Course name: Linear Algebra, Calculus & Numerical methods
Course code : MA211TA
Unit titles:
1) Elementary Linear Algebra
2) Differential Calculus
3) Multivariable Functions & partial differentiation
4) Multiple Integrals
5) Numerical methods

4/10/24

CH: ELEMENTARY LINEAR ALGEBRA

Submatrix of a Matrix ⟹
The Matrix obtained by removing some rows or columns from the given matrix is called submatrix of the given matrix.

ex: $A = \begin{bmatrix} 3 & 5 & 2 & 7 \\ 4 & 6 & 9 & 2 \\ 3 & 4 & 2 & 5 \end{bmatrix}$

$A_1 = \begin{bmatrix} 3 & 5 & 7 \\ 4 & 6 & 2 \\ 3 & 4 & 5 \end{bmatrix}$ $A_2 = \begin{bmatrix} 3 & 5 & 2 & 7 \\ 4 & 6 & 9 & 2 \end{bmatrix}$

etc are submatrices of A.

Minor of a Matrix ⇒

The determinant of square sub matrix of the given matrix A is called minor of the matrix.

ex: $A = \begin{bmatrix} 3 & 5 & 2 & 7 \\ 4 & 6 & 9 & 2 \\ 3 & 4 & 2 & 5 \end{bmatrix}$

⇒ $\begin{vmatrix} 3 & 5 & 2 \\ 4 & 6 & 9 \\ 3 & 4 & 2 \end{vmatrix}_{3\times 3}$, $\begin{vmatrix} 5 & 2 & 7 \\ 6 & 9 & 2 \\ 4 & 2 & 5 \end{vmatrix}_{3\times 3}$ etc are minors of order 3

⇒ $\begin{vmatrix} 3 & 5 \\ 4 & 6 \end{vmatrix}_{2\times 2}$, $\begin{vmatrix} 9 & 2 \\ 2 & 5 \end{vmatrix}_{2\times 2}$ etc are minors of order 2

⇒ $|3|$, $|5|$ etc are minors of order 1

Rank of a Matrix ⇒

The rank of Matrix A of order $m \times n$ is said to be 'r' if

i) there exists atleast one non-zero minor of order 'r'
ii) all the minors of order 'r+1' and above are zero

⇒ we denote the rank of A by $\boxed{\rho(A) = r}$

the order of highest order non-zero determinant or minor of the given matrix is called 'Rank of a matrix'.

Note :: i) Rank of a Null matrix is zero

ii) Rank of identity matrix of order 'n' is 'n'

iii) $\rho(A) = \rho(A^T)$

iv) if $A_{m \times n}$ is not a null matrix then $\rho(A) \leq \min(m, n)$

v) if A is a non singular matrix of order 'n' then $\rho(A) = n$

Elementary Transformations ⇒

1.) Interchange of any two rows →
interchange of i^{th} row & j^{th} row is denoted by $R_i \leftrightarrow R_j$

2.) Multiplication of any Row with a non-zero no →
multiplication of i^{th} row with a non-zero no is denoted by
$R_i \rightarrow KR_i$

3) Addition to the elements of any row the corresponding elements of other row multiplied by a non zero no →
The addition of K times j^{th} row to i^{th} row is denoted by
$R_i \rightarrow R_i + KR_j$

[note: elementary transformations do not change the rank of a matrix]

Echelon Form of a Matrix ⇒

A matrix '$A_{m \times n}$' is said to be in echelon form if

i) the leading element (pivotal element) of a non zero row must be non-zero

→ first non-zero element

ii) the no of zeroes before the first non-zero element in a non-zero row must increase with a Row no.

iii) The zero row (if exists) must be below the other non-zero rows

[note: 1) If the matrix `A' is in Echelon form, Rank of `A' =

$\rho(A)$ = The no of non-zero rows present in A

2) If leading element of each row of the echelon form of A is 1 then it is called the row reduced echelon form.

ex: $A = \begin{bmatrix} 2 & 5 & 6 & 0 & 0 & 2 \\ 0 & 4 & 3 & 5 & 6 & 7 \\ 0 & 0 & 0 & 2 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

`A' is in echelon form,

$\rho(A)$ = The no of non-zero rows = 4

ex: $A = \begin{bmatrix} 1 & 5 & 4 & 2 \\ 0 & 0 & 2 & 3 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ → can't be echelon form $\Rightarrow \begin{bmatrix} 1 & 5 & 4 & 2 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore R_2 \leftrightarrow R_3$

$\rho(A) = 3$

Solution of System of linear eqns $\Rightarrow$

consider a system of linear non-homogeneous equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad (1)$$

the above system (1) can be expressed in Matrix form as

$$AX = B$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \rightarrow$ coefficient matrix

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \rightarrow$ Solution matrix

$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \rightarrow$ Matrix of constants

Note: 1) $[A:B] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & : & b_2 \\ \vdots & & & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & : & b_m \end{bmatrix}$ is called Augumented matrix

2) the system of eqns is
(i) consistent if and only if it has a solution otherwise inconsistent
(ii)

3) the system of non-homogeneous linear eqns $AX = B$ is consistent if and only if rank of the coefficient matrix = Rank of augumented matrix

$$\boxed{\rho(A) = \rho(A:B)}$$

If all b_i's are zero, then the system $AX = B$ becomes $AX = O$ where 'O' is null matrix, such a system is called linear homogeneous eqns.

Working Rule:
To solve a non-homogeneous system $AX = B$, find $\rho(A)$, $\rho(A:B)$

(i) if $\rho(A) \neq \rho(A:B) \longrightarrow$ inconsistent (no solution) System

(ii) if $\rho(A) = \rho(A:B) = n$,
↪ no of unknowns
→ the system is consistent (unique solⁿ)

(iii) if $\rho(A) = \rho(A:B) < n$, → the system is consistent (∞ soln's)

Assuming $n-r$ variables ~~are~~ as arbitary constants we get ∞ no of solns

To solve the system of homogeneous eqns $AX=0$

i) find ~~rank~~ $\rho(A)$

↳ if $\rho(A)=n$, no of unknowns

↳ the system is consistent (always has a solution)

(trivial solⁿ) ($x_1=x_2=\dots=x_n=0$)

ii) $\rho(A)<n$ → the system has infinite solⁿ

Assuming $n-r$ as arbitary constants we get ∞ solns.

System of eqn's.

- $AX=B$
  - consistent ($\rho(A)=\rho(A:B)$)
    - unique $\rho(A)=\rho(AB)=n$
    - infinite $\rho(A)=\rho(AB)<n$
  - inconsistent $\rho(A)\neq\rho(AB)$
- $AX=0$
  - Trivial $\rho(A)=n$
  - non-trivial $\rho(A)<n$

3) $x-4y-7z=14$

$3x+8y-2z=13$

$7x-8y+28z=5$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & -4 & -7 & 14 \\ 3 & 8 & -2 & 13 \\ 7 & -8 & 28 & 5 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 7R_1$

$$\left[\begin{array}{ccc:c} 1 & -4 & -7 & 14 \\ 0 & 20 & 19 & -29 \\ 0 & 20 & 77 & -93 \end{array}\right] \Rightarrow \left[\begin{array}{ccc:c} 1 & -4 & -7 & 14 \\ 0 & 20 & 19 & -29 \\ 0 & 0 & 58 & -64 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$\rho(A:B) = 3$

$$A = \left[\begin{array}{ccc} 1 & -4 & -7 \\ 0 & 20 & 19 \\ 0 & 0 & 58 \end{array}\right] \begin{array}{c} : \cancel{14} \\ : \cancel{-29} \\ : \cancel{-64} \end{array}$$

$\rho(A) = 3$ $\qquad n = 3$

$\therefore \rho(A) = \rho(A:B) = n = 3$

the system is consistant and has a unique sol^n.

Q) $5x + 3y + 7z = 4$
$3x + 26y + 2z = 9$
$7x + 2y + 10z = 5$

$$\rho[A:B] = \left[\begin{array}{ccc:c} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right]$$

$R_1 \to 3R_1$
$R_2 \to 5R_2$
$R_3 \to 5R_3$

$$\left[\begin{array}{ccc:c} 15 & 9 & 21 & 12 \\ 15 & 130 & 10 & 45 \\ 35 & 10 & 50 & 25 \end{array}\right] \Rightarrow \left[\begin{array}{ccc:c} 15 & 9 & 21 & 12 \\ 0 & 121 & -11 & 33 \\ 0 & -11 & 1 & -3 \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - \frac{7}{3}R_1$

$R_3 \to R_3 + \frac{1}{11}R_2$

$$\left[\begin{array}{ccc:c} 15 & 9 & 21 & 12 \\ 0 & 121 & -11 & 33 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(AB) = 2$
$\rho(A) = 2$
$n = 3$

the system is consistent and has infinite solution.

Assuming one variable as arbitary constant
$\because n - r = 3 - 2 = 1$ → can be assumed as any Value

$$\begin{bmatrix} 15 & 9 & 21 \\ 0 & 121 & -11 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix}$$

$15x + 9y + 21z = 12$
$121y - 11z = 33$
let $\boxed{z = c}$ arbitary
$121y = 33 = 33 + 11z = 33 + 11c$
$$\boxed{y = \frac{3+c}{11}}$$

$15x + 9\left(\frac{3+c}{11}\right) + 21c = 12$

$15x + \frac{27}{11} + \frac{9c}{11} + 21c = 12$

$15x = 12 - \frac{27}{11} - \frac{9c}{11} - 21c = \frac{105}{11} - \frac{240c}{11} \Rightarrow \boxed{x = \frac{7}{11} - \frac{16c}{11}}$

Q) find the values of λ and μ such that the system of eqns

$$2x+3y+5z=9$$
$$7x+3y-2z=8$$
$$2x+3y+\lambda z=\mu$$

i) the system of eqns has no solⁿ
ii) unique solⁿ
iii) ∞ solution.

$6-21$ $\quad 16-63$ $\quad -4-35$

$$[A:B] = \left[\begin{array}{ccc:c} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{array}\right] \Rightarrow \left[\begin{array}{ccc:c} 2 & 3 & 5 & 9 \\ 0 & -15 & -39 & -47 \\ 0 & 0 & \lambda-5 & \mu-9 \end{array}\right]$$

$R_3 \to R_3 - R_1$

$R_2 \to 2R_2 - 7R_1$

(i) when $\lambda=5$ & $\mu \neq 9$

$\rho(A) = 2$ $\qquad \therefore \rho(A) \neq \rho(A:B)$

$\rho(A:B) = 3$

(ii) when $\lambda \neq 5$, any value of μ

$\rho(A)=3$ $\qquad \therefore \rho(A) = \rho(A:B) = n = 3$

$\rho(A:B) = 3$

(iii) when $\lambda=5$, $\mu=9$

$\rho(A)=2$ $\qquad \therefore \rho(A) = \rho(A:B) < n$

$\rho(A:B)=2$

$n=3$

Solving Homogeneous eqns.

$AX = 0$

$\rho(A) = n \rightarrow$ Trivial soln (zero)

$\rho(A) < n \rightarrow$ Non-trivial soln

Q) $x + 2y + 3z = 0$
$3x + 4y + 4z = 0$
$7x + 10y + 12z = 0$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 3R_1$
$R_3 \rightarrow R_3 - 7R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_2$

$\rho(A) = 3$
$n = 3$

the SOE have trivial soln
$x = 0, \; y = 0, \; z = 0$

Q) $4x_1 + 2x_2 + 3x_3 + 3x_4 = 0$
$6x_1 + 3x_2 + 4x_3 + 7x_4 = 0$
$2x_1 + x_2 + x_4 = 0$

$$A = \begin{bmatrix} 4 & 2 & 3 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

~~$R_1 \rightarrow R_1 - 2R$~~
$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 2 & 1 & 0 & 1 \\ 6 & 3 & 4 & 7 \\ 4 & 2 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & -8 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 3R_1$
$R_3 \rightarrow R_3 - 2R_1$

$R_3 \rightarrow 4R_3 - 3R_2$

$4 - 12$

$\rho(A) = 3$
$n = 4$

$n = \cancel{3}\,4$ $\quad n-r=1$ $\quad$ after simplifying consider which
$r=3$ $\quad\downarrow$ $\quad$ variable to take as c
take 1 arbitary constant $\quad$ don't assume $x_4 = c$

$$\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$2x_1 + x_2 + x_4 = 0$

$4x_3 + 4x_4 = 0$

$-8x_4 = 0$

$\boxed{x_4 = 0}$

$4x_3 = 0$

$\boxed{x_3 = 0}$

let $\boxed{x_2 = c}$ $\Rightarrow 2x_1 + c = 0$

$2x_1 = -c$

$\boxed{x_1 = -c/2}$

Q) determine the value of b such that the system of homogeneous eqns, has non-trivial soln, also find the soln.

$2x + y + 2z = 0$

$x + y + 3z = 0$

$4x + 3y + bz = 0$

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & b \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & b \end{bmatrix}$$

$R_2 \leftrightarrow R_1$ $\qquad$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & -1 & b-12 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & b-8 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

(quiz) in such cases, take $|A|=0$, find variable value then reduce matrix to echelon form

$\because$ since $A\cancel{X} = 0$

$\rho(A) < n \Rightarrow A$ is a singular matrix $\quad b=8$

$|A|=0$

Gauss Elimination method.

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$
$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

using ET to reduce 'A' to upper triangular matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

By back substitution, we get solution.

Gauss Jordan Method

In Gauss Jordan method we reduce the coefficient matrix 'A' to a diagonal matrix so that each equation gives the solution

Q)

Solve the following SOE using gauss Jordan method.

$x_1+3x_2-2x_3=7$
$x_1+2x_2-3x_3=10$
$2x_1-x_2+x_3=5$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 3 & -2 & 7 \\ 1 & 2 & -3 & 10 \\ 2 & -1 & 1 & 5 \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 2R_1$

$$\left[\begin{array}{ccc:c} 1 & 3 & -2 & 7 \\ 0 & -1 & -1 & 3 \\ 0 & -7 & 5 & -9 \end{array}\right] \Rightarrow \left[\begin{array}{ccc:c} 1 & 3 & -2 & 7 \\ 0 & -1 & -1 & 3 \\ 0 & 0 & 12 & -30 \end{array}\right]$$

$R_3 \to R_3 - 7R_2$

$R_3 \to R_3/12$ make this $1/-1$

$5+7$
$-9-21$

$$\left[\begin{array}{ccc:c} 1 & 3 & -2 & 7 \\ 0 & -1 & -1 & 3 \\ 0 & 0 & 1 & -30/12 \end{array}\right]$$ go reverse

$R_2 \to R_2 + R_3$
$R_1 \to R_1 + 2R_3$

$\frac{24}{12} + \frac{18}{12} = \frac{42}{12}$

$$\Rightarrow \left[\begin{array}{ccc:c} 1 & 3 & 0 & 24/12 \\ 0 & -1 & 0 & 6/12 \\ 0 & 0 & 1 & -30/12 \end{array}\right] = \left[\begin{array}{ccc:c} 1 & 0 & 0 & 42/12 \\ 0 & -1 & 0 & 6/12 \\ 0 & 0 & 1 & -30/12 \end{array}\right]$$

$R_1 \to R_1 + 3R_2$

$\therefore x_1 = 42/12 \quad x_2 = -6/12 \quad x_3 = -30/12$

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 5 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right] = \left[\begin{array}{ccc:c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$R_1 \to R_1 + R_2$
$R_2 \to R_2/-1$

$\therefore$ $\boxed{x = 2,\ y = 3,\ z = 4}$

Gauss - seidal method:
It is a Numerical method which gives approximate soln for linear equation.
This method is applicable for diagonally dominant system.

$a_{11}x + a_{12}y + a_{13}z = b_1$
$a_{21}x + a_{22}y + a_{23}z = b_2$
$a_{31}x + a_{32}y + a_{33}z = b_3$

$\left.\begin{array}{l} |a_{11}| \geq |a_{12}| + |a_{13}| \\ |a_{22}| \geq |a_{21}| + |a_{23}| \\ |a_{33}| \geq |a_{31}| + |a_{32}| \end{array}\right\}$ Diagonally dominant System

Q) using Gauss - seidel method find approx soln of SOE

$\left.\begin{array}{l} x + y + 54z = 110 \\ 27x + 6y - z = 85 \\ 6x + 15y + 2z = 72 \end{array}\right\}$ not diagonally dominant so rearrangions eqns.

perform 4 iterations.

$$X = \begin{array}{ccc} x^0 & y^0 & z^0 \\ \end{array}$$
$$X = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

(II) 1) ~~y=1~~ ~~z=1~~

$x_2 = 2.4322$

$\frac{1}{27}[85 - 6(3.5408) + 1.9132]$ (y_1, z_1)

2) $y_2 = \frac{1}{15}[72 - 6(\underset{x_2}{2.4322}) - 2(\underset{z_1}{1.9132})]$

$= 3.5720$

3) $z_2 = \frac{1}{54}[110 - (\underset{x_2}{2.4322}) - (\underset{y_2}{3.5720})]$

$= 1.9258$

(III)

1) $x_3 = 2.4257$

2) $y_3 = 3.5729$

3) $z_3 = 1.9260$

(IV) 1) $x_4 = 2.4255$

2) $y_4 = 3.5730$

3) $z_4 = 1.9260$

after 4 iterations $x = 2.4255$

$y = 3.5730$

$z = 1.9260$

Q) $x + 4y + 2z = 15$
$5x + 2y + z = 12$
$x + 2y + 5z = 20$

$$X^{0} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \begin{matrix} x^{0} \\ y^{0} \\ z^{0} \end{matrix}$$

$5x + 2y + z = 12$
$x + 4y + 2z = 15$
$x + 2y + 5z = 20$

$$x = [12 - 2y - z]\frac{1}{5}$$

$$y = [15 - x - 2z]\frac{1}{4}$$

$$z = [20 - x - 2y]\frac{1}{5}$$

(I)
$x^{1} = 1.8000$
$y^{1} = 1.8000$
$z^{1} = 2.9200$

(III) $x^{3} = 0.9987$
$y^{3} = 2.0131$
$z^{3} = 2.9950$

(V) $x^{5} = 0.9987$
$y^{5} = 2.0006$
$z^{5} = 3.000$

II $x^{2} = 1.0960$
$y^{2} = 2.0160$
$z^{2} = 2.9744$

(IV) $x^{4} = 0.9958$
$y^{4} = 2.0036$
$z^{4} = 2.9994$

H/W

1) $5x_1 - x_2 + x_3 = 10$

$2x_1 + 4x_2 = 0 \longrightarrow 2x + 4y + \overset{\substack{\text{zero} \\ \uparrow}}{0}M$

$x_1 + x_2 + 5x_3 = -1$

$X^{(0)} = [2 \quad 2 \quad -1]$

2) $5x_1 - x_2 = 9$

$-x_1 + 5x_2 - x_3 = 4$

$-x_2 + 5x_3 = 6$

Eigen Values and Eigen Vectors

CHARACTERISTIC EQN

Let 'A' be a 3X3 Sq. matrix then the eqn $|A - \lambda I| = 0$ is called characteristic eqn of the matrix A.

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - (\underbrace{\text{Trace of } A}_{a_{11}+a_{22}+a_{33}})\lambda^2 + [\text{sum of minors of } \underbrace{\quad}_{\text{diagonal elements}}]\lambda - \det A = 0$$

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Eigen Values / characteristic eqn / Latent roots

$|A-\lambda I|=0$

the roots of characteristic eqn are called eigen values.

Eigen Vector / characteristic Vector / latent Vector

in all zero vector X ($X \neq 0$) such that $AX = \lambda X$ or $(A-\lambda I)X=0$ is called an eigen vector of the matrix 'A' corresponding to the eigen value 'λ'

If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of the matrix 'A'

(i) $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$ are the eigen values of A^{-1}

(ii) $\lambda_1^m, \lambda_2^m, \lambda_3^m$ are the eigen values of A^m

(iii) $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of A^T [A same as A^T]

(iv) $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3} \rightarrow$ Adj A

(v) sum of the eigen values of A = Trace of A (sum of diagonal elements)

(vi) product of the eigen values of A = det A

(vi) if one of the eigen values of A is '0' then matrix A $\longrightarrow$ singular matrix

(vii) if 'λ' is an eigen value of 'A' then '$K\lambda$' is eigen value of 'KA'

(viii) if 'X' is the eigen vector of the matrix 'A' corresponding to the eigen value 'λ' then 'KX' is also the eigen vector corresponding to the same 'λ'.

(ix) eigen values of the diagonal matrix, upper triangular matrix, lower triangular matrix, ~~for~~ are same as principal diagonal elements.

ex: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ the eigen values are 1, 2, 3

Q) find the eigen values of the matrix $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

$|A - \lambda I| = 0$

$|A - \lambda I| = \begin{vmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{vmatrix} = 0$

$(8-\lambda)(2-\lambda) + 8 = 0$

$16 - 10\lambda + \lambda^2 + 8 = 0 \Rightarrow \lambda^2 - 10\lambda + 24 = 0$

$\lambda = 6, 4$ → eigen values of A

$(A - \lambda I) X = 0$

→ $\lambda = 6$

$\begin{bmatrix} 2 & -4 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$ $r = 1$, $n = 2$

$n - r = 1$ arbitrary constant

$2x - 4y = 0$

$y = c$, $x = 2c$.

$X_1 = \begin{bmatrix} c \\ 2c \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$

$\longrightarrow \lambda = 4$ $\quad (A-\lambda I)X = 0$

$$\begin{bmatrix} 4 & -4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$4x - 4y = 0$

$2x - 2y = 0$

$x - y = 0 \quad \Rightarrow x = y \quad X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Q) $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

opt. 1. to find characteristic eqn.

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

opt. 2.

$\lambda^3 - (1+5+1)\lambda^2 + (4-8+4)\lambda + 36 = 0$

$\lambda^3 - 7\lambda^2 + 36 = 0$

$\lambda = \underbrace{-2, 3, 6}_{\text{eigen Values}}$ (or) we have to find out one sol^n then

$\lambda = -2$
(trial & error)

$$\begin{array}{c|cccc} -2 & 1 & -7 & 0 & 36 \\ & & -2 & 18 & -36 \\ \hline & 1 & -9 & 18 & 0 \end{array}$$

$$\Rightarrow \lambda^2 - 9\lambda + 18 = 0$$
$$\lambda^2 - 6\lambda - 3\lambda + 18 = 0$$
$$\lambda = 6, 3.$$

$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Now to find eigen vector.

$\lambda = -2$ $= \begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$3x + y + 3z = 0$$
$$x + 7y + z = 0$$
$$3x + y + 3z = 0$$

take 2 non-identical eqns. (any 2)

$$\begin{array}{cccccc} 3 & 1 & 3 & 3 & 1 & 3 \\ 1 & 7 & 1 & 1 & 7 & 1 \end{array}$$

$$\frac{x}{1-21} = \frac{y}{3-3} = \frac{z}{21-1}$$

$$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -20 \\ 0 \\ 20 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} //$$

$\lambda = 3$.

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x + y + 3z = 0$$
$$x + 2y + z = 0$$
$$3x + y - 2z = 0$$

take any 2 non identical eqns

$$\begin{matrix} -2 & 1 & 3 & -2 & 1 & 3 \\ 1 & 2 & 1 & 1 & 2 & 1 \end{matrix}$$

$$\frac{x}{1-6} = \frac{y}{3+2} = \frac{z}{-4-1}$$

$$X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \\ -5 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} //$$

$\lambda = 6$

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-5x + y + 3z = 0$$
$$x - y + z = 0$$
$$3x + y - 5z = 0.$$

$$\begin{array}{cccccc} -5 & 1 & 3 & -5 & 1 & 3 \\ 1 & -1 & 1 & 1 & -1 & 1 \end{array}$$

$$\frac{x}{1+3} = \frac{y}{3+5} = \frac{z}{5-1}$$

$$X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$ //

$\therefore$ eigen values $= -2, 3, 6$

$\therefore$ corresponding eigen vectors are.

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Q) $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

$-2(-12) - 2(-6) - 3(-3)$

$24 + 12 + 9$

$$\lambda^3 - (-2+1+0)\lambda^2 + (-12-3-6)\lambda - (45) = 0$$

Rayleigh Power Method:

(even if -4 ÷ by 4)

It is a Numerical method used to find numerically largest eigen value (Dominant eigen value) and corresponding eigen vectors

let 'A' be a sq. matrix and $X^{(0)}$ be the initial eigen vectors

$$\left.\begin{aligned} AX^{(0)} &= \lambda^{(1)}X^{(1)} \\ AX^{(1)} &= \lambda^{(2)}X^{(2)} \end{aligned}\right\} \text{normalization.}$$

$$\vdots$$

continue the process till we get identical values of λ and the corresponding vector X. this process is called normalization. so final value of λ is the largest eigen value corresponding vector X is the eigen vector.

Note: If the initial vector is not mentioned in the question, you can choose the vector as

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Q) find the dominant eigen value & corresponding eigen vector using Rayleigh's power method ~~usin~~ for Matrix (5 iterations)

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad x^{0} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A x^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \lambda_1^{(1)} \overset{x^{(1)}}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}$$

$$A x^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = \overset{\lambda^{(2)}}{2} \overset{x^{(2)}}{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}$$

$$A x^{(2)} = \begin{bmatrix} 3 \\ -4 \\ 3 \end{bmatrix} = \overset{\lambda^{(3)}}{4} \overset{x^{(3)}}{\begin{bmatrix} 3/4 \\ -1 \\ 3/4 \end{bmatrix}} = 4 \begin{bmatrix} 0.75 \\ -1 \\ 0.75 \end{bmatrix}$$

$$A x^{(3)} = \begin{bmatrix} 2.5 \\ -3.5 \\ 2.5 \end{bmatrix} = \overset{\lambda^{(4)}}{3.5} \overset{x^{4}}{\begin{bmatrix} 0.7143 \\ -1 \\ 0.7143 \end{bmatrix}}$$

$$A x^{(4)} = \begin{bmatrix} 2.4286 \\ -3.4286 \\ 2.4286 \end{bmatrix} = \overset{\lambda_1^{(5)}}{3.4286} \overset{x^{(5)}}{\begin{bmatrix} 0.7083 \\ -1 \\ 0.7083 \end{bmatrix}}$$

~~$A x^{(5)}$~~ After 5 iterations, dominant eigen value $\lambda = 3.4286$

corresponding eigen vector $X = \begin{bmatrix} 0.7083 \\ -1 \\ 0.7083 \end{bmatrix}$

Q)

$$A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \qquad X^0 = [1 \; 0 \; 0]^T$$

$$X^0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$AX^{(0)} = \begin{bmatrix} 25 \\ 1 \\ 2 \end{bmatrix} = \overset{\lambda^{(1)}}{25} \overset{X^{(1)}}{\begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix}}$$

$$AX^{(1)} = \begin{bmatrix} 25.2 \\ 1.12 \\ 1.68 \end{bmatrix} = \overset{\lambda^{(2)}}{25.2} \overset{X^{(2)}}{\begin{bmatrix} 1 \\ 0.0444 \\ 0.0666 \end{bmatrix}}$$

$$AX^{(2)} = \begin{bmatrix} 25.1778 \\ 1.1333 \\ 1.7333 \end{bmatrix} = \overset{\lambda^{(3)}}{25.1778} \overset{X^{(3)}}{\begin{bmatrix} 1 \\ 0.045 \\ 0.0688 \end{bmatrix}}$$

$$AX^{(3)} = \begin{bmatrix} 25.1826 \\ 1.135 \\ 1.7246 \end{bmatrix} = 25.1826 \overset{X^4}{\begin{bmatrix} 1 \\ 0.045 \\ 0.0684 \end{bmatrix}}$$

$$AX^4 = \begin{bmatrix} 25.1821 \\ 1.1352 \\ 1.726 \end{bmatrix} = 25.1821 \overset{X^5}{\begin{bmatrix} 1 \\ 0.045 \\ 0.0685 \end{bmatrix}}$$

$$AX^5 = \begin{bmatrix} 25.1821 \\ 1.1352 \\ 1.7258 \end{bmatrix} = \overset{\lambda^6}{25.1821} \overset{X^6}{\begin{bmatrix} 1 \\ 0.045 \\ 0.0685 \end{bmatrix}}$$

After 6 iterations. dominant eigen value $\lambda = 25.1821$

eigen vector $X = [1 \; 0.045 \; 0.0685]$