unit4_tutorials

RV COLLEGE OF ENGINEERING
(An autonomous institution affiliated to VTU, Belgaum)
DEPARTMENT OF MATHEMATICS

FUNDAMENTS OF LINEAR ALGEBRA, CALCULUS AND NUMERICAL METHODS (MAT211AT)
Multiple Integrals

TUTORIAL SHEET-1

1. $\int_1^4\int_0^{\sqrt{4-x}} xy\,dy\,dx =$ ________
2. $\int_0^2\int_0^x (x+y)\,dy\,dx =$ ________
3. $\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{1-x^2}\sqrt{1-y^2}} =$ ________
4. Find the area bounded between the parabola $y^2 = 4ax$ and $x^2 = 4ay$.
5. Show that the area of one loop of the lemniscates $r^2 = a^2\cos 2\theta$ is $a^2/2$.
6. Find the area of one petal of the rose $r = a\sin 3\theta$
7. Find the area of the circle $r = a\sin\theta$ outside the cardioid $r = a(1-\cos\theta)$.
8. Find the volume of the paraboloid of revolution $x^2+y^2 = 4z$ cut off by the plane $z = 4$.
9. Find the volume of the region bounded by the paraboloid $az = x^2+y^2$ and the cylinder $x^2+y^2 = r^2$.
10. Find the volume of the portion of the sphere $x^2+y^2+z^2 = a^2$ lying inside the cylinder $x^2+y^2 = ax$.
11. Find the volume cut off the sphere $x^2+y^2+z^2 = a^2$ by the cone $x^2+y^2 = z^2$.
12. Change the order of the integration in the integrals:
 a) $\int_0^a\int_0^x \frac{\cos y}{\sqrt{(a-x)(a-y)}}\,dy\,dx$
 b) $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$
 c) $\int_0^a\int_y^a \frac{x}{x^2+y^2}\,dx\,dy$
 d) $\int_0^3\int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$

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① $\int_1^4\int_0^{\sqrt{4-x}} xy\,dy\,dx$ (x: 1→4, y: 0→$\sqrt{4-x}$)

$= \int_1^4 \frac{xy^2}{2}\Big|_0^{\sqrt{4-x}} dx$

$= \int_1^4 \frac{x(4-x)}{2}\,dx$

$= \frac{1}{2}\left[\frac{4x^2}{2} - \frac{x^3}{3}\right]_1^4$

$= \frac{1}{2}\left[2(4)^2 - \frac{4^3}{3} - \left(2 - \frac{1}{3}\right)\right]$

$= \frac{1}{2}\left[32 - \frac{64}{3} - \frac{5}{3}\right]$

$= \frac{1}{2}\times\frac{32\times 3 - 59}{3}$

$I = 37/6$

② $\int_0^2\int_0^x x+y\,dy\,dx$

$= \int_0^2 xy + \frac{y^2}{2}\Big|_0^x dx$

$= \int_0^2 x^2 + \frac{x^2}{2}\,dx$

$= \frac{3}{2}\int_0^2 x^2\,dx$

$= \frac{3}{2}\,\frac{x^3}{3}\Big|_0^2$

$= \frac{3}{2}\times\frac{8}{3}$

$I = 4$

③ $\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{1-x^2}\sqrt{1-y^2}}$ $\int\frac{1}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a)$

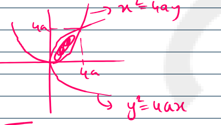
$= \int_0^1 \frac{1}{\sqrt{1-y^2}}\times\sin^{-1}(x)\Big|_0^1 dy$

$= \frac{\pi}{2}\int_0^1 \frac{1}{\sqrt{1-y^2}}\,dy$

$= \frac{\pi}{2}\sin^{-1}(y)\Big|_0^1$

$= \frac{\pi^2}{4}$

④ $y^2 = 4ax$, $x^2 = 4ay$



$y^2 = 4ax$
$\left(\frac{x^2}{4a}\right)^2 = 4ax$
$\frac{x^4}{4^2a^2} = 4ax$
$x^3 = (4a)^3$
$x = 4a$
$y = 4a$

$A = \int_0^{4a}\int_{x^2/4a}^{\sqrt{4ax}} dy\,dx$

$= \int_0^{4a} \sqrt{4ax} - \frac{x^2}{4a}\,dx$

$= 2\sqrt{a}\,\frac{x^{3/2}}{3/2} - \frac{1}{4a}\frac{x^3}{3}\Big|_0^{4a}$

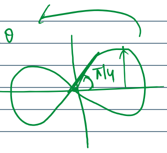
$= \frac{4\sqrt{a}\,(4a)^{3/2}}{3} - \frac{1}{4a}\frac{(4a)^3}{3}$

$= \frac{4\sqrt{a}\,(2\sqrt{a})^3}{3} - \frac{(4a)^2}{3}$

$= \frac{8\times 4\times a^2}{3} - \frac{16a^2}{3}$

$A = \frac{16a^2}{3}$

⑤ $r^2 = a^2\cos 2\theta$



$A = \int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta$

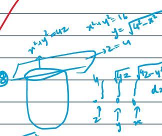
$= \int_0^{\pi/4} \frac{r^2}{2}\Big|_0^{a\sqrt{\cos 2\theta}} d\theta$

$= \frac{a^2}{2}\int_0^{\pi/4}\cos 2\theta\,d\theta$

$A = \frac{a^2}{2}\,\frac{\sin 2\theta}{2}\Big|_0^{\pi/4} = \frac{a^2}{2}\times\frac{1}{2} = \frac{a^2}{4}$

$A_{loop} = \frac{a^2}{2}$

ROUGH



$4\int_0^4\int_0^{\sqrt{4-x^2}}\int_{x^2+y^2/4}^{4} dz\,dy\,dx$

$\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{\frac{x^2+y^2}{4}}^{4} dz\,dy\,dx$

$\int\int 4 - \frac{(x^2+y^2)}{4}\,dy\,dx$

$\int \frac{1}{4}\int (16 - x^2 - y^2)$

$\int \frac{1}{4}\left(16-x^2)y - \frac{y^3}{3}\right|_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}$ $t = \sqrt{16-x^2}$

$(16-x^2)^{3/2} - \frac{(16-x^2)^{3/2}}{3} - \left(-t^3 + \frac{t^3}{3}\right)$

$t^3 - \frac{t^3}{3} + t^3 - \frac{t^3}{3}$

$\frac{2t^3}{3}$

$4\int_0^4 \frac{1}{4}\times\frac{4}{3}(4^2-x^2)^{3/2}\,dx$ let $x = 4\sin\theta$, $dx = 4\cos\theta\,d\theta$

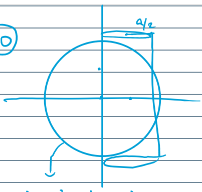
$\frac{4}{3}\int (4^2 - 4^2\sin^2\theta)^{3/2}\,4\cos\theta\,d\theta$

$\frac{4\times 4^4}{3}\int\cos^4\theta\,d\theta$

$\frac{4^4}{3}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{16\pi}{3}$

ROUGH

⑩



$x^2 + y^2 = ax$

$x^2 - 2\frac{a}{2}x + \left(\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2 + y^2 = 0$

$\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$

$\left(\frac{a}{2}, 0\right)$, $r = \frac{a}{2}$

$x^2 + y^2 + z^2 = a^2$
$\Rightarrow z = \sqrt{a^2 - x^2 + y^2}$
$z = \sqrt{a^2 - r^2}$

$r^2 = a\,r\cos\theta$
$r = a\cos\theta$

$V_I = \int_0^{\pi/2}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$ (θ, r, z)

$= \int_0\int_{a(1-\cos\theta)} \sqrt{a^2-r^2}\,r\,dr\,d\theta$ let $a^2 - r^2 = t$, $-2r\,dr = dt$

$= \int_0\int_{t=a^2} \sqrt{t}\,\frac{dt}{-2}\,d\theta$ at $r = 0$, $t = a^2$; $r = a\cos\theta$, $t = a^2(1-\cos^2\theta) = (a\sin\theta)^2$

$= \int_0 \frac{-1}{2}\,\frac{t^{3/2}}{3/2}\,d\theta$

$= \int_0 \frac{-1}{3}\left[(a\sin\theta)^{2\times\frac{3}{2}} - a^{2\times\frac{3}{2}}\right]d\theta$

$= -\frac{a^3}{3}\int_0^{\pi/2}(\sin^3\theta - 1)\,d\theta$

$= -\frac{a^3}{3}\int_0 \underbrace{\sin^2\theta}_{1-\cos^2\theta}\sin\theta\,d\theta - \int_0 1\,d\theta$

let $\cos\theta = t$, $-\sin\theta\,d\theta = dt$; at $\theta = 0$, $t = 1$; $\theta = \frac{\pi}{2}$, $t = 0$

$= -\frac{a^3}{3}\int_1^0 t^2 - 1\,dt - \theta\Big|_0^{\pi/2}$

$= -\frac{a^3}{3}\left[\frac{t^3}{3} - t\Big|_1^0 - \frac{\pi}{2}\right]$

$= -\frac{a^3}{3}\left(-\frac{1}{3} + 1 - \frac{\pi}{2}\right)$

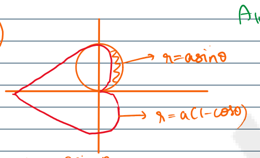
$= \frac{a^3}{3}\left(\frac{2}{3} + \frac{\pi}{2}\right)$

$V_I = \frac{a^3}{3}\left(\frac{4 + 3\pi}{6}\right)$

$V_{tot} = 4V_I = \frac{4\times a^3}{3\times 6}(3\pi + 4)$

$V_{tot} = \frac{2a^3}{9}(3\pi + 4)$

⑦



$r = a\sin\theta$
$r^2 = ar\sin\theta \longrightarrow x^2 + y^2 = ax$
$x^2 + y^2 = ay$
$x^2 + y^2 - 2\frac{a}{2}y + \left(\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2 = 0$
$x^2 + \left(y - \frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2$
centre $\left(0, \frac{a}{2}\right)$ radius $= a/2$

$A = \int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta$

$= \int_0^{\pi/2} \frac{r^2}{2}\Big|_{a(1-\cos\theta)}^{a\sin\theta} d\theta$

$= \frac{1}{2}\int a^2\left(\sin^2\theta - (1-\cos\theta)^2\right)d\theta$

$= \frac{a^2}{2}\int \sin^2\theta - (1 + \cos^2\theta - 2\cos\theta)\,d\theta$

$\frac{a^2}{2}\int \sin^2\theta - (\sin^2\theta + 2\cos^2\theta - 2\cos\theta)$

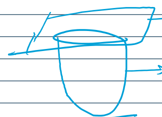
$\frac{a^2}{2}\int 2\cos\theta - 2\cos^2\theta\,d\theta$

$a^2\int \cos\theta - \left(\frac{1 + \cos 2\theta}{2}\right)d\theta$

$a^2\left[\sin\theta - \frac{1}{2}\theta - \frac{1}{2}\frac{\sin 2\theta}{2}\right]_0^{\pi/2}$

$A = a^2\left(1 - \frac{\pi}{4}\right)$

⑧



$\Rightarrow y^2 = 16 - x^2$

$V_I = \int_0^4\int_0^{\sqrt{16-x^2}}\int_{\frac{x^2+y^2}{4}}^{4} dz\,dy\,dx$ (x, y, z)

$= \int\int 4 - \frac{(x^2+y^2)}{4}\,dy\,dx$

$\int_x \frac{1}{4}\int_y 16 - x^2 - y^2\,dy$

$\int_x \frac{1}{4}\left(16 - x^2)y - \frac{y^3}{3}\right|_0^{\sqrt{16-x^2}}$

$\int_x \frac{1}{4}\left((16-x^2)^{3/2} - \frac{(16-x^2)^{3/2}}{3}\right)$

$\int_0^4 \frac{1}{6}(16-x^2)^{3/2}\,dx$ $\left\{\begin{array}{l} x = 4\sin\theta \\ dx = 4\cos\theta\,d\theta\end{array}\right\}$

$\frac{1}{6}\int_0^{\pi/2}(4^2 - 4^2\sin^2\theta)^{3/2}\,4\cos\theta\,d\theta$

$\frac{1}{6}\int (4^2)^{3/2}(\cos^2\theta)^{3/2}\times 4\cos\theta\,d\theta$

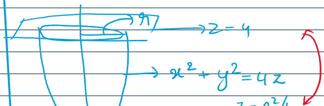
$\frac{1}{6}\times 4^3\times 4\int_0^{\pi/2}\cos^4\theta\,d\theta$

$V_I = \frac{128}{3}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = 8\pi$

$V_{tot} = 4V_I = 32\pi$

$(\cos^2\theta)^2$
$\left(\frac{1 + \cos 2\theta}{2}\right)^2$
$\frac{1}{4}(1 + \cos^2 2\theta + 2\cos 2\theta)$
$\int\frac{1}{4}\left(1 + 2\cos 2\theta + \frac{1 + \cos 4\theta}{2}\right)$

polar form:



$4 = \frac{r^2}{4}$
$z = r^2/4$
$r = 4$

$\int_0^{2\pi}\int_0^4\int_{r^2/4}^{4} r\,dz\,dr\,d\theta$ (θ, r, z)

$\int_0\int_0 r\left(4 - \frac{r^2}{4}\right)dr\,d\theta$

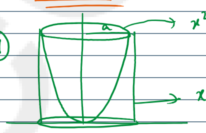
$\int_0^{2\pi} \frac{4r^2}{2} - \frac{r^4}{16}\Big|_0^4\,d\theta$

$\int_0 \frac{4^3}{2} - \frac{4^4}{16}\,d\theta$

$\int_0^{2\pi} 32 - 16\,d\theta$

$= 32\pi$

⑨



$\Rightarrow z = r^2/a$

$4\int_0^{\pi/4}\int_0^{r}\int_0^{r^2/a} r\,dz\,dr\,d\theta$ (θ, r, z)

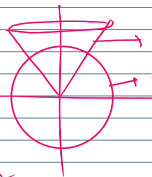
$= \int_0\int_0 \frac{r^3}{a}\,dr\,d\theta$

$= \int_0 \frac{1}{a}\,\frac{r^4}{4}\Big|_0^r d\theta$

$= \frac{r^4}{4a}\,\theta\Big|_0^{2\pi}$

$= \frac{r^4}{4a}\times 2\pi \Rightarrow V = \frac{\pi r^4}{2a}$

⑪

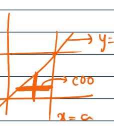


$r^2 = a^2 - r^2$
$2r^2 = a^2$
$r = a/\sqrt{2}$

$\int_0^{2\pi}\int_0^{a/\sqrt{2}}\int_r^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$ (θ, r, z)

$= \int\int\left(r - \sqrt{a^2-r^2}\right) r\,dr\,d\theta$

⑫ a) $\int_0^a\int_0^x \frac{\cos y}{\sqrt{(a-x)(a-y)}}\,dy\,dx$ (x, y)



change order: $\int_0^a\int \frac{\cos y}{\ldots}\,dx\,dy$

⑫ b) $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$

$$= 8\int_0 \int_r (a - \sqrt{a^2-r^2})\, r\, dr\, d\theta$$

$$= 8\int\left[\int_r r^2\,dr - \int_r \sqrt{a^2-r^2}\, r\,dr\right] d\theta \quad \text{let } a^2-r^2 = t,\ -2r\,dr = dt$$

$$= \int_0 \left[\frac{r^3}{3}\Big|_0^{a/\sqrt2} d\theta - \int \sqrt t \frac{dt}{-2}\right]$$

$$= \int_0 \frac{(a/\sqrt2)^3}{3} - \frac{1}{2}\frac{(a^2-r^2)^{3/2}}{3/2}\Big|_0^{a/\sqrt2} d\theta$$

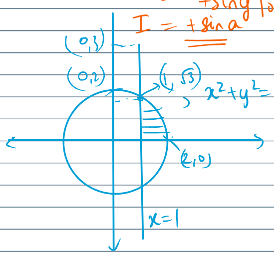
$$= \int_0 \frac{a^3}{6\sqrt2} + \frac{1}{3}\left(\frac{a^2}{2}\right)^{3/2} d\theta$$

$$= \int_0 \frac{a^3}{6\sqrt2} + \frac{a^3}{6\sqrt2}\, d\theta$$

$$= \frac{a^3}{3\sqrt2}\ \theta\Big|_0^{2\pi}$$

$$= \frac{a^3}{3\sqrt2} \times 2\pi \Rightarrow V = \frac{2\pi a^3}{3\sqrt2}$$

(12) d) $\int_1^3 \int_{x}^{\sqrt{4-y}} (x+y)\, dx\, dy$



$$= \int_{x=1}^{2} \int_{y=0}^{\sqrt{4-x}} (x+y)\, dy\, dx$$

$$= \int_1^2 xy + \frac{y^2}{2}\Big|_0^{\sqrt{4-x}} dx$$

$$= \int_1^2 \underbrace{x\sqrt{4-x}}_{} + \frac{4-x}{2}\, dx$$

how to integrate?

change order [COO]
$$= \int_{y}^{a} \int_{x}^{a} \frac{\cos y}{\sqrt{(a-x)(a-y)}}\, dx\, dy$$

$$= \int_y \frac{\cos y}{\sqrt{a-y}} \int_{x=y}^{a} \frac{1}{\sqrt{a-x}}\, dx\, dy \qquad (a-x)^{-1/2}$$

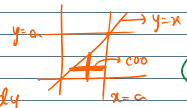
$$= \int_y \frac{\cos y}{\sqrt{a-y}}\ \frac{(a-x)^{1/2}}{(-1)}\Big|_y^a dy$$

$$= \int_0^a \frac{\cos y}{\sqrt{a-y}} \times \frac{0-\sqrt{a-y}}{-1}$$

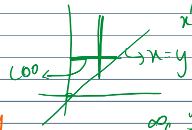
$$= +\int_0^a \cos y\, dy$$

$$= +\sin y\Big|_0^a$$

$$I = +\sin a$$



(12) b) $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$



$$= \int_{y=0}^{\infty} \int_{x=0}^{y} \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_0^\infty \frac{e^{-y}}{y}\, x\Big|_0^y dy$$

$$= \int_0^\infty e^{-y}\, dy$$

$$= -e^{-y}\Big|_0^\infty$$

$$= -e^{-\infty} - -e^{0}$$

$$= -\frac{1}{e^\infty} + 1$$

$$I = 1$$

$$\frac14 (1+\cos^2 2\theta + 2\cos 2\theta)$$

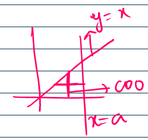
$$\int \frac14\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)$$

$$\frac14\ \theta + \frac{2\sin 2\theta}{2} + \frac{\theta}{2} + \frac{\sin 4\theta}{2\times4}\Big|_0^{\pi/2}$$

$$\frac14\left(\frac{3\times\pi}{2\ \ 2} + \sin\pi + \sin 2\pi\right)$$

$$= \frac{3\pi}{4\times4}$$

(12) c) $\int_0^a \int_y^a \frac{y}{x^2+y^2}\, dx\, dy$



$$= \int_{x=0}^{a} \int_{y=0}^{x} \frac{y}{x^2+y^2}\, dy\, dx$$

$$= \int \int \frac1t \frac{dt}{2}\, dx \qquad \text{let } x^2+y^2 = t,\ 2y\,dy = dt$$

$$= \int \frac12 \log|x^2+y^2|\Big|_0^x dx$$

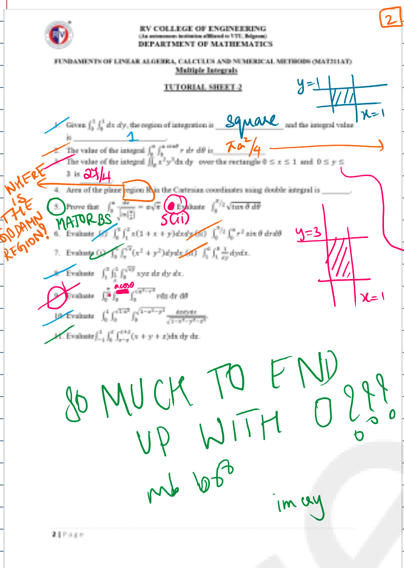
$$= \int \frac12 \log(2x^2) - \log(x^2)\, dx$$

$$= \int \frac12 \log\left(\frac{2x^2}{x^2}\right) dx$$

$$= \frac12 \log 2 \int_0^a dx$$

$$= \frac12 \log 2\ x\Big|_0^a$$

$$I = \frac{a\log 2}{2}$$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Multiple Integrals

TUTORIAL SHEET-2

1. Given $\int_0^1\int_0^1 dx\,dy$, the region of integration is square and the integral value is 1.
2. The value of the integral $\int_0^\pi \int_0^{a\sin\theta} r\,dr\,d\theta$ is $\pi a^2/4$
3. The value of the integral $\iint_R x^2y^3\,dx\,dy$ over the rectangle $0 \le x \le 1$ and $0 \le y \le 3$ is 27/4
4. Area of the plane region R in the Cartesian coordinates using double integral is ______.
5. Prove that $\int_0^a \frac{dx}{\sqrt{\ln\left(\frac{a}{x}\right)}} = a\sqrt\pi$. Evaluate $\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta$
6. Evaluate (i) $\int_0^3\int_1^2 x(1+x+y)\,dx\,dy$ (ii) $\int_0^{\pi/2}\int_0^a r^2\sin\theta\,dr\,d\theta$
7. Evaluate (i) $\int_0^1\int_x^{\sqrt x}(x^2+y^2)\,dy\,dx$ (ii) $\int_1^a\int_1^b \frac{1}{xy}\,dy\,dx$.
8. Evaluate $\int_1^2\int_{\frac1x}^{2}\int_0^{\sqrt{xy}} xyz\,dz\,dy\,dx$.
9. Evaluate $\int_0^{\pi}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} rdz\,dr\,d\theta$
10. Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dzdydx}{\sqrt{1-x^2-y^2-z^2}}$.
11. Evaluate $\int_{-1}^1\int_0^2\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz$.

WHERE IS THE GODAMN REGION?!

MAJOR BS

S(ii)

SO MUCH TO END UP WITH 0???

mb bro

im ay

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[2] $\int_0^\pi \frac{r^2}{2}\Big|_0^{a\sin\theta} d\theta$

$$\int_0^\pi \frac{a^2\sin^2\theta}{2}\, d\theta$$

$$\frac{a^2}{2}\int_0^\pi \frac{1-\cos 2\theta}{2}\, d\theta$$

$$\frac{a^2}{4}\left[\theta - \frac12 \sin 2\theta\right]_0^\pi$$

$$= \frac{\pi a^2}{4}$$

[3] $\int_{y=0}^{3}\int_{x=0}^{1} x^2y^3\, dx\, dy$

$$= \int_{y=0}^{3} \frac{x^3}{3}y^3\Big|_0^1 dy$$

$$= \int_0^3 \frac{y^3}{3}\, dy$$

$$= \frac13 \frac{y^4}{4}\Big|_0^3 = \frac{27}{4}$$

DOUBT!!

[5] $\int_{x=0}^{a} \frac{dx}{\sqrt{\ln(a/x)}}$

$$\ln\frac{a}{x} = t \Rightarrow x = ae^{-t}$$

$$\frac{x}{a}\times -\frac{1}{x^2}\,dx = dt$$

$$dx = -ax\,dt$$

$$= -a^2e^{-t}dt$$

at $x=0,\ t=\infty$
$x=a,\ t=0$

$$I = \int_{t=\infty}^{0} \frac{-a^2e^{-t}dt}{\sqrt t}$$

$$= -a^2\int_\infty^0 e^{-t}\, t^{-1/2}\, dt \qquad \text{ILATE}$$

$$= -a^2\left[t^{-1/2}\int e^{-t}dt - \int -\frac12 t^{-3/2}\int e^{-t}\,dt\right].$$

[6] i) $\int_0^3 \int_1^2 x(1+x+y)\, dx\, dy$

$$I = \int_0^3\int_1^2 (1+y)x + x^2\, dx\, dy$$

$$= \int_0^3 (1+y)\frac{x^2}{2} + \frac{x^3}{3}\Big|_1^2 dy$$

$$= \int_0^3 2(1+y) + \frac83 - \left[\frac{(1+y)}{2} + \frac13\right] dy$$

$$= \int_0^3 \frac32(1+y) + \frac73\, dy$$

$$\frac32 + \frac73 \qquad = \frac{9+14}{6} = \frac{23}{6}$$

$$= \int_0^3 \frac32 + \frac{3y}{2} + \frac73\, dy$$

$$= \int_0^3 \frac{23}{6} + \frac{3y}{2}\, dy$$

$$= \frac{23}{6}y + \frac{3y^2}{4}\Big|_0^3$$

$$= \frac{23}{6_2}\times 3 + \frac{3^3}{4}$$

$$I = \frac{46+27}{4} = \frac{73}{4}$$

(ii) $\int_0^{\pi/2}\int_0^a r^2\sin\theta\, dr\, d\theta$

$$= \int_0^{\pi/2} \frac{r^3}{3}\sin\theta\Big|_{r=0}^{a} d\theta$$

$$= \int_0^{\pi/2} \frac{a^3}{3}\sin\theta\, d\theta$$

$$= \frac{a^3}{3} -\cos\theta\Big|_0^{\pi/2}$$

$$= \frac{a^3}{3}\left[0 - -1\right]$$

$$I = a^3/3$$

S(ii) $I = \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta \longrightarrow ①$

$$I = \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta \longrightarrow ②$$

$$2I = \int_0^{\pi/2} \sqrt{\tan\theta} + \sqrt{\cot\theta}\, d\theta\ [①+②]$$

$$= \int_0^{\pi/2} \sqrt{\tan\theta} + \frac{1}{\sqrt{\tan\theta}}\, d\theta$$

[7] i) $\int_0^1\int_x^{\sqrt x} x^2+y^2\, dy\, dx$

$$= \int_0^1 x^2y + \frac{y^3}{3}\Big|_x^{\sqrt x} dx$$

$$= \int_0^1 x^{5/2} + \frac{x^{3/2}}{3} - \left(x^3 + \frac{x^3}{3}\right) dx$$

$$= \int_0^1 x^{5/2} + \frac{x^{3/2}}{3} - \frac{4x^3}{3}\, dx$$

$$= \frac{x^{7/2}}{7/2} + \frac13\frac{x^{5/2}}{5/2} - \frac43\frac{x^4}{4}\Big|_0^1$$

$$= \frac27 + \frac{2}{15} - \frac13$$

$$= \frac{15+14-35}{7\times15} = \frac{-6}{105}$$

[7] ii) $\int_1^a\int_1^b \frac{1}{xy}\, dy\, dx$

$$= \int_1^a \frac1x \log y\Big|_1^b dx$$

$$= \int_1^a \frac1x(\log b - \log 1)\, dx$$

$$= \log b\ \ \log x\Big|_1^a$$

$$I = \log a \cdot \log b$$

[9] $\int_0^{\pi/2}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$

$$= \iint z\Big|_0^{\sqrt{a^2-r^2}} r\,dr\,d\theta$$

$$= \int\int_0^{a\cos\theta} \sqrt{a^2-r^2}\ r\,dr\,d\theta$$

$$= \int -\frac12\frac{(a^2-r^2)^{3/2}}{3/2}\Big|_0^{a\cos\theta} d\theta$$

[10] $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$

$$= \iint_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{(\sqrt{1-x^2-y^2})^2 - z^2}}$$

$$= \iint \sin^{-1}\frac{z}{\sqrt{1-x^2-y^2}}\Big|_0^{\sqrt{1-x^2-y^2}} dx\,dy$$

$$= \int\int_0^{\sqrt{1-x^2}} \frac\pi2\, dy\, dx$$

$$= \int \frac\pi2 y\Big|_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \frac\pi2\sqrt{1-x^2}\, dx$$

$$= \frac\pi2\left[\frac{x}{2}\sqrt{1-x^2} + \frac12\sin^{-1}x\right]_0^1$$

$$= \frac\pi2\left(\frac12\frac\pi2\right) = \frac{\pi^2}{8}$$

[11] $\int_{-1}^{1}\int_0^2\int_{x-z}^{x+z} (x+y+z)\, dx\, dy\, dz$

$= \frac{\pi}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1$

$= \frac{\pi}{2}\left(\frac{1}{2}\cdot\frac{\pi}{2}\right) = \frac{\pi^2}{8}$

[8] $\int_1^3\int_{1/x}^1\int_0^{\sqrt{xy}} xyz\,dz\,dy\,dx$

$= \int\int xy\frac{z^2}{2}\Big|_0^{\sqrt{xy}} dy\,dx$

$= \int\int_{1/x}^1 \frac{x^2y^2}{2}\,dy\,dx$

$= \int \frac{x^2}{2}\frac{y^3}{3}\Big|_{1/x}^1 dx$

$= \int_1^3 \frac{x^2}{6}\left(1-\frac{1}{x^3}\right)dx$

$= \frac{1}{6}\int_1^3 x^2 - \frac{1}{x}\,dx$

$= \frac{1}{6}\left[\frac{x^3}{3} - \log x\right]_1^3$

$= \frac{1}{6}\left(9 - \log 3 - \left(\frac{1}{3} - \log 1\right)\right)$

$= \frac{1}{6}\left(\frac{26}{3} - \log 3\right)$

$I = \frac{13}{9} - \frac{\log 3}{6}$

$= \int_0^{\pi/2}\int_0^{a\cos\theta} \sqrt{a^2-r^2}\; r\,dr\,d\theta$

$= \int -\frac{1}{2}\frac{(a^2-r^2)^{3/2}}{3/2}\Big|_0^{a\cos\theta} d\theta$

$= -\frac{1}{3}\int_0^{\pi/2}(a^2 - a^2\cos^2\theta)^{3/2}\,d\theta$

$= -\frac{1}{3}\int_0 a^3\sin^3\theta\,d\theta$

$= \frac{1}{3}a^3 \quad \sin\theta\sin^2\theta\,d\theta$

$\sin\theta(1-\cos^2\theta)\,d\theta$

$\cos\theta = t$

$-\sin\theta\,d\theta = dt$

$= +\frac{1}{3}a^3\int_1^0 1-t^2\,dt$

$= \frac{1}{3}a^3\left[t - \frac{t^3}{3}\right]_1^0$

$= \frac{a^3}{3}\left[0 - \left(1 - \frac{1}{3}\right)\right]$

$= \left|-\frac{2a^3}{9}\right| = \frac{2a^3}{9}$

[11] $\int_{-1}^1\int_0^2\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz$

$= \int\int (x+z)y + \frac{y^2}{2}\Big|_{x-z}^{x+z} dx\,dz$

$= \int\int_0^2 (x+z)^2 - \frac{(x+z)^2}{2} - \left(x^2-z^2 + \frac{(x-z)^2}{2}\right)dx\,dz$

$= \int\int_0^2 \frac{(x+z)^2}{2} - \frac{(x-z)^2}{2} - x^2 + z^2\,dx\,dz$

$= \int\int \frac{1}{2}\left(x^2+z^2+2xz - (x^2+z^2-2xz)\right) - x^2+z^2\,dx\,dz$

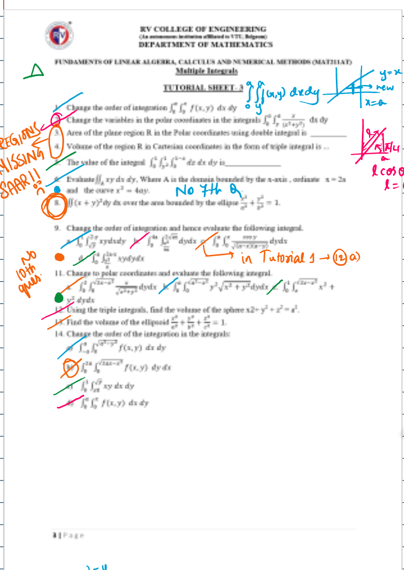
$= \int\int_0^2 \frac{1}{2}(4xz) - x^2 + z^2\,dx\,dz$

$= \int \cancel{2}z\frac{x^2}{\cancel{2}} - \frac{x^3}{3} + xz^2\Big|_0^z dz$

$= \int z^3 - \frac{z^3}{3} + z^3\,dz$

$= \int_{-1}^1 \frac{5z^3}{3}\,dz$

$= \frac{5}{3}\frac{z^4}{4}\Big|_{-1}^1 = \frac{5}{12}(1+1) = 5/6$


RV COLLEGE OF ENGINEERING

(An autonomous institution affiliated to VTU, Belgaum)

DEPARTMENT OF MATHEMATICS

FUNDAMENTALS OF LINEAR ALGEBRA, CALCULUS AND NUMERICAL METHODS (MAT211AT)

Multiple Integrals

TUTORIAL SHEET - 3

1. Change the order of integration $\int_0^{\infty}\int_0^{x} f(x,y)\,dx\,dy$ ...
2. Change the variables in the polar coordinates in the integrals $\int_0^a\int_0^{\sqrt{a^2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy$
3. Area of the plane region R in the Polar coordinates using double integral is ...
4. Volume of the region R in Cartesian coordinates in the form of triple integral is ...
5. The value of the integral $\int_0^1\int_{y^2}^1\int_0^{1-x} dz\,dx\,dy$ is ...
6. Evaluate $\iint_A xy\,dx\,dy$, Where A is the domain bounded by the x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.
8. $\iint (x+y)^2\,dy\,dx$ over the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$.
9. Change the order of integration and hence evaluate the following integral.
11. Change to polar coordinates and evaluate the following integral.
12. Using the triple integrals, find the volume of the sphere $x^2+y^2+z^2 = a^2$.
13. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$.
14. Change the order of the integration in the integrals:

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$\iint f(x,y)\,dx\,dy$ y=x new x=4

REGIONAL MISSING SAAR!!

No 7th Q

No 10th ques

in Tutorial 1 → (13)(a)

$r\cos\theta = a$

$r = a\sec\theta$

$x = r\cos\theta,\; y = r\sin\theta$

$\to x^2+y^2 = r^2$

$I = \int_0^{\pi/4}\int_0^{a\sec\theta} \frac{r\cos\theta}{r^2}\,r\,dr\,d\theta$

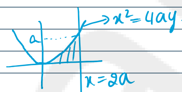
$= \int \cos\theta\,\frac{r^2}{2}\Big|_0^{a\sec\theta} d\theta$

$= \int_0^{\pi/4}\frac{a^2}{2}\cos\theta\times\sec^2\theta\,d\theta$

$= \frac{a^2}{2}\int_0^{\pi/4}\sec\theta\,d\theta$

$= \frac{a^2}{2}\ln|\sec\theta+\tan\theta|_0^{\pi/4}$

$I = \frac{a^2}{2}\left(\ln|\sqrt{2}+1| + \ln|1+0|\right)$

[6] 

$\int_0^{2a}\int_0^{x^2/4a} xy\,dy\,dx$

$= \int x\frac{y^2}{2}\Big|_0^{x^2/4a} dx$

$= \int \frac{x}{2}\left(\frac{x^2}{4a}\right)^2 dx$

$= \int_0^{2a}\frac{x^5}{2\times16a^2}\,dx$

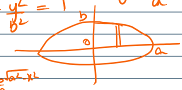
$= \frac{1}{32a^2}\frac{x^6}{6}\Big|_0^{2a}$

$= \frac{1}{32a^2}\frac{2^6a^6}{6}$

$I = \frac{a^4}{3}$

[8] $\iint (x+y)^2\,dy\,dx$

$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \to y = \frac{b}{a}\sqrt{a^2-x^2}$



$\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} (x+y)^2\,dy\,dx$

x^2+y^2+2xy

$= \int x^2y + \frac{y^3}{3} + \cancel{2}x\frac{y^2}{\cancel{2}}\Big|_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$

$= \int \frac{x^2b}{a}\sqrt{a^2-x^2} + \frac{b^3}{3a^3}(a^2-x^2)^{3/2} + \frac{xb^2}{a^2}(a^2-x^2)$

let $x = a\sin\theta$, $dx = a\cos\theta\,d\theta$; $a^2-x^2 = a^2(1-\sin^2\theta) = a^2\cos^2\theta$

$= \int_{\theta=0}^{\pi/2} a^2\sin^2\theta\frac{b}{\cancel{a}}\cancel{a}\cos\theta + \frac{b^3}{3\cancel{a^3}}\cancel{a^3}\cos^3\theta + a\sin\theta\frac{b^2}{\cancel{a^2}}\cancel{a^2}\cos^2\theta$

$a^2b\sin^2\theta\cos\theta + \frac{b^3}{3}\cos^3\theta + ab^2\sin\theta\cos^2\theta$

$\int\cos^3\theta = \cos\theta\cos^2\theta$

$\cos\theta(1-\sin^2\theta)$

$\int\cos\theta - \cos\theta\sin^2\theta$

$= \sin\theta - \int\cos\theta\sin^2\theta$

$t = \sin\theta$, $dt = \cos\theta\,d\theta$; $+\frac{b^3}{3}\sin\theta\Big|_0^{\pi/2} - \frac{b^3}{3}\int\cos\theta\sin^2\theta$; $t = \cos\theta$, $dt = -\sin\theta\,d\theta$

$= \int_0^1 a^2b\,t^2\,dt + \frac{b^3}{3} - \frac{b^3}{3}\int_0^1 t^2\,dt - \int_1^0 ab^2t^2$

$= a^2b\frac{t^3}{3}\Big|_0^1 + \frac{b^3}{3} - \frac{b^3}{3}\frac{t^3}{3}\Big|_0^1 - ab^2\frac{t^3}{3}\Big|_1^0$

$= \frac{a^2b}{3} + \frac{b^3}{3} - \frac{1}{3}\frac{b^3}{3} + \frac{ab^2}{3}$

$= \frac{a^2b}{3} + \frac{2}{3}\frac{b^3}{3} + \frac{ab^2}{3}$

$= \frac{1}{3}b\left(a^2 + \frac{2}{3}b^2 + ab\right)$

[5] $\int_0^1\int_{y^2}^1\int_0^{1-x} dz\,dx\,dy$

$= \int\int z\Big|_0^{1-x} dx\,dy$

$= \int_0^1\int_{y^2}^1 1-x\,dx\,dy$

$= \int_0^1 x - \frac{x^2}{2}\Big|_{y^2}^1 dy$

$= \int_0^1 1 - \frac{1}{2} - \left(y^2 - \frac{y^4}{2}\right)dy$

$= \int \frac{1}{2} - y^2 + \frac{y^4}{2}\,dy$

$= \frac{y}{2} - \frac{y^3}{3} + \frac{y^5}{2\times5}\Big|_0^1$

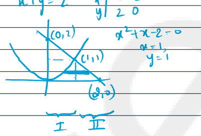
$= \frac{1}{2} - \frac{1}{3} + \frac{1}{10} \Rightarrow I = \frac{30-20+6}{60} = \frac{16}{60} = \frac{4}{15}$

[9] a) $\int_0^1\int_{\sqrt{y}}^{2-y} xy\,dx\,dy$

$x = \sqrt{y} \Rightarrow x^2 = y$

$x+y = 2$ x | 0 2 ; y | 2 0

$x^2+x-2 = 0$; $x = 1$, $y = 1$



$I_I = \int_0^1\int_0^{x^2} xy\,dy\,dx$

$I_{II} = \int_1^2\int_0^{2-x} xy\,dy\,dx$

$I = I_I + I_{II}$

$= \int_0 x\frac{y^2}{2}\Big|_0^{x^2} dx + \int_1^2 x\frac{y^2}{2}\Big|_0^{2-x} dx$

$= \int_0^1 \frac{x^5}{2}\,dx + \int_1^2 \frac{x(2-x)^2}{2}\,dx$

$= \frac{x^6}{2\times6}\Big|_0^1 + \frac{1}{2}\int_1^2 x(4+x^2-4x)\,dx$

$= \frac{1}{12} + \frac{1}{2}\;\frac{4x^2}{2} + \frac{x^4}{4} - \frac{4x^3}{3}\Big|_1^2$

$+ \frac{1}{2}\;8 + 4 - \frac{32}{3} - \left(2 + \frac{1}{4} - \frac{4}{3}\right)$

$+ \frac{1}{2}\;\frac{36-32}{3} - \left(\frac{24+3-16}{12}\right)$

$+ \frac{1}{2}\times\;\frac{4}{3} - \frac{11}{12}$

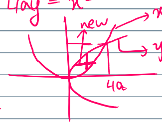
$= \frac{2}{24} + \frac{1}{2}\times\frac{5}{12}$

$= 7/24$

[9] b) $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$

$y^2 = 4ax$

$4ay = x^2$



$I = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dx\,dy$

$= \int 2\sqrt{ay} - \frac{y^2}{4a}\,dy$

$= 2\sqrt{a}\,\frac{y^{3/2}}{3/2}\Big|_0^{4a} - \frac{1}{4a}\frac{y^3}{3}\Big|_0^{4a}$

$= \frac{4a^{1/2}}{3}\times 4^{3/2}\cdot a^{3/2} - \frac{1}{4a}\frac{(4a)^3}{3}$

$= \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}$

[9] d) $\int_0^a\int_{x^2/a}^{2a-x} xy\,dy\,dx$

$y+x = 2a$

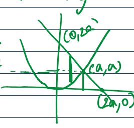
$x^2 = ay$

x | 0 2a ; y | 2a 0

$x^2 = a(2a-x)$

$x^2+ay-2a^2 = 0$

$x = a$, $y = a$



$I_I = \int_0^a\int_0^{\sqrt{ay}} xy\,dx\,dy$

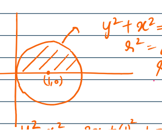
$I_{II} = \int_a^{2a}\int_0^{2a-y} xy\,dx\,dy$

$I = I_I + I_{II}$

$= \int_0^a y\frac{x^2}{2}\Big|_0^{\sqrt{ay}} + \int_a^{2a} y\frac{x^2}{2}\Big|_0^{2a-y}$

[11] b) $\int_0^a\int_0^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\,dy\,dx$

diagram is below!!



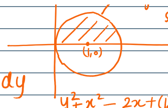
$y^2+x^2 = 2x$

$r^2 = 2r\cos\theta$

$r = 2\cos\theta$

[11] a) $\int_0^2\int_0^{\sqrt{2x-x^2}} \frac{x}{\ldots}\,dx\,dy$

$= 7/24$



$r^2 = 2r\cos\theta$
$r = 2\cos\theta$

11 a) $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy$

$y^2 + x^2 - 2x + (1)^2 - 1 = 0$
$y^2 + (x-1)^2 = 1$
$(1,0)\ ,\ r = 1$

$= \int_0^{\pi/2} \int_0^{2\cos\theta} \frac{r\cos\theta}{\cancel{r}}\, \cancel{r}\, dr\, d\theta$

$= \int_0^{\pi/2} \cos\theta \left. \frac{r^2}{2} \right|_0^{2\cos\theta} d\theta$

$= \int_0^{\pi/2} \frac{1}{2}\, 2^2 \cos^3\theta\, d\theta$

$= 2 \int_0^{\pi/2} \cos\theta \cos^2\theta\, d\theta$

$\cos\theta(1-\sin^2\theta)$
$\cos\theta - \cos\theta\sin^2\theta$

$= 2\int_0^{\pi/2} \cos\theta\, d\theta - 2\int_0^{\pi/2} \cos\theta \sin^2\theta\, d\theta$

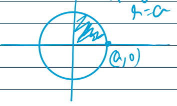
$t = \sin\theta$
$dt = \cos\theta\, d\theta$

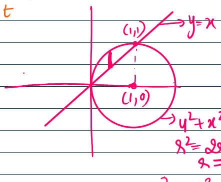
$= 2\sin\theta \Big|_0^{\pi/2} - 2\int_0^1 t^2\, dt$

$= 2(1-0) - 2 \left. \frac{t^3}{3} \right|_0^1$

$= 2 - \frac{2}{3} = \frac{4}{3}$

$y^2 + x^2 = a^2$
$r^2 = a^2$
$r = a$





$y^2 + x^2 = 2x$
$r^2 = 2r\cos\theta$
$r = 2\cos\theta$

diagram is below!!

$I = \int_0^{\pi/2} \int_0^a r^2\sin^2\theta \sqrt{r^2}\, r\, dr\, d\theta$

$= \int_\theta \int_0^a \sin^2\theta\, r^4\, dr\, d\theta$

$= \int_\theta \sin^2\theta \left. \frac{r^5}{5} \right|_0^a d\theta$

$= \frac{a^5}{5} \int_0^{\pi/2} \sin^2\theta\, d\theta$

$\cos 2x = 1 - 2\sin^2 x$
$\sin^2 x = \frac{1-\cos 2x}{2}$

$= \frac{a^5}{5} \int_0^{\pi/2} \frac{1-\cos 2\theta}{2}\, d\theta$

$= \frac{a^5}{10} \left. \theta - \frac{\sin 2\theta}{2} \right|_0^{\pi/2}$

$= \frac{a^5}{10} \left(\frac{\pi}{2} - \cancel{\frac{\sin \pi}{2}} \right) - (0)$

$I = \frac{\pi a^5}{20}$

$I = I_I + I_{II}$

$= \int_0^a y \left. \frac{x^2}{2} \right|_0^{\sqrt{ay}} + \int_a^{2a} y \left. \frac{x^2}{2} \right|_0^{2a-y}$

$= \int_0^a \frac{ay^2}{2}\, dy + \int_a^{2a} \frac{y}{2} (2a-y)^2\, dy$

$= \frac{a}{2} \left. \frac{y^3}{3} \right|_0^a + \int_a^{2a} \frac{y}{2} (4a^2 + y^2 - 4ay)\, dy$

$= \frac{a^4}{6} + \frac{1}{2} \left. \frac{4a^2 y^2}{2} + \frac{y^4}{4} - \frac{4ay^3}{3} \right|_a^{2a}$

$+ \frac{1}{2} \left[(2a^2)(2a)^2 + \frac{(2a)^4}{4} - \frac{4a\cdot(2a)^3}{3} - \left(2a^4 + \frac{a^4}{4} - \frac{4a^4}{3} \right) \right]$

$+ \frac{1}{2} \left[8a^4 + 4a^4 - \frac{32}{3}a^4 - 2a^4 - \frac{a^4}{4} + \frac{4a^4}{3} \right]$

$+ \frac{1}{2} \left[6a^4 + \frac{3a^4}{4} - \frac{28a^4}{3} \right]$

$+ \frac{1}{2} \left[\frac{72a^4 + 9a^4 - 112a^4}{12} \right]$

$\frac{12\times 6}{72}$ $\quad \frac{28\times 4}{112}$ $\quad -\frac{81}{31}$

$+ \frac{1}{2} \times \frac{31a^4}{12}$

$I = \frac{4a^4}{24} + \frac{31a^4}{24} = \frac{35a^4}{24}$

12 $V = \iiint dx\, dy\, dz$

$V_{sphere} = 8V$

$x^2 + y^2 + z^2 = a^2$

$V = \int_0^a \int^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\, dy\, dx$

$= \int_x \int_y z \Big|_0^{\sqrt{a^2-x^2-y^2}} dy\, dx$

$= \int_x \int_y \sqrt{a^2-x^2-y^2}\, dy\, dx$

$= \int_x \int_0^{\sqrt{a^2-x^2}} \sqrt{\sqrt{a^2-x^2}^{\,2} - y^2}\, dy\, dx$

$= \int_x \frac{y}{2} \sqrt{\sqrt{a^2-x^2}^{\,2} - y^2} + \frac{a^2-x^2}{2} \sin^{-1} \frac{y}{\sqrt{a^2-x^2}} \Big|_0^{\sqrt{a^2-x^2}} dx$

$= \int_x 0 + \frac{a^2-x^2}{2} \times \frac{\pi}{2} - (0)\, dx$

$= \frac{\pi}{4} \int_0^a a^2 - x^2\, dx$

$= \frac{\pi}{4} \left. a^2 x - \frac{x^3}{3} \right|_0^a$

$V = \frac{\pi}{4}\, a^3 - \frac{a^3}{3} = \frac{2\pi a^3}{12} = \frac{\pi a^3}{6}$

$V_{sphere} = 8 \times V = 8 \times \frac{\pi a^3}{6} = \frac{4\pi a^3}{3}$

11 c) $\int_0^1 \int_x^{\sqrt{2x-x^2}} x^2 + y^2\, dy\, dx$

$y^2 + x^2 = 2x$
$y^2 + (x-1)^2 = 1$
$(1,0)\ ,\ r = 1$

$I = \int_{\pi/4}^{\pi/2} \int_0^{2\cos\theta} r^2\, r\, dr\, d\theta$

$\cos^4\theta$
$(\cos^2\theta)^2$
$\left(\frac{1+\cos 2\theta}{2} \right)^2$
$\frac{1}{4} (1 + \cos^2 2\theta + 2\cos 2\theta)$
$= \frac{1}{4} \left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2} \right)$

$= \int_\theta \left. \frac{r^4}{4} \right|_0^{2\cos\theta} d\theta$

$= \int_\theta 4\cos^4\theta\, d\theta$

$= \int_\theta 1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\, d\theta$

$= \theta + \cancel{2}\frac{\sin 2\theta}{\cancel{2}} + \frac{1}{2}\theta + \frac{1}{2} \frac{\sin 4\theta}{4} \Big|_{\pi/4}^{\pi/2}$

$= \frac{3\theta}{2} + \sin 2\theta + \frac{1}{8} \sin 4\theta \Big|_{\pi/4}^{\pi/2}$

$= \frac{3\pi}{4} + 0 + 0 - \left(\frac{3\pi}{8} + 1 + 0 \right)$

$= \frac{3\pi}{4} - \frac{3\pi}{8} - 1 = \frac{3\pi}{8} - 1$

13 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad z^2 = c^2 \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right)$

$V = \int_{x=0}^{a} \int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} \int_{z=0}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\, dy\, dx$

$= \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dy\, dx$

$= \int_0^a c \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{b^2\left(1-\frac{x^2}{a^2}\right) - y^2}\, dy\, dx$

$= \int_0^a c \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{\left(b\sqrt{1-\frac{x^2}{a^2}}\right)^2 - y^2}\, dy\, dx$

$\left[\int_0^a \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}(x/a) \right]$

$= \int_0^a c \left[\frac{y}{2} \sqrt{\left(b\sqrt{1-\frac{x^2}{a^2}}\right)^2 - y^2} + \frac{\left(b\sqrt{1-\frac{x^2}{a^2}}\right)^2}{2} \sin^{-1}\left(y \Big/ b\sqrt{1-\frac{x^2}{a^2}} \right) \right]_0^{b\sqrt{1-\frac{x^2}{a^2}}}$

$= \int_0^a c \left[\frac{b^2}{2} \left(1 - \frac{x^2}{a^2} \right) \times \frac{\pi}{2} \right]$

$= \frac{\pi c b^2}{4a^2} \int_0^a a^2 - x^2\, dx$

$= \frac{\pi b^2 c}{4a^2}\, a^2 x - \frac{x^3}{3} \Big|_0^a$

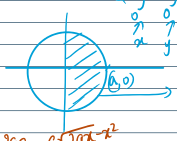
$= \frac{\pi b^2 c}{4a^2}\, a^3 - \frac{a^3}{3}$

$V_I = \frac{\pi b^2 c}{4a^2} \times \frac{2a^3}{3} = \frac{\pi abc}{6}$

$V_{tot} = 8V_I = \frac{4\pi abc}{3}$

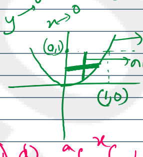
14 a) $\int_{-a}^{a} \int_0^{\sqrt{a^2-y^2}} f(x,y)\, dx\, dy$

$= 2\int_0^a \int_0^{\sqrt{a^2-x^2}} f(x,y)\, dy\, dx$



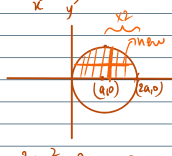
$x^2 + y^2 = a^2$

14 c) $\int_0^1 \int_0^{\sqrt{y}} xy\, dx\, dy = \int_0^1 \int_{x^2}^{1} xy\, dy\, dx$



at $y = 1$
$x = 1$

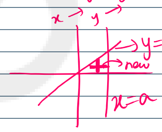
14 b) $\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} f(x,y)\, dy\, dx$



$= 2\int_0^a \int_a^{a+\sqrt{a^2+y^2}} f(x,y)\, dx\, dy$ ⇒ Is it correct??

$x^2 + y^2 - 2ax = 0$
$x^2 - 2ax + a^2 + y^2 = a^2$
$(x-a)^2 + y^2 = a^2 \Rightarrow x = \sqrt{a^2-y^2} + a$
$(a,0)\quad r = a$

14 d) $\int_0^a \int_0^x f(x,y)\, dx\, dy$



$\int_0^a \int_y^a f(x,y)\, dy\, dx$