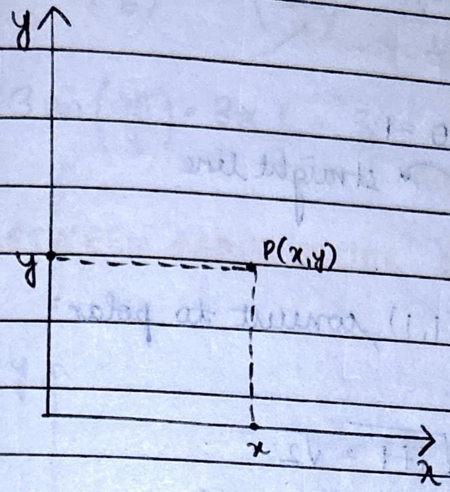
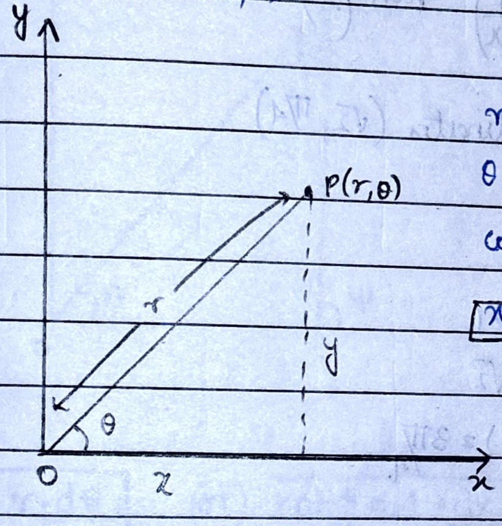
DIFFERENTIAL CALCULUS:

CARTESIAN COORDINATES:



POLAR COORDINATES: $P(r,\theta)$



r = The distance OP

θ = Angle b/w OP and OX

$\cos\theta = \dfrac{x}{r}$; $\sin\theta = \dfrac{y}{r}$

$\boxed{x = r\cos\theta \; ; \; y = r\sin\theta}$ → polar to cartesian

To eliminate θ:

$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta)$

$x^2 + y^2 = r^2$

$\dfrac{y}{x} = \dfrac{r\sin\theta}{r\cos\theta} = \tan\theta$

$$\boxed{\begin{array}{l} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\dfrac{y}{x}\right) \end{array}}$$ → Cartesian to polar

Cartesian curves $\Rightarrow y = f(x)$

Polar curves $\Rightarrow r = f(\theta)$

1. $r = a \Rightarrow r^2 = a^2 \Rightarrow x^2 + y^2 = a^2$ (Circle)
2. $\theta = k \Rightarrow \tan^{-1}\left(\dfrac{y}{x}\right) = k \Rightarrow \tan k = \dfrac{y}{x} \Rightarrow y = \underbrace{x\tan k}_{mx}$ (radial line)
3. $r = a\cos\theta \Rightarrow r^2 = 2ar\cos\theta$

   $x^2 + y^2 = 2ax$

   ↳ circle with centre at $(a,0)$ and radius (a).

$r = 5\sin\theta \rightarrow$ orientation on y axis.

$r = 3\cos\theta \rightarrow$ orientation on x axis.

(4) $(\sin\theta + \cos\theta) = \frac{1}{r}$

$\cancel{\sin\theta}\; r\sin\theta + r\cos\theta = 1$

$y + x = 1 \rightarrow$ straight line

(Q) Cartesian point (1,1), convert to polar:

$x = 1; y = 1$

$r = \sqrt{x^2+y^2} = \sqrt{1+1} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(1) = 45^\circ$

Polar coordinates $(\sqrt{2}, \pi/4)$

(Q) $(-1, 1)$

$x = -1; y = 1$

$r = \sqrt{1+1} = \sqrt{2}$

$\theta = \tan^{-1}(-1) = 3\pi/4$

Polar coordinates $(\sqrt{2}, 3\pi/4)$

(Q) $(-1,-1) \rightarrow r = \sqrt{2}; \theta = \tan^{-1}(1) = \pi/4$ (or) $5\pi/4$

Coordinate $(\sqrt{2}, 5\pi/4)$

(Q) Polar point $(2, 2\pi/3)$

$r = 2 = \sqrt{x^2+y^2}$

$\tan\left(\frac{2\pi}{3}\right) = \frac{y}{x} = -\sqrt{3} = \frac{y}{x}$

$y = -\sqrt{3}x$

$x = (-1, \sqrt{3})$

$4 = x^2+y^2$

$4 = \cancel{y^2}\, x^2+3x$

$x^2+3x-4=0$

$x^2+4x-x-4=0$

$x(x+4)-1(x+4)=0$

$(x-1)(x+4)=0$

$x=1$ or $x=-4$.

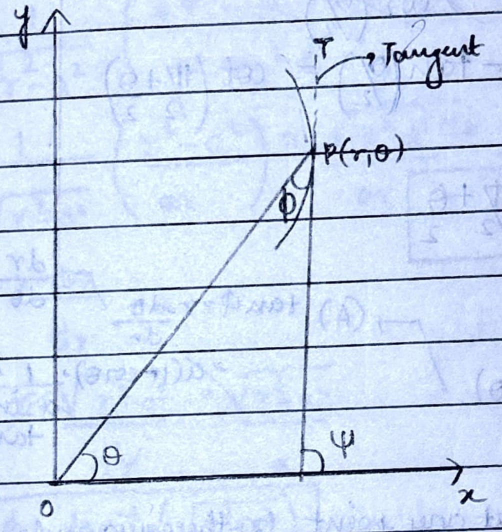
(Q) $(+3\sqrt{\ }, 5\pi/6)$

$x = 3\cos\left(\frac{5\pi}{6}\right) = \cancel{3}\times\left(\frac{-\sqrt{3}}{\cancel{2}}\right) = \frac{-\sqrt{3}\cdot 3}{2}$

$y = 3\sin\left(\frac{5\pi}{6}\right) = 3\times\frac{1}{2} = 3/2$

ANGLE BETWEEN RADIUS VECTOR AND TANGENT TO THE CURVE

$r = f(\theta)$



Let PT be the tangent to the curve at a point $P(r,\theta)$

Let ϕ be the angle b/w the radius vector OP and the tangent PT

ψ is the angle between tangent (PT) and x-axis

$\boxed{\tan\phi = r\cdot\frac{d\theta}{dr}}$ (or) $\boxed{\cot\phi = \frac{1}{r}\cdot\frac{dr}{d\theta}}$

NOTE: (1) $\psi = \theta + \phi$

Slope of $r = f(\theta) = \tan\psi = \tan(\theta+\phi)$

$$\boxed{\text{Slope} = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\cdot\tan\phi}}$$

(Q) Find the angle b/w radius vector and tangent to the curve $r = ae^{b\theta}$ (a and b are constant)

(A) $\frac{dr}{d\theta} = ae^{b\theta}\cdot b = abe^{b\theta}$

$\tan\phi = \cancel{ae^{b\theta}}\cdot\frac{1}{\cancel{abe^{b\theta}}} = \frac{1}{b} \Rightarrow \phi = \tan^{-1}\left(\frac{1}{b}\right)$

Cardioid

(Q) $r = a(1+\cos\theta)$

$\Rightarrow \log r = \log(a) + \log(1+\cos\theta)$

$\Rightarrow \frac{dr}{d\theta} \cdot \frac{1}{r} = 0 + \frac{-\sin\theta}{1+\cos\theta}$

$\frac{dr}{d\theta} \cdot \frac{1}{r} = \frac{-\sin\theta}{1+\cos\theta}$

$\cot\phi = \frac{-\cancel{2}\sin(\theta/2)\cdot\cancel{\cos(\theta/2)}}{\cancel{2}\cancel{\cos^2}(\theta/2)}$

$\cot\phi = -\tan(\theta/2) = \cot\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$

$$\boxed{\phi = \frac{\pi}{2} + \frac{\theta}{2}}$$

~~Homework~~ (1) $r = a(1-\cos\theta)$

(A) $\tan\phi = r\frac{d\theta}{dr}$ $\qquad \frac{dr}{d\theta} = a\sin\theta$

$= a(1+\cos\theta)\cdot\frac{1}{a\sin\theta}$ $\qquad \frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cdot\cos(\theta/2)}$

$\tan\phi = \tan(\theta/2)$

$\boxed{\phi = \theta/2}$

(Q) Show that at any point, on the equiangular spiral, $r = ae^{\theta\cot\alpha}$ (a and α are constants), the tangent is inclined at a constant angle to the radius vector. (ϕ).

(A) $r = ae^{\theta\cot\alpha}$

$\frac{dr}{d\theta} = a\cot\alpha\, e^{\theta\cot\alpha}$

$\tan\phi = r\cdot\frac{d\theta}{dr} = \frac{\cancel{ae^{\theta\cot\alpha}}}{a\cot\alpha\, \cancel{e^{\theta\cot\alpha}}}$

$\tan\phi = \frac{1}{\cot\alpha} \Rightarrow \tan\phi = \tan\alpha$

$\Rightarrow \boxed{\phi = \alpha}$

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(Q) Find the angle b/w radius vector and tangent to the curve, where $\theta = 1/a \sqrt{r^2-a^2} - \cos^{-1}\left(\frac{a}{r}\right)$

(A) $\frac{d\theta}{dr} = \frac{1}{a} \cdot \frac{+1 \times \cancel{2}r}{\cancel{2}\sqrt{r^2-a^2}} - \left(\frac{-1}{\sqrt{1-(a/r)^2}}\right) \cdot \frac{-a}{r^2}$

$\Rightarrow \frac{d\theta}{dr} \Rightarrow \frac{+\cancel{2}r}{a\sqrt{r^2-a^2}} - \frac{a}{r\sqrt{r^2-(a/\cancel{r})^2}}$

$= \frac{1}{\sqrt{r^2-a^2}} \left(\frac{+r}{a} - \frac{a}{r}\right)$

$= \frac{1}{\sqrt{r^2-a^2}} \left(\frac{r^2-a^2}{ar}\right) = \frac{\sqrt{r^2-a^2}}{ar} = \frac{d\theta}{dr}$

$\tan\phi = \frac{r \cdot d\theta}{dr}$

$= \cancel{r} \cdot \frac{\sqrt{r^2-a^2}}{a\cancel{r}} = \frac{\sqrt{r^2-a^2}}{a}$

$$\boxed{\phi = \tan^{-1}\left(\frac{\sqrt{r^2-a^2}}{a}\right)}$$

(Q) Show that the tangent to the curve $r = a(1+\sin\theta)$ at any point with $\theta = \pi/2$ is parallel to the initial line (x-axis)

(A) To show: $\tan\psi = 0$

$\frac{dr}{d\theta} = \cancel{a}\ a\cos\theta$

$\Rightarrow \psi = \theta + \phi$

$\tan\phi = r \cdot \frac{d\theta}{dr} = \cancel{r}.$

$\frac{d\theta}{dr} = \frac{1}{a\cos\theta}$

$\Rightarrow \cancel{a}(1+\sin\theta) \cdot \frac{1}{\cancel{a}\cos\theta} = \frac{1+\sin\theta}{\cos\theta}$

$\tan\phi = \frac{1+\sin\theta}{\cos\theta} = \frac{(\cos\theta/2 + \sin\theta/2)^{\cancel{2}}}{(\cos^2\theta/2 - \sin^2\theta/2)} = \frac{\cos\theta/2 + \sin\theta/2}{\cos\theta/2 - \sin\theta/2}$

$\tan\psi = \frac{\tan\theta + \tan\phi}{1 - \tan\theta \cdot \tan\phi} \Rightarrow \cancel{\frac{1+\sin\theta}{\cos\theta}}\ \frac{(\cos\theta/2 + \sin\theta/2) \cdot \cos\theta/2}{(\cos\theta/2 - \sin\theta/2) \cdot \cos\theta/2} \Rightarrow \frac{1/\sqrt{2} + 1/\sqrt{2}}{1/\sqrt{2} - 1/\sqrt{2}} = 0.$

$\boxed{\tan\psi = 0.}$

$$\Rightarrow \frac{1+\tan\theta/2}{1-\tan\theta/2} = \frac{1+\tan\pi/4}{1-\tan\pi/4}$$

$$\Rightarrow \tan\phi = \frac{\tan\theta + \tan\theta/2}{1-\tan\pi/4 \cdot \tan\theta/2} = \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

$$= \tan\left(\frac{\pi}{4}+\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{2}\right) = 0$$

$$\psi = \theta + \phi$$

$$= \theta + \pi/4 + \theta/2 = \pi/4 + \frac{3\theta}{2}$$

at $\theta = \pi/2$

$$= \frac{\pi}{4} + \frac{3\pi}{4} = \pi$$

$$\tan(\pi) = 0$$

$\therefore$ slope of the tangent $= \tan\psi = \tan\pi = 0$

Tangent is $\parallel^{al}$ to the initial line

(Q) Find the angle b/w the curves:

$$\frac{2a}{r} = 1 - \cos\theta \quad \text{at } \theta = \frac{2\pi}{3}$$

(A) $\cancel{2a} \Rightarrow \log(2a) - \log(r) \Rightarrow \log(1-\cos\theta)$

$$\Rightarrow -\frac{1}{r}\frac{dr}{d\theta} \Rightarrow \frac{\sin\theta}{1-\cos\theta}$$

$$\Rightarrow \frac{dr}{d\theta} = -\frac{r\sin\theta}{1-\cos\theta}$$

$$= \frac{d\theta}{dr} = \frac{-(1-\cos\theta)}{r\sin\theta}$$

$$\frac{\cos\theta - 1}{\sin\theta} \Rightarrow$$

$$\tan\phi = r\cdot\frac{d\theta}{dr} = \frac{\cancel{r}\cdot -(\cancel{\cos\theta}\,1-\cos\theta)}{\cancel{r}\sin\theta}$$

$$= \frac{-(1-\cos\theta)}{\sin\theta} \Rightarrow \frac{-(1-\cos(2\pi/3))}{\sin(2\pi/3)}$$

$$\Rightarrow \frac{-(1-(-1/2))}{\sqrt{3}/2} = -\frac{(1+1/2)}{\sqrt{3}/2}$$

$$= \frac{-3/\cancel{2}}{\sqrt{3}/\cancel{2}}$$

$$\tan\phi \Rightarrow -\sqrt{3}$$

$$\phi = \tan^{-1}(-\sqrt{3}) \Rightarrow 2\pi/3$$

$$\pi - \frac{2\pi}{3}$$

$$\frac{2\pi}{3} + \frac{1}{2}$$

$$\frac{\pi}{3} + \frac{\pi}{2}$$

$$\frac{2\pi + 3\pi}{6}$$

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ANGLE B/W TWO CURVES (OR) ANGLE OF INTERSECTION OF THE CURVES

Let α be the angle b/w two polar curves $r = f(\theta)$ and $r = g(\theta)$.

Let ϕ_1 and ϕ_2 be the angle b/w the radius vector and the tangent to the curves $r = f(\theta)$ and $r = g(\theta)$ respectively.

So $\alpha = |\phi_1 - \phi_2|$

(or) $\tan\alpha = \tan|\phi_1 - \phi_2|$

$$\tan\alpha = \frac{\tan\phi - \tan\phi_2}{1 + \tan\phi \cdot \tan\phi_2}$$

If two curves intersect orthogonally, $\tan\phi_1 \cdot \tan\phi_2 = -1$ (or) $\cot\phi_1 \cdot \cot\phi_2 = -1$

(Q) Find angle b/w the following polar curves.

(i) $r = \sin\theta + \cos\theta$; $r = 2\sin\theta$

(A) $\frac{dr}{d\theta} = \cos\theta - \sin\theta$; $\frac{dr}{d\theta} = 2\cos\theta$; $\phi_1 = \pi/4 + \theta$

$\cot\phi_1 = \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{\cos\theta - \sin\theta}{r}$ $\phi_2 = \theta$

$\phi_1 - \phi_2 = \pi/4$

$\cot\phi_2 = \frac{1}{r} \cdot 2\cos\theta = \frac{2\cos\theta}{r}$

$$\cot\alpha = \frac{\cot\phi_1 \cot\phi_2 + 1}{\cot\phi_2 - \cot\phi_1}$$

$$= \frac{\frac{\cos\theta - \sin\theta}{r} \cdot \frac{2\cos\theta}{r} + 1}{\frac{\cos\theta - \sin\theta}{r} - \frac{2\cos\theta}{r}} = \cos$$

(Q) $r = a\log\theta$; $r = \frac{a}{\log\theta}$

$$\frac{-1/\theta\,(a)}{(\log\theta)^2}$$

(A) $\frac{dr_1}{d\theta} = \frac{a}{\theta}$; $\frac{dr_2}{d\theta} = \frac{-a}{\theta(\log\theta)^2}$

$$-a/\theta\,(\log\theta)^2$$

$\tan\phi_1 = \frac{\cancel{a}\theta}{\cancel{a}} \cdot \log\theta = \theta\log\theta$

$\tan\phi_2 = \frac{\cancel{a}}{\cancel{\log\theta}} \div \frac{\theta(\log\theta)^{\cancel{2}}}{\cancel{a}} = -\theta\log\theta$

$\tan\phi_1 - \tan\phi_2 = 2\theta\log\theta$

$$\left(\log_e \boxed{e}\right)^2$$

$$a\log\theta = \frac{a}{\log\theta}$$

$$\log(\theta)^2 = 1 = (\log_e)^2$$

$$\boxed{\theta = e},\ \frac{1}{e}$$

$\boxed{\theta = e} \rightarrow$ CASE 1

$$\tan\phi_1 = e\log e = e$$

$$\tan\phi_2 = -e\log e = -e$$

$$\tan\alpha = \frac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1\tan\phi_2} = \frac{e+e}{1-e^2}$$

$$\alpha = \tan^{-1}\left(\frac{2e}{1-e^2}\right)$$

(Q) $r = \frac{a\theta}{1+\theta}$; $r = \frac{a}{1+\theta^2}$

(A) $$\frac{dr_1}{d\theta} = \frac{a(1+\theta) - a\theta(1)}{(1+\theta)^2} = \frac{a + \cancel{a\theta} - \cancel{a\theta}}{(1+\theta)^2}$$

$$\frac{dr_2}{d\theta} = \frac{-2\theta a}{(1+\theta^2)^2} = \frac{-2\theta a}{(1+\theta^2)^2}$$

$$\frac{a\theta}{1+\theta} = \frac{a}{1+\theta^2}$$

$$(1+\theta^2)\theta = (1+\theta)$$

$$\cancel{\theta} + \theta^3 = 1 + \cancel{\theta}$$

$$\theta^3 = 1$$

$$\boxed{\theta = 1}$$

$$\tan\phi_1 = \frac{a\theta}{\cancel{1+\theta}} \cdot \frac{(1+\theta)^{\cancel{2}}}{\cancel{a}} = -\theta(1+\theta) = -1(2) = -2$$

$$\tan\phi_2 = \frac{(1+\theta^2)^{\cancel{2}}}{-2\theta \cancel{a}} \times \frac{\cancel{a}}{\cancel{(1+\theta^2)}} = \frac{-(1+\theta^2)}{2\theta}$$

$$= \frac{(2)}{2} = 1$$

$$\frac{-\theta(1+\theta) + \frac{a(1+\theta^2)}{2\theta}}{1 - \theta(1+\theta)\cdot\frac{a(1+\theta^2)}{2\theta}} = \frac{\frac{-2\theta^2(1+\theta) + a(1+\theta^2)}{\cancel{2\theta}}}{\frac{2\theta - \theta(1+\theta)\cdot a(1+\theta^2)}{\cancel{2\theta}}}$$

$$= \frac{-2\theta^2 - 2\theta^3 + a + a\theta^2}{2\theta - a\theta[1+\theta^2+\theta+\theta^3]}$$

$$\tan\alpha = \frac{-2\theta^2 - 2\theta^3 + a + a\theta^2}{2\theta - a\theta - a\theta^3 - a\theta^2 - a\theta^4}$$

$$\alpha = \tan^{-1}\left[\frac{a\theta^2 + a - 2\theta^2 - 2\theta^3}{2\theta - a\theta - a\theta^3 - a\theta^2 - a\theta^4}\right]$$

$$= \tan^{-1}\left[\frac{a + a - 2 - 2}{2 - a - a - a - a}\right]$$

$$= \tan^{-1}\left[-\frac{1}{3}\right]$$

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$$\frac{\tan\phi_1 - \tan\phi_2}{1 - \tan\phi_1 \tan\phi_2} = \frac{-2-1}{1-(-2)(1)} = \frac{-3}{3} = -1$$

$$\tan\alpha = \frac{-2-1}{1-(-2)(1)} = \frac{-3}{1+2} = \frac{-\cancel{3}}{\cancel{3}} = -1$$

$$\alpha = \tan^{-1}(-1)$$

$$\boxed{\alpha = 3\pi/4}$$

Hw (Q) Show that the curves $r = a(1+\cos\theta)$ and $r^2 = a^2\cos 2\theta$ intersect at an angle given by $3\left(\sin^{-1}\left(\frac{3}{4}\right)^{1/4}\right)$.

(A) $\frac{dr}{d\theta} = -a\sin\theta$; $\cancel{2}r\frac{dr}{d\theta} = a^2 \cdot -\sin 2\theta \cdot \cancel{2}$

$$\Rightarrow r\cdot\frac{dr}{d\theta} = -a^2\sin 2\theta \Rightarrow \frac{dr}{d\theta} = \frac{-a^{\cancel{2}}\sin 2\theta}{\cancel{a}\sqrt{\cos 2\theta}}$$

$$\Rightarrow \tan\phi_1 = r\cdot\frac{d\theta}{dr} = \cancel{a}(1+\cos\theta)\cdot\frac{-1}{\cancel{a}\sin\theta} = -\frac{(1+\cos\theta)}{\sin\theta} \Rightarrow -\frac{2\cos^{\cancel{2}}(\theta/2)}{2\sin(\theta/2)\cdot\cancel{\cos(\theta/2)}}$$

$$\tan\phi_1 = -\cot(\theta/2)$$

$$\tan\phi_1 = +\tan(\pi/2 + \theta/2)$$

$$\phi_1 = \pi/2 + \theta/2$$

$$\Rightarrow \tan\phi_2 \Rightarrow r\cdot\frac{d\theta}{dr} = -\frac{\cancel{a}\sqrt{\cos 2\theta}\cdot\sqrt{\cos 2\theta}}{\cancel{a}\sin 2\theta} = -\frac{\cos 2\theta}{\sin 2\theta} = -\cot 2\theta$$

$$\tan\phi_2 = -\cot(2\theta)$$

$$\tan\phi_2 = \tan(\pi/2 + 2\theta)$$

$$\phi_2 = \pi/2 + 2\theta$$

$$\tan\alpha = \frac{\tan(\pi/2 + \theta/2) - \tan(\pi/2 + 2\theta)}{1 + \tan(\pi/2+\theta/2)\cdot\tan(\pi/2+2\theta)}$$

$$\cancel{a}(1+\cos\theta) = \cancel{a}\sqrt{\cos 2\theta}$$

$$a + a\cos\theta - a\sqrt{\cos 2\theta} = 0$$

$$a + a\cos\theta - a\sqrt{\ }$$

$$1+\cos\theta = \sqrt{\cos 2\theta}$$

$$1 + \cancel{\cos^2\theta} + 2\cos\theta = \cancel{\cos^2\theta} - \sin^2\theta$$

$$1 + \sin^2\theta + 2\cos\theta = 0$$

$$1 + (1-\cos^2\theta) + 2\cos\theta = 0$$

$$2 - \cos^2\theta + 2\cos\theta = 0$$

$$\cos^2\theta - 2\cos\theta - 2 = 0$$

$$\cancel{\cos^2\theta - r}$$

$$\cos\theta = \frac{+2 \pm \sqrt{4 - 4(1)(-2)}}{2(1)} \Rightarrow \cos\theta = 1 - \sqrt{3}$$

$$= \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$$

$$\sin\theta = 1 - (1-\sqrt{3})^2$$

$$= 1 - (1 + 3 - 2\sqrt{3})$$

$$= \cancel{1} - \cancel{1} - 3 + 2\sqrt{3} = 2\sqrt{3} - 3$$

$\phi_1 = \pi/2 + \theta/2$; $\phi_2 = \pi/2 + 2\theta$

$\alpha = |\phi_1 - \phi_2|$

$\alpha = \cancel{\pi/2} + \theta/2 - \cancel{\pi/2} - 2\theta$

$\alpha = \left|\dfrac{\theta - 4\theta}{2}\right| \Rightarrow \boxed{\alpha = \dfrac{3\theta}{2}}$

$(a(1+\cos\theta))^2 = a^2\cos 2\theta$

$\cancel{a^2}[1+\cos^2\theta + 2\cos\theta] = \cancel{a^2}\cos 2\theta$ $\quad 1+\cos\cos 2 = 2\cos^2\theta$ / $\cos$

$1 + \cos^2\theta + 2\cos\theta = 2\cos^2\theta - 1$

$\cos^2\theta - 2\cos\theta - 2 = 0$

$\cos\theta = \dfrac{+2 \pm \sqrt{4 - 4(1)(-2)}}{2(1)}$

$\cos\theta = \dfrac{2 \pm \sqrt{12}}{2} \Rightarrow \cos\theta = \dfrac{2 \pm 2\sqrt{3}}{\cancel{2}} \Rightarrow \cos\theta = 1 \pm \sqrt{3}$

$\cos\theta = 1 - \sqrt{3}$

$\theta = \cos^{-1}(1 - \sqrt{3})$

$\sin\theta = \sqrt{1 - \cos^2\theta} = 1 - (1-\sqrt{3})^2$

$= \sqrt{\cancel{1} - [\cancel{1} + 3 - 2\sqrt{3}]}$

$= \sqrt{2\sqrt{3} - 3}$

$\theta = \sin^{-1}\left((2\sqrt{3} - 3)^{1/2}\right)$

$\alpha = \dfrac{3\theta}{2} = \dfrac{3}{2}\left(\sin^{-1}(2\sqrt{3} - 3)^{1/2}\right)$

(Q) The tangents to the curve $r = a(1+\cos\theta)$ at the point $2\pi/3$ & $\pi/3$ are parellel and perpendicular with the initial line respectively.

(A) $r = a(1+\cos\theta)$

$\tan\phi = r \cdot \frac{d\theta}{dr}$ $\Rightarrow$ $\frac{dr}{d\theta} = -a\sin\theta$

$$= \cancel{a}(1+\cos\theta) \cdot \frac{-1}{\cancel{a}\sin\theta} = -\frac{(1+\cos\theta)}{\sin\theta} = -\frac{(\cancel{2}\cos^2\theta/2)}{\cancel{2}\sin\theta/2 \cdot \cos\theta/2}$$

$\therefore \tan\phi = -\cot(\theta/2)$

$\therefore \tan\phi = \tan(\pi/2 + \theta/2)$ $2\pi/3 =$; $\pi/3 =$

$\phi = \frac{\pi}{2} + \frac{\theta}{2}$

$\Rightarrow$ $\tan\phi_1 \cdot \tan\phi_2 = -1$

$\Rightarrow$ $\psi = \theta + \phi$

at $\theta = 2\pi/3$

$\psi = 2\pi/3 + \pi/2 + \cancel{2}\pi/3\times\cancel{2} = \pi + \pi/2 = 3\pi/2$

$\tan\psi = \tan 3\pi/2 = \infty$

$\Rightarrow$ The tangents are ~~parallel~~ perpendicular

$\Rightarrow$ at $\theta = \pi/3$

$\psi = \pi/3 + \pi/2 + \pi/6 = \frac{2\pi + 3\pi + \pi}{6} = \frac{\cancel{6}\pi}{\cancel{6}} = \pi$

$\tan\psi = \tan\pi = 0$

$\Rightarrow$ The tangents are parallel

(Q) Find the angle b/w the curves $r^n = a^n \sec(n\theta + \alpha)$ and $r^n = b^n \sec(n\theta + \beta)$

(A) $n\log r = n\log(a) + \log(\sec(n\theta + \alpha))$

$\frac{n}{r} \cdot \frac{dr}{d\theta} = \frac{1}{\cancel{\sec(n\theta+\alpha)}} \cdot \cancel{\sec(n\theta+\alpha)} \cdot \tan(n\theta+\alpha) \cdot n$

$\tan\phi_1 = r \cdot \frac{d\theta}{dr}$ $\Rightarrow$ $\cancel{\ldots}$ $\cdot \cancel{r} \cdot \frac{1}{\cancel{r}(\tan(n\theta+\alpha))}$

$\frac{dr}{d\theta} = \frac{r(\tan(n\theta+\alpha))\cancel{n}}{\cancel{n}}$

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$$\tan\phi_1 = \frac{1}{\tan(n\theta+\alpha)}$$

$$n\log r = n\log(b) + \log(\sec(n\theta+\beta))$$

$$\frac{n}{r}\cdot\frac{dr}{d\theta} = \frac{1}{\cancel{\sec(n\theta+\beta)}}\cdot \cancel{\sec(n\theta+\beta)}\cdot\tan(n\theta+\beta)\cdot n$$

$$\frac{dr}{d\theta} = \frac{r\cdot\tan(n\theta+\beta)\cdot\cancel{n}}{\cancel{n}}$$

$$\tan\phi_2 = r\cdot\frac{d\theta}{dr} = \frac{\cancel{r}}{\cancel{r}\tan(n\theta+\beta)}$$

$$\tan\phi_2 = \frac{1}{\tan(n\theta+\beta)}$$

$$\tan\phi_1 = \cot(n\theta+\alpha) = \tan\left(\pi/2-(n\theta+\alpha)\right)$$

$$\phi_1 = \pi/2 - n\theta - \alpha$$

$$\phi_2 = \pi/2 - n\theta - \beta$$

$$\alpha = |\phi_2-\phi_1| = \cancel{\pi/2}-\cancel{n\theta}-\beta-\cancel{\pi/2}+\cancel{n\theta}+\alpha$$

$$= (\alpha-\beta)$$

(Q) Find the angle b/w $r = a(1+\cos\theta)$ and $r = 2a\cos\theta$.

(A) $\frac{dr}{d\theta} = a\sin\theta \Rightarrow \tan\phi_1 = r\cdot\frac{d\theta}{dr} = \frac{\cancel{a}(1-\cos\theta)\cdot 1}{\cancel{a}\sin\theta} = \frac{\cancel{2}\sin^2\theta/2}{\cancel{2}\sin\theta/2\cdot\cos\theta/2}$

$$\tan\phi_1 = \tan(\theta/2)$$

$$\boxed{\phi_1 = \theta/2}$$

$\frac{dr}{d\theta} = -2a\sin\theta \Rightarrow \tan\phi_2 = \cancel{2a}\cos\theta\cdot\frac{-1}{\cancel{2a}\sin\theta} = -\cot\theta$

$$\tan\phi_2 = \tan(\pi/2+\theta)$$

$$\boxed{\phi_2 = \pi/2+\theta}$$

$$\alpha = |\phi_2-\phi_1| = \pi/2+\theta-\theta/2 = \pi/2+\theta/2$$

$\Rightarrow \cancel{a}(1-\cos\theta) = 2\cancel{a}\cos\theta$

$1-\cos\theta = 2\cos\theta$

$3\cos\theta = 1$

$\cos\theta = 1/3$

$\theta = \cos^{-1}(1/3)$

$\alpha = \frac{\pi}{2} + \frac{\theta}{2} = \frac{\pi}{2} + \frac{\cos^{-1}(1/3)}{2}$

(Q) Show that the curves intersect orthogonally. $r^2\cos 2\theta = 4$, $r^2\sin 2\theta = 9$.

(A) $\frac{dr}{d\theta} \Rightarrow 2r\cos 2\theta \mp r^2\sin 2\theta \cdot 2 = 0.$

$\Rightarrow \quad r^2 = \frac{4}{\cos 2\theta}$

$2r \cdot \frac{dr}{d\theta}$

$\cancel{2}r \cdot \frac{dr}{d\theta} \cdot \cos 2\theta = \cancel{2}r^{\cancel{2}}\sin 2\theta$

$\frac{dr}{d\theta} = \frac{r\sin 2\theta}{\cos 2\theta} = r\tan 2\theta$

$\tan\phi_1 = r \cdot \frac{d\theta}{dr} = \cancel{r} \cdot \frac{1}{\cancel{r}\tan 2\theta} = \cot 2\theta$

$\frac{dr}{d\theta} \Rightarrow 2r \cdot \frac{dr}{d\theta} \cancel{\cos 2\theta} \sin 2\theta + r^2 \cdot \cos 2\theta \cdot 2 = 0$

$\Rightarrow \frac{dr}{d\theta} = -\frac{\cancel{2}r^{\cancel{2}}\cos 2\theta}{\cancel{2r}\sin 2\theta} = -r\cot 2\theta$

$\tan\phi_2 = r \cdot \frac{d\theta}{dr} = \cancel{r} \cdot \frac{-1}{\cancel{r}\cot 2\theta} = -\tan 2\theta$

$\tan\phi_1 \cdot \tan\phi_2 = \cancel{\cot 2\theta} \cdot \frac{-1}{\cancel{\cot 2\theta}} = -1$ (Hence Proved)

CURVATURE:

Curvature is the degree of bending of the curve with respect to the arc length denoted by 'K' (Kappa),

$$K = \frac{d\psi}{ds}$$

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RADIUS OF CURVATURE

The reciprocal of curvature is called radius of curvature denoted by ρ (rho) $= \frac{ds}{d\psi}$

NOTE: (1) Curvature of the straight line is 0 and radius of curvature is ∞

(2) Curvature of the circle is the reciprocal of its radius, so the radius of the curvature is equal to its radius

Cartesian form:

Let $y = f(x)$ be the equation of the curve in cartesian form, then radius of curvature $\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$

$$K = \frac{d^2y/dx^2}{\left[1+(dy/dx)^2\right]^{3/2}}$$

Parametric form:

$x = x(t)$, $y = y(t)$ be the parametric equation of the curve, then:

$$\rho = \frac{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]^{3/2}}{\frac{dx}{dt}\cdot\frac{d^2y}{dt^2} - \frac{d^2x}{dt^2}\cdot\frac{dy}{dt}}$$

$K = \frac{1}{\rho}$

Kappa

NOTE (1) At any point dy/dx is not defined, we can find radius of curvature using the formula:

$$\rho = \frac{\left[1+(dx/dy)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}$$

(2) If $\frac{d^2y}{dx^2}$ is less than 0, then curve is concave downward.

$\therefore \rho < 0$

(3) If d^2y/dx^2 is grater than 0, then curve is concave upward.

$\therefore \rho > 0$

$\cosh x = \frac{e^x + e^{-x}}{2}$; $\sin(hx) = \frac{e^x - e^{-x}}{2}$.

$\left.\begin{array}{l} \frac{d}{dx}(\cosh x) = \sinh x \\ \\ \frac{d}{dx}(\sinh x) = \cosh x \end{array}\right]$ → all hyperbolic f^n have positive differentials

(Q) Find the radius of curvature of the curves $ay^2 = x^3$.

(A) $\frac{dy}{dx} \Rightarrow a \cdot 2y \cdot \frac{dy}{dx} = 3x^2$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{2ay}.$$

$$\Rightarrow \frac{d^2y}{dx^2} \Rightarrow \frac{6x(2ay) - 2a\, {}^{dy}/_{dx}(3x^2)}{(2ay)^2}$$

$$\Rightarrow \frac{12axy - 6ax^2 \cdot dy/dx}{4a^2y^2}$$

$$\Rightarrow \frac{12axy - \cancel{6}^{3}ax^2 \cdot \frac{3x^2}{\cancel{2}ay}}{4a^2y^2}$$

$$\Rightarrow \frac{12axy^2 - 9x^2}{4a^2y^3}$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \Rightarrow \rho = \frac{\left[1 + \frac{9x^4}{4a^2y^2}\right]^{3/2}}{\frac{12axy^2 - 9x^2}{4a^2y^3}}$$

$$\rho = \frac{\left[\frac{4a^2y^2 + 9x^4}{(4a^2y^2)^{3/2}}\right]^{3/2} \cdot 4a^2y^3}{12axy^2 - 9x^2}$$

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$$\Rightarrow \rho = \frac{(4a^2y^2+9x^4)^{3/2}}{a(12axy^2-9x^2)\cdot 2} = \frac{(4a^2y^2+9x^4)^{3/2}}{2a(12axy^2-9x^2)}$$

(Q) Find the radius of curvature of the cantenary curve $y = c\cosh\left(\frac{x}{c}\right)$.

(A) $$\frac{dy}{dx} \Rightarrow \cancel{c}\cdot\sinh\left(\frac{x}{c}\right)\cdot\frac{1}{\cancel{c}}$$

$$\frac{d^2y}{dx^2} = \cosh\left(\frac{x}{c}\right)\cdot\frac{1}{c}$$

$$\rho = \frac{\left[1+\left[\sinh\left(\frac{x}{c}\right)\right]^2\right]^{3/2}}{\cosh\left(\frac{x}{c}\right)\cdot\frac{1}{c}}$$

$$\Rightarrow \frac{\left[\cosh^2\left(\frac{x}{c}\right)\right]^{3/2}}{\cosh\left(\frac{x}{c}\right)\cdot\frac{1}{c}} = \cosh^2\left(\frac{x}{c}\right)\cdot c$$

$$\Rightarrow c\cdot\frac{y^2}{c^2} = \frac{y^2}{c}$$

(Q) Find the radius of curvature of the curve $xy = c^2$ Rectangular hyperbola

(A) $$\frac{dy}{dx} \quad y + x\frac{dy}{dx} = 0$$

$$x\cdot\frac{dy}{dx} = -y \qquad \frac{dy}{dx} = \frac{-y}{x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-\frac{dy}{dx}\cdot x - 1(-y)}{x^2} = \frac{+\frac{y}{x}\cdot x + y}{x^2} = \frac{2y}{x^2}$$

$$\rho = \frac{\left[1+\left(\frac{y}{x}\right)^2\right]^{3/2}}{2y/x^2} = \frac{\left[1+\frac{y^2}{x^2}\right]^{3/2}}{2y/x^2}$$

$$\Rightarrow \frac{[x^2+y^2]^{3/2}}{[x^2]^{3/2}} \times \frac{x^2}{2y}$$

$$\Rightarrow \frac{[x^2+y^2]^{3/2}}{2xy} = \frac{[x^2+y^2]^{3/2}}{2c^2}$$

$y = \frac{c^2}{x}$ $\quad \frac{[x^2+y^2]^{3/2}}{2c^2}$

$$\frac{\left[x^2 + \frac{c^4}{x^2}\right]^{3/2}}{2c^2} = \frac{[x^4+c^4]^{3/2}}{(x^2)^{3/2} \cdot 2c^2}$$

$$= \frac{(x^4+c^4)^{3/2}}{2x^3c^2}$$

$$= \cancel{x}$$

(Q) Find the curvature of the curve

$y = x^2 - 3x + 1$ at $(1,-1)$

(A) $\frac{dy}{dx} = 2x - 3 \quad (1,-1)$

$\hookrightarrow 2(1) - 3 = -1$

$= \frac{d^2y}{dx^2} = 2$

$$\rho = \frac{[1+1]^{3/2}}{2} = \frac{(2)^{3/2}}{2} = 2^{1/2} = \sqrt{2}$$

$$K = \frac{1}{\rho} = \frac{1}{\sqrt{2}}$$

(Q) Find the ROC of the curve $y = a \log \sec\left(\frac{x}{a}\right)$

(A) $\frac{dy}{dx} = \frac{\cancel{a}}{\cancel{\sec(x/a)}} \cdot \cancel{\sec(x/a)} \cdot \tan(x/a) \cdot \frac{1}{\cancel{a}}$

$$\frac{dy}{dx} = \tan(x/a)$$

$$\frac{d^2y}{dx^2} = \sec^2(x/a) \cdot \frac{1}{a}$$

$$\rho = \frac{[1+\tan^2(x/a)]^{3/2}}{\sec^2(x/a)} \cdot a = \frac{[\sec^2(x/a)]^{3/2}}{\sec^2(x/a)} \cdot a$$

$$\rho = \sec(x/a) \cdot a$$

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(Q) (1) $y = x^3(x-a)$ $(a,0)$ $x^4 - ax^3$

$\frac{dy}{dx} = 4x^3 - 3ax^2$

$\frac{d^2y}{dx^2} = 12x^2 - 6ax$

$\rho = \frac{[1 + (4x^3 - 3ax^2)^2]^{3/2}}{12x^2 - 6ax} = \frac{[1 + (4a^3 - 3a^3)^2]^{3/2}}{12a^2 - 6a^2}$

$= \frac{[1 + a^6]^{3/2}}{6a^2}$

(2) $y = a\sin(x/a)$ at $x = \frac{\pi a}{4}$

$\frac{dy}{dx} = +\cancel{a}\cos(x/a) \cdot \frac{1}{\cancel{a}} = +\cos(x/a) = \cos\left(\frac{\pi \cancel{a}}{4(\cancel{a})}\right) = 1/\sqrt{2}$

$\frac{d^2y}{dx^2} = -\sin(x/a) \cdot 1/a = -\sin\left(\frac{\pi \cancel{a}}{4(\cancel{a})}\right) \cdot 1/a = \frac{-1}{\sqrt{2}a}$

$\rho = -[1 + 1/2]^{3/2} \cdot \sqrt{2}a$

$\rho = -[3/2]^{3/2} \cdot \sqrt{2}a$

(3) $y = 2\log \sin(x/2)$ at $x = \pi/3$

$\frac{dy}{dx} = \cancel{2} \cdot \frac{1}{\sin(x/2)} \cdot \cos(x/2) \cdot \frac{1}{\cancel{2}} = \cot(x/2) = \cot(\pi/6) = \sqrt{3}$

$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2(x/2) = -\operatorname{cosec}^2(\pi/6) = -4$

$\rho = \frac{[1+3]^{3/2}}{-4} = \frac{4^{3/2}}{-4} = -4^{1/2}$

(Q) Find the radius of the curvature $y^2 = \frac{a^2(a-x)}{x}$ at the point $(2a, 0)$. $a^3 - a^2x$

(A) $2y \cdot \frac{dy}{dx} = \frac{-a^2(x) - [a^3 - a^2(x)](1)}{x^2} = \frac{-\cancel{a^2x} - a^3 + \cancel{a^2x}}{x^2}$

$dy/dx = \frac{-a^3}{2x^2y}$. at $(a,0)$ $dy/dx = \infty$

at $y=0$, $\frac{dy}{dx}=\infty$ then we take $\frac{dx}{dy}$

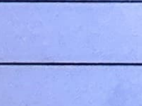
$$\frac{dx}{dy}=\frac{2x^2y}{x(-a^2)-a^3+a^2x} = \frac{2x^2y}{-a^3} \quad \text{at } (a,0)$$

$$\frac{dx}{dy}=0$$

$$\frac{d^2x}{dy^2} = \frac{-2\,(x^2)}{a^3} \quad (a,0)$$

$$= \frac{-a^2}{a^3} = \frac{-2}{a}$$

$$\rho = \frac{[1+0]^{3/2}\cdot a}{-2} = \frac{-a}{2}$$



(Q) Find the ROC of the curve $x^{2/3}+y^{2/3}=a^{2/3}$ ← Asteroid curve

(A) $\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\cdot\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-\cancel{\frac{2}{3}}x^{-1/3}}{\cancel{\frac{2}{3}}y^{-1/3}} = \frac{-x^{-1/3}}{y^{-1/3}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{1}{3}x^{-4/3}\cdot y^{-1/3} - (-x^{-1/3})\cdot -\frac{1}{3}y^{-4/3}\frac{(-x^{-1/3})}{y^{-1/3}}}{(y^{-1/3})^2}$$

$$= \frac{\frac{x^{-4/3}\cdot y^{-1/3}}{3} + \frac{x^{-2/3}\cdot y^{-1}}{3}}{(y^{-2/3})}$$

$$\rho = \left[1+\frac{x^{-2/3}}{y^{-2/3}}\right]^{3/2} = \frac{1}{3}\left[\frac{x^{2/3}+y^{2/3}}{x^{4/3}}\right] = \frac{d^2y}{dx^2}$$

$$\left[\frac{x^{-4/3}\cdot y^{-1/3}+x^{-2/3}\cdot y^{-1}}{3y^{-2/3}}\right] \Rightarrow \rho = 3(xya)^{1/3}$$

Folium of Descartes

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(Q) Find ROC of the curve $x^3+y^3=3axy$ at $(3a/2, 3a/2)$

(A) $3x^2+3y^2\cdot\frac{dy}{dx}=3ay+3ax\cdot\frac{dy}{dx}$

$3x^2-3ay=(3ax-3y^2)\cdot\frac{dy}{dx}$

$\frac{dy}{dx}=\frac{3x^2-3ay}{3ax-3y^2}=\frac{3\times\frac{9a^2}{4}-3a\times\frac{3a}{2}}{3a\cdot 3a/2-3\times\frac{9a^2}{4}}=\frac{\frac{27a^2}{4}-\frac{9a^2}{2}}{\frac{9a^2}{2}-\frac{27a^2}{4}}$

$=\frac{\frac{27a^2-18a^2}{4}}{\frac{18a^2-27a^2}{4}}=\frac{9a^2\times 4}{4\times -9a^2}=-1$

$\frac{d^2y}{dx^2}=(6x-3a\cdot dy/dx)$

$\frac{(6x-3a\cdot dy/dx)(3ax-3y^2)-(3a-6y\cdot dy/dx)(3x^2-3ay)}{(3ax-3y^2)^2}$

$=\frac{[6x-3a(-1)](3ax-3y^2)-(3a-6y(-1))(3x^2-3ay)}{(3ax-3y^2)^2}$ $\quad \frac{\frac{9a^2}{2}-\frac{27a^2}{4}}{4}$

$=\frac{[6\times 3a/2+3a][3a(3a/2)-3(3a/2)^2]-[3a+6(3a/2)][3\times\frac{9a^2}{4}-3a(3a/2)]}{[3a(3a/2)-3(3a/2)^2]^2}$

$=\frac{[12a][-\frac{9a^2}{4}]-[12a][9a^2/4]}{\left[\frac{9a^2}{2}-\frac{27a^2}{4}\right]^2}=\frac{-2\cdot 12a\cdot\frac{9a^2}{4}}{\left(-\frac{9a^2}{4}\right)^2}$

$=\frac{-24a\times 2}{\frac{9a^2}{4}}$

$\frac{-24a\times 4}{9a^2}=\frac{32}{3a}$

$=\frac{-24a\times 2}{9a^2}=\frac{+32}{3a}$

$\rho=\frac{[1+1]^{3/2}}{(+32/3a)}=+\frac{(2)^{3/2}\cdot 3a}{32}=\boxed{\rho=\frac{3a}{8\sqrt{2}}}$ → ROC

Homework (Q) $x^3-y^3=9axy$ at $(9/2, a/2)$ → Folium of Descartes

$\frac{\frac{27a^2}{4}-\frac{9a^2}{2}}{\frac{27a^2-18a^2}{4}}$

(Q) Find the ROC of the curve $y = e^{-x} \cdot x$ at the point where the curve is max^m.

(A) $\frac{dy}{dx} = e^{-x} - xe^{-x} = 0.$

$e^{-x}[1-x] = 0$

$\boxed{x=1} \rightarrow dy/dx = 0$

$\frac{d^2y}{dx^2} < 0 = e^{-x}\cdot(-1) - e^{-x} + xe^{-x}$

$= -2e^{-x} + xe^{-x} < 0$

$= -\frac{2}{e} + \frac{1}{e} = \frac{-1}{e}$

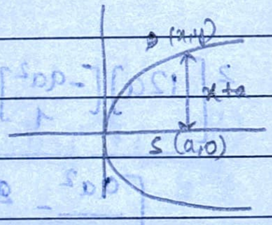
$\rho = \frac{[1+0]^{3/2}\cdot e}{-1} = -e$

$\boxed{\rho = e}$

*(Q) For the parabola $y^2 = 4ax$, the square of ROC at any point varies as the cube of the focal distance at that point.

(A) $2y \cdot \frac{dy}{dx} = 4a$ — The focal distance of the parabola $y^2 = 4ax \Rightarrow [x+a]$

$\frac{dy}{dx} = \frac{\cancel{4}^2 a}{\cancel{2}y} = \frac{2a}{y}$



$SD = \sqrt{(x-a)^2 + y^2}$

$SD = \sqrt{x^2 + a^2 - 2ax + y^2}$

$SD = \sqrt{x^2 + a^2 - 2ax + 4ax}$

$SD = \sqrt{(x+a)^2} = \boxed{x+a}$

$\boxed{\rho^2 = (SD)^3}$ → we have to prove

$\frac{d^2y}{dx^2} = \frac{-dy/dx\,(2a)}{y^2} = \frac{2a/y \cdot (-2a)}{y^2} = -\frac{4a^2}{y^3}$

$\rho = \frac{\left[1 + 4a^2/y^2\right]^{3/2}}{\frac{-4a^2}{y^3}} = \frac{\frac{(y^2+4a^2)^{3/2}}{(y^2)^{3/2}}}{\frac{-4a^2}{y^3}} = -\frac{[y^2+4a^2]^{3/2}}{4a^2}$

$= -\frac{[4ax + 4a^2]^{3/2}}{4a^2}$

$= \frac{4a^{2 \cdot 3/2}\left[x/a + 1\right]^{3/2}}{4a^2}$

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$$\rho^2 \propto \left[a\left[\frac{x}{a}+1\right]^{3/2}\right]^2$$

$$= a\left[\frac{x}{a}+1\right]^{3/2} \qquad \left[\frac{x}{a}+1\right]^3 \cdot a^2 \hookrightarrow (x+a)^3$$

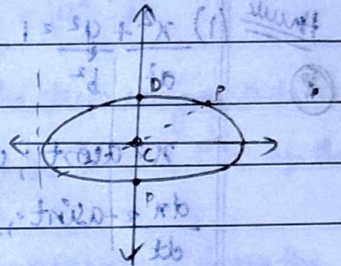
$$(x+a)^3 = x^3+a^3+3ax(x+a)$$

$$= x^3+a^3+3ax^2+3a^2x \qquad \downarrow (SD)^3$$

Hence Proved $\Rightarrow \boxed{\rho^2 \propto (SD)^3}$

Imp (Q) Show that an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the ROC at any point $P(x,y)$ is $\underline{(CD)^3}$ where C is the centre of the ellipse, D is the ab extremity on the conjugate diameter of CP.

(A) $\frac{dy}{dx} \Rightarrow \frac{2x}{a^2}+\frac{2y}{b^2}\cdot\frac{dy}{dx}=0$



$$\frac{\cancel{2}y}{b^2}\cdot\frac{dy}{dx} = \frac{-\cancel{2}x}{a^2}$$

$$\frac{dy}{dx} \Rightarrow \frac{-b^2\cdot x}{a^2\cdot y}$$

$$\frac{d^2y}{dx^2} \Rightarrow \frac{-b^2(a^2y)-(-b^2x)\cdot a^2\cdot \frac{-b^2x}{a^2y}}{(a^2y)^2}$$

$$\Rightarrow \frac{-a^2b^2y-\frac{b^4x^2}{y}}{a^4y^2} \Rightarrow \frac{-a^2b^2y^2-b^4x^2}{a^4y^3}$$

$$\rho = \frac{\left[1+\frac{b^4x^2}{a^4y^2}\right]^{3/2}}{\frac{-a^2b^2y^2-b^4x^2}{a^4y^3}} \Rightarrow \frac{\left[\frac{a^4y^2+b^4x^2}{a^4y^2}\right]^{3/2}}{\frac{-a^2b^2y^2-b^4x^2}{a^4y^3}} \Rightarrow$$

Imp (Q) $3x^2-3y^2\cdot\frac{dy}{dx} \Rightarrow 9y+9x\cdot\frac{dy}{dx}$

$$3x^2-9y \Rightarrow \frac{dy}{dx}(9x+3y^2)$$

$$\frac{dy}{dx} \Rightarrow \frac{3x^2-9y}{9x+3y^2} \qquad (a/2, a/2)$$

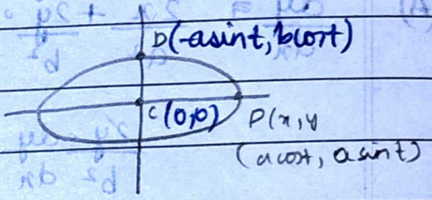
$$= \frac{3(a^2/4)-9(a/2)}{9(a/2)+3(a^2/4)} \Rightarrow \frac{\frac{3a^2-18a}{4}}{\frac{18a+3a^2}{4}} = \frac{3a^2-18a}{18a+3a^2}$$

$\rightarrow (a/2, a/2)$

$$\frac{d^2y}{dx^2} = \frac{(6x - 9\cdot dy/dx)(9x+3y^2) - (9+6y\cdot dy/dx)(3x^2-9y)}{(9x+3y^2)^2}$$

$$\Rightarrow \frac{\left(6\times a/2 - 9\left(\frac{3a^2-18a}{18a+3a^2}\right)\right)\left[9\times a/2 + 3\times \frac{a^2}{4}\right] - \left[9+6y\times a/2\left[\frac{3a^2-18a}{18a+3a^2}\right]\right]\left[3\times\frac{a^2}{4} - 9a/2\right]}{\left[9\times a/2 + 3\times \frac{a^2}{4}\right]^2}$$

Hmwk (1) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$x = a\cos t \; ; \; y = \sin t$

$\frac{dx}{dt} = -a\sin t \; ; \; \frac{dy}{dt} = b\cos t$

For conjugate diameters, the angle will always be $\pi/2$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-b\cos t}{a\sin t} = \frac{-b}{a}\cot t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \Rightarrow \frac{d}{dt}\left(\frac{dy}{dt}\right)\cdot\frac{dt}{dx}$$

$$\Rightarrow \frac{-b}{a}\left(-\operatorname{cosec}^2(t)\right)\cdot\frac{-1}{a\sin t}$$

$$\Rightarrow \frac{-b}{a^2\sin^3 t}$$

$$\rho = \frac{\left[1 + \frac{b^2\cot^2 t}{a^2}\right]^{3/2}}{-b/a^2\sin^3 t} = \left[\frac{a^2\sin^2 t + b^2\cos^2 t}{a^3\sin^3 t(-b)}\right]^{3/2} \times a^2\sin^3 t$$

$$\Rightarrow \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{-ab}$$

$$\rho = \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab} \longrightarrow \text{ROC of the curve}$$

On D, conjugate diameter of CP

$t = \pi/2 + t$

$D(a\cos(\pi/2+t), b\sin(\pi/2+t)) \Rightarrow D = (-a\sin t, b\cos t)$

$CD = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$

$(CD)^3 = (a^2\sin^2 t + b^2\cos^2 t)^{3/2}$

sub in ρ

$$\boxed{\rho = \frac{(CD)^3}{ab}}$$

(Q) Show that ρ_1 and ρ_2 are radii of curvature at two extremities on the conjugate diameters of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then

$\rho_1^{2/3} + \rho_2^{2/3} = \frac{a^2+b^2}{(ab)^{2/3}}$, let CP, CD are the two conjugate diameters of the ellipse, ρ_1, ρ_2 be the ROC of the curve respectively.

(A) $\rho_1 = \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab}$ → taking ρ_1 from the prev question.

$t = \frac{\pi}{2}+t$

$\rho_2 = \frac{(a^2\cos^2 t + b^2\sin^2 t)^{3/2}}{ab}$

To show:

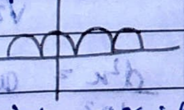
$$(\rho_1)^{2/3} + (\rho_2)^{2/3} = \frac{a^2\sin^2 t + b^2\cos^2 t}{(ab)^{2/3}} + \frac{a^2\cos^2 t + b^2\sin^2 t}{(ab)^{2/3}}$$

$$\Rightarrow \frac{a^2[\sin^2 t + \cos^2 t] + b^2[\sin^2 t + \cos^2 t]}{(ab)^{2/3}}$$

$$\Rightarrow \frac{a^2+b^2}{(ab)^{2/3}} = \text{RHS}$$

Hence Proved

(Q) Find the radius of curvature at any point of a cycloid $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$



(A) $\frac{dx}{d\theta} \Rightarrow a + a\cos\theta$

$\frac{dy}{d\theta} \Rightarrow a\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \Rightarrow \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin(\theta/2)\cdot\cos(\theta/2)}{2\cos(\theta/2)\cdot\cos(\theta/2)}$$

$$= \tan(\theta/2)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right)\cdot\frac{dt}{dt} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\cdot\frac{d\theta}{dx}$$

$$= \sec^2(\theta/2)\cdot\frac{1}{2}\cdot\frac{1}{a(1+\cos\theta)}$$

$$= \frac{\sec^2(\theta/2)}{2a\cdot 2\cos(\theta/2)\cdot\cos(\theta/2)} = \frac{1}{4a\cos^4(\theta/2)}$$

$$\rho = \left[1+\left(\tan(\theta/2)\right)^2\right]^{3/2}\cdot 4a\cos^4(\theta/2)$$

$$= \left[\sec^2(\theta/2)\right]^{3/2}\cdot 4a\cos^4(\theta/2)$$

$$= \frac{1}{\cos^3(\theta/2)}\cdot 4a\cos^4(\theta/2) = 4a\cos(\theta/2).$$

$$\rho = 4a\cos(\theta/2)$$

(Q) Find ρ at the point $\theta = -\pi/4$ on $x = a(1+\sin\theta)\cos\theta$; $y = a(1+\cos\theta)\sin\theta$

(A) $\frac{dy}{d\theta} = -a\sin\theta(1+\sin\theta) + a\cos\theta\cdot\cos\theta$

$= -a\sin\theta - a\sin^2\theta + a\cos^2\theta$

$= a[\cos\theta + \cos 2\theta] \to \theta = -\pi/4$

$\frac{dy}{d\theta} = a\left[\frac{1}{\sqrt{2}} + \cos(\pi/2)\right] = \frac{a}{\sqrt{2}}$

$\frac{dx}{d\theta} = a[(1+\sin\theta)\sin\theta + \cos\theta\cos\theta]$

$= a[-\sin\theta + \cos 2\theta] \to \theta = -\pi/4$

$\frac{dx}{d\theta} = a\left[\sin\pi/4 + 0\right]$

$= \frac{a}{\sqrt{2}}$

$\frac{d^2x}{d\theta^2} = a[-\cos\theta - 2\sin 2\theta] \to \theta = -\pi/4$

$a\left[-\frac{1}{\sqrt{2}} + 2\right]$

$\frac{d^2y}{d\theta^2} = a[-\sin\theta - 2\sin 2\theta]$

$= a\left[1/\sqrt{2} + 2\right]$

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$$\rho = \frac{\left[\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2\right]^{3/2}}{\frac{dx}{d\theta}\cdot\frac{d^2y}{d\theta^2}-\frac{dy}{d\theta}\cdot\frac{d^2x}{d\theta^2}} = \frac{\left(\frac{a^2}{2}+\frac{a^2}{2}\right)^{3/2}}{\frac{a^2}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}+2\right)-\frac{a^2}{\sqrt{2}}\left(\frac{-1}{\sqrt{2}}+2\right)}$$

$$= \frac{a^3}{\frac{a^2}{2}+\cancel{\frac{2a^2}{\sqrt{2}\cdot 2}}+\frac{a^2}{2}-\cancel{\frac{2a^2}{2\sqrt{2}}}}$$

$$= \frac{a^3}{a^2} = a$$

(Q) Find the ROC of the curve $x = a(\cos t + \log\tan(t/2))$; $y = a\sin t$

(A) $\frac{dx}{dt} = -a\sin t + \frac{a}{\tan(t/2)}\cdot\sec^2(t/2)\cdot\frac{1}{2} = -a\sin t + \frac{a\cdot\cos(t/2)}{\cos^2(t/2)}\cdot\frac{1}{\sin(t/2)\cdot 2}$

$\underbrace{\qquad}_{= \frac{a}{2\cos(t/2)\cdot\sin(t/2)}}$

$\frac{dy}{dt} = a\cos t$

$$\frac{d^2x}{dt^2} = -a\cos t + \frac{a}{2}(-)\frac{\left[-\sin(t/2)\cdot\sin(t/2) + \cos(t/2)\cdot\cos(t/2)\right]}{[\cos(t/2)\cdot\sin(t/2)]^2}$$

$$= -a\cos t - \frac{a}{2}\left[\frac{\cos(t)}{\cos^2(t/2)\cdot\sin^2(t/2)}\right]$$

$$\frac{d^2y}{dt^2} = -a\sin t$$

$$\rho = \frac{\left\{\left[a^2\sin^2 t + \frac{a^2}{4\cos^2(t/2)\sin^2(t/2)}\right] + a^2\cos^2 t\right\}^{3/2}}{\left(-a\sin t + \frac{a}{2\cos(t/2)\sin(t/2)}\right)\cdot -a\sin t - a\cos t\cdot\left(-a\cos t - \frac{1}{2}\left[\frac{\cos t}{\cos^2(t/2)\cdot\sin^2(t/2)}\right]\right)}$$

$$\rho = \frac{\left\{4a^2\sin^2 t\cos^2(t/2)\cdot\sin^2(t/2) + a^2 + a^2\cos^2 t\cdot 4\cos^2(t/2)\cdot\sin^2(t/2)\right\}^{3/2}}{a^2\sin^2 t - \frac{a^2\sin t}{2\cos(t/2)\sin(t/2)} + a^2\cos^2 t + \frac{a^2\cos^2 t}{2\cos^2(t/2)\sin^2(t/2)}}$$

$$\rho = \frac{\left[4a^2\cos^2(t/2)\cdot\sin^2(t/2) + a^2\right]^{3/2}}{a^2 + \cancel{a^2[\ldots]}} = a\cot(t)$$

=

Centre of curvature is a point that lies on the normal $P(x,y)$ on the curve with a distance ρ, denoted by $(\bar{x}, \bar{y})$

Circle of curvature: is a circle with centre as centre of curvature, radius as radius of curvature given by: $(x-\bar{x})^2+(y-\bar{y})^2=\rho^2$

NOTE: To compute the centre of curvature:

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2) \; ; \; y_1 = \frac{dy}{dx}$$

$$\bar{y} = y + \frac{1}{y_2}(1+y_1^2) \; ; \; y_2 = \frac{d^2y}{dx^2}$$

(Q) Find the circle of curvature $(y=x^2)$ at $(0,0)$

(A) $y_1 = 2x$; $y_2 = 2$

$$\rho = \frac{[1+(y_1)^2]^{3/2}}{y_2} = \frac{[1+0]^{3/2}}{2} = \frac{1}{2}$$

$$\bar{x} = x - \frac{y_1}{y_2}[1+y_1^2] = 0 - 0 = 0$$

$$\bar{y} = 0 + \frac{1}{2}[\;] = \frac{1}{2}$$

Equation of circle of curvature: $(x-0)^2+(y-1/2)^2 = \frac{1}{4}$

(Q) circle of curvature $y = x^3 - x^2$ at $(1,0)$

(A): $y_1 = 3x^2 - 2x$, $y_2 = 6x - 2$

$y_1 = 1$; $y_2 = 4$

$$\rho = \frac{[1+1]^{3/2}}{4} = \frac{1}{2}$$

$$\bar{x} = 1 - \frac{1}{4}[1+1] = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\bar{y} = 0 + \frac{1}{4}(1+1) = \frac{2}{4} = \frac{1}{2}$$

$$(x-1/2)^2+(y-1/2)^2 = 1/4$$

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(Q) Find centre of curvature of the curve $x^3+y^3=6xy$ at $(3,3)$.

(A) $3x^2+3y^2y_1 = 6y+6x\cdot y_1$

$\Rightarrow 27+27 = 18+18y_1$

$\Rightarrow 54-18 = 18y_1$

$\Rightarrow 36 = 18y_1$

$\boxed{y_1=2}$

$\boxed{y_1 = -1}$

$\boxed{y_2 = -16/3}$

$\Rightarrow 6x+6y = 6y_1+6y_1+6xy_2$

$\Rightarrow 18+18 = 12+12+18y_2$

$\Rightarrow 36 = 24+18y_2$

$\Rightarrow \frac{12}{18} = y_2$

$\rho = \frac{[1+1]^{3/2}\cdot 3}{-16} \Rightarrow -\frac{(2)^{3/2}\cdot 3}{16} = -\frac{\sqrt{8}\cdot 3}{16} \Rightarrow -\frac{\sqrt{2}\cdot 3}{8} = -\frac{3}{4\sqrt{2}}$

$\bar{x} = 3 - \frac{(-1)\cdot 3}{-16}[1+1] = 3 - \frac{3[2]}{16} = 21/8$

$\bar{y} = 3 + \frac{3}{-16}[1+1] = 3 - \frac{3\times 2}{16} = 21/8$

$(x-21/8)^2 + (y-21/8)^2 = \left(\frac{9}{32}\right)$

(Q) Find the circle of curvature of the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ $(1/2, 1/2)$

(A) $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\cdot y_1 = 0$

$\frac{1}{2\sqrt{y}}\cdot y_1 = -\frac{1}{2\sqrt{x}} \Rightarrow y_1 = -\frac{\sqrt{y}}{\sqrt{x}} = -\frac{\sqrt{1/2}}{\sqrt{1/2}} = -1$

$\boxed{y_1=-1}$

$y_2 = \frac{-\frac{1}{2\sqrt{y}}\cdot y_1\cdot\sqrt{x} + \frac{1}{2\sqrt{x}}\cdot\sqrt{y}}{(\sqrt{x})^2} = \frac{-\frac{1}{2\sqrt{1/2}}\cdot(-1)\cdot\sqrt{1/2} - \frac{1}{2\sqrt{1/2}}\cdot\sqrt{1/2}}{(\sqrt{1/2})^2}$

$\Rightarrow \frac{1/2+1/2}{1/2} = 2$

$\boxed{y_2=0}$

$\rho = \frac{[1+1]^{3/2}}{2} = \frac{(2)^{3/2}}{2}$

$\bar{x} = \frac{1}{2} + \frac{(-1)}{2}[1+1] = \frac{1}{2} - 1 = -\frac{1}{2}$

$\bar{y} = \frac{1}{2} + \frac{1}{2}[1+1] = \frac{1}{2} + 1 = \frac{3}{2}$

$\left(x+\frac{1}{2}\right)^2 + \left(y-\frac{3}{2}\right)^2 = \frac{4 \cdot (2)^3}{4} = 2$

(Q) Find COC of the curve $y = e^{3x}$ at a point where the curve crosses y axis. $(x=0)$

(A) $y_1 = e^{3x} \cdot 3 = 3$

$y_2 = 3e^{3x} \cdot 3 = 9e^{3x} = 9$

$\rho = \frac{[1+9]^{3/2}}{9} = \frac{(10)^{3/2}}{9}$

$\bar{x} = 0 - \frac{3}{9}[1+9] = \frac{-1}{3}(10) = -\frac{10}{3}$

$\bar{y} = 1 + \frac{1}{9}[1+9] = 1 + \frac{10}{9} = \frac{19}{9}$

$\left(x+\frac{10}{3}\right)^2 + \left(y-\frac{19}{9}\right)^2 = \frac{10^3}{81}$

$\left(x+\frac{10}{3}\right)^2 + \left(y-\frac{19}{9}\right)^2 = \frac{1000}{81}$

$\Rightarrow$ Radius of curvature in polar form.

Let $r = f(\theta)$ be the polar curve, the radius of curvature at any point in the curve is

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r \cdot r_2} \qquad r_1 = \frac{dr}{d\theta}$$

$$r_2 = \frac{d^2 r}{d\theta^2}$$

NOTE: If $r = 0$ and $\theta = 0 \rightarrow$ origin (or) pole. At the pole ROC:

$$\rho = \lim_{\theta \to 0}\left(\frac{r}{2\theta}\right)$$

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(Q) Find ROC at any point on the curve $r = a\theta$

(A) $r_1 = a$

$r_2 = 0$

$$\rho = \frac{(\cancel{a\theta}\ a^2\theta^2 + a^2)^{3/2}}{a^2\theta^2 + 2a^2 - \cancel{a\theta}} = \frac{(a^2\theta^2 + a^2)^{3/2}}{a^2\theta^2 + 2a^2}$$

$$= \frac{a^{2\times 3/2}[\theta^2+1]^{3/2}}{a^2[\theta^2+2]} = \frac{a[\theta^2+1]^{3/2}}{[\theta^2+2]}$$

(Q) Find ROC for $r = a/\theta$

(A) $r_1 = -\frac{a}{\theta^2}$

$r_2 = +\frac{a(2)}{\cancel{2}\theta^3}$

$$\rho = \frac{[a^2/\theta^2 + a^2/\theta^4]^{3/2}}{a^2/\theta^2 + 2a^2/\theta^4 \cancel{+ a^2/\theta \cdot 2a/\theta^3}}$$

$$= \frac{a^{2\times 3/2}[1/\theta^2 + 1/\theta^4]^{3/2}}{a^2[1/\theta^2 + \cancel{2/\theta^4} + \cancel{2/\theta^4}]} = \frac{a[1/\theta^2 + 1/\theta^4]^{3/2}}{a[1/\theta^2 + 4/\theta^4]}$$

$$= \cancel{\frac{a}{\theta^{2\times 3/2}}[1+1/\theta^2]}$$

$$\cancel{\alpha} = \frac{a/\theta\,[1+1/\theta^2]^{3/2}}{[1+4/\theta^2]}$$

$1/\theta^2\,[1 + 1/\theta^2 \cdot 1/\theta^2]$

(Q) Show that for a parabola $2a/r = 1+\cos\theta$, the ROC is $2\sqrt{\frac{r^3}{a}}$

(A) $\log 2a - \log r = \log(1+\cos\theta)$

$\Rightarrow -\frac{1}{r}\cdot\frac{dr}{d\theta} = -\frac{\sin\theta}{1+\cos\theta}$

$\frac{dr}{d\theta} = \frac{r\sin\theta}{1+\cos\theta}$

$\frac{d^2r}{d\theta^2} = (1+\cos\theta)\left(\frac{dr}{d\theta}\sin\theta + \cos\theta\cdot r\right)$

$$= \frac{-r\sin\theta(-\sin\theta)}{(1+\cos\theta)^2}$$

$$= \frac{(1+\cos\theta)\left(\frac{r\sin\theta}{1+\cos\theta} \times \sin\theta + r\sin\theta\right)}{(1+\cos\theta)^2}$$

$$= \frac{(1+\cos\theta)(r\sin^2\theta + r\sin\theta(1+\cos\theta))}{(1+\cos\theta)^3}$$

$$= \frac{r(\sin^2\theta + \sin\theta + \sin\theta\cos\theta)}{(1+\cos\theta)^2}$$

DIFFERENTIAL CALCULUS

(Q) $r = a(1+\sin\theta)$

$\Rightarrow \tan\psi = 0 = \tan(\theta+\phi)$

$\Rightarrow \tan\phi = r \cdot \frac{d\theta}{dr} = \cancel{a}(1+\sin\theta) \cdot \frac{1}{\cancel{a}\cos\theta} = \frac{[\cos(\theta/2)+\sin(\theta/2)]^2}{[\cos^2(\theta/2)-\sin^2(\theta/2)]}$

$\Rightarrow \frac{\cos(\theta/2)+\sin(\theta/2)}{\cos(\theta/2)-\sin(\theta/2)}$ → divide by $\cos(\theta/2)$

$\tan\phi \Rightarrow \frac{1+\tan(\theta/2)}{1-\tan(\theta/2)}$ → Putting $\theta = \pi/2$

$\tan(\pi/4+\theta/2)$

$\phi = \pi/4+\theta/2$

$\psi = \theta+\phi$

$\psi \Rightarrow \theta + \frac{\pi}{4} + \frac{\theta}{2}$

$\psi \Rightarrow \pi/2 + \pi/4 + \pi/4 = \pi/2 + \pi/2 = \pi \Rightarrow \tan(\pi) = 0$

(Q) $r = \sin\theta + \cos\theta$

$\frac{dr}{d\theta} \Rightarrow \cos\theta - \sin\theta = \tan\phi_1 = r \cdot \frac{d\theta}{dr} = \frac{\sin\theta+\cos\theta}{\cos\theta-\sin\theta}$

$\tan\phi_1 \Rightarrow \frac{1+\tan\theta}{1-\tan\theta} \Rightarrow \tan(\pi/4+\theta)$

$\phi_1 = \pi/4+\theta$

$r = 2\sin\theta$

$\frac{dr}{d\theta} = 2\cos\theta \Rightarrow \tan\phi_2 = r \cdot \frac{d\theta}{dr} = \cancel{2}\sin\theta \cdot \frac{1}{\cancel{2}\cos\theta} = \tan\theta$

$\phi_2 \Rightarrow \theta$

$|\phi_1 - \phi_2| = \alpha = \pi/4$

(Q) $r = \frac{a\theta}{1+\theta}$; $r = \frac{a}{1+\theta^2}$ → find the point of intersection

$\frac{\cancel{a}\theta}{1+\theta} = \frac{\cancel{a}}{1+\theta^2}$ $\quad \theta(1+\theta^2) = 1+\theta$

$\cancel{\theta}+\theta^3 = 1+\cancel{\theta}$

$\boxed{\theta = 1}$

in the 3th plane

TUTORIAL SHEET (1)

(1) $x = -1;\ y = -\sqrt{3}$

$r = \sqrt{x^2+y^2} = \sqrt{(1)+3} = \sqrt{4} = 2$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) = \frac{4\pi}{3} = \pi + \frac{\pi}{3}$

(2) $r = \sqrt{2};\ \theta = \frac{5\pi}{4} \rightarrow 3^{rd}$ quadrant

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{y}{x}\right) = \frac{5\pi}{4} \Rightarrow \frac{y}{x} = \tan\left(\frac{5\pi}{4}\right) \Rightarrow \frac{y}{x} = 1$

$\boxed{y = x} \rightarrow y = -1;\ x = -1$

$(-1, -1)$

(3) $x^2+y^2 = 2ax \rightarrow g = a, 0$

$x^2+y^2-2ax = 0 \rightarrow r = (a, 0)$

$x = r\cos\theta$

$y = r\sin\theta$

$x = a;\ y = 0.$

$r = \sqrt{a^2} = a$

$\theta = \tan^{-1}\left(\frac{0}{a}\right) =$

$\Rightarrow x^2+y^2 = r^2;\ 2a(r\cos\theta)$

$r^2 - 2a(r\cos\theta) = 0$

$r^{\cancel{2}} = 2a\cancel{r}\cos\theta$

$\boxed{r = 2a\cos\theta}$

(4) $\theta - k = 0$

$\tan^{-1}\left(\frac{y}{x}\right) = k \Rightarrow \frac{y}{x} = \tan(k)$

$y = \underbrace{x\tan(k)}_{mx} \Rightarrow$ straight line

(5) $\cot(\phi_1)\cdot\cot(\phi_2) = -1$

(6) $r_1 = \frac{k\theta}{1+\theta};\ \frac{dr_1}{d\theta} \Rightarrow \frac{k(1+\theta) - k\theta(1)}{(1+\theta)^2} = \frac{k + \cancel{k\theta} - \cancel{k\theta}}{(1+\theta)^2} = \frac{k}{(1+\theta)^2}$

$\tan\phi_1 = r_1\cdot\frac{d\theta}{dr_1} = \frac{\cancel{k}\theta}{\cancel{1+\theta}} \times \frac{(1+\theta)^{\cancel{2}}}{\cancel{k}} = \theta(1+\theta) = \theta + \theta^2$

$r_2 = \frac{k}{1+\theta^2};\ \frac{dr_2}{d\theta} = \frac{(1+\theta^2) - 2\theta(k)}{(1+\theta^2)^2}$

$\tan\theta_2 = r_2\cdot\frac{d\theta}{dr} = \frac{k}{\cancel{1+\theta^2}}\cdot\frac{(1+\theta^2)^{\cancel{2}}}{-2\theta\cancel{k}} = \frac{-(1+\theta^2)}{2\theta}$

$\frac{\cancel{k}\theta}{1+\theta} = \frac{\cancel{k}}{1+\theta^2}$; $\theta(1+\theta^2) = 1+\theta$

$\cancel{\theta}+\theta^3 = 1+\cancel{\theta}$

$\boxed{\theta = 1}$

$\tan\phi_1 = 2$

$\tan\phi_2 = -1$

$\alpha = |\phi_1 - \phi_2|$ → modulus

$\tan\alpha = \tan(\phi_1-\phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1 \cdot \tan\phi_2}$

$= \frac{2+1}{1-2} = -3$

$\boxed{\tan^{-1}(3) = \alpha}$

(7) $\psi = \cancel{2}r \cdot \frac{dr}{d\theta} = a^2\sin 2\theta \cdot \cancel{2}$

$r \cdot \frac{dr}{d\theta} = a^2 \sin 2\theta \Rightarrow \frac{dr}{d\theta} = \frac{a^2\sin 2\theta}{r}$

$\tan\phi = r \cdot \frac{d\theta}{dr} = r \cdot \frac{r}{a^2\sin 2\theta} = \frac{r^2}{a^2\sin 2\theta} \Rightarrow \frac{\cancel{a^2}\cos 2\theta}{\cancel{a^2}\sin 2\theta}$

$\tan\phi = \cot(2\theta)$

$\tan\phi = \tan(\pi/2 - 2\theta)$

$\phi = \pi/2 - 2\theta$

$\psi = \theta + \phi$

$\psi = \pi/2 - 2\theta + \theta$

$\psi_1 = \pi/2 - \theta - 2\theta + \pi$

$\psi_2 = 3\pi/2 - 3\theta$

$\alpha = \psi_1 - \psi_2 =$

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(Q) Show that the ROC of the curve $r = a\sin(n\theta)$ at the pole is $(na/2)$

(A) $\rho = \lim_{\theta \to 0}\left(\frac{r}{2\theta}\right)$

$= \lim_{\theta \to 0}\left(\frac{a\sin(n\theta)}{2\theta}\right)$

$= \frac{a}{2}\lim_{\theta \to 0}\left(\frac{\sin(n\theta)}{\cancel{n\theta}} \times \frac{n}{n}\right) \leftarrow \left(\frac{\sin n\theta}{n\theta} = 1\right) \to \lim_{\theta \to 0}$

$= \frac{an}{2}$.

(Q) Show that the each point of ~~hemisphere~~ lemmiscrate $r^2 = a^2\cos 2\theta$, the curvature is proportional to the radius vector at that point.

(A) $k \propto r$.

$2\log(r) = 2\log a + \log(\cos 2\theta)$

$\frac{\cancel{2}}{r}\cdot\frac{dr}{d\theta} = \frac{-\sin 2\theta \cdot \cancel{2}}{\cos 2\theta}$

$\frac{dr}{d\theta} = -r\tan 2\theta$

$\frac{d^2r}{d\theta^2} = -\frac{dr}{d\theta}\cdot\tan 2\theta - r\cdot\sec^2(2\theta)\cdot 2$

$= -r\tan^2 2\theta - r\sec^2(2\theta)\cdot 2$

$= -r[\tan^2 2\theta + \sec^2(2\theta)\cdot 2]$

$= -r[3\sec^2(2\theta) - 1]$ $\quad r^2/a^2 = \cos 2\theta$

$\Rightarrow \cancel{a}\ -r[3\sec^2 2\theta - 1]$ $\quad a^2/r^2 = \sec 2\theta$

$\rho = \frac{[1 + \cancel{r^2}\tan^2 2\theta]^{3/2}}{-r\cdot[3\sec^2 2\theta - 1]} = \frac{[\cancel{r}^2\sec^2 2\theta]^{3/2}}{r\cdot[3\sec^2 2\theta - 1]}$

$\frac{\cancel{r^3}\sec^3 2\theta}{} = \left(\frac{r^2\sec^3 2\theta}{3\sec^2 2\theta - 1}\right)$

$k = 1/\rho$.

$k = \frac{\cancel{r}^2\sec^3 2\theta}{3\sec^2 2\theta - 1} \Rightarrow k \propto r$

$k = \frac{3\sec^2 2\theta - 1}{a^2\sec^2 2\theta}$. $\quad \frac{a^2\cos 2\theta\cdot\sec^3 2\theta}{3\sec^2 2\theta - 1} = \left(\frac{a^2\cdot\sec^2 2\theta}{3\sec^2 2\theta - 1}\right) \Rightarrow$

$k \propto r$

(Q) Show that for the curve $(x^2+y^2)^2 = a^2(y^2-x^2)$ at the point $(0,a)$ $\rho = a/3$.

(A) $x = r\cos\theta;\ y = r\sin\theta$

$x^2+y^2 = r^2$

$(r^2)^2 = a^2(r^2\sin^2\theta - r^2\cos^2\theta)$

$r^{\cancel{4}} = a^2 \cancel{r^2}(-\cos 2\theta)$

$r^2 = -a^2\cos 2\theta$

$x=0;\ y=a$

$r\cos\theta = 0;\ r\sin\theta = a$

$\theta = \pi/2 \qquad r=a;\ \sin\theta = 1 \Rightarrow \theta = \pi/2$

At $\theta = \pi/2 \rightarrow$ find ρ.

Hmwk (Q) Find ROC of the curve $\theta = \sqrt{r^2-a^2} - \cos^{-1}\left(\frac{a}{r}\right)$

TAYLOR'S THEOREM AND MCLAURIN'S THEOREM

If $f(x)$ is continuous and differentiable at the point $x=a$ (in the neighbourhood of a), then the taylor series expansion of $f(x)$ at $x=a$ is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots$$

When $a=0$, the taylor's series reduces to Mclaurin series.

The Mclaurin series expansion of $f(x)$ at $x=0$ is

$$f(x) = f(0) + f'(0)\cdot x + \frac{x^2}{2!}f''(0) + \dots$$

(Q) Find the Taylor and Maclaurin of $f(x) = \cos x$ at $x = \pi/3$

(A) $f(x) = (x-\pi/2)\cdot -\sin x + \frac{(x-\pi/2)^2}{2!} - \cos x + \dots$

$f(x) = \frac{1}{2} + (x-\pi/3)\cdot \cancel{\sin x}\, \frac{-\sqrt{3}}{2} + \frac{(x-\pi/3)^2}{2!}\cdot\frac{-1}{2} + \dots$

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Mclaurin's series:

$f(x) =$

(Q) Express $f(x) = e^{-x/3}$ as a power series in terms of x upto 3^{rd} degree term.

(A) Mclauren's series

$\boxed{x=0}$

$$f(x) = f(0) + f'(0)(x) + f''(0)\frac{x^2}{2!} + f'''(0)\cdot\frac{x^3}{3!} + \cdots$$

$$= 1 + \frac{-1}{3} \times \cancel{e^{-x/3}} + \frac{1}{9}\frac{e^{-2x/3}}{2} - \frac{1}{27}\cdot\frac{e^{-3x/3}}{6}$$

$$= 1 - \frac{1}{3}x + \frac{1}{18}x^2 - \frac{1}{162}x^3$$

$-\frac{1}{3}$

$-\frac{1}{3}\cdot -\frac{1}{3}$

$\frac{1}{9}$

$27 \times 6 = 162$

(Q) $f(x) = a^x$, $f'(x) = a^x \log(a)$, $f''(x) = a^x \log(a)^2$

(A) $f(x) = 1 + x\log(a) + \frac{x^2}{2!}(\log a)^2 + \frac{x^3 (\log a)^3}{3!}$

$$= 1 + x\log(a) + \frac{x^2(\log a)^2}{2!} + \frac{x^3(\log a)^3}{3!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

(Q) Find Mclauren series expression $\log(1+x)$. & deduce $\log\sqrt{\frac{1+x}{1-x}}$

(A) $f(x) = \log(1+x)$

$f'(x) = \frac{1}{1+x}$ $\quad f(0) = 0$

$f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2}$ $\quad f''(0) = -1$

$f'''(x) = -(-1)(1+x)^{-2}$

$= \frac{2}{(1+x)^3}$ $\quad f'''(0) = 2$

$f^{IV}(x) = \frac{-6}{(1+x)^4}$; $f^{IV}(0) = -6$

$f^{V}(x) = \frac{24}{(1+x)^5}$; $f^{V}(x) = 24$

$f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \frac{x^4 f^{IV}(0)}{4!} + \frac{x^5 f^{V}(0)}{5!}$

$= 0 + x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!} + \frac{24x^5}{5!}$

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3!} - \frac{x^4}{4} + \frac{x^5}{5} \ldots$

Substitute $x \to -x$

$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5}$

$\log\left(\frac{1+x}{1-x}\right)^{1/2} = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\left[\log(1+x) - \log(1-x)\right]$ ① − ②

$= \frac{1}{2}\left[2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \ldots\right]$

$= x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots$

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(Q) Express $\log(\sec x)$ in powers of x. Hence, find the series expression for $\tan x$.

(A) $f(x) = \log(\sec x)$ $\qquad f(0) = 0$

$f'(x) = \frac{1}{\sec x} \cdot \sec x \cdot \tan x = \tan x$ $\qquad f'(x) = 0$

$\qquad f''(x) = 1$

$f''(x) = \sec^2 x$ $\qquad f'''(x) = 2 \times 1 \times 0 = 0$

$f'''(x) = 2\sec x \cdot \sec x \tan x$

$= 2\sec^2 x \cdot \tan x$

$= 2f''(x) \cdot f'(x)$ easy to write the derivative

$f^{IV}(x) = 2f'''(x) \cdot f'(x) + 2f''(x) \cdot f''(x)$

$= 2 \times 0 \times 0 + 2 \times 1 \times 1 = 2$

$f^{V}(x) = 2f^{IV}(x) \cdot f'(x) + 2f'''(x) \cdot f''(x) + 2f'''(x) \cdot f''(x) + 2f''(x) \cdot f'''(x)$

$= 2 \times 2 \times 0 + 2 \times 0 \times 1 + 2 \times 0 \times 1 + 2 \times 1 \times 0$

$= 0$

$f^{VI}(x) = 2[f^{V}f' + 3f''f''']$

$= 2[f^{V}(x) \cdot \overset{0}{f'(x)} + f^{IV}(x)^{2} \cdot f''(x) + 3f'''(x)^{2} \cdot f'''(x) + 3 f''(x) \cdot f^{IV}(x)]$

$= 2[2 + 6] = 2 \times 8 = 16$

$f(x) = \log(\sec x) = \frac{x^2}{2!} \times 1 + \frac{x^3}{3!} \times 0 + \frac{x^4}{4!} \times 2 + \frac{x^5}{5!} \times 0 + \frac{x^6}{6!} \times 16$

$= \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45}$

$\frac{\cancel{720}}{\cancel{16}} \quad 360 \quad 180$

differentiate both sides w.r.t x,

$\tan x = \frac{\cancel{2}x}{\cancel{2}} + \frac{\cancel{4}x^3}{\cancel{12}_3} + \frac{6x^5}{45}$

$f^{IV}(0) =$

(Q) Express root of $1+\sin 2x = \sqrt{1+\sin 2x}$ in Mclaurren series!

(A) $f(x) = \sqrt{1+\sin 2x} = \sqrt{(\sin x+\cos x)^2} = \sin x+\cos x$ $\quad f(0) = 1$

$\Rightarrow f'(x) = \cos x - \sin x \quad f'(0) = 1$

$\Rightarrow f''(x) = -\sin x - \cos x \quad f''(0) = -1$

$\Rightarrow f'''(x) = -\cos x + \sin x \quad f'''(0) = -1$

$\Rightarrow f^{IV}(x) = \sin x + \cos x \quad f^{IV}(0) = 1$

$\Rightarrow f(x) = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

(Q) Express $\log$ of $(1+e^{x})$ in terms of x. Hence deduce the series for $\frac{e^x}{1+e^x}$.

(A) $f(x) = \log(1+e^{x}) \quad f(0) = \log(2)$

$f'(x) = \frac{e^x}{1+e^x} \quad f'(0) = \frac{1}{2}$

$\quad f''(0) = \frac{1}{4}$

$f''(x) = \frac{e^x(1+e^x) - e^x(e^x)}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2} \quad f'''(0) = 0$

$f'''(x) = \frac{e^x(1+e^x)^2 - 2(1+e^x).e^x(e^x)}{(1+e^x)^4} = 0$

$\Rightarrow \frac{e^x - e^{2x}}{(1+e^x)^3}$

$(1-2)(8)-3(4)$

$(e^x - e^{2x}).2(1+e^x)$

$\frac{1(4)-2(2)}{(2)^4}$

$f(x) = \log(2) + \frac{1}{2}x + \frac{x^2}{2!}\cdot\frac{1}{4}x + \cdots \quad f^{IV}(0) = -\frac{1}{8}$

$-8/64^2$

$\log\left(\frac{1+e^x}{\cancel{1+e^x}}\right) = \log(2) + \frac{x}{2} + \frac{x^2}{8} \mp \frac{x^4}{4!}\cdot\frac{1}{8}$

$\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{2x}{8\,4} + \frac{4x^3}{4!\times 16} = \frac{1}{2} + \frac{x}{4} \mp \frac{x^3}{4!\cdot 8} + \cdots$

(Q) Express $f(x) = \tan^{-1}(x)$. $\quad f(0) = 0$.

(A) $\frac{+1}{1+x^2} = f'(x) \Rightarrow f'(0) = 1$

$\frac{-2x}{(1+x^2)^2} \Rightarrow f''(0) = 0 \quad \rightarrow -2x(f'(x))^{-2}$.

$(1+x^2)^{-2}$

$-2(1+x^2)^{-3}\, 2x$

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$$f'''(0) = \frac{-2(1+x^2)^2 - 2x\cdot 2(1+x^2)\cdot 2x}{(1+x^2)^4} = \frac{-2}{1} = -2$$

$$f^{iv}(0) = \left[\frac{1}{(1+x^2)^2} - \frac{4x^2}{(1+x^2)^3}\right]' = \frac{-2(2x)}{(1+x^2)^3} - \frac{8x(1+x^2)^3}{}$$

$$= \frac{-4x}{(1+x^2)^3} - \frac{8x(1+x^2)^{\cancel{3}} + 4x^2\,3(1+x^2)^2(2x)}{(1+x^2)^6}$$

$$= 0$$

$$f^{v}(0) = -2\left[2[x f^{iv}(x)] + f''(x)\right] + f'''(x) + 2\left[f'(x)\cdot f'''(x) + f''(x)\cdot f''(x)\right]$$

$$f^{v}(0) = -2\left[0 + 2(-2) + 2(0+0)\right]$$

$$= \cancel{8}\,24$$

$$f(x) = x - \frac{2x^3}{3!} + \frac{x^5}{5!}(24) + \cdots$$

(1) $\log(1+\sin x)$; (2) $e^{\sin x}$

$-8+12/(2)^6 = 4/64 = 1/16$

$\frac{3(1+e^x)^2\cdot e^x(e^x - e^{2x})}{(1+e^x)^6}$