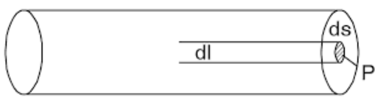

Fig. 19.2

If dI is the current through a small area ds containing a point P in a direction normal to that of the current, then

Current density at P, $$J = \frac{dI}{ds}$$

Current density-a vector. Current density at a point is a *vector* quantity and its direction is that in which a positive charge carrier would move at that point. It is denoted as $\vec{J}$.

Dimensions. The current density has dimensions of

$$\frac{\text{current}}{\text{area}} = \frac{\text{amp}}{\text{met}^2} = \frac{\text{coulomb}}{\text{met}^2\text{sec}} \qquad \text{[in S.I. units]}$$

and $$\frac{\text{stat Amp}}{\text{cm}^2} = \frac{\text{stat coulomb}}{\text{cm}^2\text{sec}}$$ [in C.G.S. units]

19.3 DISTINCTION BETWEEN CURRENT AND CURRENT DENSITY

(*i*) Current density is the current per unit area normal to the direction of drift velocity of the electrons.

(*ii*) Current density is a vector quantity whereas current is a scalar quantity.

(*iii*) Current I is the scalar product of current density vecotr $\vec{J}$ and area vector $d\vec{s}$ at a point

$$\mathrm{I} = \vec{J}.d\vec{s}$$

(*iv*) The unit of current density is current per unit area. The S.I. unit is Amp per m^2.

19.4 RELATION BETWEEN CURRENT, CURRENT DENSITY AND DRIFT VELOCITY

(*i*) **Current and current density.** If dI is the current through a small area ds containing a point P in a direction normal to that of the current then current density at P

$$J = \frac{dI}{ds} \quad \text{or} \quad dI = J\,ds$$

but when the cross-sectional are ds is not normal to the direction of the current at that point, then

$$dI = J\,ds\cos\theta = \vec{J}.d\vec{s}$$

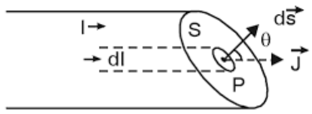

Fig. 19.3

where θ is the angle between direction of current or the vector $\vec{J}$ and the outward drawn normal to the area element $d\vec{s}$.

∴ Flux of the vector $\vec{J}$ over the whole area S = current through the conductor $I = \iint_S \vec{J}.d\vec{s}$

(*ii*) **Current density and drift velocity.** Consider a medium through which a macroscopic electric current if flowing. We suppose that this current is only due to one type of charge carriers. The average of the random thermal velocities of these particles will be zero and they will posses an average drift velocity say $\vec{v}$ in the direction of the applied electric field. To find the value of $\vec{J}$ at a point P consider a small vector area $d\vec{s}$ surrounding the point, then all the charge carriers lying in a cylinder of length v will pass through the area in one second.

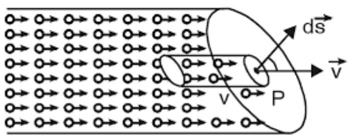

Fig. 19.4

The volume of the cylinder = $\vec{v}.d\vec{s}$

If N is the number density of charge carriers *i.e.*, the number of charge carriers per unit volume, then

Number passing through the area $d\vec{s}$ in one second = $N\vec{v}.d\vec{s}$.

If e is the charge per carrier, then

Charge passing through the area $d\vec{s}$ in one second $= dq = Ne\vec{v}.d\vec{s}$

The charge dq passing per second through the area $d\vec{s}$ measures the current dI through it

$$\therefore \quad dI = Ne\,\vec{v}\,.d\vec{s} = \vec{J}\,.d\vec{s} \qquad \text{... } (i)$$

where $\vec{J} = ne\,\vec{v}$ is called the *current density vector.*

$$\therefore \quad \vec{J} = ne\,\vec{v} \qquad \text{... } (ii)$$

The direction of current density vector is along the direction of drift velocity $\vec{v}$.

(*iii*) **Current and drift velocity.** According to relation (*i*) $dI = \vec{J}\,.d\vec{s} = Ne\,\vec{v}\,.d\vec{s}$

$$\therefore \quad \text{Current I} = \iint_s dI = \iint_s Ne\,\vec{v}\,.d\vec{s}$$

It should be noted that in the absence of any external electric field, the electrons in a metallic conductor do not contribute anything towards current density. A metallic conductor can be supposed to be a *lattice* of atoms in fixed positions and a large number of free electrons known as *conduction* electrons. These electrons come from the atoms of the metal leaving the atoms as positively charged ions. The conduction electrons move in all random directions like the molecules of a gas confined to a vessel. The random motion arises because of collisions between the electrons and the ions of the metal lattice. In the absence of an electric field the average of these random thermal velocities is zero and they do not have an organised motion in any particular direction. Hence they do not contribute anything towards current density.

Moreover, a conductor (*i.e.*, wire) carrying a current is not charged at all. When a wire is carrying current, electrons flow in the wire from one end to the other. The number of electrons entering per second at one end is equal to the number of electrons leaving at the other end. As there is no accumulation of charge at any point, the wire remains neutral.

19.5 MICROSCOPIC AND MACROSCOPIC CURRENTS

Microscopic current. *Microscopic currents are those currents which are confined within the boundaries of the atoms.*

The most familiar example is that of an electron revolving round the nucleus in an orbit. This orbital motion of the electron is equivalent to a current. The current being localised within the volume of the atom is a microscopic current. This microscopic current is also responsible for the magnetic properties of atoms.

Macroscopic current. *Macroscopic currents are those in which there is an organised motion of electric charge over distances large as compared to atomic dimensions.*

The current passing through a conductor connected to a battery is an example of macroscopic current.

Current density vector. The current density vector is *not* microscopic. The current density vector $\vec{J} = Ne\,\vec{v}$ where $\vec{v}$ is the drift velocity of the electron. The drift velocity of an electron is the average of the microscopic velocities of all electrons over a volume large compared to the molecular size. The drift velocity is therefore, a macroscopic quantity.

Hence the current density vector is also a macroscopic concept.

19.6 EQUATION OF CONTINUITY

Consider a small area elements $d\vec{s}$ of a closed surface S.

Discussion. The term $\vec{\nabla} \cdot \vec{J}$ represent the limiting value of net outward flow of electric current per unit area while the term $\frac{\partial \rho}{\partial t}$ gives the rate of change of charge per unit volume.

According to the equation of continuity the two terms are equal and opposite and their sum is zero.

19.7 EQUATION OF CONTINUITY IMPLIES CONSERVATION OF CHARGE IN SPACE

The equation of continuity

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

can be put in the form

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

$$\therefore \qquad (\vec{\nabla} \cdot \vec{J})dv = -\frac{\partial \rho}{\partial t} dv$$

Integrating both sides over the whole volume V, we have

$$\iiint_V (\vec{\nabla} \cdot \vec{J})\, dv = -\iiint_V \frac{\partial \rho}{\partial t} dv \qquad \text{... }(i)$$

According to Gauss's divergence theorem

$$\iiint_V (\vec{\nabla} \cdot \vec{J})\, dv = \oiint_s \vec{J} . d\vec{s} \qquad \text{... }(ii)$$

Comparing (*i*) and (*ii*), we gave

$$\oiint_s \vec{J} . d\vec{s} = -\iiint_V \frac{\partial \rho}{\partial t} dv = -\frac{\partial}{\partial t} \iiint_V \rho\, dv = -\frac{\partial q}{\partial t}$$

where q is the total charge in the volume V.

$\oiint_s \vec{J} . d\vec{s} = I$ the rate of flow of charge out of the volume V through the closed surface S.

Hence the rate of flow of charge out of a volume V through the closed surface S = rate of decrease of charge in the volume V.

In other words *the total charge in space remains constant and is therefore conserved.*

19.8 STEADY CURRENT

A current flowing through a conductor is said to be stationary or steady if the current passing through any two sections of the conductor say A and B is the same. In other words, the total charge entering the volume V through the section at A = the total charge leaving the volume V through the section at B. Thus, the charge density ρ in any volume V of the conductor remains constant.

$$\text{or} \qquad \frac{\partial \rho}{\partial t} = 0$$

Substituting in the equation of continuity $\vec{\nabla}.\vec{J}+\frac{\partial\rho}{\partial t}=0,$ we have

$$\vec{\nabla}.\vec{J} = 0$$

$\therefore$ For stationary current $\vec{\nabla}.\vec{J} = div\ J = 0$

This is the *basic* equation or *continuity* equation for *steady* or *stationary* current. *This equation also expresses the conservation of charge in space.*

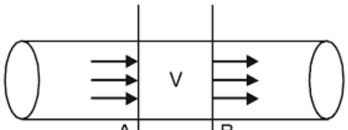

Fig. 19.6

But $\vec{J} = \sigma\vec{E}$ (pl. see article 19.9) where $\vec{E}$ is the electric field vector and σ the conductivity. For a homogeneous conductor σ is constant everywhere.

$$\therefore \qquad \vec{\nabla}.\vec{J} = \vec{\nabla}.\left(\sigma\vec{E}\right) = 0$$

or $$\sigma\vec{\nabla}\cdot\vec{E} = 0 \quad \text{or} \quad \vec{\nabla}.\vec{E} = 0 \qquad ...(i)$$

19.9 VECTOR FORM OF OHM'S LAW

For a very large number of solid homogeneous materials known as ohmic media *the current density vector* $\vec{J}$ *at any point is proportional to the electric field strength so long as the field is low.*

$$\therefore \qquad \vec{J} \propto E$$

or $$\vec{J} = \sigma\vec{E}$$

The statement $\vec{J} = \sigma\vec{E}$ is known as the vector form of Ohm's law. σ is a constant characteristic of the medium and is known as its *conductivity*.

Derivation of conventional form of Ohm's law. To derive the conventional form of Ohm's law from the vector form consider a conductor AB to which a steady electric field $\vec{E}$ is applied as shown in the direction from A to B. The current density vector $\vec{J}$ will have the same value at each point and also the same direction *i.e.*, from A to B. Let a be the area of cross-section of the conductor and $d\vec{s}$ a small vector element of this area.

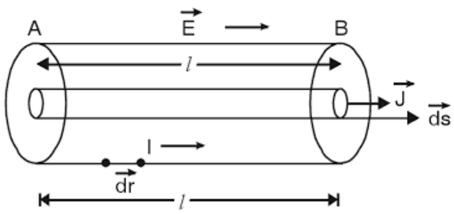

Fig. 19.7

Electric current *I*. The electric current flowing through the conductor

$$I = \iint_a \vec{J}.d\vec{s}$$

As $\vec{J}$ and $d\vec{s}$ have the same direction, $\vec{J}.d\vec{s} = J\,ds$

$$\therefore \qquad I = \iint_a J.ds = J\iint_a ds = Ja \qquad ...(i)$$

as J = a constant.

As $J = nev$ according to Eq. (*ii*) of article 19.4

$$I = Ja = naev \qquad \text{... } (i)\ (a)$$

Potential difference *V*. If V represents the potential difference between the point A and B and $d\vec{r}$ a small vector element of its length, then

$$V = \int_A^B \vec{E}.d\vec{r}$$

Again the vectors $\vec{E}$ and $d\vec{r}$ have the same direction

$$\therefore \qquad \vec{E}.d\vec{r} = Edr$$

and

$$V = \int_A^B Edr = E\int_0^l dr = El$$

or

$$E = \frac{V}{l} \qquad \text{... } (ii)$$

as E = a constant and $AB = l$

According to vector form of Ohm's law $\vec{J} = \sigma\vec{E}$

Taking magnitudes only when $\vec{J}$ and $\vec{E}$ are in the same direction, we have

$$J = \sigma E$$

Substituting in (*i*), we have

$$I = \sigma Ea = V\frac{\sigma a}{l}$$

or

$$\frac{V}{I} = \frac{l}{\sigma a} = R \text{ (}a\text{ constant)} \qquad \text{... } (iii)$$

Thus starting from the vector form of Ohm's law $\vec{J} = \sigma\vec{E}$ we obtain the conventional form of Ohm's law $\frac{V}{I} = R$ or $V = RI$

The relation $V = RI$ shows that *the potential difference between two ends of a conductor is directly proportional to the current flowing through it.* The constant of proportionally $R = \frac{l}{\sigma a}$ is called the resistance of the conductor.

The relation $I = \sigma Ea$ can be put in the form $I = -\sigma a\frac{dV}{dl}$.

as $E = -\frac{dV}{dl}$ the potential gradient.

19.10 OHM'S LAW IN VECTOR FORM $\vec{J} = \sigma \vec{E}$

In article 19.9 we have derived the conventional form of Ohm's law from its vector form. This is the converse of the same.

Proceeding as in article 19.9, we have

$$I = Ja \qquad \text{... }(i)$$

and
$$E = \frac{V}{l} \text{ or } V = El \qquad \text{... }(ii)$$

The conventional form of Ohm's law states that

$$V = IR \text{ where } R = \rho\frac{l}{a}$$

Substituting the value of $V = El$ and $I = Ja$, we have

$$El = JaR = Ja\rho\frac{l}{a} = J\rho l$$

or
$$E = J\rho$$

$$\therefore \quad \frac{J}{E} = \frac{1}{\rho} = \sigma$$

Hence
$$\vec{J} = \sigma\vec{E}$$

which is the vector form of Ohm's law.

19.11 ELECTRICAL CONDUCTIVITY (σ)

Electrical conductivity σ is defined as the ratio of current density vector $\vec{J}$ to the electric field vector $\vec{E}$.

or
$$\sigma = \frac{\vec{J}}{\vec{E}} = \frac{|\vec{J}|}{|\vec{E}|}$$

Units. The practical C.G.S. unit of $\sigma = \frac{\text{ampere/cm}^2}{\text{Volt/cm}} = (\text{ohm cm})^{-1}$ known as 'reciprocal Ohm cm' and the S.I. units $(\text{ohm met.})^{-1}$ or Siemens per metre (Sm^{-1}).

Resistivity. According to the conventional form of Ohm's law the ratio of the potential difference V applied to the ends of a conductor to the current I flowing through it is a constant.

or
$$\frac{V}{I} = R$$

where R is a constant known as the resistance of the conductor.

Thus resistance of a conductor is defined as the ratio of the potential difference applied at the ends of a conductor to the current flowing through it.

The unit of resistance is an *Ohm. It is the resistance of a conductor through which a current of 1 ampere flows when a potential difference of 1 volt is applied across it.* Its value depends upon the length of the conductor l, the area of cross-section a and the volume resistivity or specific resistance ρ and

$$R = \rho \frac{l}{a} \qquad \dots (i)$$

or

$$\rho = R \frac{a}{l}$$

In C.G.S. units. *Resistivity* is, therefore, *defined as the resistance of a cm cube of the material or* the resistance of a conductor of length 1 cm and area of cross-section 1 sq cm.

The unit of resistivity is ohm cm.

In S.I. units *the resistivity is the resistance of a metre cube of the material* or the resistance of a conductor of length 1 metre and area of cross-section one square metre.

The unit of resistivity in S.I. is ohm-metre.

Relation. According to Eq. (*i*) $R = \rho \frac{l}{a}$ and according to relation (*iii*) article 19.9 $R = \frac{l}{\sigma a}$

$$\therefore \qquad \frac{l}{\sigma a} = \rho \frac{l}{a} \quad \text{or} \quad \sigma = \frac{1}{\rho}$$

Thus conductivity is the reciprocal of resistivity.

$$\therefore \qquad \rho = \frac{|\vec{E}|}{|\vec{J}|}$$

Its dimensions are $\dfrac{\text{charge}}{\text{distance}^2} \Big/ \dfrac{\text{charge}}{\text{sec. distance}^2} = \text{sec.}$

Thus resistivity has dimensions of time.

Variation of conductivity with temperature. Conductivity σ is the reciprocal of resistivity ρ *i.e.,* $\sigma = \frac{1}{\rho}$. It has been explained in article 19.14 that resistivity of a good conductor increases with temperature. This therefore, means that conductivity decreases with temperature. It is for this reason that the conductivity becomes very large at very low temperatures. At temperatures near the absolute zero, the conductor becomes *super conducting*.

19.12 ATOMIC VIEW OF OHM'S LAW

The vector form of Ohm's law states that $\vec{J} = \sigma \vec{E}$. If there are only *one* type of charge carriers, N their number density, e the charge on each and $\vec{v}$ the average drift velocity, then according to atomic view $\vec{J} = N e \vec{v}$.

Average drift velocity. In the absence of the electric field the charge carriers have velocities in all possible directions and their average value taken over a sufficient time is zero. When the electric field $\vec{E}$ is applied a force $\vec{E} e$ acts on each charge carrier. If m is the mass of a charge carrier, then acceleration $= \dfrac{\vec{E} e}{m}$.

If T is the *mean free time* between collisions, then the *additional* momentum (or impulse) acquired by the particle

$$= \text{Force} \times \text{time} = \vec{E}\,e\,T$$

This is an *ordered* contribution and is the same for each particle. As the charge carriers have random velocities and are moving about in all possible directions, they suffer collisions in the process. At each collision the momentum gained by the charge carrier gets altered. If, however, we take an average over a *large* number of collisions the additional ordered momentum gained by the charge carriers due to the electric field gets completely destroyed by the collisions. We can, therefore, suppose that just after a collision the velocity of a charge carrier is zero and it has a constant acceleration $\frac{Ee}{m}$ and, therefore, its velocity after a time T is given by

$$\vec{v}_r = 0 + \frac{\vec{E}\,e\,T}{m} = \frac{\vec{E}\,e\,T}{m}$$

∴ Average drift velocity during the mean free time

$$\vec{v} = \frac{0 + \vec{E}\,e\,\frac{T}{m}}{2} = \frac{1}{2}\frac{\vec{E}\,e\,T}{m}$$

Ohm's law. $\vec{J} = \sigma\vec{E}$. If we consider the charge carriers to be free electrons, then since an electron carries a negative charge.

Average drift velocity $$\vec{v} = -\frac{1}{2}\frac{\vec{E}\,e\,T}{m} \qquad \ldots (i)$$

The current density vector corresponding to the flow of electrons

$$\vec{J} = -N\,e\,\vec{v} \qquad (ii)$$

Substituting the value of $\vec{v}$ from (i) in (ii), we have

$$\vec{J} = \frac{N\,e^2\,T}{2m}\vec{E}$$

or $$\vec{J} = \sigma\vec{E} \qquad \ldots (iii)$$

where $$\sigma = \frac{N\,e^2\,T}{2m}$$

Now, N, e and m are constant quantities independent of $\vec{E}$. If T is also a constant, then

$$\sigma = \frac{\vec{J}}{\vec{E}} = \text{a constant} \qquad \ldots (iv)$$

More than one charge carrier. If we have more than one type of charge carriers and N_i, e_i m_i and T_i represents the number density, charge, mass and mean free time for one type (say ith) then current density due to this type.

$$\vec{J}_i = \frac{N_i\,e_i^2\,T_i}{2m_i}\vec{E}$$

The total current density $\vec{J}$ is given by the summation of current densities due to all types of particles

$$\vec{J} = \sum \frac{N_i e_i^2 T_i}{2m_i} \vec{E} = \sigma \vec{E}$$

19.13 LIMITATIONS OF OHM'S LAW

Ohm's law in vector form states that $\vec{J} = \sigma \vec{E}$ where σ is the electrical conductivity of the medium and is given by $\sigma = \frac{Ne^2T}{2m}$.

A medium will obey Ohm's law as long as σ is a constant *i.e.,* independent of $\vec{E}$.N, e and m are constant which do not depend upon $\vec{E}$. Therefore, *if T becomes a function of $\vec{E}$, then σ no longer remains constant and Ohm's law breaks down.*

Suppose λ is the mean free path of the electron, the λ gives the distance travelled by the electron between two collisions.

$\therefore$ Average time between two collisions $T = \frac{\lambda}{v+u}$

where v is the drift velocity and u the thermal velocity of the electron. We shall now discuss the validity of Ohm's law for low as well as high electric fields.

Low electric fields. For small values of $\vec{E}$ say 100 volt/metre the drift speed of free electrons in metals is about 0.08 metres/sec. which is very low as compared to their thermal velocities of the order of 10^5 metres/sec. T is then determined only by the thermal velocities, the average value of which for a metal reamins constant at a given temperature.

$\therefore$ T is a constant and $\sigma = \frac{Ne^2T}{2m}$ = a constant

Hence Ohm's law is obeyed.

High electric fields. For large values of $\vec{E}$ of the order of 10^8 V/m the drift velocity becomes 0.8×10^5 m/s which is comparable to the thermal velocity. This changes the time between collisions. The mean free time will not, therefore, be a constant but will now be a function of $\vec{E}$. The current density $\vec{J}$ will no longer be proportional to $\vec{E}$.

Hence Ohm's law breaks down.

The failure of Ohm's law occurs at electric fields of the order of 10^8 V/m.

19.14 VARIATION OF RESISTANCE WITH TEMPERATURE

The resistance of pure metals increases as the temperature is raised. If R_0 is the resistance at 0° C and R_t the resistance at t° C, then

$$R_t = R_0(1 + \alpha t)$$

where α is called the *temperature coefficient of resistance.*

or
$$\alpha = \frac{R_t - R_0}{R_0 t}$$

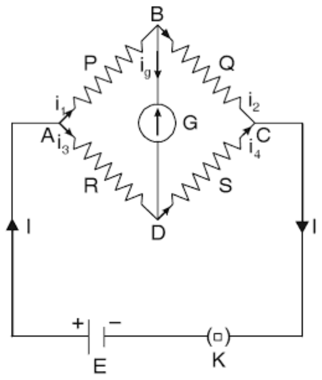

Fig. 19.10

Now applying Kirchhoff's voltage law to the meshes ABDA and ABCDA, we get

$$i_1P + i_gG - i_3R = 0 \quad \text{... } (iii)$$

and

$$i_1P + i_2Q - i_4S - i_3R = 0 \quad \text{... } (iv)$$

When the bridge is balanced, galvanometer shows "***zero deflection***" and points B and D are at the same potential and $i_g = 0$. Hence equation (*i*), (*ii*) and (*iii*)

$$i_1 = i_2 \quad \text{... } (v)$$

$$i_3 = i_4 \quad \text{... } (vi)$$

$$i_1P = i_3R \quad \text{... } (vii)$$

Putting the values of (*v*) and (*vi*) in equation (*iv*), we get,

$$i_1P + i_1Q - i_3S - i_3R = 0$$

$$i_1(P + Q) = i_3(R + S) \quad \text{... } (viii)$$

Now dividing euqation (*viii*) by (*vii*), we get,

$$\frac{i_1(P+Q)}{i_1P} = \frac{i_3(R+S)}{i_3R}$$

$$\frac{(P+Q)}{P} = \frac{R+S}{R}$$

$$1+\frac{Q}{P} = 1+\frac{S}{R}$$

$$\frac{Q}{P} = \frac{R}{S}$$

or

$$\frac{P}{Q} = \frac{R}{S}$$

Thus, at balance point, *i.e.*, where the galvanometer shows zero deflection, we have $\frac{P}{Q} = \frac{R}{S}$.

If P, Q and R are known, S can be calculated.

19.17 GROWTH OF CURRENT IN LR CIRCUIT (Helmholtz Equation)

Consider a circuit having an inductance L and a resistance R placed in series with a battery of e.m.f. E Fig. 19.11. When the key K is in position 1, the current slowly increase from zero to a

Time constant : The fraction $\frac{L}{R}$ is called the time current of the circuit.

If $t = \frac{L}{R}$, then

$$I = I_0 (1 - e^{-t/t}) = I_0 (1 - e^{-1})$$

$$= I_0\left(1 - \frac{1}{e}\right) = I_o\left(\frac{e-1}{e}\right)$$

$$= I_o\left(\frac{1.718}{2.718}\right)$$

$\therefore$ $I = 0.6321 I_0$

Hence, time constant of LR ciruit is defined as time during which the current rises to 0.6321 of its maximum value.

19.18 DECAY OF CURRENT IN LR CIRCUIT (HELMHOLTZ EQUATION)

After the steady current has been established, if the source of steady e.m.f. is now removed (by disconnecting position 1 of key K and connecting now to position 2), the current begins to decay (fall) exponentially in the LR circuit as shown in fig. 19.13.

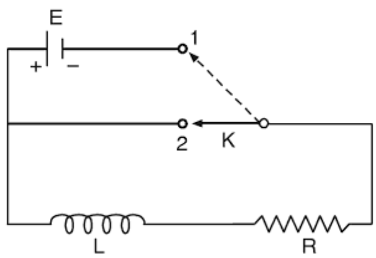

Fig 19.13

During the time the current is decreasing, there is an e.m.f. induced in the inductance having a value $e = -L\frac{dI}{dt}$. The differential loop equation using Kirchhoff's second law (voltage law) as

$$\left[0 - L\frac{dI}{dt}\right] = RI$$

or $$\frac{dI}{I} = -\frac{R}{L}t$$

Integrating both sides, we get

$$\int \frac{dI}{I} = -\frac{R}{L}\int dt + B$$

where B is a constant of integration

$$\therefore \quad \log_e I = -\frac{R}{L}t + B \qquad ...(i)$$

or $$-\log_e (q_0 - q) = \frac{t}{RC} + A \quad \text{... }(ii)$$

where A is the constant of integration.

When $t = 0$ $\quad q = 0$

$\therefore$ $\quad -\log_e q_0 = A$

Substituting the value of A in (ii), we get

$$-\log_e (q_0 - q) = \frac{t}{RC} - \log_e q_0$$

or $$\log_e\left(\frac{q_0 - q}{q_0}\right) = -\frac{t}{RC}$$

or $$\frac{q_0 - q}{q_0} = e^{\frac{t}{RC}}$$

or $$q_0 - q = q_0^{-\frac{t}{RC}}$$

$\therefore$ $$q = q_0 (1 - e^{-t/RC}) \quad \text{... }(iii)$$

Current $$I = \frac{dq}{dt} = \frac{d}{dt}[q_0(1 - e^{-t/RC})]$$

or $$I = \frac{q_0}{RC} e^{-t/RC}$$

The steady (maximum) current $I_0 = \dfrac{E}{R} = \dfrac{q_0}{RC}$

$\therefore$ $$I = I_0\, e^{-t/RC} \quad \text{... }(iv)$$

The variation of charge with time at growth is shown in Fig 19.17. Fig 19.18 shows the variation of current during charging and discharging of C through R.

Time constant. The time constant = RC is known as *capacitative* time constant.

Taking $RC = t$; then $\dfrac{t}{RC} = 1$

$\therefore$ $$q = q_0 (1 - e^{-1}) = q_0\left(1 - \frac{1}{e}\right) = 0.6321 q_0$$

Hence time constant RC is defined as the time taken by the capacitor to get charged to $\left(1 - \dfrac{1}{e}\right)$ *or (0.6321) of the maximum value of charge.*

19.21 DISCHARGE OF CAPACITOR THROUGH RESISTANCE (R)

Let a capacitor of capacitance C, having a charge q_0 be connected to a resistance R. A current flows through the resistance. Let the charge left after a time t be q and let it decreases at the rate $\dfrac{dq}{dt}$,

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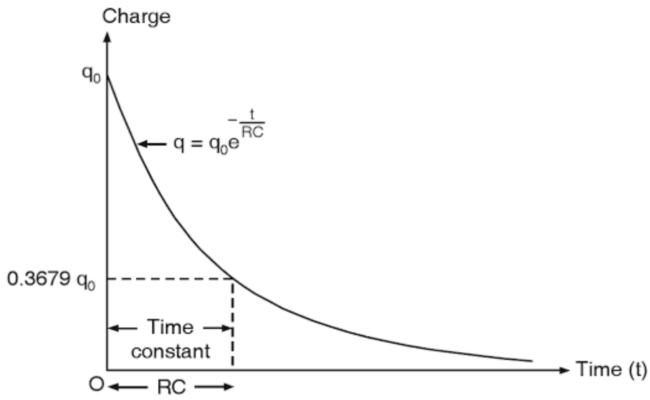

Fig. 19.19 : Decay of charge of condenser

Current $$I = \frac{dq}{dt} = \frac{d}{dt}\left(q_0\, e^{-t/RC}\right) = \frac{-q_0}{RC} e^{-t/RC} = -I_0\, e^{-t/RC} \qquad \ldots (iv)$$

where $I_0 = \dfrac{q_0}{RC}$ = the maximum current.

At $t = 0$, the current during charging [from eq. (iv) article 19.20], is I_0 and during discharging [from Eq. (*iv*), article 19.21] is $-I_0$. The variation of I during charging and discharging of condenser is shown in fig. 19.18. The current I falls exponentially with time (t)

Sum of the currents during charging and discharging. At any time t, the current during charging of a capacitor

$$I = \frac{q_0}{RC} e^{-\frac{t}{RC}}$$

and during discharging

$$I' = -\frac{q_0}{RC} e^{-\frac{t}{RC}}$$

$\therefore$ Sum of the currents $I + I' = \dfrac{q_0}{RC} e^{-\frac{t}{RC}} - \dfrac{q_0}{RC} e^{-\frac{t}{RC}} = 0$

Thus, the sum of currents at any time during charging and discharging of the condenser is zero.

19.22 ENERGY STORED IN THE CAPACITOR OF RC CIRCUIT

To dervive the relation for energy stored in the capacitor of RC circuit, we put extra, small amount of charge of dq on the capacitor. The amount of additional work done will be

$$dU_E = \frac{q}{C} \times dq$$

This work done is stored in the form of potential energy. The process of increase of $P.E.$ of the system continues untill a total charge Q has been transferred. The total work done will be

$$U_E = \int dU_E = \int_0^Q \frac{q}{c} dq = \frac{1}{2}\frac{Q^2}{C}$$

$$R = Ri = \frac{dq}{dt}$$

Back *e.m.f.* in the inductance

$$L = -L\frac{di}{dt} = -L\frac{d^2q}{dt^2}$$

As the applied *e.m.f.* is E, we have

$$\frac{q}{C} + R\frac{dq}{dt} = E - L\frac{d^2q}{dt^2}$$

or $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \left(\frac{q}{C} - E\right) = 0$$

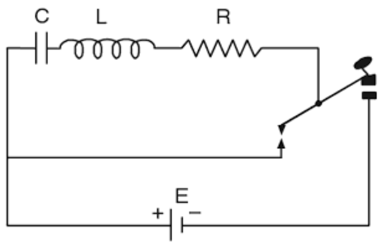

Fig. 19.20

or $$\frac{d^2q}{dt^2} + \frac{R\,dq}{L\,dt} + \left(\frac{q}{LC} - \frac{E}{L}\right) = 0$$

Put $$\frac{R}{L} = 2b \text{ and } \frac{1}{LC} = k^2$$

$\therefore$ $$\frac{d^2q}{dt^2} + 2b\frac{dq}{dt} + \left(k^2q - \frac{E}{L}\right) = 0$$

or $$\frac{d^2q}{dt^2} + 2b\frac{dq}{dt} + k^2\left(q - \frac{E}{k^2L}\right) = 0 \qquad \dots (i)$$

Again put $q - \frac{E}{k^2L} = x$ so that $\frac{dx}{dt} = \frac{dq}{dt}$ and $\frac{d^2x}{dt^2} = \frac{d^2q}{dt^2}$.

Substituting in (*i*), we have

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + k^2x = 0 \qquad \dots (ii)$$

Let a trial solution of this differential equation be $x = e^{\alpha t}$, then

$$\frac{dx}{dt} = \alpha e^{\alpha t}, \text{ and } \frac{d^2x}{dt^2} = \alpha^2 e^{\alpha t}$$

Substituting in Eq. (*ii*), we have

$$\alpha^2 e^{\alpha t} + 2b\alpha e^{\alpha t} + k^2\alpha\, e^{\alpha t} = 0$$

or $$\alpha^2 + 2b\alpha + k^2 = 0$$

This is a quadratic equation in α. Its roots are

$$\alpha = -b \pm \sqrt{b^2 - k^2}$$

As there are two values of α, the general solution of differential equation (*ii*) is

$$x = Ae^{(-b+\sqrt{b^2-k^2})t} + Be^{(-b-\sqrt{b^2-k^2})t} \qquad \dots (iii)$$

In circuit for which R is small the amplitude will die slowly. For $R = 0$, the amplitude is constant and the oscillations become simple harmonic. The maximum charge is much greater than the steady value q_0. It is possible that the maximum charge may raise the potential of the capacitor so high that the insulation may breakdown.

Current. The current in the circuit at any instant is obtained by differentiating the expression for charge, obtained in Eq. (*vii*).

$$I = \frac{dq}{dt} = q_0 k e^{-bt}\sin(\omega t - \theta) + q_0\frac{kb}{\omega}e^{-bt}\cos(\omega t - \theta)$$

$$= q_0\, e^{-bt}\frac{k^2}{\omega}\left[\frac{\omega}{k}\sin(\omega t - \theta) + \frac{b}{k}\cos(\omega t - \theta)\right]$$

$$= q_0\, e^{-bt}\frac{k^2}{\omega}[\sin(\omega t - \theta)\cos\theta + \cos(\omega t - \theta)\sin\theta]$$

$$= q_0\, e^{-bt}\frac{k^2}{\omega}\sin\omega t$$

Period of oscillation. The time period of oscillation is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k^2 - b^2}} = \frac{2\pi}{\sqrt{\dfrac{1}{LC} - \dfrac{R^2}{4L^2}}}$$

If R is negligible, then $\quad T = 2\pi\sqrt{LC}$

and then frequency $\quad n = \dfrac{1}{T} = \dfrac{1}{2\pi\sqrt{LC}}$

19.25 DISCHARGING A CONDENSER THROUGH INDUCTANCE AND RESISTANCE

Condenser C is charged by pressing key K (1 and 3 are in contact) for a sufficient time, and then released (Now 1 and 2 are in contact). Thus the source of *e.m.f.* E is withdrawn and the charge is allowed to pass through L and R as shown is Fig. 19.22. Let us investigate the transient condition here.

Initially, the condenser C is charged to the *e.m.f.* E of the battery by pressing the key K downwards. Let q_0 be the charged on it. When the key K is released the capacitor circuit is completed through the inductance and the resistance. The capacitor slowly losses its charge due to which there is a current I at any instant. The current varies at the rate $\dfrac{dI}{dt}$. If q is the charge on the capacitor some time later, then

Potential difference across the capacitor $= \dfrac{q}{C}$

Potential difference across the resistance $= RI$

Back *e.m.f.* in the inductance $= -L\dfrac{dI}{dt}$

As there is no source of *e.m.f.* in the circuit

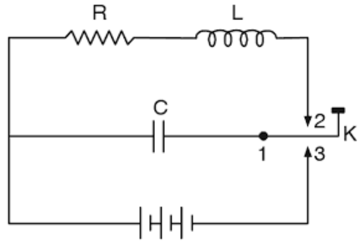

Fig. 19.22

$$\therefore \qquad \frac{q}{C}+RI = -L\frac{dI}{dt}$$

or
$$\frac{q}{C}+RI+L\frac{dI}{dt} = 0$$

But as
$$I = \frac{dq}{dt} \text{ and } \frac{dI}{dt}=\frac{d^2q}{dt^2}$$

$$\therefore \qquad L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C} = 0$$

or
$$\frac{d^2q}{dt^2}+\frac{R}{L}\frac{dq}{dt}+\frac{1}{LC}q = 0 \qquad \text{... } (i)$$

Put
$$\frac{R}{L} = 2b \text{ and } \frac{1}{LC} = k^2, \text{ then}$$

$$\frac{d^2q}{dt}+2b\frac{dq}{dt}+k^2q = 0 \qquad \text{... } (ii)$$

Let a trial solution of differential equation be

$$q = e^{\alpha t}$$

Then
$$\frac{dq}{dt} = \alpha\, e^{\alpha t} \text{ and } \frac{d^2q}{dt^2} = \alpha^2\, e^{\alpha t}$$

Substituting the values in (*ii*) we have

$$\alpha^2\, e^{\alpha t} + 2b\alpha e^{\alpha t} + k^2\, e^{\alpha t} = 0$$

or
$$\alpha^2 + 2b\alpha + k^2 = 0$$

This is a quadratic equation in α. Hence its roots are

$$\alpha = -b\pm\sqrt{b^2-k^2}$$

As there are two values of α, the general solution of the differential equation (*ii*) is

$$q = Ae^{(-b+\sqrt{b^2-k^2})t}+Be^{(-b-\sqrt{b^2-k^2})t} \qquad \text{... } (iii)$$

where A and B are arbitrary constants, the value of which can be determined from boundary conditions.

Now at $\qquad t = 0 \qquad q = q_0$

Substituting in (*iii*) we have

$$q_0 = A + B$$

The value of current i at any time is obtained by differentiating equation (*iii*) and we have

$$i = \frac{dq}{dt} = A(-b+\sqrt{b^2-k^2})e^{\sqrt{(b^2-k^2)t}}+B(-b-\sqrt{b^2-k^2})e^{(-b-\sqrt{b^2-k^2})t}$$

Again at $t = 0 \qquad i = 0$

$$\therefore \qquad 0 = A(-b+\sqrt{b^2-k^2}) + B(-b-\sqrt{b^2-k^2})$$

$$= -b(A+B) + \sqrt{b^2-k^2}(A-B)$$

$$= -bq_0 + \sqrt{b^2-k^2}(A-B)$$

or
$$A - B = \frac{bq_0}{\sqrt{b^2-k^2}} \qquad \text{... } (v)$$

Adding (iv) and (v), we get

$$2A = q_0\left(1+\frac{b}{\sqrt{b^2-k^2}}\right)$$

or
$$A = \frac{1}{2}q_0\left(1+\frac{b}{\sqrt{b^2-k^2}}\right)$$

Subtracting (v) from (iv) we get

$$2B = q_0\left(1-\frac{b}{\sqrt{b^2-k^2}}\right)$$

or
$$B = \frac{1}{2}q_0\left(1-\frac{b}{\sqrt{b^2-k^2}}\right)$$

Substituting the values of A and B in (iii), we have

$$q = \frac{1}{2}q_0\left[\left(1+\frac{b}{\sqrt{b^2-k^2}}\right)e^{(-b+\sqrt{b^2-k^2})t} + \left(1-\frac{b}{\sqrt{b^2-k^2}}\right)e^{(-b-\sqrt{b^2-k^2})t}\right] \qquad \text{... } (vi)$$

Substituting the values of b and k in we have

$$q = \frac{1}{2}q_0\left[1+\frac{\frac{R}{2L}}{\sqrt{\frac{R^2}{2L^2}-\frac{1}{LC}}}\right]e^{\left(-\frac{R}{2L}+\sqrt{\frac{R^2}{2L^2}-\frac{1}{LC}}\right)t} + \left[1-\frac{\frac{R}{2L}}{\sqrt{\frac{R^2}{2L^2}-\frac{1}{LC}}}\right]e^{\left(-\frac{R}{2L}-\sqrt{\frac{R^2}{2L^2}-\frac{1}{LC}}\right)t} \qquad \text{... } (vii)$$

The exponential factor $e^{-\frac{Rt}{2L}}$ gives us a charge decaying exponentially with time. Superimposed upon this is the effect due to the factor under the root sign. This factor has three possible cases.

Case 1 : When $\dfrac{R^2}{4L^2} > \dfrac{1}{LC}$

and charges in electric and magnetic field give rise to an electromagnetic radiation which is propagated through space with the velocity of light. These electromagnetic waves form the basis of wireless telegraphy and the help of a code enable the transmission of message from one place to the other.

SOLVED EXAMPLES

Example 19.1 A current of 10 Amp. exists in 10 ohm resistance for 4 minutes. Find (*i*) how many coulombs and (*ii*) how many electrons pass through any cross-section of the resistance in this time? (*M.D.U.* 2000)

Solution. Current I = 10 Amp; time $t = 4$ min $= 4 \times 60$ s.

Charge $Q = It = 10 \times 4 \times 60 = 2400$ Coulombs.

Charge on the electron $e = 1.6 \times 10^{-19}\ C$

$\therefore$ Number of electron passing through and any cross-section of the resistance in 4 min

$$= \frac{Q}{e} = \frac{2400}{1.6\times10^{-19}} = 1.5\times10^{22} \text{ electrons}$$

Example 19.2 A copper wire is carrying a current of 2*A* and is having an area of cross-section equal to 10^{-6} m^2. If the number of electron per m^3 be 8×10^{28}, calculate the current density and averaged drift velocity. The charge on the electron is 1.6×10^{-19} *C*.

Solution. Current density $\vec{J} = \dfrac{\text{current}}{\text{area}} = \dfrac{2A}{10^{-6}m^2}$

$= 2 \times 10^6$ A/m^2 $= 2 \times 10^6$ C/m^2 s

Now $\vec{J} = Ne\vec{v}$ where $\vec{v}$ is the average drift velocity, N the number of electrons per m^3 and e the electronic charge.

$$\therefore \qquad v = \frac{J}{Ne} = \frac{2\times10^6}{8\times10^{28}\times1.6\times10^{-19}} = 15.6\times10^{-5} \text{ m/sec.}$$

Example 19.3 A aluminium wire whose radius is 1 mm is welded to a copper wire whose radius is 2 mm. The composite wire carries a steady current of 5 Amp. What would be the ratio of their current densities? (*Gharwal U.* 2000)

Solution. Radius of Al wire $r_{Al} = 1$ mm $= 1 \times 10^{-3}$ m

Radius of Cu wire $r_{Cu} = 2$ mm $= 2 \times 10^{-3}$ m

Current $I = 5$ Amp.

$$\text{Current density in Al wire } J_{Al} = \frac{I}{\pi r_{Al}^2} = \frac{5}{\pi(1\times10^{-3})^2} = \frac{5}{\pi\times10^{-6}}$$

$$\text{Current density in Cu wire } J_{Cu} = \frac{I}{\pi r_{Cu}^2} = \frac{5}{\pi(2\times10^{-3})^2} = \frac{5}{4\pi\times10^{-6}}$$

$$\therefore \qquad \frac{J_{Al}}{J_{Cu}} = \frac{5}{\pi\times10^{-6}}\times\frac{4\pi\times10^{-6}}{5} = 4$$

Example 19.6 A current of 10 amp is flowing through a copper conductor of area 1 cm^2. Find out the

(*i*) electric field in the conductor

(*ii*) potential drop across its one km length. Resistivity of copper 1.7 × 10^{-6} ohm cm.

(*Pbi. U.* 2003; *H.P.U.* 2002)

Solution. Current density $\vec{J}$ = 10 amp/sec cm.

$$\text{Resistivity } \rho = 1.7 \times 10^{-6} \text{ ohm cm.}$$

$$\vec{J} = \sigma \vec{E}$$

$$\therefore \quad \vec{E} = \frac{\vec{J}}{\sigma} = \vec{J}\rho = 10 \times 1.7 \times 10^{-6} = 1.7 \times 10^{-5} \text{ volt/cm}$$

(*ii*) Potential difference $V = \int_l \vec{E}.\vec{dl} = El$

$$l = 1 \text{ km} = 10^5 \text{ cm}$$

$$\therefore \quad V = 1.7 \times 10^{-5} \times 10^5 = 1.7 \text{ volt.}$$

Example 19.7 An electric current of 1 Ampere is flowing in a wire of copper 0.01 cm^2 cross-sectional area. What is the electrical field in the wire? take resistivity of copper = 1.6 × 10^{-8} ohm meter. (*K.U.*, 2002)

Solution. $a = 0.01 \text{ cm}^2 = 10^{-2} \text{ cm}^2 = 10^{-6} \text{ cm}^2$

$$\vec{J} = \frac{I}{a} = \frac{1}{10^{-6}} = 10^6 \text{ A/m}^2;$$

$$\rho = 1.6 \times 10^{-8} \text{ ohm meter}$$

$$\therefore \quad \sigma = \frac{1}{\rho} = \frac{1}{1.6 \times 10^{-8}} = \frac{10^8}{1.6} \text{ (ohm m)}^{-1}$$

$$\vec{E} = \frac{\vec{J}}{\sigma} = \frac{10^6 \times 1.6}{10^8} = 1.6 \times 10^{-2} = 0.16 \times 10^{-3} \text{ volt/m}$$

Example 19.8 An electric field of intensity 0.01 Vm^{-1} exists between two points on a conductor of cross-sectional area 10^{-6} m. Calculate the current density and current if resistivity of the material of conductor is 1.75 × 10^{-8} ohm. m.

(*H.P.U.* 2000; *M.D.U.* 2002; *Nagpur Uni.* 2008)

Solution. Here $\vec{E} = 0.01 \text{ Vm}^{-1}$ $\rho = 1.75 \times 10^{-8}$ ohm–m ; $a = 10^{-6} \text{ m}^2$

Now
$$\vec{J} = \sigma\vec{E} = \frac{\vec{E}}{\rho} \qquad \left[\because \sigma = \frac{1}{\rho}\right]$$

$$= \frac{0.01}{1.75 \times 10^{-8}} = 0.571 \times 10^6 \text{ Amp/m}^2$$

Also $m = 9 \times 10^{-31}$ kg $e = 1.6 \times 10^{-19}$ C

$$\therefore \quad T = \frac{2\times 9\times 10^{-31}}{1.7\times 10^{-8}\times 10^{29}\times(1.6\times 10^{-19})^2} = \frac{18}{1.7\times 1.6\times 1.6}\times 10^{-14}$$

or $T = 4.13 \times 10^{-14}$ S.

Example 19.12 A copper wire having a cross-section 1 cm² carries a current of 1.5 ampere. Assuming that each copper atom contributes one free electron, calculate the drift velocity of free electron. Given that atomic weight and density of copper are 63 and 9 gms/cc respectively.

Solution. Current density $\vec{J} = \frac{1.5A}{10^{-4}\,m^2} = 1.5\times 10^4\,\text{A/m}^2 = 1.5\times 10^4\,\text{C/m}^2\text{s}$

Number of atoms per c.c. $= \frac{6.02\times 10^{23}}{7}$

as gm atomic volume $= \frac{63}{9}$ c.c.

Number of atoms per m³ $= N = \frac{6.02\times 10^{23}\times 10^6}{7}$

$e = 1.6 \times 10^{19}$ C

$$\therefore \quad \vec{v} = \frac{\vec{J}}{Ne} = \frac{7\times 1.5\times 10^4}{6.02\times 10^{23}\times 10^6\times 1.6\times 10^{-19}} = 10.9\times 10^{-5}\ \text{ms}^{-1}$$

Example 19.13 What is the equivalent resistance between the terminal points A and B in the network shown in figure 19.25? Assume that the resistance of each resistor is 10 ohm.

(*Nagpur Uni*. 2004)

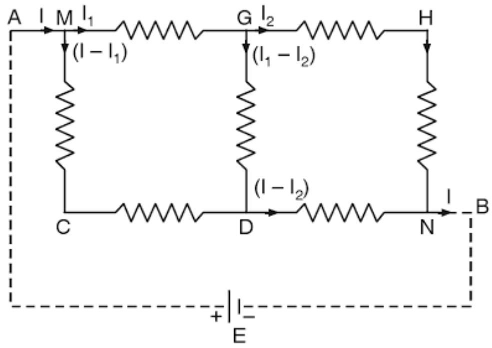

Fig. 19.25

Solution. The distribution of current in different branches is shown in Fig. 19.25

Let R be the resistance of each resistor.

Applying Kirchhoff's first law (current law) at D and N respectively, the current along DN is $(I - I_1 + I_1 - I_2) = I - I_2$.

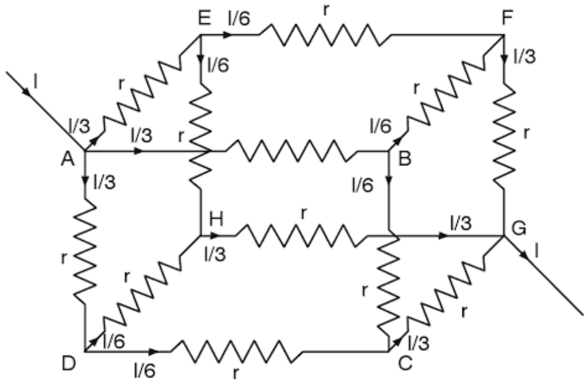

Fig. 19.26

$$= \mathrm{I}r\left(\frac{\mathrm{I}}{3}+\frac{\mathrm{I}}{6}+\frac{\mathrm{I}}{3}\right)$$

$$= \frac{5}{6}\mathrm{I}.r \qquad \text{... } (i)$$

Let R_t be the equivalent resistance between A and G, then

$$V = \mathrm{I}R_t \qquad \text{... } (ii)$$

Equating Eq. (*i*) and (*ii*), we get

$$\mathrm{I}R_t = \frac{5}{6}\mathrm{I}\,r$$

$$\mathrm{R}_t = \frac{5}{6}r$$

Example 19.15 A battery of 6 volts *emf* and 0.5 ohm internal resistance is joined in parallel with another of 10 volts *emf* and 1 ohm internal resistance. The combination is used to send a current through an external resistance of 12 ohm. Calculate, by application of Kirchhoff's law the current thorough each battery.

Solution. Applying Kirchhoff's second law to the mesh ABFEA of Fig. 19.27,

$$I_1 \times 0.5 + (I_1 + I_2)\,12 = 6$$

or $$12.5\,I_1 + 12\,I_2 = 6$$

or $$25\,I_1 + 24\,I_2 = 12 \qquad \text{... } (i)$$

Similarly, applying Kirchhoff's second law to the mesh CDFEC,

$$I_2 \times 1 + (I_1 + I_2)\,12 = 10$$

$$12\,I_1 + 13\,I_2 = 10 \qquad \text{... } (ii)$$

Solving Eq. (*i*) and (*ii*), we get

$$I_1 = -2.27 \text{ amp}$$

$$I_2 = 2.865 \text{ amp}$$

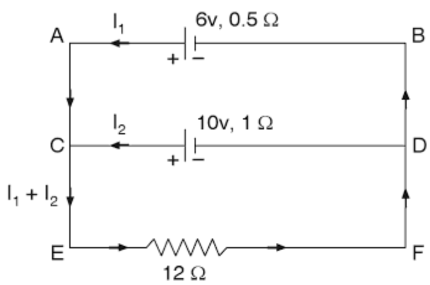

Fig. 19.27

Solution. Time constant $= \dfrac{L}{R}$

I*st case*. $$\frac{L}{R} = 2.5 \times 10^{-3} \text{ sec. or } L = 2.5 \times 10^{-3} R \quad ...(i)$$

2*nd case*. $$\frac{L}{R+80} = 0.5 \times 10^{-3} \text{ sec or } L = 0.5 \times 10^{-3} R + 40 \times 10^{-3} \quad ...(ii)$$

From (*i*) and (*ii*), we have

$$0.5 \times 10^{-3} R + 40 \times 10^{-3} = 2.5 \times 10^{-3} R$$

or $$R = 20 \text{ ohm}$$

Substituting in (*i*), we have

$$L = 2.5 \times 10^{-3} \times 20 = 50 \times 10^{-3} H = 50 \text{ mH}$$

Example 19.21 A capacitor charged by a D.C. source through a resistance of 2 mega ohm takes 0.5 see to charge to $\frac{3}{4}$ of its final value. Show that the capacitance of the capacitor is nearby 0.18 microfarad. (*Gauhati U*., 2000; *K.U*. 2002)

Solution. When the capacitor is being charged,

$$q = q_0 (1 - e^{-t/RC})$$

$\therefore$ $$\frac{q}{q_0} = 1 - e^{-t/RC}$$

or $$1 - \frac{q}{q_0} = e^{-t/RC}$$

or $$\log_e\left(1 - \frac{q}{q_0}\right) = -\frac{t}{RC}$$

Now $$\frac{q}{q_0} = \frac{3}{4}, R = 2 \times 10^6 \text{ ohm}, t = 0.5 \text{ sec}$$

$\therefore$ $$\log_e\left(1 - \frac{3}{4}\right) = \frac{-0.5}{2 \times 10^6 C}$$

or $$\log_e \frac{1}{4} = -\frac{0.25 \times 10^{-6}}{C}$$

or $$-1.386\, C = -0.25 \times 10^{-6}$$

$\therefore$ $$C = \frac{0.25}{1.386} \times 10^{-6} = 0.18 \times 10^{-6} \text{ farad} = 0.18 \text{ microfarad}$$

Example 19.22 A condenser of capacity 1μF is discharge through 1MΩ resistance find the time in which the charge on it falls to 36.8% of its initial value.

(*Nagpur Uni. s*/2009, 2004, *Gauhati U*. 2007)

Solution. The instantaneous charge during discharge is given by

PA693

$$q = q_0 e^{\frac{-t}{RC}}$$

or

$$t = 2.303 RC \log_{10} \frac{q_0}{q}$$

$$= 2.303 \times 1 \times 10^6 \times 1 \times 10^{-6} \log_{10} \frac{100}{36.8}$$

$$= 2.303 \times \log_{10} 2.7173$$

$$= 0.999 \text{ sec.}$$

Example 19.23 Find the maximum value of resistance in *LCR* circuit so that the circuit can just oscillate. (*Nagpur Uni*. 2007)

Solution. It is a transition state from dead beat to oscillatory *i.e.* $b^2 = k^2$

$$\frac{R^2}{4L^2} = \frac{1}{LC}$$

$$R^2 = \frac{4L^2}{LC} = 4\frac{L}{C}$$

$$\therefore \quad R_{\text{max}} = 2\sqrt{\frac{L}{C}}$$

Example 19.24 A capacitor of capacitance 1μF is allowed to discharge through an inductance of 0.2 Henry and the resistance of 800 ohm connected in series. Prove that the discharge is oscillatory. Find its frequency. (*Napur Uni*. 2006)

Solution. Given C = 1$\mu F = 10^{-6}$ F, $L = 0.2$ Henry, $R = 800\ \Omega$

$$\frac{R^2}{4L^2} = \frac{800 \times 800}{4 \times 0.04} = \frac{800 \times 800}{0.16} = 4 \times 10^6$$

$$\frac{1}{LC} = \frac{1}{0.2 \times 10^{-6}} = 5 \times 10^6$$

Since, $\frac{R^2}{4L^2} < \frac{1}{LC}$ the discharge is oscillatory

$$\therefore \quad \text{Frequency} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

$$= \frac{1}{2 \times 3.14} \times \sqrt{5 \times 10^6 - 4 \times 10^6}$$

$$= \frac{10^3}{6.28} = 159 \text{ Hz.}$$

EXERCISE CH. 19

LONG QUESTIONS

1. **(*a*)** Define the terms drift velocity, current and current density vector $\vec{J}$. Differentiate between electric current and current density.

(*P.U.* 2001; *Kerala U.* 2001; *Pbi. U.* 2000, *H.P.U.* 2000, *H.P.U.* 2003)

(*b*) Derive a relation between (*i*) current density and current $\left(I = \iint_S \vec{J} \, . \vec{ds} \right)$.

(*ii*) Current density and drift velocity $(\vec{J} = ne\,\vec{v})$.

(*G.N.D.U.* 2004; P.U. 2001, 2000, *H.P.U.* 2002)

(*iii*) Current and drift velocity $\left(I = \iint_S Ne\,\vec{v} \, . \vec{ds} \right)$. (*M.DU*. 2002, 2001)

2. Derive and discuss the continuity equation

$$\vec{\nabla} . \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

where $\vec{J}$ is the current density vector and ρ is the charge density.

(*K*.U. 2001, *P.U.* 2002, 2001; *H.P.U.* 2001; *Indore U.* 2001; *Kerala U.* 2001, *Nagpur U. s*/2009; *G.N.D.U.* 2001, 2000; *Pbi. U.* 2003, 2000, *M.D.U.* 2002, 2000)

3. (*a*) Define the terms electrical conductivity (σ) and restively (ρ). Express conductivity in term of $|\vec{E}|$ and $|\vec{J}|$. Establish a relation between the two state and define the S.I. unit of electrical conductivity and resistivity. (*H.P.U.* 2002; *Pbi. U.* 2002; *M.D.U.* 2001)

(*b*) How does electrical conductivity depend upon temperature of conductor?

4. **(*a*)** Discuss the validity of Ohm's law from atomic viewpoint and derive the microscopic form of Ohm's law $\vec{J} = \sigma \vec{E}$ from consideration of motion of free electrons in a conductor.

(*b*) Give the limitations of Ohm's law and discuss the situation where Ohm' law fails.

(*H.P.U.*, 2002, 2000; *K.U.*, 2000; *G.N.D.U.*, 2001; *P.U.*, 2003, 2001)

5. Derive Helmholtz's equations for the growth and decay of an electric current in circuit with resistance and self-inductance. What is meant by the time constant of the circuit?

(*Nag. U.*, 2001, *D.A.U. Agra* 2007, *Kerala U.*, 2001; *Gauhati, U.*, 2000)

6. Derive Helmholtz equation for the decay of current in a *L–R* circuit

(*Nagpur Uni.* 2007, 2006)

7. **(*a*)** Why is the self induced e.m.f. stronger when the current in a circuit is cut off than when it is started? (*Pbi. U.*, 2003)

(*b*) A lamp connected in parallel with a large inductor glows brilliantly when the current is switched off. Explain.

20

ALTERNATING CURRENT (A.C.)

INTRODUCTION

When a d.c. source viz. battery or electric cell (Leclanche, Daniel etc.) is connected to a resistance, a negative charge (an electron) flows from electrode of lower potential to an electrode at higher potential. The effect, of course, is the same as that of positive charge flowing from positive to negative electrode, which constitutes, what we call it as "conventional d.c. current". However, a.c. source is that in which current is not unidirectional but has a periodic function with time. The phase terminal alternately changes its polarity from +ve to –ve and vice-versa. The frequency of a.c. supplied in India is 50 cycles per second. It means, phase terminal changes its polarity from +ve to –ve 50 times in one second. In other words, the alternating current is one which passes through a cycle of changes, each cycle contains positive half and negative half cycle. In general, an a.c. has sinusoidally oscillating nature without change in amplitude and has a definite frequency. Resistance (R) plays the same role for a.c. as well as d.c. but inductance (L) and capacitance (C) play altogether different roles, and is discussed in detail in this chapter.

20.1 REPRESENTATION OF ALTERNATING ELECTRIC CURRENT

The variation of alternating *e.m.f.* is given by $E = E_0 \sin \omega t$ and the corresponding current as $I = I_0 \sin \omega t$ where E_0 is the amplitude or peak value of voltage, E represents the instantaneous voltage at time t and $\omega = 2\pi f$ is the angular frequency of the voltage applied, periodic time of oscillation is $T = \dfrac{2\pi}{\omega}$. The terms ωt is called the phase angle. The *e.m.f.* (and the current) first rises to maximum or peak value E_0 (or I_0) in one direction and falls to zero. The direction then quickly reverses so that *e.m.f.* (or current) rises to maximum but in opposite direction $-E_0$ (or $-I_0$) and again falls to zero. Both these actions combined together gives one cycle. The number of such cycles per second is called frequency (f).

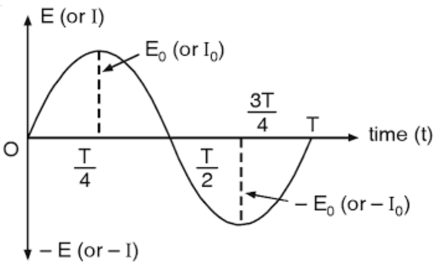

Fig. 20.1

20.2 A.C. CIRCUITS

In previous chapter 19, we have studied the transient currents in a circuit using d.c. source. The problem is relatively much simpler, if the circuit contains d.c. source as compared to the a.c. source. For example, a circuit containing resistance R and capacitance C in series with an alternating current generator (a.c. source), then the steady state current is not zero, as in the case of direct current (d.c. source). In one cycle, with a.c. source, the capacitor is charged, discharged, recharged oppositely, and then discharged once more. This happens again and again. The objective of this chapter is to determine the current in such a circuit. Basically, to analyse a.c. circuit, two methods are popularly used:

1. The vector method, and
2. The complex number method.

20.3 CIRCUIT ANALYSIS USING COMPLEX NUMBER REPRESENTATION

The A.C. network analysis becomes simple and more convenient by using complex number representation. *A combination of real and imaginary quantity is called a complex quantity.* A complex number can be written as

$$\vec{Z} = x + jy \qquad \ldots (i)$$

where $\vec{Z}$ is a complex number consisting of x as a real part and jy as imaginary part, where $j = \sqrt{-1}$. The complex number $\vec{Z}$ is represented in Cartesian coordinates by a vector in a complex plane by choosing x-axis as real axis and y-axis as imaginary axis as shown in Fig (20.2). The complex number $\vec{Z}$ is completely defined by point P. since the projection along the real axis of line $\vec{OP}$ gives real part and its projection on y-axis gives the imaginary part of the complex number.

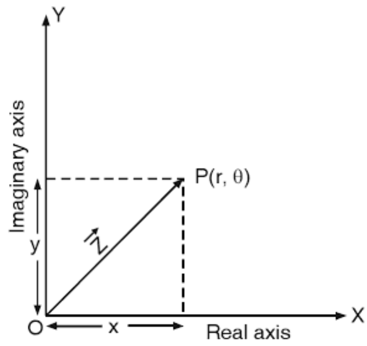

Fig. 20.2

Suppose the point P has polar coordinates (r, θ), then, we have

$$x = r\cos\theta \qquad \ldots. (ii)$$
$$y = r\sin\theta$$

Substituting in Eq. (i), we get

$$\vec{Z} = r\cos\theta + jr\sin\theta$$
$$= r(\cos\theta + j\sin\theta)$$
$$= r\,e^{j\theta} \qquad \ldots (iii)$$

where
$$r = \sqrt{x^2 + y^2} \text{ and} \qquad \ldots (iv)$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad \ldots (v)$$

Moreover, any vector in a complex plane can be completely specified by complex number. This indicates that analysis of A.C. network can be easily simplified, if the sinusoidal voltages and currents are expressed in complex number representation than vector method representation. Thus, using complex number representation A.C. voltage and current is expressed as $E = E_0\,e^{j\omega t}$ and $I = I_0\,e^{j\omega t}$ respectively, where $\theta = \omega t$.

20.4 A.C. APPLIED TO PURE RESISTANCE ONLY

Fig. 20.3 (a) shows a circuit containing a pure resistance R and an alternating *e.m.f.* $E_0 \sin \omega t$ applied across it. This voltage is the imaginary part of the complex number $E_0\,e^{j\omega t}$. Thus, in complex number representation, we write

$$E = E_0\, e^{j\omega t}. \qquad \text{... }(i)$$

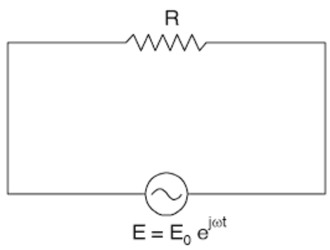

Fig. 20.3 (*a*)

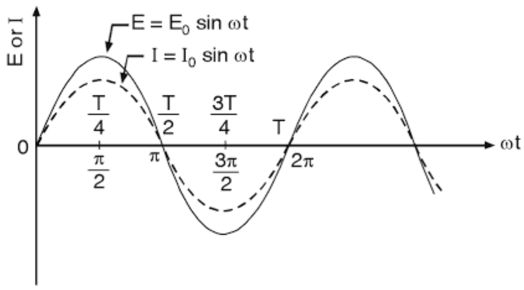

Fig. 20.3 (*b*)

It I is the instantaneous current in the circuit at any time t, then e.m.f. equation of the circuit is

$$RI = E_0\, e^{j\omega t} \qquad \text{... }(ii)$$

$$\therefore \qquad I = \frac{E_0\, e^{j\omega t}}{R} = \frac{E_0}{R}\, e^{j\omega t}$$

or,

$$I = I_0\, e^{j\omega t} \qquad \text{... }(iii)$$

where $I_0 = \dfrac{E_0}{R}$, the maximum value of the current in the circuit.

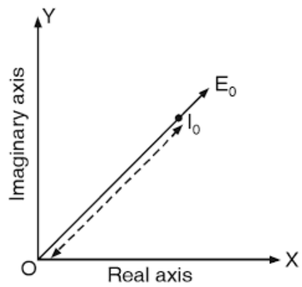

Fig. 20.4

Eq. (*iii*) represents the variation of current with time t in the circuit. Fig. 20.3 (*b*) shows the variation of *e.m.f.* and current through resistance R. Comparing the two waveforms and corresponding equations [Eq. (*i*) and (*iii*)], we conclude that the current has the same waveform and frequency as that of the applied voltage. Thus, the current is in phase with the applied e.m.f as shown in Fig. 20.3 (*b*). Fig. 20.4 shows the complex number representation of voltage and current through resistance R. Here E_0 and I_0 are different in magnitude ($E_0 > I_0$) but have the same phase, *i.e.* phase difference between E and I is zero. The impedance Z is real and is equal to R ohm. The admittance Y is given by

$$\text{Admittance,} \qquad Y = \frac{1}{Z} = \frac{1}{R} \qquad \text{... }(iv)$$

20.5 A.C. APPLIED TO PURE INDUCTANCE ONLY

An inductor of self-inductance L is connected to an alternating voltage $E = E_0 \sin \omega t$ as shown in Fig. 20.5 (*a*). This voltage is the imaginary part of the complex number $E_0 e^{j\omega t}$. Thus, in complex number representation, we write as

$$E = E_0\, e^{j\omega t} \qquad \text{... }(i)$$

Let I be instantaneous value of the current at any instant t and $\dfrac{dI}{dt}$, the rate of change of

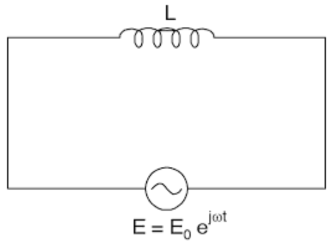

Fig. 20.5 (*a*)

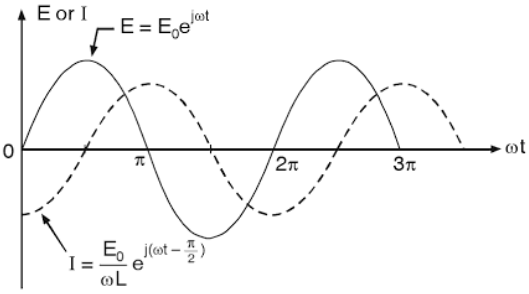

Fig. 20.5 (*b*)

current at the same instant t, then *e.m.f.* induced in the inductor is given by $-L\frac{dI}{dt}$. The negative sign indicates that the induced *e.m.f.* opposes the changes of current. The two seats of voltages balance each other. Hence

$$E_0\, e^{j\omega t} - L\frac{dI}{dt} = 0$$

$$\therefore \qquad dI = \frac{E_0}{L} e^{j\omega t}.\, dt$$

On integration, we get

$$\int dI = \frac{E_0}{L}\int e^{j\omega t}\, dt + k$$

Where k is constant of integration and $k = 0$

$$I = \frac{E_0}{L}\cdot\frac{e^{j\omega t}}{j\omega}$$

$$= \frac{E_0}{j\omega L} e^{j\omega t}$$

$$= E\big/ j\omega L \qquad \ldots (ii)$$

The quantity $j\omega L$ is the complex inductive reactance of the inductance L.

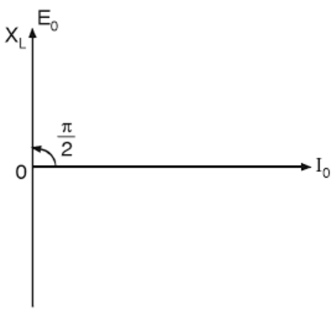

Fig. 20.6

Also,

$$I = \frac{E}{j\omega L} = \frac{E_0 \,.\, e^{j\omega t}}{\omega L\left(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}\right)}$$

$$= \frac{E_0 \,.\, e^{j\omega t}}{\omega L \,.\, e^{j\pi/2}}$$

Impedance. The complex impedance of the A.C. circuit.

$$Z = R + jL\omega = R + j\,X_L$$

$\sqrt{R^2 + X_L^2}$ gives the *modulus* of Z

i.e., $$|\vec{Z}| = Z = \sqrt{R^2 + X_L^2}$$

$\phi = \tan^{-1}\dfrac{\omega L}{R} = \tan^{-1}\dfrac{X_L}{R}$ gives the *argument* of Z.

Fig. 20.12 represents the phasor diagram in complex number representation.

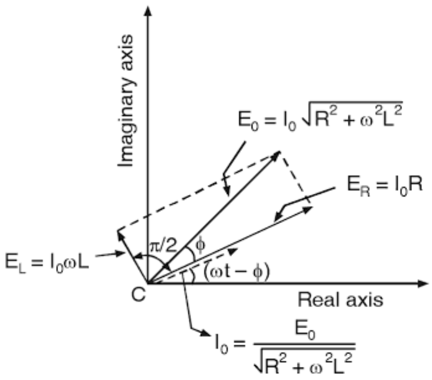

Fig. 20.12

20.8 A.C. CIRCUIT WITH CAPACITANCE AND RESISTANCE IN SERIES

Let the A.C. circuit contains a pure resistance R and a pure capacitance C to which a complex *e.m.f.* $E = E_0\, e^{j\omega t}$ is applied as shown in Fig. 20.13 (*a*).

If I is the instantaneous current through the circuit, then

Potential drop across $R = E_R = RI$

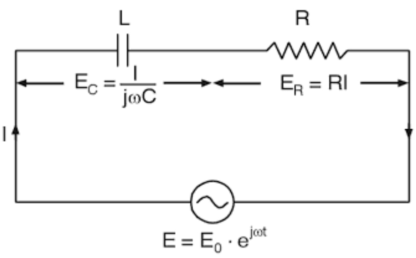

Fig. 20.13 (*a*)

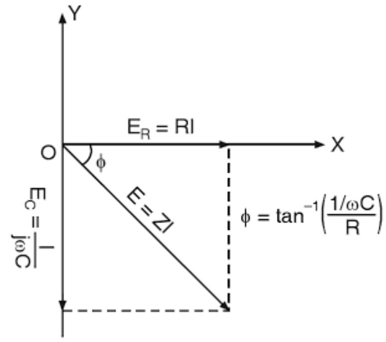

Fig. 20.13 (*b*)

The current in R is in phase with *e.m.f.*

Potential drop across $C = E_C = \dfrac{1}{j\omega C} I$

where $\dfrac{1}{\omega C}$ is the reactance due to capacitance and is denoted by X_C.

Multiplication by $\dfrac{1}{j}$ implies that E_C lags with respect to the current by a phase angle $\pi/2$.

Hence we can represent $E_R = RI$ along the real axis and $E_C = \dfrac{1}{j\omega C} I = -\dfrac{j}{\omega C} I$ along the negative direction of the imaginary axis of the complex plane as in Fig. 20.13 (*b*)

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$$\therefore \qquad E = E_R + E_C = RI - \frac{jI}{\omega C} = I\left(R - \frac{j}{\omega C}\right)$$

$$I = \frac{E}{R - j/\omega C} = \frac{E_0\, e^{j\omega t}}{R - j/\omega C} \qquad \ldots (ii)$$

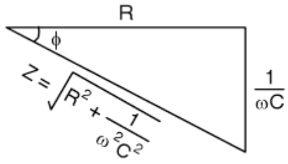


Fig. 20.13 (*c*)

Put $R = Z\cos\phi$ and $\frac{1}{\omega C} = Z\sin\phi$, then

$$\tan\phi = \frac{1/\omega C}{R} \text{ and } Z = \sqrt{R^2 + \frac{1}{\omega^2 C^2}}$$

$$\therefore \qquad R - \frac{j}{C\omega} = Z\cos\phi - j\,Z\sin\phi = Z\,e^{-j\phi}$$

Substituting in (*ii*), we have

$$I = \frac{E_0\, e^{j\omega t}}{Ze^{-j\phi}} = \frac{E_0}{Z} e^{j(\omega t + \phi)} = I_0\, e^{j(\omega t + \phi)}$$

where $$I_0 = \frac{E_0}{Z}$$

Phase difference. When an alternating *e.m.f.* $E = E_0\, e^{j\omega t}$ is applied to a circuit containing a resistor and a capacitor in series, the current through the circuit is given by $I = I_0\, e^{j(\omega t + \phi)}$

The current, therefore, Leads the *e.m.f.*

On the other hand if a current $I = I_0\, e^{j\omega t}$ flows through an RC circuit, the E.M.F is given by

$$E = E_0\, e^{j(\omega t - \phi)}$$

i.e., the. *e.m.f.* lags with respect to the current by an angle ϕ as shown in Fig. 20.14.

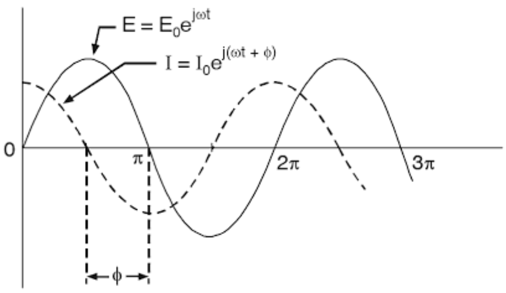


Fig. 20.14

Impedance. The complex impedance of the A.C. circuit

$$Z = R - \frac{j}{\omega C} = R - j\,X_C$$

The modulus of $\vec{Z}$ will be

$$Z = \frac{E_0}{I_0} = \frac{E_0/\sqrt{2}}{I_0/\sqrt{2}} = \frac{E_0(rms)}{I_0(rms)} \quad \ldots (xii)$$

Thus, the impedance of a LCR circuit can be defined as the ratio of the r.m.s value of e.m.f applied to the r.m.s. current flowing through it.

Eq. (x) shows that the current lags the applied voltage in phase by an angle. given by

$$\phi = \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right) \quad \ldots (xiii)$$

This phase difference is shown in fig 20.17 (b) in the complex plane. The voltage across the resistance E_R is in phase with the current while, $E_X = E_L - E_C$ which is the net voltage across the combination of L and C leads the current by an angle $\pi/2$. The total voltage E_0 is shown leading the current by an angle ϕ given by Eq. ($xiii$).

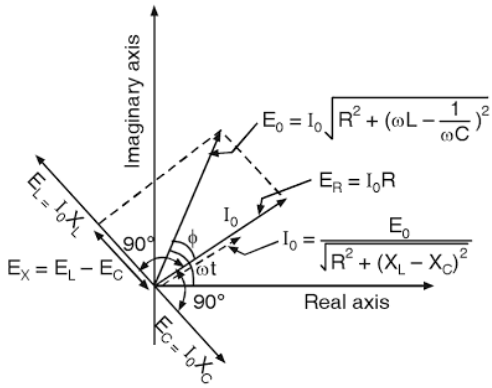


Fig. 20.17 (*b*)

Special Cases: The phase difference ϕ (either leading or lagging) depends on the relative values of X_L and X_C. Accordingly the following three cases arise:

1. If $\omega L > \frac{1}{\omega C}$ *i.e.* when the impressed frequency $\left(f = \frac{\omega}{2\pi}\right)$ is very large, the phase angle ϕ (see Eq. $xiii$) will be positive and the current will lag behind the applied e.m.f. The potential difference across the inductor $I_0X_L = I_0\omega L$ will be greater than that across the capacitor $I_0/\omega C$ and therefore, the circuit behaves as an inductive circuit. The vector voltage in complex plane is shown in Fig. 20.17(b).

2. If $\omega L < \frac{1}{\omega C}$ In this case the angle ϕ will be negative and the current will leads the impressed e.m.f. The circuit, therefore, behaves as capacitive circuit. The vector diagram in complex plane is shown in Fig. 20.18.

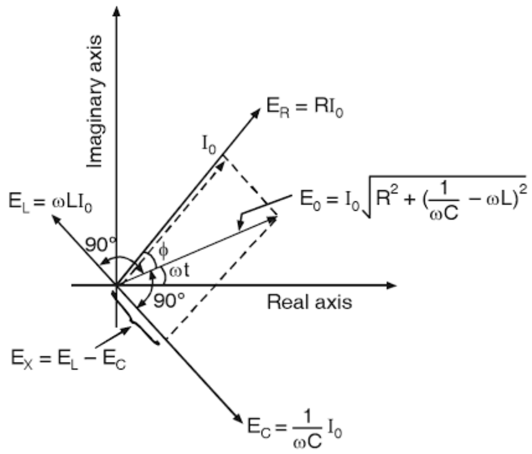


Fig. 20.18

3. If $\omega L = \dfrac{1}{\omega C}$ This is very important case which leads to a resonance condition. In this case,

$$\phi = \tan^{-1}\left(\frac{\omega L - \dfrac{1}{\omega C}}{R}\right) = 0$$

The phase angle ϕ becomes zero. i.e. the impressed *e.m.f.and* the current are in phase. the potential difference across the inductance ($E_L = \omega_r L I_0$) and that across the capacitance $\left(E_C = \dfrac{I_0}{\omega_r C}\right)$ are equal in magnitude but opposite in direction (phase) and therefore cancel out each other. The whole voltage drops across the resistance R. Thus, the circuit behaves as purely resistive circuit as shown by vector diagram in complex plane in Fig. 20.19.

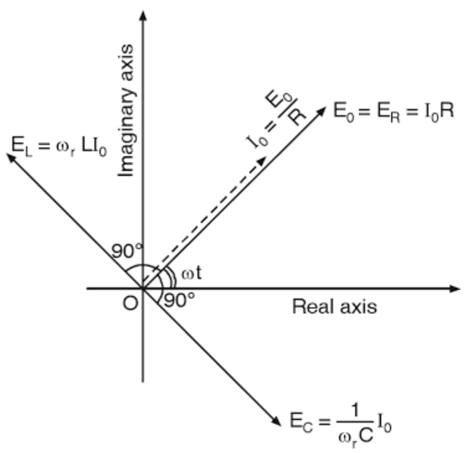


Fig. 20.19

Resonance : This is an interesting case, when $\omega L = \dfrac{1}{\omega C}$ in which impressed voltage and current are in phase and the current is maximum. The circuit is said to be a series resonant circuit and the frequency, at which this occurs is called resonant frequency denoted by ω_r. Thus, at resonance,

$$\omega_r L = \frac{1}{\omega_r C} \qquad \ldots (xiv)$$

or

$$\omega_r = \frac{1}{\sqrt{LC}}$$

Putting $\omega_r = 2\pi f_r$, we get

$$f_r = \frac{1}{2\pi\sqrt{LC}} \qquad \ldots (xv)$$

Fig. 20.20 shows the variation of current with frequency and the current is maximum i.e peak current, at frequency equal to resonant frequency (f_r).

In other words, when the frequency of the applied *e.m.f.* is equal to the natural frequency of the LC circuit, the current is maximum and the resonance is said to take place. Moreover, under this condition (condition of resonance), phase angle ϕ between current and applied *e.m.f.* $\phi = \tan^{-1}\left(\dfrac{\omega L - \dfrac{1}{\omega C}}{R}\right) = 0$. *i.e.* at resonance current and applied *e.m.f.* are in phase. However it should be noted that the phase difference between E_L and E_c is 180° and their magnitudes are equal $E_L = E_C$.

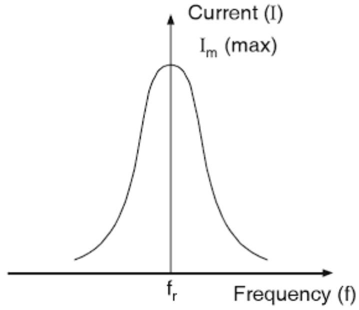


Fig. 20.20: Resonance Curve

The maximum amplitude of current at resonance is given by

$$(I_0)_{res} = \frac{E_0}{R}$$

because at resonance e $X_L - X_C = L\omega - \dfrac{1}{C\omega} = 0,$ the impendance $Z = R$ and is purely resistive. The phase difference between current and voltage

$$(\phi)_{res} = \tan^{-1}\left[\frac{L\omega - \dfrac{1}{C\omega}}{R}\right]_{res} = 0$$

Thus the current is in phase with applied voltage at resonance frequency $\omega = \omega_r = \dfrac{1}{\sqrt{LC}}$. If $R = 0$ then for all values of $\omega < \omega_r$, the current leads the voltage by a phase angle $\dfrac{\pi}{2}$ and for all values of $\omega > \omega_r$ the current lags behind the voltage by $\frac{\pi}{2}$. For finite values of R the angle ϕ by which the current leads for values of $\omega < \omega_r$ and lags for values of $\omega > \omega_r$ with respect to the voltage, slowly increase from 0 to $+\frac{\pi}{2}$ as ω decreases from ω_r to zero and from 0 to $-\frac{\pi}{2}$ as ω increases from ω_r to infinity as shown in Fig. 20.21

20.11 SHARPNESS OF RESONANCE IN SERIES LCR CIRCUIT

(Effect of Resistance)

The frequency of the applied *e.m.f.* must be equal to the natural frequency of the *LC* circuit. Since, at resonances $X_L = X_C$ *i.e.* $\omega L = \frac{1}{\omega C}$, out of L, C and R, it is the resistance that controls the current.

The amplitude of the current at resonance $(I_0)_{res} = \frac{E_0}{R}$. Thus smaller the value of R, larger is the value of $(I_0)_{res}$. If $R = 0$; $(I_0)_{res} = \infty$. A graph showing the variation of I with frequency ω of the applied voltage for $R = 0$, R very low and R large is shown in Fig. 20.22 It is seen from the graph that smaller the value of R sharper is the resonance peak. The resonance is said to be sharp if the current amplitude at resonance falls off quickly for a small change in frequency on either side of the resonant frequency ω_r. Quantitatively, '*The sharpness of resonance for a series resonant circuit is defined as the ratio of the resonant frequency to the difference in frequencies taken on both sides of the resonant frequency for which the power dissipation in the circuit becomes half the value at resonance frequency.*' The two points on the resonance curve for which the power dissipation is half of its value at resonance are called *half power points*.

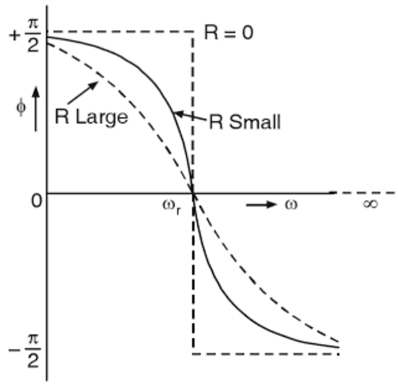


Fig. 20.21

As electric power = $I^2 R$, for power to fall to half its value the current amplitude will become $\frac{1}{\sqrt{2}}$ the amplitude at resonance. Therefore, *sharpness of resonance may also be defined as the ratio of the resonance frequency to the difference of frequencies on both sides of the resonance frequency at which the current in the circuit falls to $\frac{1}{\sqrt{2}}$ of its value at resonance.*

Let the corresponding values of the frequencies be

$$\omega_r - \Delta\omega \text{ and } \omega_r + \Delta\omega.$$

If $(I_0)_{res}$ is the current at resonance, the current at a frequency $\omega_r \pm \Delta\omega$

$$= \frac{1}{\sqrt{2}}(I_0)_{res} = \frac{1}{\sqrt{2}}\frac{E_0}{R}$$

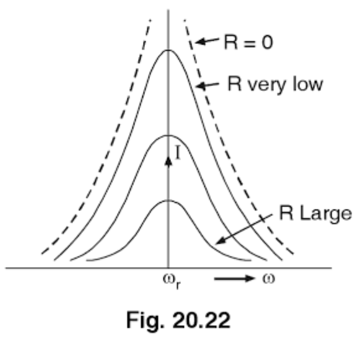


Fig. 20.22

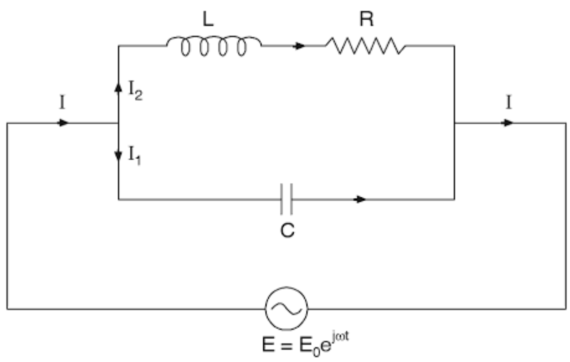


Fig. 20.24

then $$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}$$

or $$Y = Y_1 + Y_2 \text{ where } Y = \frac{1}{Z}$$

Now, the current, $$I_2 = \frac{E}{Z_2}$$

and $$I_1 = \frac{E}{Z_1}$$

$\therefore$ $$I = I_1 + I_2$$

Now from Fig. 20.14,we write impedance of C is Z_1 and is given by $Z_1, = \frac{1}{j\omega C}$ and Z_2 is for R and L in series, i.e. $$Z_2 = R + j\omega L.$$

$\therefore$ $$\frac{1}{Z_1} = j\omega C \text{ and } \frac{1}{Z_2} = \frac{1}{R + j\omega L} \quad \text{... } (ii)$$

Multiplying by $R - j\omega L$ i.e. rationalising, we get

$$\frac{1}{Z_2} = \frac{R - j\omega L}{(R + j\omega L)(R - j\omega L)} = \frac{R - j\omega L}{R^2 + \omega^2 L^2} \quad (\because j^2 = -1)$$

$\therefore$ Total Impedance, $$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} \text{ or } Y = Y_1 + Y_2$$

or $$Y = j\omega c + \frac{R - j\omega L}{R^2 + \omega^2 L^2}$$

$$= j\omega C + \frac{R}{R^2 + \omega^2 L^2} - \frac{j\omega L}{R^2 + \omega^2 L^2}$$

$\therefore$ $$Y = \frac{R}{R^2 + \omega^2 L^2} + \frac{j\omega C\,(R^2 + \omega^2 L^2) - j\omega L}{R^2 + \omega^2 L^2}$$

$$= \frac{R}{R^2+\omega^2 L^2} + \frac{j\omega(CR^2 + C\omega^2 L^2 - L)}{R^2+\omega^2 L^2} \qquad \text{... (iii)}$$

This can be written as $Y = G + jB$ where $G = \dfrac{R}{R^2+\omega^2 L^2}$ is the conductance and

$$\text{B} = \frac{\omega[CR^2 + C\omega^2 L^2 - L]}{R^2+\omega^2 L^2} \qquad \text{... (iv)}$$

Now $\tan\theta = \dfrac{B}{G}$, At resonance $\theta = 0$

i.e. $$B = 0 \quad \because \quad G \neq 0$$

$$\therefore \quad CR^2 + C\omega^2 L^2 - \text{L} = 0 \qquad \text{... (v)}$$

$$C\omega^2 L^2 = L - CR^2$$

or $$\omega^2 = \frac{1}{LC} - \frac{R^2}{L^2}$$

or $$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}} \qquad \text{... (vi)}$$

This gives the frequency at parallel resonace.

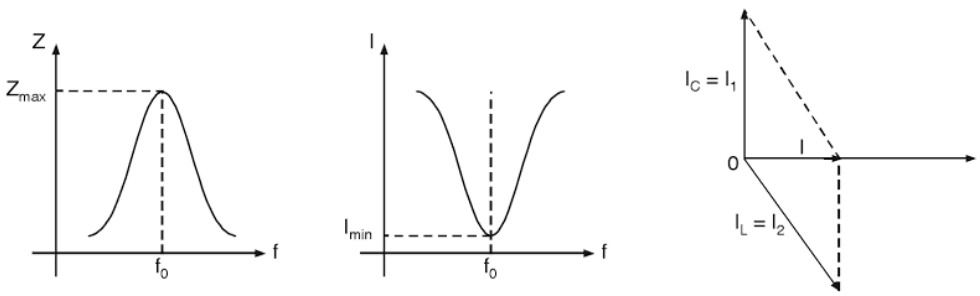
$\therefore$ Natural frequency, $$f_r = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}} \qquad \text{... (vii)}$$

The impedance at parallel resonance is $\dfrac{1}{G} = R$

$\because$ B $\neq 0$ or $$\frac{1}{G} = \frac{R^2+\omega^2 L^2}{R} \qquad [\because \text{Eq. } (iii)] \qquad \text{... (viii)}$$

But $$R^2 + \omega^2 L^2 = \frac{L}{C} \qquad [\because \text{Eq. } (v)]$$

$$\therefore \quad \text{Z at resonance} = \frac{L}{CR} \qquad \text{... (ix)}$$



(a) (b) (c)

Fig. 20.25

Thus, from Fig. 20.25 (*a*), the impedance is maximum (but not infinite) at the resonance and the current is minimum. In Fig. 20.25 (b), the variation of current is plotted against frequency (*f*). I is in phase with E at resonance and I_L and I_C having equal values. Now from Fig. 20.25 (*c*),

We write
$$\frac{I_1}{I} = \omega C Z_p = \frac{\omega L}{R}$$

Q of the circuit is defined as ratio of capacitor current to the line current,

$$Q = \frac{I_1}{I} = \frac{\omega L}{R} \quad \text{... (x)}$$

Rejector circuit. As the current in a parallel resonant circuit at resonant frequency is minimum and has almost zero value (Impedance being maximum) such a circuit is known as **rejector circuit**. The circuit does not accept *i.e.* rejects the incoming signals of the frequency which is equal to its natural frequency.

Applications. Parallel resonant circuits produce oscillations and are used as *transmitting circuits.*

In such a circuit the current through the inductance is equal to the current through the capacitance but these have a phase difference of 180° between them. Such circuits, therefore, reject (or cut off) the currents corresponding to parallel resonant frequencies and allow other frequencies to pass through. These are, therefore known as **filter circuit**s or *rejector circuits.* Some times these are also called *anti-resonant circuits.*

Distinction. (*i*) In series resonant circuit impedance is minimum at resonance whereas in a parallel resonant circuit, it is maximum.

(*ii*) In a series resonant circuit the current through the circuit is maximum whereas in a parallel resonant circuit maximum voltage is set up across A and B, the two ends of the circuit having L and C in parallel.

20.16 POWER CONSUMED BY A.C. CIRCUIT AND POWER FACTOR

In an $A.C.$ circuit, in general the applied *e.m.f.* and the current are not in phase. Let the applied *e.m.f.* and the current be represented by

$$E = E_0 \sin \omega t$$

or
$$i = I_0 \sin (\omega t - \phi)$$

where ϕ is the angle of lag or lead.

The power Ei at any instant is given by

$$Ei = E_0 I_0 \sin \omega t \sin (\omega t - \phi)$$

$\therefore$ Work done in a very small time dt

$$Eidt = E_0 I_0 \sin \omega t \sin (\omega t - \phi)\, dt$$

Hence the total work done in time T, equal to half the time period

$$= \int_0^T E_0 I_0 \sin \omega t \sin (\omega t - \phi)\, dt$$

$$= \frac{E_0 I_0}{2} \int_0^T \cos \phi\, dt - \frac{E_0 I_0}{2} \int_0^T \cos (2\omega t - \phi)\, dt$$

$\because \qquad \sin A \sin B = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\}$

Now $\qquad \int_0^T \cos(2\omega t - \phi)\, dt = 0$ and $\int_0^T \cos\phi\, dt = \cos\phi T$

$\therefore$ Work done in time $T = \frac{E_0 I_0}{2}\cos\phi \times T$

$\therefore$ Average power $= \frac{E_0 I_0 \cos\phi}{2\times T}\times T = \frac{E_0 I_0}{2}\cos\phi$

$$= \frac{E_0}{\sqrt{2}}\frac{I_0}{\sqrt{2}}\cos\phi$$

$$= E_{0\,(rms)} \cdot I_{0\,(rms)} \cos\phi. \qquad \text{... } (i)$$

Power factor. From Eq. (*i*) we find that

Average power = virtual current × virtual *e.m.f.* × cosine of the angle of lag (or lead)

The factor cos ϕ is called the *power factor* and $E_{0(\text{rms})} \times I_{0(\text{rms})} \cos\phi$ is called the *true power*.

Hence true power = apparent power × power factor.

For a circuit containing resistance, inductance and capacitance in series, the value of ϕ is given by

$$\tan\phi = \frac{X_L - X_C}{R}$$

where X_L is the reactance due to inductance, X_C the reactance due to capacitance and R the ohmic resistance. The value of the power factor cos ϕ as shown in Fig. 20.26 is given by

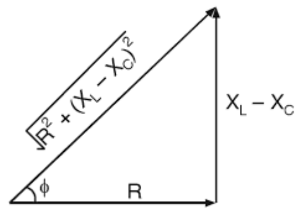


Fig. 20.26

$$\cos\phi = \frac{R}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{R}{Z}$$

$$= \frac{\text{Resistance}}{\text{impedance}}$$

This relation holds good for a circuit containing only resistance and inductance, only resistance and capacitance as well as for a circuit containing resistance, inductance and capacitance.

For circuits having only resistance and inductance

$$\cos\phi = \frac{R}{\sqrt{R^2 + X_L^2}}$$

For circuits having only resistance and capacitance

$$\cos\phi = \frac{R}{\sqrt{R^2 + X_C^2}}$$

Wattless currents. *Current is said to be wattless when the power factor cos* $\phi = 0$ *and the circuit does not consume any power.*

Therefore the condition for the current to be wattless is that

$$\cos\phi = 0$$

or

$$\phi = \frac{\pi}{2}$$

i.e., the phase difference between the current and the voltage should be $\frac{\pi}{2} = 90°$.

20.17 CAPACITOR IMPROVES POWER FACTOR IN ELECTRIC MOTOR

A capacitor of large value is connected across an electric motor to improve the power factor. The armature and the field magnets of an electric motor consist of a large number of windings of copper *wire* wound over a magnetic core and therefore, have a *large inductance*. The power factor in an A.C. circuit is given by

$$\cos\phi = \frac{R}{\sqrt{R^2 + (X_L - X_C)^2}}$$

When X_L is large and X_C is small (or negligible) cos ϕ has a *low* value. But when X_C is increased by connecting a capacitor of large value across the motor $(X_L - X_C)$ has a very small value and the power factor cos ϕ approaches nearly *unity* thereby giving a high useful power.

20.18 CHOKE COIL (SUPERIOR ELEMENT TO REDUCE A.C. CURRENT)

For many purposes it is required to reduce the current in a given circuit when the supply voltage is constant. In a direct current circuit this is done by using a rheostat but in this case there is a loss of power equal to i^2R. Thus actually we deliberately allow a part of the available power to be wasted across an idle resistance and maintain the useful power in the apparatus within the desired limit.

In an alternating current circuit a **choke coil** is used for this purpose. It consists of an inductance which is made by having a large number of turns of an insulated copper wire wound over a closed soft iron laminated core. The presence of an inductance in an A.C. circuit reduces the current which is given by the relation

$$I_{\text{rms}} = \frac{E_v}{\sqrt{R^2 + L^2\omega^2}}$$

The power in an A.C. circuit is given by $E_{rms}\, I_{rms} \cos\phi$. The controlling action of the choke coil is due to the presence of the power factor

$$\cos\phi = \frac{E_v}{\sqrt{R^2 + L^2\omega^2}}$$

As the inductance L is increased the value of cos ϕ decreases and the circuit draws a lesser power from the source. Since the ohmic resistance of the inductance is small the loss of power across the choke coil is negligible, the only loss being due to hysteresis. Choke coils used on low frequency have an iron core and are known as low frequency or audio frequency (*A.F.*) chokes whereas chokes used on high frequency have air cores and are known as high frequency or radio (*R.F.*) chokes.

Reason. The core of low frequency choke (or *A.F.* choke) is made of soft iron to increase its inductance. The co-efficient of self induction of a coil is given by $L = \mu_r \mu_0 N^2 a/l$ when N is the total number of turns in the choke, l its length, a the area of cross-section, μ_0 the permeability of air or free space and μ_r the relative permeability of the material of the core. As μ_r for soft iron is very high, a choke with soft iron core has an inductance μ_r times the inductance of the same choke with air core. As reactance due to inductance $X_L = L\omega = 2\pi fL$, X_L also becomes very large for a choke with soft iron.

A soft iron core is not required for high frequency (*R.F.*) chokes as the frequency and hence ω being very large the reactance is very high even with an air core.

Superiority of choke coil. A choke coil is considered superior to a rheostat and a transformer because in it the only wastage of energy is due to hysteresis loss in the iron core which is much less than the wastage of energy in the resistance that would reduce the current to the same extent. A choke coil draws nearly the same energy from the source as is required in the circuit by changing the power factor of the circuit. Whereas in a rheostat the energy drawn from the source remains uncharged - the rheostat wasting more energy if the circuit requires less. On the other hand in a transformer in addition to the hysteresis loss there are many other losses, the most important being the loss due to leakage of magnetic flux. Moreover the primary circuit consumes energy due to its ohmic resistance even when the secondary circuit is open.

20.19 COMPARATIVE STUDY OF SERIES AND PARALLEL RESONANT CIRCUITS

The following table describes the results of series and parallel resonant circuits discussed in this chapter. The comparative features are:

Series Resonant circuit	**Parallel Resonant circuit**
1. Resonant frequency is $f_r = \dfrac{1}{2\pi\sqrt{LC}}$	1. Resonant frequency is $f_r = \dfrac{1}{2\pi}\sqrt{\dfrac{1}{LC} - \dfrac{R^2}{L^2}}$
2. At resonance, the impedance of the circuit is minimum while the admittance is maximum	2. At resonance, the impedance of the circuit is maximum while the admittance is minimum.
3. At resonance, the power factor is unity and the impedence is purely resistive Z_r = R	3. At resonance, here also the power factor is unity but the impedance is given by $Z_r = \dfrac{L}{RC}$
4. At resonance, the current is maximum.	4. At resonance, the current is minimum.
5. Series resonant circuit is called as acceptor circuit because it accepts a particular frequency and rejects others.	5. The circuit is called as rejector circuit because it rejects only one frequency and accepts others.
6. At resonance, the circuit gives voltage amplification equal to Q, the quality factor of the circuit.	6. At resonance, the circuit gives current magnification equal to quality factor (Q) of the circuit.

20.20 ELECTRICAL NETWORK

Many electrical circuit are complicated, and analysis for current, voltage, power or frequecny performance become difficult unless systematic methods are developed and employed.

20.21 Y AND Δ NETWORKS

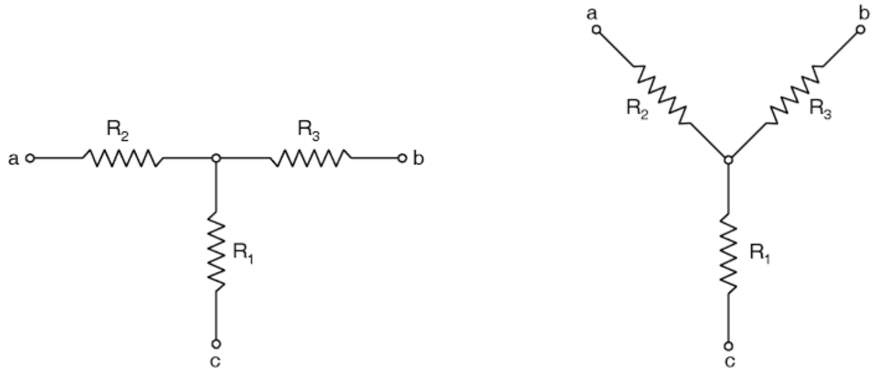


Fig. 20.30. T or Y network

The shape of the networks shown in Fig. 20.30 is similar to letters T and Y respectively. The resistances in the arms are R_2, R_3 and R_1 as shown in figure.

Another networks are shown in Fig. 20.31 whose shapes are similar to the Greek letters π (*pi*) or Δ (delta). R_A can be either on the top or the bottom between R_B and R_C. These tow shapes are identical in the sense that the point C in Δ network is separated into two points c' and c'' for π network, the connections remain unchanged.

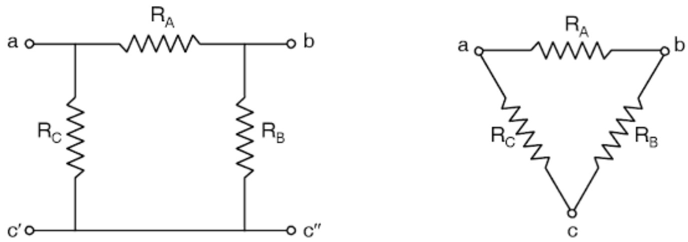


Fig. 20.31: π or Δ network

The formulae for conversion T to π or Y to Δ are derived by using Kirchhoff's laws. The formulae are:

$$R_A = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_1} \quad \ldots (i)$$

$$R_B = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_2} \quad \ldots (ii)$$

$$R_C = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_3} \quad \ldots (iii)$$

or

$$R_\Delta = \frac{\sum \text{All cross products in } Y}{\text{Opposite } R \text{ in } Y}$$

Note that resistors R_1 is opposite to closed side R_A, R_2 to R_B and R_3 is opposite to closed side R_C.

For the opposite conversion, i.e. conversion from Δ to Y or π to T, the formulae derived are:

$$R_1 = \frac{R_B R_C}{R_A + R_B + R_C} \quad \ldots (iv)$$

$$R_2 = \frac{R_C R_A}{R_A + R_B + R_C} \quad \ldots (v)$$

$$R_3 = \frac{R_A R_B}{R_A + R_B + R_C} \quad \ldots (vi)$$

or $$R_Y = \frac{\text{Product of two adjacent } R \text{ in } \Delta}{\Sigma \text{ all } R \text{ in } Y}$$

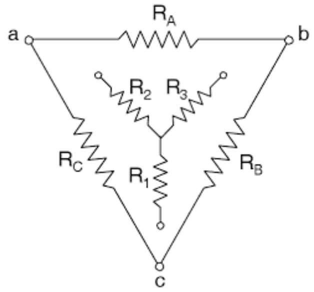


Fig. 20.32

Here each resistor in open arm (*i.e.* in Y network) has two adjacent resistors, i.e. R_1 has R_B and R_C; R_2 has R_A and R_C while R_3 has R_A and R_B as adjacent resistors. To remember the formula, put Y in Δ as shown in Fig. 20.32

20.22 TRANSMISSION OF ELECTRIC POWER

Polyphase system: Polyphase or multiphase system consists of two or more number of equal voltages with fixed phase difference, which supply power to loads connected by the lines, called transmission lines. In the two-phase system, the two equal voltages differ in phase by 90°, while in the three-phase system, the equal voltages differ in phase by 120°. Three-phase system is generally used for generation and transmission of electric power.

Three phase system. When the alternator has three generator windings equally spaced on rotation in a constant magnetic field results in induced output voltages 120° out of phase with each other as shown in Fig. 20.33 (*a*).

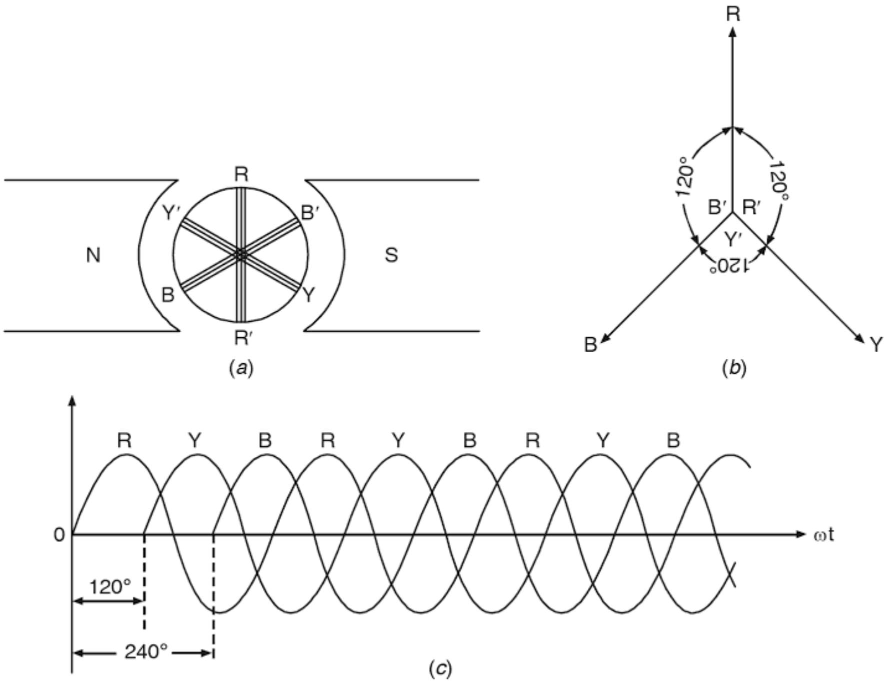


Fig. 20.33: Three phase system

R.M.S. value of current $I_v = \frac{E_v}{Z} = \frac{E_0}{\sqrt{2}Z} = \frac{.707E_0}{Z}$

$$= \frac{7.07}{3025} = 2.34 \times 10^{-3}\ \text{A}.$$

$$= 2.34\ \text{mA}$$

Phase difference between *e.m.f.* and current

$$\phi = \tan^{-1}\frac{X_L - X_C}{R} = \tan^{-1}\frac{3142 - 159}{500} = \tan^{-1} 5.966$$

or $\phi = 80.48°$

Example 20.6. Find the natural frequency of a circuit containing inductance 50 micro-henry and a capacity of 0.005 micro farad. Find the wavelength to which this corresponds.

Solution. Here Inductance L = 50×10^{-6} henry

capacitance C = 0.0005 microfarad = 5×10^{-10} farad

Natural frequency of the circuit

$$n = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{50\times10^{-6}\times5\times10^{-10}}}$$

$$= \frac{10^8}{10\pi\sqrt{10}} = 1007000 \text{ cycles/sec.}$$

$$= 1.007 \times 10^6\ \text{Hz} = 1.007\ \text{MHz}$$

Example 20.7. A capacitor of 50μF and inductance of 0.2025 *H* are connected in series. If the resistance of the circuit is negligible find the frequency at which resonance occurs

(*Nagpur Uni.* 2002)

Solution. Resonant frequency, $n = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\times3.142\times\sqrt{0.2025\times50\times10^{-6}}}$

$$= 50\ \text{Hz}.$$

Example 20.8. An A.C. circuit has L = 10 mH, C = 10 μF, R = 10 ohm, calculate (*i*) Natural frequency (*ii*) resonant frequency (*iii*) impedance of the circuit at resonance (*iv*) *Q*-factor. (*Nagpur Uni. s*/2009)

Solution. (*i*) Natural frequency $n = \frac{1}{2\pi\sqrt{LC}}$

$$= \frac{1}{2\pi\sqrt{10\times10^{-3}\times10\times10^{-6}}} = \frac{10^3\times\sqrt{10}}{2\pi}$$

$$= 0.502 \times 10^3 = 502\ \text{Hz}$$

(*ii*) Resonant frequency = natural frequency = 502 Hz

(*iii*) Impedance $Z = \sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}$

Solution. (*a*) Reactance due to inductance $X_L = 2\pi nL = 2\times\frac{22}{7}\times 50\times 0.7 = 220\,\Omega$

Resistance $R = 50\ \Omega$

Impedance $Z = \sqrt{X_L^2 + R^2} = \sqrt{220^2 + 50^2} = 225.6\,\Omega$

Total current $I_{rms} = \frac{E_{rms}}{Z} = \frac{200}{225.6} 0.8865 A$

The power through the inductance is wattless as the voltage and the current in it have a phase difference of $\frac{\pi}{2}$.

$\therefore$ Wattless component of power $= I_{rms} \times$ potential drop across the inductance

$$= I_{rms} \times I_{rms} \times X_L$$

$$= I_{rms}^2 \times X_L = (0.8865)^2 \times 220$$

$$= 172.9 \text{ watt}$$

Power component is the power through the resistance in which the current and voltage are in phase.

$\therefore$ Power component $= I_{rms} \times$ potential drop across R

$$= I_{rms}^2 R = (0.8865)^2 \times 50$$

$$= 39.29 \text{ watt}$$

$$\text{Power factor} = \cos\phi = \frac{R}{Z} = \frac{50}{225.6} = 0.2216.$$

Note. In the above question true power $= E_{rms}\, I_{rms} \cos\phi$

$$= 200 \times 0.8863 \times 0.2216 = 39.29 \text{ watt}$$

which is the same as the power component across R showing thereby that the power component across the inductance does not contribute towards the power in the circuit i.e., it is wattless.

(*b*) Here $X_L = \sqrt{3}R$

$\therefore$ Power factor $\cos\phi = \frac{R}{\sqrt{R^2 + X_L^2}} = \frac{R}{\sqrt{R^2 + 3R^2}} = \frac{1}{2}$

Example 20.14. An A.C. voltage of r.m.s. value 10 *V* is applied to a parallel combination of *L* and *C* in which *L* = 200 μ*H* and *C* = 0.5 μ*F*. Calculate (*i*) the resonant frequency (*ii*) the currents in *L* and *C* at resonance.

Solution. Here $E_{rms} = 10\ V,\ L = 200\ \mu\text{H} = 200 \times 10^{-6}\ \text{H} = 2 \times 10^{-4}\ \text{H}$

$C = 0.5\ \mu\text{F} = 0.5 \times 10^{-6}\ \text{F} = 5 \times 10^{-7}\ \text{F}$

(*i*) the resonant frequency,

$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\times 3.142\sqrt{2\times 10^{-4}\times 5\times 10^{-7}}}$$

$$= \frac{10^5}{6.284} = 1.5913\times 10^4 Hz = 15.913\text{K Hz}.$$

PA732

(*ii*) The r.m.s. current in the inductive branch at resonance,

$$= \frac{E_{rms}}{\omega_r L} = \frac{E_{rms}}{2\pi f_r L} = \frac{10}{2 \times 3.142 \times 1.5913 \times 10^4 \times 2 \times 10^{-4}}$$

$$= \frac{10}{12.568 \times 1.5913} = 0.5 \text{ A}$$

(*iii*) The r.m.s. current in the capacitive branch at resonace

$$= \frac{E_{rms}}{\frac{1}{\omega r C}} = E_{rms} \times \omega_r C = E_{rms} \times 2\pi f_r C$$

$$= 10 \times 2 \times 3.142 \times 1.5913 \times 10^4 \times 5 \times 10^{-7}$$

$$= 3.142 \times 0.15913 = 0.5 \text{ A}$$

Thus, the current in the two branches are equal but opposite, at resonance.

Example 20.13. An a.c. voltage of r.m.s value 0.1 *V* is applied is an LCR series circuit in which $L = 100\ \mu H$, $C = 200\ pf$ and $R = 2\Omega$. Calculate

(*i*) resonant frequency,

(*ii*) the current in the circuit at resonance and

(*iii*) the voltage drops across *L* and *C* at resonace. Draw vector diagram at resonance.

Solution. Here,

$$E_{rms} = 0.1\ V, \text{ L} = 100\ \mu\text{H} = 100 \times 10^{-6} \text{ H} = 10^{-4} \text{ H}$$

$$C = 200 \times 10^{-12}\ F = 2 \times 10^{-10} \text{ F, R} = 2\Omega$$

(*i*) The resonance frequency, $f_r = \frac{1}{2\pi\sqrt{LC}}$

$$= \frac{1}{2 \times 3.142 \times \sqrt{10^{-4} \times 2 \times 10^{-10}}}$$

$$= \frac{10^7}{6.284 \times 1.414} = 1.125 \times 10^6 \text{ Hz.}$$

(*ii*) The current in the circuit at resonance

$$I_{rms} = \frac{E_{rms}}{R}$$

$$= \frac{0.1}{2} = 0.05\text{A}$$

(*ii*) r.m.s. voltage drop across *L* at resonance

$$= I_{rms} \times \omega_r L$$

$$= I_{rms} \times 2\pi f_r L$$

$$= 0.05 \times 2 \times 3.142 \times 1.125 \times 10^6 \times 10^{-4}$$

$$= 3.142 \cdot 1.125 \times 10 = 35.35\ V$$

(*iv*) r.m.s. voltage drop across C at resonance

$$= I_{rms} \times \frac{1}{\omega_r C}$$

$$= I_{rms} \times \frac{1}{2\pi f_r C}$$

$$= \frac{0.05}{2 \times 3.142 \times 1.125 \times 10^6 \times 2 \times 10^{-10}}$$

$$= \frac{500}{12.568 \times 1.125} = 35.35 \text{ V}$$

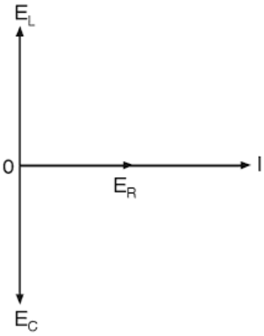


Fig. 20.35

Thus, the voltage drops across each L and C is the same *i.e.* 35.35 V. Moreover, this voltage drop is 353.5 times greater than the applied voltage at resonance. Hence the series resonance LCR circuit produces voltage amplification. Fig. 20.35 shows the vector diagram at resonance.

Example 20.14. A.C. circuit contains 100 mH inductance and 10 ohm is resistance. Calculate the power factor if the frequency of a.c. is 50 Hz. (*Nagpur Uni.* 2004)

Solution. Power factor is

$$\cos\phi = \frac{R}{\sqrt{R^2 + X_L^2}} = \frac{R}{\sqrt{R^2 + (2\pi fL)^2}}$$

$$= \frac{10}{\sqrt{100 + (2 \times 3.14 \times 50 \times 100 \times 10^{-3})^2}}$$

$$= \frac{10}{\sqrt{100 + (3.14)^2}} = 0.3035$$

Example 20.15. In Fig. 20.36, calculate the values of R_1, R_2 and R_3. Given $R_A = 4\Omega$, $R_B = 6\Omega$ and $R_C = 10\Omega$.

Solution.
$$R_1 = \frac{R_B R_C}{R_A + R_B + R_C} = \frac{6 \times 10}{4 + 6 + 10} = 3\Omega$$

$$R_2 = \frac{R_C R_A}{R_A + R_B + R_C} = \frac{10 \times 4}{4 + 6 + 10} = 2\Omega$$

$$R_3 = \frac{R_A R_B}{R_A + R_B + R_C} = \frac{4 \times 6}{4 + 6 + 10} = 1.2\Omega$$

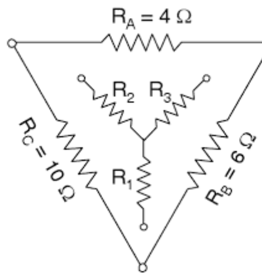


Fig. 20.36

Example 20.16. Estimate the total resistance of the following circuit using T and Δ network.

Solution. The given network shown in Fig. 20.37 consists of two Δs connected between p_1, and p_2, upper Δ and lower Δ. For simplicity, we replace one of the two Δs. Say lower one by Y with R_A on its top, as shown in Fig. 20.38 (*a*).

From the lower Δ between $p_1\ p_2\ p_4$, and replacing it by Y network, we get

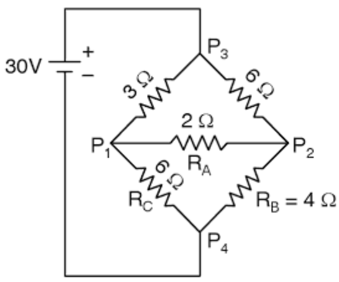


Fig. 20.37

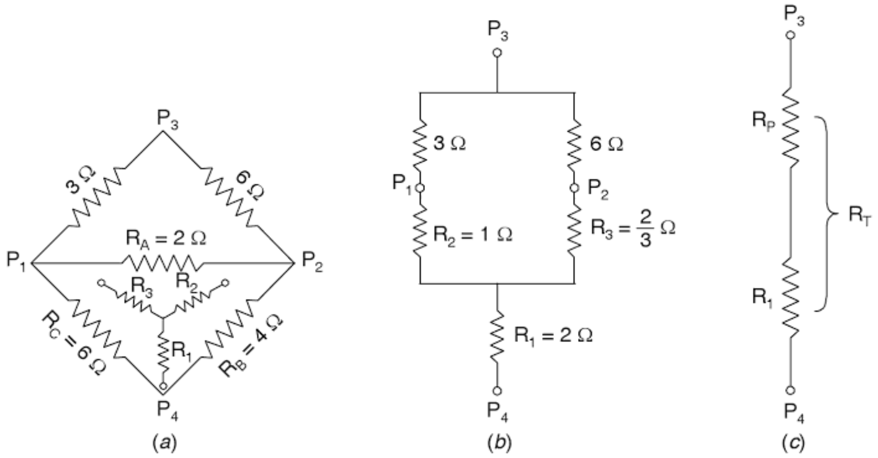


Fig. 20.38

$$R_1 = \frac{R_B \cdot R_C}{R_A + R_B + R_C} = \frac{4 \times 6}{2+4+6} = \frac{24}{12} = 2\Omega$$

$$R_2 = \frac{R_C \cdot R_A}{R_A + R_B + R_C} = \frac{6 \times 2}{2+4+6} = \frac{12}{12} = 1\Omega$$

$$R_3 = \frac{R_A \cdot R_B}{R_A + R_B + R_C} = \frac{2 \times 4}{2+4+6} = \frac{8}{12} = \frac{2}{3}\,\Omega$$

Using these values in equivalent Y [Fig. 20.38 (*b*)], these form series parallel circuit. The series resistance (3 + 1 =4 Ω) and $\left(6 + \frac{2}{3} = \frac{20}{3}\Omega\right)$ form a parallel combination whose equivalent resistance R_p will be

$$\frac{1}{R_p} = \frac{1}{4} + \frac{1}{20/3}$$

$$\frac{1}{R_p} = \frac{5+3}{20} = \frac{8}{20}$$

or
$$R_p = \frac{20}{8} = 2.5\Omega$$

This R_p (= 2.5 Ω) is in series with R_1 (= 2Ω) gives the total resistance R_T as shown in Fig. 20.38 (c).

$\therefore$
$$R_T = R_p + R_1 = (2.5 + 2) = 4.5\Omega$$

Example 20.19. Find the total resistance between A and B. All quantities are in ohm.

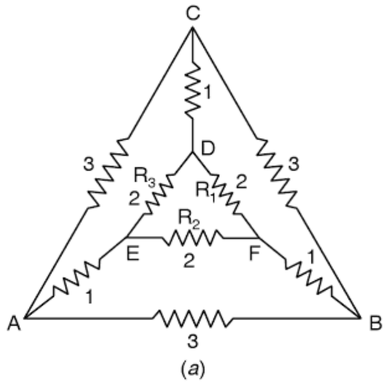


Fig. 20.43 (*a*)

Solution.

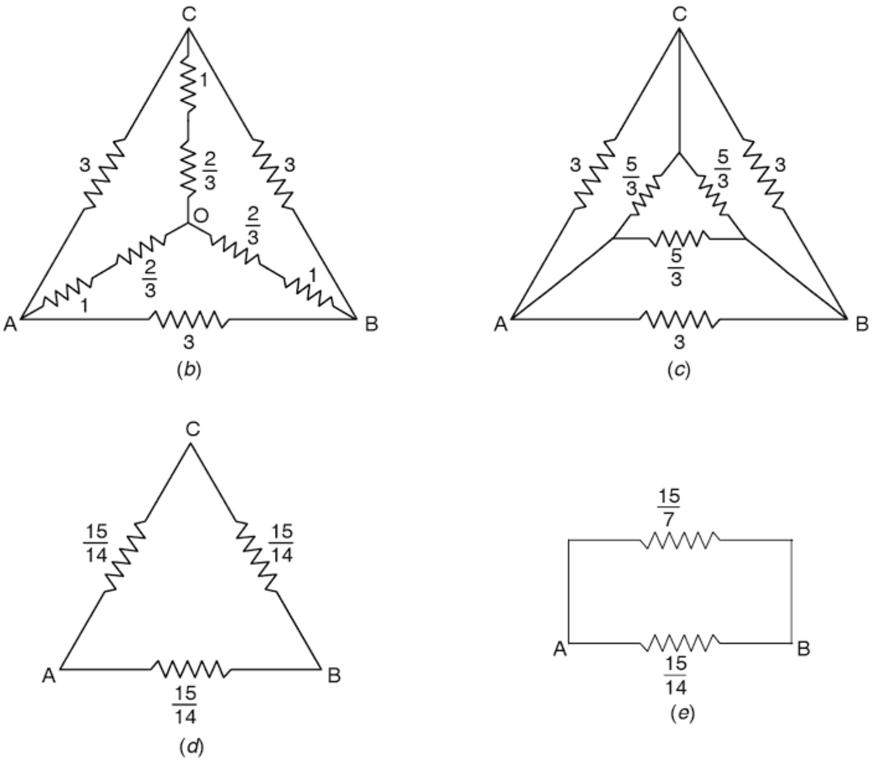


Fig. 20.43

Δ (delta) network inside between *EFD* is converted to *T* network by $R_1 = R_2 = R_3 = \dfrac{2 \times 2}{2+2+2}$ $= \dfrac{2}{3}\Omega$. This $\dfrac{2}{3}\Omega$ is in series with 1 gives $\dfrac{5}{3}\Omega$. This form again another Δ (delta) network of each side $\frac{5}{3}\,\Omega$. This inner Δ with outer Δ of each side 3 and $\frac{5}{3}$ gives $\frac{3\times5/3}{3+5/3}=\frac{5}{14/3}=\frac{15}{14}\Omega$. Finally, we get Δ of each side $\frac{15}{14}\Omega$. Upper two resistances in series gives $\frac{15}{14}+\frac{15}{14}=\frac{15}{7}\Omega$ is parallel with base AB of $\frac{15}{14}\Omega$.

$$\therefore \qquad R_T\,(\text{parallel}) = R_{AB} = \frac{\frac{15}{7}\times\frac{15}{14}}{\frac{15}{7}+\frac{15}{14}} = \frac{15}{7}\times\frac{15}{14}\times\frac{14}{45}$$

$$= \frac{5}{7}\Omega$$

EXERCISE CH. 20

LONG QUESTIONS

1. (***a***) Define complex number. Explain why complex numbers can be used to express sinusoidal quantities.

(b) Explain the phase relationship between *e.m.f.* and current through pure resistance only. (*M.S.U. Tiruneveli* 2007)

2. Define pure inductance. Show that the current in an inductive circuit lags behind the voltage by 90°. Plot the current, self induced *e.m.f.* and voltage curve and construct the vector diagram for an inductive circuit.

3. (***a***) Why a capacitor has practically infinite d.c. resistance? Explain the passage of alternating current through a purely capacitive circuit.

(***b***) Deduce equation for current in a capacitive circuit. Plot the current and voltage curves for the capacitive circuit.

4. With the help of j-operator method, obtain an expression for the impedance and phase in each case when a sinusoidal *e.m.f.* is applied to:

(*i*) Pure resistance (R)

(*ii*) Pure Inductance (L)

(*iii*) Pure Capacitance (C)

5. An A.C. voltage $E = E_0\, e^{j\omega t}$ is applied to a circuit having (*i*) an inductace L and a resistance R and (*ii*) a capacitance C and a resistance R in series. Derive and expression for the current that flows in the circuit. Also determine the impedance and phase difference between current and e.m.f. using j-operator.

(*K.U.* 2002; *Nagpur U. s/*2009; *M.D.U.*, 2002)

6. An alternating e.m.f. $E_0 \sin \omega t$ is applied to the ends of circuit containing resistance R, self-inductance L and capacitance C. Calculate the impedance of the circuit, phase angle and the current at any instant. (*Meerut U.* 2003, 2000; *G.N.D.U.* 2003; *Nag. U.* 2002)

7. (***a***) What is (*i*) a series resonant circuit and (*ii*) a parallel resonant circuit? Distinguish between the two. Why a series resonant circuit is known as an acceptor circuit and parallel resonant circuit as rejector circuit? What are their practical applications? Discuss the resonance condition in parallel LCR Circuit.

(*P.U.* 2002; *M.D.U.* 2002, 2001; *G.N.D.U.* 2000; *Gharwal. U.* 2000; *Meerut U.* 2005)

21 MAGNETOSTATICS - I

FORCE ON A MOVING CHARGE

INTRODUCTION

Magnetostatics deals with the force that a test charge experiences in the presence of both the static electric field as well as the static magnetic field. In 1820, Orested discovered that a current in a wire produces magnetic field around it, similar to an electric field around a static charge. The basic magnetic field vector $\vec{B}$ is called magnetic induction. It is represented by lines of induction. It should be noted that a static magnetic field cannot change the kinetic energy of a moving charge. It can only deflect it sideways, whereas when a charge moves in static electric field, its kinetic energy changes ($\frac{1}{2}mv^2 = eV$, where V is potential). The combined effect of static magnetic field and electric field on a moving charged particle was studied by Lorentz and force experienced is known as 'Lorentz force' and is discussed in this chapter. We will extend the idea to the atomic dipole, its dipole moment, angular momentum and gyromagnetic ratio.

21.1 MAGNETIC INDUCTION FIELD $\vec{B}$

The electric field vector $\vec{E}$ is defined as the force per unit positive stationary test charge. The direction of the electric field is the same as the direction of the force acting on the test charge. A test charge q at rest in the electric field $\vec{E}$ experiences a force $\vec{F_S}$ given by

$$\vec{F_S} = q\vec{E}$$

Suppose the test charge begins to move with a velocity $\vec{v}$ through a point P and experiences an additional force $\vec{F_m}$ (neglecting the force of gravity) which changes with magnitude as well as direction of $\vec{v}$ then a magnetic induction field $\vec{B}$ is a said to exist at P.

If we vary the direction of $\vec{v}$ through the point P, keeping the magnitude of $\vec{v}$ constant, then in general, the magnitude of the force $\vec{F_m}$ changes but the direction always remains perpendicular to the direction of $\vec{v}$. For a particular direction of $\vec{v}$ as also for the opposite direction $-\vec{v}$, the force $\vec{F_m} = 0$. This direction gives the direction of $\vec{B}$.

If now the direction of $\vec{v}$ is taken at right angles to this direction the force $\vec{F}_m$ is a maximum. The magnitude of the force depends upon the angle θ that the direction of $\vec{v}$ makes with the direction of $\vec{B}$, the charge q and the velocity v. It is given by

$$|\vec{F}_m| = qvB \sin\theta$$

Evidently for $\theta = 0$ (or π) *i.e.*, when the direction of $\vec{v}$ is the same as that of $\vec{B}$ (or opposite to it) $F_m = 0$ and for $\theta = \pi/2$ ($\sin\theta = 1$) F_m has a maximum value given by

$$(F_m)_{\max} = qvB$$

The magnitude of $\vec{B}$ is, therefore, given by

$$B = \frac{(F_m)_{\max}}{qv}$$

In the vector form the force $\vec{F}_m$ acting on test charge q moving with velocity $\vec{v}$ in a magnetic inducting field $\vec{B}$ is given, in magnitude and direction by

$$\vec{F}_m = q\vec{v} \times \vec{B}$$

Hence a magnetic induction field is defined as that field in which a moving charge experiences a velocity dependent force by

$$\vec{F}_m = q\vec{v} \times \vec{B}$$

Units. The magnitude of the force on a charge q moving with a velocity $\vec{v}$ in a direction perpendicular to the magnetic induction field $\vec{B}$ is given by

$$F = qvB$$

or
$$B = \frac{F}{qv}$$

when $F = 1$ Newton; $q = 1$ coulomb; $v = 1$ met/sec; $B = 1$.

The S.I. (or M.K.S.) unit of magnetic induction is known as a Tesla of Weber per metre^2.

Tesla. *A magnetic induction field has a strength of 1 Tesla if it exerts a force of 1 newton on a charge of 1 coulomb moving with a velocity of 1 meter per second in a direction perpendicular to that of the field.*

The C.G.S. unit of magnetic induction field is a *Gauss.*

Gauss. *A magnetic induction field has a strength of 1 Gauss if it exerts of force of 1 dyne on a charge of one ab-coulomb (or e.m.u) moving with a velocity 1 cm per second in a direction perpendicular to that of the field.*

Relation
$$\text{Tesla} = \frac{\text{Newton}}{\text{coulomb} \times \text{metre sec}^{-1}}$$

$$\text{Gauss} = \frac{\text{dyne}}{\text{ab-coulomb} \times \text{cm sec}^{-1}}$$

$$\therefore \quad \frac{\text{Tesla}}{\text{Gauss}} = \frac{\text{Newton/dyne}}{(\text{coulomb/ab-coulomb}) \times (\text{metre/cm})} \quad \frac{10^5}{\frac{1}{10} \times 100} = 10^4$$

$$\therefore \quad 1 \text{ Tesla} = 10^4 \text{ Gauss}$$

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21.2 LORENTZ FORCE LAW

The force on a test charge q when it moves with a velocity $\vec{v}$ through a region in which both electric field $\vec{E}$ and magnetic field $\vec{B}$ are present, is given by

$$\vec{F} = q_0\vec{E} + q_0(\vec{v} \times \vec{B})$$

$$= q_0[\vec{E} + \vec{v} \times \vec{B}] \quad \text{... }(i)$$

This relation is know as Lorentz force law and the force $\vec{F}$ is know as Lorentz force.

When $\vec{E} = 0$, *i.e.*, only a magnetic field $\vec{B}$ is present, then

$$\vec{F} = q(\vec{v} \times \vec{B}) \quad \text{... }(ii)$$

The equation gives the experimental law of interaction between the magnetic induction of $\vec{B}$ and the moving charge q.

The direction of vectors $\vec{F}$, $\vec{v}$ and $\vec{B}$ are shown in Fig. 21.1. It is clear that the direction of $\vec{F}$ is perpendicular to the plane containing the magnetic field vector $\vec{B}$ and the velocity vector $\vec{v}$. The magnitude of $\vec{F}$ is given by

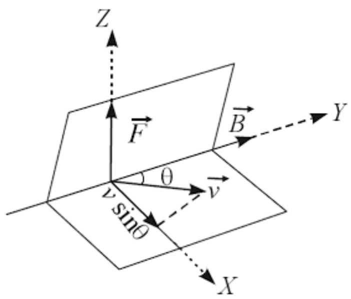

Fig. 21.1

$$F = qvB \sin\theta$$

where $v \sin\theta$ is the component of $\vec{v}$ normal to $\vec{B}$.
When $\theta = 0$, $F = 0$ and when $\theta = \pi/2$,

$$F = qvB \quad \text{... }(iii)$$

Thus, there is no force acting on the charged particle when it moves along (*i.e.* parallel) the direction of magnetic field. However the charged experience a force (given by Eq. *iii*), when it moves along the direction perpendicular to the magnetic induction $\vec{B}$.

21.3 WORK DONE BY MAGNETIC FIELD ON A MOVING CHARGE

From the relation $\vec{F} = q(\vec{v} \times \vec{B})$ we find that when the charge moves in a steady magnetic field the magnetic force $\vec{F}$ is always at right angles to the velocity vector $\vec{v}$ as well as the field vector $\vec{B}$.

The work done by the force $\vec{F} = \vec{F}.d\vec{s} = \vec{F}.\vec{v}\,dt = 0$ as $\vec{F}.\vec{v} = 0$ because $\vec{F}$ is always perpendicular to $\vec{v}$.

No work is, therefore done by a static magnetic field on a moving charge or a moving charged particle

Hence kinetic energy of a moving charge remains uncharged in a static magnetic field. Thus an applied magnetic field can only alter the direction of the velocity vector but it cannot change the speed of the charged particle and therefore cannot increase its kinetic energy.

21.4 FORCE ON A MOVING CHARGE

Consider the case when $\vec{E} = 0$ and only magnetic field of magnetic induction $\vec{B}$ is present. The force acting on a positive charge q moving with a velocity $\vec{v}$ in a magnetic field $\vec{B}$ is given by $\vec{F} = q(\vec{v} \times \vec{B})$. The direction of the force $\vec{F}$ is given by *the right hand screw rule* for vector cross product or *Fleming's left hand rule* as stated below:

Fleming's left hand rule. *If we stretch the first finger, the middle finger and the thumb of left hand mutually perpendicular to each other such that the first finger points in the direction of magnetic field. The middle finger in the direction of motion of positive charge (or direction of electric current) then the thumb represent the direction of force experienced by the charged particle.*

Thus if $\vec{v}$ acts along the $+X$ axis, $\vec{B}$ along $+Y$ axis, then according to the rule for vector cross product $\vec{F}$ will act along $+Z$ axis as shown in Fig. 21.1 for a positive charge.

As the electron carries a negative charge, the direction of the force on it is along– Z-axis.

This force always acts at right angles, to the direction of motion of the particle. The particle, therefore, moves along a circular path of radius r so that the centripetal force $mr\omega^2$ is equal to qvB

$$\therefore \qquad mr\omega^2 = qvB = qr\omega B \qquad (\because v = r\omega)$$

$$\text{or} \qquad \omega = \text{B}\, q/\text{m}$$

21.5 MAGNETIC FLUX

Magnetic flux through a surface is defined as the total number of lines of magnetic induction passing through the surface.

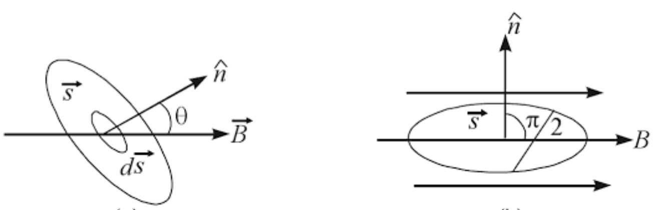

Fig. 21.2

Consider an element of surface area $d\vec{s}$ of a surface area $\vec{s}$. Let $\hat{n}$ be a unit vector normal to the area element $d\vec{s}$, then $\hat{n}$ represent the direction of the area vector $d\vec{s}$ as shown in Fig.

$$\vec{F} = \int_l I\, d\vec{l} \times \vec{B}$$

$$= I\left(\int_l d\vec{l}\right) \times \vec{B} = I\,\vec{l} \times \vec{B}$$

as $\vec{B}$ is uniform through the length of the wire the element $\vec{dl}$ will have the same direction at every point of the length $\vec{l}$.

The magnitude this force is given by

$$F = IlB \sin\theta$$

where θ is the angle between the vector $\vec{l}$ and $\vec{B}$. When $\theta = 0$, $F = 0$ and when $\theta = \frac{\pi}{2}$; $F = IlB$ and is a maximum.

The direction of the force $\vec{F}$ is perpendicular to the place containing the vector $\vec{l}$ and $\vec{B}$ given by right hand screw rule for vector products.

21.7 RECTANGULAR CURRENT LOOP IN EXTERNAL MAGNETIC FIELD

Consider a rectangular current loop *ABCD* placed in a uniform magnetic field of induction $\vec{B}$ carrying a current I as shown in Fig. 21.4. The force acting on the side *BC*

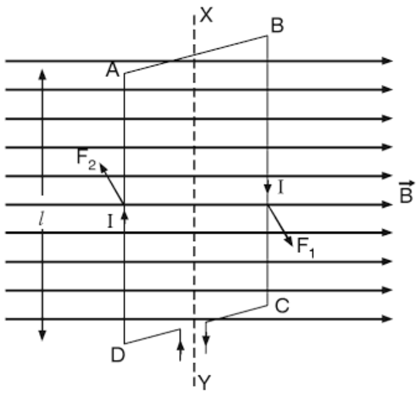

Fig. 21.4

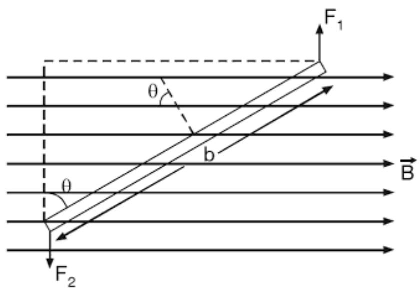

Fig.21.5

is $$F_1 = I\,\vec{\ell} \times \vec{B} = BI\ell \quad \text{... }(i)$$

Since $\vec{l}$ is perpendicular to $\vec{B}$. The force $\vec{F_1}$ is perpendicular to $i\,\vec{l}$ and $\vec{B}$.

Similarly, there is a force $\vec{F_2}$ acting on side *AD* which is equal and opposite to $\vec{F_1}$. The sides *AB* and *CD* experience a net force which is zero.

The two forces beings equal, opposite and parallel constitute a couple about an axis *XY*.

The perpendicular distance called as moment arm between the two forces is $b \sin\theta$.

Theory of ballistic galvanometer. Let n be the number of turns in the coil; l the length of its vertical side, b its breadth and B the magnetic field in which suspended.

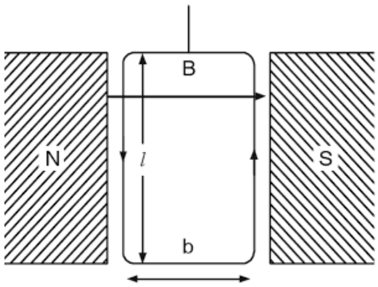

Fig. 21.7

Let there be a current i at any instant, then

Force on each vertical wire $= i\ l\ B$

$\therefore$ Force on each vertical side $= nilB$

If this current remains constant for a very small time dt, then

Impulse of force $= nlB\ idt$

$\therefore$ Total change in momentum during the time the whole charge q passes through it is

$$= \int nl\,B\,i\,dt = nl\,B\int idt = nl\,Bq$$

This change in momentum causes a rotation of the coil about the axis of suspension producing an angular momentum given by

Angular momentum $= nl\,Bq\,b = nBAq$

where A = area of the coil $= lb$

The angular momentum $= I\omega$ where I is the momentum of inertia and ω the angular velocity

$$I\omega = n\,BA\,q \qquad \ldots (i)$$

Due to the angular velocity the coil possesses a kinetic energy $\frac{1}{2}I\omega^2$ and is brought to rest by performing work in twisting the suspension wire. If c is the restoring couple per unit angular twist, then

Couple for a twist $\theta = c\theta$

and work done for a further small deflection $d\theta = c\theta \cdot d\theta$

$\therefore$ Total work done in twisting the suspension wire from 0 to θ.

$$= \int_0^\theta c\theta.d\theta = \frac{1}{2}c\theta^2$$

where θ is the deflection of the coil when whole of the kinetic energy has been used up in producing the twist.

$$\therefore \qquad \frac{1}{2}I\omega^2 = \frac{1}{2}c\theta^2.$$

$$\text{or} \qquad I\omega^2 = c\theta^2 \qquad \ldots (ii)$$

If T is the time period of torsional vibrations of the coil when no current passes through it, then

$$T = 2\pi\sqrt{\frac{I}{c}}$$

$$\text{or} \qquad I = \frac{T^2 c}{4\pi^2} \qquad \ldots (iii)$$

Multiplying (*ii*) and (*iii*), we have

$$I^2 \omega^2 = \frac{c^2 T^2 \theta^2}{4\pi^2}$$

or
$$I \omega = \frac{cT\theta}{2\pi} \quad \ldots (iv)$$

Comparing (*i*) and (*iv*), we have

$$nABq = \frac{cT\theta}{2\pi}$$

or
$$q = \frac{cT}{2\pi n\, AB}\theta = \frac{T}{2\pi}\frac{c}{nAB}\theta = k_b\theta \quad \ldots (v)$$

The quantity $\frac{c}{nAB}$ is known as the *current sensitivity* and the quantity $\frac{T}{2\pi}\cdot\frac{c}{nAB}$ is known as the *charge sensitivity* of the ballistic galvanometer.

$\therefore$ Charge sensitivity $= \frac{T}{2\pi}$ current sensitivity.

The charge sensitivity of a ballistic galvanometer is also denoted as k_b and is called the *constant of the ballistic galvanometer.*

Moreover, we can write ... (*vi*)

Charge sensitivity $= \frac{T}{2\pi} \times$ current sensitivity.

When charge sensitivity is equal to current sensitivity, $\frac{T}{2\pi} = 1$ or period time (Time period) $T = 2\pi$ sec.

Dead beat galvanometer. In a moving coil *dead beat* galvanometer the current carrying coil moves around a soft iron core, thereby inducing an eddy current in it. This induced current opposes the motion of the coil thereby causing damping and making the galvanometer dead beat.

No such iron core is used in a ballistic galvanometer.

21.9 CORRECTION FOR DAMPING IN B.G.

Damping. In the theory of the ballistic galvanometer it has been supposed that the whole of the kinetic energy of the moving system is used up in twisting the suspension fibre. In practice, however, the motion of the coil is opposed by (*i*) resistance of air, and (*ii*) induced *e.m.f.* in the coil. The amplitude of successive oscillations goes on decreasing and this effect is known as *damping*.

Need for damping correction. The first throw of the ballistic galvanometer is always smaller than what it would have been in the ideal case when there is no damping and a correction must be applied to it to get the correct value.

When successive values of θ are observed to the left and right as the coil oscillates, it is found that

$$\frac{\theta_1}{\theta_2} = \frac{\theta_2}{\theta_3} = \frac{\theta_3}{\theta_4} = \ldots\ldots = d$$

where the constant d is called the *decrement* of the swing and is defined as the ratio of two successive deflections, one to the right and the other to the left.

If $$d = e^{\lambda}$$

the $$\lambda = \log_e d$$

here λ is known as the **logarithmic decrement.**

Damping correction. When there is no damping, the motion of the coil is simple harmonic and can be represented by the projection of a point P moving in a circle on a fixed straight line, say AB.

In this case the amplitude of successive oscillation will be

$$OA,\ OB,\ OA, \text{ etc.}$$

When there is damping, the amplitude of successive oscillations goes on decreasing at a constant rate. It is clear from Fig. 21.8 that the shrinkage (decrease in amplitude) from θ_1 to θ_2 takes place in half of a vibration or one oscillation. Since the impulse is given to the moving system when it was in the mean position, the shrinkage that actually takes place before the first throw θ_1 is observed, is in a quarter vibration or half an oscillation.

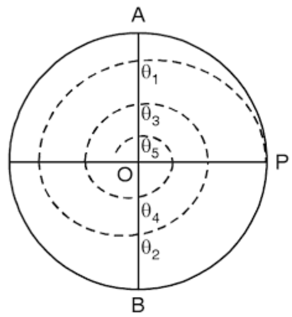

Fig. 21.8

Now for one oscillation, the shrinkage is given by the relation

$$\frac{\theta_1}{\theta_2} = \frac{\theta_2}{\theta_3} = \frac{\theta_3}{\theta_4} = e^{\lambda}$$

$\therefore$ For two oscillations the shrinkage is

$$\frac{\theta_1}{\theta_3} = \frac{\theta_1}{\theta 2} \times \frac{\theta_2}{\theta_3} = e^{\lambda} \times e^{\lambda} = e^{2\lambda}$$

Hence for half an oscillation the shrinkage is $e^{\lambda/2}$.

If θ_0 is the corrected throw, then

$$\frac{\theta_0}{\theta_1} = e^{\lambda/2}$$

or $$\theta_0 = \theta_1 e^{\lambda\backslash 2} = \theta_1 \left(1 + \frac{\lambda}{2} + \frac{\lambda^2}{4(2!)} + \ldots\right)$$

$$= \theta_1 \left(1 + \frac{\lambda}{2}\right)$$

as λ is small and its squares and higher powers can be neglected.

Hence the corrected formula for the moving coil ballistic galvanometer is

$$q = \frac{T}{2\pi} \frac{c}{nAB} \theta \left(1 + \frac{\lambda}{2}\right)$$

In practice the value of λ is found by observing θ_1 and θ_{11}, *i.e.*, the first throw and the 11th throw, *i.e.*, after 10 oscillations. Then

$$\frac{\theta_1}{\theta_{11}} = e^{10\lambda}$$

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Now $\oint ydy = 0$ and $\oint zdz = 0,$ for it we plot a curve of y against y or z against z it is a straight line and the area included by it for a complete cycle is zero, $\oint ydx$ gives the component of the vector area $\vec{A}$ of the loop in the xy plane taken in the clockwise direction *i.e.*, the area is directed along the $-Z$ axis

$$\therefore \qquad \oint ydx = -A_z$$

Similarly $\oint zdx$ gives the component of the area in the zx, plane taken in the anticlockwise direction *i.e.,* the area is directed along the $+ Y$ axis.

$$\therefore \qquad \oint z\,dx = A_y$$

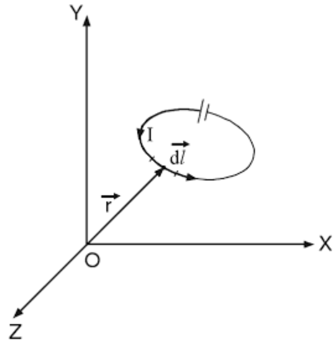

Fig. 21.9

Since $\vec{B}$ is uniform, its competent B_x, B_y and B_z are constant.

$$\therefore \qquad \oint \{\vec{r} \times (d\vec{l} \times \vec{B})\}_x = -A_z B_y + A_y B_z = [\vec{A} \times \vec{B}]_x$$

Similarly

$$\oint \{\vec{r} \times (d\vec{l} \times \vec{B})\}_y = [\vec{A} \times \vec{B}]_y$$

and

$$\oint \{\vec{r} \times (d\vec{l} \times \vec{B})\}_z = [\vec{A} \times \vec{B}]_z$$

Combining, we get

$$\oint \vec{r} \times (d\vec{l} \times \vec{B}) = \vec{A} \times \vec{B}$$

$$\therefore \qquad \text{Torque } \vec{\tau} = I(\vec{A} \times \vec{B})$$

Thus the torque on a current loop depends only on the area of the loop. It does not depend upon the shape of the loop. The magnitude of the torque $= IAB \sin\theta$ where θ is the angle between the area vector $\vec{A}$ and field vector $\vec{B}$.

Current loop of *n* turns. If the loop has n turns, each turn experiences the same torque.

$\therefore$ Torque on whole coil of n turns $= nI(\vec{A} \times \vec{B})$

2.11 MAGNETIC MOMENT OF A CURRENT LOOP

The torque on a current loop (single coil) of area A through which a current I flows is given by

$$\vec{\tau} = I\vec{A} \times \vec{B} \qquad \ldots (i)$$

where $\vec{B}$ is the magnetic induction field.

The torque acting on a electric dipole having electric diploe moment $\vec{M}_E$ in a electric field $\vec{E}$ is given by

$$\vec{\tau} = \vec{M}_E \times \vec{E} \qquad \text{... } (ii)$$

Comparing (i) and (ii), we find that *the current loop in a uniform magnetic field behaves similar to an electric dipole in a uniform electric field i.e. a small current loop has the properties of a magnetic dipole.*

The vector $I\vec{A} = \vec{M}$ is called the magnetic moment of the current loop. Hence if a current loop of magnetic moment $\vec{M}$ is placed in a uniform magnetic field $\vec{B}$, the torque acting on it is given by

$$\vec{\tau} = \vec{M} \times \vec{B}$$

In other words, a current carrying loop of wire behaves as a *magnetic dipole* of magnetic moment $\vec{M} = I\vec{A}$ where I is the current and $\vec{A}$ the area of the loop.

Unit of magnetic moment. The unit of moment of a current loop, $\vec{M}$ is IA = Amp m^2 $= \frac{\text{Coulumb}}{\text{sec}} \text{m}^2 = \text{Cs}^{-1}\,\text{m}^2$. Also torque $\vec{\tau} = \vec{M} \times \vec{B}$. The unit of torque is $(\vec{\tau} = \vec{r} \times \vec{F})$ that of work.

$\therefore$ Unit of $\vec{M} = \frac{\text{work}}{B}$ = Joule per Tesla (JT^{-1})

Pole strength. In the study of magnetism, the magnetic moment of a magnetic dipole (say a bar magnet) is defined as the product of the pole strength m and the distance two opposite poles (one north pole and the other south) the distance between the two poles known as magnetic *length* and denoted as $2\vec{l}$.

$$\therefore \text{ Magnetic moment} \qquad \vec{M} = m \times 2\vec{l}$$

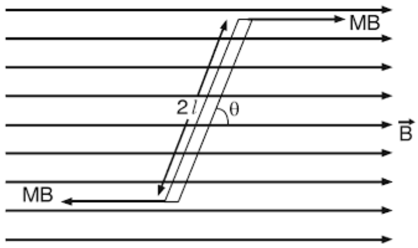

Fig. 21.10

Unit of pole strength. The unit of pole strength $m = \frac{\text{Unit of } \vec{M}}{\text{Unit of length}} = \frac{\text{Amp m}^2}{\text{m}} = \text{Amp m}$

Direction of magnetic moment of a current loop. The magnitude of the magnetic moment of current loop is the product of the current and the area of the loop. The direction of the dipole moment is given by the following rule.

Let the fingers of the right hand curl round the loop in the direction of the current, the extended right thumb will then point in the direction of magnetic dipole moment vector.

21.12 EQIVALENCE OF CURRENT COIL TO A BAR MAGNET

The above equivalence in true for a coil of any number of turns and the any shape. Therefore, Eq. (ii) in article 21.7 is valid even if the coil is not rectangular. Let us consider a coil (irregular) as shown in Fig. 21.11. The plane may be divided into large number of narrow rectangular circuits. The current in the coil (or loop) remains same because in the common boundary of two adjacent loops, the two are having equal but oppositely directed currents. Therefore, they mutually cancel each other. The torque due to ith rectangular circuit is

$$\tau_i = IA_i B \sin\theta$$

Total torque τ is given by

$$\tau = \sum_{i=1}^{i=n} \tau_i = IB\sin\theta \sum_{i=1}^{i=n} A_i$$

$$= IAB \sin\theta$$

where, $A = \sum_{i=1}^{i=n} A_i$ is the total area

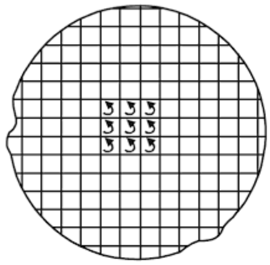

Fig. 21.11

of the coil. Thus any current carrying coil is equivalent to a bar magnet or magnetic dipole.

21.13 MAGNETIC MOMENT OF ATOMIC DIPOLE

An atomic dipole consists of an electron of charge $-e$ and m revolving around an atomic nucleus with a linear velocity v in an orbit of radius r (Fig. 21.12). The revolving electron is

equivalent to a current $\quad I = \dfrac{e}{2\pi r/v} = -\dfrac{ev}{2\pi r}$

The magnitude of the magnetic dipole moment of a current loop of area A carrying a current I is given by

$$M = IA$$

$\therefore$ Magnetic moment of the atomic dipole $= -\dfrac{ev}{2\pi r} \times \pi r^2$

$$= -\frac{1}{2} evr = -\frac{1}{2}\frac{e}{m} mvr = -\frac{1}{2}\frac{e}{m} L$$

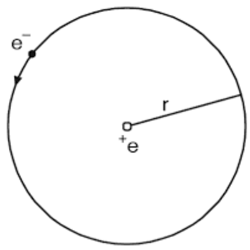

Fig. 21.12

where $L = mvr$ = the magnitude of the angular momentum of the electron revolving around the nucleus.

21.14 ANGULAR MOMENTUM AND GYROMAGNETIC RATIO

All substances are composed of atoms and an atom consists of a central positively charged nucleus where the whole mass of the atom is supposed to be concentrated, with suitable number of electrons revolving round it in more or less circular orbits. An electron carries a *negative* charge of 1.6×10^{-19} coulomb. The revolution of an electron in a clockwise direction is equivalent to a conventional current in the anticlockwise current in the anticlockwise direction and the electronic orbit behaves like a *magnetic dipole*.

Consider the simplest atom of hydrogen. It consists of one proton in the nucleus and one electron revolving round it. If m is the mass of the electron, $-e$ the charge on it, v its velocity of revolution and r the radius of the orbit, then the electron is maintained in the orbit due to the electrostatic force of attraction between the electron and the nucleus which provides the necessary centripetal force. Taking magnitudes only

$$\therefore \quad \frac{mv^2}{r} = \frac{1}{4\pi\varepsilon_0}\frac{e.e}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}$$

where n is an integer.

Substituting the value of L in (i), we have

$$\vec{M} = \frac{1}{2}\frac{e}{m}\frac{nh}{2\pi} = \frac{neh}{4\pi m}$$

$$= \frac{ne\hbar}{2m} \qquad \ldots (ii) \quad \left[\text{where } \hbar = \frac{h}{2\pi}\right]$$

From relation (ii) we find that the magnetic moment due to orbital motion of an electron must be an integral multiple of $\frac{e\hbar}{2m}$.

Bohr magneton. The smallest value of orbital magnetic moment of the electron for $n = 1$ is given by

$$M = \frac{e\hbar}{2m} = \frac{eh}{4\pi m} \qquad \ldots (iii)$$

This is the unit of magnetic moment and is known as a *Bohr Magneton*. It is denoted by μ_B.

Numerical value of Bohr magneton. Now $e = 1.6 \times 10^{-19}$ coul

$h = 6.6 \times 10^{-24}$ Joule sec; m $= 9 \times 10^{-31}$ kg

Substituting these values in Eq. (iii), we get

$$\mu_B = \frac{eh}{4\pi m} = \frac{1.6 \times 10^{-19}\,\text{coul} \times 6.6 \times 10^{-34}\,\text{Joule sec}}{4\pi \times 9 \times 10^{-31}\,\text{kg}}$$

$$= 9.27 \times 10^{-24}\,\text{Joule}\,\frac{\text{coul met sec}^{-1}}{\text{kg met sec}^{-2}}$$

$$= 9.27 \times 10^{-24}\,\text{Joule}\,\frac{\text{coul met sec}^{-1}}{\text{Newton}}$$

$$= 9.27 \times 10^{-24}\,\frac{\text{Joule}}{\text{Tesla}} = 9.27 \times 10^{-24}\,\text{Joule/Tesla}$$

[The unit of Bohr magneton is also the same as that of M. As $M = IA$, the unit of Bohr magneton is also Am^2]

Bohr magneton is used as a unit of measurement of atomic magnetic moments.

Orbital gyromagnetic ratio. It is defined as *the ratio of magnetic moment of the atomic dipole to its angular momentum.*

From relation (i) we have,

$$\text{Orbital gyromagnetic ratio} = \frac{M}{L} = \frac{e}{2m}.$$

The gyromagnetic ratio is also know as magneto-mechanical ratio. Here e is the charge and m, the mass of the charged particle.

Example 21.3. A test charge $q = 3.2 \times 10^{-19}$ C is moving with linear velocity $\vec{v} = (2\hat{i} + 2\hat{j}) ms^{-1}$ in a combined electric and magnetic field of intensity $\vec{E} = (3\hat{i} + 6\hat{j} + \hat{k})NC^{-1}$ and $\vec{B} = (2\hat{j} + 3\hat{k})T$ respectively. Calculate the force experienced by the test charge.

(*Pbi. U.*, 2001)

Solution. Here $q = 3.2 \times 10^{-19}$ C; $\vec{v} = (2\hat{i} + 2\hat{j})\,\text{ms}^{-1}$;

$$\vec{E} = (3\hat{i} + 6\hat{j} + \vec{k})NC^{-1}; \qquad \vec{B} = (2\hat{j} + 3\hat{k})T$$

The force experienced by the test charge q moving with velocity $\vec{v}$ in a combined electric and magnetic field $\vec{E}$ and $\vec{B}$ respectively is given by

$$\vec{F} = q\vec{E} + q(\vec{v} \times \vec{B})$$

$$\therefore \qquad \vec{F} = 3.2 \times 10^{-19}(3\vec{i} + 6\hat{j} + \hat{k}) + 3.2 \times 10^{-19}[(2\hat{i} + 2\hat{j}) \times (2\vec{j} + 3\vec{k})]$$

$$= 3.2 \times 10^{-19}[(3\hat{i} + 6\hat{j} + \hat{k}) + (4\hat{k} - 6\hat{j} + 6\hat{i})]$$

$$= \mathbf{3.2 \times 10^{-19}(9\hat{i} + 5\hat{k})N}$$

Example 21.4. A particle of charge 5 μC having velocity $8 \times 10^6 \hat{i}$ enters a combined electric field $10^6 \hat{j}$ and magnetic field $0.2\hat{k}$.. What is the force acting on it?

(*Kerala, U.*, 2001)

Solution. $\vec{F} = q\vec{E} + q(\vec{v} \times \vec{B}) = 5 \times 10^{-6}(10^6 \hat{j}) + 5 \times 10^{-6}(8 \times 10^6 \hat{i} \times 0.2\hat{k})$

$= 5\hat{j} - 8\hat{j} = \mathbf{-3\hat{j}}$ *i.e.*, a force of 3 N will act along ($-Y$) direction.

Example 21.5. A proton and a deutron have equal kinetic energies. Compare the radii of their paths when a magnetic field is applied normal to their orbits.

Solution. Let mass of the proton $= m$

$\therefore$ Mass of deutron $= 2m$ [Supposing mass of proton = mass of neutron]

Velocity of the proton $= v_p$

Velocity of the deutron $= v_d$

As their kinetic energies are equal; $\frac{1}{2}mv_p^2 = \frac{1}{2}2mv_d^2$

or $\qquad v_p^2 = 2v_d^2$ or $v_p = \sqrt{2}v_d \;\therefore\; \frac{v_d}{v_p} = \frac{1}{\sqrt{2}}$

Charge on the proton $=$ charge on the deutron $= e$

Magnetic field $= \vec{B}$

5. (***a***) Calculate the torque on a current loop in a steady magnetic induction field. Hence obtain the expression for the torque on a rectangular current loop suspended in a uniform magnetic field . (*Gauhati U.* 2000)

(***b***) How will you explain that a current carrying loop is equivalent to a magnet? Hence define the magnetic moment of a current loop. Also find the *S.I.* unit of pole strength. (*Pbi. U.* 2002, 2001; *Gauhati U.* 2000)

6. (***a***) Prove that the magnetic moment of an electron is given by $\vec{M} = -\frac{e}{2m}\vec{L}$ where e = charge on the electron, m = electron mass and $\vec{L}$ = orbital angular momentum due to orbital motion. Hence show that magnetic moment due to orbital motion of an electron must be an integral multiple of $\frac{eh}{2m}$. How do you define a Bohr magneton? Give its numerical value. (*P.U.* 2000; *G.N.D.U* 2000; *Gauhati U.* 2000; *H.P.U.* 2001, 2000; *Pbi. U.* 2003, 2002, 2001; *M.D.U.* 2002)

(***b***) What is orbital gyromagnetic ratio? (*Pbi.* U. 2001; *P.U.* 2000)

7. (***a***) Give the construction, theory and working of a ballistic galvanometer. (*Gauhati U.* 2007)

(***b***) Under what condition does a moving coil galvanometer behaves as a ballistic.

(***c***) What is dead beat galvanometer. In which instruments it is used (*Meerut U.* 2005, 2004)

SHORT QUESTIONS

1. (*a*) Define magnetic induction field without the concept of magnetic poles. What are the C.G.S. and M.K.S. units of magnetic field and how are they related? (*M.D.U.*, 2001; *Pbi. U.*, 2002)

2. Show that no work is done by a magnetic field on a charged particle moving in it and that the magnetic force acting on a moving charged particle cannot increase its kinetic energy. (*D.A.U. Agra* 2008; *P.U.*, 2001; *Kerla U.*, 2001)

3. Calculate the torque acting on a current loop placed in a uniform magnetic field. (*Nagpur U. s*/2009)

4. What is the function of iron core in a moving coil galvanometer? (*Pbi. U.* 2003)

5. Define current sensitivity of a moving coil galvanometer. (*Meerut U.* 2003)

6. Show mathematically how the observed throw can be corrected for damping in a ballistic galvanometer. (*Nagpur U.*, 2001; *Kerala U.*, 2001)

7. Calculate the magnetic moment of an atomic dipole. (*P.U.* 2003)

8. When a current carrying rectangular wire is placed in uniform magnetic field, show that torque acting on it is

$$\vec{\tau} = \vec{M} \times \vec{B}$$

where $\vec{M}$ is the magnetic dipole moment.

22 MAGNETOSTATICS - II

MAGNETIC FIELD DUE TO STEADY CURRENTS

INTRODUCTION

In the previous chapter, we have introduced the concept of magnetic flux density and the magnetic induction field $\vec{B}$ in which a moving charge experiences velocity dependent force. We will discuss here how $\vec{B}$ in related with the current carrying coil with varying distance from it. In fact, we are often interested not in the field of moving charge, but in that of an 'element of current' as a length *dl* of wire carrying a current *I*. This concept leads us to calculate the field due to many different geometrical current configurations.

Oersted, in 1820, discovered first the significant relationship between a current and its magnetic field by demonstrating that magnetic needle sets itself perpendicular to the steady current carrying straight wire. In the same year, Biot and Savart were successful to propose the mathematical equation for the field due to a single current carrying element. This was the prime footprint in the field of magnetostatics.

22.1 BIOT AND SAVART'S LAW

Statement: The law state that the magnitude $\delta\vec{B}$ of the magnetic induction at a point *P* due to a small element of length δl of a conductor carrying a current *I* is given by

$$\delta B \propto \frac{I\delta\ell\sin\theta}{r^2} \qquad ...(i)$$

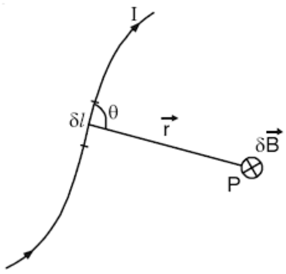

Fig. 22.1

where *r* is the distance of *P* from the element, θ is the angle between the element δl and the line joining it to *P*, Eq. (*i*) may be written as

$$\delta B = k\frac{I\delta\ell\sin\theta}{r^2} \qquad ...(ii)$$

where *k* is a constants of proportionality, whose value depends upon the system of units and the medium in which the conductor is situated. If the conductor is situated in air or vacuum, then in the SI units the constant *k* is written as $k = \frac{\mu_0}{4\pi}$ where μ_0 is called *permeability* of free space.

Therefore, Biot and Savart's law in SI units when the conductor is situated in air or vacuum is

$$\delta B = \frac{\mu_0}{4\pi} \cdot \frac{I\delta\ell \sin\theta}{r^2} \qquad ...(iii)$$

Thus, the magnetic field due to a current element a point P is

(*i*) directly proportional to the magnitude of the current element

$$\delta B \propto Idl$$

(*ii*) inversely proportional to the square of the distance of the point P from the current element

$$\delta B \propto \frac{1}{r^2}$$

(*iii*) directly proportional to the sine of the angle between the direction of current element $(Id\vec{l})$ and vector $\vec{r}$

$$\delta B \propto \sin\theta$$

The constant of proportionality in SI unit is $\frac{\mu_0}{4\pi}$.

The value of the μ_0 is $\mu_0 = 4\pi \times 10^{-7}$ weber per ampere-metre (Wb/Am).

$= 12.56 \times 10^{-7}$ (Wb/Am).

The direction of $\delta\vec{B}$ at P is determined by the vector product $\delta\vec{l} \times \vec{r}$. In Fig. 22.1 the direction of $\delta\vec{B}$ at P is perpendicular to the plane containing $\delta\vec{\ell}$ and $\vec{r}$. This is determined by the right hand thumb rule.

In vector form, the Biot and Savart's law is then written as

$$\delta\vec{B} = \frac{\mu_0}{4\pi} \frac{I(\delta\vec{l} \times \hat{r})}{r^3}$$

where $\hat{r}$ is unit vector along position vector $\vec{r}$.

or, $$\delta\vec{B} = \frac{\mu_0}{4\pi} \frac{I(\delta\vec{\ell} \times \hat{r})}{r^3} \qquad ...(iv)$$

where $\delta\vec{l}$ is a vector length pointing in the direction of the current I, and $\vec{r}$ is the position vector from the centre of the element $\delta\vec{\ell}$ to the point P.

Principle of Superposition

The resultant field at P is found by integrating equation (*iv*) as the principle of superposition also holds here as in the case of electric field,

$$\therefore \qquad \vec{B} = \oint d\vec{B}$$

or, $$\vec{B} = \frac{\mu_0}{4\pi} \oint \frac{I}{r^3} (d\vec{\ell} \times \vec{r}) \qquad (v)$$

where $\oint$ stands for close path.

Right hand thumb rule

Grasp the wire with right hand, the thumb pointing the direction of the current, the fingers will curl around the wire in the direction of $\vec{B}$.

For surface and volume currents, the Biot and Savart's law becomes:

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{K} \times \vec{r}}{r^3} da' \text{ and } \vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{J} \times \vec{r}}{r^3} d\tau'$$

where da' and $d\tau'$ be the area and volume elements respectively and $\vec{K}(r')$ and $\vec{J}(r')$ are the functions of r'.

22.2 FORCE BETWEEN TWO CURRENT CARRYING ELEMENTS

Consider two current carrying elements $I_1\ell_1$ and $I_2\ell_2$ flowing in opposite directions separated by a distance r as shown in Fig. 22.2.

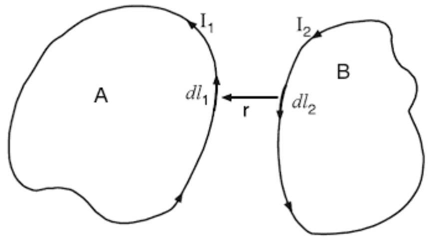

Fig. 22.2

The magnetic induction $d\vec{B}$ due to current I_2 through dl_2 on dl_1 at distance r is given give by

$$d\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{I_2 (d\vec{\ell}_2 \times \vec{r})}{r^3} \qquad (i)$$

The force exerted due to $d\vec{B}$ on another conductor $d\ell_1$ carrying current I_1 is obtained by general relation $\vec{F} = I\,\vec{\ell}_1 \times \vec{B}$ Using this relation, the force between the two adjacent current elements will be

$$d\vec{F} = \frac{\mu_0 I_1 I_2}{4\pi}\left(\frac{d\vec{\ell}_1 \times d\vec{\ell}_2 \times \vec{r}}{r^3}\right) \qquad ...(iii)$$

This formula for current elements is similar to Coulomb's law of force in electrostatics between two static charges. The force varies inversely as square of the distance between the two current elements. However, the direction of the force between two current carrying elements is *normal* to the line joining the element (Idl) while in Coulomb's law, force acts along the line joining the two charges.

22.3 APPLICATIONS OF BIOT AND SAVART'S LAW

Some of the important cases are discussed here where Biot and Savart's law in used :

22.3.1 Magnetic Field due to a Long Straight Conductor Carrying a Steady Current

Let AB be a straight conductor carrying a current I as shown in Fig. 22.3 The conductor is taken along the Y-axis and the current is supposed to flow along $-Y$ direction. Consider a point P at a perpendicular distance $x = OP$ from the conductor where O represents the origin and OP produced represents the X-axis.

Let $d\vec{l} = -\hat{j}\,dy$ be a small current element at a distance y from the origin O and let the vector distance of P from the centre of the element be $\vec{r}$, then

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$\vec{r} = -\hat{j}y + \hat{i}x$

The magnetic field at P due to current elements $d\vec{l}$ is given by Biot and Savart's law

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{l} \times \vec{r}}{r^3} = \frac{\mu_0 I}{4\pi}\frac{(-\hat{j}\,dy) \times (-\hat{j}y + \hat{i}\,x)}{r^3}$$

$$= \frac{\mu_0 I}{4\pi}\frac{xdy}{r^3}\hat{k}$$

i.e., the magnetic induction field $d\vec{B}$ along the $+Z$ axis perpendicular to plane containing the conductor AB and the point P

∴ Total magnetic field at P due to whole length of the wire AB

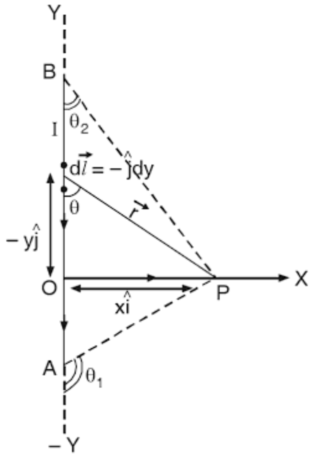

Fig. 22.3

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int_A^B \frac{xdy}{r^3}\vec{k} \qquad \text{... }(i)$$

If θ is the angle between the direction of current element $Id\vec{l}$ and vector $\vec{r}$, then

$$x = r \sin\theta \text{ or } r = x \operatorname{cosec}\theta$$

$$y = r \cot\theta$$

∴ $$dy = -x \operatorname{cosec}^2\theta \, d\theta$$

Substituting in (i) we have $\vec{B} = \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2} \frac{-x^2 \operatorname{cosec}^2\theta d\theta}{x^3 \operatorname{cosec}^3\theta}\hat{k}$

where θ_1 is the value of θ at the end A and θ_2 that at the end B.

∴ $$\vec{B} = \frac{\mu_0 I}{4\pi x}\int_{\theta_1}^{\theta_2} -\frac{d\theta}{\operatorname{cosec}\theta}\hat{k}$$

$$= \frac{\mu_0}{4\pi}\frac{I}{x}\int_{\theta_1}^{\theta_2} -\sin\theta\, d\theta\,\hat{k} = \frac{\mu_0 I}{x}[\cos]_{\theta_1}^{\theta_2}\hat{k}$$

or $$\vec{B} = \frac{\mu_0}{4\pi}\frac{I}{x}(\cos\theta_2 - \cos\theta_1)\hat{k} \qquad \text{...}(ii)$$

Infinitely long conductor. If the conductor is infinitely long

$$\theta_1 = \pi \text{ and } \theta_2 = 0$$

$$\cos\theta_2 = 1 \text{ and } \cos\theta_1 = -1$$

Hence $$\vec{B} = \frac{\mu_0}{4\pi}\frac{2I}{x}\hat{k} \qquad \text{...}(iii)$$

22.3.2 Force Between Two Parallel Straight Conductors

Let *AB* and *CD* be two *long* straight conductors carrying currents I_1 and I_2 respectively and lying parallel to each other along the *Y*-axis at a distance *x*. The current in both is flowing along the –*Y* direction as shown in Fig. 22.4. Then according to Biot's and Savart's law the intensity of the magnetic field $\vec{B}$ at a point on the wire *CD* due to the current I_1 in *AB* is given by

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{2I_1}{x}\hat{k} \qquad \text{... } (i)$$

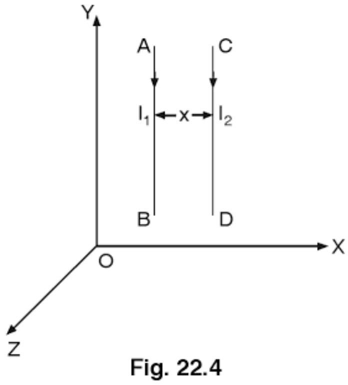

Fig. 22.4

The direction of magnetic field $\vec{B}$ is along the +*Z* direction.

Force experienced by length $d\vec{l}$ of the wire *CD* due to the field $\vec{B}$

$$= I_2\, d\vec{l} \times \vec{B} = I_2(-\,dl\,\hat{j}) \times (B\hat{j}) \qquad \text{...}(ii)$$

As the current flows in *CD* along –*Y* direction, the current elements is taken as $-I_2\, dl\hat{j}$

Substituting the value of $\vec{B}$ from (*i*) in (*ii*), we have Force on a length *dl* of *CD*

$$= I_2(-dl\,\hat{j}) \times \left(\frac{\mu_0}{4\pi}\frac{2I_1}{x}\right)\hat{k} = -\frac{\mu_0}{4\pi}\frac{2I_1I_2dl}{x}(\hat{j}\times\hat{k})$$

$$= -\frac{\mu_0}{4\pi}\frac{2I_1I_2\,dl}{x}\hat{i} = \frac{\mu_0}{4\pi}\frac{2I_1I_2dl}{x}(-\hat{i})$$

$\therefore$ Force on a length *l*
$$= \frac{\mu_0}{4\pi}\frac{2I_1I_2}{x}(-\hat{i})\int_0^l dl$$

$$= \frac{\mu_0}{4\pi}\frac{2I_1I_2}{x}(-\hat{i})$$

Hence force on a unit length $= \frac{\mu_0}{4\pi}\frac{2I_1I_2}{x}(-\hat{i}) = \frac{\mu_0}{2\pi}\frac{I_1I_2}{x}(-\hat{i})$

Thus the force due to the current in *AB* on the conductor *CD* acts along –*X* direction *i.e.*, it is a force of *attraction* and its magnitude is $\frac{\mu_0}{2\pi}\frac{I_1I_2}{x}$. ...(*iii*)

Currents in opposite directions: The force is mutual *i.e.*, the force on a unit length of *AB* due to the current in *CD* is also same. This is in accordance with Newton's third law of motion. Two conductors *attract* each other if the currents flow in the *same direction* and they *repel* each other if the currents flow in *opposite directions.*

Force between two equal parallel co-axial coils. Let *A* and *B* be two co-axial circular coils of one turn each of radius *a* with their centres a small distance *x* apart (Fig. 22.5). Let I_1 be the current in the coil *A* and I_2 that in *B*. As *x* is small as compared to *a* the magnitude of the force on unit length of *B* due to *A* is

The lines of magnetic induction are endless circles concentric with the toroid as shown in Fig. 22.12. The direction of the magnetic field $\vec{B}$ will be tangent to the circle at every point.

If we consider a toroid having inner radius r_1, outer radius r_2 with centre O and total number of turns N as shown in Fig. 22.12, the magnetic field due to the toroid will depends upon the location of the observation point. Three cases arise:

(*i*) *The observation point P_i lies at a distance r_i less than r_1.* To find the magnetic field at P_i draw a circle with radius r_i and centre O. It is clear from the figure that no turn of the current carrying wire of the toroid passes through the area enclosed by the path *i.e.* $I = 0$

The line integral of the magnetic field along the closed path *i.e.*,

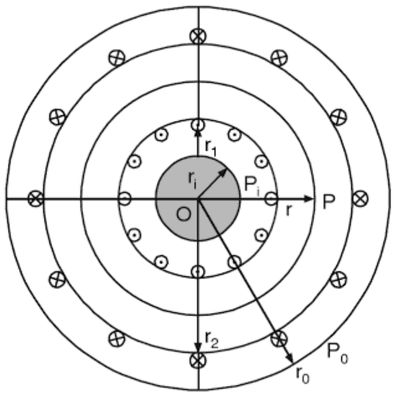

Fig. 22.12

$$\oint \vec{B} \cdot d\vec{r} = 2\pi r_i B = \mu_0 NI = 0 \qquad [\because I = 0]$$

$\therefore \quad B = 0$

Thus the magnetic field is zero inside the ideal toroid.

(*ii*) *The observation point P_0 lies at a distance r_0 greater than r_2.* To find the magnetic field at P_0 draw a circle with radius r_0 and centre O. It is clear from the figure that as many turns of the current carrying wire emerge out of the outer boundary of the toroid (shown as ⊗) as enter into its inner boundary (shown as ⊙) *i.e.* the net current I linked with the whole area enclosed by the path is zero $I = 0$.

The line integral of the magnetic field along closed path

$$i.e. \quad \oint \vec{B} \cdot d\vec{r} = 2\pi r_0 B = \mu_0 NI = 0 \qquad (\because I = 0)$$

Thus the magnetic field is zero outside the ideal toroid.

(*iii*) *The observation point P lies at a distance r where $r > r_1$ but $r < r_2$.* To find the magnetic field at P draw a circle with radius r and centre O. It is clear from the figure that all the N turns of the current carrying wire enter into its boundary (shown as ⊙) i.e. the current linked with the whole area enclosed by the path $= NI$.

The line integral of the magnetic field along the closed path *i.e.*

$$\oint \vec{B} \cdot d\vec{r} = 2\pi r B = \mu_0 NI$$

or $$B = \frac{\mu_0 NI}{2\pi r}$$

As r varies from r_1 to r_2 as we move from the inner edge to the outer edge of the toroid *the magnetic field inside the core of the ideal toroid is non-uniform*

Magnetic field on the inner edge of the toroid of radius r_1

$$B_1 = \frac{\mu_0 NI}{2\pi r_1}$$

and Magnetic field on the outer edge of toroid of radius r_2

$$B_2 = \frac{\mu_0 NI}{2\pi r_2}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \iint_S \vec{J} \cdot d\vec{s} \qquad \text{... (iii)}$$

But according to Stoke's theorem in vector analysis

$$\oint \vec{B} \cdot d\vec{l} = \iint_S (Curl\ \vec{B} \cdot d\vec{s}$$

$$= \iint_s (\vec{\nabla} \times \vec{B}) \cdot d\vec{s} \qquad \text{... (iv)}$$

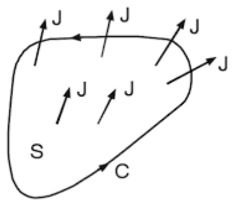

Fig. 22.14

From (*iii*) and (*iv*), we have

$$\iint_s (\vec{\nabla} \times \vec{B}) \cdot d\vec{s} = \mu_0 \iint_S \vec{J} \cdot d\vec{s}$$

or
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \qquad \text{... (v)}$$

The equation $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ *is known as Ampere's law in differential form*. It is a point equation and gives the relation between magnetic field at a point and the current density at the same point in space.

22.6 AMPERE'S CIRCUITAL LAW INDEPENDENT OF SHAPE OF THE PATH

The relation $\oint \vec{B} \cdot d\vec{l} = \mu_0 I$ holds not only for a circular path but for any arbitrary closed path which encircles the wire carrying current as shown in Fig. 22.15 As the wire carries a steady current in the + Z direction, the magnetic field due to it lies entirely in XY plane. We shall, therefore, consider the closed path that also lie the XY plane

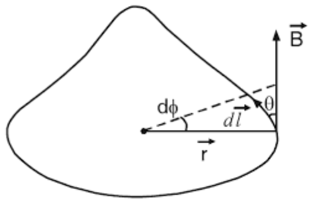

Fig. 22.15

$$\therefore \qquad \vec{B} \cdot \vec{dl} = B\,dl \cos\theta$$

where $dl \cos\theta$ is the component of the line element $d\vec{l}$ along the direction of $\vec{B}$, θ being the angle between the vector $d\vec{l}$ and the vector $\vec{B}$ which has a direction perpendicular to the radius vector $\vec{r}$. Thus $dl \cos\theta$ is perpendicular to $\vec{r}$ and if $d\vec{l}$ subtends an angle $d\phi$ at the wire, then

$$d\phi = \frac{dl \cos\theta}{r}$$

$$\therefore \qquad \vec{B} \cdot \vec{dl} = Bdl \cos\theta = Br\,d\phi = \frac{\mu_0}{4\pi} \frac{2I}{r} r d\phi = \frac{\mu_0}{4\pi} 2I d\phi \qquad \text{... (ii)}$$

Hence
$$\oint \vec{B} \cdot \vec{dl} = \frac{\mu_0}{4\pi} 2I \oint d\phi = \frac{\mu_0}{4\pi} 2I . 2\pi = \mu_0 I \qquad \text{... (iii)}$$

as $\oint d\phi$, the angle subtended by a closed path at point inside it is 2π.

The relation $\oint \vec{B} \cdot \vec{d} l = \mu_0 I$ is known as *Ampere's line integral theorem or Ampere's circuital law*. It is similar to Gauss's law in electrostatics.

22.13 MAGNETIC SCALAR AND VECTOR POTENTIALS

In the case of an electric field, we have

$$\vec{E} = -\text{grad V} = -\vec{\nabla} V_E$$

where $\vec{E}$ is the electric field intensity and V_E the electric potential, which is a scalar quantity. The electric field is a *conservative* field and hence

$$\vec{\nabla} \times \vec{E} = 0 \text{ or } \vec{\nabla} \times \vec{\nabla} V_E = 0$$

But *curl* $\vec{B} = \vec{\nabla} \times \vec{B} = 0$ only in the special case when the line integral $\oint \vec{B}.dl = 0$ *i.e.*, when the line integral does not enclose a current. When the line integral encloses a current of current density $\vec{J}$ then $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

In a current free space $\vec{J} = 0$ and therefore $\vec{\nabla} \times \vec{B} = 0$

Thus for a current free space we can write

$$\vec{B} = -\vec{\nabla} V_m$$

where V_m is a scalar function called *magnetic scalar potential.*

$\vec{B}$ is the negative gradient of scalar potential only in current free space. $\vec{B}$ is not, in general, the negative gradient of a scalar potential.

The condition for magnetic scalar potential to exist is that the current density vector $\vec{J} = 0$ *i.e., it is a current free space.*

Magnetic vector potential. The divergence of a magnetic induction field $\vec{\nabla} \cdot \vec{B}$ is always zero.

$$\therefore \qquad \vec{\nabla} . \vec{B} = 0$$

If $\vec{A}$ is another vector such that $\vec{B} = \vec{\nabla} \times \vec{A}$, then

$$\vec{\nabla} . \vec{B} = \vec{\nabla} . \vec{\nabla} \times \vec{A} \text{ is always equal to zero.}$$

The vector $\vec{A}$ is called *magnetic vector potential.*

Hence magnetic vector potential is defined as a vector function the curl of which is equal to $\vec{B}$ *the magnetic induction field.*

22.14 MAGNETIC FIELD DUE TO A CURRENT LOOP

Consider an arbitrary current distribution (or current loop) through which a current I is flowing. This current carrying loop behaves as magnetic dipole. Let us evaluate magnetic field $\vec{B}(\vec{r})$ at any point P located near to it and having position vector $\vec{r}$.

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$$\vec{B}(\vec{r}) = -\frac{\mu_0}{4\pi}\iiint \vec{J}\,dv' \times \vec{\nabla}\left[\frac{1}{|\vec{r}-\vec{r}\,'|}\right]$$

$$= \frac{\mu_0}{4\pi}\iiint \vec{\nabla}\left[\frac{1}{|\vec{r}-\vec{r}\,'|}\right]\times \vec{J}\,dv' \qquad \dots (iv)$$

But $$\vec{\nabla}\times\frac{\vec{J}}{|\vec{r}-\vec{r}\,'|} = (\vec{\nabla}\times\vec{J})\frac{1}{|\vec{r}-\vec{r}\,'|} + \vec{\nabla}\frac{1}{|\vec{r}-\vec{r}\,'|}\times\vec{J}$$

and for steady current $\vec{\nabla}\times\vec{J} = 0$

$$\therefore \qquad \vec{\nabla}\times\frac{\vec{J}}{|\vec{r}-\vec{r}\,'|} = \vec{\nabla}\frac{1}{|\vec{r}-\vec{r}\,'|}\times\vec{J} \qquad \dots (v)$$

From (*iv*) and (*v*), we get

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\iiint \vec{\nabla}\left[\frac{1}{|\vec{r}-\vec{r}\,'|}\right]\times\vec{J}\,dv'$$

$$\therefore \qquad \vec{B}(\vec{r}) = \vec{\nabla}\times\frac{\mu_0}{4\pi}\iiint\frac{\vec{J}\,dv'}{|\vec{r}-\vec{r}\,'|} \qquad \dots (vi)$$

The equation (*vi*) gives the magnetic field due to dipole (*i.e.,* current loop)

But $\vec{B}(\vec{r}) = curl\,\vec{A}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r})$ where $\vec{A}(\vec{r})$ is the magnetic vector potential at the position vector $\vec{r}$.

$$\therefore \qquad \vec{\nabla}\times\vec{A}(\vec{r}) = \vec{\nabla}\times\frac{\mu_0}{4\pi}\iiint\frac{\vec{J}}{\vec{r}-\vec{r}\,'}\,dv'$$

or $$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\iiint\frac{\vec{J}\,dv'}{|\vec{r}-\vec{r}\,'|}$$

If $\vec{r}$ is the vector distance of the point P from the current element taken to be of a small vector length $d\vec{r}$, the replacing $\vec{r}-\vec{r}\,'$ by $|\vec{r}|$ and $dv' = S\,dr'$ by $S\,dr = dv$, we get, the magnetic vector potential $\vec{A}$ as

$$\vec{A} = \frac{\mu_0}{4\pi}\iiint\frac{\vec{J}}{r}\,dv \qquad \dots (vii)$$

22.15 MAGNETIC FIELD AT ANY POINT DUE TO MAGNETIC DIPOLE

Fig. 22.19. shows a magnetic dipole *SN*. *P* is any point in air at distance *r* from the centre O of the dipole. Let $\vec{\mu}$ be the magnetic moment of the dipole. Let θ be the angle between *OP* and the axis *SN* of the dipole. The dipole $\vec{\mu}$ is a vector quantity, hence we resolve it into two components :

(*i*) the component along $OP = \mu \cos \theta$

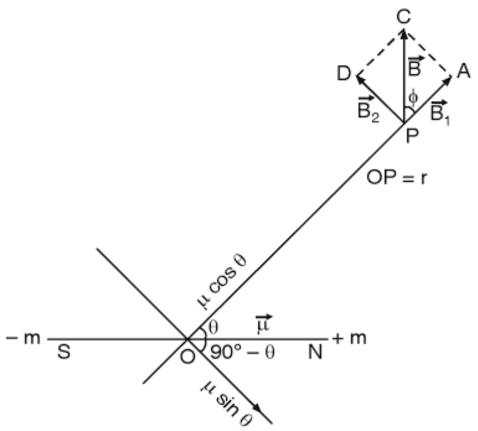

Fig. 22.19

(*ii*) the component perpendicular to $OP = \mu \sin \theta$

With respect to $\mu \cos \theta$ the point *P* is on the axis. Since the dipole is short, the magnitude of the magnetic induction $\vec{B}_1$ at *P* due to this component is given by

$$B_1 = \frac{\mu_0}{4\pi} \cdot \frac{2\mu \cos\theta}{r^3} \qquad \text{... } (i)$$

where μ_0 (= $4\pi \times 10^{-7}$ Wb/Am) is the permeability of free space.

The direction of $\vec{B}_1$ is along PA. With respect to $\mu \sin \theta$, the point P is on the equator. Since the dipole is short the magnitude of the magnitude of the magnetic induction $\vec{B}_2$ at P due to this component is given by

$$B_2 = \frac{\mu_0}{4\pi} \cdot \frac{2\mu \sin\theta}{r^3} \qquad \text{... } (ii)$$

The direction of $\vec{B}_2$ is opposite to that of $\mu \sin \theta$, *i.e.* along PD which is perpendicular to *OP*. The magnitude and the direction of the resultant magnetic induction $\vec{B}$ at *P* is represented by the diagonal of the rectangle *PACD* in which *PA* represent $\vec{B}_1$ and *PD* represents $\vec{B}_2$. The magnitude of $\vec{B}$ is given by

22.16 MAGNETISATION

Free and bound currents. Ordinary conduction currents which arise because of the presence of batteries or other sources of *e.m.f.* in electrical circuits that actually involve motion of electrons in a macroscopic path are called *free currents*. Such currents can be started or stopped with a switch and measured with the help of an ammeter.

The currents associated with molecular or atomic magnetic dipole moments are known as *bound currents.* These arise due to orbital or the spin motion of the electrons within the atom.

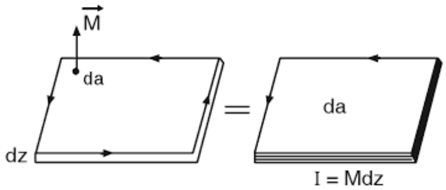

Fig. 22.20

The electron in an atom have a magnetic dipole moment due to their orbital motion around the nucleus as well as due to their spin. Due to this fact certain atoms and molecules have a *net* magnetic dipole moment. As a result when a substance is magnetised all its magnetic diploes are oriented in the same direction. A block of material is said to be *uniformly magnetised* if it contains a large number of atomic dipoles all pointing in the same direction evenly distributed throughout its volume. If the dipole moment of each atomic dipole is $\vec{m}$ and the number of such dipoles per unit volume is n, then

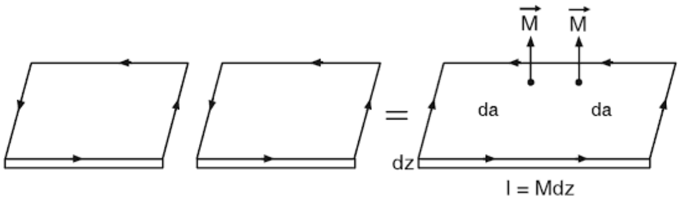

Fig. 22.21

Magnetic moment per unit volume $\vec{M} = \vec{m}\, n$

The magnetic moment per unit volume is known as magnetic polarisation or intensity of magnetisation or simply magnetisation.

Consider a small rectangular piece of uniformly magnetised material of magnetisation $\vec{M}$ having surface area da and thickness dz, then its

$$\text{Magnetic moment} \quad = \quad \vec{M}\, da\, dz$$

Now a current loop has a magnetic moment I × area of the loop = Ida

$$\therefore \qquad I = Mdz$$

Hence a magnetic dipole of moment $\vec{M}\, da\, dz$ is equivalent to a loop of area da through the boundary of which is flowing a current Mdz in the *anticlockwise* direction as shown in Fig. 22.20.

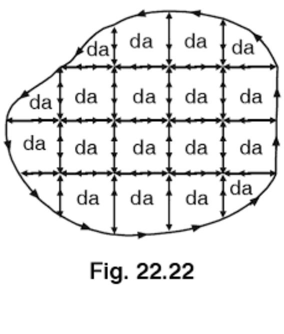

Fig. 22.22

Now, place two identical pieces side by side so that the rim currents in the boundary where they touch each other are *equal and opposite* and thus *cancel out* [Fig. 22.21]. The two pieces are thus equivalent to a single slab or area 2 da through the boundary of which current $I = Mdz$ is flowing.

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In this way, we can have any number of pieces and construct a slab of an arbitrary shape of thickness dz and area $A = \Sigma\, da$ and uniform magnetision $\vec{M}$ which is equivalent to a current circuit of area A through the boundary of which a current $I = Mdz$ is flowing.

By placing a number of slabs each of thickness dz one above the other we can have a magnetised material of any shape and thickness with an equivalent boundary current $I = \Sigma\, Mdz$. Conversely a material of any thickness or shape can be divided into slabs each of thichness dz and each slab can be further subdivided into rectangular piece of area da. The result derived by placing two such pieces in contact with common boundary can be generalised for the whole material.

This is the situation when the magnetisation is uniform so that the currents cancel out at the common boundary of the two pieces of the slab and the magnetisation is only due to the current in the outermost boundary or rim of the slab, as shown in Fig. 22.22.

The equivalent current is purely theoretical concept and it is a fictitious current that cannot be measured.

As M is uniform, $\qquad I = \Sigma M\, dz = M\, \Sigma dz = Mz$

Now $\dfrac{I}{z}$ = current through unit length is denoted by J_z. It is known as *surface current density*.

$\therefore \qquad J_z = M$

22.17 MAGNETISATION CURRENT DENSITY VECTOR $\vec{J}$

Let us consider the variation of M_z along the Y-axis and the current density is given by

$$J_{x_1} = \frac{\partial Mz}{\partial y} \text{ along } x\text{-direction.}$$

Similarly, if we consider the variation of M_y along Z-axis, the current density J_{x_2} in the –ve X-direction as shown in Fig. 22.23 given by

$$Jx_2 = -\frac{\partial M_y}{\partial z} \text{ along } x\text{-direction.}$$

Hence, total contribution to current density vector along x-direction

$$J_x = J_{x_1} + J_{x_2}$$

$$= \left(\frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}\right)$$

In general, the value of J_y and J_z will also involved and given by

$$J_y = \left(\frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x}\right) \text{ and } J_z = \left(\frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y}\right)$$

Hence $\qquad \vec{J} = \hat{i}\, J_x + \hat{j}\, J_y + \hat{k}\, J_z$

$$= \hat{i}\left(\frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}\right) + \hat{j}\left(\frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x}\right) + \hat{k}\left(\frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y}\right)$$

$$= \vec{\nabla} \times \vec{M} = Curl\, \vec{M}$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_{free} \qquad \text{... }(vi)$$

Thus the vector $\vec{H}$ is determined only by free currents and not by the magnetisation of the material. $\vec{H}$, therefore, represents the *magnetising field*. The vector $\vec{B}$ is due to the total current and represent the total field or *induction field*

$$\vec{B} = \mu_0 (\vec{H} + \vec{M}) \qquad \text{... }(vii)$$

The relation $\vec{\nabla} \times \vec{H} = \vec{J}_{free}$ is known as *differential form of Ampere's law in the presence of magnetic material.*

Units. The magnetic induction $\vec{B}$ and the magnetising field $\vec{H}$ are different fields. Therefore, they are measured in different units.

The unit of $\vec{B}$ is a Tesla. (1 Tesla = 1 Wb/m^2). As $\vec{B} = \mu_0 (\vec{H} + \vec{M})$, the unit of $\vec{H}$ and $\vec{M}$ is the same as that of $\dfrac{\vec{B}}{\mu_0}$ *i.e.*, Ampere turns per metre. The unit of $\vec{M}$ and $\vec{H}$ is also Ampere turns per metre.

22.18 MAGNETIC SUSCEPTIBILITY, PERMEABILITY AND THEIR RELATIONSHIP

Magnetic susceptibility. *The ratio of the magnitude of magnetic dipole moment per unit volume $\vec{M}$ and the magnitude of the magnetising field $\vec{H}$ is known as magnetic susceptibility* χ_m.

$$\therefore \qquad \chi_m = \frac{|\vec{M}|}{|\vec{H}|} = \frac{M}{H} \qquad \text{... }(i)$$

The magnetic susceptibility of vacuum (or free space) is zero. Magnetic susceptibility is a pure number as the unit $\vec{M}$ and H are the same.

Permeability. *The ratio of magnitude of magnetic induction field $\vec{B}$ inside the material and the magnitude of magnetising field $\vec{H}$ is known as magnetic permeability* μ.

$$\therefore \qquad \mu = \frac{|\vec{B}|}{|\vec{H}|} = \frac{B}{H} \qquad \text{... }(ii)$$

Unit of μ. unit of μ $= \dfrac{\text{Unit of } B}{\text{Unit of } H} = \dfrac{\text{Tesla}}{A/m}$

But $\qquad \text{Tesla} = \dfrac{\text{Newton}}{\text{Amp metre}} = \dfrac{N}{Am}$

i.e., the field $\vec{B}$ acts along the $+Z$ direction.

Distance of the moving electron from the long wire $x = 0.1$ m

$$I = 2\,A \text{ and } \frac{\mu_0}{4\pi} = 10^{-7} \text{ S.I units}$$

$$\therefore \qquad \vec{B} = 10^{-7} \times \frac{2\times 2}{0.1} = 4\times 10^{-6}\,\hat{k} \text{ Tesla}$$

Velocity of the electron $= 4 \times 10^4$ ms^{-1}

As the electron travels in a direction opposite to the current, its direction of motion is along $+Y$ direction and

$$\vec{v} = 4\times 10^4\,\hat{j}$$

Charge on the electron $q = -1.6 \times 10^{-19}$ C

$\therefore$ Force on the moving electron

$$= q(\vec{v}\times\vec{B})$$

$$= -1.6\times 10^{-19}(4\times 10^4\,\hat{j}\times 4\times 10^{-6}\,\hat{k})$$

$$= -2.56\times 10^{-20}\,\hat{i} = 2.56\times 10^{-20} \text{ N along } -X \text{ direction}$$

The electron is, therefore, attracted towards the current carrying wire.

Example 22.2. Calculate of the magnitude of the magnetic field due to a long thin wire carrying current of 15 Amp at distance of 1 cm from the wire, given $\mu_0 = 4\pi \times 10^{-7}$ N/A^2.

Solution. Current $I = 15$ Amp; Distance $x = 1$ cm $= 0.01$ m

$$\text{Magnitude of magnetic field } |\vec{B}| = \frac{\mu_0}{4\pi}\frac{2I}{x} = 10^{-7}\frac{2\times 15}{0.01}$$

$$= 3 \times 10^{-4} \text{ Tesla}$$

Example 22.3. Calculate the magnetic field at a distance of 5 m from an infinite straight conductor carrying current of 100 A. (*P.U.*, 2002)

Solution. Current $I = 100$ A; Distance $x = 5$ m

$$\text{Magnitude of magnetic field } |\vec{B}| = \frac{\mu_0}{4\pi}\frac{2I}{x} = 10^{-7}\frac{2\times 100}{5} = 4\times 10^{-6} \text{ Tesla}$$

Example 22.4. Two parallel straight wires are placed at 2 cm and 6 cm mark at right angles to the metre scale. The currents in them are 1 A and 3 A respectively. Find the mark at which they will produce zero magnetic field.

Solution. Suppose the two wires are A and B and current flows through them in $-Y$ direction as shown.

Let P be the point at the mark x at the mark which the two currents produce zero magnetic field.

For wire A kept at 2 cm = 0.02 m mark the point P lies in the $+X$ direction. Hence

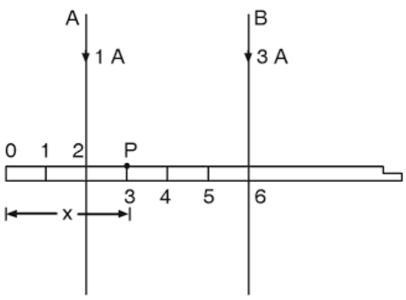

Fig. 22.25

$$\text{Magnetic field at } P = \frac{\mu_0}{4\pi}\frac{2I}{r}\hat{k}$$

$$= \frac{\text{Charge}}{\text{time}} = \text{charge} \times \text{frequency} = ev$$

Now $e = 1.6 \times 10^{-19}$ C; $v = 6.8 \times 10^{15}$ rev/sec

$\therefore$ $I = 1.6 \times 10^{-19} \times 6.8 \times 10^{15} = 10.88 \times 10^{-4}$ Amp.

(a) Magnetic field at the centre of the orbit $B = \dfrac{\mu_0 I}{2a}$

Now $\dfrac{\mu_0}{4\pi} = 10–7$ Weber/A–m or $\mu_0 = 4\pi \times 10^{-7}$ Weber/A-m

$a = 5.1 \times 10^{-11}$ m

$$\therefore \quad B = \frac{4\pi \times 10^{-7} \times 10.88 \times 10^{-4}}{2 \times 5.1 \times 10^{-11}} = 13.4 \text{ Weber/m}^2 = 13.4 \text{ Tesla}$$

(b) Magnetic dipole moment $M = IA = I\pi r^2$

$$= 10.88 \times 10^{-4} \times \pi \times (5.1 \times 10^{-11})^2$$

$$= 8.89 \times 10^{-24} \text{ Am}^2$$

Example 22.11. A Helmholtz galvanometer has coils of radius 11 cm and the number of turns $70\sqrt{5}$. Calculate the current through the coil which produces a deflection of 45°.

$$\mathbf{H = \frac{0.32}{4\pi \times 10^{-3}}} \textbf{ Amp. turns/m.}$$ (*Gharwal U.,* 2000)

Solution. Here $a = 11 \text{ cm} = 0.11\, m$; $n = 70\sqrt{5}$, $\theta = 45°$

$$H = B_H = \frac{0.32}{4\pi \times 10^{-3}} \text{ Amp. turns/m} = \frac{0.32 \times 4\pi \times 10^{-7}}{4\pi \times 10^{-3}} = 0.32 \times 10^{-4} \text{ T}$$

$$I = \frac{5\sqrt{5}\, a B_H \tan\theta}{8\mu_0 n} = \frac{5\sqrt{5} \times 0.11 \times 0.32 \times 10^{-4} \times 1}{8 \times 4\pi \times 10^{-7} \times 70\sqrt{5}}$$

$$= \frac{5\sqrt{5} \times 0.11 \times 0.32 \times 10^{-4} \times 1 \times 7}{8 \times 4 \times 22 \times 10^{-7} \times 70\sqrt{5}} = 25 \times 10^{-3} \text{ A} = 25 \text{ mA}$$

Example 22.12. Two similar coils of wire having a radius of 7 cm and 60 turns have a common axis and are 18 cm apart. Find the strength of the magnetic field at a point midway between them on their common axis, when a current of 0.1 Amp is passed through them.

(*G.N.D.U.,* 2002)

Solution. Here $n = 60$; $a = 7 \text{ cm} = 0.07$ m; $x = 9 \text{ cm} = 0.09$ m

Current $I = 0.1$ Amp $\mu_0 = 4\pi \times 10^{-7}$

$$\text{Field due to either coil} = \frac{\mu_0}{2} \frac{nIa^2}{(a^2 + x^2)^{3/2}}$$

$$= \frac{1}{2} \times 4\pi \times \frac{10^{-7} \times 60 \times 0.1 \times (0.07)^2}{(0.09^2 + 0.07^2)^{3/2}}$$

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$$= 0.01247 \times 10^{-4} \text{ Tesla}$$

$\therefore$ Field due to the two coils midway between them

$$= 0.1247 \times 10^{-4} \times 2 = 0.2494 \times 10^{-4} \text{ Tesla.}$$

Example 22.13. A solenoid 4 m long and mean diameter 8 cm has 10^4 turns. If a current of 5A is flowing through it, calculate the magnetic field at its centre.

Solution. As the solenoid has a length 100 times its radius, it may be taken to be solenoid of infinite length.

Number of turns $= 10^4$; Length = 4 m

$\therefore$ Number of turns per meter $n = \dfrac{10^4}{4} = 2.5 \times 10^3$

Current $I = 5\text{A}$

$\therefore$ Magnetic field at the centre of the solenoid $B = \mu_0 nI$

$$= 4\pi \times 10^{-7} \times 2.5 \times 10^3 \times 5 = 15.7 \times 10^{-3} \text{ Tesla.}$$

Example 22.14. A solenoid of 1200 turns is wound uniformly in single layer on a glass tube 24 cms long and 10 cms in diameter. Find the strength of the field (*a*) at the centre and (*b*) at the end when 0.1 ampere current flows through it.

Solution. Length of the solenoid = 24 cms = 0.24 m

Total number of turns = 1200

Number of turns per metre $n = \dfrac{1200}{0.24} = 5000$

Current through the solenoid $I = 0.1$ amp

Let θ_1 and θ_2 be the angle subtended at P by the ends X and Y, then

$$\cos\theta_2 = \cos(180 - \theta_1) = -\cos\theta_1 = \frac{12}{13}$$

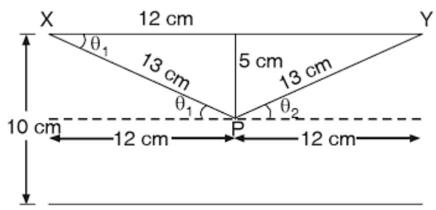

Fig. 22.26

(*a*) Magnitude of magnetic field at any point P inside the solenoid

$$B = \frac{\mu_0}{2} nI(\cos\theta_1 - \cos\theta_2) = \mu_0 nI\cos\theta_1 \qquad [\because \cos\theta_2 = -\cos\theta_1]$$

$$= 4\pi \times 10^{-7} \times 5000 \times 0.1 \times \frac{12}{13} = 5.8 \times 10^{-4} \text{ Tesla}$$

(*b*) If the point P lies at the end Y of the solenoid, then let $\theta_1 = \theta$ be the angle *subtended* by X at P, then

$$PX = \sqrt{24^2 + 5^2} = \sqrt{601} \qquad \therefore \cos\theta = \frac{24}{\sqrt{601}}$$

or $$\oint \vec{B}.\vec{dl} = \mu_0 I = 4\pi \times 10^{-7} \times 5 \times 10^{-3} = 62.9 \times 10^{-10} \text{ Tesla metre.}$$

Example 22.17. A toroid of small cross sectional area has 10^4 turns and outer radius of 9 cm and inner radius of 5 cm. The current in the toroid winding is 0.14 A. Find the magnetic field inside to toroid. (*P.U.*, 2003)

Solution. Total number of turns on the toroid N = 10^4

Mean radius of the toroid (of small cross-sectional area)

$$r = \frac{r_1 + r_2}{2} = \frac{5+9}{2} = 7\,\text{cm} = 7\times10^{-2}\,\text{m}$$

$$\text{current} = 0.14 \text{ A.}$$

Magnetic field inside the toroid $B = \dfrac{\mu_0 NI}{2\pi r}$

$$= \frac{\mu_0}{4\pi} \cdot \frac{2NI}{r} = \frac{10^{-7}\times2\times10^4\times0.14}{7\times10^{-2}}$$

$$= 4 \times 10^{-3} \text{ T}$$

Example 22.18. A current in a solenoid produces a magnetising field of 167 Amp met^{-1}. What is the magnetic induction inside it if it has an iron core of magnetic susceptibility 5000?

Solution Magnetising field $H = 167 \text{ Am}^{-1}$

Succeseptibility $\chi_m = 5000$

Permeability $\mu = \mu_0 (1 + \chi_m) = 4\pi \times 10^{-7} (1 + 5000)\ N/A^2$

Now $$\frac{B}{H} = \mu$$

$\therefore$ $$B = \mu H = 167\times4\pi\times10^{-7}\times5001\ N/Am$$

$$= 1.05\ N/Am = 1.05 \text{ Tesla.}$$

Example 22.19. The magnetic susceptibility of a medium is 940 × 10^{-4}. Calculate its absolute and relative permeability. (*P.U.* 2000)

Solution. Susceptibility $\chi_m = 940 \times 10^{-4} = 0.094$

Relative permeability $$\mu_r = \frac{\mu}{\mu_0} = 1 + \chi_m$$

$\therefore$ $$\mu_r = 1.094$$

and $$\mu = \mu_r \mu_0 = 1.094 \times 4\pi \times 10^{-7} = 13.75 \times 10^{-7}\ N/A^2$$

Example 22.20. Magnetic susceptibility of aluminium is 2.3 × 10^{-5} Find its permeability and relative permeability. (*M.D.U.* 2003)

Solution. $$\mu_r = 1 + \chi_m = 1 + 2.3 \times 10^{-5} = 1.000023$$

$$\mu = \mu_0 \mu_r \times 4\pi \times 10^{-7} \times 1.000023 = 12.57 \times 10^{-12}\ N/A^2$$

Example 22.21. A sample of iron develops a magnetic moment of 8000 Am^2. If the area of cross-section of the sample is 16 sq. cm, and its length is 5 cm, calculate

(*i*) Intensity of magnetisation (*ii*) Magnetic induction

(*iii*) permeability and (*iv*) susceptibility of the sample when the magnetising field intensity is 2 × 10^7 Am^{-1}. (*K.U.* 2003)

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Solution. Magnetic moment $= 8000\ Am^2$

Area of cross-section $= 16$ Sq. cm $= 16\times 10^{-4}\ m^2$

Length $= 5$ cm $= 5 \times 10^{-2}$ m

$\therefore$ Volume of the sample $= 16 \times 10^{-4} \times 5 \times 10^{-2} = 80 \times 10^{-6}\ m^3$

$$\text{Intensity of magnetisation } \vec{M} = \frac{\text{Magnetic moment}}{\text{Volume}}$$

$$= \frac{8000}{80 \times 10^{-6}} = 10^8 Am^{-1}$$

Magnetising field $H = 2 \times 10^7\ Am^{-1}$

$$\therefore \text{ Susceptibility } \chi_m = \frac{\vec{M}}{H} = \frac{10^8}{2 \times 10^7} = 5$$

Permeability $\mu = \mu_0 (1+\chi_m) = 4\pi \times 10^{-7} (1+5)$

$= 24\ \pi \times 10^{-7} = 75.4 \times 10^{-7}\ N/A^2$

Magnetic induction $B = \mu H = 24\ \pi \times 10^{-7} \times 2 \times 10^7$

$= 48\ \pi = 150.8$ Tesla.

Example 22.22. The maximum value of permeability a material is 0.126 N/A². What is relative permeability and magnetic susceptibility? (*P.U.* 2003)

Solution. $\mu = 0.126\ N/A^2,\ \mu_0 = 4\pi \times 10^{-7}\ N/A^2$

$$\therefore \text{ Relative permeability } \mu_r = \frac{\mu}{\mu_0} = \frac{0.126}{4\pi \times 10^{-7}} = 10^5$$

Susceptibility χ_m is given by

$\mu_r = 1+ \chi_m$

$\therefore\ \chi_m = \mu_r - 1 = 10^5 - 1 = 99999$

EXERCISE CH. 22

LONG QUESTIONS

1. **(*a*)** State and explain Biot and Savart's law in vector form in S.I. units.
(*G.N.D.U.*, 2004, 2003, 2000, *Kerla U.*, 2001; *M.D.U.*, 2000; *Meerut U.*, 2001 *Agra U.*, 2008)

(*b*) Apply Biot-Savart's law to determine the magnetic field due to a steady current I in a long straight wire. (*Nagpur Uni.*, 2009, 2007, 2006)

2. **(*a*)** State Biot-Savart's law in vector form, in S.I. system
(*Nagpur Uni.*, 2007, 2005)

(*b*) Using Biot and Savart's law find the magnetic field due to an infinite straight wire carrying current. (*H.P.U* 2003; *Meerut U.,* 2001, 2000; *Gauhati U.,* 2000)

3. **(*a*)** Using Biot and Savart's law find an expression for the intensity of magnetic induction field at a point on the axis of a circular coil carrying a steady current. Will this field be uniform? Show graphically the variation of the field with distance from the centre.

(*b*) From the result so obtained calculate the field at the center of a circular coil having n turns. (*Meerut U.*, 2005, 2003; *G.N.D.U.*, 2003, 2001; *Nagpur U.*, 2000; *D.A.U. Agra* 2005, 2003; *Kerala U.*, 2001; *Purvanchal.*, 2007, 2006)

$$Curl\,\vec{B} = \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

where symbols have usual meaning.

(*G.N.D.U.*, 2001, 2000; *M.D.U.*, 2001, 2000; *P.U.*, 2002, 2001; *Pbi.U.* 2000; *K.U.*, 2002, 2000)

5. Obtain Ampere's law in differential form

$$Curl\,\vec{B} = \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

6. Show that the line integral of the magnetic field around any path that does not enclose a current carrying wire is always zero.

7. Using Ampere's law obtain an expression for the magnetic field due to a current carrying straight conductor of infinite length. (*Meerut U.*, 2001)

8. What is the value of $\vec{\nabla}.\vec{B}$ and $\vec{\nabla}\times\vec{B}$ for points inside and outside a current loop?

(*G.N.D.U.* 2002, 2000)

9. Explain why the relation $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ and $\vec{\nabla}.\vec{B} = 0$ is not sufficient to derermine $\vec{B}$ at a point even if $\vec{J}$ is known at this point. (Agra U., 2000)

10. Compare and contrast Biot-Savart's law and Coulomb's law

11. Differentiate between the terms scalar and vector potentials as applied in magnetism. Derive an expression for the vector potential and show that $\vec{B} = \vec{\nabla}\times\vec{A}$ where $\vec{B}$ is the magnetic induction and $\vec{A}$ the magnetic vector potential.

(*G.N.D.U., 2001, 2000; Kerala U. 2001; K.U., 2001; H.P.U., 2003, 2002; M.D.U., 2001; Pbi. U., 2001, 2000; P.U., 2001*)

12. Using Ampere's law, calculate the magnetic field at a point inside a long current carrying solenoid (*Agra U.* 2004)

13. Give interpretation of a bar magnet as a surface distribution of solenoid current.

14. Explain magnetisation vector $\vec{M}$ and magnetisation current density vector $\vec{J}$.

15. Define magnetic susceptibility and permeability. Find their relationship.

(*Meerut U.* 2000)

16. Derive an expression for the magnetic induction $\vec{B}$ at any point due to a magnetic dipole.

17. Using Ampere's circutal law evaluate magnetic field inside a solenoid having n turns and infinite length.

18. A current is sent through a hanging coiled spring. What changes do you expect and why.

(*Kolkata U.* 2007)

19. State Ampere's law. Explain the concept of Maxwell's displacement current and show that how it led to a modification of Ampere's law.

(*Meerut U.* 2005, 2004*, M.S.U. Tiruneveli* 2007*; Agra* 2007)

20. Explain the magnetic field, flux and poles. (*M.S.U. Tiruneveli* 2007)

21. State and explain Ampere's circuital law. (*Nagpur U. s*/2009)

23 TIME VARYING FIELDS

ELECTROMAGNETIC INDUCTION

INTRODUCTION

So long as a steady (d.c.) and constant current flows through a conductor or a coil, the magnetic induction $\vec{B}$ at a point near to it remains constant. However, if the current is made to fluctuate, the magnetic induction also changes with time. Accordingly, the lines of magnetic induction called as *magnetic flux* changes. Whenever, the number of lines of magnetic induction or magnetic flux through a conducting circuit changes, an induced e.m.f. is produced in it. If the circuit is closed, a current flows. The e.m.f. or current produced lasts only for the time the change in flux continues. This phenomenon is known as *electromagnetic induction*. The magnitude of the induced e.m.f. or current is directly proportional to the rate at which the magnetic flux changes. The magnetic flux through a circuit can be changed by any one of the following methods:

(*i*) By motion of a magnet near a circuit or away from it or the motion of a circuit towards or away from a magnet.

(*ii*) By rotation of a coil in a magnetic field.

(*iii*) By starting, stopping or changing the current in a neighbouring circuit (Mutual induction).

(*iv*) By starting, stopping or changing the current in the circuit itself (Self induction).

This phenomenon of electromagnetic induction is completely studied by two laws namely, the Faraday's law of elctromagnetic induction and Lenz's law.

23.1 FARADAY'S LAW OF ELECTROMAGNETIC INDUCTION

Whenever the number of magnetic lines of induction or magnetic flux through a circuit changaes, an induced e.m.f. is produced in the circuit. The magnitude or induced e.m.f. is proportional to the rate of change of magnetic flux and lasts so long as the change continues.

Lenz's law. *The direction of the induced e.m.f. or current is such that it tends to stop the movement which produces it.*

Induced e.m.f. If $d\phi$ is the change in magnetic flux in a time dt, then according to Faradays' law the induced e.m.f.

$$e \propto \frac{d\phi}{dt}$$

But according to Lenz's law, it opposes the cause that produces it. As the cause of induced *e.m.f.* is change in flux $d\phi$, the two have opposite signs.

$$\therefore \qquad e \propto -\frac{d\phi}{dt}$$

or

$$e = -k\frac{d\phi}{dt}$$

where k is the constant of proportionality.

In S.I units $k = 1$ and

$$e = -\frac{d\phi}{dt} \qquad ...(i)$$

But $\qquad d\phi = \vec{B} \cdot d\vec{s} \qquad$ [Refer 21.5 Eq (i)]

where $\vec{B}$ is the magnetic flux density (or magnetic induction field) and $d\vec{s}$ a small area surrounding the point where magnetic flux density is $\vec{B}$.

$$\therefore \qquad e = -\frac{d\phi}{dt} = -\vec{B}\frac{d\vec{s}}{dt} \qquad ...(ii)$$

23.2 LENZ'S LAW IS IN ACCORDANCE WITH THE LAW OF CONSERVATION OF ENERGY

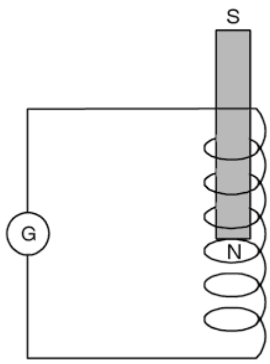

Fig. 23.1

Connect the terminals of a coil of wire through a galvanometer as shown in Fig. 23.1 Introduce the north pole of a bar magnet, into the coil and note that there is a deflection showing thereby that an induced *e.m.f.* (or current) has been set up. If we withdraw the magnet the galvanometer again shows a deflection in the opposite direction indicating thereby that the induced *e.m.f.* (or current) is now in the opposite direction.

When the north pole of a magnet is brought near a coil, the upper face of the coil should acquire a north polarity due to the induced *e.m.f.* or current because it will then be able to oppose the movement of the magnet. Similarly when the north pole of the magnet is taken away, the upper face should acquire a south polarity so that it again opposes the movement of the magnet by attracting the north pole. This is in accordance with the law of conservation of energy. When the north pole of the magnet is brought near the coil, the upper face, which acquires a north polarity due to the induced current in it, exerts a force of repulsion. Work is done in moving the magnet into the coil against the force of repulsion. It is this mechanical work which is converted into electrical energy.

Similarly on taking away the north pole, the upper face becomes a south pole and work has to be done in moving the magnet away from the coil against the force of attraction.

Thus we find that Lenz's law is in accordance with the law of conservation of energy.

23.3 INDUCED CHARGE IN A CIRCUIT OF RESISTANCE R

When the magnetic flux linked with a coil varies, an induced *e.m.f.* is set up in it. If e is the induced *e.m.f.* in volts at an instant and R the resistance of the circuit in ohms (*S.I. units*), then

$$e = -\frac{d\phi}{dt}$$

and current in amperes in the circuit at that time, $i = \frac{e}{R} = -\frac{1}{R}\frac{d\phi}{dt}$.

If the current remains constant for a very small time dt, then

Charge dq passing through the circuit $= -\frac{1}{R}\frac{d\phi}{dt}.dt = -\frac{d\phi}{R}$

$\therefore$ Total charge through the circuit, if the flux increases from ϕ_1 to ϕ_2 is given by

$$q = \int_{\phi 1}^{\phi_2} -\frac{d\phi}{R}$$

$$= \frac{1}{R}\left[-\phi\right]_{\phi 1}^{\phi 2} = \frac{\phi_1 - \phi_2}{R}$$

It is thus seen that the total charge is independent of the time during which the change in magnetic flux takes place.

$\therefore$ Charge in coulombs $= \dfrac{\text{Change in magnetic flux in webers}}{\text{Resistance in ohms}}$

23.4 PROOF OF INDUCED E.M.F. $e = -\frac{d\phi}{dt}$

(Using straight conductor)

Consider a straight conducting rod ab lying with its length along the Z-axis in a uniform magnetic field $\vec{B}$ directed along the Y-axis and moving with a velocity $\vec{V}$ along + X-axis.

According to Lorentz force equation a charged particle in the conductor carrying a charge q will experience a force $\vec{F}$ given by

$$\vec{F} = q\ (\vec{v} \times \vec{B})$$

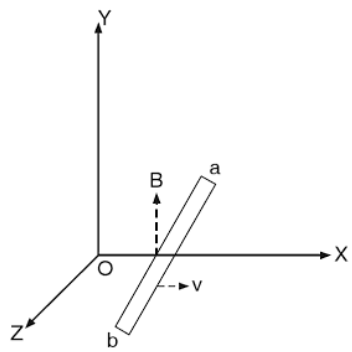

Fig. 23.2

Thus if q is *positive*, the force in +Z direction and if q is *negative*, it is in – Z direction. A positive charge in the conductor will, therefore, move from a to b under the effect of this force (fig. 23.2).

$$\therefore \quad \frac{\vec{F}}{q} = (\vec{v} \times \vec{B})$$

But $\dfrac{\vec{F}}{q} = \vec{E}$, the induced electric field.

$$\therefore \quad \vec{E} = (\vec{v} \times \vec{B}) \qquad ...(i)$$

Since the conducting rod possesses negatively charged electrons, which are free to move, are forced to move from end b to end a , making b end positively charged and end a negatively charged. This accumulation continues till both the forces acting *i.e.* $\vec{E}$ due to accumulation of charges and $\vec{E}$ produced due to motion of the conductor become equal. This is known as induced electric field $\vec{E}$ given by

$$\vec{E} = -\ (\vec{v} \times \vec{B}) \quad ...(ii)$$

Consider an elemental length $d\vec{l}$ of the rod *ab*. The induced *e.m.f.* across the ends of the conductor due to the electric field is given by

$$e = \int_a^b \vec{E} \cdot d\vec{l} = \int_a^b (\vec{v} \times \vec{B}) \cdot d\vec{l}$$

$$= \int_a^b -\vec{B} \cdot (\vec{v} \times d\vec{l})$$

$[\because\ (\vec{A} \times \vec{B}) \cdot \vec{C} = -\vec{B} \cdot (\vec{A} \times \vec{C})]$

or
$$e = -\vec{B} \cdot \left[\int_a^b \vec{v} \times d\vec{l}\right] \quad ...(iii)$$

$[\because \vec{B}$ is a uniform field]

The conductor *ab* moves through a distance $\vec{v}\,dt$ along the X-axis in a time dt and in this time the area swept by the element $d\vec{l}$ is

$$\vec{v}\,dt \times d\vec{l}$$

$\therefore$ Total area $d\vec{s}$ swept by the conductor *ab* in time dt is given by

$$d\vec{s} = \int_a^b \vec{v}\,dt \times d\vec{l}$$

or
$$\frac{d\vec{s}}{dt} = \int_a^b \vec{v} \times d\vec{l}$$

Substituting in (*iii*), we have

$$e = -\vec{B} \cdot \left[\frac{d\vec{s}}{dt}\right] = -\frac{\vec{B} \cdot d\vec{s}}{dt}$$

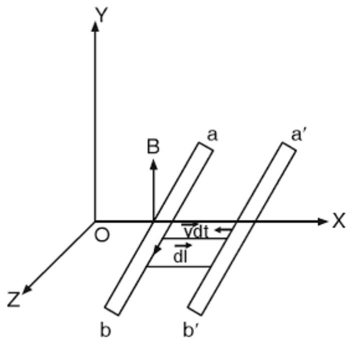

Fig. 23.3

But
$$\vec{B}\ .\ d\vec{s} = d\phi$$

$\therefore$
$$e = -\frac{d\phi}{dt} \quad .... (iv)$$

If $d\phi$ = 1 Weber = 1 Tesla met^2 and dt = 1 sec, then e = 1 volt

Magnetic flux of $\vec{B_1}$ through the circuit C_2

$$\phi_{21} = \iint_s \vec{B_1} \cdot d\vec{s_2}$$

where $\vec{ds_2}$ is a small vector area element of the surface S_2. If the shape and relative position of the two circuits is fixed, then the magnetic flux ϕ_{21} is proportional to the current I_1.

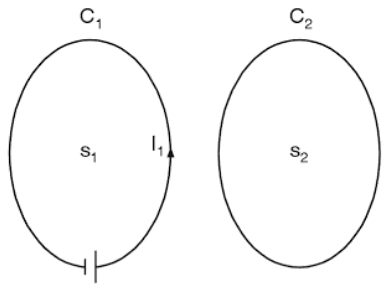

Fig. 23.7

$\therefore \qquad \phi_{21} = \text{constant } I_1 = M_{21} I_1 \qquad ...(i)$

Suppose the current in C_1 changes very slowly so that ϕ_{21} remains proportional to I_1, then an electromotive force e_{21} will be induced in the circuit C_2 given by

$$e_{21} = -\frac{d\phi_{21}}{dt} = -M_{21}\frac{dI_1}{dt} \qquad ...(ii)$$

where M_{21} is a constant known as *co-efficient of mutual induction of coil C_2 with respect to the coil C_1*. Conversely if a current I_2 is made to flow in coil C_2 instead of current I_1 in the coil C_1 then

$$\phi_{12} = \text{constant } I_2 = M_{12} I_2 \qquad ...(iii)$$

If the current in C_2 changes at the rate $\frac{dI_2}{dt}$, then *e.m.f.* induced in the coil C_1 is given by

$$e_{12} = -\frac{d\phi_{12}}{dt} = -M_{12}\frac{dI_2}{dt} \qquad ...(iv)$$

where M_{12} is the *co-efficient of mutual induction of coil C_1 with respect to the coil C_2*.

But, according to reciprocity theorem in mutual induction

$$M_{12} = M_{21}$$

$\therefore$ From relations (*i*) and (*iii*), we have that if a current I flows in one coil the magnetic flux ϕ linked with the other is given by

$$\phi = MI$$

where M is the co-efficient of mutual induction between the two circuits.

If $\qquad I = 1, M = \phi$

Hence the co-efficient of mutual induction between two circuits is defined as the magnetic flux linked with one due to a unit current flowing through the other.

From relations (*ii*) and (*iv*), we have that if a current I flow in one coil and changes at the rate $\frac{dI}{dt}$ then *e.m.f.* induced in the other coil

$$e = -M\frac{dI}{dt}$$

where M is the co-efficient of mutual induction between the two coils and e, M and I are selected in relevant units.

23.11 CO-EFFICIENT OF SELF-INDUCTION OF A CURRENT LOOP

Let C be the current loop through which a current i is flowing. Consider two small elements of length $d\vec{l_1}$ and $d\vec{l_2}$ separated by a vector distance $\vec{r}$, then magnetic field produced by the element $d\vec{l_1}$ at a point on the element $d\vec{l_2}$ according to Biot and Savart's law is given by

$$B = \frac{\mu_0}{4\pi}\oint \frac{i d\vec{l_1} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi}\oint \frac{i d\vec{l_1} \times \vec{r}}{r^3}$$

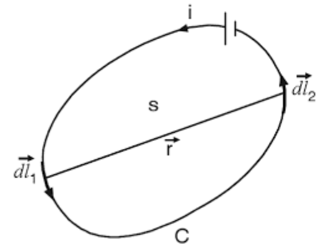

Fig. 23.8

But $$\frac{\vec{r}}{r^3} = -\vec{\nabla}\frac{1}{r}$$

$$\therefore \quad \oint \frac{i d\vec{l_1} \times \vec{r}}{r^3} = -\oint i d\vec{l_1} \times \vec{\nabla}\frac{1}{r} = \oint \vec{\nabla}\frac{1}{r} \times i\, d\vec{l_1}$$

Since $i d\vec{l_1}$ is not a function of position co-ordinates, we have $\vec{\nabla} \times i d\vec{l_1} = 0$

or $$\frac{1}{r}\vec{\nabla} \times i d\vec{l_1} = 0$$

We can therefore, put

$$\frac{\mu_0}{4\pi}\oint \frac{i d\vec{l_1} \times \vec{r}}{r^3} = \frac{\mu_0}{4\pi}\left[\oint \vec{\nabla}\frac{1}{r} \times i d\vec{l_1} + \oint \frac{1}{r}\vec{\nabla} \times i d\vec{l_1}\right]$$

$$= \frac{\mu_0}{4\pi}\oint \vec{\nabla} \times \frac{i d\vec{l_1}}{r}$$

Hence $$\vec{B} = \frac{\mu_0}{4\pi}\oint \vec{\nabla} \times \frac{i d\vec{l_1}}{r} \qquad \ldots(i)$$

The magnetic flux linked with the current loop

$$\phi = \iint_s \vec{B}.d\vec{s} = \iint_s d\vec{s}.\vec{B} \qquad \ldots(ii)$$

$[\because \vec{A}.\vec{B} = \vec{B}.\vec{A}]$

where $\vec{s}$ is the area of the surface bounded by the current loop and $d\vec{s}$ a small element of this area.

Substituting the value of $\vec{B}$ from (i) in (ii), we have

$$\phi = \iint_s d\vec{s}.\frac{\mu_0}{4\pi}\oint \vec{\nabla} \times \left(\frac{i d\vec{l_1}}{r}\right)$$

If the current i remains unchanged for a small time dt, then

Work done in time $dt = -e\ i\ dt$

$\therefore$ Total work done W establishing a current I in a time t or total energy required is given by

$$W = \int_0^t -ei\,dt = \int_0^t L\frac{di}{dt}i\,dt = \int_0^I Li\,di$$

$$= \left[\frac{1}{2}Li^2\right]_0^I = \frac{1}{2}LI^2. \qquad ..(i)$$

This expression gives the energy required to build up a current I in the inductor of inductance L.

If $I = 1$, then equation (ii), becomes

$$W = \frac{1}{2}L \quad \text{or} \quad L = 2\,W$$

Thus the co-efficient of self-induction is numerically equal to twice the work done in establishing a unit current in an inductor.

23.15 ELECTRICAL INERTIA

Self-inductance is called *electrical inertia* because self-inductance plays the same role in an electrical circuit as inertia (or mass) does in mechanical motion. The purpose of both is to slow down the change. At make of an electrical circuit self-inductance induces an opposing *e.m.f.* in the circuit and thus slows down the growth of current. At *break* it sets up an *e.m.f.* in the same direction as the original current and thus slows down the decay of the current.

Similarly, the moment of inertia slows down the motion of a body at start and prevents its stoppage when the body tends to stop. The energy associated with an inductance is $\frac{1}{2}Li^2$ and the energy due to linear motion is $\frac{1}{2}mv^2$ and in rotational motion is $\frac{1}{2}I\omega^2$. The magnetic flux ϕ linked with inductance is Li and it corresponds to linear momentum mv and angular momentum $I\omega$. The induced *e.m.f* $L\frac{di}{dt}$ corresponds to force $m\frac{dv}{dt}$ or torque $I\frac{d\omega}{dt}$.

23.16 TRANSFORMER

A transformer is a device for converting alternating current at low voltages to high voltages and *vice-versa*. The transformer which converts low potentials to high potentials is called a ***step-up transformer***, where as the one which converts high potentials to low potentials is called a ***step-down transformer***. A step up transformer is shown in Fig. 23.10.

Construction. A transformer usually consists of a rectangular (or circular) core of soft iron in the form of laminas insulated from one another. The *primary* and the *secondary* coils are wound up on the iron core so as to avoid leakage of magnetic flux and are well insulated from one another.

In the step-up transformer the number of turns in the secondary coil is greater than the number of turns in the primary coil while opposite is the case in a step-down transformer.

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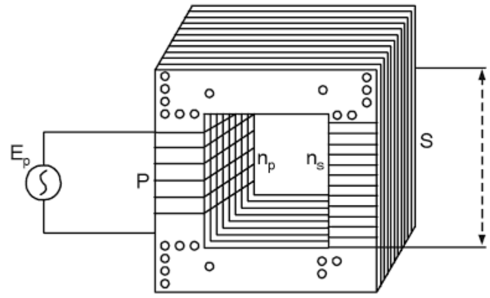

Fig. 23.10

Principle. Let L_p and R_p be the inductance and resistance of the primary coil and L_s and R_s the corresponding values of the secondary coil. When an alternating *e.m.f.* E_p is applied to the primary a current I_p flows in it. This develops a magnetic flux in the core due to which an induced *e.m.f.* is set up in the primary and

$$E_p = R_p I_p + j\omega L_p I_p \quad \text{... }(i)$$

The magnetic flux developed in the primary is also linked with the secondary in which an induced *e.m.f.* is set up due to mutual induction. If the co-efficient of mutual induction between the primary and the secondary is M, and if the secondary circuit is *open*, we have

$$E_s = j\omega MI_p \quad \text{... }(ii)$$

as under ideal conditions R_p may be assumed to be zero. Further if there is no leakage of flux $M = \sqrt{L_p L_s}$ where L_s is the co-efficient of self induction of the secondary. Thus we have

$$\frac{E_s}{E_p} = \frac{M}{L_p} = \frac{\sqrt{L_p L_s}}{L_p} = \sqrt{\frac{L_s}{L_p}}$$

If the primary coil is assumed to be infinite solenoid, then its inductance

$$L_p = \frac{\mu_0 \mu_r n_p^2 a}{l}$$

where n_p is the total number of turns in the primary, l its length, a the area of cross-section, μ_0 the permeability of free space and μ_r the relative permeability of the material of the core. Similarly the self inductance L_S of the secondary is given by

$$L_s = \frac{\mu_0 \mu_r n_s^2 a}{l}$$

assuming that the secondary has n_s turns and is wound over primary *i.e.*, it has the same length and area of cross-section

$$\therefore \quad \frac{L_s}{L_p} = \frac{n_s^2}{n_p^2}$$

Hence $$\frac{E_s}{E_p} = \sqrt{\frac{L_s}{L_p}} = \frac{n_s}{n_p} = k$$

23.17 LOSSES IN TRANSFORMER

Although a transformer is a very efficient machine as there are no moving parts in it, yet it suffers from a number of losses.

(*i*) Copper losses. As the current flows through the primary and the secondary coil which generally consists of copper wires and have some resistance, heat is produced. The heat produced is given by I^2R and brings about a loss of energy.

(*ii*) Iron losses. Induced currents are also produced in the **iron** core of the transformer and utilise a part of energy. This loss is reduced by making the core of laminated sheets of soft iron insulated from each other.

(*iii*) Magnetic leakage. The entire magnetic flux produced by the primary does not link with the secondary, thereby allowing a certain amount of energy supplied to the primary to go waste.

(*iv*) Hysteresis loss. The core of the transformer is taken through a cycle of magnetisation as may times per second as the frequency of the *A.C.* supply. In each cycle a loss of energy equal to the area of the P_m H loop takes place per unit volume of the core. This loss can be reduced by selecting a core for which the P_m H loop has a very small area, where P_m is magnetic moment of atomic magnet of the specimen.

23.18 USES OF TRANSFORMER

(*i*) A transformer is used to step-up or step down the A.C. voltage. If it raises the voltage, the current is automatically lowered in the same ratio. Such a transformer is called a *step-up transformer.* If it lowers the voltage, the current automatically rises. Such a transformer is called a *step down transformer* and is used in welding.

(*ii*) It is used to convert a variable audio-frequency current from lower voltage to high voltage as in telephone or in radio communication.

(*iii*) It is used to change the range of *A.C.* measuring instruments. Such transformers are known as instrument transformers.

(*iv*) In addition to these uses there are some special purpose transformers:

(*a*) *Impedance transformer.* It is used to match the impedance of two independent circuits.

(*b*) *Constant voltage transformer*. It is designed to give a constant output voltage even when the input voltage varies considerably.

(*c*) *Constant current transformer*. It is designed to give a constant output current.

23.19 ADVANTAGE OF A.C. IN LONG DISTANCE TRANSMISSION

When electrical energy is to be transmitted over a long distance the following disadvantage may be experienced:

(*i*) The line wire being very long has an appreciable resistance. A large amount of energy is dissipated in the wire in the form of heat.

(*ii*) There is a large fall of potential along the line wire so that the voltage at the receiving station is much below the voltage at the generating station.

(*iii*) In order to keep the line resistance low and to carry large currents thick wires have to be used which have a high initial cost. Thick wires being heavy require stronger poles to support them.

In *A.C.* transmission the use of a transformer removes all these disadvantages. The efficiency of a transformer is very high and there are practically no losses except for slight loss of energy due to *magnetic leakage, eddy currents* and *hysteresis*. The following example will make it clear:

or $$\vec{\nabla}.\vec{J}+\frac{\partial\rho}{\partial t} = 0 \qquad ...(iii)$$

This is known as the *equation of continuity* and is true for varying currents.

A *steady current* produces a magnetic field, given by

$$Curl\,\vec{B} = \vec{\nabla}\times\vec{B} = \mu_0\vec{J} \qquad ...(iv)$$

This is the differential form of Ampere's law for steady currents.

Incosistency with time varying fields From relation (*iv*), we have

$$\mu_0\vec{\nabla}.\vec{J} = \vec{\nabla}.\vec{\nabla}\times\vec{B} = 0 \qquad ...(v)$$

since the divergence of the curl of a vector is identically zero.

Thus the equation of continuity gives $\vec{\nabla}.\vec{J} = -\frac{\partial\rho}{\partial t}$ and from differential form of Ampere' s law equation (*v*) we have $\vec{\nabla}.\vec{J} = 0$

Thus the equation of continuity [equation (*ii*)] $\vec{\nabla}.\vec{J} = -\frac{\partial\rho}{\partial t}$ gives $\vec{\nabla}.\vec{J} \neq 0$ for charge density varying in time. On the other hand the equation *curl* $\vec{B} = \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ leads to $\vec{\nabla}\cdot\vec{J} = 0$ [Equation (*v*)].

There is, therefore, an apparent contradiction between relation (*ii*) and (*iv*) which means relation (*iv*) needs to be modified for varying currents and should be written as

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \text{an additional term}$$

We shall discuss it in detail.

23.22 CONCEPT OF DISPLACEMENT CURRENT

Consider a parallel plate capacitor *ab* and *a* resistance *R* as shown in fig 23.11 (*a*).

If the plate *a* of the capacitor carries a positive charge and the plate *b* an equal negative charge then capacitor will discharge itself through the resistance *R* and a *varying* current will flow through the circuit with the key *K* is closed.

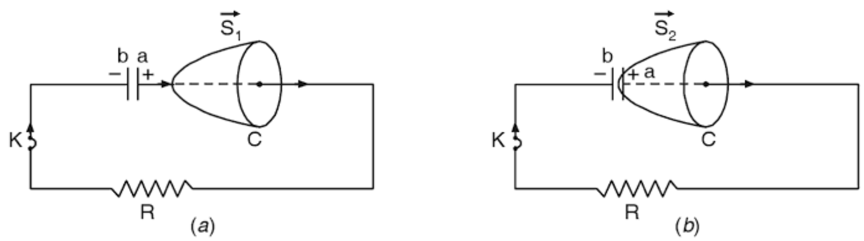

Fig. 23.11

$$Curl\,\vec{B} = \vec{\nabla}\times\vec{B} = \mu_0\left[\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right] \quad \text{... } (i)$$

The quantity $\varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{\partial \vec{D}}{\partial t}$ where $\vec{D}$ is the *electric displacement field vector*, is known as *vacuum displacement current density*. It has dimensions of current per unit area and is denoted by J_d to distinguish it from *conduction current density* $\vec{J}$.

$$\therefore \quad \vec{\nabla}\times\vec{B} = \mu_0\left[\vec{J} + \vec{J}_d\right] \quad \text{... } (ii)$$

Special case. In empty space where there are no currents

$$\vec{J} = 0$$

and equation (ii) becomes

$$\vec{\nabla}\times\vec{B} = \mu_0 J_d = \mu_0\,\varepsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \text{... } (iii)$$

The term $\frac{\partial \vec{E}}{\partial t}$ is the rate at which electric field is changing between the plates of the capacitor.

This equation shows that a varying electric field between the capacitor plates gives rise to magnetic field. Hence varying electric field is also a source of magnetic field.

Dimensions of $\frac{\partial \vec{D}}{\partial t}$

We have $\quad \frac{\partial \vec{D}}{\partial t} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

The dimensions of ε_0 are Coulomb^2/Newton m^2 = [C^2 N^{-1} m^{-2}]

The dimensions of E are Newton/Coulomb = [NC^{-1}]

$\therefore$ Dimensions of $D = \varepsilon_0 E$ are [C^2 N^{-1} m^{-2}] [NC^{-1}] = [Cm^{-2}]

and dimension of $\frac{\partial D}{\partial t}$ = [Cm^{-2}] [S^{-1}] = [CS^{-1}m^{-2}] = Amp/m^2

= Dimensions of current per unit area of current density

Displacement current density

When a capacitor is charged, there exists an electric field between the plates of the capacitor.

In a parallel plate capacitor the magnitude of the electric field $\vec{E}$ is given by

$$E = \frac{\sigma}{\varepsilon_0} = \frac{1}{\varepsilon_0}\frac{q}{A} \quad \text{... } (iv)$$

where σ is the surface charge density, q is the charge on the plates and A its area. When the plates of the capacitor are connected through a resistance, the charge slowly flows from the positive to the negative plate in the outside circuit and the current i is given by

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(*i*) It is time dependent equation and relates the space variation of $\vec{B}$ with time variation of $\vec{D}$ or $\vec{E}$.

(*ii*) It relates the magnetic field vector $\vec{B}$ with electric displacement vector $\vec{D}$ and current density vector $\vec{J}$.

(*iii*) It is a statement of Ampere's law.

(*iv*) According to this equation magnetic field vector $\vec{B}$ can be generated by current density vector $\vec{J}$ and time variation of $\vec{D}$ (or $\vec{E}$) jointly as well as separately.

23.27 ELECTRIC AND MAGNETIC FIELD ENERGY DENSITY

The energy density is defined as the energy stored per unit volume. The magnetic field energy is given by the equation

$$U_M = \frac{1}{2}\int_v \vec{H}.\vec{B}\, dV \qquad \text{... } (i)$$

and the electric field energy by

$$U_E = \frac{1}{2}\int_v \vec{E}.\vec{D}\, dV \qquad \text{... } (ii)$$

(*I*) Energy stored in a magnetic field

When a current flows through an inductor in the form of a solenoid, the work done in establishing the current is stored as energy of the magnetic field linked with the solenoid. If the solenoid has a length l, total number of turns n, and a current I flows through it, then magnetic field within the solenoid

$$= B = \mu_0 \frac{nI}{l} \qquad \text{... } (iii)$$

If a is the area of cross-section of the solenoid, then co-efficient of self-induction

$$L = \frac{\mu_0 n^2 a}{l} \qquad \text{... } (iv)$$

$\therefore$ Energy stored in the magnetic field $W = \frac{1}{2}LI^2$

But from (*i*) $\quad I = \dfrac{Bl}{\mu_0 n}$

$$\therefore \qquad W = \frac{1}{2}\frac{\mu_0 n^2 a}{l}\frac{B^2 l^2}{\mu_0^2 n^2} = \frac{1}{2}\frac{B^2}{\mu_0} al$$

Example 23.2 The magnetic flux through a circular loop is given by 0.02 t^3 *Wb*. What is the induced *e.m.f* in the loop at t = 1 millisecond? (*P.U.,* 2002)

Solution. $$e = -\frac{d\phi}{dt} = -0.02 \times 3t^2. \text{ At } t = 1 \text{ millisecond} = 10^{-3} \text{ sec}$$

$$e = 0.06 \times 10^{-6} \text{ Volt} = 6 \times 10^{-8} \text{ Volt.}$$

Example 23.3 A coil of 100 turns is pulled in 0.04 sec from between the poles of a magnet where its area includes a flux of 40 × 10^{-6} Wb. Calculate the induced *e.m.f.* in the coil.

Solution. Number of turns in the coil = 100

Magnetic flux included in the coil area = 40 × 10^{-6} Wb

∴ Total magnetic flux linked with the coil $\phi = 100 \times 40 \times 10^{-6} = 4 \times 10^{-3}$ Wb

Time in which flux is removed t = 0.04 sec

∴ Induced *e.m.f*. $$e = -\frac{d\phi}{dt} = -\frac{\text{Change in magnetic flux}}{\text{Time}}$$

$$= \frac{4 \times 10^{-3}}{0.04} = 0.1 \text{ volt.}$$

Example 23.4 A coil of 100 turns of area 3.0 cm² is jerked out of a magnetic field. The charge thus induced in the coil which is connected to a circuit with a total resistance of 600 ohm is 5 × 10 $^{-5}$ Coulomb. Find the flux density of the field. (*Meerut U.*, 2001)

Solution. Number of turns n = 100; Area $A = 3 \text{ cm}^2 = 3 \times 10^{-4} \text{ m}^2$

Magnetic flux $\phi = nAB = 100 \times 3 \times 10^{-4} \times B$

where B is the flux density.

Change in magnetic flux when the coil is jerked out of the magnetic field = $\phi_1 - \phi_2 = \phi$

Charge induced $q = 5 \times 10^{-5}$ C; $R = 600\ \Omega$

Now $$q = \frac{\phi_1 - \phi_2}{R} = \frac{\phi}{R}$$

or $$5 \times 10^{-5} = \frac{100 \times 3 \times 10^{-4} \times B}{600}$$

$$B = \frac{5 \times 600 \times 10^{-5}}{100 \times 3 \times 10^{-4}} = 1 \text{Tesla}$$

Example 23.5 A loop of wire of area *A* and resistance *R* is kept perpendicular to a uniform magnetic induction *B*. If the loop is rotated uniformly through 180° in a time *t*, find the amount of charge which flows through the loop in this time. (*G.N.D.U* 2003)

Solution. Magnetic flux linked with the wire loop $\phi = BA$

Change in magnetic flux, when rotated through 180° = $+\phi - (-\phi) = 2\phi = 2BA$

Charge which flows through the loop

$$= \frac{\text{Change in magnetic flux}}{R} = \frac{2BA}{R}$$

Example 23.6 A field of 0.02 Tesla acts at right angles to a coil of area 0.01 sq. metre with 50 turns. The coil is removed from the field in 1/10 th of a second. Find the average *e.m.f.* produced in it.

Solution. Strength of the field B = 0.02 Tesla

Area of the coil $S = 0.01$ sq. metre; Number of turns $n = 50$

As the coil is at right angles to the field, the direction of the area vector and field vector is the same and the angle between them is zero.

$\therefore$ Magnetic flux linked with the coil, when placed in the magnetic field

$$\phi = n\iint_S \vec{B}.d\vec{s} = nBS = 50 \times 0.02 \times 0.01 \text{ Weber}$$

Time during which the flux is removed = 0.1 sec

$$\therefore \quad \frac{d\phi}{dt} = \frac{50 \times 0.02 \times 0.01}{0.1} = 0.1$$

Hence induced *e.m.f.* $\quad e = -\dfrac{d\phi}{dt} = -0.1 \text{ volt}$

The negative sign only indicates the direction of induced *e.m.f.*

Example 23.7 A wire of length 200 cms held perpendicular to *XY* plane is moved with a velocity $\vec{v} = 2\hat{i} + 3\hat{j} + \hat{k}$ metres/sec through a region of uniform induction $\vec{B} = \hat{i} + 2\hat{k}$ weber/meter^2. Calculate the electric field $\vec{E}$ developed in the wire and potential difference between its ends.

Solution. The electric field $\vec{E}$ developed in the wire is independent of its length and is given by

$$\vec{E} = -(\vec{v} \times \vec{B}) \qquad \text{[in S.I. units]}$$

where $\vec{E}$ is in coulombs/ metre^2, $\vec{v}$ is in metres/sec and $\vec{B}$ in webers/metre^2.

$$\vec{E} = \vec{v} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 1 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= \hat{i}(6-0) + \hat{j}(1-4) + \hat{k}(0-3)$$

$$= 6\hat{i} - 3\hat{j} - 3\hat{k} = 3(2\hat{i} - \hat{j} - \hat{k})$$

Now potential difference between the ends of the wire is equal to the *e.m.f.* induced in it.

or $$e = \int_a^b \vec{E} \cdot d\vec{l} = \int_0^l (\vec{E}.\hat{k})\,dl$$

where l is the length of the wire and $d\vec{l} = \hat{k}\,dl$ as the wire is perpendicular to the X–Y plane, l is along the Z- axis and its small vector element $d\vec{l} = \hat{k}\,dl$.

$$\therefore \qquad e = \int_0^l (6\hat{i} - 3\hat{j} - 3\hat{k}).\hat{k}\,dl = \int_0^l -3dl = -3[l]_0^l = -3l$$

Since $\qquad l = 200$ cm = 2 met

$\therefore \qquad e = -6$ volt

The negative sign indicates that the *e.m.f.* acts along the $-Z$ direction.

Example 23.8 An aeroplane is flying with a speed of 1000 km/hour. Calculate the *e.m.f.* generated between the tips of the wings of the plane if the length of each wing is 20 metres and vertical component of earth's magnetic field is 3.1 × 10^{-5} Weber/m^2.

Solution. Let the aeroplane fly along the X-axis.

$$\therefore \quad \text{Velocity of the aeroplane} = v\hat{i} = 1000\,\text{km/hr} = \frac{1000\times10^3}{60\times60} = \frac{10^4}{36}\,\text{ms}^{-1}$$

Then, the wings of the aeroplane will be along the Y-axis.

Length of each wing = 20 metres

$\therefore \quad$ Distance between the tips of the wing = $l\hat{j} = 2\times20 = 40\,\text{m}$

Vertical component of earth's magnetic filed, therefore, acts along the Z-axis

$$= B\hat{k} = 3.1\times10^{-5}\ \text{Weber/m}^2$$

Induced *e.m.f.*, $\qquad e = -\vec{B}.\left[\int_a^b \vec{v}\times d\vec{l}\right]$

As $\vec{v}$ is a constant, $\int_a^b \vec{v}\times d\vec{l} = \vec{v}\times\int_a^b d\vec{l} = \vec{v}\times\vec{l} \qquad \left[\because \int_a^b d\vec{l} = l\right]$

Now $\qquad \vec{v}\times\vec{l} = v\hat{i}\times l\hat{j} = vl\hat{k} = \dfrac{10^4\times40}{36} = 1.11\times10^4\,\hat{k}$

Hence $\qquad e = -\vec{B}.\left[\int_a^b \vec{v}\times d\vec{l}\right] = -B\hat{k}\times1.11\times10^4\,\hat{k}$

$$= -3.1\times10^{-5}\times1.11\times10^4 = -0.344\text{ volt}$$

Example 23.9 Calculate the co-efficient of mutual induction of a pair of two coils if a current of 10 Amp. in one coil produces a flux of 10^{-4} Wb per turn in the second coil of 1000 turns.

(*M.D.U.*, 2001)

Solution. Magnetic flux per turns = 10^{-4} Wb

$\therefore \quad$ Total magnetic flux linked with 1000 turns $\phi = 1000\times10^{-4} = \dfrac{1}{10}$ Wb

Current $I = 10$ A

$\therefore$ Co-efficient of mutual induction $M = \dfrac{\phi}{I} = \dfrac{10}{10} = 1$ Wb/ Amp = 1 Henry.

Example 23.10 Calculate the co-efficient of self-induction of a coil of 1000 turns when a current of 2.5 Amp produces a magnetic flux of 0.5 micro-Weber.

Solution. Magnetic flux $= 0.5 \times 10^{-6}$ Weber

Number of turns $= 1000$

$\therefore$ Total magnetic flux linked with the coil $\phi = 1000 \times .5 \times 10^{-6}$ Weber

Now $\phi = LI$

and $I = 2.5$ Amp

$$\therefore \quad L = \frac{\phi}{I} = \frac{1000 \times 0.5 \times 10^{-6}}{2.5} = 2 \times 10^{-4} \text{ Henry.}$$

Example 23.11 Calculate the self inductance of a solenoid of 200 turns and length 25 cm radius 5 cm having an air core. (*Meerut* 2003, *Pbi, U.*, 2003)

Solution. Total number of turns $n = 200$

Length of the solenoid $l = 25$ cm = 0.25 m

Radius $r = 5$ cm = 0.05 m

$\therefore$ Area of cross-section $a = \pi r^2 = \pi \times 0.05 \times 0.05$ sq.m

$$\text{Coefficient of self-induction } L = \frac{\mu_0 n^2 a}{l}$$

$$= \frac{4\pi \times 10^{-7} \times 200 \times 200 \times \pi \times 0.05 \times 0.05}{0.25} = 1.58 \times 10^{-3} \text{ Henry}$$

Example 23.12 A solenoid has a length of 50 cms and a radius of 1 cm. If the number of turns in the solenoid is 500, relative permeability of the material on which the turns are wound is 800, calculate the co-efficient of self-inductance.

Solution. Number of turns $n = 500$

Length of the solenoid $l = 500$ cm = 0.5 m

Radius = 1 cm = 0.01 m

$\therefore$ Area of cross-section $a = \pi \times .01 \times .01$ sq m

Relative permeability $\mu_r = 800$

$$\text{Co-efficient of self-induction } L = \frac{\mu_0 \mu_r n^2 a}{l}$$

$$= \frac{4\pi \times 10^{-7} \times 800 \times 500 \times 500 \times \pi \times 0.01 \times 0.01}{0.5}$$

$= 0.158$ Henry.

Example 23.13 An *e.m.f.* of 200 V is applied to an inductance of 10 H. It has a resistance of 4 Ω. If the current attains the maximum value, then find the energy stored in the inductance. (*K.U.*, 2001)

Solution. When the current attains a maximum *steady* value there is no effect of inductance.

$\therefore$ Maximum steady current $I = \frac{E}{R} = \frac{200}{40} = 5A$

Energy stored in the inductance,

$$W = \frac{1}{2}LI^2 = \frac{1}{2} \times 10 \times 25 = 125 \text{ Joule}$$

Example 23.14 The ratio of number of turns in primary to secondary is 1 : 20. It is connected to a supply of 200 volts ac. Find voltage across secondary and find the ratio of $\frac{I_p}{I_s}$. (*Nagpur Uni.* 2004)

Solution. The transformation ratio K is given by

$$K = \frac{N_s}{N_p} = \frac{V_s}{V_p} = \frac{I_p}{I_s}$$

$$\therefore \quad V_s = \frac{N_s}{N_p} . V_p$$

Given $\frac{N_p}{N_s} = \frac{1}{20}$, $V_p = 200$ volts a.c

$$\therefore \quad V_s = \frac{20}{1} \times 200 = 4000 \text{ volt (a.c)}$$

and $$\frac{I_p}{I_s} = \frac{V_s}{V_p} = \frac{4000}{200} = \frac{20}{1} = 20$$

Example 23.15 Show that the energy stored in the magnetic field of a coil of 1000 turns, length 100 cm, area of cross-section 7 sq. cm. and wound over a core of magnetic permeability 1000 is nearly 0.44 Joule when 1 Amp current is passed through it.

Solution. Magnetic field at a point on the axis of a long coil (solenoid) = $\mu_0 nI$ where n is the number of turns per metre and the core is air. For a core of relative permeability μ_r

Magnetic field $B = \mu_r \mu_0 nI$

In such a case energy per unit volume = $\frac{B^2}{2\mu_r \mu_0}$ instead of $\frac{B^2}{2\mu_0}$

Now $B = \mu_r \mu_0 nI = \mu_0 \times 1000 \times 1000 \times 1 = 10^6 \mu_0$ Tesla

$$\therefore \quad \text{Energy per unit volume} = \frac{B^2}{2\mu_r \mu_0} = \frac{\mu_0^2 \times 10^{12}}{2 \times 1000 \times \mu_0} = \frac{\mu_0}{2} \times 10^9 \text{ Joule}$$

Now $\frac{\mu_0}{4\pi} = 10^{-7}$ $\quad \therefore \mu_0 = 4\pi \times 10^{-7}$

$$\therefore \quad \text{Energy per unit volume} = \frac{4\pi \times 10^{-7}}{2} \times 10^9 = 2\pi \times 10^2 \text{ Joule}$$

8. According to the equation of continuity div. $\vec{J} = -\frac{\partial \rho}{\partial t}$ which gives div $\vec{J} \neq 0$ for charge density varying in time but curl $\vec{B} = \mu_0 \vec{J}$ leads to div. $\vec{J}$ = div $(1/\mu_0$ curl $\vec{B}) = 0$. Explain the contradiction. (*H.P.U.* 2003)
9. State Maxwell's equations for electromagnetic field. Discuss each equation critically. (*Meerut U.* 2003, 2001, 2000, *Pbi U.* 2001, 2000, *G.N.D.U.* 2001, *K.U.* 2000)
10. (*i*) Write Maxwell's equations in electromagnetic theory.
    (*ii*) Explain the physical significance of each of these equations giving the basic laws from which these are derived.. (*M.D.U.* 2003, 2002, 2001, *G.N.D.U.* 2004, 2002, 2001, 2000, *Kerala, U.* 2001, *Luck. U.* 2002; *K.U.* 2000; *Pbi. U.* 2000)
11. State and explain self and mutual inductance. (*Nagpur U. s*/2009)

SHORT QUESTIONS

1. Explain the fact that Lenz's law is in accordance with the law of conservation of energy. (*M.D.U.* 2002, 2001)
2. What are the various methods of changing magnetic flux linked with the coil. (*G.N.D.U* 2003)
3. Derive an expression for the quantity of electricity induced in a coil through which the magnetic flux is varying.
4. Establish Faraday's law $e = -\frac{d\phi}{dt}$, when a conducting loop moves with a velocity $\vec{v}$ in a non-uniform magnetic field. (*Pbi, U.*, 2001)
5. What is mutual induction? Define co-efficient of mutual induction between two coils. Give the units in which it is measured. (*Nagpur U.* 2007, 2006, 2001; *Kerala U.* 2001; *Meerut U.* 2004, 2003, 2001; *M.S.U. Tiruneveli* .2007; *Pbi. U.* 2001, *M.D.U.* 2001; *K.U.* 2000)
6. What is the difference between the self-induction and mutual-induction? (*H.P.U.* 2003)
7. Derive an expression for the co-efficient of self-induction of a current loop.
8. (*a*) What is a solenoid? Derive an expression for the co-efficient of self-inductance of a long uniformly wound solenoid. Hence find the co-efficient of self-induction of a toroidal solenoid of radius *r*. (*K.U.* 2001, *G.N.D.U.* 2001, *M.D.U.* 2000; *P.U.* 2003; *M.S.U. Tiruneveli* 2007, *H.P.U.* 2000)
9. Why inductance is called electrical inertia? (*H.P.U.* 2000)
10. Write the basic equations of Electricity and magnetism
11. Write the Maxwell's equations for free space. Explain the physical significance of each equation. (*Nagpur Uni s*/2009, 2008, 2007, 2006, *K.U.* 2000, P*bi U,* 2003, 2000)
12. Explain the concept of Maxwell's displacement current and derive an expression for it. (*Agra U.* 2004, *Pbi. U*, 2002, *G.N.D.U.* 2004, *Meerut U.* 2003)
13. Distinguish between conduction and displacement current (*G.N.D.U.* 2003)
14. What are the requirements of an ideal transformer. (*Nagpur Uni. s*/2009, 2008, 2007, 2006)
15. What are the losses in transformer? How are they minimised. (*Nagpur. U.* 2007, 2006, 2003)

24

ELECTROMAGNETIC WAVES

INTRODUCTION

Oersted's experiment in 1820 was the milestone in the branch of electricity and magnetism in physics which categorically stated that electric charges in motion through a conductor is always accompanied with magnetic field around it. In 1831, Michael Faraday performed number of experiments and discovered that whenever a magnetic field is changed around a closed circuit, an induced e.m.f. is generated. This means that electric current and magnetic fields cannot be separated from each other, rather they co-exist and are co-related with each other.

In the early of the 19^{th} century, two different units (e.s.u. and e.m.u) of electric charge were used, one for electrostatic and the other for magnetic phenomena involving currents. These two units of charge had different physical dimensions. Their ratio has units of velocity and measurement showed that the ratio had a numerical value that was precisely equal to the speed of light (c). It was regarded as an extraordinary coincidence that had no explanation.

James Clerk Maxwell (1831-1879) in search for an explanation of this coincidence, brought the important laws together and formulated a unified theory in 1861. He found that all the basic principles of electric and magnetic phenomena (*i.e.* electro-magnetism) can be formulated in terms of four fundamental equations, now called Maxwell's field equations.Purely on mathematical basis, Maxwell predicted the existence of electromagnetic (E.M.) waves and such waves can be radiated by accelerating charges.

The above prediction came true when Hertz discovered electromagnetic waves in 1887. But the potential of E.M. waves remained unnoticed till the end of 19^{th} century. In the beginning of 20^{th} century. *i.e.* in 1901 Marconi brothers in Bell laboratory were successful to transmit letter 'S' using morse code over the Atlantic sea. This was the beginning of wireless-communication; popularly known as radio communication.

In Maxwell time, the visible light, infrared and ultraviolet light rays were only known as electromagnetic waves. Later on radio waves, T.V. waves, microwaves X-ray, gamma rays, cosmic rays etc were identified as electromagnetic waves. All electromagnetic waves travel through free space (vacuum) with the same speed c, where $c = 3 \times 10^8$ m/s the velocity of light.

24.1 ORIGIN OF ELECTROMAGNETIC WAVES

Faraday's law shows that a changing magnetic field gives rise to an electrical field. Ampere-Maxwell law shows that a changing electric field gives rise to a magnetic field. It means that when either electric or magnetic field varies with time, the other field is induced in space around it. The net effect is that an electromagnetic disturbance is generated due to changing electric and magnetic fields. The disturbance propagates in the form of an electromagnetic wave.

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The concept of **displacement current** forms the basis of origin of electromagnetic waves. To understand the concept of displacement current, let us consider an electrical circuit containing a parallel plate condenser (Fig. 24.1).

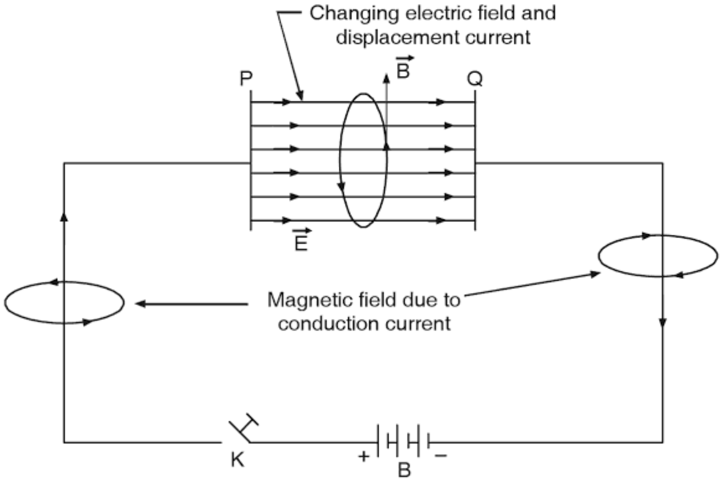

Fig. 24.1. Conduction and displacement current.

In the absence of the condenser, if the key K is closed an electric current is set up in the circuit called as **conduction current** given by Ohm's law and in vector form $\vec{J} = \sigma \vec{E}$. A magnetic field is produced around the conductor wire due to this conduction current. If the circuit is discontinuous (key K open), the current becomes zero. But if a condenser is inserted in the gap as shown in Fig 24.1 and the current is made and broken (by pressing and releasing key K), the condenser is charged and discharged (or we may apply a.c. source). Although this circuit is discontinuous for a direct current, a medium between the plates, due to change in the electric field between the plates, resulting in the production of a **displacement current**. The displacement current is the current which is set up in a dielectric medium ($\sigma = 0$) due to variation of induced displacement charge produced by the changing electric field, applied across the dielectric. Displacement charge density is given by $\frac{\partial \vec{D}}{\partial t} = \in \frac{\partial \vec{E}}{\partial t}$. Thus, *there is a conduction current in the wires and displacement current between the plates, both are accompanied with magnetic fields.*

The existence of magnetic field between the plates P and Q of the condenser can be experienced by keeping a compass needle in the gap. The compass needle shows deflection. This magnetic field between the two plates of a condenser is due to the flow of some current, Maxwell named it as displacement current. Displacement current originates due to changing electric field between the plates of condenser. This displacement current produces changing magnetic field between the plates and so compass needle shows deflection.

The conduction current produces magnetic field due to motion of charges. Whereas the displacement current produces magnetic field due to time rate of change of electric field. Thus in the space between the plates of capacitor, both the electric and magnetic fields exist. Here, the time varying electric field $(\vec{E})$ and time varying magnetic field $(\vec{B})$ are perpendicular to each other.

These two mutually perpendicular fields constitute an electromagnetic wave and propagate in the direction perpendicular to both $\vec{E}$ and $\vec{B}$.

Common examples of electromagnetic wave generators are : An electric spark produced in the air gap between two electrodes of a spark plug of motor cycle is a simple example of a generator of electromagnetic waves. A spark produced between oppositely charged clouds generates electromagnetic (EM) waves, a disturbance received in a receiver in rainy days. Any oscillating circuit, say LCR resonating circuit in electronics can be constructed for generating EM waves of different frequencies for radio and TV communication.

24.2 CHARACTERISTICS OF ELECTROMAGNETIC WAVES

The most important characteristic of electromagnetic waves are :

1. Electromagnetic waves are produced whenever electric charges are accelerated. viz. accelerating charges in an antenna.

2. Electromagnetic waves propagate in the form of varying electric field and magnetic field. These two field $(\vec{E} \text{ and } \vec{B})$ are mutually perpendicular to each other and also perpendicular to the direction of propagation of electromagnetic waves. Electromagnetic waves are transverse in nature.

3. EM waves do not require material medium for their propagation. They can travel through vacuum.

4. EM waves may be reflected, refracted, transmitted, or absorbed (or partially reflected, transmitted or absorbed) depending on the nature of surface and the frequency of wave.

5. EM waves, travel in vacuum with a velocity c given by

$$c = \frac{1}{\sqrt{\mu_0 \in_0}} = 3 \times 10^8 \text{ m/s}$$

and in the material medium, its velocity is

$$v = \frac{1}{\sqrt{\mu \in}}$$

where μ is absolute permeability and $\in$ is absolute permittivity of that medium.

6. The energy in EM waves is divided equally between the electric and magnetic field vectors.

7. The cross product $\vec{E} \times \vec{B}$ always give the direction of propagation of EM waves.

8. EM waves carry momentum. Hence they are able to exert pressure on the incident surface. Tail of comet is always directed away from the sun. This is due to the pressure exerted by the sun rays on the molecules evaporating from the comet.

9. EM waves transports energy from one region to another. The rate of energy flow per unit area or power flow per unit area (vector $\vec{S}$) is given by Poynting vector as

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \vec{E} \times \vec{H}$$

where μ_0 is permeability of free space. (Refer article 24.9)

10. The ratio of the amplitudes of electric and magnetic fields is always constant and it is equal to velocity of the electromagnetic waves. Mathematically,

$$c = \frac{E}{B}$$

24.3 E.M. WAVE EQUATIONS IN A CONDUCTING MEDIUM

Let us consider a conducting medium of electrical conductivity σ and having finite permeability μ and permittivity $\in$. Suppose the medium is free from a charge source, hence charge density $\rho = 0$. In such a case, Maxwell's four field equations are :

$$\vec{\nabla}\cdot\vec{D} = \in\vec{\nabla}\cdot\vec{E} = 0 \text{ or } \vec{\nabla}.\vec{E} = 0 \qquad \text{... } (i)$$

$$\vec{\nabla}.\vec{B} = 0 \qquad \text{... } (ii)$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \text{... } (iii)$$

$$\vec{\nabla}\times\vec{B} = \mu\left(\vec{J}+\in\frac{\partial\vec{E}}{\partial t}\right) = \mu\left(\sigma\vec{E}+\in\frac{\partial\vec{E}}{\partial t}\right) \qquad \text{... } (iv)$$

As $\qquad \vec{J} = \sigma\vec{E}$

1. For electric field $\vec{E}$: Let us eliminate $\vec{B}$ from eq. (*iii*) and (*iv*). Taking curl of Eq. (*iii*), we get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$

Substituting for $\vec{\nabla}\times\vec{B}$ from Eq. (*iv*), we get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{\partial}{\partial t}\left(\mu\sigma\vec{E}+\mu\in\frac{\partial\vec{E}}{\partial t}\right)$$

$$= -\mu\sigma\frac{\partial\vec{E}}{\partial t} - \mu\in\frac{\partial^2\vec{E}}{\partial t^2} \qquad \text{... } (v)$$

Applying the triple vector identity.

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B})$$

we get,

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\vec{\nabla}.\vec{E}) - \vec{E}(\vec{\nabla}.\vec{\nabla})$$

$$= 0 - \nabla^2\vec{E} \qquad (\because \text{Eq. } i)$$

$$= -\nabla^2 \vec{E} \qquad \text{... }(vi)$$

Substituting for $\vec{\nabla}\times(\vec{\nabla}\times\vec{E})$ in Eq. (v), we get

$$-\nabla^2 \vec{E} = -\mu\sigma\frac{\partial \vec{E}}{\partial t} - \mu\epsilon\frac{\partial^2 \vec{E}}{\partial t^2}$$

or,
$$\nabla^2 \vec{E} = \mu\sigma\frac{\partial \vec{E}}{\partial t} + \mu\in\frac{\partial^2 \vec{E}}{\partial t^2} \qquad \text{... }(vii)$$

This is the general wave equation for electric field $\vec{E}$.

If the medium is non-conducting, $\sigma = 0$ then we have

$$\nabla^2 \vec{E} = \mu\in\frac{\partial^2 \vec{E}}{\partial t^2} \qquad \text{... }(viii)$$

Comparing Eq. (vii) and ($viii$), we find that $\left(\mu\sigma\frac{\partial \vec{E}}{\partial t}\right)$ is the **"dissipative terms"** which allows the current to flow.

2. For magnetic field $\vec{B}$:

Let us eliminate $\vec{E}$ from Eq. (iii) and (iv). Taking curl of Eq. (iv), we get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\times\left(\mu\sigma\vec{E}+\mu\in\frac{\partial \vec{E}}{\partial t}\right)$$

$$= \mu\sigma(\vec{\nabla}\times\vec{E}) + \mu\in\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

$$= \mu\sigma\left(-\frac{\partial \vec{B}}{\partial t}\right) + \mu\in\frac{\partial}{\partial t}\left(-\frac{\partial \vec{B}}{\partial t}\right) \qquad (\because \text{ Eq } (iii))$$

$$= -\mu\sigma\frac{\partial \vec{B}}{\partial t} - \mu\in\frac{\partial^2 \vec{B}}{\partial t^2} \qquad \text{... }(ix)$$

Applying the triple vector identity,

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}\,(\vec{A}.\vec{C}) - \vec{C}\,(\vec{A}.\vec{B})$$

We get,

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\,(\vec{\nabla}.\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{\nabla})$$

$$= 0 - \nabla^2 \vec{B} = -\nabla^2 \vec{B}$$

Substituting this in Eq. (*ix*), we get

$$-\nabla^2 \vec{B} = -\mu\sigma \frac{\partial \vec{B}}{\partial t} - \mu \in \frac{\partial^2 \vec{B}}{\partial t^2}$$

or

$$\nabla^2 \vec{B} = \mu\sigma \frac{\partial \vec{B}}{\partial t} + \mu \in \frac{\partial^2 \vec{B}}{\partial t^2} \quad \text{... } (x)$$

This is the general wave equation for magnetic field vector $\vec{B}$.

It the medium is non-conducting, $\sigma = 0$, then we have

$$\nabla^2 \vec{B} = \mu \in \frac{\partial^2 \vec{B}}{\partial t^2} \quad \text{... } (xi)$$

Comparing Eq. (*x*) and (*xi*), we find that $\left(\mu\sigma \frac{\partial \vec{B}}{\partial t}\right)$ is the **" dissipative term"** which allow the current to flow.

24.4 E.M WAVE EQUATION IN FREE SPACE

For free space,

Electrical conductivity, $\sigma = 0$

Charge density, $\rho = 0$

Current density $\vec{J} = 0$

Relative permittivity(K *i.e.* $\in_r$) $= 1$

Relative permeability $(\mu_r) = 1$

Hence, $\in = \in_0 . \in_r = \in_0 .1 = \in_0$

and $\mu = \mu_0 . \mu_r = \mu_0 .1 = \mu_0$

In such a case, Maxwell's field equations are :

$$\vec{\nabla} . \vec{D} = \in_0 \vec{\nabla} . \vec{E} = 0 \quad \text{or} \quad \vec{\nabla} . \vec{E} = 0 \quad \text{... } (i)$$

$$\vec{\nabla} . \vec{B} = 0. \quad \text{.. } (ii)$$

$$\vec{\nabla} . \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{... } (iii)$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \in_0 \frac{\partial \vec{E}}{\partial t}\right) = \mu_0 \left(\sigma \vec{E} + \in_0 \frac{\partial \vec{E}}{\partial t}\right)$$

$$= 0 + \mu_0 \in_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \in_0 \frac{\partial \vec{E}}{\partial t} \quad (\because \sigma = 0) \text{ ... } (iv)$$

Applying the triple vector identity,

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}.\vec{C})-\vec{C}(\vec{A}.\vec{B})$$

We get,

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}(\vec{\nabla}.\vec{B})-\vec{B}(\vec{\nabla}.\vec{\nabla})$$

$$= -\nabla^2\vec{B} \qquad \text{... }(ix)$$

Since $\vec{\nabla}.\vec{B} = 0$ [$\because$ Eq (ii)]

From Eq. ($viii$) and (ix), we have

$$-\nabla^2\vec{B} = -\mu_0\in_0\frac{\partial^2\vec{B}}{\partial t^2}$$

or,

$$\nabla^2\vec{B} = \mu_0\in_0\frac{\partial^2\vec{B}}{\partial t^2} \qquad \text{... }(x)$$

Eq. (vii) and (x), the relation between the space and time variation of $\vec{E}$ and $\vec{B}$ are called **"three dimensional wave equation"** for $\vec{E}$ and $\vec{B}$ respectively.

Discussion: Any function satisfying equations like (vii) or (x) describes a wave. The square root of the quantity that is the reciprocal of the co-efficient of the time derivative gives the phase velocity of the wave. These equations, therefore indicate that electromagnetic fields changing with time, propogate in the form of electromagnetic waves with a velocity $\frac{1}{\sqrt{\mu_0\in_0}}$.

Substituting the values of μ_0 and $\in_0$, we find that

$$\frac{1}{\sqrt{\mu_0\in_0}} = \frac{1}{\sqrt{(4\pi\times10^{-7}\,\text{wb/A.m}^2)(8.9\times10^{-12}\,\text{c}^2/\text{N.m}^2)}} = 3.0\times10^8 \text{ m/s}$$

Thus,

$$\frac{1}{\sqrt{\mu_0\in_0}} = c, \text{ the velocity of light.}$$

The emergence of the speed of light from electromagnetic considerations in the most achievement of Maxwell's field equation. Maxwell predicted that electromagnetic disturbance propagates in free space with a speed equal to that of light and hence light waves were confirmed to be electromagnetic in nature.

24.5 MAXWELL'S EQUATIONS FOR UNIFORM PLANE ELECTROMAGNETIC WAVES IN FREE SPACE (VACUUM)

Maxwell's equations for a charge free ($\rho = 0$) non-conducting medium ($\sigma = 0$) having finite permeability μ_0 and permittivity $\in_0$ for free space are expressed in vector form as

$$\therefore \qquad \frac{\partial E}{\partial y} = \frac{\partial E}{\partial z} = \frac{\partial B}{\partial y} = \frac{\partial B}{\partial z} = 0$$

Now $\qquad \vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$

and $\qquad \vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

$$\therefore \qquad \frac{\partial E_y}{\partial y} = 0,\ \frac{\partial E_z}{\partial z} = 0,\ \frac{\partial B_y}{\partial y} = 0;\ \frac{\partial B_z}{\partial z} = 0$$

As $\frac{\partial E_y}{\partial y} = 0$ and $\frac{\partial E_z}{\partial z} = 0$, Maxwell's first relation given by Eq. (1) becomes,

$$\frac{\partial E_x}{\partial x} = 0 \qquad \text{... }(i)$$

As $\frac{\partial B_y}{\partial y} = 0$ and $\frac{\partial B_z}{\partial z} = 0$, Maxwell's second relation given by Eq. (2) becomes,

$$\frac{\partial B_x}{\partial x} = 0 \qquad \text{... }(ii)$$

Considering the x-component Eq (3A) of Maxwell's third relation and Eq. (4A) of Maxwell's fourth relation we have

$$-\frac{\partial B_x}{\partial t} = 0 \quad \text{or} \quad \frac{\partial B_x}{\partial t} = 0 \qquad \text{... }(iii) \left[\because \frac{\partial E_z}{\partial y} = \frac{\partial E_y}{\partial z} = 0\right]$$

and

$$\mu_0 \in_0 \frac{\partial E_x}{\partial t} = 0 \quad \text{or} \quad \frac{\partial E_x}{\partial t} = 0 \qquad \text{... }(iv) \left[\because \frac{\partial B_z}{\partial y} = \frac{\partial B_y}{\partial z} = 0\right]$$

From Eq. (i), (ii) (iii)and (iv) we conclude that the x-components of either $\vec{E}$ or $\vec{B}$ do not vary with space (co-ordinate) or time *i.e.* E_x and B_x are constant.

Since, we are considering only the oscillatory nature of either $\vec{E}$ or $\vec{B}$, a constant E_x and B_x will have no effect on the wave propagation. Hence they can be assumed to be zero, *i.e.* $E_x = B_x = 0$ Substituting these values in Eq. (3B), (3C), (4B) and (4C) we have

$$\frac{\partial E_z}{\partial x} = \frac{\partial B_y}{\partial t} \qquad \text{... }(v)$$

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t} \qquad \text{... }(vi)$$

$$\frac{\partial B_z}{\partial x} = -\mu_0 \in_0 \frac{\partial E_y}{\partial t} \qquad \text{... }(vii)$$

$$\frac{\partial^2 E_z}{\partial x \partial t} = \frac{\partial^2 B_y}{\partial t} \qquad \text{... (vi)}$$

and
$$\frac{\partial^2 B_y}{\partial x^2} = \mu_0 \in_0 \frac{\partial^2 E_z}{\partial x \partial t} \qquad \text{... (vii)}$$

From Eq. (*vi*) and (*vii*), we have

$$\frac{\partial^2 B_y}{\partial x^2} = \mu_0 \in_0 \frac{\partial^2 B_y}{\partial t^2} \qquad \text{... (viii)}$$

This is the wave equation of B_y of a plane polarized electromagnetic wave.

The solution of Eq. (*v*) of a plane polarized E.M wave is

$$E_z = E_0 \sin \frac{2\pi}{\lambda}(vt - x) \qquad \text{... (ix)}$$

and that of Eq. (*viii*) is

$$B_y = B_0 \sin \frac{2\pi}{\lambda}(vt - x) \qquad \text{... (x)}$$

where E_0 and B_0 are the maximum values of $\vec{E}$ and $\vec{B}$ respectively.

Recalling Eq. (*ix*), as
$$\frac{\partial E_z}{\partial x} = \frac{\partial B_y}{\partial t}$$

From Eq. (*ix*),
$$\frac{\partial E_z}{\partial x} = -E_0 \frac{2\pi}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x)$$

From Eq. (*x*),
$$\frac{\partial B_y}{\partial t} = -B_0 \frac{2\pi v}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x)$$

Substituting, we get

$$-E_0 \frac{2\pi}{\lambda} = -B_0 \frac{2\pi v}{\lambda}$$

$$E_0 = B_0 . v$$

or,
$$\frac{E_0}{B_0} = v \qquad \text{or,} \qquad \frac{E_0}{H_0} = \mu v \qquad \text{... (xi)}$$

but
$$v = \sqrt{\frac{1}{\mu \in}}$$

$\therefore$
$$\frac{E_0}{B_0} = \sqrt{\frac{1}{\mu \in}}$$

or
$$\frac{E_0}{H_0} = \frac{\mu}{\sqrt{\mu \in}} = \sqrt{\frac{\mu}{\in}} \qquad \text{... (xii)}$$

24.8 CHARACTERISTIC IMPEDANCE OF DIELECTRIC

The electromagnetic wave, during its propogation experiences a definite characteristic impedance (Z) of the dielectric medium. We define electric circuit impedance $Z = \frac{V}{I}$, where V is voltage applied and I, the current flowing through it.

In the case of electromagnetic waves, the ratio of *the instantaneous electric field intensity to the instantaneous magnetic field intensity gives the impedance of the dielectric medium.*

$\because$ For electromagnetic waves $Z = \frac{E}{H}$

The dimensions of E are $\frac{\text{volts}}{\text{metre}}$ and H are $\frac{\text{amperes}}{\text{metre}}$

$$\therefore \quad \frac{E}{H} = \frac{\text{volts}}{\text{metre}} \times \frac{\text{metre}}{\text{amperes}} = \frac{\text{volts}}{\text{amperes}} = \text{ohms}$$

But
$$\frac{E_0}{H_0} = \frac{E_z}{H_y} = \sqrt{\frac{\mu}{\epsilon}}$$

or, characteristic impedance,
$$Z = \left|\frac{\vec{E}}{\vec{H}}\right| = \sqrt{\frac{\mu}{\epsilon}}$$

Electric and magnetic fields are in phase

As the ratio $\left|\frac{\vec{E}}{\vec{H}}\right|$ is real and positive, the electric and magnetic fields are in phase with each other.

$\therefore$ Characteristic impedance of a dielectric medium to electromagnetic waves $Z = \sqrt{\frac{\mu}{\epsilon}}$

Characteristic impedance of free space

For free space

$$\mu_0 = 4\,\pi \times 10^{-7} \text{ kg m coulomb}^2$$

$$\epsilon_0 = \frac{1}{36\pi \times 10^9} \text{ coulomb}^2 \text{ sec}^2/\text{kg m}^3 \text{ and}$$

$$\sqrt{\frac{\mu_0}{\epsilon_0}} = \sqrt{4\pi \times 10^{-7} \times 36\pi \times 10^9} = 377 \text{ ohm}$$

$\boldsymbol{\vec{E} = 377\,\vec{H}}$. In free space $\left|\frac{\vec{E}}{\vec{H}}\right| = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377$ or $|\vec{E}| = 377|\vec{H}|$

$\therefore$ In free space the value of $|\vec{E}|$ the electric field vector at any instant is 377 time, the value of $|\vec{H}|$ the magnetic field vector.

24.9 THE POYNTING VECTOR

The most remarkable characteristic of an electro-magnetic wave is that it transports energy from one region to another. We can describe the energy transfer in terms of the rate of energy flow per unit area or power per unit area by a vector $\vec{S}$, called as ***Poynting vector***, which was introduced by the British Physicist John H. Poynting (1852 -1914).

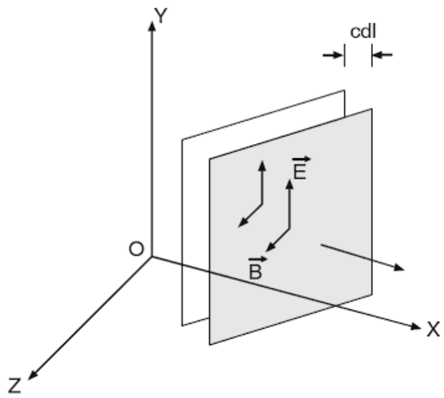

Fig. 24.3

Let us calculate energy dW passing during time dt through a unit area held perpendicular to the direction of propagation of the wave as shown in Fig. 24.3. In time dt, the wave front moves a distance $dx = c\ dt$.

$$\therefore \quad dW = wc\ dt \qquad \text{... (i)}$$

where w is the energy density and is given by $w = \epsilon_0 E^2$

$$\therefore \quad dW = \epsilon_0 E^2 c\ dt \qquad \text{... (ii)}$$

For an electromagnetic wave,

$$\epsilon_0 E^2 = B^2 = \mu_0 H^2 \qquad \text{... (iii)}$$

which implies that the electric energy density in the E.M wave at any instant is equal to the magnetic energy density at the same point.

$$\therefore \quad \sqrt{\epsilon_0}\, E = \sqrt{\mu_0}\, H$$

$$\therefore \quad dW = \sqrt{\epsilon_0 \mu_0}\, EHc\, dt \qquad \text{... (iv)}$$

Since $\quad c = \dfrac{1}{\sqrt{\epsilon_0 \mu_0}}$,

$$dW = EH\, dt \qquad \text{... (v)}$$

The term EH represents the magnitude of energy flux density vector

i.e. $$|\vec{S}| = EH. \qquad \text{... (vi)}$$

$-\iiint \vec{E}.\vec{J}\, dV$ = Work done by the field on the source because $\vec{E}.\vec{J}$ is the energy consumed per unit volume due to joule heating.

Equation (*viii*) is the statement of Poynting theorem. It gives the law of conservation of energy.

In free space $\vec{J} = 0$ and equation (*viii*) becomes

$$\iint \vec{S}.d\vec{A} = -\frac{\partial}{\partial t}\iiint_V \left(\frac{\in E^2}{2} + \frac{\mu H^2}{2}\right) dV \qquad \text{... } (ix)$$

This equation shows that the flow of energy per unit time across the boundary of the closed volume is equal to the rate of charge of total energy of the electromagnetic field.

Equation of Continuity. Again substituting in Eq. (*ix*)

$$\iint \vec{S}.d\vec{A} = \iiint_V \vec{\nabla}.\vec{S}\, dV \text{ from Eq } (vii)\ (a)$$

and
$$\left(\frac{\in E^2}{2} + \frac{\mu H^2}{2}\right) = U_E + U_M = U \qquad \text{... } (x)$$

where U_E is electric energy per unit volume, U_M the magnetic energy per unit volume and U the total energy per unit volume, we have.

$$\iiint_V \vec{\nabla}.\vec{S}\, dV = -\iiint_V \frac{\partial}{\partial t} U\, dV$$

or
$$\iiint_V \vec{\nabla}.\vec{S}\, dV + \iiint_V \frac{\partial}{\partial t} U\, dV = 0$$

which gives
$$\vec{\nabla}.\vec{S} + \frac{\partial U}{\partial t} = 0. \qquad \text{... } (xi)$$

This is called *equation of continuity*

24.11 TOTAL ENERGY DENSITY IN ELECTROMAGNETIC WAVES

Since an electromagnetic wave consists of both components, the electric field vector $\vec{E}$ and magnetic field vector $\vec{B}$ (or $\vec{H}$), the total energy density is the sum of electrostatic and magnetic energy density.

For a plane polarized electromagnetic wave travelling along X- direction, the two components E_z and H_y are related by the equation

$$\frac{Ez}{Hy} = \sqrt{\frac{\mu}{\in}} \qquad \text{... } (i)$$

where μ is the magnetic permeability and $\in$, the electric pemittivity of the dielectric medium. Squaring Eq. (*i*), we get

$$\in E_z^2 = \mu H_y^2$$

Now the quantity $\in E_z^2$ has the dimension of $\frac{\text{Farads}}{\text{metre}} \times \frac{\text{Volts}^2}{\text{metre}^2} = \frac{\text{Joule}}{\text{metre}^3}$ *i.e.* energy per unit volume.

Therefore, the quantity $\frac{1}{2} \in E_z^2$ is the electrostatic energy per unit volume for a dielectric and $\frac{1}{2}\mu H_y^2$ is the magnetic energy per unit volume. We know that energy stored per unit volume is called the energy density (U).

Hence,

$$\text{Electrostatic energy density}, U_e = \frac{1}{2} \in E_z^2$$

$$\text{Magnetic energy density}, \quad U_m = \frac{1}{2}\mu H_y^2$$

$\therefore$ Total energy density of a plane-polarized E.M wave will be

$$U_p = \frac{1}{2} \in E_z^2 + \frac{1}{2}\mu H_y^2 \qquad \text{... } (ii)$$

In general, for an electromagnetic wave

$$U = \frac{1}{2} \in E^2 + \frac{1}{2}\mu H^2 \qquad \text{... } (iii)$$

Moreover, it is observed that electrostatic energy density U_e is equal to the magnetic energy density U_m.

$$\therefore \qquad \frac{U_e}{U_m} = \frac{\frac{1}{2} \in E^2}{\frac{1}{2}\mu H^2} = \frac{\in}{\mu}\frac{E^2}{H^2}$$

but

$$\frac{E}{H} = \sqrt{\frac{\mu}{\in}} \quad \text{or} \quad \frac{E^2}{H^2} = \frac{\mu}{\in}$$

$$\therefore \qquad \frac{U_e}{U_m} = \frac{\in}{\mu}.\frac{\mu}{\in} = 1$$

or

$$U_e = U_m \qquad \text{... } (iv)$$

From Eq (iii), we have

$$\text{U} = U_e + U_m = \frac{1}{2} \in E^2 + \frac{1}{2}\mu H^2$$

$$= \frac{1}{2} \in E^2 + \frac{1}{2} \in E^2 = \in E^2$$

$$= (\in_r)(\in_0 E^2) \qquad [\because \in = \in_0 . \in_r]$$

$$= \in_r \times [\text{energy density in free space}] \qquad \text{... } (v)$$

Thus, the energy density in a dielectric is $\in_r$ times the energy denisty of the same wave in vacuum (*i.e.* free space).

$$\frac{E_r}{E_i} = \frac{Z_2 - Z_1}{Z_2 + Z_1} \qquad \text{... }(v)$$

Adding 1 to both sides, we have

$$\frac{E_r}{E_i} + 1 = \frac{Z_2 - Z_1}{Z_2 + Z_1} + 1 \text{ or } \frac{E_r + E_i}{E_i} = \frac{2Z_2}{Z_2 + Z_1}$$

or

$$\frac{E_t}{E_i} = \frac{2Z_2}{Z_2 + Z_1} \qquad \text{... }(vi)\ [\because E_r + E_i = E_t]$$

$\frac{E_r}{E_i}$ is known as **amplitude reflection co-efficient**. It is denoted by R and is given by

$$R = \frac{E_r}{E_i} = \frac{Z_2 - Z_1}{Z_2 + Z_1} \qquad \text{... }(vii)$$

$\frac{E_t}{E_i}$ is known as **amplitude transmission co-efficient**. It is denoted by T and is given by

$$T = \frac{E_t}{E_i} = \frac{2Z_2}{Z_2 + Z_1} \qquad \text{... }(viii)$$

24.13 REFLECTION CO-EFFICIENT FOR POLARIZED E.M. WAVE

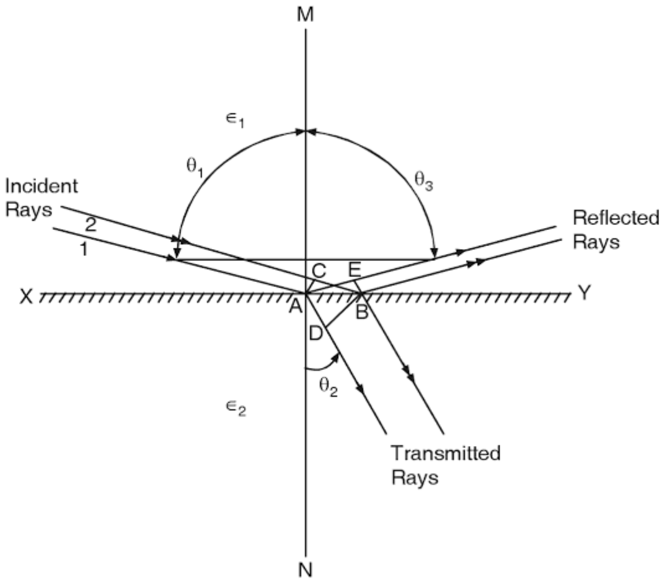

Fig. 24.5

Consider a plane electromagnetic wave incident obliquely on a plane surface XY of a dielectric. A part of the plane wave will be transmitted and part of it reflected, but in this case, the transmitted wave will be refracted *i.e.* the direction of propagation is altered.

$$= \sqrt{\frac{\epsilon_2}{\epsilon_1}}\frac{\cos\theta_1}{\cos\theta_2} - \sqrt{\frac{\epsilon_2}{\epsilon_1}}.\frac{E_r}{E_i}\frac{\cos\theta_1}{\cos\theta_2}$$

$$\frac{E_r}{E_i}\left(1+\sqrt{\frac{\epsilon_2}{\epsilon_1}}\frac{\cos\theta_1}{\cos\theta_2}\right) = \sqrt{\frac{\epsilon_2}{\epsilon_1}}.\frac{\cos\theta_1}{\cos\theta_2} - 1$$

$$\frac{E_r}{E_i} = \frac{\sqrt{\epsilon_2}\cos\theta_1 - \sqrt{\epsilon_1(1-\sin^2\theta_2)}}{\sqrt{\epsilon_2}\cos\theta_1 + \sqrt{\epsilon_1(1-\sin^2\theta_2)}}$$

But , we know $\sin^2\theta_2 = \dfrac{\epsilon_1 \sin^2\theta_1}{\epsilon_2}$

$$\therefore \quad \frac{E_r}{E_i} = \frac{\sqrt{\epsilon_2}\cos\theta_1 - \sqrt{\epsilon_1}\sqrt{(1-\sin^2\theta_2)}}{\sqrt{\epsilon_2}\cos\theta_1 + \sqrt{\epsilon_1}\sqrt{\epsilon_1(1-\sin^2\theta_2)}}$$

$$= \frac{\sqrt{\epsilon_2}\cos\theta_1 - \sqrt{\epsilon_1}\sqrt{1-\dfrac{\epsilon_1\sin^2\theta_1}{\epsilon_2}}}{\sqrt{\epsilon_2}\cos\theta_1 + \sqrt{\epsilon_1}\sqrt{1-\dfrac{\epsilon_1\sin^2\theta_1}{\epsilon_2}}}$$

$$= \frac{\sqrt{\epsilon_2}\cos\theta_1 - \sqrt{\epsilon_1}\sqrt{\dfrac{\epsilon_1}{\epsilon_2}\left(\dfrac{\epsilon_2}{\epsilon_1}-\sin^2\theta_1\right)}}{\sqrt{\epsilon_2}\cos\theta_1 + \sqrt{\epsilon_1}\sqrt{\dfrac{\epsilon_1}{\epsilon_2}\left(\dfrac{\epsilon_2}{\epsilon_1}-\sin^2\theta_1\right)}}$$

$$= \frac{\sqrt{\epsilon_2}\cos\theta_1 - \dfrac{\epsilon_1}{\sqrt{\epsilon_2}}\sqrt{\left(\dfrac{\epsilon_2}{\epsilon_1}-\sin^2\theta_1\right)}}{\sqrt{\epsilon_2}\cos\theta_1 + \dfrac{\epsilon_1}{\sqrt{\epsilon_2}}\sqrt{\left(\dfrac{\epsilon_2}{\epsilon_1}-\sin^2\theta_1\right)}}$$

$$= \frac{\sqrt{\epsilon_2}\dfrac{\sqrt{\epsilon_2}}{\epsilon_1}\cos\theta_1 - \sqrt{\left(\dfrac{\epsilon_2}{\epsilon_1}-\sin^2\theta_1\right)}}{\sqrt{\epsilon_2}\dfrac{\sqrt{\epsilon_2}}{\epsilon_1}\cos\theta_1 + \sqrt{\left(\dfrac{\epsilon_2}{\epsilon_1}-\sin^2\theta_1\right)}}$$

Reflection Co-efficient
$$R = \frac{E_r}{E_i} = \frac{\left(\dfrac{\epsilon_2}{\epsilon_1}\right)\cos\theta_1 - \sqrt{\left(\dfrac{\epsilon_2}{\epsilon_1}-\sin^2\theta_1\right)}}{\left(\dfrac{\epsilon_2}{\epsilon_1}\right)\cos\theta_1 + \sqrt{\left(\dfrac{\epsilon_2}{\epsilon_1}-\sin^2\theta_1\right)}} \quad \text{... }(viii)$$

Equation (*viii*) gives the reflection co-efficient for vertical polarization.

24.14 TOTAL INTERNAL REFLECTION.

Its $\in_1$ is greater than $\in_2$ both the reflection co-efficients for vertical and horizontal polarization become complex numbers, when

$$\sin\theta_1 > \sqrt{\frac{\in_2}{\in_1}}$$

Both the co-efficient take the form $\frac{(a+jb)}{(a-jb)}$ and thus have unit magnitude. In other words, the reflection is total, called as total internal reflection.

(*i*) **For horizontal polarization**, the reflection co-efficient

$$\frac{E_r}{E_i} = \frac{\sqrt{\in_1}\cos\theta_1 - \sqrt{\in_2}\cos\theta_2}{\sqrt{\in_1}\cos\theta_1 + \sqrt{\in_2}\cos\theta_2}$$

In case of total internal reflection, the value of $\cos\theta_2$ takes the form

$$\cos\theta_2 = -j\sqrt{\frac{\in_1}{\in_2}\sin^2\theta_1 - 1}$$

On substitution, we get the co-efficient of reflection as

$$R = \frac{E_r}{E_i} = \frac{\cos\theta_1 + j\sqrt{\sin^2\theta_1 - \left(\frac{\in_2}{\in_1}\right)}}{\cos\theta_1 - j\sqrt{\sin^2\theta_1 - \left(\frac{\in_2}{\in_1}\right)}}$$

(*ii*) **For vertical polarization**, the reflection coefficient is given by

$$\frac{E_r}{E_i} = \frac{\sqrt{\in_2}\cos\theta_1 - \sqrt{\in_1}\cos\theta_2}{\sqrt{\in_2}\cos\theta_1 + \sqrt{\in_1}\cos\theta_2}$$

In case of total internal reflection, on substituting the value of

$$\cos\theta_2 = -j\sqrt{\frac{\in_1}{\in_2}\sin^2\theta_1 - 1}.$$

We get the reflection co-efficient as

$$R = \frac{E_r}{E_i} = \frac{\left(\frac{\in_2}{\in_1}\right)\cos\theta_1 + j\sqrt{\sin^2\theta_1 - \left(\frac{\in_2}{\in_1}\right)}}{\left(\frac{\in_2}{\in_1}\right)\cos\theta_1 - j\sqrt{\sin^2\theta_1 - \left(\frac{\in_2}{\in_1}\right)}}$$

24.15 REFLECTION AND REFRACTION OF E.M. WAVES BY IONOSPHERE

The phenomenon of reflection and refraction of electromagnetic waves (radio waves) in ionosphere is frequency dependent. At low frequencies, say 100 KHz, a change in an electron and ion density is so great that the bottom of ionosphere acts virtually an abrupt discontinuity in the medium and refractive index of each layer goes on changing. On the otherhand, at high end of the high frequency band (nearly 30 MHz), the length of wavelength is sufficiently short that the ionization density changes only slightly. Thus, within this range of frequency, the ionosphere may be treated as a continuously varying refractive index dielectric region. The incident wave gets refracted continuously by penetrating into upper ionospheric layers. When the angle of incidence becomes greater than critical angle, the wave gets internally reflected and starts travelling towards the earth. This is how, radio communication is established.

The phase velocity of a radio wave as it travel through the ionosphere *i.e.* ionised medium is expressed as

$$v_p = \frac{c}{\epsilon_r} \qquad \text{... }(i)$$

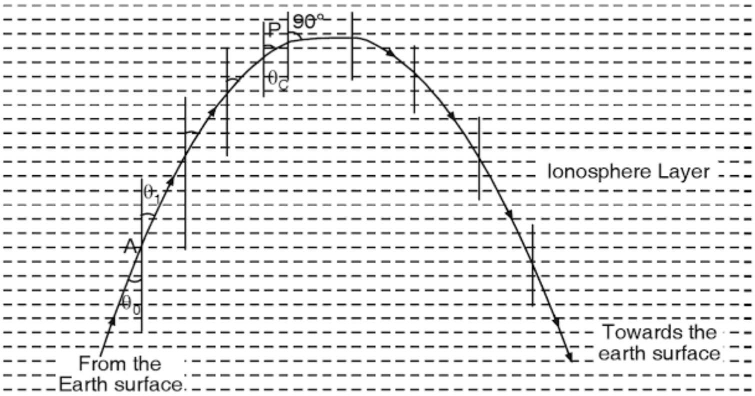

Fig. 24.6

As the waves reaches a particular height, where $\in_r$ becomes zero, v_p, the phase velocity becomes infinite. We know that energy associated with a wave travels with group velocity v_g. For the ionised layer, we write

$$v_p . v_g = c^2 \qquad \text{... }(ii)$$

Thus, when phase velocity becomes infinite, the group velocity v_g becomes zero, with the consequence that energy ceases to be propagated upward and starts travelling downward. The group velocity in its downward journey gathers speed as it comes downward towards the earth.

Using Snells law, $\mu_1 \sin \theta_1 = \mu_0 \sin \theta_0$

where

μ_1 = Refractive index at point A.

θ_0 = Angle of incidence at lower edges of the ionosphere.

θ_1 = Angle of refraction at point A.

μ_0 = Refractive index of the free space, assumed to be unity.

The radio waves, when enter the ionized region of the ionosphere, encounter increasing electron density (decreasing refractive index), hence the path of propagation of this wave keeps on bending more and more. It ultimately reaches point P, where the electron density is so high, that the angle of refraction becomes 90° and the wave then travels downwards back towards the earth. Let μ_m be the refractive index at point P, N be the electron density at point P,

Then,

$$\mu_m \sin 90° = \mu_0 \sin \theta_0$$

Example 24.3 A radio station radiates power of 10^5 watts uniformly over a hemisphere concentric with the station. Find the magnitude of Poynting vector and the amplitude of electric and magnetic field at a point 10 kms from the radio station. $\in_0 = 9 \times 10^{-12}$ farads per metre and $\mu_0 = 4\pi \times 10^{-7}$ henry per metre. [*G.N.D.U.* 2004]

Solution. Total average power radiated by the radio station

$$P = 10^5 \text{ W}$$

Radius of the hemisphere=10 kms = 10^4m

Let S_{AV} denote the average Poynting vector over the surface of the hemisphere, then

$$S_{AV} = \frac{P}{2\pi r^2} = \frac{10^5}{2\pi \times 10^8} = 1.592 \times 10^{-4} \text{ W/m}^2$$

For free space, average Poynting vector

$$S_{AV} = \frac{1}{2} c \in_0 E_0^2$$

$$\therefore \qquad E_0 = \left[\frac{2S_{AV}}{c \in_0}\right]^{\frac{1}{2}} = \left[\frac{2 \times 1.592 \times 10^{-4}}{3 \times 10^8 \times 9 \times 10^{-12}}\right]^{\frac{1}{2}}$$

$$= 0.3434 \text{ volt/metre}$$

Now

$$\vec{H}_0 = \frac{\vec{B}_0}{\mu_0} = \frac{\vec{E}_0}{\mu_0 c}$$

or

$$H_0 = \frac{E_0}{\mu_0 c} = \frac{0.3434}{4\pi \times 10^{-7} \times 3 \times 10^8} = 9.1 \times 10^{-4} \text{ ampere turns per metre}$$

Example 24.4 Calculate the value of Poynting vector on the surface of the sun if the power radiated by it is 3.8×10^{26} watt (radius of sun = 7×10^8)m. If the average distance between the sun and the earth is 1.5×10^{11} m show that the value of solar constant is 1.34×10^3 watts/m^3 sec (*H.P.U.* 2001; *M.D.U.* 2000)

Solution. Total average power radiated by the sun

$$P = 3.8 \times 10^{26} \text{ W}$$

Average distance between sun and earth

$$R = 1.5 \times 10^{11} \text{ m.}$$

$\therefore$ Solar constant = Average value of Poynting vector on the surface of earth (S_E)

or Solar constant

$$S_E = \frac{3.8 \times 10^{26}}{4\pi (1.5 \times 10^{11})^2} = 1.34 \times 10^3 \text{ Watt/m}^2$$

Example 24.5 On the surface of earth the energy received is 1.33 kW/m² from the sun. Calculate the electric field associated with sunlight (on surface of earth) assuming that it is essentially monochromatic (λ = 6000 Å). Given $\in = 9 \times 10^{-12}$ S.I units.

Solution. Energy received per second per square meter on the surface of the earth = Average value of Poynting vector on Earth's surface (*b*) for a charge free conducting medium

$\rho = 0, \sigma \neq 0$ (*G.N.D.U.* 2001; *H.P.U.* 2003; *P.U.* 2000; *Pbi. U.* 2003, 2001; *M.D.U.* 2002; *Nagpur U.* 2007)

11. (*i*) Write the wave equations for plane polarized electromagnetic waves in a dielectric medium having finite values of μ and $\in$ but $\sigma = 0$ and write solutions to these equations

(*ii*) What is the phase difference between $\vec{E}$ and $\vec{H}$ Explain.

(*iii*) For propagation of uniform plane wave in free space, prove that $\dfrac{\vec{E}}{\vec{H}} = \sqrt{\dfrac{\mu}{\in}}$

(*Nagpur Uni.*, 2003; *G.N.D.U.* 2000)

12. State Maxwell's equations in electromagnetic theory for a non-conducting medium. Derive the component form of the four relations. (*G.N.D.U.* 2004)

13. State Maxwell's equations in electromagnetic theory. Show by mathematical treatment that E.M. waves are transverse in nature with electric and magnetic field vectors at right angles to the direction of propagation.

(*Nagpur U.* 2007, 2003; *Gauhati U.* 2002; *M.D.U.* 2002; *Purvanchal U.* 2004)

14. Explain the phenomenon of reflection and refraction of electromagnetic waves (radio waves) by ionosphere.

SHORT QUESTION

1. What is the phase difference between $\vec{E}$ and $\vec{H}$? Explain with diagram.

(*Nagpur Uni*. 2007)

2. Derive the equation of speed of electromagnetic waves in free space.

(*Nagpur uni*. 2007, *D.A.U. Agra* 2004)

3. State Maxwell's field equations for electromagnetic waves in free space and hence set up wave equations for the same. (*Pbi. U.* 2003, 2002; *Purvanchal* 2007, 2004; *D.A..U. Agra* 2004)

4. Show by mathematical treatment that E.M. waves are transverse in nature.

(*Nagpur Uni*. 2008, 2007, *D.A.U. Agra*, 2004)

5. Discuss the origin of electromagnetic waves.

6. Give the characteristics of electromagnetic waves in free space.

7. What do you mean by plane polarized E.M. wave obtain the solution for electric and magnetic component.

8. Show that the value of electric field vector at any instant is 377 times the value of magnetic field vector in free space. (*P.U.* 2003, 2001, *G.N.D.U.* 2002, *M.D.U.* 2002)

9. Explain the reflection and transmission of an electromagnetic wave incident normally on a plane between media of impedance Z_1 and Z_2. Find out the expression for the reflection and transmission co-efficients and express these in terms of the refractive index of the two media. (*P.U.* 2004, 2003,2001; *G.N.D.U.* 2004; *Kolkata .U.* 2002)

10. Calculate the reflection co-efficient for a horizontally polarized E.M. wave.

11. Calculate the relation between field strengths of reflected and incident E.M. wave when it horizontally polarized due to reflection at the plane boundary of a dielectric.

12. Calculate the reflection co-efficient when total internal reflection takes place of an electromagnetic wave at the boundary of a dielectric.

13. Calculate reflection coefficient *i.e.* $\frac{E_r}{E_i}$ for an electromagnetic wave when it is vertically polarized due to reflection at the boundary of a dielectric.

14. Define characteristic impedance of a medium to electromagnetic waves.
(*Nagpur U. s*/2009)

15. Explain the phenomenon of reflection and refraction of electromagnetic waves (radio waves) by ionosphere.

16. Show that the displacement term in Maxwell equations has the dimensions of current per unit area.

17. Distinguish between conduction and displacement current. What made Maxwell suggest the presence of displacement current. (*K.U.* 2000)

18. A changing electric field induces magnetic field. Write the mathematical relation to this effect.

(**Hint.** $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \vec{J} + \in \frac{\partial \vec{E}}{\partial t}$ for conducting medium

and $\vec{\nabla} \times \vec{H} = \in \frac{\partial \vec{E}}{\partial t}$ for non-conducting medium)

19. Prove that the velocity of an E.M. wave in medium of magnetic permeability μ, electric susceptibility $\in$ and conductivity $\sigma = 0$ is given $v = \sqrt{\frac{1}{\mu \in}}$.
(*Nagpur Uni.* 2007; *Meerut U.* 2004)

20. Show that electrostatic energy per unit volume in an electromagnetic wave is equal to magnetic energy per unit volume and total energy density in the dielectric = $\in_r \times$ total energy density in free space. (*Kolkata U.* 2004; *D.A.V. Agra* 2005)

21. Derive the law of conservation of charge from Maxwell's field equations
(*Kolkata U.* 2004)

22. By using following Maxwell's equations

$$-\mu \frac{\partial H_y}{\partial t} = \frac{\partial E_z}{\partial x} \quad \text{and}$$

$$-\mu \frac{\partial H_z}{\partial t} = \frac{\partial E_y}{\partial x}$$

Show that the electromagnetic waves are transverse in nature. (*Nagpur U.* 2008)

23. Is dielectric constant (K) the same as relative permittivity ($\in$)? Explain
(*Nagpur U.* 2008)

24. Show that the electromagnetic wave travels with velocity of light in free space.
(*Nagpur U.* 2006, *Purvanchal U.* 2004)

25. Derive the relation $\nabla^2 \vec{H} = \mu_0 \in_0 \frac{\partial^2 \vec{H}}{\partial t^2}$ (*Nagpur U.* 2007)

26. For a plane electromagnetic wave, show that $\vec{E}.\vec{H} = 0$

27. Discuss reflection and refraction of a plane electromagnetic wave at a plane boundary of two dielectrics and hence establish the laws of reflection.
(*Meerut Uni* 2005; *Agra* 2007)

28. Solving E.M. wave equations for plan wave, show that $\vec{E} \times \vec{H}$ points in the direction of propogation of the wave. (*Purvanchal U.* 2007)

29. Find out wave equation of $\vec{E}$ and $\vec{B}$ in E-M waves and prove that speed of wave in vacuum is given by $\frac{1}{\sqrt{\mu_0 \in_0}}$. (*Nagpur U. s*/2009; *D.A.U. Agra* 2008)

30. Obtain equation of continuity from Maxwell's equations of electromagnetic field.
(*D.A.U. Agra* 2007, 2006)

NUMERICALS

1. If a 2KW laser beam is concentrated by a lens into a cross-sectional area of 10^{-10} m^2, find the value of Poynting vector and the amplitude of electric field. (*P.U.* 2000)

[**Ans.** $\vec{S} = 2 \times 10^{13}$ W/m^2; $E_0 = 1.21 \times 10^8$ volt/m]

2. If a 200 Watt laser beam is concentrated by a lens into a cross-sectional area of about 10^{-4} sq. cm, find the value of Poynting vector and amplitude of electric field.

[**Ans.** 2×10^{10} W/m^2, 3.85×10^6 volt/m]

3. If for vacuum $\mu_0 = 4\pi \times 10^{-7}$ kg m/coul^2 and $\in_0 = \frac{1}{36\pi \times 10^9}$ coul^2 sec^2/kg m^3, find the velocity of electro-magnetic waves and show that it is equal to the speed of light.
(*Nagpur Uni.* 2008; *Pbi U.* 2002) [**Ans.** $c = 3 \times 10^8$ m/s]

4. If the amplitude of $\vec{H}$ in a plane wave is 1amp/ metre, find the magnitude of $\vec{E}$ for a plane wave in free space. (*Nagpur Uni.* 2006)

[**Hint.** $\frac{E_z}{H_y} = \sqrt{\frac{\mu}{\in}}$]

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