# vector integration_tutorials

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vector integratio...

RV College of Engineering

Department of Mathematics

**VECTOR CALCULUS, LAPLACE TRANSFORM & NUMERICAL METHODS (MA221TA)**

**UNIT-II**

**VECTOR INTEGRATION**

**TUTORIAL SHEET**

1. Find the total work done by the force represented by $\vec F = 3xy\hat i - y\hat j + 2zx\hat k$ in moving a particle round the circle $x^2+y^2=4$, $x = 2\cos\theta$, $y = 2\sin\theta$ & $z = 0$, $0 \le \theta \le 2\pi$.
2. Evaluate $\int_c y^2dx - 2x^2dy$ along the parabola $y = x^2$ from (0,0) to (2,4).
3. Evaluate the line integral $\int_C (x^2+xy)dx + (x^2+y^2)dy$, where $C$: square: $x = \pm1$, $y = \pm1$ *Ans*: 0
4. Verify Green's theorem for $\int_C (e^{-x}\sin y)dx + (e^{-x}\cos y)dy$, where $C$ is the rectangle, whose vertices are $(0,0), (\pi, 0), (\pi, \frac{\pi}{2})$ and $(0, \frac{\pi}{2})$. Ans: $[2(e^{-\pi}-1)]$
5. Using Green's theorem, evaluate $\oint_C (x^2 - \cosh y\,dx + (y + \sin x)dy)$ where $C$ is the boundary of the rectangle $0 \le x \le \pi$, $0 \le y \le 1$. *Ans*: $\pi(\cosh 1 - 1)$
6. Using the Green's theorem, find the area enclosed between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$
7. If $S$ is the surface of the sphere $x^2+y^2+z^2 = d^2$ and $\vec A = ax\hat i + by\hat j + cz\hat k$, evaluate $\iint_S \vec A\cdot\hat n\,ds$. *Ans*: $\frac{2\pi d^3}{3}(a+b+c)$
8. If $\vec F = 2y\hat i - 3\hat j + x^2\hat k$ and $S$ is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$, show that $\iint_S \vec F\cdot\hat n\,ds = 132$.
9. Find the surface integral over the parallelepiped $x = 0, y = 0, x = 1, y = 2, z = 3$ when $\vec A = 2xy\hat i + yz^2\hat j + xz\hat k$ *Ans*: 33.
10. Using divergence theorem, evaluate $\iint_S \vec r\cdot\hat n\,ds$ where $S$ is the surface of the sphere $x^2+y^2+z^2 = 9$. *Ans*: $108\pi$
11. Verify divergence theorem for $\vec F = 4xz\hat i - y^2\hat j + yz\hat k$ taken over the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.
12. Using divergence theorem, evaluate $\iint_S \vec F\cdot\hat n\,ds$ over the entire surface $S$ of the region above $xy$ plane bounded by the cone $x^2+y^2 = z^2$ the plane $z = 4$ where $\vec F = 4xz\hat i - xyz^2\hat j + 3z\hat k$ *Ans*: ~~$704\pi$~~ $320\pi$
13. Verify Stokes's theorem where $\vec A = (2x-y)\hat i - yz^2\hat j - y^2z\hat k$ and $S$: upper half of the surface of the sphere $x^2+y^2+z^2 = 1$ *Ans*: $\pi$
14. Evaluate $\oint_C xy\,dx + xy^2\,dy$ by Stoke's theorem where $C$ is the square in the $xy$ plane with vertices $(1,0)$ $(-1,0)$ $(0,1)$ $(0,-1)$.
15. Evaluate $\oint_C 4z\,dx - 2x\,dy + 2x\,dz$ by Stoke's theorem where $C$ is the ellipse $x^2+y^2 = 1$, $z = y+1$. *Ans*: $-4\pi$

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1] $\vec F = 3xy\hat i - y\hat j + 2zx\hat k$

→ $x^2+y^2=4$, $x = 2\cos\theta$, $y = 2\sin\theta$, $z=0$

$dx = -2\sin\theta\,d\theta$, $dy = 2\cos\theta\,d\theta$, $dz = 0$

→ $W = \int_c \vec F\cdot d\vec s$

$= \int_c [(3xy\hat i - y\hat j + 2zx\hat k)\cdot(dx\hat i + dy\hat j + dz\hat k)]$

$= \int 3xy\,dx - y\,dy + 2zx\cdot dz$

$= \int 3\cdot 2\cos\theta\cdot 2\sin\theta\cdot(-2\sin\theta\,d\theta) - 2\sin\theta\cdot 2\cos\theta\,d\theta + 0$

$= \int_0^{2\pi}(-24\cos\theta\sin^2\theta - 4\sin\theta\cos\theta)\,d\theta$

$= -24\int\cos\theta\sin^2\theta\,d\theta - 4\int\sin\theta\cos\theta\,d\theta$

$\sin\theta = t$ $\quad$ $\sin\theta = t$

$\cos\theta\,d\theta = dt$ $\quad$ $\cos\theta\,d\theta = dt$

$= -24\int t^2\,dt - 4\int t\,dt$

$= -\cancel{24}^{8}\frac{t^3}{\cancel{3}} - \cancel{4}^{2}\frac{t^2}{\cancel{2}}$

$= -8\sin^3\theta - 2\sin^2\theta\Big|_0^{2\pi}$

$W = 0$

2] $I = \int_c y^2dx - 2x^2dy$, $y = x^2$, $dy = 2x\,dx$ (1) $\quad (0,0)\to(2,4)$

$= \int_0^2 (x^2)^2\cdot dx - 2x^2\cdot 2x\,dx$ from (1)

$= \int_0^2 (x^4 - 4x^3)\,dx$

$= \frac{x^5}{5} - \frac{4x^4}{4}\Big|_0^2$

$I = \frac{32}{5} - 16 \Rightarrow I = \frac{-48}{5}$

3] $I = \int_c \underbrace{(x^2+xy)}_{P}dx + \underbrace{(x^2+y^2)}_{Q}dy$ $\quad x = \pm1$, $y = \pm1$

→ $P = x^2+xy$ ; $Q = x^2+y^2$

$\frac{dP}{dy} = x$ $\quad$ $\frac{dQ}{dx} = 2x$

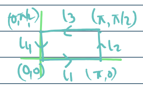
→ $I = \int_c P\,dx + Q\,dy = \iint\left(\frac{dQ}{dx} - \frac{dP}{dy}\right)dx\,dy$

$= \int_{-1}^{1}\int_{-1}^{1} 2x - x\,dx\,dy$

$= \int_{-1}^{1}\int_{-1}^{1} x\,dx\,dy = \int_{-1}^{1}\frac{x^2}{2}\Big|_{-1}^{1}dy$

$I = \int_{-1}^{1}\frac{1}{2} - \left(\frac{1}{2}\right)dy \Rightarrow I = 0$

4] $I = \int_c \underbrace{e^{-x}\sin y}_{P}\,dx + \underbrace{e^{-x}\cos y}_{Q}\,dy$




→ $P = e^{-x}\sin y$ $\quad$ $Q = e^{-x}\cos y$

$\frac{dP}{dy} = e^{-x}\cos y$ $\quad$ $\frac{dQ}{dx} = -e^{-x}\cos y$

→ $I = \int_c P\,dx + Q\,dy = \iint \frac{dQ}{dx} - \frac{dP}{dy}\,dx\,dy$

$= \int_0^{\pi/2}\int_0^{\pi} -e^{-x}\cos y - e^{-x}\cos y\,dx\,dy$

$= \int_0^{\pi/2}\int_0^{\pi} -2e^{-x}\cos y\,dx\,dy$

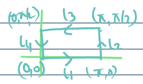
$= \int_0^{\pi/2} +2e^{-x}\cos y\Big|_{x=0}^{\pi}dy$

$= \int_0^{\pi/2} 2(e^{-\pi} - e^0)\cos y\,dy$

$= 2(e^{-\pi}-1)\sin y\Big|_0^{\pi/2}$

$= 2(e^{-\pi}-1)[\sin\pi/2 - \sin 0]$

$I = 2(e^{-\pi}-1)$




→

| line | eqn | limits | $\vec F\cdot d\vec r$ | $\int_c e^{-x}\sin y\,dx + e^{-x}\cos y\,dy$ $\int_c \vec F\cdot d\vec r$ |
|---|---|---|---|---|
| $l_1$ | $y=0$, $dy=0$ | $x(0\to\pi)$ | 0 | 0 |
| $l_2$ | $x=\pi$, $dx=0$ | $y(0\to\pi/2)$ | $e^{-\pi}\cos y\,dy$ | $\int_0^{\pi/2} e^{-\pi}\cos y\,dy = e^{-\pi}\sin y\Big|_0^{\pi/2} = e^{-\pi}$ |
| $l_3$ | $y=\pi/2$, $dy=0$ | $x(\pi\to0)$ | $e^{-x}dx$ | $\int_\pi^0 e^{-x}dx = -e^{-x}\Big|_\pi^0 = -(e^0 - e^{-\pi}) = -1 + e^{-\pi}$ |
| $l_4$ | $x=0$, $dx=0$ | $y(\pi/2\to0)$ | $\cos y\,dy$ | $\int_{\pi/2}^0 \cos y\,dy = \sin y\Big|_{\pi/2}^0 = \sin 0 - \sin\pi/2 = -1$ |

$\int_c P\,dx + Q\,dy = 0 + e^{-\pi} - 1 + e^{-\pi} - 1$

$= 2(e^{-\pi}-1)$ $\quad\therefore$ verified.

5] $I = \int_c \underbrace{x^2 - \cosh y}_{P}\,dx + \underbrace{(y + \sin x)}_{Q}dy$ $\quad 0\le x\le\pi$, $0\le y\le 1$

$P = x^2 - \cosh y$ ; $Q = y + \sin x$

$\frac{dP}{dy} = -\sinh y$ $\quad$ $\frac{dQ}{dx} = \cos x$

→ $I = \int_c P\,dx + Q\,dy = \iint \frac{dQ}{dx} - \frac{dP}{dy}\,dx\,dy$

$= \int_0^1\int_0^{\pi} \cos x + \sinh y\,dx\,dy$

$= \int_0^1 \sin x + x\sinh y\Big|_0^{\pi}dy$

$= \int_0^1 \cancel{\sin\pi}^{0} + \pi\sinh y - (\cancel{\sin 0}^{0} + 0)\,dy$

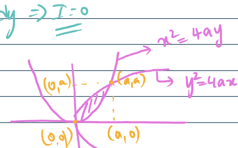
$= \int_0^1 +\pi\sinh y\,dy$

$= +\pi\cosh y\Big|_0^1 = \pi(\cosh 1 - \cos 0)$

$I = \pi(\cosh 1 - 1)$

6) $y^2 = 4ax$

$x^2 = 4ay$




7) $S \Rightarrow x^2+y^2+z^2 = d^2 \Rightarrow$

$\vec A = ax\hat i + by\hat j + cz\hat k$

$\phi = x^2+y^2+z^2 - d^2$

$\nabla\phi = \frac{\partial\phi}{\partial x}\hat i + \frac{\partial\phi}{\partial y}\hat j + \frac{\partial\phi}{\partial z}\hat k$

$\nabla\phi = 2x\hat i + 2y\hat j + 2z\hat k$

$\hat n = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat i + 2y\hat j + 2z\hat k}{\sqrt{(2x)^2 + (2y)^2 + (2z)^2}}$

$\hat n = \frac{x\hat i + y\hat j + z\hat k}{d}$

$\hat n\cdot\hat k = \frac{z}{d}$

$\vec A\cdot\hat n = (ax\hat i + by\hat j + cz\hat k)\left(\frac{x\hat i + y\hat j + z\hat k}{d}\right)$

$= \frac{ax^2 + by^2 + cz^2}{d}$

→ $\iint_S \vec A\cdot\hat n\,ds$

$= \iint_R \vec A\cdot\hat n\,\frac{dx\,dy}{|\hat n\cdot\hat k|}$

$= \int_{y=0}^{d}\int_{x=0}^{d} \frac{ax^2 + by^2 + cz^2}{\cancel{d}}\,\frac{dx\,dy}{\frac{z}{\cancel{d}}}$

$\int_y\int_x \frac{ax^2 + by^2}{\sqrt{d^2 - x^2 - y^2}} + c\sqrt{d^2 - x^2 - y}\;dx\,dy.$

7) → $\nabla\cdot\vec A = \left(\frac{\partial}{\partial x}\hat i + \frac{\partial}{\partial y}\hat j + \frac{\partial}{\partial z}\hat k\right)(ax\hat i + by\hat j + cz\hat k)$

$\nabla\cdot\vec A = a + b + c$

→ $\iiint \nabla\cdot\vec A\,dV = \iiint \nabla\cdot\vec A\cdot r\,dr\,d\theta\,dz$

8) 8. If $\vec F = 2y\hat i - 3\hat j + x^2\hat k$ and $S$ is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$, show that $\iint_S \vec F\cdot\hat n\,ds = 132$.

$\phi \Rightarrow y^2 - 8x = 0$

$\nabla\phi = \frac{\partial\phi}{\partial x}\hat i + \frac{\partial\phi}{\partial y}\hat j + \frac{\partial\phi}{\partial z}\hat k$

first octant bounded by the planes $y = 4$ and $z = 6$, show that $\iint_S \vec{F}\cdot\hat{n}\,ds = 132$.

$\phi \Rightarrow y^2 - 8x = 0$

$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$

$= -8i + 2yj$

$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-8i+2yj}{\sqrt{8^2+(2y)^2}} = \frac{2(-4i+yj)}{\sqrt{4(16+y^2)}}$ $\quad 64+4y^2$

$\hat{n} = \frac{-4i+yj}{\sqrt{16+y^2}} \Rightarrow \hat{n}\cdot\hat{i} = \frac{-4}{\sqrt{16+y^2}}$

$\rightarrow \bar{F}\cdot\hat{n} = (2yi - 3j + x^2k)\left(\frac{-4i+yj}{\sqrt{16+y^2}}\right)$

$= \frac{-8y-3y}{\sqrt{16+y^2}} = \frac{-11y}{\sqrt{16+y^2}}$

$\rightarrow \iint_S \bar{F}\cdot\hat{n}\,ds = \iint_{z\,y} \bar{F}\cdot\hat{n}\,\frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$

$= \int_{z=0}^{6}\int_{y=0}^{4} \frac{-11y}{\cancel{\sqrt{16+y^2}}} \cdot \frac{dy\,dz}{\frac{-4}{\cancel{\sqrt{16+y^2}}}}$

$= \frac{11}{4}\int_z \frac{y^2}{2}\Big|_0^4 dz$

$= \frac{11}{4}\int_z \frac{16}{2}\,dz = \frac{11}{\cancel{4}}\times\cancel{8}^{\,2}\; z\Big|_0^6$

$= 22\times 6 = 132$

$\rightarrow \iiint_V \nabla\cdot\bar{A}\,dV = \iiint \nabla\cdot\bar{A}\cdot r\,dr\,d\theta\,dz$

$= \int_{z=-d}^{d}\int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{d^2-z^2}} (a+b+c)\,r\,dr\,d\theta\,dz$

$x^2+y^2+z^2 = d^2$

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2$

$r = \sqrt{d^2-z^2}$

$= \int_z\int_\theta (a+b+c)\frac{r^2}{2}\Big|_0^{\sqrt{d^2-z^2}} d\theta\,dz$

$= \int_z\int_\theta \frac{(d^2-z^2)}{2}(a+b+c)\,d\theta\,dz$

$= \int_z \frac{(a+b+c)}{2}[d^2-z^2]\,\theta\Big|_0^{2\pi} dz$

$= \int_z \frac{2\pi(a+b+c)}{2}(d^2-z^2)\,dz$

$= \pi(a+b+c)\; d^2z - \frac{z^3}{3}\Big|_{-d}^{d}$

$= \pi(a+b+c)\left[d^3 - \frac{d^3}{3} - \left(-d^3 + \frac{d^3}{3}\right)\right]$

$= \pi(a+b+c)\left(2d^3 - \frac{2d^3}{3}\right)$

$= \frac{4\pi d^3}{3}(a+b+c)$

10) Using divergence theorem, evaluate $\iint_S \vec{r}\cdot\hat{n}\,ds$ where $S$ is the surface of the sphere $x^2+y^2+z^2 = 9$. *Ans*: $108\pi$

$\rightarrow$ Spherical coordinates:

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$\rightarrow x^2+y^2+z^2 = 9$

$r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta = 9$

$r^2 = 9$

$r = 3$

$\rightarrow r(0\to 3)$

$\theta(0\to\pi)\quad \phi(0\to 2\pi)$

$\rightarrow \bar{r} = xi + yj + z\hat{k}$

$\rightarrow \nabla\cdot\bar{r} = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)(xi + yj + zk)$

$= 1+1+1$

$\nabla\cdot\bar{r} = 3$

$\rightarrow \iint_S \bar{r}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\bar{r}\,dV$

$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{3} 3\,r^2\sin\theta\,dr\,d\theta\,d\phi$

$= \int_\phi\int_\theta \cancel{3}\sin\theta\,\frac{r^3}{\cancel{3}}\Big|_0^3 d\theta\,d\phi$

$= \int_\phi 27 - \cos\theta\Big|_0^\pi d\phi$

$= \int_\phi -27(\cos\pi - \cos 0)\,d\phi$

$= \int_\phi -27(-1-1)\,d\phi$

$= 54\,\phi\Big|_0^{2\pi}$

$= 54\times 2\pi = 108\pi$

alternatively,

$\iiint_V \nabla\cdot\bar{r}\,dV = 3(\text{volume of sphere})$

$V = 3\times\frac{4}{3}\pi r^3 = \cancel{3}\times\frac{4}{\cancel{3}}\times\pi\times 3^3 = 108\pi$

9) 9. Find the surface integral over the parallelepiped $x = 0, y = 0, x = 1, y = 2, z = 3$ when $\vec{A} = 2xy\hat{i} + yz^2\hat{j} + xz\hat{k}$ *Ans*: 33.

$\rightarrow \nabla\cdot\bar{A} = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)(2xyi + yz^2j + xzk)$

$\nabla\cdot\bar{A} = 2y + z^2 + x$

$\iint_S \bar{A}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\bar{A}\,dV$

$= \int_{z=0}^{3}\int_{y=0}^{2}\int_{x=0}^{1}(2y + z^2 + x)\,dx\,dy\,dz$

$= \int_z\int_y (2y+z^2)x + \frac{x^2}{2}\Big|_0^1 dy\,dz$

$= \int_z\int_y 2y + z^2 + \frac{1}{2}\,dy\,dz$

$= \int_z \cancel{2}\frac{y^2}{\cancel{2}} + \left(z^2+\frac{1}{2}\right)y\Big|_0^2 dz$

$= \int_z 2^2 + \left(z^2+\frac{1}{2}\right)2\,dz$

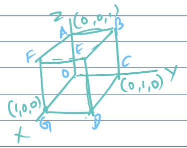
$= \int_z 4 + 2z^2 + 1\,dz$

$= 5z + \frac{2z^3}{3}\Big|_0^3$

$= 15 + \frac{2\cdot\cancel{3^3}^{\,9}}{\cancel{3}}$

$= 15 + 18 = 33$

11) 11. Verify divergence theorem for $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\,\hat{k}$ taken over the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.




$x\,(0\to 1)$

$y\,(0\to 1)$

$z\,(0\to 1)$

$\rightarrow$ Surface $S_1$: OABC $\quad S_2$: FGDE $\quad S_3$: AOGF $\quad S_4$: BODC $\quad S_5$: OCDG $\quad S_6$: AFEB

| plane | $yz\ (x=0)$ | $yz\ (x=1)$ | $xz\ (y=0)$ | $xz\ (y=1)$ | $xy\ (z=0)$ | $xy\ (z=1)$ |
|---|---|---|---|---|---|---|
| $\hat{n}$ | $-\hat{i}$ | $\hat{i}$ | $-\hat{j}$ | $\hat{j}$ | $-\hat{k}$ | $\hat{k}$ |
| $ds$ | $dy\,dz$ | $dy\,dz$ | $dx\,dz$ | $dx\,dz$ | $dx\,dy$ | $dx\,dy$ |
| $\bar{F}\cdot\hat{n}$ | $-4xz$ | $4xz$ | $y^2$ | $-y^2$ | $-yz$ | $yz$ |
| $\iint \bar{F}\cdot\hat{n}$ | $\int_z\int_y -4xz\,dy\,dz$ | $\int_z\int_y 4xz\,dy\,dz$ | $\int_z\int_x y^2\,dx\,dz$ | $\int_z\int_x -y^2\,dx\,dz$ | $\int_y\int_x -yz\,dx\,dy$ | $\int_y\int_x yz\,dx\,dy$ |
| | $S_1 = 0\ (\because x=0)$ | $=\int_z 4(1)z\,y\|_0^1\,dz$ $= 4\frac{z^2}{2}\|_0^1$ $S_2 = 2$ | $S_3 = 0\ (\because y=0)$ | $=\int_z -(1)^2\cdot x\|_0^1\,dz$ $= -1\,z\|_0^1$ $S_4 = -1$ | $S_5 = 0\ (\because z=0)$ | $=\int_y y\cdot(1)\cdot x\|_0^1\,dy$ $= \frac{y^2}{2}\|_0^1$ $S_6 = 1/2$ |

$$\iint_S \bar{F}\cdot\hat{n}\,ds = S_1+S_2+S_3+S_4+S_5+S_6$$
$$= 0+2+0-1+0+\frac{1}{2}$$
$$= \frac{3}{2}$$

$\rightarrow$ $\nabla\cdot\bar{F} = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)(4xz\,i - y^2 j + yz\,k)$

$$\nabla\cdot\bar{F} = 4z - 2y + y = 4z - y$$

$\rightarrow$ $\iiint_V \nabla\cdot\bar{F}\,dV = \int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} 4z - y\ dx\,dy\,dz$

$$= \int_z\int_y (4z-y)\,x\Big|_0^1\,dy\,dz$$
$$= \int_z 4zy - \frac{y^2}{2}\Big|_0^1\,dz$$
$$= \int_z 4z - \frac{1}{2}\,dz$$
$$= \frac{4z^2}{2} - \frac{1}{2}z\Big|_0^1 = 2 - \frac{1}{2} = \frac{3}{2}$$

$\therefore$ $\iint_S \bar{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\bar{F}\,dV$ Verified

12) 12. Using divergence theorem, evaluate $\iint_S \vec{F}.\hat{n}\,ds$ over the entire surface $S$ of the region above $xy$ plane bounded by the cone $x^2+y^2=z^2$ the plane $z=4$ where $\vec{F} = 4xz\hat{\imath} - xyz^2\hat{\jmath} + 3z\,\hat{k}$ *Ans*: $704\pi$

$\rightarrow$ $\nabla\cdot\bar{F} = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)(4xz\,i - xyz^2 j + 3z\,k)$

$$\nabla\cdot\bar{F} = 4z - xz^2 + 3$$

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$
$\{ x^2+y^2 = r^2$

$\rightarrow$ $x^2+y^2 = z^2 \Rightarrow r^2 = z^2 \Rightarrow r = z$

$\rightarrow$ $r\ (0\to z)$
$\theta\ (0\to 2\pi)$
$z\ (0\to 4)$

$\rightarrow$ $\iint_S \bar{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\bar{F}\,dV$

$$= \int_{z=0}^{4}\int_{\theta=0}^{2\pi}\int_{r=0}^{z} (4z - xz^2 + 3)\,r\,dr\,d\theta\,dz$$
$$= \int_z\int_\theta\int_r (4z - r\cos\theta\,z^2 + 3)\,r\,dr\,d\theta\,dz$$
$$= \iiint_{z\,\theta\,r} \left[(4z+3)r - r^2\cos\theta\,z^2\right] dr\,d\theta\,dz$$
$$= \iint_{z\,\theta} (4z+3)\frac{r^2}{2} - \cos\theta\,z^2\,\frac{r^3}{3}\Big|_0^z\,d\theta\,dz$$
$$= \iint_{z\,\theta} \frac{4z^3}{2} + \frac{3z^2}{2} - \cos\theta\,\frac{z^5}{3}\,d\theta\,dz$$
$$= \int_z \left(2z^3 + \frac{3z^2}{2}\right)\theta - \sin\theta\,\frac{z^5}{3}\Big|_0^{2\pi}\,dz$$
$$= \int_z 2\pi\left(2z^3 + \frac{3z^2}{2}\right)dz$$
$$= 2\pi\left(\frac{2z^4}{4} + \frac{3}{2}\frac{z^3}{3}\right)\Big|_0^4$$
$$= \frac{2\pi}{2}(z^4+z^3)\Big|_0^4$$
$$= (4^4+4^3)\pi = 320\pi$$

13) 13. Verify Stokes's theorem where $\vec{A} = (2x-y)\hat{\imath} - yz^2\hat{\jmath} - y^2z\,\hat{k}$ and $S$: upper half of the surface of the sphere $x^2+y^2+z^2=1$ *Ans*: $\pi$

T.P. $\oint_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\hat{n}\,ds$.


$\rightarrow$ $\oint_C \bar{F}\cdot d\bar{r} = \int_C \left((2x-y)i - \cancel{yz^2}\,j - \cancel{y^2}z\,k\right)(dx\,i + dy\,j + dz\,k)$

$\int_C \bar{F}\cdot d\bar{r} = \int_C (2x-y)\,dx$ $\quad(\because z=0\ (xy\text{ plane}).$

$\begin{cases} x = \cos\theta & dx = -\sin\theta\,d\theta \\ y = \sin\theta \end{cases}$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C (2\cos\theta - \sin\theta)(-\sin\theta\,d\theta)$$
$$= \int_{\theta=0}^{2\pi} (\sin^2\theta - \sin 2\theta)\,d\theta$$

$\cos 2\theta = 1 - 2\sin^2\theta$
$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$= \int_0^{2\pi} \frac{1}{2}(1-\cos 2\theta) - \sin 2\theta\ d\theta$$
$$= \frac{\theta}{2} - \frac{1}{2}\frac{\sin 2\theta}{2} + \frac{\cos 2\theta}{2}\Big|_0^{2\pi}$$
$$= \frac{2\pi}{2} - 0 + \frac{\cos 4\pi}{2}$$

$\int_C \bar{F}\cdot d\bar{r} = \pi$

$\rightarrow$ $\nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$

$$= (\cancel{-2yz} + \cancel{2yz}) - 0\hat{j} + 1\hat{k} = \hat{k}$$

$\rightarrow$ $\phi = x^2+y^2+z^2-1$

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$
$$= 2x\,i + 2y\,j + 2z\,k$$
$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(xi+yj+zk)}{\sqrt{2^2(x^2+y^2+z^2)}}$$
$$\hat{n} = \frac{xi+yj+zk}{1}$$

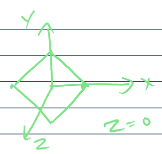
$\rightarrow$ $\iint (\nabla\times\bar{F})\cdot\hat{n}\,ds$

$$= \iint \cancel{\hat{k}\cdot\hat{n}}\cdot\frac{dx\,dy}{\cancel{|\hat{n}\cdot\hat{k}|}}$$
$$= \iint dx\,dy$$
$$= \text{Area of circle}$$

$\therefore$ $\int_C \bar{F}\cdot d\bar{r} = \iint (\nabla\times\bar{F})\cdot\hat{n}\,ds$
$\therefore$ Stokes verified.

14) 14. Evaluate $\oint_C xy\,dx + xy^2\,dy$ by Stoke's theorem where $C$ is the square in the $xy$ plane with vertices $(1,0)$ $(-1,0)$ $(0,1)$ $(0,-1)$.

$\rightarrow$ $\nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & xy^2 & 0 \end{vmatrix}$




$$\nabla\times\bar{F} = 0\hat{i} - 0\hat{j} + (y^2 - x)\hat{k}$$

$$\begin{vmatrix} xy & xy^2 & 0 \end{vmatrix} \quad z=0$$

$$\nabla \times \bar{F} = 0\hat{i} - 0\hat{j} + (y^2 - x)\hat{k}$$

$$\to \iint (\nabla \times \bar{F}) \cdot \hat{n}\, ds = \iint (y^2 - x)\hat{k} \cdot \hat{k}\ dx\, dy$$

$$= \int_{y=-1}^{1} \int_{x=-1}^{1} (y^2 - x)\, dx\, dy$$

$$= \int_y y^2 x - \frac{x^2}{2} \Big|_{-1}^{1} dy$$

$$= \int_y \left(y^2 - \frac{1}{2}\right) - \left(-y^2 - \frac{1}{2}\right) dy$$

$$= \int_y 2y^2\, dy$$

$$= \frac{2y^3}{3}\Big|_{-1}^{1} = \frac{2}{3}(1 - (-1)) = \frac{4}{3}$$

$$= \iint dx\, dy$$

$$= \text{Area of circle}$$

$$= \pi a^2 = \pi (1)^2 = \pi$$

15. Evaluate $\oint_C 4z\,dx - 2x\,dy + 2x\,dz$ by Stoke's theorem where $C$ is the ellipse $x^2 + y^2 = 1,\ z = y + 1$. *Ans*: $-4\pi$

$$\to \nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4(y+1) & -2x & 2x \end{vmatrix}$$

$$= 0\hat{i} - (2-4)\hat{j} + (-2-4)\hat{k}$$

$$\nabla \times \bar{F} = 2\hat{j} - 6\hat{k}$$

$$\to \phi = x^2 + y^2 - 1 \qquad (x^2 + y^2 = 1)$$

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

$$\nabla\phi = 2x\hat{i} + 2y\hat{j}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2xi + 2yj}{\sqrt{2^2(x^2+y^2)}} = x\hat{i} + y\hat{j}$$

$$\to \iint (\nabla \times F) \cdot \hat{n}\, ds = \iint (2j - 6k) \cdot (x\hat{i} + y\hat{j})$$