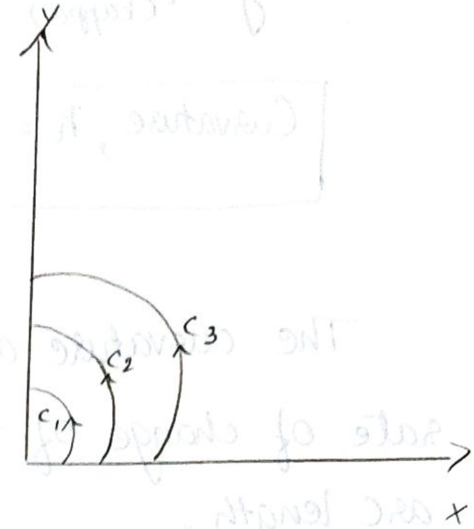
3-11-10

Curvature

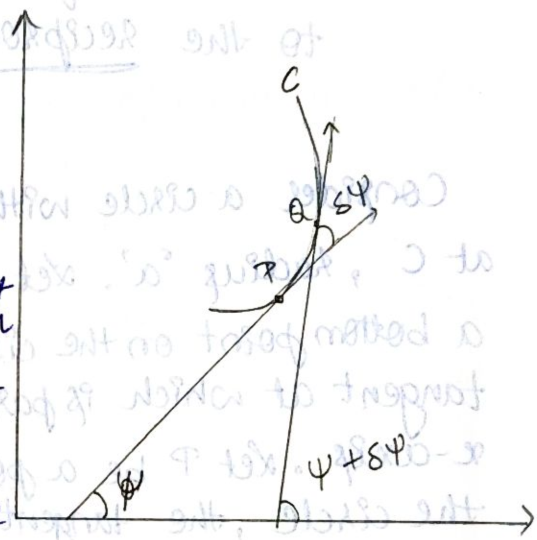
Let the particle moves along the semi-circles C_1, C_2, C_3 which are from origin. Then the angle of rotation is same but the bending of circle depends on the arc length



Here, C_1 is sharper than C_2 and C_2 is sharper than C_3.

Curvature

Let P be a point on the curve C. Let the tangent at P makes an angle ψ in the +ve direction of x-axis. Let Q be another point near to P, tangent at which makes an angle $\psi+\delta\psi$ with x-axis.



$\delta\psi$ is angle between the tangents at P, Q which gives the change with angle ψ or more of bending of the curve from P to Q.

$\frac{\delta\psi}{\delta s}$ is the ratio of bending of the curve wrt the arc length.

The rate of change of bending of the curve wrt arc length s from P to Q is

$$\boxed{\frac{d\psi}{ds} = \lim_{P\to Q} \frac{\delta\psi}{\delta s}}$$

This is rate is called curvature at the point P &

denoted by κ (kappa)

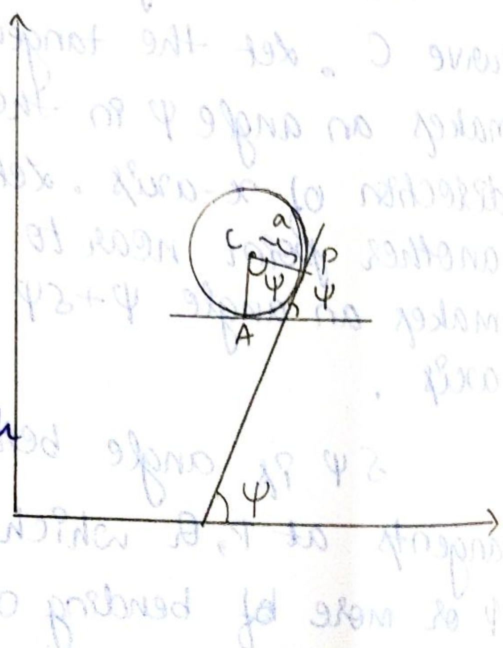
$$\boxed{\text{Curvature, } \kappa = \frac{d\psi}{ds}}$$

The curvature at anypoint on the curve is the rate of change of the angle turned through wrt arc length.

Note :

1) The curvature of straight line is zero.

2) Curvature of circle is constant and is equal to the reciprocal of the radius of the circle.



Consider a circle with centre at C, radius 'a'. Let A be a bottom point on the circle, the tangent at which is parallel to x-axis. Let P be a point on the circle, the tangent at which makes angle ψ with x-axis then

$$A\hat{C}P = \psi$$

$$s = \widehat{AP} = a\psi$$

$$\frac{ds}{d\psi} = a$$

$$\boxed{\kappa = \frac{d\psi}{ds} = \frac{1}{a}}$$

$\therefore$ Curvature of circle is reciprocal of radius

Radius of Curvature :

Radius of curvature at any point on the curve is the reciprocal of curvature at that point.

Denoted by $\rho_{(\text{rho})}$

$$\boxed{\rho = \frac{ds}{d\psi}}$$

Note : → Radius of curvature of a circle is constant and is equal to the radius of the circle.

Radius of Curvature in Cartesian Form :-

Let $y = f(x)$ be the equation of curve in cartesian form then

$$y' = \frac{dy}{dx} = \tan\psi$$

wkt

$$\boxed{\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

$$y'' = \frac{d^2y}{dx^2} = \sec^2\psi \, \frac{d\psi}{dx}$$

$$= \sec^2\psi \, \frac{d\psi}{ds} \cdot \frac{ds}{dx}$$

$$\frac{d^2y}{dx^2} = \sec^2\psi \cdot \frac{1}{\rho} \cdot \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$= (1+\tan^2\psi)\, \frac{1}{\rho}\, \sqrt{1+\tan^2\psi}$$

$$= [1+(y')^2]\, \frac{1}{\rho}\, \sqrt{1+(y')^2}$$

$$\therefore \boxed{\rho = \frac{[1+(y')^2]^{3/2}}{y''}}$$

Note : If the tangent is parallel to y-axis, to find the radius of curvature we use $\boxed{\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{d^2x/dy^2}}$

Radius of Curvature in Parametric Form :-

Let $x = x(t)$ and $y = y(t)$ where t is a parameter be the equation of the curve in parametric form.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy/dt}{dx/dt}\right)$$

$$= \frac{d}{dt}\left[\frac{dy/dt}{dx/dt}\right] \cdot \frac{dt}{dx}$$

$$= \frac{\frac{dx}{dt} \cdot \frac{d^2y}{dt^2} - \frac{dy}{dt} \cdot \frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2} \cdot \frac{dt}{dx}$$

$$= \left[\frac{dx}{dt} \cdot \frac{d^2y}{dt^2} - \frac{d^2x}{dt^2} \cdot \frac{dy}{dt}\right]\left(\frac{dt}{dx}\right)^3$$

$$\frac{d^2y}{dx^2} = \frac{\frac{dx}{dt} \cdot \frac{d^2y}{dt^2} - \frac{dy}{dt} \cdot \frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$$

wkt

$$\rho = \frac{[1+(y')^2]^{3/2}}{y''}$$

$$= \frac{\left[1 + \left(\frac{dy/dt}{dx/dt}\right)^2\right]^{3/2}}{\left[\frac{dx}{dt} \cdot \frac{d^2y}{dt^2} - \frac{dy}{dt} \cdot \frac{d^2x}{dt^2}\right]}\left(\frac{dx}{dt}\right)^3$$

$$\boxed{\rho = \frac{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{3/2}}{\frac{dx}{dt} \cdot \frac{d^2y}{dt^2} - \frac{dy}{dt} \cdot \frac{d^2x}{dt^2}}}$$

Note : Writing $\frac{dx}{dt} = x'$, $\frac{dy}{dt} = y'$, $\frac{d^2x}{dt^2} = x''$ and $\frac{d^2y}{dt^2} = y''$

$$\boxed{\rho = \frac{[(x')^2+(y')^2]^{3/2}}{x'y''-y'x''}}$$

Problems :-

1) Find the radius of curvature of the curve $y = c\cosh\frac{x}{c}$.

$$\frac{dy}{dx} = c\sinh\frac{x}{c}\cdot\frac{1}{c} = \sinh\frac{x}{c}$$

$$\frac{d^2y}{dx^2} = \cosh\frac{x}{c}\cdot\frac{1}{c}$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{[1+\sinh^2 x/c]^{3/2}}{\frac{1}{c}\cosh\frac{x}{c}}$$

$$= \frac{c\cosh^3 x/c}{\cosh x/c}$$

$$\boxed{\rho = c\cosh^2 x/c = \frac{y^2}{c}}$$

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2) Find the radius of curvature at any point on the curve $ay^2 = x^3$.

$$2ay\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{2ay} = \frac{3x^2}{2a\left(\frac{x^3}{a}\right)^{1/2}} = \frac{3}{2}\sqrt{\frac{x}{a}}$$

$$\frac{d^2y}{dx^2} = \frac{3}{2\sqrt{a}}\,\frac{1}{2\sqrt{x}} = \frac{3}{4\sqrt{ax}}$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left(1+\frac{9x}{4a}\right)^{3/2}}{\frac{3}{4\sqrt{ax}}}$$

$$= \frac{(4a+9x)^{3/2}\,\cancel{4}\,.(ax)^{1/2}}{2\cancel{8}a^{3/2}(3)}$$

$$\boxed{\rho = \frac{(4a+9x)^{3/2}\sqrt{x}}{6a}}$$

3) Show that the radius of curvature at any point on the curve $x^{2/3}+y^{2/3}=a^{2/3}$ is $3(axy)^{1/3}$.

$$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}y'=0$$

$$\boxed{y' = -\left(\frac{y}{x}\right)^{1/3}}$$

$$y'' = -\frac{\left(x^{1/3}\frac{1}{3y^{2/3}}y' - y^{1/3}\frac{1}{3x^{2/3}}\right)}{x^{2/3}}$$

$$= \frac{-1}{x^{2/3}}\left[\frac{-1}{3y^{1/3}} - \frac{y^{1/3}}{3x^{2/3}}\right]$$

$$= \frac{1}{3x^{2/3}}\left[y^{-1/3}+y^{1/3}x^{-2/3}\right]$$

$$= \frac{1}{3x^{2/3}}\left[\frac{1}{y^{1/3}}+\frac{y^{1/3}}{x^{2/3}}\right]$$

$$= \frac{1}{3x^{2/3}}\frac{(x^{2/3}+y^{2/3})}{(x^2y)^{1/3}}$$

$$y'' = \frac{a^{2/3}}{3(x^4y)^{1/3}}$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left[1+\left(\frac{y}{x}\right)^{2/3}\right]^{3/2}}{\frac{a^{2/3}}{3(x^4y)^{1/3}}}$$

$$= \frac{a^{\cancel{2/3 \times 3/2}}}{x\,a^{2/3}}\,3(x^4y)^{1/3}$$

$$= 3\, a^{1/3} x^{4/3} y^{4/3}$$

$$\boxed{\rho = 3(axy)^{1/3}}$$

Q) Find radius of curvature of the curve $x^3+y^3 = 3axy$ at point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$.

$$\cancel{3}x^2 + \cancel{3}y^2\frac{dy}{dx} = \cancel{3}a\left[y + x\frac{dy}{dx}\right]$$

$$(y^2 - ax)\frac{dy}{dx} = ay - x^2$$

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

$$\left(\frac{dy}{dx}\right)_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} = \frac{\frac{3a^2}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - \frac{3a^2}{2}}$$

$$\left(\frac{dy}{dx}\right)_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} = -1$$

$$\frac{d^2y}{dx^2} = \frac{(y^2 - ax)\left[a\frac{dy}{dx} - 2x\right] - (ay - x^2)\left[2y\frac{dy}{dx} - a\right]}{(y^2 - ax)^2}$$

$$\left(\frac{d^2y}{dx^2}\right)_{(3a/2, 3a/2)} = \frac{\frac{3a^2}{4}(-a - 3a) + \frac{3a^2}{4}(-3a - a)}{\left(\frac{3a^2}{4}\right)^2}$$

$$= \frac{2 \times \cancel{3a^2}(-4a)\, 16^{4}}{\cancel{4}\, (3a^2)^2}$$

$$\left(\frac{d^2y}{dx^2}\right)_{(3a/2, 3a/2)} = \frac{-32a}{3a^2} = \frac{-32}{3a}$$

$$\rho = \frac{[1 + (y')^2]^{3/2}}{y''}$$

$$= \frac{(1 + 1)^{3/2}(3a)}{(-32)} = -2^{3/2 - 5}(3a)$$

$$= -2^{-7/2}\, 3a = \frac{-3a}{\sqrt{128}}$$

$$\therefore \boxed{\rho = \frac{3a}{\sqrt{128}}} = \boxed{\frac{3\sqrt{2}a}{16}}$$

5) Find the radius of curvature of the curve $y^2 = \frac{a^2(a-x)}{x}$ at the point $(a, 0)$

$$y^2 = \frac{a^3}{x} - a^2$$

$$2y\frac{dy}{dx} = \cancel{\alpha}\ \frac{a^3}{x} - a^2 \qquad \boxed{\frac{dy}{dx} = \frac{-a^3}{2x^2y}}$$

$$\frac{dx}{dy} = \frac{-2x^2y}{a^3}$$

$$\left(\frac{dx}{dy}\right)_{(a,0)} = 0$$

$$\frac{d^2x}{dy^2} = -\frac{2}{a^3}\left[x^2 + y\,2x\frac{dx}{dy}\right]$$

$$\left(\frac{d^2x}{dy^2}\right)_{(a,0)} = -\frac{a^2(2)}{a^3} = -\frac{2}{a}$$

$$\rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}$$

$$= \frac{(1+0)^{3/2}}{\left(\frac{-2}{a}\right)} = -\frac{a}{2}$$

$$\therefore \boxed{\rho = +\frac{a}{2}}$$

6) Find the radius of curvature at $(a, 0)$ of the curve $xy^2 = a^3 - x^3$.

$$y^2 = \frac{a^3}{x} - x^2$$

$$2y\frac{dy}{dx} = a^3\left(-\frac{1}{x^2}\right) - 2x$$

$y'_{(a,0)}$ is not defined.

$$\therefore \frac{dx}{dy} = \frac{-2y}{\left(\frac{a^3}{x^2} + 2x\right)}$$

$$\left(\frac{dx}{dy}\right)_{(a,0)} = 0$$

$$\frac{d^2x}{dy^2} = -2\left[\frac{\left(\frac{a^3}{x^2}+2x\right)(1) - y\left[-2a^3\frac{1}{x^3}\frac{dx}{dy}+\frac{2dx}{dy}\right]}{\left(\frac{a^3}{x^2}+2x\right)^2}\right]$$

$$\frac{d^2x}{dy^2} = -\frac{2}{3a}$$

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}} = \frac{1(3a)}{(-2)} = \frac{-3a}{2}$$

$$\boxed{\therefore \rho = -3a/2}$$

Show that for parabola $y^2 = 4ax$, ρ^2 varies as $(SP)^3$ where SP is the distance from point $P(x,y)$ to the focus $S(a,0)$.

$$2y\frac{dy}{dx} = 4a$$

$$\left(\frac{dy}{dx}\right) = \frac{2a}{y}$$

$$\left(\frac{d^2y}{dx^2}\right) = \frac{2a(-1)}{y^2}\frac{dy}{dx}$$

$$= \frac{-4a^2}{y^3}$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left[1+\frac{4a^2}{y^2}\right]^{3/2} y^3}{-4a^2}$$

$$\rho = \frac{(y^2+4a^2)^{3/2}}{(-4a^2)}$$

$$= \frac{(4ax+4a^2)^{3/2}}{-4a^2}$$

$$= \frac{8a^{3/2}(x+a)^{3/2}}{-4a^2}$$

$$= \frac{-2(x+a)^{3/2}}{\sqrt{a}}$$

$$\boxed{\rho^2 = \frac{4(x+a)^3}{a}}$$

The distance between P and S is $SP = (x+a)$

$\boxed{\therefore \rho^2 \propto (SP)^3}$

$\therefore \rho^2$ varies as $(SP)^3$

8) Find the radius of the curvature of the curve

$x = a\left(\cos t + \log \tan t/2\right)$

$y = a \sin t.$

$$\frac{dy}{dx} = \frac{a \cos t}{a\left(-\sin t + \frac{1 \sec^2 t/2}{2 \tan t/2}\right)}$$

$$= \frac{\cos t}{\left(-\sin t + \frac{1}{\sin t}\right)}$$

$$= \frac{\cos t (\sin t)}{\cos^2 t}$$

$$\frac{dy}{dx} = \tan t$$

$$\frac{d^2y}{dx^2} = \sec^2 t \frac{dt}{dx} = \frac{\cancel{a}\sec^2 t}{a} \frac{\sin t}{(\cos^2 t)}$$

$$= \frac{\sin t \sec^4 t}{a}$$

$$\frac{d^2y}{dx^2} = \frac{\cancel{a}\sin t \sec^4 t}{a}$$

$$\rho = \frac{a[1 + (y')^2]^{3/2}}{\cancel{a} y''}$$

$$= \frac{a(1 + \tan^2 t)^{3/2}}{\cancel{a}\sin t \sec^4 t}$$

$$= \frac{a \cancel{\sec^3 t}}{\cancel{a}\sin t \cancel{\sec^4 t}}$$

$$\boxed{\rho = \frac{a \cot t}{\cancel{a}}}$$

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The ρ

Centre of curvature :-

The point C which lies on the normal at P with the distance ρ from P is called centre of curvature at the point P.

Circle of curvature :-

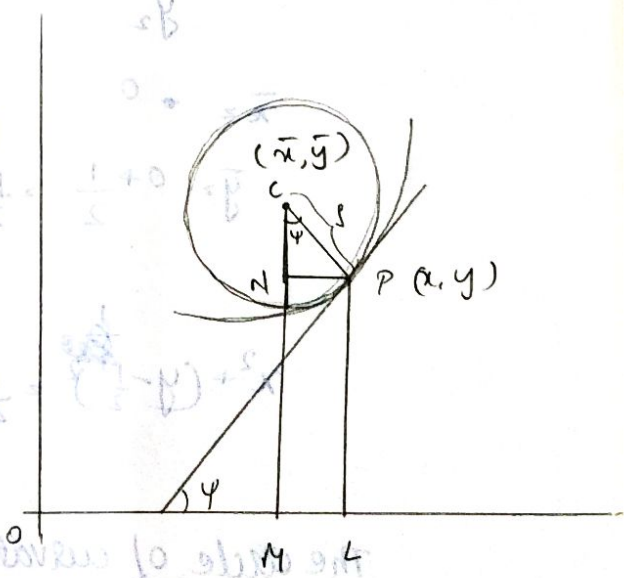
The circle with centre as centre of curvature at point P, radius as radius of curvature at point P is called circle of curvature at point P.

If $(\bar{x}, \bar{y})$ is centre of curvature, ρ is radius of curvature, then the equation of circle of curvature is

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

Formulae for centre of curvature :-

Let $C(\bar{x}, \bar{y})$ be centre of curvature at the point $P(x,y)$



From figure,

$$\bar{x} = OM = OL - ML$$

$$\bar{x} = x - NP$$

$$\boxed{\bar{x} = x - \rho\sin\psi}$$

$$\bar{y} = CM = CN + MN$$

$$= CN + y$$

$$\boxed{\bar{y} = y + \rho\cos\psi}$$

wkt $\boxed{\tan\psi = y_1}$

$$\cos\psi = \frac{1}{\sec\psi} = \frac{1}{\sqrt{1+\tan^2\psi}} = \frac{1}{\sqrt{1+y_1^2}}$$

$$\sin\psi = \tan\psi\cos\psi = \frac{y_1}{\sqrt{1+y_1^2}}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\therefore \boxed{\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)}$$

and

$$\therefore \boxed{\bar{y} = y + \frac{(1+y_1^2)}{y_2}}$$

Problems :-

1) Find circle of curvature at the point $(0,0)$ for $y = x^2$.

sol^n:

$y_1 = 2x \qquad y_{1(0,0)} = 0$

$y_2 = 2 \qquad y_{2(0,0)} = 2$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{1}{2}$$

$\bar{x} = 0$

$\bar{y} = 0 + \frac{1}{2} = \frac{1}{2}$

$$x^2 + \left(y - \tfrac{1}{2}\right)^2 = \frac{1}{4}$$

The circle of curvature is the circle with centre $(0, 1/2)$ and radius $\underline{1/2}$.

i.e $\boxed{x^2 + (y - 1/2)^2 = 1/4}$

2) Find the circle of curvature at the point (1,0) for the curve $y = x^3 - x^2$.

$$y_1 = 3x^2 - 2x \qquad y_1 = 3 - 2 = 1$$

$$y_2 = 6x - 2 \qquad y_2 = 4.$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{(1+1)^{3/2}}{4} = \frac{2\sqrt{2}}{4\,2} = \frac{1}{\sqrt{2}}$$

$$\bar{x} = 1 - \frac{1}{4\,2}(1+1) = \frac{1}{2}$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2} = 0 + \frac{1}{4}(1+1) = 1/2$$

$$\therefore \boxed{(x-1/2)^2 + (y-1/2)^2 = (1/\sqrt{2})^2}$$ is eq^n of circle of curvature.

with centre $(1/2, 1/2)$ and radius $1/\sqrt{2}$

3) If (α, β) is centre of curvature at any point (x,y) for the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ then show that $\alpha + \beta = 3(x+y)$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\, y' = 0$$

$$y' = -\sqrt{\frac{y}{x}}$$

$$y'' = -\frac{\left(\sqrt{x}\,\frac{1}{2\sqrt{y}}\,y' - \sqrt{y}\,\frac{1}{2\sqrt{x}}\right)}{x}$$

$$y'' = +\frac{1}{2x} + \frac{\sqrt{y}}{2x\sqrt{x}} = \frac{1}{2x}\left(1 + \sqrt{\frac{y}{x}}\right)$$

$$\alpha = x + \sqrt{\frac{y}{x}} \cdot \frac{2x}{\left(1+\sqrt{\frac{y}{x}}\right)}\left(1 + \frac{y}{x}\right)$$

$$= x + \sqrt{\frac{y}{x}}\;\frac{\left(\frac{x+y}{x}\right) 2x\sqrt{x}}{(\sqrt{x}+\sqrt{y})}$$

$$\boxed{\alpha = x + 2\sqrt{y}\,\frac{(x+y)}{(\sqrt{x}+\sqrt{y})}}$$

$$\beta = y + \frac{\left(1+\frac{y}{x}\right)2x}{\left(1+\sqrt{\frac{y}{x}}\right)}$$

$$= y + \frac{\left(\frac{x+y}{x}\right)2x\sqrt{x}}{(\sqrt{x}+\sqrt{y})}$$

$$\boxed{\beta = y + 2\sqrt{x}\,\frac{(x+y)}{(\sqrt{x}+\sqrt{y})}}$$

$$\alpha+\beta = x + 2\sqrt{y}\frac{(x+y)}{(\sqrt{x}+\sqrt{y})} + y + 2\sqrt{x}\frac{(x+y)}{(\sqrt{x}+\sqrt{y})}$$

$$= (x+y) + 2\frac{(x+y)}{\cancel{(\sqrt{x}+\sqrt{y})}}\cancel{(\sqrt{x}+\sqrt{y})}$$

$$\boxed{\alpha+\beta = 3(x+y)}$$

Hence proved.

4) Show that for the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ the radius of curvature at any point $P(x,y)$ is $\frac{CD^3}{ab}$ where C is the centre of the ellipse, D is the extremity on the conjugate diameter of CP.

soln: On the ellipse $x = a\cos\theta$, $y = b\sin\theta$

$$\frac{dy}{dx} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$$

$$\frac{d^2y}{dx^2} = +\frac{b}{a}\frac{\operatorname{cosec}^2\theta}{\cancel{(-a)\sin\theta}\ (-a)\sin\theta}$$

$$= -\frac{b}{a^2}\operatorname{cosec}^3\theta$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\rho = \frac{\left(1+\frac{b^2}{a^2}\cot^2\theta\right)^{3/2}}{-\frac{b}{a^2}\operatorname{cosec}^3\theta}$$

$$= \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{(-ba)}$$

$$\boxed{\therefore \rho = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab}}$$

The distance between the point $C(0,0)$, $D(a\sin\theta, b\cos\theta)$

is

$$\boxed{CD = (a^2\sin^2\theta + b^2\cos^2\theta)^{1/2}}$$

$$\boxed{\therefore \rho = \frac{(CD)^3}{ab}}$$

5) Show that if ρ_1, ρ_2 are radii of curvature at two extremities on the conjugate diameters of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ then $(\rho_1)^{2/3}+(\rho_2)^{2/3} = \frac{a^2+b^2}{(ab)^{2/3}}$

$$\rho_1 = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab}$$

The radius of curvature at an extremity $P(a\cos\theta, b\sin\theta)$

is $\rho_1 = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab}$

The radius of curvature at the extremity. D

$(a\sin\theta, b\cos\theta)$ is $\rho_2 = \frac{(a^2\cos^2\theta + b^2\sin^2\theta)^{3/2}}{ab}$

$$(\rho_1)^{2/3} + (\rho_2)^{2/3}$$

$$= \frac{a^2\sin^2\theta + b^2\cos^2\theta}{(ab)^{2/3}} + \frac{a^2\cos^2\theta + b^2\sin^2\theta}{(ab)^{2/3}}$$

$$\boxed{(\rho_1)^{2/3}+(\rho_2)^{2/3} = \frac{a^2+b^2}{(ab)^{2/3}}}$$

Hence proved.

6) Show that radius of curvature at any point on the cycloid $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$ is $4a\cos\theta/2$

$$\frac{dy}{dx} = \frac{a\sin\theta}{a(1+\cos\theta)}$$

$$= \frac{a \cdot 2\sin\theta/2 \cos\theta/2}{a \cdot 2\cos^2\theta/2}$$

$$\frac{dy}{dx} = \tan\theta/2$$

$$\frac{d^2y}{dx^2} = \frac{1}{2}\frac{\sec^2\theta/2}{a\,2\cos^2\theta/2}$$

$$\frac{d^2y}{dx^2} = \frac{1}{4a}\sec^4\theta/2$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{(1+\tan^2\theta/2)^{3/2}}{1} 4a\cos^4\theta/2$$

$$= \sec^3\theta/2 \cdot 4a\cos^4\theta/2$$

$$\therefore \boxed{\rho = 4a\cos\theta/2}$$

Hence showed.