UNIT 3: MULTIVARIABLE CALCULUS:

Let $Z = f(x,y)$

kept constant

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

geometrically represent slope

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$ provided the limit exists

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \cdot \partial y}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \cdot \partial z}$$

2° derivatives

NOTE: $$\frac{\partial^2 z}{\partial x \cdot \partial y} = \frac{\partial^2 z}{\partial x \cdot \partial y}$$ holds true → partial differentials are differentiable and continuous

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial x^2}\right) = \frac{\partial^3 z}{\partial y \cdot \partial x^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y^2}\right) = \frac{\partial^3 z}{\partial x \cdot \partial y^2}$$

(Q) Find $f(x)$ & $f(y)$ of the following

(A) (1) $f(x,y) = x^y$

$$\frac{\partial f}{\partial x} = yx^{y-1} \; ; \; \frac{\partial f}{\partial y} = x^y \cdot \log(x)$$

(2) $\sin^{-1}(2x-3y)$

$$\frac{\partial f}{\partial x} = \frac{+1}{\sqrt{1-(2x-3y)^2}} \cdot 2 \qquad \frac{\partial f}{\partial y} = \frac{1}{\sqrt{1-(2x-3y)^2}} (-3)$$

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(3) $\tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial f}{\partial x} = \frac{1}{1+(y/x)^2}\left(\frac{-y}{x^2}\right)$$

$$\Rightarrow \frac{-y}{x^2+y^2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

$$\Rightarrow \frac{\partial^2 f}{\partial x \cdot \partial y} = \frac{\partial^2 f}{\partial y \cdot \partial x}$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) \Rightarrow -\frac{\{x^2+y^2(1) - y(2y)\}}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial^2 f}{\partial y \cdot \partial x} \Rightarrow \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \Rightarrow \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$$

$$\Rightarrow \frac{x^2+y^2(1) - x(2x)}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{y^2-x^2}{(x^2+y^2)^2} \Rightarrow \therefore \frac{\partial^2 f}{\partial x \cdot \partial y} = \frac{\partial^2 f}{\partial y \cdot \partial x}$$

(1) $f = x\cos y + ye^x$

(2) $f = e^{xy} \cdot \ln y$

(3) $f = \frac{2y}{y+\cos x}$

(4) $f = \sqrt{x^2+y^2}$

(1) If $u = \log(x^3+y^3+z^3-3xyz)$;

then prove that (i) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$

(1) $\frac{\partial f}{\partial x} = \cos y + ye^{x}$

$\frac{\partial f}{\partial y} =$

(1) Let $u = \log(x^3 + y^3 + z^3 - 3xyz)$

differentiate 'u' partially w.r.t. x, y, z respectively

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}(3x^2-3yz)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz}(3y^2-3xz)$$

$$\frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz}(3z^2-3xy)$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(\cancel{x^2+y^2+z^2-xy-yz-zx})}{\cancel{x^3+y^3+z^3-3xyz}}$$

$$= \frac{3}{x+y+z}$$

LHS $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u.$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u.$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2}$$

(Q) If $z = f(x+ct) + \phi(x-ct)$; prove that $\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}$

(A) Let $z = f(x+ct) + \phi(x-ct)$

$$\frac{\partial z}{\partial x} = f'(x+ct)(1) + \phi'(x-ct)\cdot(1)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ct)c^2 + \phi''(x-ct)(-c)^2$$

$$= c^2\{f''(x+ct) + \phi''(x-ct))$$

From eq (1)

$$\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial z^2}{\partial x^2}$$

(Q) If $\theta = t^n e^{-r^2/4t}$, what value of 'n' will make $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t}$

(A) $\theta = t^n e^{-r^2/4t}$

$$\frac{\partial\theta}{\partial r} = t^n e^{-r^2/4t}\left(\frac{-2r}{4t}\right)$$

$$= \frac{-1}{2} t^{n-1} r e^{-r^2/4t}$$

$$r^2\frac{\partial\theta}{\partial r} = \frac{-1}{2} t^{n-1} r^3 e^{-r^2/4t}$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{-1}{2}t^{n-1}\{r^3 e^{-r^2/4t}\cdot\left(\frac{-2r}{4t}\right) + e^{-r^2/4t}\cdot 3r^2\}$$

$$= \frac{-1}{2}t^{n-1}\{\frac{-1}{2}r^4 t^{-1} e^{-r^2/4t} + e^{-r^2/4t}\cdot 3r^2\}$$

$$= e^{-r^2/4t}\left[\frac{1}{4}t^{n-2}r^4 - \frac{3}{2}t^{n-1}\cdot r^2\right]$$

$$\frac{1}{\partial^2}\left\{\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)\right\}$$

$$= e^{-r^2/4t}\left[\frac{1}{4}t^{n-2}r^2 - \frac{3}{2}t^{n-1}\right]$$

$$\frac{\partial\theta}{\partial t} = t^n e^{-r^2/4t}\left(\frac{+r^2}{4t^2}\right) + e^{-r^2/4t} \cdot n t^{n-1}$$

$$= e^{-r^2/4t}\left(\frac{t^{n-2}}{4} \cdot r^2 + n t^{n-1}\right) \longrightarrow \boxed{n = -3/2}$$

NOTE: The f^n which satisfies laplace eq^n

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \longrightarrow \text{laplace equation 2-D}$$

$u \rightarrow$ harmonic function.

(Q) Show that $\theta = \tan^{-1}(y/x)$ satisfies the laplace equation.

(A) $$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}$$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+(y/x)^2} \cdot \left(\frac{-y}{x^2}\right)$$

$$= \frac{x^2}{x^2+y^2} \cdot \frac{-y}{\cancel{x^2}} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right)$$

$$= \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial\theta}{\partial x}\right) = -y\left(\frac{-1 \times 2x}{(x^2+y^2)^2}\right)$$

$$= \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial\theta}{\partial y}\right) = \frac{-2y \cdot x}{(x^2+y^2)^2}$$

$$= \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0$$

θ satisfies laplace equation

(Q) Show that $v = e^x \cos y$ satisfies the laplace equation.

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(Q) show that $v = e^{x-at}\cos(x-at)$ is the solution of one dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \cdot \frac{\partial^2 y}{\partial x^2}$$

(A) $$\frac{\partial y}{\partial t} = e^{x-at}\cdot\cos(x-at)\cdot(-a) \mp e^{x-at}\cdot\sin(x-at)(-a)$$

$$= -a\left[e^{x-at}\cdot\cos(x-at) - e^{x-at}\sin(x-at)\right]$$

$$= -a\left[y - e^{x-at}\sin(x-at)\right]$$

$$\frac{\partial^2 y}{\partial t^2} = -a\left[\frac{\partial y}{\partial t} - e^{x-at}\sin(x-at)\cdot(-a)\right] - e^{x-at}\cos(x-at)(-a)\Big]$$

$$= a^2\Big[e^{x-at}\cancel{\cos(x-at)}(-a) + e^{x-at}\sin(x-at) + e^{x-at}$$
$$\sin(x-at) + \cancel{a e^{x-at}\cos(x-at)}\Big]$$

$$= e^{\cancel{x-at}}\cancel{\cos(x-at)}(-a^2\cancel{a}) + \cancel{e}^{x-at}\cancel{\sin}(x-at)\cancel{(a^2+a)}$$

$$= 2a^2 e^{x-at}\sin(x-at) \quad \text{--- (1)}$$

$$\frac{\partial y}{\partial x} = e^{x-at}\cos(x-at) - e^{x-at}\sin(x-at)$$

$$\frac{\partial^2 y}{\partial x^2} = e^{x-at}\cancel{\cos(x-at)} - e^{x-at}\sin(x-at) - \cancel{e^{x-at}\cos(x-at)}$$
$$- e^{x-at}\sin(x-at)$$

$$= -2e^{x-at}\sin(x-at) \quad \text{--- (2)}$$

sub (2) in (1)

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$$

(Q) Show that D-Alembert solution $y = \phi(y+ax) + \psi(y-ax)$ setisfy the wave equation, $y_{xx} = a^2 y_{yy}$

(A) $$\frac{\partial y}{\partial x} = \phi(y+ax)\cdot a + \psi(y-ax)\cdot(-a)$$

(Q) Show that $y = \frac{1}{\sqrt{t}} e^{-x^2/4a^2t}$ is the solution of one dimensional heat equation.

(A) $\frac{\partial y}{\partial t} = a^2 \frac{\partial^2 y}{\partial x^2}$

$\Rightarrow \frac{\partial y}{\partial t} = \frac{-1}{2(t)^{3/2}} e^{-x^2/4a^2t} + \frac{1}{\sqrt{t}} e^{-x^2/4a^2t} \cdot \frac{-x^2}{4a^2}$

$= \frac{-1}{2(t)^{3/2}} e^{-x^2/4a^2t} + \frac{1}{\sqrt{t}} e^{-x^2/4a^2t} \cdot \frac{x^2}{4a^2}$

$\frac{\partial y}{\partial t} = \frac{-1}{2(t)^{3/2}} e^{-x^2/4a^2t} + \frac{1}{\sqrt{t}} e^{-x^2/4a^2t} \cdot \frac{x^2}{4a^2}$

$\frac{\partial^2 y}{\partial x^2}$, $\frac{\partial y}{\partial x} = \frac{1}{\sqrt{t}} e^{-x^2/4a^2t} \cdot \frac{-2x}{4a^2t}$

① $= \frac{1}{\sqrt{t}} e^{-x^2/4a^2t} \cdot \frac{-2x}{4a^2t}$

$\frac{\partial^2 y}{\partial x^2} \Rightarrow \frac{1}{\sqrt{t}} e^{-x^2/4a^2t} \cdot \frac{-2x}{4a^2t} \cdot -2x$

$(\sqrt{t})^{-1/2}$

$-1(\sqrt{t})^{-1-1}$

$-1(\sqrt{t})^{-2}$

t^{-1}

$\frac{-1}{t}$

$t^{-1/2}$

$-\frac{1}{2}t^{-1-1/2}$

$-\frac{1}{2}t^{-3/2}$

t^{-1}

$-1t^{-2}$

$\frac{-1}{t^2}$

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(Q) Show that $v = e^{a\theta} \cos(a \log r)$ satisfies the laplace equation

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0 \longrightarrow \text{polar}$$

(A)

$$\frac{\partial v}{\partial r} = \left[-\sin(a \log r)\frac{a}{r}\right] e^{a\theta}$$

$$\frac{d^2 v}{dr^2} = e^{a\theta}\left\{\left[-\cos(a \log r)\left(\frac{a}{r}\right)^2\right] - \left[\sin(a \log r)\left(\frac{-a}{r^2}\right)\right]\right\}$$

$$\frac{dv}{d\theta} = a e^{a\theta} (\cos)[a \log r]$$

$$\frac{d^2 v}{d\theta^2} = a^2 e^{a\theta} (\cos(a \log r))]$$

$$\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr} + \frac{1}{r^2}\frac{d^2 v}{d\theta^2} = 0$$

$$\Rightarrow -\frac{a^2}{r^2} e^{a\theta} \cancel{\cos(a \log r)} + \frac{a}{r^2}\left[\cancel{e^{a\theta}\sin(a \log r)}\right] - \frac{a}{r^2}$$

$$\cancel{e^{a\theta} \sin(a \log r)} + \cancel{\frac{a^2}{r} e^{a\theta} \cos(a \log r)}$$

$$= 0$$

(Q) Show that $u = e^{r\cos\theta} \cos(r\sin\theta)$, $v = e^{r\cos\theta} \sin(r\sin\theta)$ satisfy the CR eqn (Cauchy-Riemann)

(A) $u_r = \frac{1}{r} V_\theta$; $v_r = -\frac{1}{r} u_\theta$

$$= u_r = \frac{du}{dr} = e^{r\cos\theta} \cdot \cos\theta \cdot \cos(r\sin\theta) + e^{r\cos\theta}(-\sin(r\sin\theta) \cdot \sin\theta)$$

$$= e^{r\cos\theta}[\cos\theta\cos(r\sin\theta) - \sin\theta\sin(r\sin\theta)]$$

$$= e^{r\cos\theta} \cos(\theta + r\sin\theta)$$

$$v_r = \frac{du}{d\theta}$$

(*) (Q) given $x^x y^y z^z = c$, then show that at $x=y=z$,

$$\frac{\partial^2 z}{\partial x \cdot \partial y} = -(x \log ex)^{-1}$$

(A) $x\log x + y\log y + z\log z = \log c$

Diff wrt y partially

$$y \cdot \frac{1}{y} + \log y + z \cdot \frac{1}{z} \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = -\frac{(1+\log y)}{(1+\log z)} \rightarrow \text{diff wrt } x$$

$$\frac{\partial^2 z}{\partial y \cdot \partial x} = -\left[\frac{(1+\log z)\cdot 0 - (1+\log y)\cdot \frac{1}{z}\frac{\partial z}{\partial x}}{(1+\log z)^2}\right]$$

$$= \frac{1}{z}\left[\frac{1+\log y}{(1+\log z)^2}\right] \cdot \frac{\partial z}{\partial x} \rightarrow (1)$$

$$\frac{\partial z}{\partial x} = \frac{-(1+\log x)}{1+\log z} \rightarrow (2)$$

sub (1) in (2)

$$\frac{\partial^2 z}{\partial x \cdot \partial y} = \frac{1}{z} \cdot \frac{(1+\log y)(1+\log x)}{(1+\log z)^3}$$

at $x=y=z$; $z=x$; $y=x$

$$\frac{\partial^2 z}{\partial x \cdot \partial y} = \frac{-1}{x} \frac{(1+\log x)^2}{(1+\log x)^3} = \frac{-1}{x(1+\log x)}$$

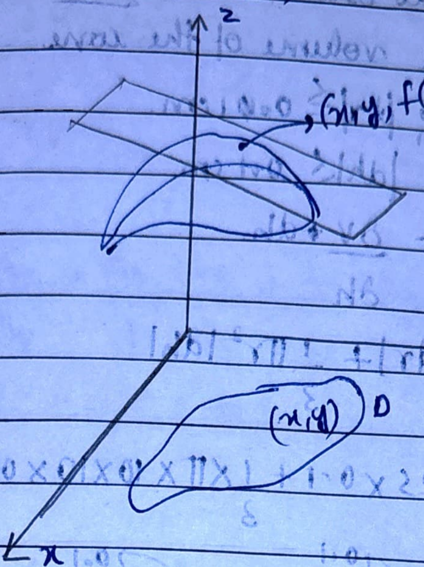
$$= \frac{-1}{x(\log_e + \log x)} = -(x\log ex)^{-1}$$

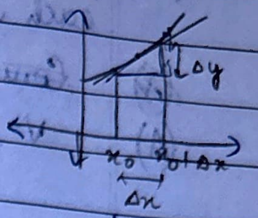
Total differential:

let $z = f(x,y)$ → equation of a surface.

$z = 4x - y + 3$ ↳ equation of a plane

$\Delta y = f(x_0 + \Delta x) - f(x_0)$





$dy = f'(x) \cdot dx$

$z = f(x,y)$, $D \subseteq R^2$ ↓ set of

Approximation of z when x and y are changed

$$\boxed{dz = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} dy}$$

Ex1 Find total differentials of

(a) $z = e^{-2x} \cos(2\pi t)$

$dz = -e^{-2x} \cdot 2 \cos(2\pi t) \cdot dx + -\sin(2\pi t) e^{-2x} \cdot dt \cdot 2\pi$

⇒

(b) $l = xze^{-y^2 - z^2}$

$dl = ze^{-y^2-z^2} \cdot dx + \left(xe^{-y^2-z^2} + xz \cdot e^{-y^2-z^2} \cdot (-2z)\right) \cdot dz$
$+ \left(xze^{-y^2-z^2} \cdot (-2y)\right) \cdot dy$

(2) If $z = x^2 - xy + 3y^2$; (x,y) changes from $(3,-1)$ to $(2.96, -0.95)$

(A) $dz = (2x - y)\big|_{(3,-1)} (-0.04) + (x + 6y)\big|_{(3,-1)} (0.05)$

Compare the values of Δz and dz.

$\Delta z = (2.96)^2 + (2.96)(0.95) + 3(0.95)^2 - (3)^2 + (3)(-1) - 3(1)^2$

$= (2.96+3)(\cancel{2.96}\ 0.04) + 3((0.95+1)(0.95-1)) +$

$-(6+1)(0.04) - 9(0.05)$

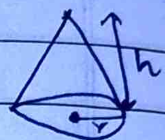
$= -0.73$

$(2.96)(0.95) + (3)(-1) = -0.1189$

(3) The base radius and height of a right circular cone are measured as 10cm and 25cm respectively with possible error in measurement is as much as 0.1cm each. Use the diff to estimate the max error in the calculated volume of the cone.

(A) $V = \frac{1}{3}\pi r^2 h$; $|dr| \le 0.\cancel{0}1$ cm.

$|dh| \le 0.1$ cm



$$dV = \frac{\partial V}{\partial r}\cdot dr + \frac{\partial V}{\partial h}\cdot dh \quad (10, 25)$$

$$= \frac{2\pi r h}{3}\cdot|dr| + \frac{1}{3}\pi r^2 |dh|$$

$$= \frac{2\pi \times 10 \times 25 \times 0.1}{3} + \frac{1 \times \pi \times 10 \times 10 \times 0.1}{3}$$

$$= 523.5 \times \underbrace{|dr|}_{0.1} + 104.7\,\underbrace{|dh|}_{0.1}$$

$$= 62.82\ cm^3$$

(Q) The dimension of a rectangle box is measured to be 75cm, 60cm and 40cm and each measurement correct to within 0.2cm. Use differentials to estimate the largest possible error when the volume of the box is calculated for these measurements.

(A) $V = lbh$

$$dV = bh(dl) + lh(db) + lb(dh)$$

$$= 60 \times 40 \times 0.2 + 75 \times 40 \times 0.2 + 75 \times 60 \times 0.2$$

$$= \cancel{0.2}$$

$$= 1980\ cm^3$$

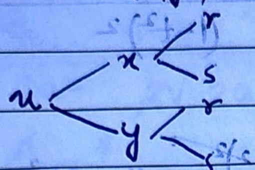
Total derivative $\to u \langle {x \atop y} \rangle t$ $\quad \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{}$

$$\frac{du}{dt} = \frac{du}{\partial x} \cdot \frac{dx}{dt}$$

Differentiation of composite f^n

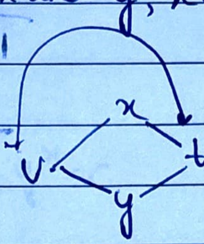
If u and v are f^n of x and y, x and y are f^n of r and ~~x~~ s, then u and v are composite f^n of r and s.

$[u = u(x,y);\ v = v(x,y),\ x = x(r,s);\ y = y(r,s)]$



$\frac{\partial u}{\partial r} =$

NOTE: If u and v are f^n of x and y, x and y are f^n of t, i.e. $u = u(x,y)$; $x = x(t)$; $y = y(t)$.



then total derivative of u wrt t is given by $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$

$u \to$ depends on (x and y) $\to$ two independent variables

differentiation of implicit f^n ( )

(Q) $f(x,y) = c \Rightarrow$ An implicit f^n of x and y.

by total derivative $df =$ ~~$f'(x,y)$~~ $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0$ derivative of constant is 0.

$$\boxed{\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}} = -\frac{f_x}{f_y}$$

(Q) Hmwk $u = xy + yz + zx$

$x = t\cos t$; $y = t\sin t$; $z = t$

Find du (rate of change of u wrt t at $t = \pi/4$)

(Q) If $u = x^2 + y^2 + z^2$; $x = e^{2t}$; $y = e^{2t}\cos 3t$; $z = e^{2t}\sin 3t$ find du/dt as a total derivative of t.

(A) $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$

$= 2x \cdot e^{2t} \cdot 2 + 2y [e^{2t} \cdot 2\cos 3t - e^{2t}\sin 3t \cdot 3] + 2z [e^{2t} \cdot 2 \cdot \sin 3t + e^{2t} \cdot \cos 3t \cdot 3]$

$= 2e^{4t} \cdot 2 + 2e^{2t}\cos 3t [e^{2t} \cdot 2\cos 3t - e^{2t} \cdot \sin 3t \cdot 3] + 2e^{2t} \sin 3t [e^{2t} \cdot 2\sin 3t + e^{2t} \cdot 3\cos 3t]$

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(Q) $Z = \tan^{-1}\left(\frac{x}{y}\right)$; $x = 2t$; $y = 1-t^2$

$\frac{dz}{dt} = \frac{2}{1+t^2}$ using total derivative

(A) $z = \tan^{-1}\left(\frac{2t}{1-t^2}\right)$, $\frac{dz}{dt} = \frac{1}{1+\left(\frac{2t}{1-t^2}\right)^2} \cdot \left[\frac{2(1-t^2) - 2t(-2t)}{(1-t^2)^2}\right]$

$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$

$$\frac{\cancel{(1-t^2)^2}}{(1-t^2)^2 + (2t)^2} \cdot \frac{2 - 2t^2 + 4t^2}{\cancel{(1-t^2)^2}}$$

$$\frac{2 + 2t^2}{1 + t^4 - 2t^2 + 4t^2}$$

$$\frac{2(1+t^2)}{1+t^4+2t^2} = \frac{2\cancel{(1+t^2)}}{(1+t^2)^{\cancel{2}}}$$

$$\frac{2}{(1+t^2)}.$$

HW $Z = xy^2 + yx^2$, $x = at^2$, $y = 2at$ find $\frac{dz}{dt}$ using total derivatives at $t=1$.

(Q) $U = \sin(x^2+y^2)$, $ax^2+by^2 = c^2$, find $\frac{du}{dx}$

(A) $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$

$\cos(x^2+y^2) \cdot 2x + \cos(x^2+y^2) \cdot 2y \cdot \frac{dy}{dx}$ —— ①

$f(x,y) = ax^2 + by^2 = c^2$

$\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-\cancel{2}ax}{\cancel{2}by} = \frac{-ax}{by}$

$$\frac{du}{dx} = \cos(x^2+y^2) \cdot 2x + \cos(x^2+y^2) \cdot 2y \cdot \left(\frac{-ax}{by}\right).$$

(Q) $z = \sqrt{x^2+y^2}$ and $x^3+y^3+3axy = 5a^2$, S.T. the rate of change of z wrt x at $x=y=a$ is 0 using total derivative.

(A) $\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x + \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y \cdot \frac{dy}{dx}$

$3x^2 + 3y^2 \cdot \frac{dy}{dx} + 3ay + 3ax \cdot \frac{dy}{dx} = 0$

$(3y^2 + 3ax)y' = -3ay - 3x^2$

$$y' = \frac{-3ay-3x^2}{3y^2+3ax}$$

$\sqrt{x}$

$(x)^{1/2-1}$

$x^{-1/2}$

$\frac{1}{2\sqrt{x}}$

$$\frac{2x}{2\sqrt{x^2+y^2}} + \frac{2y}{2\sqrt{x^2+y^2}} \cdot \frac{(-3ay-3x^2)}{3y^2+3ax} =$$

$$\frac{\cancel{2}a}{\cancel{2}\sqrt{2a^2}} + \frac{\cancel{2}a}{\cancel{2}\sqrt{2a^2}} \cdot \frac{(-3a^2-3a^2)}{3a^2+3a^2}$$

$$\frac{\cancel{a}}{\cancel{a}\sqrt{2}} + \frac{\cancel{a}}{\cancel{a}\sqrt{2}} \cdot \frac{-\cancel{6a^2}}{\cancel{6a^2}}$$

$$\frac{1}{\sqrt{2}} - \frac{\cancel{a}}{\sqrt{2}} = 0$$

ay.

(Q) Find $\frac{dy}{dx}$ if $x^y + y^x = c$.

(A) $$\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{yx^{y-1} + y^x \cdot \log(y)}{x^y \log(x) + xy^{x-1}}$$

(Q) $\frac{dy}{dx}$ if $(\cos y)^x = (\sin(x))^y$

$f(x,y) = (\cos y)^x - (\sin x)^y$ $\qquad -f_x$

$$\frac{dy}{dx} = \frac{(\cos y)^x \log(\cos y) - y(\sin x)^{y-1} \cdot \cos x}{x(\cos y)^{x-1} \cdot -\sin y - (\sin x)^y \cdot \log(\sin x)}$$

differentiation of composite f^n

If u and v are f^n of x and y → x and y are f^n of r and s, then u and v are composite f^n of r and s.

$u = u(x,y)$; $x = x(r,s)$; $y = y(r,s)$

$$\begin{tikzcd} & x \arrow[r] \arrow[dr] & r \\ u \arrow[ur] \arrow[dr] & & s \\ & y \arrow[uur] \arrow[ur] & \end{tikzcd}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$$

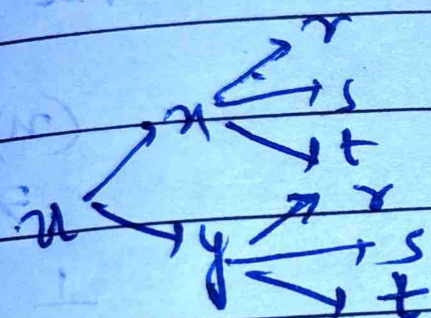
$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}.$$

$$\begin{tikzcd} & x \arrow[r] \arrow[dr] & r \\ v \arrow[ur] \arrow[dr] & & s \\ & y \arrow[uur] \arrow[ur] & \end{tikzcd}$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial v}{\partial s} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial s}.$$

(2) $u = u(x,y)$; $x = x(r,s,t)$; $y = y(r,s,t)$



$$\Rightarrow \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\Rightarrow \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}$$

similarly do for v.

(Q) $z = f(x,y)$; $x = e^u \sin v$; $y = e^u \cos v$, show that $\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = e^{2u}\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]$

(A)
$$\begin{tikzcd} & x \arrow[r] \arrow[dr] & u \\ z \arrow[ur] \arrow[dr] & & v \\ & y \arrow[r] \arrow[ur] & v \end{tikzcd}$$

LHS: $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$

$= \frac{\partial z}{\partial x}\cdot e^u \sin v + \frac{\partial z}{\partial y}\cdot e^u \cos v$

$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$

$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot e^u \cos v + \frac{\partial z}{\partial y}\cdot(-e^u \sin v)$

LHS $\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\cdot e^{2u}\cdot \sin^2 v + \left(\frac{\partial z}{\partial y}\right)^2 e^{2u}\cdot\cos^2 v + \cancel{2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\cdot} e^u \sin v\cdot e^u \cos v +$

$\left(\frac{\partial z}{\partial x}\right)^2 e^{2u}\cos^2 v + \left(\frac{\partial z}{\partial y}\right)^2 e^{2u}\cdot\sin^2 v - \cancel{2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}}\cdot e^u \cos v\cdot e^u \sin v$

$= e^{2u}\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]$

(Q) $u = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$; ST $u_x^2 + u_y^2 = u_r^2 + \frac{1}{r^2}u_\theta^2$

(A) $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cdot\cos\theta + \frac{\partial u}{\partial y}\cdot\sin\theta$

$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}\cdot(-r\sin\theta) + \frac{\partial u}{\partial y}(+r\cos\theta)$

$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial u}{\partial y}\right)^2\sin^2\theta + \cancel{2\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}\sin\theta\cdot\cos\theta}$

$+ \cancel{\frac{1}{r^2}}\left[\left(\frac{\partial u}{\partial x}\right)^2\cdot \cancel{r^2}\sin^2\theta + \left(\frac{\partial u}{\partial y}\right)^2 \cancel{r^2}\cos^2\theta - \cancel{2\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}\cdot r^2\sin\theta\cos\theta}\right]$

$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 =$ RHS

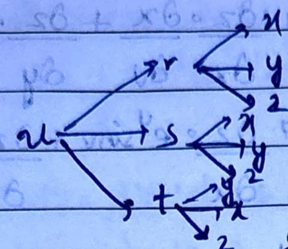
$\frac{\partial u}{\partial x} \rightarrow$ means. diff u with x keeping y and z constant

$u = x+y+z$

(Q) $u = f(x-y, y-z, z-x)$, find $u_x + u_y + u_z$.

(A) $u_x = \frac{\partial u}{\partial x} \Rightarrow$

Let $r = x-y$, $s = y-z$; $t = z-x$



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r}\cdot(1) + \cancel{\frac{\partial u}{\partial s}}\cdot(0) + \frac{\partial u}{\partial t}\cdot(-1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r}\cdot(-1) + \frac{\partial u}{\partial s}(1) + \cancel{\frac{\partial u}{\partial t}}\cdot(0)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

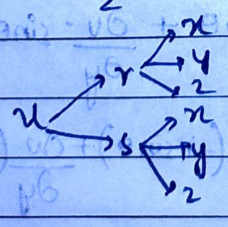
$$= \cancel{\frac{\partial u}{\partial r}}(0) + \frac{\partial u}{\partial s}\cdot(-1) + \frac{\partial u}{\partial t}\cdot(1)$$

$$u_x + u_y + u_z = \cancel{\frac{\partial u}{\partial r}} - \cancel{\frac{\partial u}{\partial t}} - \cancel{\frac{\partial u}{\partial r}} + \cancel{\frac{\partial u}{\partial s}} - \cancel{\frac{\partial u}{\partial s}} + \cancel{\frac{\partial u}{\partial t}}$$

$= 0$

(Q) $u = f\left(\frac{x}{z}, \frac{y}{z}\right)$, find the value $xu_x + yu_y + zu_z$

(A) Let $r = \frac{x}{z}$, $s = \frac{y}{z}$



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\left(\frac{1}{z}\right) + \cancel{\frac{\partial u}{\partial s}}(0)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}(x) + \frac{\partial u}{\partial s}(y)$$

$$\frac{\partial u}{\partial y} = \cancel{\frac{\partial u}{\partial r}}(0) + \frac{\partial u}{\partial s}\left(\frac{1}{z}\right)$$

$$\cancel{\frac{x}{z}}\cdot\frac{\partial u}{\partial r} + \cancel{\frac{y}{z}}\frac{\partial u}{\partial s} - \frac{x z}{z}\frac{\partial u}{\partial r} - \frac{yz\,\partial u}{\partial s} = 0$$

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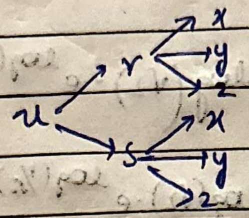
(Q) $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, $x^2u_x + y^2u_y + z^2u_z$

(A) $r = \frac{y-x}{xy}$; $s = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z}$

↳ $1/x - 1/y$



$\Rightarrow$ $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}$

$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}$

x^{-1}

$\frac{-1}{x^2}$

$\Rightarrow$ $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\left(\frac{-1}{x^2}\right) + \frac{\partial u}{\partial s}\left(\frac{-1}{x^2}\right)$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\left(\frac{1}{y^2}\right) + \frac{\partial u}{\partial s}(0)$

$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}\left(\frac{1}{z^2}\right)$

$\cancel{x^2\cdot\frac{1}{x^2}\left(-\frac{\partial u}{\partial r}\right)} - \cancel{x^2\cdot\frac{1}{x^2}\left(\frac{\partial u}{\partial s}\right)} + \cancel{y^2\cdot\frac{1}{y^2}}\cdot\frac{\partial u}{\partial r} + \cancel{z^2\cdot\frac{1}{z^2}}\frac{\partial u}{\partial s}$

$= 0$

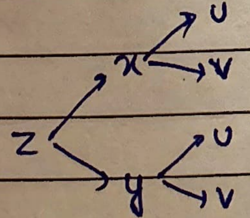
(Q) If $z = 4e^x \log y$, $x = \log(\cos v)$; $y = u\sin v$. → (u and v.)

$\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}$ at $(2, \pi/4)$

$4e^x\cdot\frac{1}{y}\cdot\sin v.$

$4e^2\cdot 1$

(A)



$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$

$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$

$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot(0) + \frac{\partial z}{\partial y}\cdot\sin v$ → $(2, \pi/4)$

$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial y}\cdot$

$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{-\sin v}{\cos v} + \frac{\partial z}{\partial y}\cdot u\cos v.$

$\frac{\partial z}{\partial u} = 4e^x\cdot\frac{1}{y}\cdot\frac{1}{\sqrt{2}}$

$x = \log(1/\sqrt{2})$; $y = 2\times 1/\sqrt{2}$

$\partial z/\partial y = 4e^x \cdot 1/y$ $\partial z/\partial x = 4\log y\, e^x$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot \cancel{-\sec}\tan v + \frac{\partial z}{\partial y}\cdot u\cos v$$

$$= -\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\cdot 2\times\frac{1}{\sqrt{2}}$$

$$= +\frac{4e^x}{y}\cdot 2\times\frac{1}{\sqrt{2}} - 4\log y\, e^x$$

$$= \frac{4e^{\log(1/\sqrt{2})}}{\cancel{2\times\frac{1}{\sqrt{2}}}}\cdot\cancel{2\times\frac{1}{\sqrt{2}}} - 4\log(\sqrt{2})\cdot e^{\log(1/\sqrt{2})}$$

$$4e^{\log(1/\sqrt{2})} - 4\log(\sqrt{2})\cdot e^{\log(1/\sqrt{2})}$$

$$\frac{\partial z}{\partial v} = 4e^{\log(1/\sqrt{2})}\left[1-4\log(\sqrt{2})\right]$$

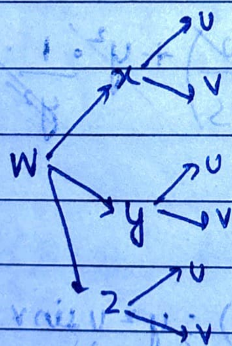
$$\frac{\partial z}{\partial u} = \frac{4e^x}{y}\cdot\frac{1}{\sqrt{2}} = \frac{\cancel{4}^{2}e^{\log(1/\sqrt{2})}}{\cancel{\sqrt{2}}}\cdot\frac{1}{\cancel{\sqrt{2}}} = 2e^{\log(1/\sqrt{2})}$$

$\frac{\partial w}{\partial u}, \frac{\partial w}{\partial v}$

Hw (Q) $w = xy + yz + zx$; $x = u+v$; $y = u-v$; $z = uv$ at $(1/2, 1)$

(Q) $v = f(2x-3y, 3x-4z, 4z-2x)$ find $v_x + v_y + v_z$

(A)



$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}$$