unit3_tutorials

R V COLLEGE OF ENGINEERING
(An autonomous institution affiliated to VTU, Belgaum)
DEPARTMENT OF MATHEMATICS

FUNDAMENTS OF LINEAR ALGEBRA, CALCULUS AND NUMERICAL METHODS (MAT211AT)
Multivariable Functions and Partial Differentiation

TUTORIAL SHEET-1

1. If $x = r\cos\theta, y = r\sin\theta$, then $\left(\frac{\partial r}{\partial \theta}\right)^2 + \left(\frac{\partial x}{\partial \theta}\right)^2 =$ r^2 **Ans**: r ?
2. If $z = x\sin y + y\sin x$, then $\frac{\partial^2 z}{\partial x \partial y} =$ $\cos y + \cos x$ **Ans**: $\cos y + \cos x$
3. If $z = e^{2x^2+xy}$, then $\frac{\partial z}{\partial y} =$ xe^{2x^2+xy} **Ans**: xe^{2x^2+xy}
4. The steady state temperature of a metal sheet is $T(x,y) = x^2 - a^2y^2$. The values of 'a' for which $T(x,y)$ satisfies the Laplace equation $T_{xx} + T_{yy} = 0$ are ± 1. **Ans**: $\pm a$
5. If $u = y\cos(xy)$ then $\frac{\partial u}{\partial y}$ at the point $(1,\pi)$ is -1. **Ans**: -1
6. If $V = f(x-ct) + g(x+ct)$ where f and g are arbitrary functions of $x - ct$ and $x + ct$ respectively and c is a constant, then show that $\frac{\partial^2 V}{\partial t^2} = c^2 \frac{\partial^2 V}{\partial x^2}$
7. If $u = \frac{x+y}{x-y}$, verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$
8. If $u = ae^{-gx}\sin(nt - gx)$, where a, g and n are positive constants and $\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial x^2}$, show that $g = \sqrt{\frac{n}{2\mu}}$.
9. If V is the volume and S is the total surface area of rectangular box of length x, breadth y and height z, find
   (i) the rate of change of V with respect to x if y = 4 and z = 12,
   (ii) the rate of change of S with respect to z if x = 3 and y = 4.
   Ans: (i) 48 (ii) 14
10. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, then show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$.

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TUTORIAL SHEET-2

1. Given $z = xy^2 + x^2y$ where x and y are functions of t with $x(1) = 1, y(1) = 2, x'(1) = 3$ and $y'(1) = 4$. The value of $\frac{dz}{dt}$ at t = 1 is 50. **Ans**: 16 ?
2. For the implicit function $(\cos x)^y = (\sin y)^x$, $\frac{dy}{dx} =$ ______. **Ans**: $-\left[\frac{y(\cos x)^{y-1}(-\sin x) - (\sin y)^x \log\sin y}{(\cos x)^y \log(\cos x) - x(\sin y)^{x-1}\cos y}\right]$
3. Given 't' represents time and $u = x^2 - y^2, x = \frac{1}{t}, y = e^t$ then the rate of change of u with respect to 't' is ______. **Ans**: $\frac{-2}{t^3} - 2e^{2t}$
4. For the implicit function $e^x - e^y = 2xy$, $\frac{dy}{dx} =$ ______. **Ans**: $\left[\frac{e^x - 2y}{e^y + 2x}\right]$
5. Given, $x^2 + y^2 + 3xz = 1$ and $x + y = 1$, then $\frac{dz}{dx} =$ ______ **Ans**: $\frac{-(2x+3)}{3} + \frac{2z}{3x}$
6. If $z = z(x,y), x = e^u\sin v, y = e^u\cos v$, then $\frac{\partial z}{\partial u}$ ______. **Ans**: $\frac{\partial z}{\partial x}e^u\sin v$
7. If $u = xyz$ where $x = e^{-t}, y = e^{-t}\sin^2 t, z = \sin t$, then find $\frac{du}{dt}$.
   Ans: $e^{-t}\sin^2 t(3\cos t - 2\sin t)$ e^{-2t}
8. If $z = x^2 + 2xy + 4y^2$ and $y = e^{2x}$, find $\frac{dz}{dx}$. **Ans**: $2(x + e^{2x}) + 2(x + 4e^{2x})2e^{2x}$
9. If z is a function of x and y and if $x = e^u\sin v, y = e^u\cos v$, prove that
   (i) $\frac{\partial z}{\partial u} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$
   (ii) $\frac{\partial z}{\partial x} = e^{-u}\left(\sin v\frac{\partial z}{\partial u} + \cos v\frac{\partial z}{\partial v}\right)$
10. If $= f(x-y, y-z, z-x)$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.
11. If $u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$, determine $\frac{\partial^2 u}{\partial x \partial y}$.

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TUTORIAL SHEET-3

1. Match the following:

i)	If $x = e^u\sin v, y = e^v\cos v$ then $J\left(\frac{x,y}{u,v}\right) =$ ______.	a)	$\frac{1}{4v\sin 2u}$
ii)	If (a,b) is a stationary point of f(x,y) and $f_{xx} = 3$, $f_{xy} = 2$ and $f_{yy} = 2$ at this point then the nature of (a,b) is ____.	b)	minimum
iii)	The nature of the point (1, -1) to the function $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 7$ is ______.	c)	$e^{2u+v}(\sin v\cos v - \sin^2 v)$
		d)	$\frac{v\sin 2u}{4}$
iv)	If $\frac{\partial(x,y)}{\partial(u,v)} = v\sin 2u$ and $\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{1}{4}$ then $\frac{\partial(u,v)}{\partial(r,\theta)} =$ ______.	e)	Neither maximum nor minimum
		f)	Saddle point
		g)	maximum
		h)	$e^{u+v}(\sin v\cos v - \sin^2 v) - e^u\cos v$

Ans: (i) - (c) (ii) -(g) (iii) - (e) (iv) - (a)

2. Find the extreme values of $\sin x + \sin y + \sin(x+y)$. **Ans**: Maximum value at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, maximum value is $\frac{3\sqrt{3}}{2}$
3. Find the volume of largest rectangular parallelepiped that can be inscribed in an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Ans: Maximum volume $= \frac{8abc}{3\sqrt{3}}$
4. Find the maximum and minimum distances of the point (1, 2, 3) from the sphere $x^2 + y^2 + z^2 = 56$ using Lagrange's Method of undetermined multipliers. **Ans**: Minimum distance at (2, 4, 6) = $\sqrt{14}$, Maximum distance at (-2,-4,-6), maximum distance = $\sqrt{126}$
5. Show that $u = \frac{x^2-y^2}{x^2+y^2}, v = \frac{2xy}{x^2+y^2}$ are functionally dependent and find the relation between them. Ans: $u^2 + v^2 = 1$.
6. For $u = xyz, v = yz + zx + xy, w = x + y + z$, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$. Ans: $(y-z)(z-x)(x-y)$.
7. If $x = e^v\sec u, y = e^v\tan u$, then verify that $JJ' = 1$.

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① $y = r\sin\theta$; $\left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2$
$\frac{dy}{d\theta} = r\cos\theta$ $= r^2\cos^2\theta + r^2\sin^2\theta$
$x = r\cos\theta$ $= r^2$
$\frac{dx}{d\theta} = -r\sin\theta$

② $z = x\sin y + y\sin x$
$\frac{d}{dx}\left(\frac{dz}{dy}\right) = \frac{d}{dx}(x\cos y + \sin x)$
$= \cos y + \cos x$

③ $z = e^{2x^2+xy}$
$\frac{dz}{dy} = xe^{2x^2+xy}$

④ $T(x,y) = x^2 - a^2y^2 = y$
$T_{xx} + T_{yy} = 0$
$\frac{d^2y}{dx^2} + \frac{d^2y}{dy^2} = 0$
$\frac{d}{dx}(2x) + \frac{d}{dy}(-a^2(2y)) = 0$
$2 - 2a^2 = 0$
$2 = 2a^2$
$a^2 = 1 \Rightarrow a = \pm 1$

⑤ $u = y\cos(xy)$
$\frac{\partial u}{\partial y} = (1)\cos(xy) + y(-\sin(xy))\cdot x$
at $x = 1, y = \pi$
$= \cos\pi - \pi\sin\pi$
$= -1$

⑥ $V = f(x-ct) + g(x+ct)$
$\rightarrow \frac{dV}{dt} = f'(x-ct)\cdot(-c) + g'(x+ct)\cdot c$
$\frac{d^2V}{dt^2} = +c^2f''(x-ct) + c^2g''(x+ct)$
$\rightarrow \frac{dV}{dx} = f'(x-ct) + g'(x+ct)$
$\frac{d^2V}{dx^2} = f''(x-ct) + g''(x+ct)$
$\Rightarrow \frac{d^2V}{dt^2} = c^2\frac{d^2V}{dx^2}$

⑦ $u = \frac{x+y}{x-y}$; $\frac{d^2u}{dydx} = \frac{d}{dy}\left(\frac{du}{dx}\right)$
$\frac{d^2u}{dxdy} = \frac{d}{dx}\left(\frac{du}{dy}\right)$
$= \frac{d}{dx}\left(\frac{(1)(x-y) - (x+y)(-1)}{(x-y)^2}\right)$
$= \frac{d}{dx}\left(\frac{x - \cancel{y} + x + \cancel{y}}{(x-y)^2}\right)$
$= \frac{d}{dx}\left(\frac{2x}{(x-y)^2}\right)$
$= 2\left(\frac{(1)(x-y)^2 - x(2(x-y))}{(x-y)^4}\right)$
$= 2\left(\frac{x^2 + y^2 - \cancel{2xy} - 2x^2 + \cancel{2xy}}{(x-y)^4}\right)$
$= 2\frac{(y^2 - x^2)}{(x-y)^4}$ (=)

$= \frac{d}{dy}\left(\frac{(1)(x-y) - (x+y)(1)}{(x-y)^2}\right)$
$= \frac{d}{dy}\left(\frac{\cancel{x} - y - \cancel{x} - y}{(x-y)^2}\right)$
$= \frac{d}{dy}\left(\frac{-2y}{(x-y)^2}\right)$
$= -2\left(\frac{(1)(x-y)^2 - y(2(x-y)(-1))}{(x-y)^4}\right)$
$= -2\left(\frac{x^2 + y^2 - \cancel{2xy} + \cancel{2xy} - 2y^2}{(x-y)^4}\right)$
$= \frac{-2(x^2 - y^2)}{(x-y)^4} = \frac{2(y^2 - x^2)}{(x-y)^4}$

⑧ $u = ae^{-gx}\sin(nt - gx)$
$\rightarrow \frac{\partial u}{\partial t} = ae^{-gx}\cos(nt - gx)\cdot n$
$\rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$
$= \frac{\partial}{\partial x}\left(a\left(e^{-gx}\cos(nt-gx)(-g) + e^{-gx}(-g)\sin(nt-gx)\right)\right)$
$= \frac{\partial}{\partial x}\left(-age^{-gx}\left(\cos(nt-gx) + \sin(nt-gx)\right)\right)$
$= -ag\left[e^{-gx}(-g)(\cos x + \sin x) + e^{-gx}(-\sin x(-g) + \cos x(-g))\right]$
$= ag^2e^{-gx}\left(\cos x + \cancel{\sin x} - \cancel{\sin x} + \cos x\right)$
$= 2ag^2e^{-gx}\cos(nt - gx)$

$\frac{du}{dt} = \mu\frac{d^2u}{dx^2}$
$\cancel{ae^{-gx}}\cancel{\cos(nt-gx)}\cdot n = \mu\cdot 2\cancel{a}g^2\cancel{e^{-gx}}\cancel{\cos(nt-gx)}$
$n = \mu(2g^2)$
$g = \sqrt{\frac{n}{2\mu}}$

⑨ $V = xyz$ $S = 2(xy + yz + xz)$
(i) $\left.\frac{\partial V}{\partial x}\right|_{y=4, z=12} = yz = 4 \times 12 = 48$
(ii) $\left.\frac{\partial S}{\partial z}\right|_{x=3, y=4} = 2(y + x)$
$= 2(4 + 3)$
$= 14$

⑩ $u = \log(x^3 + y^3 + z^3 - 3xyz)$
$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$
$\rightarrow \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$
$\rightarrow \frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz} \times (3x^2 - 3yz)$
$\rightarrow \frac{\partial u}{\partial y} = \frac{1}{''} \times (3y^2 - 3xz)$
$\rightarrow \frac{\partial u}{\partial z} = \frac{1}{''} \times (3z^2 - 3xy)$
$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{1}{''}\left(3x^2 + 3y^2 + 3z^2 - 3yz - 3xz - 3xy\right)\right)$
$\left(\frac{3(x^2 + y^2 + z^2 - xy - yz - xz)}{x^3 + y^3 + z^3 - 3xyz}\right)$
$\left(\frac{3(\;'' \;)}{(x+y+z)(\;''\;)}\right)$
$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$
$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \ldots$
$= 3\left(\frac{-1}{(x+y+z)^2}\right) + \ldots$
$= -\frac{3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2} + \ldots$
$= \frac{-9}{(x+y+z)^2}$

$\frac{x+y}{x-y}$

[1] $z = xy^2 + x^2y$ $z < \frac{x}{y} > t$
$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$
$= [y^2 + 3x^2y](x'(t)) + [x(2y) + x^3](y'(t))$
at $t = 1$, $x = 1$, $x'(1) = 3$, $y = 2$, $y'(1) = 4$
$\frac{dz}{dt} = [2^2 + 3(1)^2(2)](3) + [(1)(2\times 2) + (1)^3](4)$
$= 30 + 20$
$= 50$

[2] $(\cos x)^y = (\sin y)^x$ → implicit
$f = (\cos x)^y - (\sin y)^x$ $\langle a^x = a^x\log a\rangle$
$\frac{dy}{dx} = -\frac{f_x}{f_y}$
$= -\frac{\left[y(\cos x)^{y-1}x(-\sin x) - \sin y^x\cdot\log(\sin y)\right]}{\left[\cos x^y\cdot\log(\cos x) - x(\sin y)^{x-1}\cos y\right]}$

[3] $u = x^2 - y^2$; $x = \frac{1}{t}$, $y = e^t$
$\frac{du}{dt} = \frac{du}{dx}\cdot\frac{dx}{dt} + \frac{du}{dy}\cdot\frac{dy}{dt}$
$= 2x\left(\frac{-1}{t^2}\right) + -2y\cdot e^t$
$= \frac{2}{t}\left(\frac{-1}{t^2}\right) - 2e^{2t}$
$= \frac{-2}{t^3} - 2e^{2t}$

[4] $e^x - e^y = 2xy$
$f: e^x - e^y - 2xy = 0$
$\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-[e^x - 2y]}{-e^y - 2x}$
$= \frac{e^x - 2y}{e^y + 2x}$

[5] $x^2 + y^2 + 3xz = 1$ $x + y = 1$
$2x + 2y\frac{dy}{dx} + 3\left(x\frac{dz}{dx} + z\right) = 0$ $\frac{dx}{dx} + \frac{dy}{dx} = 0$
$2x - 2y + 3x\frac{dz}{dx} + 3z = 0$ $\frac{dy}{dx} = -1$
$3x\frac{dz}{dx} = 2y - 2x - 3z$ $\frac{1 - x^2 - y^2}{3x}$ $(x+y)^2 = x^2 + y^2 + 2xy$
$\frac{dz}{dx} = \frac{2y}{3x} - \frac{2}{3} - \frac{3z}{3x}$ $1 - (x^2 + y^2)$
$= \frac{2u}{} - \frac{2}{} - z$ $1 - [(x+y)^2 + 2xy)]$

[6] $z < \frac{x}{y}$ $x < \frac{u}{v}$ $y < \frac{u}{v}$

[7] $u = xyz$
$x = e^{-t}$, $y = e^{-t}\sin^2 t$, $z = \sin t$
$u < x\; y > t$

$$\frac{dz}{dx} = \frac{2y}{3x} - \frac{2}{3} - \frac{\cancel{3}z}{\cancel{3}x}$$

$$= \frac{2y}{3x} - \frac{2}{3} - \frac{z}{x}$$

$$\frac{1-[(x+y)^2+2xy)]}{3x}$$

$$\frac{x^2}{x^2} + \frac{y^2}{x^2} + \frac{3xz}{x^2} = \frac{1}{x^2}$$

$$1 + \frac{y^2}{x^2} + \frac{3z}{x} = \frac{1}{x^2}$$

$$\frac{3z}{x} = \frac{1-y^2}{x^2} - 1$$

$$-\frac{z}{x} = \frac{1}{3} - \frac{(1-y^2)}{3x^2}$$

$$\frac{2y}{3x}\left(-\frac{2}{3} + \frac{1}{3}\right) - \frac{(1-y^2)}{3x^2}$$

$$\frac{2y}{3x} - \frac{1}{3} - \frac{(1-y^2)}{3x^2}$$

$$-\frac{(2x-x^2)}{3x^2} - \frac{1}{3}$$

$$\frac{2y}{3x} - \frac{(2-x)}{3x} - \frac{1}{3}$$

$$\frac{2y-2}{3x} + \frac{\cancel{x}}{\cancel{3x}} - \cancel{\frac{1}{3}}$$

$$\frac{2(y-1)}{3x}$$

$x + y = 1$

$y = 1 - x$

$y^2 = 1 + x^2 - 2x$

$1 - y^2 = 2x - x^2$

$\boxed{6}$ $z \to x, y$; $x \to u, v$; $y \to u, v$

$x = e^u \sin v \qquad y = e^v \cos v$

$$\frac{dz}{du} = \frac{\partial z}{\partial x}\cdot\frac{dx}{du} + \frac{\partial z}{\partial y}\cdot\frac{dy}{du}$$

$$= \frac{\partial z}{\partial x}\cdot e^u \sin v + \cancel{\frac{\partial z}{\partial y}(0)}$$

$$= \frac{\partial z}{\partial x}(\sin v\, e^u)$$

$\boxed{8}$ $z = x^2 + 2xy + 4y^2, \quad y = e^{3x}$

$$\frac{dy}{dx} = 3e^{3x}$$

$$\frac{dz}{dx} = 2x + 2\left(x\frac{dy}{dx} + y\right) + 4(2y)\frac{dy}{dx}$$

$$= 2x + 2(x(3e^{3x}) + e^{3x}) + 8(e^{3x})(3e^{3x})$$

$$= 2x + 2e^{3x}(3x+1) + 24e^{6x}$$

$$= 2x + 6xe^{3x} + 2e^{3x} + 24e^{6x}$$

$$= 2(x + e^{3x}) + 6e^{3x}(x + 4e^{3x})$$

$\boxed{10}$ $u = f(x-y,\ y-z,\ z-x)$

$r = x - y, \quad s = y - z, \quad t = z - x$

$u \to r, s, t$; $r \to x, y$; $s \to y, z$; $t \to x, z$

$$\to \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r}(1) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}(-1)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t}$$

$$\to \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r}(-1) + \frac{\partial u}{\partial s}(1) + \frac{\partial u}{\partial t}(0)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial s} - \frac{\partial u}{\partial r}$$

$$\to \frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}(-1) + \frac{\partial u}{\partial t}(1)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} - \frac{\partial u}{\partial s}$$

$$\to \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \cancel{\frac{\partial u}{\partial r}} - \cancel{\frac{\partial u}{\partial t}} + \cancel{\frac{\partial u}{\partial s}} - \cancel{\frac{\partial u}{\partial r}} + \cancel{\frac{\partial u}{\partial t}} - \cancel{\frac{\partial u}{\partial s}} = 0$$

$x = e^{-t}, \quad y = e^{-t}\sin^2 t, \quad z = \sin t$

$u \to x, y, z \to t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial t}$$

$$= yz(-e^{-t}) + xz(-e^{-t}\sin^2 t + e^{-t}(2\sin t\cos t)) + xy\cos t$$

$$= -e^{-2t}\sin^3 t + e^{-2t}\sin^2 t(2\cos t - \sin t) + e^{-2t}\sin^2 t\cos t$$

$$= e^{-2t}\sin^2 t(-\sin t + 2\cos t - \sin t + \cos t)$$

$$\frac{du}{dt} = e^{-2t}\sin^2 t(3\cos t - 2\sin t)$$

$\boxed{9}$ $z \to x, y$; $x \to u, v$; $y \to u, v$

$x = e^u \sin v \qquad y = e^u \cos v$

PT:

(i) $\frac{\partial z}{\partial u} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$

$$\to \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x}\cdot e^u \sin v + \frac{\partial z}{\partial y}\cdot e^u \cos v$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot x + \frac{\partial z}{\partial y}\cdot y$$

proved

ii) $\frac{\partial z}{\partial x} = e^{-u}\left(\sin v\frac{\partial z}{\partial u} + \cos v\frac{\partial z}{\partial v}\right)$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot e^u \cos v + \frac{\partial z}{\partial y}\cdot e^u(-\sin v)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot y + \frac{\partial z}{\partial y}\cdot(-x)$$

$$\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(x+y) + \frac{\partial z}{\partial y}(y-x)$$

RHS =

$$e^{-u}\left\{\sin v\left(\frac{\partial z}{\partial x}e^u s v + \ldots\right.\right.$$

$$+\cos v\left(\frac{\partial z}{\partial x}e^u c v \ldots\right.$$

$$= e^{-u}\left[\frac{\partial z}{\partial x}e^u s^2 v + \frac{\partial z}{\partial y}\ldots\right.$$

$$+ \frac{\partial z}{\partial x}e^u c^2 v - \ldots$$

$$= \cancel{e^{-u}}\,\frac{\partial z}{\partial x}\cancel{e^u}(s\ldots$$

$$= \frac{\partial z}{\partial x} = LHS$$

$\boxed{11}$ $u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(x^{\cancel{2}}\cdot\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{\cancel{x}}\right) - 2\cancel{y}\cdot\frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{-x}{y^2}\right.$$

$$= \frac{\partial}{\partial x}\left(\frac{x^3}{x^2+y^2} + \frac{2xy}{x^2+y^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x^3+2xy}{x^2+y^2}\right)$$

$$= \frac{(3x^2+2y)(x^2+y^2) - (x^3+2xy)(2x+y^2)}{(x^2+y^2)^2}$$

$$= \frac{3x^4 + 3x^2y^2 + 2x^2y + 2y^3 - 2x^4 - x^3y^2 - 4x^2y - 2xy^3}{(x^2+y^2)^2}$$

$$= \frac{x^4 + 3x^2y^2 - 2x^2y + 2y^3 - x^3y^2 - 2xy^3}{(x^2+y^2)^2}$$

$\frac{\partial z}{\partial y} \cdot e^u cv)$

$+ \frac{\partial z}{\partial y} e^u(-sv))$

$\cancel{e^u cvsv}$

$\left. \frac{z}{y} e^u \cancel{cvsv} \right]$

$^2v + c^2v)$