UNIT - I

VECTOR DIFFERENTIATION

Topic Learning Objectives:

- Understand the existence of vector functions, derivatives of vector functions and rules of differentiation. Geometrical and physical interpretation of derivative of vector functions.
- The importance of defining vector differential operator ∇ and the operations- Gradient of scalar point functions, Divergence and Curl of vector point functions.
- Some important vector identities and applications.

Note: In all the vectors wherever i, j, k are used they have to be treated as unit vectors $\hat{i}, \hat{j}, \hat{k}$ along x, y, z directions respectively.

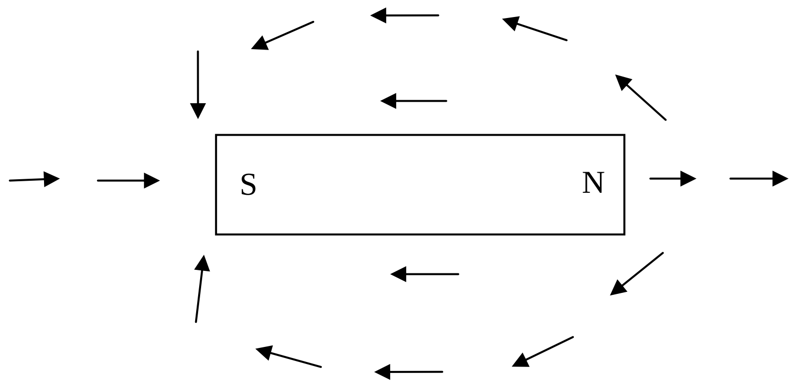
Vector calculus plays an important role in differential geometry and in the study of partial differential equations. Vector calculus originated in the 19th century in connection with the needs of mechanics and physics, when operations on vectors began to be performed directly, without their previous conversion to coordinate form. More advanced studies of the properties of mathematical and physical objects which are invariant with respect to the choice of coordinate systems led to a generalization of vector calculus. It is used extensively in physics and engineering, especially in the description of electromagnetic fields, gravitational fields and fluid flow.

Vector Fields:

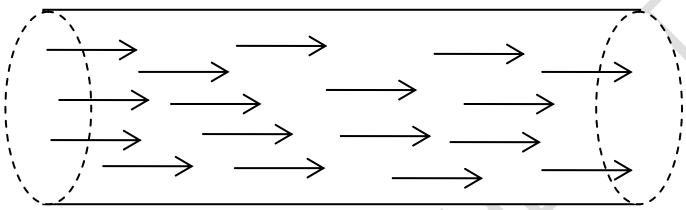
If at each point (x, y, z) there is an associated vector $\vec{v}(x,y,z) = v_1(x,y,z)i + v_2(x,y,z)j + v_3(x,y,z)k$, then $v(x,y,z)$ is a vector function and the field processing such a vector function is called a vector field.

Examples:

(i) A magnetic field B in a region of space, $B = B_1 i + B_2 j + B_2 k$



(ii) The velocity field of water flowing in a pipe, $v(x, y, z)$.



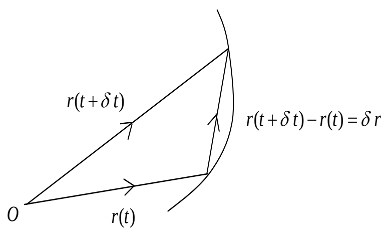
Vector function is a function whose domain is set of real numbers and whose range is a set of vectors.

Derivative of a Vector Function:

Let the position vector of a point P (x, y, z) in space be $\vec{r}(t) = x\,i + y\,j + z\,k$.

If *x, y, z* are all functions of *t*, then ***r*** is said to be a vector function of *t*. As the parameter *t* varies the point P traces a curve in space. Therefore $\vec{r}(t) = x(t)\,i + y(t)\,j + z(t)\,k$ is the vector equation of the curve, where $x(t), y(t)$ and $z(t)$ are real functions of the real variable t.

This function can be viewed as describing a space curve. Intuitively, it can be regarded as a position vector, expressed as a function of '*t*' that traces out a space curve with increasing values of *t*.



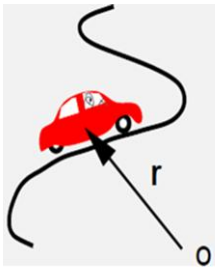
If $\vec{r}(t) = x(t)i + y(t)j + z(t)k$ is a vector function of a scalar variable t then the derivative of $\vec{r}(t)$ with respect to t is

$$\frac{d}{dt}\vec{r}(t) = \lim_{\delta t \to 0}\left[\frac{\vec{r}(t+\delta\, t) - \vec{r}(t)}{\delta\, t}\right]$$

$$= \lim_{\delta t \to 0}\frac{x(t+\delta\, t) - x(t)}{\delta\, t}i + \lim_{\delta t \to 0}\frac{y(t+\delta\, t) - y(t)}{\delta\, t}j + \lim_{\delta t \to 0}\frac{z(t+\delta\, t) - z(t)}{\delta\, t}k$$

$$= \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k$$

• For example, suppose you were driving along a wiggly road with position $r(t)$ at time t.



• Differentiating $r(t)$ should give velocity $v(t)$.

• Differentiating $v(t)$ should yield acceleration $a(t)$.

• Differentiating $a(t)$should yield the jerk $j(t)$

Velocity and Acceleration:

If $\vec{r}(t) = x(t)i + y(t)j + z(t)k$ is the position vector of a particle moving along a smooth curve in space, then $v(t) = \frac{d\vec{r}}{dt}$ is the particle's **velocity vector**, tangent to the curve. At any time t, the direction of ***v(t)*** is the **direction of motion**, the magnitude of ***v(t)*** is the particle's **speed**, and the derivative $a(t) = \frac{dv}{dt}$, when it exists, is the particle's **acceleration vector**.

In summary,

- Velocity is the derivative of position vector: $v(t) = \frac{d\vec{r}}{dt}$
- Speed is the magnitude of velocity: $speed = |v(t)|$
- Acceleration is the derivative of velocity: $a(t) = \frac{dv}{dt} = \frac{d^2\vec{r}}{dt^2}$

- Unit Tangent vector $\hat{T} = \frac{v(t)}{|v(t)|}$ is the direction of motion at time t.
- Component of velocity along a given vector $\vec{C}$ is $v(t) \cdot \hat{C}$
- Component of acceleration along a given vector $\vec{C}$ is $a(t) \cdot \hat{C}$

Differentiation rules for vector functions:

(1) $\frac{d}{dt}[\vec{a}(t) + \vec{b}(t)] = \frac{d}{dt}\vec{a}(t) + \frac{d}{dt}\vec{b}(t)$

(2) $\frac{d}{dt}[c\vec{a}(t)] = c\left[\frac{d}{dt}\vec{a}(t)\right]$, where c is constant

(3) $\frac{d}{dt}[\vec{a}(t) \cdot \vec{b}(t)] = \left[\frac{d}{dt}\vec{a}(t)\right] \cdot \vec{b}(t) + \vec{a}(t) \cdot \left[\frac{d}{dt}\vec{b}(t)\right]$

(4) $\frac{d}{dt}[\vec{a}(t) \times \vec{b}(t)] = \left[\frac{d}{dt}\vec{a}(t)\right] \times \vec{b}(t) + \vec{a}(t) \times \left[\frac{d}{dt}\vec{b}(t)\right]$

Examples:

1. A particle moves such that its position vector at time t is $\vec{r} = e^{-t}\vec{\imath} + 2\cos 3t\,\vec{\jmath} + 3\sin 3t\,\vec{k}$. Determine its velocity, acceleration and their magnitude, direction at time $t = 0$.

Solution: velocity : $\vec{v} = \frac{d\vec{r}}{dt} = -e^{-t}i - 6\sin 3t\,\hat{\jmath} + 9\cos 3t\,k$

$\vec{v}(0) = -i + 9\text{k}$, magnitude = $\sqrt{82}$, direction is $\frac{1}{\sqrt{82}}(-i + 9k)$

acceleration: $\vec{a} = \frac{d^2\vec{r}}{dt^2} = e^{-t}i - 18\cos3t\,\text{j} - 27\sin3t\,\text{k}$, $\vec{a}(0) = i - 18\text{j}$, magnitude $=\sqrt{325}$, direction is $\frac{1}{\sqrt{325}}(i - 18\text{j})$.

2. For the curves whose equations are given below, find the unit tangent vectors:

(i) $x = t^2 + 1, y = 4t - 3, z = 2(t^2 - 3t)$ at $t = 0$.

(ii) $\vec{r} = a\cos3t\,\text{i} + a\sin3t\,\text{j} + 4at\,\text{k}$ at $t = \frac{\pi}{4}$

Solution: (i) In the vector form equation of the given curve is

$$\vec{r} = x\,\text{i} + y\,\text{j} + z\,\text{k} = (t^2 + 1)\,\text{i} + (4t - 3)j + 2(t^2 - 3t)k$$

$$\Rightarrow \frac{d\vec{r}}{dt} = 2t\,\text{i} + 4\,\text{j} + 2(2t - 3)\,\text{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{4t^2 + 16 + (4t - 6)^2} = \sqrt{20t^2 + 52 - 48t} = 2\sqrt{5t^2 + 13 - 12t}$$

∴ Unit tangent vector to the given curve at a point '$\hat{n}$' is given by

$$\hat{n} = \frac{d\vec{r}/\mathrm{dt}}{|d\vec{r}/\mathrm{dt}|} = \frac{2[\mathrm{t\ i} + 2\ \mathrm{j} + (2\mathrm{t}-3)\ \mathrm{k}]}{2\sqrt{5\mathrm{t}^2 + 13 - 12\mathrm{t}}}$$

At t = 0, $\hat{n} = \frac{(2\mathrm{j}-3\mathrm{k})}{\sqrt{13}}$

(ii) $\vec{r} = \mathrm{a\ cos3t\ i + a\ sin3t\ j + 4at\ k}$

$$\frac{d\vec{r}}{\mathrm{dt}} = -3\ \mathrm{asin3t\ i} + 3\ \mathrm{acos3t}\ \vec{j} + 4\mathrm{a\ k}$$

$$\left|\frac{d\vec{r}}{\mathrm{dt}}\right| = \sqrt{9\mathrm{a}^2\sin^2 3\mathrm{t} + 9\mathrm{a}^2\cos^2 3\mathrm{t} + 16\mathrm{a}^2} = 5\mathrm{a}$$

$$\hat{n} = \frac{d\vec{r}/\mathrm{dt}}{|d\vec{r}/\mathrm{dt}|} = \frac{a[-3\sin 3\mathrm{t\ i} + 3\cos 3\mathrm{t\ j} + 4\ \mathrm{k}]}{5\mathrm{a}}$$

At t=$\frac{\pi}{4}$, $\frac{1}{5}\left[\frac{-3}{\sqrt{2}}i - \frac{-3}{\sqrt{2}}j + 4\ \mathrm{k}\right]$=$\frac{1}{5\sqrt{2}}\left[-3\hat{i} - 3\mathrm{j} + 4\sqrt{2}\hat{k}\right]$

3. Find the angle between the tangents to the curve $\vec{r} = t^2 i + 2tj - t^3 k$ at the points $t = \pm 1$.

Solution: $\frac{\mathrm{d}\vec{r}}{\mathrm{dt}} = 2\mathrm{t\ i} + 2\ \mathrm{j} - 3\mathrm{t}^2\ \mathrm{k}$

$$\left|\frac{d\vec{r}}{\mathrm{dt}}\right| = \sqrt{4\mathrm{t}^2 + 4 + 9\mathrm{t}^4}$$

$$\hat{n} = \frac{d\vec{r}/\mathrm{dt}}{|d\vec{r}/\mathrm{dt}|} = \frac{2\mathrm{t\ i} + 2\ \ - 3\mathrm{t}^2\ \mathrm{k}}{\sqrt{4\mathrm{t}^2 + 4 + 9\mathrm{t}^4}}$$

At $t = 1$,$\hat{n}_1 = \frac{2\ \mathrm{i}+2\mathrm{j}-3\mathrm{k}}{\sqrt{17}}$

At $t = -1$,$\hat{n}_2 = \frac{-2\ \mathrm{i}+2\mathrm{j}-3\mathrm{k}}{\sqrt{17}}$

Angle between unit tangent vectors at the points $t = \pm 1$ is given by

$$\cos\theta = \frac{\hat{n}_1 . \hat{n}_2}{\hat{n}_1 \hat{n}_2} = \frac{9}{17}$$

$\theta = cos^{-1}(9/17)$.

4. A particle moves along the curve $x = cos(t-1)$, $y = sin(t-1), z = at^3$ where a is a constant. Find a so that acceleration is perpendicular to position vectors at $t = 1$.

Solution: At time t position vector of particle is

$$\vec{r} = \cos(t-1)i + \sin(t-1)\,j + at^3k$$

$$\vec{v} = \frac{d\vec{r}}{dt} = -\sin(t-1)i + \cos(t-1)\,j + 3at^2\,k$$

$$[v]_{t=1} = j + 3ak$$

Acceleration $=\frac{d\vec{v}}{dt} = -\cos(t-1)\,i - \sin(t-1)\,j + 6at\,k$

$\frac{d\vec{v}}{dt}_{t=1} = -i + 6a\,k,\ \vec{r}_{t=1} = i + ak$

Given acceleration is perpendicular to position vector,

$$\vec{r}.\frac{d\vec{v}}{dt} = 0$$

$$\Rightarrow -1 + 6a^2 = 0 \Rightarrow a^2 = \frac{1}{6} \Rightarrow a = \pm 1/\sqrt{6}$$

5. A particle moves along the curve $\vec{r} = 2t^2\,i + (t^2-4t)\,j + (3t-5)\,k$. Find the component of velocity and acceleration in the direction of vector$c = i - 3j + 2k$ at $t = 1$.

Solution: Given $\vec{r} = 2t^2\,i + (t^2-4t)\,j + (3t-5)\,k$

Velocity $\vec{v} = \frac{d\vec{r}}{dt} = 4t\,i + (2t-4)\,j + 3\,k$

Acceleration$=\frac{d\vec{v}}{dt} = 4\,i + 2\,j$

At $t = 1$, $\vec{v} = 4\,i - 2\,j + 3\,k$

$$\frac{d\vec{v}}{dt} = 4\,i + 2\,j$$

Also $c = i - 3j + 2k$

$$|c| = \sqrt{14}$$

$$\hat{c} = \frac{\vec{c}}{|c|} = \frac{i - 3j + 2k}{\sqrt{14}}$$

$\therefore$ Component of velocity at $t = 1$along the given vector $\vec{c}$is,

$$\vec{v}.\hat{c} = \frac{1}{14}(4+6+6) = \frac{16}{\sqrt{14}}$$

$\therefore$Component of acceleration at$t = 1$along the given vector $\vec{c}$is,

$\frac{d\vec{v}}{dt}.\hat{c} = \frac{1}{14}(4-6) = \frac{-2}{\sqrt{14}}$.

6. A particle moves along the curve $x = (1 - t^3), y = (1 + t^2),\ z = (2t - 5)$ determine its velocity and acceleration. Also find the component s of velocity and acceleration at t=1, in the direction of $2\hat{\imath} + j + 2k$.

Solution: Let $\vec{r} = x(t)i + y(t)j + z(t)k$

$$\vec{r} = (1 - t^3)\hat{\imath} + (1 + t^2)j + (2t - 5)$$

$$v = \frac{dr}{dt} = -3t^2\hat{\imath} + 2tj + 2k$$

$$a = \frac{dv}{dt} = -6t\hat{\imath} + 2j$$

At *t=1*, $v = -3\hat{\imath} + 2j + 2k,\quad a = -6\hat{\imath} + 2j$

Let $\vec{D} = 2\hat{\imath} + j + 2k$ be the given direction.

$$\hat{n} = \frac{\vec{D}}{|\vec{D}|} = \frac{2\hat{\imath} + j + 2k}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{2\hat{\imath} + j + 2k}{3}$$

Component of velocity along the given direction $\vec{D}$ is

$$\vec{v}.\hat{n} = (-3\hat{\imath} + 2j + 2k).\frac{2\hat{\imath} + j + 2k}{3} = \frac{-6 + 2 + 4}{3} = 0$$

Component of acceleration along the given direction $\vec{D}$ is

$$\vec{a}.\hat{n} = (-6\hat{\imath} + 2j).\frac{2\hat{\imath} + j + 2k}{3} = \frac{-12 + 2}{3} = \frac{-10}{3}$$

Exercise:

1. A person on a hang glider is spiralling upward due to rapidly rising air on a path having position vector $r(t) = (3\cos t)i + (3\sin t)j + t^2\ k$. Find

(a) The velocity and acceleration vectors

(b) The glider's speed at any time *t*.

2. Given the curve $x = t^2 + 2, y = 4t - 5, z = 2t^2 - 6t$ find the unit tangent vector at the point $t = 2$.

Answers

1. $\vec{v} = -(3\sin t)i + (3\cos t)j + 2t\ k;\ \vec{a} = (-3cost)i + (-3sint)j + 2k;\quad |\vec{v}| = \sqrt{9 + 4t^2}$

2. $\frac{2i+2j+k}{3}$

Scalar and Vector Point Functions:

A physical quantity that can be expressed as a continuous function and which can assume definite values at each point of a region of space is called a point function in that region, and the region containing the point function is called a field.

There are two types of point functions namely **scalar point function** and **vector point function**.

Scalar point function:

At each point (x, y, z) of a region R in space if there corresponds a definite scalar $\phi(x, y, z)$, then such a function $\phi(x, y, z)$ is called a scalar point function and the region is called a scalar field.

Examples: Functions representing the temperature, density of a body, gravitational potential etc. are scalar point functions.

Vector point function: At each point (x, y, z) of a region R in space if there corresponds a definite vector $\overrightarrow{f}(x, y, z) = f_1(x, y, z)\mathrm{i} + f_2(x, y, z)j + f_3(x, y, z)k$, then such a function $\overrightarrow{f}(x, y, z)$ is called a vector point function, the region is called a vector field.

Examples: Functions representing the velocity of moving fluid particle, gravitational force, etc. are vector point functions.

Level surface: The scalar point function $\phi(x, y, z)$ is usually called the potential function and $\phi(x, y, z) = c$ represents the family of surfaces in the scalar field. If at each point on the surface, $\phi(x, y, z) = c$ has the same value then the surface is called the level surface.

Definition: The vector differential operator denoted by ∇ read as **del** or **nabla** is defined by $\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$ is called **vector differential operator**. This operator has no meaning on itself but assumes specific meaning depending on how it operates on a scalar or vector point function.

Gradient of a scalar point function: Let $\phi(x, y, z)$ be any scalar point function defined at some point (x, y, z) of a scalar field so that the function is continuously differentiable. Then the vector function $\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$ is called a gradient of scalar function t $\phi(x, y, z)$ and it is denoted by $\nabla\phi$ or $grad\ \phi$. Thus $grad\ \phi = \nabla\phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$

Note:

1. If ϕ is a scalar point function, then $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}$ are called components of $grad\ \phi$.

2. $\nabla\phi = \sqrt{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2}$ is called the magnitude of $grad\ \phi$.

Geometrical interpretation of gradient: $grad\ \phi$ is a vector normal to the surface $\phi = constant$ and has a magnitude equal to the rate of change of *f* along this normal.

Properties of Gradient

1. The differential $d\phi$ of ϕ is given by $d\phi = \nabla\phi . dr$ where $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$
2. For any scalar function ϕ and ψ and any scalar α and β $\nabla(\alpha\ \phi \pm \beta\ \psi) = \alpha\ \nabla\phi \pm \beta\ (\nabla\psi)$
3. For any scalar function φ and ψ

i) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\ \nabla\phi$ ii) $\nabla\left(\frac{\phi}{\psi}\right) = \frac{(\psi\ \nabla\phi - \phi\ \nabla\psi)}{\psi^2}$ if $\psi \neq 0$

Unit normal vector:

Since $\nabla\varphi$ is normal vector to surface $\phi(x, y, z) = c$ then unit vector is denoted by $\hat{n}$ and is defined as, $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\nabla\phi}{|\nabla\phi|}$ where $\hat{n} = \nabla\phi$ = normal vector.

Note: The angle between the normal to the surfaces is given by $\cos\theta = \widehat{n_1}\widehat{n_2}$.

Directional derivative:

If $\vec{a}$is any vector incline at an angle θ to the direction of $\nabla\varphi$ where $\varphi$is scalar point function then

$$\nabla\phi . \hat{a} = \left(\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k\right) \cdot \frac{(a_1 i + a_2 j + a_3 k)}{|\vec{a}|} = a_1 \frac{\partial \phi}{\partial x} + a_2 \frac{\partial \phi}{\partial y} + a_3 \frac{\partial \phi}{\partial z}.$$

It represents component of $\nabla\varphi$ in the direction of $\vec{a}$ which is known as directional derivative of φ in the direction of $\vec{a}$.

Maximum Directional Derivative:

The direction derivative will be maximum in the direction of $\nabla\varphi$ $(\vec{a} = \nabla\phi)$ and maximum value of the directional derivative $= \frac{\nabla\phi.\nabla\phi}{|\nabla\phi|} = \frac{|\nabla\phi|^2}{|\nabla\phi|} = |\nabla\phi|$

Maximum directional derivative is also called normal derivative.

$\therefore$ normal derivative $= |\nabla\phi|$

Problems:

1. If $\phi(x,y,z) = \log(x^2+y^2+z^2)$ then find $\nabla\phi$ and $|\nabla\phi|$ $at(1,2,1)$.

Solution: Given $\phi(x,y,z) = \log(x^2+y^2+z^2)$

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial x}j + \frac{\partial\phi}{\partial x}k = i\left[\frac{2x}{x^2+y^2+z^2}\right] + j\left[\frac{2y}{x^2+y^2+z^2}\right] + k\left[\frac{2z}{x^2+y^2+z^2}\right]$$

$$= (xi+yj+zk)\ \frac{2}{x^2+y^2+z^2}$$

At (1,2,1), $\nabla\phi = \frac{1}{3}(i+2j+k)$, $|\nabla\phi| = \frac{\sqrt{1^2+2^2+1^2}}{3} = \sqrt{\frac{2}{3}}$

2. If $\phi(x,y,z) = xy^2z^3 - x^3y^2z$ then find $\nabla\phi$ and $|\nabla\phi|$ at (1, -1, 1).

Solution: Given $\phi(x,y,z) = xy^2z^3 - x^3y^2z$

$$\frac{\partial\phi}{\partial x} = y^2z^3 - 3x^2y^2z, \frac{\partial\phi}{\partial y} = 2xyz^3 - 2x^3yz, \frac{\partial\phi}{\partial z} = 3xy^2z^3 - x^3y^2$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k = (y^2z^3 - 3x^2y^2z)i + (2xyz^3 - 2x^3yz)j + (3xy^2z^3 - x^3y^2)k$$

At $(1,-1,1), \nabla\phi = -2i + 0j + 2k$

$|\nabla\phi| = \sqrt{(-2)^2+0^2+2^2} = \sqrt{8} = 2\sqrt{2}$.

3. Find the directional derivative of $\phi(x,y,z) = x^2yz + xz^2$ at the point $(1,-2,1)$ in the direction of $2i - j + 2k$.

Solution: $\phi(x,y,z) = x^2yz + xz^2$

$$\text{grad}\phi = \nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k = (2xyz + z^2)i + (x^2z)j + (x^2y + 2xz)k$$

At $(1,-2,1), \nabla\phi = -3i + j + 0k$

Given $\vec{a} = (2i - j + 2k) \Rightarrow |\vec{a}| = 3$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{3}(2i - j + 2k)$$

The directional derivative of $\phi(x,y,z)$ is $\nabla\emptyset \cdot \hat{a} = (-3i + j + 0k) \cdot \frac{1}{3}(2i - j + 2k)$

The directional derivative $\nabla\emptyset \cdot \hat{a} = \frac{1}{3}(-6 - 1 + 0) = -\frac{7}{3}$.

4. Find the directional derivative of $\phi(x,y,z) = xy^2 + yz^3$ at $(2,-1,1)$ in the direction $2i + j + 2k$

Solution: $\phi(x,y,z) = xy^2 + yz^3$

$$\text{grad}\phi = \nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k = (y^2)i + (2xy + z^3)j + (3yz^2)k$$

At $(2,-1,1)$ $\nabla\phi = i - 3j - 3k$

Given $\vec{a} = (2i + j + 2k) \Rightarrow |\vec{a}| = 3$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{3}(2i + j + 2k)$$

The directional derivative of $\phi(x,y,z)$ is $\nabla\emptyset \cdot \hat{a} = (i - 3j - 3k) \cdot \frac{1}{3}(2i + j + 2k)$

The directional derivative $\nabla\emptyset \cdot \hat{a} = \frac{1}{3}(2 - 3 - 6) = -\frac{7}{3}$.

5. Find the directional derivative of $\phi(x,y,z) = x^4 + y^4 + z^4$ at the point $(-1,2,3)$ in the direction towards the point $(2,-1,-1)$.

Solution: Given $\phi(x,y,z) = x^4 + y^4 + z^4$

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k = 4x^3i + 4y^3j + 4z^3k$$

At $(-1,2,3)$ $\nabla\phi = -4i + 32j + 108k$

Let $P = (-1,2,3)$ and $Q = (2,-1,-1)$

$\vec{a} = \overrightarrow{OQ} - \overrightarrow{OP} = (3i - 3j - 4k) \Rightarrow |\vec{a}| = \sqrt{34}$

The directional derivative is $\nabla\emptyset \cdot \hat{a} = (-4i + 32j + 108k) \cdot \frac{1}{\sqrt{34}}(3i - 3j - 4k)$

The directional derivative $\nabla\emptyset \cdot \hat{a} = \frac{1}{\sqrt{34}}(-12 - 96 - 432) = -\frac{540}{\sqrt{34}}$.

6. Find the directional derivative of $\phi(x,y,z) = \frac{1}{x^2+y^2+z^2}$ at the point $(1,2,-3)$ in the direction of $2i-3j+k$.

Solution: Let $\phi(x,y,z) = \frac{1}{x^2+y^2+z^2}$

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k = \frac{-2x}{(x^2+y^2+z^2)^2}i + \frac{-2y}{(x^2+y^2+z^2)^2}j + \frac{-2z}{(x^2+y^2+z^2)^2}k$$

At the point $(1,2,-3)\nabla\phi = \frac{-2}{(14)^2}i + \frac{-4}{(14)^2}j + \frac{6}{(14)^2}k \Rightarrow \nabla\phi = \frac{-2}{(14)^2}[i+2j-3k]$

$\vec{a} = (2i-3j+k) \Rightarrow |\vec{a}| = \sqrt{14},$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{14}}(2i-3j+k)$$

The directional derivative $\nabla\emptyset \cdot \hat{a} = \frac{-2}{(14)^2}(i+2j-3k) \cdot \frac{1}{\sqrt{14}}(2i-3j+k)$

The directional derivative $\nabla\emptyset \cdot \hat{a} = \frac{-2}{(14)^2\sqrt{14}}(2-6-3) = \frac{1}{14\sqrt{14}}.$

7. Find the maximum directional derivative of $\phi(x,y,z) = x^3y^2z$ at $(1,-2,3)$.

Solution: Given $\phi(x,y,z) = x^3y^2z$

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial x}j + \frac{\partial\phi}{\partial x}k = (3x^2y^2z)i + (2x^3yz)j + (x^3y^2)k$$

At $(1,-2,3)$ $\nabla\phi = 36i - 12j + 4k$

The maximum directional derivative $= |\nabla\phi| = 4\sqrt{(9^2)+(-3)^2+1^2} = 4\sqrt{91}$

8. Find the maximum directional derivative of $\phi(x,y,z) = x^2y + yz^2 - xz^3$ at $(-1,2,1)$.

Solution: Given $\phi(x,y,z) = x^2y + yz^2 - xz^3$

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k = (2xy - z^3)i + (x^2+z^2)\cdot j + (2yz - 3xz^2)k$$

At $(-1,2,1)\nabla\phi = -5i + 2j + 7k$

The maximum directional derivative $= |\nabla\phi| = \sqrt{(-5)^2+2^2+7^2} = \sqrt{78}$

9. Find the unit normal vector to the surface $x^2 - y^2 + z = 3$ at $(1,0,2)$.

Solution: Let $\phi(x,y,z) = x^2 - y^2 + z$

$\nabla\phi = 2xi - 2yj + k$

At $(1,0,2)$ $\nabla\phi = 2i + k$

$|\nabla\phi| = \sqrt{2^2+1^2} = \sqrt{5}$

The unit normal vector $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{1}{\sqrt{5}}(2i + k)$

10. Find the unit normal vector to the surface $\phi(x, y, z) = x^2y + y^2z + z^2x = 5$ at the point (1, -1, 2).

Solution: $\phi(x, y, z) = x^2y + y^2z + z^2x$

$\nabla\phi = (2xy + z^2)\hat{\imath} + (x^2 + 2yz)\hat{\jmath} + (y^2 + 2xz)\hat{k}$

At $(1, -1, 2)$

$\nabla\phi = 2i - 3j + 5k$ and $|\nabla\phi| = \sqrt{2^2 + (-3)^2 + 5^2} = \sqrt{38}$

The unit normal vector $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{1}{\sqrt{38}}(2i - 3j + 5k)$

11. Find the angle between the normals to the surface $2x^2 + 3y^2 = 5z$ at the points $(2, -2, 4)$ and (-1, -1, 1).

Solution: Let $\phi(x, y, z) = 2x^2 + 3y^2 - 5z$

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

$\therefore \nabla\phi = 4xi + 6yj - 5k$

Now at $(2, -2, 4), \nabla\phi_1 = 8i - 12j - 5k \Rightarrow |\nabla\phi_1| = \sqrt{8^2 + (-12)^2 + (-5)^2} = \sqrt{233}$

At $(-1, -1, 1), \nabla\phi_2 = -4i - 6j - 5k \Rightarrow |\nabla\phi_2| = \sqrt{(-4)^2 + (-6)^2 + (-5)^2} = \sqrt{77}$

Unit normal vector to the surface at $(2, -2, 4)$ is $\hat{n_1} = \frac{\nabla\phi_1}{|\nabla\phi_1|} = \frac{1}{\sqrt{233}}(8i - 12j - 5k)$

Unit normal vector to the surface at $(-1, -1, 1)$ is $\hat{n_2} = \frac{\nabla\phi_2}{|\nabla\phi_2|} = \frac{1}{\sqrt{77}}(-4i - 6j - 5k)$

Angle between the normals is given by $\cos\theta = \hat{n_1} \cdot \hat{n_2}$

$$\cos\theta = \frac{1}{\sqrt{233}}(8i - 12j - 5k) \cdot \frac{1}{\sqrt{77}}(-4i - 6j - 5k) = \frac{1}{\sqrt{17941}}(-32 + 72 + 25)$$

$$= \frac{65}{\sqrt{17941}}$$

$\theta = \cos^{-1}\left(\frac{65}{\sqrt{17941}}\right)$ is the angle between the normals.

12. Find the angle between the normals to the surface $xy = z^2$ at the points $(1, 9, -3)$ and $(-2, -2, 2)$.

Solution: Let $\phi(x,y,z) = xy - z^2$

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

$$\nabla\phi = yi + xj - 2zk$$

Now at $(1,9,-3)$ $\nabla\phi_1 = 9i + j + 6k \Rightarrow |\nabla\phi_1| = \sqrt{9^2 + 1^2 + 6^2} = \sqrt{118}$

$$\therefore \hat{n_1} = \frac{\nabla\phi_1}{|\nabla\phi_1|} = \frac{1}{\sqrt{118}}(9i + j + 6k)$$

At $(-2,-2,2), \nabla\phi_2 = -2i - 2j - 4k \Rightarrow |\nabla\phi_2| = \sqrt{(-2)^2 + (-2)^2 + (-4)^2} = \sqrt{24}$

$$\therefore \hat{n_2} = \frac{\nabla\phi_2}{|\nabla\phi_2|} = \frac{1}{\sqrt{24}}(-2i - 2j - 4k)$$

Angle between the normal is $\cos\theta = \hat{n_1} \cdot \hat{n_2}$

$$\cos\theta = \frac{1}{\sqrt{118}}(9i + j + 6k) \cdot \frac{1}{\sqrt{24}} \cdot (-2i - 2j - 4k) = \frac{1}{\sqrt{2832}}(-18 - 2 - 24) = \frac{-44}{4\sqrt{117}}$$

$$= \frac{-11}{\sqrt{117}}$$

Hence the acute angle $\theta = \cos^{-1}\left(\frac{11}{\sqrt{117}}\right)$.

13. Find the angle between the surfaces $x^2 + y^2 + z^2 = 16$ and $x^2 + y^2 - z = 4$ at the point $(2,-1,2)$ common to them.

Solution: The angle between the two surfaces at common point is angle between the normals drawn to the surfaces at that point.

Let $\phi_1(x,y,z) = x^2 + y^2 + z^2, \nabla\phi_1 = 2xi + 2yj + 2zk$

At $(2,-1,2)$ $\nabla\phi_1 = 4i - 2j + 4k \Rightarrow |\nabla\phi_1| = \sqrt{4^2 + (-2)^2 + 4^2} = 6$

Now $\hat{n_1} = \frac{\nabla\phi_1}{|\nabla\phi_1|} = \frac{1}{6}(4i - 2j + 4k)$

Let $\phi_2(x,y,z) = x^2 + y^2 - z, \nabla\phi_2 = 2xi + 2yj - k$

At $(2,-1,2)$ $\nabla\phi_2 = 4i - 2j - k \Rightarrow |\nabla\phi_2| = \sqrt{4^2 + (-2)^2 + (-1)^2} = \sqrt{21}$

Now $\hat{n_1} = \frac{\nabla\phi_2}{|\nabla\phi_2|} = \frac{1}{\sqrt{21}}(4i - 2j - k)$

Angle between the normals is $\cos\theta = \hat{n_1} \cdot \hat{n_2}$

$$\cos\theta = \frac{1}{6}(4i - 2j + 4k) \cdot \frac{1}{\sqrt{21}} \cdot (4i - 2j - k) = \frac{1}{6 \cdot \sqrt{21}}(16 + 4 - 4) = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$

14. Find the angle between the surface $xlogz = y^2 - 1$ and $x^2y = 2 - z$ at the point (1,1,1)

Solution: Let $\phi_1 = y^2 - x\log z - 1, \nabla\phi_1 = \frac{\partial\phi_1}{\partial x}i + \frac{\partial\phi_1}{\partial x}j + \frac{\partial\phi_1}{\partial x}k = (-\log z)i + 2yj - \frac{x}{z}k$

$\nabla\phi_1(1,1,1) = 2j - k, |\nabla\phi_1| = \sqrt{5}$

$\widehat{n_1} = \frac{\nabla\phi_1}{|\nabla\phi_1|} = \frac{2j - k}{\sqrt{5}}$

Let $\phi_2 = x^2y - 2 + z, \nabla\phi_2 = \frac{\partial\phi_2}{\partial x}i + \frac{\partial\phi_2}{\partial x}j + \frac{\partial\phi_2}{\partial x}k = (2xy)i + (x^2)j + k$

$\nabla\phi_2(1,1,1) = 2i + j + k, |\nabla\phi_2| = \sqrt{6}$

$\widehat{n_2} = \frac{\nabla\phi_2}{|\nabla\phi_2|} = \frac{2i + j + k}{\sqrt{6}}$

$\cos\theta = \frac{1}{\sqrt{30}}, \theta = \cos^{-1}\left(\frac{1}{\sqrt{30}}\right)$

15. Find whether the surfaces $4x^2 - z^3 = 4$ and $5x^2 - 2yz = 7x$ intersect orthogonally at the point (1,-1,-2).

Solution: Let $\phi_1(x,y,z) = 4x^2 - z^3 - 4, \nabla\phi_1 = 8xi + 0J - 3z^2\mathrm{k}$

At $(1,-1,-2), \nabla\phi_1 = 8i + 0j - 12k \Rightarrow |\nabla\phi_1| = \sqrt{64 + 144} = \sqrt{208}$

$\hat{n}_1 = \frac{\nabla\phi_1}{|\nabla\phi_1|} = \frac{1}{\sqrt{208}}(8i + 0j - 12k)$

$\phi_2(x,y,z) = 5x^2 - 7x - 2yz, \nabla\phi_2 = (10x - 7)i - 2zj - 2yk$

At $(1,-1,-2), \nabla\phi_2 = 3i + 4j + 2k \Rightarrow |\nabla\phi_2| = \sqrt{3^2 + 4^2 + 2^2} = \sqrt{29}$

$\hat{n_2} = \frac{\nabla\phi_2}{|\nabla\phi_2|} = \frac{3i + 4j + 2k}{\sqrt{29}}$

Angle between two normals is $\cos\theta = \hat{n_1} \cdot \hat{n_2}$

$\cos\theta = \frac{1}{\sqrt{208}}(8i + 0j - 12k) \cdot \frac{1}{\sqrt{29}}(3i + 4j + 2k) = \frac{1}{\sqrt{6032}}(24 + 0 - 24) = 0$

$\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

Therefore the surfaces intersect orthogonally.

16. Find the constants a and b so that the surface $3x^2 - 2y^2 - 3z^2 + 8 = 0$ is orthogonal to the surface $ax^2 + y^2 = bz$ at the point $(-1,2,1)$.

Solution: Let $\phi_1(x,y,z) = 3x^2 - 2y^2 - 3z^2 + 8$

$\Rightarrow \nabla\phi_1 = 6xi - 4yj - 6zk$

At $(-1,2,1)\nabla\phi_1 = -6i - 8j - 6k \Rightarrow |\nabla\phi_1| = \sqrt{(-6)^2 + (-8)^2 + (-6)^2} = \sqrt{136} = 2\sqrt{34}$

Now $\hat{n}_1 = \frac{\nabla\phi_1}{|\nabla\phi_1|} = \frac{1}{2\sqrt{34}}(-6i - 8j - 6k) = -\frac{1}{\sqrt{34}}(3i + 4j + 3k)$

$\phi_2(x, y, z) = ax^2 + y^2 - bz, \nabla\phi_2 = 2axi + 2yj - bk$

At $(-1,2,1)$ $\nabla\phi_2 = -2ai + 4j - bk \Rightarrow |\nabla\phi_2| = \sqrt{4a^2 + 16 + b^2}$

$$\hat{n_2} = \frac{\nabla\phi_2}{|\nabla\phi_2|} = \frac{-2ai + 4j - bk}{\sqrt{4a^2 + b^2 + 16}}$$

Since the surfaces intersect orthogonally

$$\hat{n_1} \cdot \hat{n_2} = 0 \Rightarrow -\frac{1}{\sqrt{34}}(3i + 4j + 3k) \cdot \frac{1}{\sqrt{4a^2 + b^2 + 16}}(-2ai + 4j - bk) = 0$$

$$\Rightarrow (3i + 4j + 3k) \cdot (-2ai + 4j - bk) = 0 \Rightarrow -6a + 16 - 3b = 0\#(1)$$

i.e. $6a + 3b = 16$

Also the point $(-1,2,1)$ lies on the surface $ax^2 + y^2 = bz \Rightarrow a + 4 = b$

i.e. $a - b = -4$

Solving the equation (1) and (2) $a = \frac{4}{9}$ and $b = \frac{40}{9}$

17. Find a and b such that the surface $ax^2 - byz = (a + 2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at (1,-1,2).

Solution:

Let $\phi_1 = ax^2 - byz - (a + 2)x,$ (1)

$$\nabla\phi_1 = \frac{\partial\phi_1}{\partial x}i + \frac{\partial\phi_1}{\partial x}j + \frac{\partial\phi_1}{\partial x}k = \big(2ax - (a + 2)\big)i + (-bz)j + (-by)k$$

$\nabla\phi_1(1, -1,2) = (a - 2)i - 2bj + bk$

Let $\phi_2 = 4x^2y + z^3 - 4, \nabla\phi_2 = \frac{\partial\phi_2}{\partial x}i + \frac{\partial\phi_2}{\partial y}j + \frac{\partial\phi_2}{\partial z}k = 8xyi + 4x^2j + 3z^2k$

$\nabla\phi_2(1, -1,2) = -8i + 4j + 12k$

$\nabla\phi_1 . \nabla\phi_2 = 0$

$[(a - 2)i - 2bj + bk]. [-8i + 4j + 12k\,] = 0$

$-8(a - 2) - 8b + 12b = 0$

$-8a + 16 + 4b = 0$

$2a - b - 4 = 0$ (2)

Since the point (1,-1,2) lies on the surface $\phi_1(x,y,z)$ $Eq.(1)$

$a(1)^2 - b(-1)(2) = (a+2)1 \;=>\; 2b-2=0 \;=>\; b=1$

Now Eq. (2) => $2a-(1)-4=0 => a=\frac{5}{2}$

Exercise:

1. If $\phi(x,y,z) = x^2 + \sin y + z$ then find $\nabla\phi$ at $\left(0,\frac{\pi}{2},1\right)$.
2. Find the directional derivative of $\phi(x,y,z) = xyz - xy^2z^3$ at $(1,2,-1)$ in the direction of $i-j-3k$.
3. Find the maximum directional derivative of $\phi(x,y,z) = x^3y^2z$ at the point (1, -2, 3).
4. Find the unit normal vector to the surface $3x^2+2y^2+4z^2=9$ at $(1,-1,1)$.
5. Find the unit normal vector to the surface $x^3+y^3+z^3=14+3xyz$ at $(1,-1,2)$.
6. Find the angle between the normals to the surface $z^2-xy=0$ at the points $(4,1,2)$ and $(3,3,-3)$.
7. Find the angle between the normals to the surface $2x^2+3y^2=3z$ at the points $(2,-2,4)$ and $(-1,-1,1)$.
8. Find the angle between the normals to the surface $x\log z = y^2-1$ at the points $(1,1,1)$ and $(2,1,1)$.
9. Find the angle between the normals to the surface $x\log z = y^2-1$ and $x^2y=2-z$ at the point $(1,1,1)$ common to them.
10. Find the constants a and b so that the surfaces $x^2+ayz=3x$ and $bx^2y+z^3=(b-8)y$ intersect orthogonally at the point (1, 1,-2).
11. Find the angle between the normals to the surface $xy-z^2=0$ at the points $(1,1,1)$ and $(4,1,2)$.
12. Find the angle between the normals to the surface $xy^2z=3x+z^2$ and $3x^2-y^2+2z=1$ at the point $(1,-2,1)$ common to them.

Answers:

1. k	2. $\frac{29}{\sqrt{11}}$	3. $4\sqrt{91}$
4. $\frac{1}{\sqrt{29}}(3i-2j+4k)$	5. $\frac{1}{\sqrt{35}}(3i-j+5k)$	6. $\cos^{-1}\left(\frac{1}{\sqrt{22}}\right)$
7. $\cos^{-1}\left(\frac{65}{\sqrt{17941}}\right)$	8. $\cos^{-1}\left(\frac{3}{\sqrt{10}}\right)$	9. $\cos^{-1}\left(\frac{1}{\sqrt{30}}\right)$
10. $a=\frac{5}{2}, b=1$	11. $\cos^{-1}\left(\frac{13}{\sqrt{6}\sqrt{33}}\right)$	12. $\cos^{-1}\left(\frac{-3}{7\sqrt{6}}\right)$

Divergence of a vector function:

Let $\overrightarrow{f}(x,y,z) = f_1(x,y,z)i + f_2(x,y,z)j + f_3(x,y,z)k$ be a continuously differentiable vector function, then divergence of a vector point function is denoted by $\nabla \cdot \vec{f}$ or $div\overrightarrow{f}$ and defined as

$$\nabla \cdot \vec{f} \text{ or } div\overrightarrow{f} = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right) \cdot (f_1 i + f_2 j + f_3 k) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Clearly divergence of a vector point function is a scalar point function.

Physical interpretation: If $\vec{f}$ represents a velocity field of a gas or fluid then $div\overrightarrow{f}$ represents the **rate of expansion per unit volume under the flow of gas or fluid**.

Definition: A vector function $\vec{f}$ is said to be a **Solenoidal** if $div\overrightarrow{f} = 0$.

Clearly constant vector function is a solenoidal vector function.

Curl of vector function: Let $\overrightarrow{f}(x,y,z) = f_1(x,y,z)i + f_2(x,y,z)j + f_3(x,y,z)k$ be a continuously differentiable vector function, then∇ operating vectorially on $\vec{f}$ is denoted by$curl\,\overrightarrow{f}$ or $\nabla \times \vec{f}$ is given by

$$\nabla \times \vec{f} = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right) \times (f_1 i + f_2 j + f_3 k) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\nabla\times\vec{f}=\left[\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right]i+\left[\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right]j+\left[\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right]k$$

Clearly curl of a vector function is a vector function.

Physical interpretation: The curl of a vector function represents **rotational motion.**

Definition: A vector function $\vec{f}$ is said to be irrotational vector function if $curl\ \vec{f}=\vec{0}$.

Laplacian of a scalar field

Let $\varphi=\varphi(x,y,z)$ be a given scalar field. Then $\nabla\varphi$is a vector field given by,

$$\nabla\varphi=\frac{\partial\varphi}{\partial x}\hat{i}+\frac{\partial\varphi}{\partial y}\hat{j}+\frac{\partial\varphi}{\partial z}\hat{k}$$

$\therefore$ divergence of $\nabla\varphi$ is given by

$$div(\nabla\varphi)=\nabla.(\nabla\varphi)=\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial z}\right)$$

$$div(\nabla\varphi)=\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}$$

The RHS is call Laplacian of $\varphi$and denoted by $\nabla^2\varphi$.

$\therefore$ By definition $\nabla^2\varphi=\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}$ …. (1)

$\therefore div\ \left(\nabla\phi\right)=\nabla.\nabla\varphi=\nabla^2\varphi$

Equation (1) can be rewritten as,

$$\nabla^2\varphi=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\varphi$$

∇^2 is the differential operator given by, $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ and is called Laplacian operator.

Problems:

1. If $\vec{f}=x^2yi-2xzj+2yzk$ then find $div\ \vec{f}$.

Solution: $div\ \vec{f}=\nabla\vec{f}=\frac{\partial\vec{f_1}}{\partial x}+\frac{\partial\vec{f_2}}{\partial x}+\frac{\partial\vec{f_3}}{\partial x}=\frac{\partial}{\partial x}(x^2y)+\frac{\partial}{\partial y}(-2xz)+\frac{\partial}{\partial z}(2yz)$

$$\nabla\cdot\vec{f}=2xy+0+2y=2y(x+1).$$

2. If $\vec{f}=3xyi+x^2zj-y^2e^{2z}k$ then find $\nabla\cdot\vec{f}$ at (1, 2, 0).

Solution: $div\ \vec{f}=\nabla.\vec{f}=\frac{\partial\vec{f_1}}{\partial x}+\frac{\partial\vec{f_2}}{\partial x}+\frac{\partial\vec{f_3}}{\partial x}$

$$=\frac{\partial}{\partial x}(3xy)+\frac{\partial}{\partial y}(x^2z)+\frac{\partial}{\partial z}(-y^2e^{2z})=3y+0-2y^2e^{2z}$$

At (1, 2, 0) $\nabla \cdot \vec{f} = -2.$

3. If $\vec{f} = \frac{xi+yj}{x+y}$ then find $div\, \vec{f}$.

Solution: $\vec{f} = \frac{xi}{x+y} + + \frac{yj}{x+y}$

$$div\, \vec{f} = \frac{\partial}{\partial x}\left(\frac{x}{x+y}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x+y}\right) = \frac{y}{(x+y)^2} + \frac{x}{(x+y)^2} = \frac{x+y}{(x+y)^2} = \frac{1}{x+y}$$

4. If $\phi(x,y,z) = 2x^3y^2z^4$ then find $div(grad\, \phi)$.

Solution: $\phi(x,y,z) = 2x^3y^2z^4$

$$grad\, \phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k => grad\, \phi = 6x^2y^2z^4 i + 4x^3yz^4 j + 8x^3y^2z^3 k$$

$$div\, (grad\, \phi) = \frac{\partial}{\partial x}(6x^2y^2z^4) + \frac{\partial}{\partial y}(4x^3yz^4) + \frac{\partial}{\partial z}(8x^3y^2z^3)$$

$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^3$$

5. If $\vec{f} = 3xyi + 20yz^2j - 15xzk$ and $\phi = y^2 - xz$ then find $div\,(\phi\, \vec{f})$.

Solution: $\phi\vec{f} = (y^2 - xz)(3xy\, i + 20yz^2 j - 15z\, k)$

$\phi\vec{f} = (3xy^3 - 3x^2yz)\, i + (20y^3z^2 - 20xyz^3)\, j - (15xy^2z - 15x^2z^2\,)k$

$div\,(\phi\vec{f}) = \frac{\partial}{\partial x}(3xy^3 - 3x^2yz) + \frac{\partial}{\partial y}(20y^3z^2 - 20xyz^3) - \frac{\partial}{\partial z}(15xy^2z - 15x^2z^2\,)k$

$div\,(\phi\vec{f}) = 3y^3 - 6xyz + 60y^2z^2 - 20xz^3 - 15xy^2 - 30x^2z$

6. Show that the vector function $\vec{f} = 2xyzi + (xy - y^2z)j + (x^2 - zx)k$ is solenoidal.

Solution: Consider $div\, \vec{f} = \frac{\partial}{\partial x}(2xyz) + \frac{\partial}{\partial y}(xy - y^2z) + \frac{\partial}{\partial z}(x^2 - zx)$

$$div\, \vec{f} = 2yz + x - 2yz - x = 0$$

$\therefore \vec{f}$ is solenoidal vector function.

7. If $\vec{f} = (ax + 3y + 4z)i + (x - 2y + 3z)j + (3x + 2y - z)k$ is solenoidal vector field, then find the value of a.

Solution: If $\vec{f}$ is solenoidal then $div\vec{f} = 0$

Hence $\frac{\partial}{\partial x}(ax + 3y + 4z) + \frac{\partial}{\partial y}(x - 2y + 3z) + \frac{\partial}{\partial z}(3x + 2y - z) = 0$

$=> a - 2 - 1 = 0 \; => a = 3$

8. If $\vec{r} = xi + yj + zk$ and $\mathrm{r} = |\vec{r}|$ then show that div $(r^n\vec{r}) = (\mathrm{n} + 3)r^n$

Solution: Given $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$, $r^2 = x^2 + y^2 + z^2 = \sum x^2$

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$, Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

Now $r^n\,\vec{r} = r^n(xi + yj + zk) = r^n xi + r^n yj + r^n zk$

$div\,(r^n\,\vec{r}) = \sum r^n + \sum nr^{n-2}x^2 = 3r^n + nr^{n-2}\sum x^2 = 3r^n + nr^{n-2}r^2 = 3r^n + nr^n$

$$\therefore div\,(r^n\,\vec{r}) = (3+n)r^n$$

9. If $\vec{r} = xi + yj + zk$ and r = $|\vec{r}|$ then show that $\frac{\vec{r}}{r^3}$ is solenoidal.

Solution: Given , $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$, $r^2 = x^2 + y^2 + z^2 = \sum x^2$

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$, Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\frac{\vec{r}}{r^3} = \frac{1}{r^3}(xi + yj + zk)$$

Consider that $div\left(\frac{\vec{r}}{r^3}\right) = \sum \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{\sum r^3.1 - 3r^2 x\frac{\partial r}{\partial x}}{(r^3)^2}$

$div\left(\frac{\vec{r}}{r^3}\right) = \frac{\sum r^3.1 - 3x^2 r}{r^6} = \sum\frac{r^3}{r^6} - \frac{3r}{r^6}\sum x^2 = \sum\frac{1}{r^3} - \frac{3r}{r^6}.r^2 = \frac{3}{r^3} - \frac{3}{r^3} = 0$

$\therefore \frac{\vec{r}}{r^3}$ is solenoidal vector field.

10. If $\vec{r} = xi + yj + zk$ and $r = |\vec{r}|$ then prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$.

Solution: Given $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}, r^2 = x^2 + y^2 + z^2 = \sum\ x^2$

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$, Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\nabla^2 f(r) = \sum\frac{\partial^2}{\partial x^2}(f(r)) = \sum\frac{\partial}{\partial x}\left(f'(r)\frac{\partial r}{\partial x}\right) = \sum\frac{\partial}{\partial x}\left(f'(r)\frac{x}{r}\right) = \sum\frac{\partial}{\partial x}\left(\frac{xf'(r)}{r}\right)$$

$$\nabla^2 f(r) = \sum\left[\frac{r\left\{f'(r) + xf''(r)\frac{\partial r}{\partial x}\right\} - xf'(r)\frac{\partial r}{\partial x}}{r^2}\right] = \sum\left[\frac{r\{f'(r) + xf''(r)^x\} - xf'(r)\frac{x}{r}}{r^2}\right]$$

$$\nabla^2 f(r) = \sum\frac{f'(r)}{r} + \frac{f''(r)}{r^2}\sum x^2 - \frac{f'(r)}{r^3}\sum x^2 = 3\frac{f'(r)}{r} + \frac{f''(r)}{r^2}\cdot r^2 - \frac{f'(r)}{r^3}\cdot r^2$$

$$\nabla^2 f(r) = \frac{2f'(r)}{r} + f''(r)$$

11. If $\vec{r} = xi + yj + zk$ and $r = |\vec{r}|$ then prove that $\nabla^2(r^{n+1}) = (n+1)(n+2)r^{n-1}$.

Solution: Given $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}, r^2 = x^2 + y^2 + z^2 = \sum x^2$

$$\nabla^2(r^{n+1}) = (n+1)\sum\frac{\partial}{\partial x}(r^{n-1}\cdot x) = (n+1)\sum\left[r^{n-1}\cdot 1 + x\cdot(n-1)r^{n-2}\cdot\frac{\partial r}{\partial x}\right]$$

$$\nabla^2(r^{n+1}) = (n+1)\sum\left[r^{n-1} + x\cdot(n-1)\cdot r^{n-2}\cdot\frac{x}{r}\right]$$

$$\nabla^2(r^{n+1}) = (n+1)[\sum r^{n-1} + (n-1)r^{n-3}\sum x^2] = (n+1)[3\cdot r^{n-1} + (n-1)r^{n-3}\cdot r^2]$$

$$\nabla^2(r^{n+1}) = (n+1)[3r^{n-1} + (n-1)\cdot r^{n-1}]$$

$$= (n+1)[3+n-1]r^{n-1} = (n+1)(n+2)r^{n-1}.$$

12. If $\vec{r} = xi + yj + zk$ and $r = |\vec{r}|$ then prove that $\text{div}(r^n\vec{r}) = (\text{n}+3)r^n$ and hence show that $\nabla^2(r^n\vec{r}) = n(n+3)r^{n-2}\vec{r}$.

Solution: From the problem number 12 we already proved that $\nabla\cdot(r^n\vec{r}) = (\text{n}+3)r^n$

Now $\nabla^2(r^n\vec{r}) = \nabla[\nabla\cdot(r^n\vec{r})] = \nabla[(n+3)r^n] = \sum\frac{\partial}{\partial x}(n+3)r^n i = (n+3)\sum nr^{n-1}\frac{\partial r}{\partial x}i$

$$\nabla^2(r^n\vec{r}) = n(n+3)\sum r^{n-1}\frac{x}{r}i \Rightarrow \nabla^2(r^n\vec{r}) = n(n+3)r^{n-2}\sum xi$$

$$\Rightarrow \nabla^2(r^n\vec{r}) = n(n+3)r^{n-2}\vec{r}$$

13. If $\vec{f} = xy^2 i + 2x^2yzj - 3y^2zk$ then find curl $\vec{f}$.

Solution: curl $\vec{f} = \nabla\times\vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3y^2z \end{vmatrix} = i(-6yz - 2x^2y) - j(0-0) +$

$k(4xyz - 2xy)$

14. Show that $\vec{f} = (\sin x + z)i + (\cos y - z)j + (x-y)k$ is irrotational.

Solution: Consider curl $\vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin x + z & \cos y - z & x - y \end{vmatrix} = i(-1+1) - j(1-1) +$

$k(0-0)$ curl $\vec{f} = 0 \Rightarrow \vec{f}$ is irrotational.

15. If $\vec{f} = x^2 i - 2xzj + 2yzk$ then find curl(curl$\vec{f}$).

Solution: curl $\vec{f} = \nabla \times \vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -2xz & 2yz \end{vmatrix} = i(2z+2x) - j(0-0) + k(-2z)$

$$\text{curl}(\text{curl}\vec{f}) = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (2x+2z) & 0 & -2z \end{vmatrix} = i(0-0) - j(0-2) + k(0-0) = 2j$$

16. Prove that curl $\vec{r} = 0$

Solution: Let $\vec{r} = xi + yj + zk$

$$\text{curl } \vec{r} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = i(0-0) - j(0-0) + k(0-0) = \vec{0}$$

$\therefore$ $\vec{r}$ is irrotational.

17. For any differentiable vector function $\vec{f}$ prove that $\text{div}(\text{curl}\vec{f}) = 0$.

Solution: Let $\vec{f} = f_1 i + f_2 j + f_3 k$

$\therefore$ curl $\vec{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) i + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) j + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) k$

$$\text{div}(\text{curl}\vec{f}) = \frac{\partial}{\partial x}\left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right] + \frac{\partial}{\partial y}\left[\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right] + \frac{\partial}{\partial z}\left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right]$$

$$\text{div}(\text{curl}\vec{f}) = \frac{\partial^2 f_3}{\partial x \partial y} - \frac{\partial^2 f_2}{\partial x \partial z} + \frac{\partial^2 f_1}{\partial y \partial z} - \frac{\partial^2 f_3}{\partial y \partial x} + \frac{\partial^2 f_2}{\partial z \partial x} - \frac{\partial^2 f_1}{\partial z \partial y}$$

But mixed partial derivatives are equal.

$\therefore$ $\text{div}(\text{curl}\vec{f}) = 0 \Rightarrow \text{curl}\vec{f}$ is solenoidal.

18. For what value of a the vector field $\vec{f} = (axy - z^3)i + (a-2)x^2 j + (1-a)xz^2 k$ is irrotational.

Solution: If $\vec{f}$ is irrotational, then $\text{curl}\vec{f} = 0$

Hence $\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy - z^3 & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = 0$

$\Rightarrow [0-0]i - [(1-a)z^2 - (0-3z^2)]j + [2x(a-2) - ax]k = 0$

$\Rightarrow 0i - [4-a]z^2 j + [a-4]xk = 0 \Rightarrow a = 4$

19. Find the constants a, b, c so that,

$\vec{f} = (x + 2y + az)i + (bx - 3y - z)j + (4x + cy + 2z)k$ is irrotational.

Solution: If $\vec{f}$ is irrotational, then curl$\vec{f} = 0$

Hence $\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = 0$

$\Rightarrow [c + 1]i - [4 - a]j + [b - 2]k = 0$

$\Rightarrow c + 1 = 0, 4 - a = 0, b - 2 = 0 \Rightarrow a = 4, b = 2, c = -1$

20. Show that $\vec{f} = (2xy + z^3)i + x^2 j + 3xz^2 k$ is irrotational and find the function ϕ such that $\vec{f} = \text{grad}\emptyset$.

Solution: Given that $\vec{f} = (2xy + z^3)i + x^2 j + 3xz^2 k$

$$\text{curl}\vec{f} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$\Rightarrow [0 - 0]i - [3z^2 - 3z^2]j + [2x - 2x]k = 0$

curl$\vec{f} = 0 \Rightarrow \vec{f}$ is irrotational vector field.

We have to find the function ϕ such that $\vec{f} = \text{grad}\emptyset$

i.e. $(2xy + z^3)i + x^2 j + 3xz^2 k = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$

$$\Rightarrow \frac{\partial\phi}{\partial x} = 2xy + z^3, \frac{\partial\phi}{\partial y} = x^2, \frac{\partial\phi}{\partial z} = 3xz^2$$

We have $d\phi = \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz$

$d\phi = (2xy + z^3)dx + x^2 dy + 3xz^2 dz$

Regrouping the terms we get

$d\phi = 2xydx + x^2 dy + z^3 dx + 3z^2 xdz \Rightarrow d\phi = d(x^2 y) + d(xz^3)$

$\Rightarrow d\phi = d(x^2 y + xz^3) \Rightarrow \phi = x^2 y + xz^3 + c$

21. Show that $\vec{f} = (\sin y + z)i + (x\cos y - z)j + (x - y)k$ is irrotational. Find the function ϕ such that $\vec{f} = \text{grad}\emptyset$.

Solution: Given that $\vec{f} = (\sin y + z)i + (x\cos y - z)j + (x - y)k$

Consider curl $\vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y + z & xcoy - z & x - y \end{vmatrix}$

$\text{curl}\vec{f} = i[-1+1] - j[1-1] + k[\cos y - \cos y] = i(0) + j(0) + k(0) = 0$

Therefore $\vec{f}$ is irrotational.

Find the function ϕ such that $\vec{f} = \text{grad}\phi$

$(\sin y + z)i + (x\cos y - z)j + (x-y)k = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$

$\Rightarrow \frac{\partial\phi}{\partial x} = \sin y + z, \frac{\partial\phi}{\partial y} = x\cos y - z, \frac{\partial\phi}{\partial z} = x - y$

$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

$d\phi = (\sin y + z)dx + (x\cos y - z)dy + (x-y)dz$

Regrouping the terms

$\text{d}\phi = \sin y dx + x\cos y dy + zdx + xdz - zdy - ydz$

$\text{d}\phi = \text{d}(\text{x}\sin y) + d(xz) - d(yz) \Rightarrow d\emptyset = \text{d}(\text{x}\sin y + xz - yz)$

$\phi = \text{x}\sin y + xz - yz + c.$

23. If $\vec{r} = xi + yj + zk$ and $r = |\vec{r}|$ then show that $r^n\vec{r}$ is irrotational for all values of n and solenoidal for $n = -3$.

Solution: Given, $r = |\vec{r}| = \sqrt{x^2+y^2+z^2} r^2 = x^2 + y^2 + z^2 = \sum x^2$

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$

$r^n\vec{r} = r^n(xi + yj + zk) = r^nxi + r^nyj + r^nzk$

$\text{curl}\vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^nx & r^ny & r^nz \end{vmatrix} = \sum\left[\frac{\partial}{\partial y}(r^nz) - \frac{\partial}{\partial z}(r^ny)\right]i$

$\text{curl}\vec{f} = \sum\left[nr^{n-1}\frac{\partial r}{\partial y}.z - nr^{n-1}\frac{\partial r}{\partial z}y\right] = \sum\left[nr^{n-1}\frac{y}{r}z - nr^{n-1}\frac{z}{r}y\right]i$

$\text{curl}\vec{f} = \sum[nr^{n-2}yz - nr^{n-2}yz] = \sum 0i = \vec{0}$

$\therefore r^n\vec{r}$ is irrotational for all values of n.

Consider $\text{div}[r^n\vec{r}] = \sum\frac{\partial}{\partial x}(r^n x) = \sum\left[r^n\cdot 1 + x\cdot nr^{n-1}\cdot\frac{\partial r}{\partial x}\right] = \sum\left[r^n + nr^{n-1}\cdot x\cdot\frac{x}{r}\right]$

$\text{div}[r^n\vec{r}] = \sum[r^n + n\cdot r^{n-2}\cdot x^2] = \sum r^n + n\cdot r^{n-2}\sum x^2 = 3r^n + n\cdot r^{n-2}\cdot r^2$

$\text{div}[r^n\vec{r}] = 3r^n + n\cdot r^n = (3+n)r^n$

$r^n\vec{r}$ is solenoidal implies $\text{div}[r^n\vec{r}] = 0 \Rightarrow (n+3)r^n = 0 \Rightarrow n+3=0, \therefore n=-3$

24. Show that $\vec{F} = \frac{xi+yj}{x^2+y^2}$ is both Solenoidal and Irrotational.

Solution: Given, $\vec{F} = \frac{xi+yj}{x^2+y^2}$

Now, $\nabla\cdot\vec{F} = \text{div}\vec{F} = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\cdot\left(\frac{xi}{x^2+y^2} + \frac{yi}{x^2+y^2}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$
$$= \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}$$
$$= \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2} = 0$$

$\nabla\cdot\vec{F} = 0 \Rightarrow \vec{F}$ is Solenoidal

$$\text{Curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} & 0 \end{vmatrix}$$
$$= i\{0-0\} + j\{0-0\} + k\left\{\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\right\}$$
$$= k\left\{\frac{-2xy+2xy}{(x^2+y^2)^2}\right\} = 0$$
$$\text{Curl}\vec{F} = \nabla\times\vec{F} = 0 \Rightarrow \vec{F} \text{ is Irrotational}$$

Hence $\vec{F}$ is both soleniodal and irrotational.

25. Prove that $\vec{f} = (2x+yz)i + (4y+zx)j - (6z-xy)k$ is solenoidal as well as irrotational. Also find the scalar potential of $\vec{f}$.

Solution: Given $\vec{f} = (2x+yz)i + (4y+zx)j - (6z-xy)k$

$$\nabla \vec{f} = \frac{\partial}{\partial x}(2x+yz) + \frac{\partial}{\partial y}(4y+zx) - \frac{\partial}{\partial z}(6z-xy) = 2+4-6 = 0$$

$\vec{f}$ is Solenoidal.

$$\nabla \times \vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+yz & 4y+zx & -6z+xy \end{vmatrix} = (x-x)i - (y-y)j + (z-z)k = 0$$

$\vec{f}$ is irrotational

Now, to find ϕ such that $\vec{f} = \nabla\phi$

$$(2x+yz)i + (4y+zx)j - (6z-xy)k = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

$$\frac{\partial\phi}{\partial x} = 2x+yz \quad \ldots.(1), \qquad \frac{\partial\phi}{\partial y} = 4y+zx \quad \ldots..(2), \qquad \frac{\partial\phi}{\partial z} = -(6z-xy) \quad \ldots..(3)$$

Integrating (1), (2), (3) partially w.r.t x,y,z respectively, we get

$\phi(x,y,z) = x^2 + xyz + f_{1(y,z)} \qquad \ldots..(4)$

$\phi(x,y,z) = 2y^2 + xyz + f_{2(x,z)} \qquad \ldots..(5)$

$\phi(x,y,z) = -3z^2 + xyz + f_{3(x,y)} \qquad \ldots..(6)$

Combining (4), (5), (6) we get, $\phi(x,y,z) = x^2 + y^2 - 3z^2 + xyz + k$, where k is an arbitrary constant. Therefore ϕ is the scalar potential of $\vec{f}$.

26. Prove that $\vec{f} = (y^2\cos x)i + (2y\sin x - 4)j + (3xz^2)k$ is irrotational. Also find the scalar potential of $\vec{f}$.

Solution: Given $\vec{f} = (y^2\cos x)i + (2y\sin x - 4)j + (3xz^2)k$

$$\nabla \times \vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2\cos x & 2y\sin x - 4 & 3xz^2 \end{vmatrix}$$

$$= (0-0)i - (3z^2 - 3z^2)j + (2ycos\ x - 2y\cos x)k = 0$$

$\vec{f}$ is irrotational

Now, to find ϕ such that $\vec{f} = \nabla\phi$

$$(y^2\cos x)i + (2y\sin x - 4)j + (3xz^2)k = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

$$\frac{\partial \phi}{\partial x} = y^2 \cos x \quad \ldots.(1), \qquad \frac{\partial \phi}{\partial y} = 2y \sin x - 4 \quad \ldots..(2), \qquad \frac{\partial \phi}{\partial z} = 3xz^2 \quad \ldots..(3)$$

Integrating (1), (2), (3) partially w.r.t x,y,z respectively, we get

$\phi(x, y, z) = y^2 \sin x + z^3 x + f_{1(y,z)}$ $\quad \ldots..(4)$

$\phi(x, y, z) = (\sin x)\, y^2 - 4y + f_{2(x,z)}$ $\quad \ldots..(5)$

$\phi(x, y, z) = xz^3 + f_{3(x,y)}$ $\quad \ldots..(6)$

Combining (4), (5), (6) we get, $\phi(x, y, z) = y^2 \sin x + xz^3 - 4y^2 + k$, where k is an arbitrary constant. Therefore ϕ is the scalar potential of $\vec{f}$.

27. Prove that $\vec{f} = (6xy + z^3)i + (3x^2 - z)j + (3xz^2 - y)k$ is solenoidal as well as irrotational. Also find the scalar potential of $\vec{f}$.

Solution: Given$\vec{f} = (6xy + z^3)i + (3x^2 - z)j + (3xz^2 - y)k$

$$\nabla \vec{f} = \frac{\partial}{\partial x}(6xy + z^3) + \frac{\partial}{\partial y}(3x^2 - z) - \frac{\partial}{\partial z}(3xz^2 - y) = 2 + 4 - 6 = 0$$

$\vec{f}$ is Solenoidal.

$$\nabla \times \vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix} = 0$$

$\vec{f}$ is irrotational

Now, to find ϕ such that $\vec{f} = \nabla \phi$

$$(6xy + z^3)i + (3x^2 - z)j + (3xz^2 - y)k = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

$$\frac{\partial \phi}{\partial x} = 6xy + z^3 \quad \ldots.(1), \qquad \frac{\partial \phi}{\partial y} = 3x^2 - z \quad \ldots..(2), \qquad \frac{\partial \phi}{\partial z} = 3xz^2 - y \quad \ldots..(3)$$

Integrating (1), (2), (3) partially w.r.t x, y, z respectively, we get

$\phi(x, y, z) = 3x^2 y + xz^3 + f_{1(y,z)}$ $\quad \ldots..(4)$

$\phi(x, y, z) = 3x^2 y - yz + f_{2(x,z)}$ $\quad \ldots..(5)$

$\phi(x, y, z) = xz^3 - yz + f_{3(x,y)}$ $\quad \ldots..(6)$

Combining (4), (5), (6) we get, $\phi(x, y, z) = 3x^2 y + xz^3 - yz + k$, where k is an arbitrary constant. Therefore ϕ is the scalar potential of $\vec{f}$.

Exercise:

1. If $\vec{f} = 3x^2 i + 5xy^2 j + xyz^3 k$ then find div$\vec{f}$ at (1,2,3)
2. If $\vec{f} = (y^2 + z^2 - x^2)i + (z^2 + x^2 - y^2)j + (x^2 + y^2 - z^2)k$ then find div$\vec{f}$, curl$\vec{f}$
3. If $\vec{f} = zx^3 i + x \cdot y^3 j + yz^3 k$ then find $\nabla \cdot \vec{f}$, and $\nabla \times \vec{f}$
4. Show that the vector field $\vec{f} = (x + 3y)i + (y - 3z)k + (x - 2z)k$ is solenoidal.
5. Show that $\vec{f} = 2x^2 zi - 10xyzj + 3xz^2 k$ is solenoidal.
6. Determine a so that the vector field $\vec{f} = (x + 2)i + (y - 2z)j + (x - az)k$ is solenoidal.
7. Determine the constant a such that the vector field $\vec{f} = (x + 3y)i + (y - 2z)j + (x - az)k$ is solenoidal.
8. If $\vec{f} = x(y + z)i + y(z + x)j + z(x + y)k$ then find curl $\vec{f}$
9. Show that $\vec{f} = (2xy + z^3)i + x^2 j + 3xz^2 k$ is irrotational.
10. Show that $\vec{f} = (6xy + z^3)i + (3x^2 - z)j + (3xz^2 - y)k$ is irrotational.
11. Show that the vector field $\vec{f} = \frac{xi+yj+zk}{(x^2+y^2+z^2)^{\frac{1}{2}}}$ is irrotational.
12. If $\vec{f} = x^2 i + y^2 j + z^2 k$ and $\vec{g} = yzi + zxj + xyk$ then show that $f \times g$ is solenoidal.
13. If $\vec{f} = (2x + 3y + az)i + (bx + 2y + 3z)j + (2x + cy + 3z)k$ is irrotational vector Field, then find the constants a, b, c.
14. If $\phi = x^2 y + 2xy + z^2$ then show that $\nabla\phi$ is irrotational.
15. If $\phi = x^2 - y^2$ then show that ϕ satisfies the Laplacian equation.
16. If $\phi = 2x^2 yz^3$ then find $\nabla^2\phi$ at (1,1,1).
17. If $\phi = x^2 - y^2 + 4z$ then find $\nabla^2\phi$.
18. If $\vec{r} = xi + yj + zk$ and $r = |\vec{r}|$ then find grad $\left[\text{div}\left(\frac{\vec{r}}{r}\right)\right]$
19. If $\vec{r} = xi + yj + zk$ and $r = |\vec{r}|$ then prove that grad $\left(\frac{1}{r}\right)$ is solenoid.
20. If $\vec{r} = xi + yj + zk$ and $r = |\vec{r}|$ then show that $\nabla(r^3 \cdot \vec{r}) = 6r^3$
21. If $\vec{r} = xi + yj + zk$ and $r = |\vec{r}|$ then find $\nabla^2(r^n)$ and hence show that $\nabla^2\left(\frac{1}{r}\right) = 0$
22. If $\vec{v} = \vec{w} \times \vec{r}$ where $\vec{w}$ is a constant vector. Then prove that $\vec{w} = \frac{1}{2}\text{curl}\vec{v}$
23. Prove that gradϕ is irrotational and curl $\vec{f}$ is solenoidal.
24. Show that $\vec{f} = 2xyzi + x^2 zj + x^2 yk$ is irrotational and find the function ϕ such that $\vec{f} = \text{grad } \phi$.

25. Show that $\vec{f} = (\sin y + z\cos x)i + (x\cos y + \sin z)j + (y\cos z + \sin x)k$ is irrotational and find the function ϕ such that $\vec{f} = \nabla\phi$
26. Show that $\vec{r}|\vec{r}|$ is irrotational and find the function ϕ such that $\vec{r}|\vec{r}| = \nabla\phi$
27. Show that divr $= \frac{2}{r}$, where $\vec{r} = xi + yj + zk$
28. Find the scalar function ϕ such that $\nabla\phi = y^2z^3i + 2xyz^3j + 3xy^2z^2k$ given that $\phi(x, y, z) = 0$ at the origin.
29. Show that $\nabla^2(\log r) = \frac{1}{r^2}$
30. Show that $\frac{\vec{r}}{r^3}$ is both solenoidal and irrotational.
31. Show that $\vec{f} = (3x^2 + 2y^2 + 1)i + (4xy - 3y^2z - 3)j + (2 - y^3)k$ is solenoidal as well as irrotational. Also find the scalar potential of $\vec{f}$.
32. Show that $\vec{f} = (y^2 + 2xz^2)i + (2xy - z)j + (2x^2z - y + 2z)k$ is solenoidal as well as irrotational. Also find the scalar potential of $\vec{f}$.

Answers:

1. 80
2. div$\vec{f} = -2(x + y + z)$, curl$\vec{f} = 2\{(y - z)i + (z - x)j + (x - y)k\}$
3. divf $= 3(x^2z + y^2x + z^2y)$, curlf $= z^3i + x^3j + y^3k$
4. $\frac{-2\vec{r}}{r^3}$
5. $a = 2$
6. $\phi = 2x^2yz + c$
7. $a = 2$
8. $\phi = x\sin y + y\sin z + z\sin x + c$
9. $(z - y)i + (z - x)j + (y - x)k$
10. $\phi = \frac{1}{3}(x^2 + y^2 + z^2)^{\frac{3}{2}} + c$
11. $a = 2, b = 3, c = 3$
12. $x^3 + 2y^2x + x - y^3z - 3y + 2z + k$
13. $\phi = xy^2z^2$
14. $xy^2 + x^2z^2 - yz + z^2 + k$

Cylindrical Polar Coordinates System:

In a cylindrical coordinate system a point P is specified by (r, θ, z)

The coordinate surfaces are

$r =$ constant [Family of co-axial and concentric cylinders]

$\theta =$ constant [Family of planes passing through z-axis]

$z =$ constant [Planes perpendicular to z-axis]

The intervals for the parameters are

$$r \geq 0, 0 \leq \theta < 2\pi, -\infty < z < \infty$$

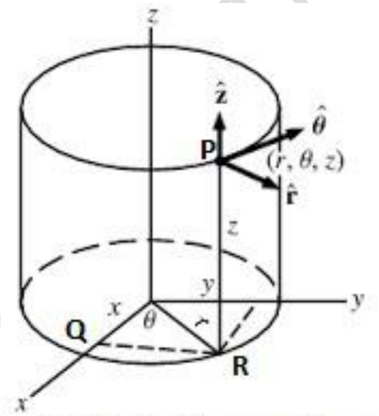
The transformations that relate cylindrical coordinates to cartesian coordinates are

$$x = r\cos\theta, y = r\sin\theta, z = z$$

The transformations that relate cartesian coordinates to cylindrical coordinates are

$$r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}\left(\frac{y}{x}\right), z = z$$

In cylindrical coordinate system $\hat{e}_r, \hat{e}_\theta$, and $\hat{e}_z$ are the unit vectors along the coordinate axes. These unit vectors are mutualy orthogonal unit vectors.
That is $\hat{e}_r \cdot \hat{e}_\theta = \hat{e}_\theta \cdot \hat{e}_z = \hat{e}_z \cdot \hat{e}_r = 0$.



Cylindrical Polar Coordinates

Vector Differential Operator in Cylindrical Coordinate System:

$$\nabla = \frac{\partial}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{e}_\theta + \frac{\partial}{\partial z}\hat{e}_z$$

Gradient $\nabla\psi$ in cylindrical polar coordinates:

For a scalar potential $\psi(r, \theta, z)$ in cylindrical coordinates,

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_\theta + \frac{\partial\psi}{\partial z}\hat{e}_z$$

Divergence $\nabla \cdot \vec{f}$ in cylindrical polar coordinates:

For a vector field $\vec{f}(r,\theta,z) = f_1\hat{e}_r + f_2\hat{e}_\theta + f_3\hat{e}_z$ in cylindrical coordinates,

$$\operatorname{div}(\vec{f}) = \nabla \cdot \vec{f} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rf_1) + \frac{\partial}{\partial \theta}(f_2) + \frac{\partial}{\partial z}(rf_3)\right]$$

Curl $\nabla \times \vec{f}$ in cylindrical polar coordinates:

For a vector field $\vec{f}(r,\theta,z) = f_1\hat{e}_r + f_2\hat{e}_\theta + f_3\hat{e}_z$ in cylindrical coordinates,

$$\operatorname{curl}\vec{f} = \nabla \times \vec{f} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ f_1 & rf_2 & f_3 \end{vmatrix}$$

Laplacian $\nabla^2\psi$ in cylindrical polar coordinates:

For a scalar potential $\psi(r,\theta,z)$ in cylindrical coordinates,

$$\begin{aligned}\nabla^2\psi = \nabla \cdot (\nabla\psi) &= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial\psi}{\partial z}\right)\right] \\ &= \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}\end{aligned}$$

Examples:

1. Compute gradient and Laplacian of the scalar field $\psi(r,\theta,z) = r + z\cos\theta$ in the cylindrical coordinates (r,θ,z).

   Sol: grad$\psi = \frac{\partial\psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{e}_\theta + \frac{\partial\psi}{\partial z}\hat{e}_z = \hat{e}_r - \frac{1}{r}z\sin\theta\,\hat{e}_\theta + \cos\theta\,\hat{e}_z$

$$\begin{aligned}\nabla^2\psi &= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial\psi}{\partial z}\right)\right] = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2} \\ &= 0 + \frac{1}{r} - \frac{1}{r^2}z\cos\theta + 0 = \frac{1}{r} - \frac{1}{r^2}z\cos\theta\end{aligned}$$

2. Compute divergence and curl of the vector field $\vec{A} = \sin\theta\,\hat{e}_r + \frac{1}{r}\cos\theta\,\hat{e}_\theta - rz\hat{e}_z$ in the cylindrical coordinates (r,θ,z).

Sol: $\operatorname{div}\vec{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_1) + \frac{\partial}{\partial \theta}(A_2) + \frac{\partial}{\partial z}(rA_3)\right] = \frac{1}{r}\left[\frac{\partial}{\partial r}(r\sin\theta) + \frac{\partial}{\partial \theta}(\cos\theta) + \frac{\partial}{\partial z}(r^2 z)\right]$

$= \frac{1}{r}[\sin\theta - \sin\theta + r^2] = r$

$\operatorname{curl}\vec{A} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_1 & rA_2 & A_3 \end{vmatrix} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ \sin\theta & r\cos\theta & rz \end{vmatrix}$

$= (0-0)\hat{e}_r - (z-0)\hat{e}_\theta + (\cos\theta - \cos\theta)\hat{e}_z = -z\hat{e}_\theta$

Spherical Polar Coordinate System:

In a spherical coordinate system a point P is specified by (r, θ, ϕ)

The coordinate surfaces are

$r = c_1$ concentric spheres centered at origin O.

$\theta = c_2$ circular half angle cones with z-axis as axis and vertex at origin.

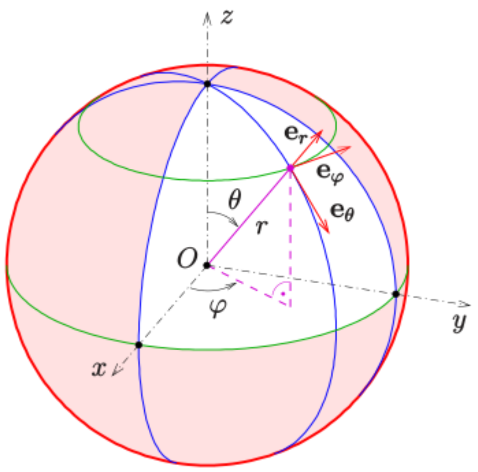
$\phi = c_3$ Family of planes intersecting along z-axis.

The transformations from Cartesian coordinates to spherical coordinates areas follows:

$$x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$$

The transformations from Cartesian coordinates to spherical coordinates areas follows:

$$r = \sqrt{x^2+y^2+z^2}, \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right), \phi = \tan^{-1}\left(\frac{y}{x}\right)$$



In spherical coordinate system $\hat{e}_r, \hat{e}_\theta$, and $\hat{e}_\phi$ are the unit vectors along the coordinate axes.

These unit vectors are mutualy orthogonal unit vectors.

That is $\hat{e}_r \cdot \hat{e}_\theta = \hat{e}_\theta \cdot \hat{e}_\phi = \hat{e}_\phi \cdot \hat{e}_r = 0$.

Vector Differential Operator in Cylindrical Coordinate System:

$$\nabla = \frac{\partial}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{e}_\phi$$

Gradient $\nabla\psi$ in spherical polar coordinates:

For a scalar potential $\psi(r, \theta, \phi)$ in spherical coordinates,

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial \phi}\hat{e}_\phi$$

Divergence $\nabla \cdot \vec{f}$ in spherical polar coordinates:

For a vector field $\vec{f}(r, \theta, \phi) = f_1\hat{e}_r + f_2\hat{e}_\theta + f_3\hat{e}_\phi$ in spherical coordinates,

$$\text{div}(\vec{f}) = \nabla \cdot \vec{f} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta f_1) + \frac{\partial}{\partial \theta}(r\sin\theta f_2) + \frac{\partial}{\partial \phi}(r f_3)\right]$$

Laplacian $\nabla^2\psi$ in spherical polar coordinates:

For a scalar potential $\psi(r, \theta, \phi)$ in spherical coordinates,

$$\nabla^2\psi = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{r\sin\theta}{r}\frac{\partial\psi}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{r}{r\sin\theta}\frac{\partial\phi}{\partial z}\right)\right]$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial \phi^2}$$

Curl $\nabla \times \vec{f}$ in spherical polar coordinates:

For a vector field $\vec{f}(r, \theta, \phi) = f_1\hat{e}_r + f_2\hat{e}_\theta + f_3\hat{e}_\phi$ in spherical coordinates,

$$\text{curl}\vec{f} = \nabla \times \vec{f} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ f_1 & rf_2 & r\sin\theta f_3 \end{vmatrix}$$

1. Determine the curl and divergence of the vector field $\vec{f} = \frac{2\cos\theta}{r^3}\hat{e}_r + \frac{\sin\theta}{r}\hat{e}_\theta$ in spherical coordinates (r, θ, ϕ).

$$\text{curl}\vec{f} = \nabla \times \vec{f} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ f_1 & rf_2 & r\sin\theta f_3 \end{vmatrix} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ \frac{2\cos\theta}{r^3} & \sin\theta & 0 \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[(0-0)\hat{e}_r - (0-0)\hat{e}_\theta + \left(0 + \frac{2\sin\theta}{r^3}\right)r\sin\theta\hat{e}_\phi\right] = \frac{1}{r^4}\hat{e}_\phi$$

$$\text{div}(\vec{f}) = \nabla\cdot\vec{f} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta f_1) + \frac{\partial}{\partial \theta}(r\sin\theta f_2) + \frac{\partial}{\partial \phi}(rf_3)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{2\cos\theta}{r^3}\right) + \frac{\partial}{\partial \theta}\left(r\sin\theta\frac{\sin\theta}{r}\right) + \frac{\partial}{\partial \phi}(0)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[-\frac{\sin 2\theta}{r^2} + \sin 2\theta\right]$$

Video Links:

1. Vector differentiation
https://www.khanacademy.org/math/multivariable-calculus/multivariable-derivatives/position-vector-functions/v/differential-of-a-vector-valued-function

2. Gradient
https://www.youtube.com/watch?v=fZ231k3zsAA
https://www.youtube.com/watch?v=GkB4vW16QHI

3. Directional derivative
https://www.youtube.com/watch?v=Dcnj1bYEZlY

4. Applications of Gradient, Divergence and curl
https://www.youtube.com/watch?v=qOcFJKQPZfo
https://www.youtube.com/watch?v=vvzTEbp9lrc

DIVERGENCE AND CURL
https://www.youtube.com/watch?v=rB83DpBJQsE