UNIT-III

LAPLACE TRANSFORM

Topic Learning Objectives:

Upon Completion of this unit, students will be able to:

- Understand the existence, uniqueness and basic concepts of Laplace transform (LT).
- Determine the Laplace transform of elementary functions.
- Describe the properties of the Laplace transform such as linearity, time shifting, scaling, differentiation in the s-domain, division by t, differentiation and integration in the time domain.
- Develop the Laplace transform of periodic functions, Heaviside (unit step) function, Dirac Delta function and its applications.

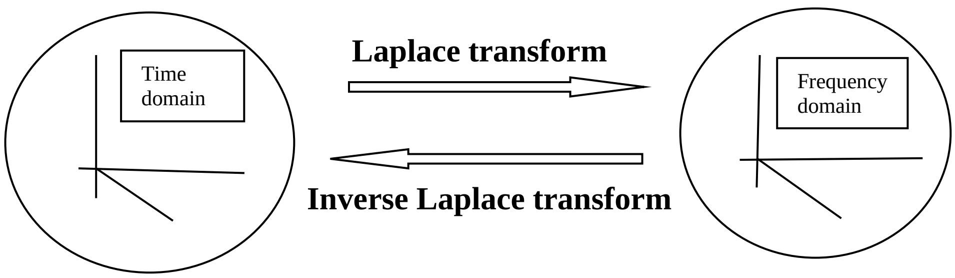
Transforms are used in Science and Engineering as a tool for simplifying analysis and look at data from different angle.

The Laplace Transform method is a technique for solving linear differential equations with initial conditions. It is commonly used to solve electrical circuit and systems problems.

The purpose of Laplace transformation is to solve different differential equations. There are a number of methods to solve homogeneous and non-homogeneous equations, but Laplace transform comes in to use when to solve the equations that cannot be solved by any of the regular methods developed.

The Laplace transform converts integral and differential equations into algebraic equations. Although it is a different and beneficial alternative of variations of parameters and undetermined coefficients, the transform is most advantageous for input force functions that are piecewise, periodic or pulsive.

In simple words, Laplace transform converts time domain signal into frequency domain, which is depicted more clearly in below figure:



Examples:

The problem of solving the equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x$ with conditions $y(0) = y'(0) = 1$ is an initial value problem.

The problem of solving the equation $3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \cos x$ with $y(1) = 1, y(2) = 3$ is called boundary value problem.

Laplace transform is essentially employed to solve initial value problems. This technique is of great utility in applications dealing with mechanical systems and electric circuits. It possesses a powerful set of properties for analysis of signals and systems. Besides, this technique may also be employed to find the solution of certain integral.

The transform is named after the French Mathematician Pierre-Simon, marquis de Laplace (1749 – 1827).

Pierre-Simon, marquis de Laplace was a prominent French mathematical physicist and astronomer of the 19th century, who made crucial contributions in the arena of planetary motion by applying Sir Isaac Newton's theory of gravitation to the entire solar system. His work regarding the theory of probability and statistics is considered pioneering and has influenced a whole new generation of mathematicians. Laplace heavily contributed in the development of differential equations, difference equations, probability and statistics. His work was important to the development of engineering, mathematics, statistics, physics, astronomy and philosophy.

Definition:

Let $f(t)$ be a real valued function defined for all $t \geq 0$ and s be a parameter, real or complex. Then the integral

$$\int_0^\infty e^{-st} f(t)dt$$

Is said to be the **Laplace transform** of f, provided the integral converges and is denoted by $L\{f(t)\}$.

Thus $\boldsymbol{L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt}$ (1)

Observe that the value of the integral on the right-hand side of (1) depends on s. Hence

$L\{f(t)\}$ is a function of s denoted by $F(s)$ or $\overline{f}(s)$.

Thus $\boldsymbol{L\{f(t)\} = F(s)}$ (2)

In relation (2), $f(t)$ is called the Inverse Laplace transform of F(s) and is denoted by $L^{-1}[F(s)]$.

Thus $L^{-1}[F(s)] = f(t)$. (3)

Suppose $f(t)$ is defined as follows:

$$f(t) = \begin{cases} f_1(t), & 0 < t < a \\ f_2(t), & a < t < b \\ f_3(t), & t > b \end{cases}$$

Note that f (t) is piecewise continuous. The Laplace transform of $f(t)$. is defined as

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)$$

$$= \int_0^a e^{-st} f_1(t)\,dt + \int_a^b e^{-st} f_2(t)\,dt + \int_b^\infty e^{-st} f_3(t)\,dt$$

Note:

In a practical situation the variable t represents time and s represents frequency. Hence the Laplace transform converts time domain into the frequency domain.

Uses of Laplace Transformation in Control System

Laplace transform is useful mathematical tool to explain the integrals in the interval from 0 to infinity. It is also used for analysing and designing the analog signals.

Also, we know that control systems mainly deal with analog systems. So usually, it is obligatory to take integral from 0 to infinity unless initial conditions are not provided. Laplace transform converts time domain into frequency domain which makes evaluation easy.

We can get the time response of the given system by taking inverse Laplace transform that is ratio of Laplace of output to the Laplace of input.

Examples:

1. Find $\boldsymbol{L\{f(t)\}}$ given $\boldsymbol{f(t) = \begin{cases} t, & 0 < t < 3 \\ 4, & t > 3 \end{cases}}$

$I = t\frac{e^{-st}}{-s} + \frac{1}{s}\int e^{-st}$

$= -\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\Big|_0^3$

$= -\frac{3}{s}e^{-3t} - \frac{1}{s^2}e^{-3t} + \frac{1}{s^2}$

Solution: ILATE

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \int_0^3 e^{-st}t\,dt + \int_3^\infty 4e^{-st}dt.$$

Integrating the terms on the RHS to get $L\{f(t)\} = \frac{1}{s}e^{-3s} + \frac{1}{s^2}(1 - e^{-3s})$

+

2. Find $L\{f(t)\}$ given $f(t) = \begin{cases} \sin 2t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$

long

Solution:

$i_2 = 4\frac{e^{-st}}{-s}\Big|_3^\infty = +\frac{4}{s}e^{-3t}$

$$\int e^{at}\sin(bt)\,dt = \frac{e^{at}(a\sin bt - b\cos bt)}{a^2+b^2}$$

$L\{f\ (t)\} = \int_0^{\pi} e^{-st} f(t)dt + \cancel{\int_{\pi}^{\infty} e^{-st} f(t)dt}^{\to 0} = \int_0^{\pi} e^{-st} \sin 2t\ dt$

ILATE

$= \left[\frac{e^{-st}}{s^2+4}\{-s\ \sin 2t - 2\cos 2t\}\right]_0^{\pi} = \frac{2}{s^2+4}[1 - e^{-\pi s}]$

3. Find $\{f\ (t)\}$ given $f(t) = \begin{cases} e^t, & 0 < t < 5 \\ 3, & t > 5 \end{cases}$

Solution:

$L\{f\ (t)\} = \int_0^5 e^{-st} f(t)dt + \int_5^{\infty} e^{-st} f(t)dt = \int_0^5 e^{-st} e^t dt + \int_5^{\infty} e^{-st}.3dt$

$= \int_0^5 e^{(1-s)t} dt + 3\int_5^{\infty} e^{-st} dt = \left[\frac{e^{(1-s)t}}{1-s}\right]_0^5 + \left[\frac{3e^{-st}}{s}\right]_0^5$

$= \frac{e^{5(1-s)}-1}{1-s} + \frac{3e^{-5s}}{s}.$

Existence and Uniqueness of Laplace Transform

The sufficient condition for existence of Laplace transform is based on the concepts of Piecewise continuous function and exponential order.

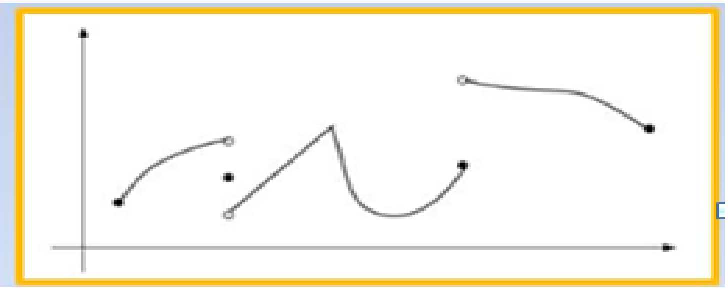
Piecewise Continuous function

If an interval [a, b] can be partitioned by a finite number of points $a_0 = t_0 < t_1 < t_2 < \ldots\ldots t_n = b$ such that

(i) f is continuous on each sub interval $(t_i,\ t_{i+1})$.

(ii) $\left|\lim_{t\to t_i^+} f(t)\right| < \infty\ \ \forall i = 0,1,2,\ldots,n-1$

(iii) $\left|\lim_{t\to t_{i+1}^-} f(t)\right| < \infty\ \ \forall i = 1,2,\ldots,n$

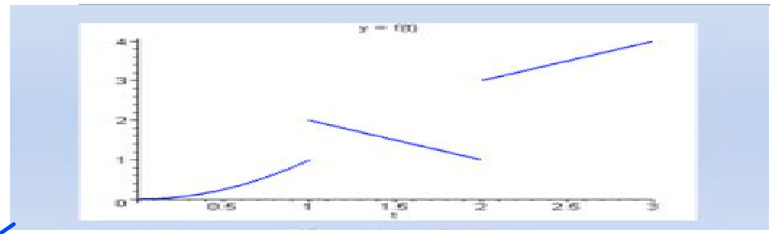


Then the function f is piecewise continuous or f is piecewise continuous on [a, b] if it is continuous there except for a finite number of jump discontinuities.

Examples:

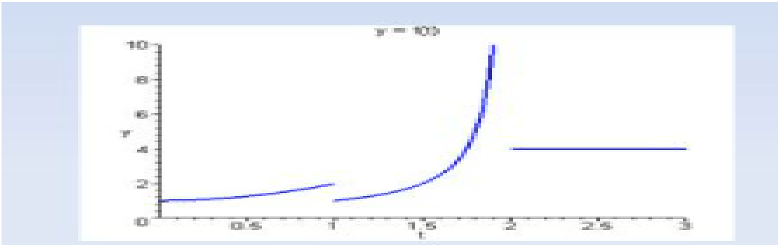
1. The function defined by

$$f\ (t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 3 - t, & 1 < t \le 2 \\ t + 1, & 2 < t \le 3 \end{cases}$$ is piecewise continuous in $[0, 3]$.



2. The function defined by

$$f\ (t)\ =\ \begin{cases} t^2+1, & 0 \le t \le 1 \\ \frac{1}{2-t}, & 1 < t \le 2 \\ 4, & 2 < t \le 3 \end{cases}$$ is not a piecewise continuous in $[0, 3]$.



Exponential order

A function $f(t)$ is said to be of exponential order c, if there exist constants M and c such that $|f(t)| \le Me^{ct}$ for sufficiently large t.

Example:

Any Polynomial is of exponential order. This is clear from the fact that

$e^{at} = \sum_{n=0}^{\infty} \frac{t^n a^n}{n!} = t^n \le \frac{n!}{a^n} e^{at}$, but $f(t) = e^{t^2}$ is not of exponential order.

Sufficient condition for the existence of Laplace transform

Let $f(t)$ be a piecewise continuous function in $[0, \infty)$ and is of exponential order then Laplace transform $F(s)$ of $f(t)$ exists for $s > c$, where c is a real number that depends on $f(t)$.

Uniqueness of Laplace transform

Let $f(t)$ and $g(t)$ be continuous functions such that $F(s) = G(s)$ for all $s > k$ then $f(t) = g(t)$ at all t.

Transform of elementary functions

1. Let ‘a’ be a constant then

$$L\ (e^{at}) = \int_0^{\infty} e^{-st} e^{at} dt = \int_0^{\infty} e^{-(s-a)t} dt$$

$= \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty = \frac{1}{s-a},$

Thus $L\,(e^{at}) = \frac{1}{s-a}, s > a$

In particular when $a = 0$ to get $L(1) = \frac{1}{s}, s > 0.$

$L(\cosh at) \;= L\left(\frac{e^{at}+e^{-at}}{2}\right) = \frac{1}{2}\int_0^\infty e^{-st}[e^{at} + e^{-at}]dt$

$= \frac{1}{2}\int_0^\infty [e^{-(s-a)t} + e^{-(s+a)t}dt]$

$L(\cosh at) = \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \frac{s}{s^2-a^2}$

Thus, $(\cosh at) \;= \frac{s}{s^2-a^2}, s \;>\; |a|\;,$

$L\,(\sinh at) \;=\; L\left(\frac{e^{at}-e^{-at}}{2}\right) = \frac{a}{s^2-a^2}, s \;>\; |a|$

Thus, $L\,(\sinh at) = \frac{a}{s^2-a^2}$, $s > |a|$

2. $L\,(\sin at) \;=\; \int_0^\infty e^{-st} \sin at\;\, dt$

Suppose $s > 0$ then integrate by using the formula

$\int e^{ax} \sin bt\, dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx]$

$L\,(\sin at) = \frac{a}{s^2+a^2}, s > 0.$

3. $L\,(\cos at) \;=\; \int_0^\infty e^{-st} \cos at\;\, dt$

Suppose $s > 0$ and integrate by using the formula

$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}[a \cos bx + b \sin bx]$

$L\,(\cos at) \;= \frac{s}{s^2+a^2}, s \;>\; 0.$

4. Let n be a constant, which is a non-negative real number or a negative non-integer

Then $L\,(t^n) \;=\; \int_0^\infty e^{-st} t^n dt$

Let $s > 0$ and put $st = x$ then $L(t^n) = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \frac{dx}{s} = \frac{1}{s^{n+1}}\int_0^\infty e^{-x} x^n dx$

The integral $\int_0^\infty e^{-x}\, x^n dx$ is called gamma function of $(n + 1)$ denoted by $\Gamma(n + 1)$.

Thus $L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$

In particular if n is a non-negative integer then $\Gamma(n + 1) = n!$. Hence $L(t^n) = \frac{n!}{s^{n+1}}.$

Table of Laplace Transform

$f(t)$	F(s)
1	$\frac{1}{s}$, s > 0
e^{at}	$\frac{1}{s-a}$, s > a
$\cos hat$	$\frac{s}{s^2-a^2}$, s > |a|
$\sin hat$	$\frac{a}{s^2-a^2}$, s > |a|
$\sin at$	$\frac{a}{s^2+a^2}$, s > 0
$\cos at$	$\frac{s}{s^2+a^2}$, s > 0
t^n	$\frac{n!}{s^{n+1}}$, s > 0 n is a positive integer
t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$, s > 0 $n \neq -1,-2,-3, \ldots ..$

Region of Convergence (RoC)

Laplace transform F(s) of a function $f(t)$ converges provided that the limit $\lim_{R\to\infty}\int_0^R f(t)e^{-st}dt$ exists. The Laplace transform converges absolutely if the integral $\int_0^\infty |f(t)e^{-st}|dt$ exists. The set of values of s for which $F(s)$ exists is known as the region of convergence.

Examples:

1. Evaluate: (i) $L(\sin 3t \sin 4t)$ (ii) $L(\cos^2 4t)$ (iii) $L(\sin^3 2t)$.

Solution:

(i) $L\,(\sin 3t \sin 4t) = L\left[\frac{1}{2}(\cos t - \cos 7t)\right]$

$=\frac{1}{2}[L(\cos t) - L(\cos 7t)]$ (linearity property)

$= \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+49}\right] = \frac{24s}{(s^2+1)(s^2+49)}.$

(ii) $L\,(\cos^2 4t) = L\left[\frac{1}{2}(1+\cos 8t)\right] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+64}\right].$

(iii) We have $sin^3 t = \frac{1}{4}(3\, sin\, t - sin\, 3t)$

$$sin^3 2t = \frac{1}{4}(3 \sin 2t - \sin 6t)$$

So $L(sin^3 2t) = \frac{1}{4}\left[\frac{6}{s^2+4} - \frac{6}{s^2+36}\right] = \frac{48}{(s^2+4)(s^2+36)}$

2. Find $L\,(\cos t \cos 2t \cos 3t)$.

Solution:

$\cos 2t \cos 3t = \frac{1}{2}[cos\, 5t + cos\, t]$

So $\cos t \cos 2t \cos 3t \;= \frac{1}{2}[cos\, 5t\, cos\, t + cos^2 t]$

$$= \frac{1}{4}[\cos 6t + \cos 4t + 1 + \cos 2t]$$

Thus $L\,(\cos t \cos 2t \cos 3t) \;= \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{s}{s^2+16} + \frac{1}{s} + \frac{s}{s^2+4}\right]$.

3. Find $L\,(\cosh^2 2t)$.

Solution:

Using $\cosh^2 t = \frac{1+\cosh 2t}{2}$

For $\cosh^2 2t = \frac{1+\cosh 4t}{2}$

Thus$L(\cosh^2 2t) = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2-16}\right]$.

4. Evaluate (i) $L(\sqrt{t})$ (ii) $L\left(\frac{1}{\sqrt{t}}\right)$ (iii) $L\,(t^{-\frac{3}{2}})$.

Solution:

We have $L\,(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$

(i) For $n = \frac{1}{2}$, $\;\mathrm{L}\,(t^{1/2}) = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{3/2}}$

Since$\Gamma(n+1) = n\Gamma(n)$, we have $\Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$

Thus $\;L(\sqrt{t}) = \frac{\sqrt{\pi}}{2s^{3/2}}$.

(ii) For $n \;=\; -\frac{1}{2}$ to get $L(t^{-1/2}) = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}}$

(iii) For $n = -\frac{3}{2}$ to get $L(t^{-3/2}) = \frac{\Gamma\left(-\frac{1}{2}\right)}{s^{-1/2}} = \frac{-2\sqrt{\pi}}{s^{-1/2}} = -2\sqrt{\pi s}$.

5. Evaluate: (i) $L(t^2)$ (ii) $\;L(t^3)$.

Solution: $L\,(t^n) = \frac{n!}{s^{n+1}}$

(i) For $n = 2$, $L(t^2) = \frac{2!}{s^3} = \frac{2}{s^3}$.

(ii) For $n = 3$, $L(t^3) = \frac{3!}{s^4} = \frac{6}{s^4}$.

6. Find $L\,(e^{-5t} + 5e^{2t})$.

Solution: $L\,(e^{at}) = \frac{1}{s-a}$

Thus, $L\,(e^{-5t} + 5e^{2t}) = L(e^{-5t}) + L(5e^{2t}) = \frac{1}{s+5} + \frac{5}{s-2}$

7. Find the Laplace transform of a^{kt}

Solution:

$a^{kt} = e^{\log a^{kt}}$

$a^{kt} = e^{kt\,\log a}$

$L\{a^{kt}\} = \frac{1}{s-k\log a}$

Exercise:

Find the Laplace transform of the following functions:

(i) $\sin(3t + 4)$ (ii) $\cos 2t \sin 3t$ (iii) $\sin t \sin 2t \sin 3t$ (iv) $\cos^3 t$ (v) $(\sin t - \cos t)^2$

Answers:

(i) $\frac{3\cos 4 + s\sin 4}{s^2+9}$ (ii) $\frac{1}{2}\left(\frac{5}{s^2+25} + \frac{1}{s^2+1}\right)$ (iii) $\frac{1}{4}\left(\frac{2}{s^2+4} + \frac{4}{s^2+16} - \frac{6}{s^2+36}\right)$ (iv) $\frac{s(s^2+7)}{(s^2+1)(s^2+9)}$ (v) $\frac{1}{s} - \frac{2}{s^2+4}$

Properties of Laplace transform

1. **Linearity**

For any two functions $f(t)$ and $g(t)$ whose Laplace transform exists and any two constants a and b we have

$$L\,[a\,f(t) + b\,g(t)] = a\,L\{f\,(t)\} + b\,L\{g(t)\}.$$

Proof

By definition

$$L\,[a\,f(t) + b\,g(t)] = a\int_0^\infty e^{-st} f(t)dt + b\int_0^\infty e^{-st} g(t)dt$$
$$= a\,L\,\{f\,(t)\} + b\,L\,\{g(t)\}$$

In particular,

for $a = b = 1$, $L\,[\,f(t) + g(t)] = L\{f(t)\} + L\{g(t)\}$ and

for $a = -b = 1$, $L\,[\,f(t) - g(t)] = L\,\{f(t)\} - L\{g(t)\}$.

The linearity of the Laplace transform follows from its definition as an integral and the fact that integration is a linear operation.

2. Scaling

If $L\{f\ (t)\} = F(s)$, then $L\ [f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$, where a is a positive constant.

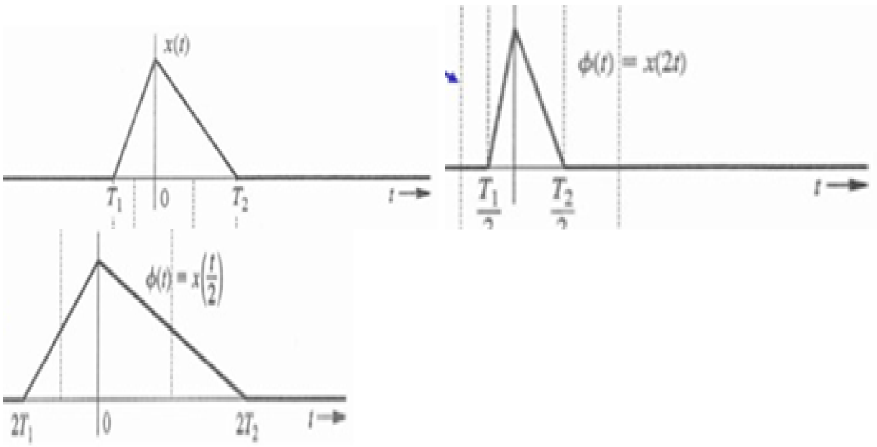
Proof

By definition $L\{f(at)\}\ = \int_0^{\infty} e^{-st} f(at) dt$ (1)

Put $at\ =\ x$ the expression (1) becomes,

$$L\ \{f(at)\}\ = \frac{1}{a}\int_0^{\infty} e^{-\left(\frac{s}{a}\right)x} f(x) dx = \frac{1}{a} F\left(\frac{s}{a}\right)$$

Scaling in time introduces the inverse scaling in s. (It depends on the sign of 'a')



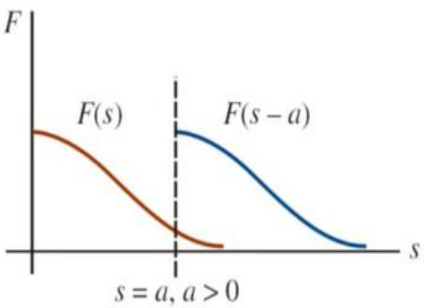
3. s**- domain shift (First shifting property)**

Let a be any real constant then $L[e^{at} f(t)]\ =\ F(s-a)$

Proof

By definition $L[e^{at}\ f(t)] =\ \int_0^{\infty} e^{-st}[e^{at} f(t)] dt =\ \int_0^{\infty} e^{-(s-a)} f(t) dt = F(s-a)$

Multiplication by an exponential in time introduces a shift in frequency s to the Laplace transform of $f(t)$ i.e. Laplace transform of $e^{at} f(t)$ can be written down directly by changing s to $s-a$ in $F(s)$.

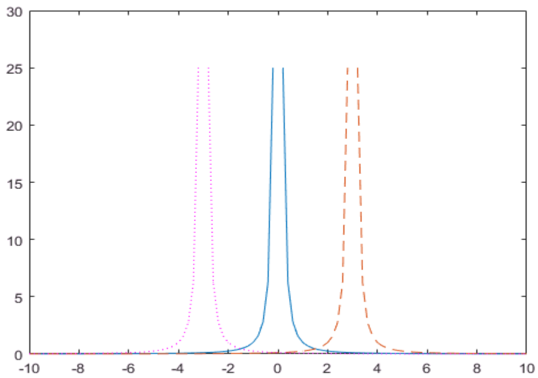


Example:

If $f(t)\ =\ t,$ then $F(s) = \frac{1}{s^2}$

If $f(t) = e^{3t}\, t$ then $F(s) = \frac{1}{(s-3)^2}$ (Shifting towards right)

If $f(t) = e^{-3t}\, t$ then $F(s) = \frac{1}{(s+3)^2}$ (Shifting towards left)



Examples:

1. Find $L[e^{-3t}\,(2\cos 5t - 3\sin 5t)]$.

Solution:

Consider $f(t) = 2\cos 5t - 3\sin 5t$

$$L\{f(t)\} = L[(2\cos 5t - 3\sin 5t)]$$

$$F(s) = 2\frac{s}{s^2+25} - \frac{3(5)}{s^2+25}$$

Given $L\left[e^{-3t}\left(2\cos 5t - 3\sin 5t\right)\right] = 2\frac{(s+3)}{(s+3)^2+25} - \frac{15}{(s+3)^2+25}$ (s- domain shift)

$$= \frac{2s-9}{s^2+6s+34}.$$

3. Find $L\,[\cosh at\ \sin at]$.

Solution:

$L\,[\cosh at\ \sin at] = L\left[\frac{(e^{at}+e^{-at})}{2}\sin at\right]$

$$= \frac{1}{2}\left[\frac{a}{(s-a)^2+a^2} + \frac{a}{(s+a)^2+a^2}\right]$$

$$= \frac{a(s^2+2a^2)}{[(s-a)^2+a^2][(s+a)^2+a^2]}$$

3. Find $L(\cosh t \sin^3 2t)$.

Solution:

Given $L\left[\left(\frac{e^t+e^{-t}}{2}\right)\left(\frac{3\sin 2t - \sin 6t}{4}\right)\right]$

$=\frac{1}{8}[3L(e^t\sin 2t)-L(e^t\sin 6t)+3L(e^{-t}\sin 2t)-L(e^{-t}\sin 6t)]$

$=\frac{1}{8}\left[\frac{6}{(s-1)^2+4}-\frac{6}{(s-1)^2+36}+\frac{6}{(s+1)^2+4}-\frac{6}{(s+1)^2+36}\right]$

$=\frac{3}{4}\left[\frac{1}{(s-1)^2+4}-\frac{1}{(s-1)^2+36}+\frac{1}{(s+1)^2+24}-\frac{1}{(s+1)^2+36}\right].$

4. Find $L(e^{-4t}t^{-5/2})$.

Solution:

$L(t^n)=\dfrac{\Gamma(n+1)}{s^{n+1}}$

Put $n=\ -5/2$, $L(t^{-5/2})=\frac{\Gamma(-3/2)}{s^{-3/2}}=\frac{4\sqrt{\pi}}{3s^{-3/2}}$. Change s to $s+4$.

$\therefore\ L(e^{-4t}t^{-5/2})=\dfrac{4\sqrt{\pi}}{3(s+4)^{-3/2}}.$

5. Find $L\{(e^t+1)^2\cos t\}$.

Solution:

$(e^t+1)^2\cos t=(e^{2t}+1+2e^t)\cos t$

Therefore, $L\{(e^t+1)^2\cos t\}=L\{e^{2t}\cos t\}+L\{\cos t\}+L\{e^t\cos t\}$

$=\frac{s-2}{(s-2)^2+1}+\frac{s}{s^2+1}+\frac{2(s-1)}{(s-1)^2+1}$

Exercise:

Find the Laplace transform of the following functions.

(i) $e^{-2t}\ \cos^2 2t$ (ii) $e^{2t}\sin 3t\cos 2t$

Solutions:

(i) $\dfrac{(s+2)^2+8}{(s+2)((s+2)^2+16)}$ (ii) $\dfrac{3(s-2)^2+15}{((s-2)^2+25)((s-2)^2+1)}$

Differentiation in the s-domain

If $\boldsymbol{L\{f(t)\}=F(s)}$, then $\boldsymbol{L[t^n f(t)]=(-1)^n\frac{d^n}{ds^n}F(s), n}=1, 2, 3,...$

Suppose that n is positive integer by definition

$F(s)=\int_0^\infty e^{-st}f(t)dt$

Differentiating 'n' times on both sides w.r.t. s,

$$\frac{d^n}{ds^n}F(s)=\frac{\partial^n}{\partial s^n}\int_0^\infty e^{-st}f(t)dt$$

Performing differentiation under the integral sign,

$$\frac{d^n}{ds^n}F(s) = \int_0^\infty (-t)^n e^{-st} f(t)dt$$

Multiplying on both sides by $(-1)^n$,

$$(-1)^n \frac{d^n}{ds^n}F(s) = \int_0^\infty (t^n f(t)e^{-st}dt = L[t^n f(t)], \text{ by definition}$$

Thus, $L\,[t^n f(t)] \;= (-1)^n \frac{d^n}{ds^n}F(s)$. This is the transform of $t^n f(t)$.

Also $L^{-1}\left[\frac{d^n}{ds^n}F(s)\right] = (-1)^n t^n f(t)$.

Differentiation in s-domain corresponds to multiplication by $-t$ in the time domain

In particular for $n = 1, L\,[t\,f(t)] = -\frac{d}{ds}F(s)$,

for $n = 2,\; L[t^2\,f(t)] = \frac{d^2}{ds^2}F(s)$ etc.

Transform of $\frac{f(t)}{t}$

By definition, $F(s) = \int_0^\infty e^{-st} f(t)dt$

Therefore $\int_s^\infty F(s)ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)dt\right]ds = \int_0^\infty f(t)\left[\int_s^\infty e^{-st}ds\right] dt$

$$= \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt = \int_0^\infty e^{-st}\left[\frac{f(t)}{t}\right]dt = L\left(\frac{f(t)}{t}\right)$$

Thus $L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(s)ds.$

Examples:

1. Find $L\,(t^2 \sin 3t)$.

Solution:

$$L(\sin 3t) = \frac{3}{s^2+9}$$

So that $L\,(t^2 \sin 3t) = \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right) = -6\frac{d}{ds}\frac{s}{(s^2+9)^2} = \frac{18(s^2-3)}{(s^2+9)^3}$.

2. Find $L\,[t\,e^{-t}\,\sin 4t]$.

Solution:

$$L\,(\sin 4t) \;= \frac{4}{s^2+16}$$

So that $L(t\;\sin 4t) = -\frac{d}{ds}\frac{4}{s^2+16} = \frac{8s}{(s^2+16)2}$.

By s – shifting property

So that $L\,[t\,e^{-t}\ \sin 4t] = \frac{8(s+1)}{(s+1)^2+16)^2} = \frac{8(s+1)}{(s^2+2s+17)^2}$.

3. Find $L\{(t\cos 2t)^2\}$

Solution:

$L\{(t\cos 2t)^2\} = L\{t^2\cos^2 2t\}$

$$L\{\cos^2 2t\} = \frac{1}{2}L\{1+\cos 4t\} = \frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2+16}\right]$$

$$L\{(t\cos 2t)^2\} = \frac{1}{2}\frac{d^2}{ds^2}\left[\frac{1}{s}+\frac{s}{s^2+16}\right] = \frac{1}{2}\frac{d}{ds}\left[-\frac{1}{s^2}+\frac{16-s^2}{(s^2+16)^2}\right]$$

$$= \frac{1}{2}\left[\frac{2}{s^3}+\frac{(s^2+16)^2(-2s)-(16-s^2)2.(s^2+16)(2s)}{(s^2+16)^4}\right] = \frac{1}{s^3}+\frac{s(s^2-48)}{(s^2+16)^3}$$

4. Find $L\left(\frac{e^{-t}\sin t}{t}\right)$

Solution:

$L(\sin t) = \frac{1}{s^2+1}$

$L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{ds}{s^2+1} = [\tan^{-1}s]_s^\infty$

$= \frac{\pi}{2} - \tan^{-1}s\ = \cot^{-1}s$

By s – shifting property , $L\left(\frac{e^{-t}\sin t}{t}\right) = \cot^{-1}(s+1)$.

5. Find $L\left(\frac{\sin t}{t}\right)$ and hence evaluate $L\left(\frac{\sin at}{t}\right)$.

Solution:

We have $L(\sin t) = \frac{1}{s^2+1}$

So $L\{f(t)\} = L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{ds}{s^2+1} = [\tan^{-1}s]_S^\infty$

$= \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s = F(s)$.

Consider $\mathrm{L}\left(\frac{\sin at}{t}\right) = a\,\mathrm{L}\left(\frac{\sin at}{at}\right) = a\,L\{f(at)\}$

$= a\left[\frac{1}{a}F\left(\frac{s}{a}\right)\right]$ (scaling)

$= \cot^{-1}\left(\frac{s}{a}\right)$.

6. Find $L\left[\frac{\cos at-\cos bt}{t}\right]$

Solution:

$\int \frac{s}{s^2+a^2}\, ds$

$x = s^2+a^2$

$dx = 2s\, ds$

$\int \frac{dx}{2x} = \frac{1}{2}\log x$

We have $L[\cos at - \cos bt] = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

So that $L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty \left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right] ds = \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty$

$$= \frac{1}{2}\left[\underset{s\to\infty}{Lt}\log\left(\frac{s^2+a^2}{s^2+b^2}\right) - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\left[0 + \log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right] = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right).$$

7. Find $L\left[\frac{e^{-at}-e^{-bt}}{t}\right]$

Solution:

$$L\{e^{-at} - e^{-bt}\} = \frac{1}{s+a} - \frac{1}{s+b}$$

$$L\left[\frac{e^{-at} - e^{-bt}}{t}\right] = \int_s^\infty \left[\frac{1}{s+a} - \frac{1}{s+b}\right] ds = [\log(s+a) - \log(s+b)]_s^\infty$$

$$= \left[\log\frac{s+a}{s+b}\right]_s^\infty = \underset{s\to\infty}{Lt}\log\frac{s+a}{s+b} - \log\frac{s+a}{s+b} = \log\frac{s+b}{s+a}$$

Exercise:

Find the Laplace transform of the following functions.

(i) $t\sin 3t\cos 2t$ (ii) $t^2\, e^{-t}\cos t$ (iii) $\frac{1-e^{-2t}}{t}$ (iv) $\frac{e^{-at}-e^{-bt}}{t}$ (v) $\frac{\sin^2 t}{t}$.

Answers:

(i) $\frac{5s}{(s^2+25)^2} + \frac{s}{(s^2+1)^2}$ (ii) $\frac{2(s+1)(s^2+2s-2)}{((s+1)^2+1)^3}$ (iii) $log\left(\frac{s+2}{s}\right)$ (iv) $log\left(\frac{s+b}{s+a}\right)$

(v) $\frac{1}{2}\, log\left(\frac{\sqrt{s^2+4}}{s^2}\right)$.

$\frac{1}{s^2+1} \to + \frac{2s}{(s^2+1)^2} \to \frac{2(s+3)}{((s+3)^2+1)^2}$

$s \to s+3$

8. Prove that $\int_0^\infty e^{-3t}\, t\sin t\, dt = \frac{3}{50}$.

$\frac{2\times 3}{(9+1)^2} = \frac{6}{10\times 10} = \frac{3}{50}$

Solution:

By $\int_0^\infty e^{-st}\, t\sin t\, dt = L(t\sin t) = -\frac{d}{ds}L(\sin t) = -\frac{d}{ds}\left[\frac{1}{s^2+1}\right] = \frac{2s}{(s^2+1)^2}$

Put $s = 3$ to get $\int_0^\infty e^{-3t}\, t\sin t\; dt = \frac{3}{50}$.

9. Evaluate $\int_0^\infty e^{3t}t^3\sin t\, dt$

Solution:

By definition, $\int_0^\infty e^{-st}t^3\sin t\, dt = L(t^3\sin t)$

$L\{\sin t\} = \dfrac{1}{s^2+1} = F(s)$ $-\dfrac{2s}{(s^2+1)^2}$

$L\{t^3 \sin t\} = (-1)^3 F'''(s)$

$F'(s) = -\frac{2s}{(s^2+1)^2}$; $F''(s) = \frac{6s^2-2}{(s^2+1)^3}$; $F'''(s) = \frac{-24s^3+24s}{(s^2+1)^4}$

Therefore, $L\{t^3 \sin t\} = \frac{24s^3-24s}{(s^2+1)^4}$

Put $s = -3$, to get $\int_0^\infty e^{3t} t^3 \sin t \, dt = -\frac{576}{10^4}$

10. Evaluate $\int_0^\infty e^{2t} t \cos^2 t \, dt$ $\dfrac{1+\cos 2t}{2}$

Solution:

By definition, $\int_0^\infty e^{-st} t \cos^2 t \, dt = L(t \cos^2 t)$

$$L\{\cos^2 t\} = \frac{1}{2} L\{1+\cos 2t\} = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$$

$$L\{t\cos^2 t\} = -\frac{1}{2}\frac{d}{ds}\left[\frac{1}{s} + \frac{s}{s^2+4}\right] = \frac{1}{2}\left[\frac{1}{s^2} + \frac{s^2-4}{(s^2+4)^2}\right]$$

Put $s = -2$, to get $\int_0^\infty e^{2t} t \cos^2 t \, dt = \frac{1}{8}$

Exercise:

Evaluate the following integrals using Laplace transforms:

(i) $\int_0^\infty t e^{-2t} \sin 3t \, dt$ **(ii)** $\int_0^\infty e^{3t} t^3 \cos t \, dt$ **(iii)** $\int_0^\infty \left(\frac{e^{-3t}-e^{-6t}}{t}\right) dt$ **(iv)** $\int_0^\infty \frac{e^{-t} \sin\sqrt{3}t}{t} dt$

Answers: (i) $\frac{12}{169}$ (ii) $\frac{168}{10^4}$ (iii) $\log 2$ (iv) $\frac{\pi}{3}$

Differentiation in the time domain

If $\boldsymbol{L\{f(t)\} = F(s)}$, then

$\boldsymbol{L\{f^{(n)}(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots\ldots - f^{(n-1)}(0)}$.

Consider

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t) dt$$
$$= [e^{-st} f(t)]_0^\infty - \int_0^\infty (-s) e^{-st} f(t) dt \text{ (integration by parts)}$$
$$= \left[\underset{t\to\infty}{Lt} (e^{-st} f(t)) - f(0)\right] + sLf(t)$$
$$= 0 - f(0) + s L\{f(t)\}$$

Thus

$$L\{f'(t)\} = s L\{f(t)\} - f(0)$$

Similarly

$$L\{f''(t)\} = s^2 L\{f(t)\} - s f(0) - f'(0)$$

In general

$$L\{f^{(n)}(t)\} = s^n L\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots\ldots - f^{(n-1)}(0).$$

Integration property

If $L\{f(t)\} = F(s)$, then $L\int_0^t f(t)dt = \frac{1}{s}F(s)$

Let $\varphi(t) = \int_0^t f(t)dt$ then $\varphi(0) = 0$ and $\varphi'(t) = f(t)$

Now $L\{\varphi(t)\} = \int_0^\infty e^{-st}\varphi(t)dt = \left[\varphi(t)\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \varphi'(t)\frac{e^{-st}}{-s}dt$

$= (0-0) + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt$

Thus $L\int_0^t f(t)dt = \frac{1}{s}F(s)$

Examples:

1. By using the Laplace transform of $\sin at$, find the Laplace transform of $\cos at$.

Solution:

Let $f(t) = \sin at$, then $L\{f(t)\} = \frac{a}{s^2+a^2}$ ✓

We note that $f'(t) = a\cos at$

Taking Laplace transforms, $L\{f'(t)\} = L(a\cos at) = aL(\cos at)$

$s^n L\{f(t)\} - s^{n-1} L\{f(0)\}$.

Or $L\{\cos at\} = \frac{1}{a}L\{f'(t)\} = \frac{1}{a}[sL\{f(t)\} - f(0)]$

$= \frac{1}{a}\left[\frac{sa}{s^2+a^2} - 0\right]$. $n=1$

Thus $L\{\cos at\} = \frac{s}{s^2+a^2}$.

2. Given $L\left[2\sqrt{\frac{t}{\pi}}\right] = \frac{1}{s^{3/2}}$, show that $L\left[\frac{1}{\sqrt{\pi t}}\right] = \frac{1}{\sqrt{s}}$.

$L\left[2\sqrt{\frac{t}{\pi}}\right] = \frac{1}{s^{3/2}}$

Solution:

$L(f(t)) =$

Let $f(t) = 2\sqrt{\frac{t}{\pi}}$, given $L[f(t)] = \frac{1}{s^{3/2}}$

$f(t) = 2\sqrt{\frac{t}{\pi}}$

Note that $f'(t) = \frac{2}{\sqrt{\pi}}\frac{1}{2\sqrt{t}} = \frac{1}{\sqrt{\pi t}}$

$f'(t) = \frac{2}{\sqrt{\pi}}\frac{1}{2\sqrt{t}}$

Taking Laplace transform, to get $L\{f'(t)\} = L\left[\frac{1}{\sqrt{\pi t}}\right]$

Hence $L\left[\frac{1}{\sqrt{\pi t}}\right] = Lf'(t) = sLf(t) - f(0) = s\left(\frac{1}{s^{3/2}}\right) - 0$

$f'(t) = \frac{1}{\sqrt{\pi t}}$

Thus $L\left[\frac{1}{\sqrt{\pi t}}\right] = \frac{1}{\sqrt{s}}$.

$L(f'(t)) = L\left[\frac{1}{\sqrt{\pi t}}\right]$

3. Find $L\int_0^t \left(\frac{\cos at - \cos bt}{t}\right)dt$.

Solution:

$sLf(t) - f(0)$.

$\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

$\frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2}\Big|_s^\infty$

$s \cdot \frac{1}{s^{3/2}} = \frac{1}{s^{1/2}}$

Here $L\{f\ (t)\} = L\left(\frac{\cos at - \cos bt}{t}\right) = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$

Using the result $L\{\int_0^t f(t)dt = \frac{1}{s}L\{f(t)\}$

$$L\left\{\int_0^t \left(\frac{\cos at - \cos bt}{t}\right) dt\right\} = \frac{1}{2s}\log\left(\frac{s^2+b^2}{s^2+a^2}\right).$$

4. Find $L\{\int_0^t te^{-t} \sin 4t\ dt\}$.

Solution:

$$L[te^{-t} \sin 4t] = \frac{8(s+1)}{(s^2+2s+17)^2} = 2$$

$\frac{1}{s}$ 2

Thus$L\{\int_0^t te^{-t} \sin 4t\ dt\} = \frac{8(s+1)}{s(s^2+2s+17)^2}$.

5. Find $L\{\int_0^t (e^{-t} \sin 4t + t \cos 2t)\ dt\}$.

Solution:

$$L\{sin\ 4t\} = \frac{4}{s^2+16}$$

$$L\{e^{-t} \sin 4t\} = \frac{4}{(s+1)^2+16} = \frac{4}{s^2+2s+17}$$

$$L\{\cos 2t\} = \frac{s}{s^2+4}$$

$$L\{t\ cos\ 2t\} = -\frac{d}{ds}\left(\frac{s}{s^2+4}\right) = \frac{s^2-4}{(s^2+4)^2}$$

Therefore $L\{e^{-t} \sin 4t + t \cos 2t\} = \frac{4}{s^2+2s+17} + \frac{s^2-4}{(s^2+4)^2}$ $= X$ 2

Hence $L\left\{\int_0^t (e^{-t} \sin 4t + t \cos 2t)\ dt\right\} = \frac{4}{s(s^2+2s+17)} + \frac{s^2-4}{s(s^2+4)^2}$ $\frac{1}{s} \cdot X$

Exercise:

Find the Laplace transform of the following functions.

(i) $L\int_0^t e^t \frac{\sin t}{t} dt$ (ii) $L\int_0^t e^t\ cosh\ t\ dt$ (iii) $L\int_0^t t^2 \sin at\ dt$

Answers:

(i) $\frac{1}{s}\cot^{-1}(s-1)$ (ii) $\frac{(s-1)}{s(s^2-2s)}$ (iii) $\frac{2a(3s^2-a^2)}{s(s^2+a^2)^3}$

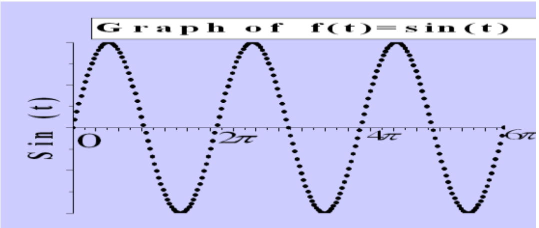
Periodic function

Definition

A function $f(t)$ is said to be a periodic function of period $T > 0$ if $f(t) = f(t + nT)$ where $n = 1, 2, 3, \dots$. The graph of the periodic function repeats itself in equal intervals.

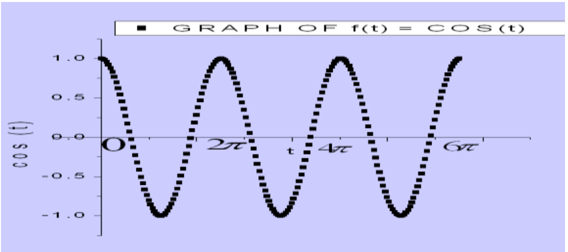
For example $\sin t, \cos t$ are periodic functions of period 2π since $\sin(t + 2n\pi) = \sin t$, $\cos(t + 2n\pi) = \cos t$.

The graph of $f(t) = \sin t$ is shown below :



Note that the graph of the function between 0 and 2π is the same as that between 2π and 4π and so on.

The graph of $f(t) = \cos t$ is shown below :



Note that the graph of the function between 0 and 2π is the same as that between 2π and 4π and so on.

Laplace Transform of a periodic function

Let $f(t)$ be a periodic function of period T then $L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt.$

Proof:

Using definition, we have $L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = \int_0^\infty e^{-su} f(u) du$

$$= \int_0^T e^{-su} f(u) du + \int_T^{2T} e^{-su} f(u) du + \ldots\ldots + \int_{nT}^{(n+1)T} e^{-su} f(u) du + \ldots. + \infty$$

$$= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-su} f(u) du$$

Let us set, $u = t + nT$, then $L\{f(t)\} = \sum_{n=0}^{\infty} \int_{t=0}^{T} e^{-s(t+nT)} f(t + nT) dt$

Here $f(t + nT) = f(t)$, by periodic property

Hence, $L\{f(t)\} = \sum_{n=0}^{\infty} (e^{-sT})^n \int_0^T e^{-st} f(t) dt$

$$= \left[\frac{1}{1-e^{-sT}}\right] \int_0^T e^{-st} f(t) dt,$$ identifying the above series as a geometric series.

Thus, $L\{f(t)\} = \left[\frac{1}{1-e^{-sT}}\right] \int_0^T e^{-st} f(t) dt.$

Examples

1. For the periodic function $f(t)$ of period 4, defined by $f(t) = \begin{cases} 3t, & 0 < t < 2 \\ 6, & 2 < t < 4 \end{cases}$, find $L\{f(t)\}$.

Solution:

Here period of $f(t) = T = 4$

By definition $L\{f(t)\} = \left[\frac{1}{1-e^{-sT}}\right]\int_0^T e^{-st}f(t)dt = \left[\frac{1}{1-e^{-4s}}\right]\int_0^4 e^{-st}f(t)dt$

$$= \frac{1}{1-e^{-4s}}\left[\int_0^2 3te^{-st}dt + \int_2^4 6e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-4s}}\left[3\left\{\left[t\left(\frac{e^{-st}}{-s}\right)\right]_0^2 - \int_0^2 1.\frac{e^{-st}}{-s}dt\right\} + 6\left(\frac{e^{-st}}{-s}\right)_2^4\right]$$

$$= \frac{1}{1-e^{-4s}}\left[\frac{3(1-e^{-2s}-2se^{-4s})}{s^2}\right].$$

Thus $L\{f(t)\} = \frac{3(1-e^{-2s}-2se^{-4s})}{s^2(1-e^{-4s})}$

ILATE

$t\frac{e^{-st}}{-s} - \int 1.\frac{e^{-st}}{-s}$

$-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\Big|_0^2$

2. $f(t) = t^2, 0 < t < 2$ and $f(t+2) = f(t)$ for $t > 2$. Find $L\{f(t)\}$.

Solution:

Here period of $f(t) = T = 2$

By definition $L\{f(t)\} = \left[\frac{1}{1-e^{-sT}}\right]\int_0^T e^{-st}f(t)dt = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}t^2dt$

Applying Bernoulli's rule of integration by parts,

$t^2\frac{e^{-st}}{-s} - \int 2t\frac{e^{-st}}{-s} \ldots$

$$L\{f(t)\} = \frac{1}{1-e^{-2s}}\left[t^2\frac{e^{-st}}{-s} - 2t\frac{e^{-st}}{(-s)^2} + 2\frac{e^{-st}}{(-s)^3}\right]_0^2$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{-4e^{-2s}}{-s} - \frac{4e^{-2s}}{s^2} - \frac{2e^{-2s}}{s^3} + \frac{2}{s^3}\right]$$

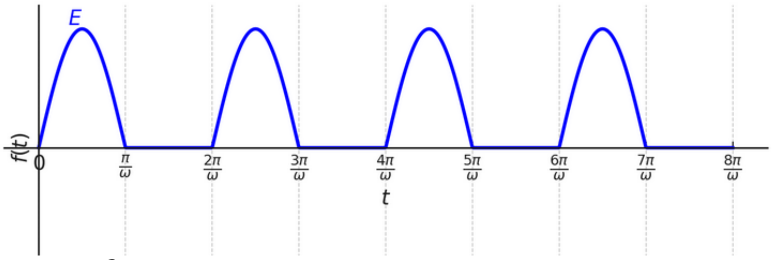
$$= \frac{2}{s^3(1-e^{-2s})}(1 - e^{-2s} - 2s^2e^{-2s} - 2se^{-2s})$$

$$= \frac{2}{s^3(1-e^{-2s})}[1 - e^{-2s}(1 + 2s + 2s^2)]$$

3. A periodic function of period $\frac{2\pi}{\omega}$ is defined by $f(t) = \begin{cases} E\sin\omega t, & 0 \le t < \frac{\pi}{\omega} \\ 0, & \frac{\pi}{\omega} \le t \le \frac{2\pi}{\omega} \end{cases}$

where E and ω are positive constants. Show that $L\{f(t)\} = \frac{E\omega}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$.

Solution:



Here $T = \frac{2\pi}{\omega}$.

Therefore $L\{f(t)\} = \frac{1}{1-e^{-s(2\pi/\omega)}}\int_0^{2\pi/\omega} e^{-st} f(t)dt$

$$= \frac{1}{1-e^{-s(2\pi/\omega)}}\int_0^{\pi/\omega} E e^{-st} \sin\omega t dt$$

$$= \frac{E}{1-e^{-s(2\pi/\omega)}}\left[\frac{e^{-st}}{s^2+\omega^2}\{-s\sin\omega t - \omega\cos\omega t\}\right]_0^{\pi/\omega}$$

$$= \frac{E}{1-e^{-s(2\pi/\omega)}}\frac{\omega(e^{-s\pi/\omega}+1)}{s^2+\omega^2}$$

$$= \frac{E\omega(1+e^{-s\pi/\omega})}{(1-e^{-s\pi/\omega})(1+e^{-s\pi/\omega})(s^2+\omega^2)}$$

$$= \frac{E\omega}{(1-e^{-s\pi/\omega})(s^2+\omega^2)}.$$

ILATE

$I = \sin\omega t \frac{e^{-st}}{-s} - \int \omega\cos\omega t \frac{e^{-st}}{-s}$

$+ \frac{\omega}{s}\left[\cos\omega t \frac{e^{-st}}{-s} + \int \omega \sin\omega t \frac{e^{-st}}{-s}\right]$

$I = X\sin\omega + \frac{\omega}{s}\left[-\cos\omega X - \frac{\omega I}{s}\right]$

$I = X\sin\omega t - \frac{\omega}{s}\cos\omega t X - \frac{\omega^2 I}{s^2}$

$I + \frac{\omega^2 I}{s^2} = X\left(\sin\omega t - \frac{\omega}{s}\cos\omega t\right)$

$\frac{(s^2+\omega^2)}{s^2} I = e^{-\omega t}\frac{s\sin\omega t - \omega\cos\omega t}{-s}$

$I = e^{-\omega t}(s\, s$

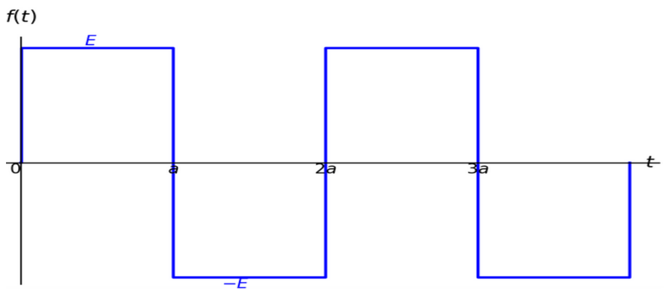
4. A periodic function $f(t)$ of period $2a,\ a > 0$ is defined by

$$f(t) = \begin{cases} E, & 0 < t \le a \\ -E, & a < t \le 2a. \end{cases}$$

a 2a

Show that $L\{f(t)\} = \frac{E}{s}\tanh\left(\frac{as}{2}\right)$.

Solution:



Here $T = 2a$. Therefore $L\{f(t)\} = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)dt$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a E e^{-st}dt + \int_a^{2a} -E e^{-st}dt\right]$$

$$= \frac{E}{s(1-e^{-2as})}[(1-e^{-sa}) + (e^{-2as} - e^{-as})]$$

$$= \frac{E}{s(1-e^{-2as})}[(1-e^{-as})^2] = \frac{E(1-e^{-as})^{\cancel{2}}}{s\cancel{(1-e^{-as})}(1+e^{-as})}$$

$$= \frac{E}{s}\left[\frac{e^{as/2}-e^{-as/2}}{e^{as/2}+e^{-as/2}}\right] = \frac{E}{s}\tanh\left(\frac{as}{2}\right).$$

Exercise:

1. Find $L\{f(t)\}$ of periodic function given $f(t) = \begin{cases} t & 0 \le t \le a \\ 2a - t & a < t \le 2a \end{cases}$ and $f(2a+t) = f(t)$.

2. Find $L\{f(t)\}$ for given function
$$f(t) = \begin{cases} 1, & 0 < t < \frac{a}{2} \\ -1, & \frac{a}{2} < t < a \end{cases}, \text{ where } f(2a+t) = f(t).$$

3. Show that the Laplace transform of the periodic function defined by $f(t) = \frac{Kt}{T}; 0 < t < T;$ $f(t+T) = f(t)$ is $-\frac{Ke^{-sT}}{s(1-e^{-sT})} + \frac{K}{s^2T}$.

Solution:

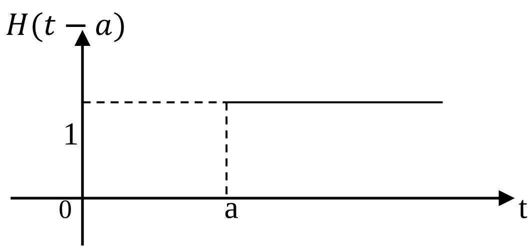
1. $\frac{1}{s^2}tanh\left(\frac{as}{2}\right)$ 2. $\frac{1}{s}tanh\left(\frac{as}{4}\right)$

Step function

In many engineering applications, we deal with an important discontinuous function $H(t-a)$ defined as follows :

$$\mathrm{H(t-a)} = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$$, where a is a non-negative constant.

The Heaviside function is also denoted by $u(t-a)$. The graph of the function is shown below:



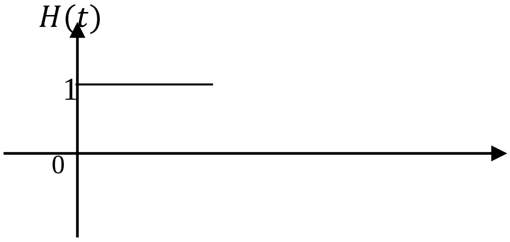
This function is known as the unit step function or the Heaviside function. The function is named after the British electrical engineer Oliver Heaviside (1850 –1925)

Oliver Heaviside was an English self-taught electrical engineer, mathematician and physicistwho adapted complex numbers to the study of electrical circuits, invented mathematical techniques for the solution of differential equations (equivalent to Laplace transforms), reformulated Maxwell's field equations in terms of electric and magnetic forces and energy flux, and independently co-formulated vector analysis.

Note that the value of the function suddenly jumps from value zero to the value 1 as $t \to a$ from the left and retains the value 1 for all $t > a$. Hence the function H(t-a) is called the unit step function.

In particular, when a = 0 the function $H(t-a)$ become H(t) where

$$\mathrm{H(t)} = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$



Transform of Heaviside unit step function

By definition $L\,[H\,(t-a)] = \int_0^{\infty} e^{-st} H(t-a)dt$

$= \int_0^{a} e^{-st} 0 dt + \int_a^{\infty} e^{-st}(1)dt$

$= \dfrac{e^{-as}}{s}$

In particular, $a = 0$, $L\,\{H\,(t)\} = \frac{1}{s}$

Heaviside shift theorem / t - shifting Property (second shifting Property)

If $L\{f(t)\} = F(s)$, then $L\,[f(t-a)\,H(t-a)] = e^{-as}\,F(s)$

Proof: We have

$$L\,[f\,(t-a)\,H\,(t-a)] = \int_0^{\infty} f(t-a)H(t-a)e^{-st}dt$$
$$= \int_a^{\infty} e^{-st} f(t-a)dt$$

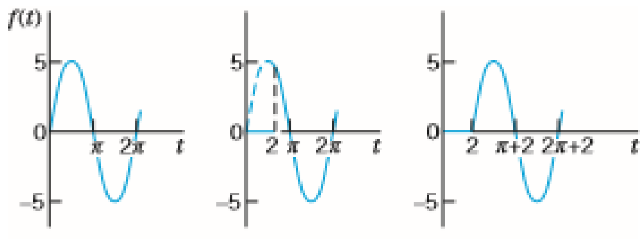
Setting $t - a = u$ then

$L\,[f(t-a)\,H(t-a)] = \int_0^{\infty} e^{-s(a+u)} f(u)du = e^{-as}\; L\{f(t)\} = e^{-as}\; F(s).$

Also $L^{-1}\,[e^{-as}\,L\,\{f(t)\}] = f(t-a)\,H(t-a)$

The unit step function is a typical ‘engineering function’ made to measure for engineering applications, which often involve functions (mechanical or electrical driving forces) that are either “off” or “on”. Functions $f(t)$ multiply with $u(t-a)$ produce all sorts of effects.

The simple basic idea is illustrated in the following figures.



(A) $f(t) = 5 \sin t$ (B) $f(t)u(t-2)$ (C) $f(t-2)u(t-2)$

In figures

(A) Given function $f(t) = 5 \sin t$

(B) It is switched off between $t = 0$ and $t = 2$ $(u(t-2) = 0$ when $t < 2)$ and switched on beginning at $t = 2$.

(C) It is shifted to the right by 2 units.

Examples:

1. Find $L[e^{t-2} + \sin(t-2)]\, H(t-2)$.

$L\left[f(t-2)\, H(t-2) \right] = e^{-2s} f(t)$

Solution:

Let $f(t-2) = e^{t-2} + \sin(t-2)$

Then $f(t) = [e^{t} + \sin t]$ so that $L\{f(t)\} = \frac{1}{s-1} + \frac{1}{s^2+1}$

$e^{-2s}\left(\frac{1}{s-1} + \frac{1}{s^2+1}\right)$

By Heaviside shift theorem $L[f(t-2)H(t-2) = e^{-2s}L\{f(t)\}$

Thus$L[e^{(t-2)} + sin(t-2)]H(t-2) = e^{-2s}\left[\frac{1}{s-1} + \frac{1}{s^2+1}\right]$

2. Find $L(3t^2 + 2t + 3)\, H(t-1)$.

Solution:

Let $f(t-1) = 3t^2 + 2t + 3$ so that $f(t) = 3(t+1)^2 + 2(t+1) + 3 = 3t^2 + 8t + 8$

Hence, $L\{f(t)\} = \frac{6}{s^3} + \frac{8}{s^2} + \frac{8}{s}$

Thus $L[3t^2 + 2t + 3]\, H(t-1) = L[f(t-1)\, H(t-1)] = e^{-s} f(t)$

$$= e^{-s}\left[\frac{6}{s^3} + \frac{8}{s^2} + \frac{8}{s}\right]$$

3. Find $L\{e^{-t}\, H(t-2)\}$.

Solution:

Let $f(t-2) = e^{-t}$ so that $f(t) = e^{-(t+2)}$

$e^{at} = \frac{1}{s-a}$

Thus $L\{f(t)\} = \frac{e^{-2}}{s+1}$

By shift theorem $L[f(t-2)H(t-2)] = e^{-2s}L\{f(t)\} = \frac{e^{-2(s+1)}}{s+1}$

Thus $L[e^{-t}H(t-2)] = \frac{e^{-2(s+1)}}{s+1}$

4. Find $L[\sin^3 t\, H(t-2\pi)]$

Solution:

Let $F(t-2\pi) = \sin^3 t$. Then $F(t) = \sin^3(t+2\pi) = \sin^3 t$.

$L\{F(t)\} = \frac{1}{4}\, L\{3\sin t - \sin 3t\} = \frac{1}{4}\left[\frac{3}{s^2+1} - \frac{3}{s^2+9}\right] = \frac{6}{(s^2+1)(s^2+9)}$.

Thus $L\{F(t-2\pi)H(t-2\pi)\} = e^{-2\pi s}L\{F(t) = \frac{6e^{-2\pi s}}{(s^2+1)(s^2+9)}$

5. Let $f(t) = \begin{cases} f1\ (t), & t \le a \\ f2\ (t), & t > a \end{cases}$

Verify that $f(t) = f_1(t) + [f_2(t) - f_1(t)]H(t-a)$.

Solution:

Consider $f_1(t) + [f_2(t) - f_1(t)]H(t-a) = \begin{cases} f_1(t) + f_2(t) - f_1(t), & t > a \\ f_1(t) + 0, & t \le a \end{cases}$

$$= \begin{cases} f_2(t), & t > a \\ f_1(t), & t \le a \end{cases}$$

6. Let

$$f(t) = \begin{cases} f1\ (t), & t \le a \\ f2\ (t), & a < t \le b \\ f3\ (t), & t > b \end{cases}$$

Verify that $f(t) = f_1(t) + [f_2(t) - f_1(t)]H(t-a) + [f_3(t) - f_2(t)]\, H(t-b)$

7. Express the following functions in terms of unit step function and hence find their Laplace transform.

(i). $f(t) = \begin{cases} t^2, & 1 < t \le 2 \\ 4t, & t > 2 \end{cases}$

Solution:

Here $f(t) = t^2 + (4t - t^2)\, H(t-2)$.

Hence $L\{f\ (t)\} = L(\ t^2) + L(4t - t^2)H(t-2)$ -- (i)

Let $\varphi(t-2) = 4t - t^2$ so that $\varphi(t) = 4(t+2) - (t+2)^2 = -t^2 + 4$

Now $L\{\varphi(t)\} = -\frac{2}{s^3} + \frac{4}{s}$

Expression (i) reduces to $L\{f(t)\} = \frac{2}{s^3} + L[\varphi(t-2)H(t-2)]$

$$= \left(\frac{2}{s^3}\right) + e^{-2s}L\{\varphi(t)\}$$

$$= \left(\frac{2}{s^3}\right) + e^{-2s}\left(\frac{4}{s} - \frac{2}{s^3}\right).$$

(ii). $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \sin t, & t > \pi \end{cases}$

Solution:

Here $f(t) = \cos t + (\sin t - \cos t)H(t-\pi)$.

Hence $L\{f(t)\} = \frac{s}{s^2+1} + L(\sin t - \cos t)H(t-\pi)$ -- (ii)

Let $\varphi(t-\pi) = \sin t - \cos t$

Then $\varphi(t) = \sin(t+\pi) - \cos(t+\pi) = -\sin t + \cos t$

So that $L\{\varphi(t)\} = -\frac{1}{s^2+1} + \frac{s}{s^2+1}$

Expression (ii) reduces to $L\{f(t)\} = \frac{s}{s^2+1} + L[\varphi(t-\pi)H(t-\pi)]$

$$= \frac{s}{s^2+1} + e^{-\pi s}L\{\varphi(t)\}$$

(iii). Express the following function in terms of unit step function and hence find its Laplace transform.

$$f(t) = \begin{cases} 1, & 0 < t \leq 1 \leftarrow a \\ t, & 1 < t \leq 2 \leftarrow b \\ t^2, & t > 2 \end{cases}$$

$1 + (t-1)u(t-1) + (t^2-t)u(t-2)$

Solution:

We know that $f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a) + [f_3(t) - f_2(t)]u(t-b)$

Therefore, $f(t) = 1 + (t-1)u(t-1) + (t^2-t)u(t-2)$

$L\{f(t)\} = L(1) + L\{(t-1)u(t-1)\} + L\{(t^2-t)u(t-2)\}$

$L(1) = \frac{1}{s}$

Consider $L\{(t-1)u(t-1)\}$

Let $F(t-1) = t-1$. Then $F(t) = t+1-1 = t$

$L\{F(t)\} = \frac{1}{s^2}$

$L\{F(t-1)u(t-1)\} = e^{-s}L\{F(t)\} = \frac{e^{-s}}{s^2}$.

Now consider $L\{(t^2-t)u(t-2)\}$

Let $G(t-2) = t^2 - t$. Then $G(t) = (t+2)^2 - (t+2) = t^2 + t + 2$

$L\{G(t)\} = \frac{2}{s^3} + \frac{1}{s^2} + \frac{2}{s}$

$L\{G(t-2)u(t-2)\} = e^{-2s}L\{G(t)\} = e^{-2s}\left[\frac{2}{s^3} + \frac{1}{s^2} + \frac{2}{s}\right]$.

Hence, $L\{f(t)\} = \frac{1}{s^2} + \frac{e^{-s}}{s^2} + e^{-2s}\left[\frac{2}{s^3} + \frac{1}{s^2} + \frac{2}{s}\right]$.

$$= \frac{s}{s^2+1} + e^{-\pi s}\left[\frac{s-1}{s^2+1}\right]$$

Exercise:

1. Find the Laplace transform of the following functions:

(i) $t^2\, H(t-3)$ (ii) $(e^{-t}\cos 2t)\, H(t-\pi)$

#

Answers:

(i) $\frac{e^{-3s}}{s^3}(9s^2+6s+2)$ (ii)$e^{-\pi(s+1)}\left[\frac{s+1}{s^2+2s+5}\right]$

2. Express the following functions in terms of unit step function and hence find their Laplace transform:

(i) $f(t) = \begin{cases} 2t, & 0 < t \le \pi \\ 1, & t > \pi \end{cases}$ (ii) $f(t) = \begin{cases} t^2, & 0 < t < 2 \\ 8t, & t > 2 \end{cases}$

(iii) $f(t) = \begin{cases} \pi - t, & 0 < t \le \pi \\ \sin t, & t > \pi \end{cases}$ (iv) $f(t) = \begin{cases} \cos t, & 0 < t \le \pi/2 \\ \sin t, & t > \pi/2 \end{cases}$

Answers:

(i) $\frac{2}{s^2} - e^{-\pi s}\left(\frac{1-2\pi}{s} - \frac{2}{s^2}\right)$ (ii) $\frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} - \frac{4}{s^2} - \frac{12}{s}\right)$

(iii) $\frac{\pi}{s} - \frac{1}{s^2} - e^{-\pi s}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right)$ (iv) $\frac{s}{s^2+1} + e^{\frac{-\pi}{2}s}\left(\frac{s+1}{s^2+1}\right)$

3. Express the following functions in terms of unit step function and hence find their Laplace transform.

(i) $f(t) = \begin{cases} \sin t, & 0 < t \le \pi \\ \sin 2t, & \pi < t \le 2\pi \\ \sin 3t, & t > 2\pi \end{cases}$

(ii) $f(t) = \begin{cases} t^2, & 0 < t \le 2 \\ 4, & 2 < t \le 4 \\ 0, & t > 4 \end{cases}$

Answer:

(i) $\frac{1}{s^2+1} + e^{-\pi s}\left(\frac{2}{s^2+4} + \frac{1}{s^2+1}\right) + e^{-2\pi s}\left(\frac{3}{s^2+9} - \frac{2}{s^2+4}\right)$

(ii)$\frac{2}{s^3} - \left(\frac{2}{s^3} + \frac{4}{s^2}\right)e^{-2s} - \frac{4}{s}e^{-4s}$

Unit Impulse function

Phenomena of an impulsive nature, such as the action of forces or voltages over short interval of time, arise in various applications, for instance, if a mechanical system is hit by a hammer blow, an airplane makes 'hard' landing, a ship is hit by a single high wave, hit a tennis ball by a racket and so on. These problems are modeled by '**Dirac's delta function**' (unit impulse function) $\delta(t)$ and efficiently solved by Laplace transform.

The unit impulse function may be regarded as the limiting form of the function

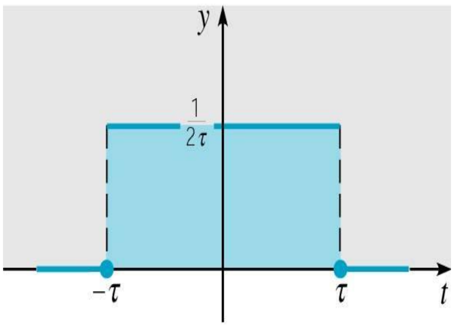
$$\delta_\varepsilon(t-a) = \begin{cases} \frac{1}{\varepsilon}, & a < t < a+\varepsilon \\ 0, & otherwise \end{cases} \quad \text{as } \varepsilon \to 0.$$

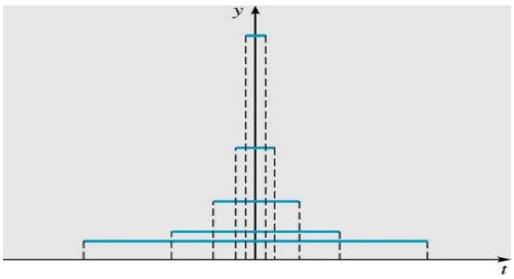
As $\varepsilon \to 0$, the width of the strip shown in fig becomes smaller and smaller, the height of the strip increases indefinitely in such a way that the area remains unity in $(\varepsilon, \frac{1}{\varepsilon})$.
Thus the unit impulse function or Dirac delta function $\delta(t-a)$ is defined as

$$\delta(t-a) = \begin{cases} \infty, & t = a \\ 0, & t \neq a. \end{cases}$$

Such that $\int_0^\infty \delta(t-a)dt = 1 \ (a \geq 0)$.

The unit impulse function $\delta(t)$ is defined to have the properties
$\delta(t) = 0$ for $t \neq 0$ and $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$
The unit impulse function is usually called the **Dirac delta function**.





Laplace Transform of Unit Impulse function

If $f(t)$ be a continuous function at $t = a$, then $\int_0^\infty f(t)\delta(t-a)dt = f(a)$
Proof:

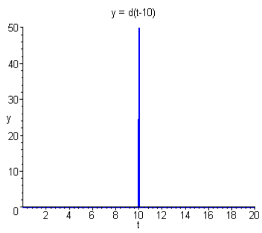
$$\int_0^\infty f(t)\delta_\varepsilon(t-a)dt = \int_a^{a+\varepsilon} f(t)\frac{1}{\varepsilon}dt = (a+\varepsilon-a)f(\tau)\frac{1}{\varepsilon} = f(\tau)$$

As $\varepsilon \to 0$, $\int_0^\infty f(t)\delta(t-a)dt = f(a)$

Taking $f(t) = e^{-st}$ we get, $\int_0^\infty e^{-st}\delta(t-a)dt = e^{-as}$

ie., Note: 1. $L[f(t)\delta(t-a)] = e^{-as}f(a)$.
2. $L[\delta(t-a)] = e^{-as}$.
In particular, if $a = 0$, $L[\delta(t)] = 1$.

For example when $a = 10$, $L[\delta(t-10)] = e^{-10s}$



Examples:

1. Find $L\{(t-1)^2\delta(t-a)\}$.

Solution:

Here $f(t) = (t-1)^2$ so that $f(a) = (a-1)^2 = a2 + 1 - 2a$

Thus $L\{(t-1)^2\delta(t-a)\} =$ e^{-as} f(a)

$= e^{-as}[a^2 + 1 - 2a]$

2. Find $L\left\{\sin t \ \delta\left(t-\frac{\pi}{2}\right)\right\}$.

Solution:

Here $f(t) = \sin t$ so that $f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

Thus $L\{\sin t \ \delta\left(t-\frac{\pi}{2}\right)\} = e^{-\pi s/2} f\left(\frac{\pi}{2}\right)$

$= e^{-\pi s/2}[1]$

3. Evaluate $\int_0^\infty t^m(\log t)^n \delta(t-3)$

Solution:

$f(t) = t^m(\log t)^n$ and $a = 3$

$f(a) = f(3) = 3^m(\log 3)^n$

Therefore $\int_0^\infty t^m(\log t)^n \delta(t-3) = 3^m(\log 3)^n$

Exercise:

Find the Laplace transform of the following functions:

(i) $t^2 \delta(t-3)$ (ii) $(e^{-t}\cos 2t)\delta(t - \pi)$. (iii) $\frac{e^{-t}+\log t}{t}\delta(t-3)$

Answer:

(i) $e^{-3s}(9)$ (ii) $e^{-\pi s}(e^{-\pi})$ (iii) $\frac{e^{-3s}}{3}(e^{-3} + \log 3)$

Video links:

https://www.youtube.com/watch?v=6MXMDrs6ZmA
https://www.youtube.com/watch?v=wnnnv4wt-Lw
https://www.youtube.com/watch?v=N-zd-T17uiE
https://www.youtube.com/watch?v=l7nzLD3t4Uc