(Q) $x = u(1-v)$; $y = uv$; show $JJ' = 1$

(A) $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$

$= \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u - \cancel{uv} + \cancel{uv}$

$= u$

$x = u - uv$

$= u - y$

$x + y = u$

$v = y/u = \frac{y}{x+y}$

$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{(x+y)(1) - y(1)}{(x+y)^2} \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix}$

$= \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2} = \frac{x+y}{(x+y)^2} = \frac{1}{x+y} = \frac{1}{u}$

$\therefore J \cdot J' = u \cdot 1/u = 1$

(Q) If $x = e^u \cos v$; $y = e^u \sin v$, prove that $J \cdot J' = 1$

(A) $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \partial x/\partial v \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$x^2 + y^2 = e^{2u}$

$\ln(x^2+y^2) = 2u$

$u = \frac{\ln(x^2+y^2)}{2}$

$= \begin{vmatrix} \cos v\, e^u & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{vmatrix} = e^{2u}\cos^2 v + e^{2u}\sin^2 v$

$= e^{2u}$

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$u = \frac{\ln(x^2+y^2)}{2}$

$y/x = \tan v$

$\tan^{-1}(y/x) = v.$

$$= \begin{vmatrix} \frac{\cancel{2}x}{(x^2+y^2)\cdot\cancel{2}} & \frac{\cancel{2}y}{\cancel{2}(x^2+y^2)} \\ \frac{-y}{1+(y/x)^2\cdot x^2} & \frac{1}{1+(y/x)^2}\cdot\frac{1}{x} \end{vmatrix}$$

$$= \frac{x}{(x^2+y^2)}\cdot\frac{y^2}{x(x^2+y^2)} + \frac{y}{1+(y/x)^2\cdot x^2}\cdot\frac{y}{x^2+y^2}$$

$\frac{x^2}{x^2+y^2}\cdot\frac{1}{x}$

$$= \frac{y^2}{(x^2+y^2)^2} + \frac{y^2}{(x^2+y^2)^2}$$

$\frac{xy\cdot x^2}{x^2+y^2\cdot x}$

$$= \frac{2y^2}{(x^2+y^2)^2} = \frac{2(}{(e^{2u})^2} \longrightarrow \frac{x^2+y^2}{(x^2+y^2)^2} \qquad x^2+y^2 = e^{2u}$$

$J\cdot J' = e^{2u}\cdot 1/e^{2u} = 1$

(Q) $x = r\cos\theta$; $y = r\sin\theta$, prove that $J\cdot J' = 1$

(Q) Show that $u = \frac{x+y}{1-xy}$; $v = \tan^{-1}x + \tan^{-1}y$ are functionally dependent also find the relation between them.

(A) $$\frac{\partial u}{\partial x} = \frac{(1-xy)(1) - (-y)(x+y)}{(1-xy)^2}$$

$$= \frac{1-\cancel{xy}+\cancel{xy}+y^2}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1+x^2}{(1-xy)^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0.$$ ∴ so u and v are functionally dependent

$v = \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}(u)$

∴ $\boxed{\tan v = u}$

(Q) ST $u = x+y+z$; $v = x^2+y^2+z^2$; $w = xy+yz+zx$ are functionally dependent and find the relation b/w them.

(A) $\frac{\partial u}{\partial x} = 1$; $\frac{\partial v}{\partial x} = 2x$; $\frac{\partial w}{\partial x} = y+z$

$\frac{\partial u}{\partial y} = 1$; $\frac{\partial v}{\partial y} = 2y$; $\frac{\partial w}{\partial y} = x+z$

$\frac{\partial u}{\partial z} = 1$; $\frac{\partial v}{\partial z} = 2z$; $\frac{\partial w}{\partial z} = x+y$

$$\Rightarrow J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & x+y \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & x+z & x+y \end{vmatrix} (2) \qquad R_1 \longrightarrow R_1+R_2+R_3$$

$$= \begin{vmatrix} x+y+z+1 & x+y+z+1 & x+y+z+1 \\ x & y & z \\ y+z & x+z & x+y \end{vmatrix} = 0.$$

$u = (x+y+z)^2 \Rightarrow x^2+y^2+z^2+2(xy+yz+zx)$

$$\boxed{u^2 = v+2w}$$

(Q) $x = r\cos\theta;\ y = r\sin\theta;\ z = z$, find Jacobian of $J\left(\frac{x,y,z}{r,\theta,z}\right)$

$\underbrace{x = r\cos\theta;\ y = r\sin\theta}$ → cylindrical polar curves.

(A) $$J\left(\frac{x,y,z}{r,\theta,z}\right) = \begin{vmatrix} dx/dr & dy/dr & dz/dr \\ dx/d\theta & dy/d\theta & dz/d\theta \\ dx/dz & dy/dz & dz/dz \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(\cos\theta\, r\cos\theta + r\sin\theta\cdot\sin\theta)$$
$$= r\cos^2\theta + r\sin^2\theta$$
$$= r(1) = r$$

(Q) $x = r\sin\theta\cos\phi;\ y = r\sin\theta\sin\phi;\ z = r\cos\theta$ → spherical polar curves.

(A) $dx/dr = \sin\theta\cos\phi \qquad dy/dr = \sin\theta\sin\phi \qquad \partial z/\partial r = \cos\theta$

$dx/d\theta = r\cos\theta\cos\phi \qquad \partial y/\partial\theta = r\cos\theta\sin\phi \qquad \partial z/\partial\theta = -r\sin\theta$

$dx/d\phi = -r\sin\theta\sin\phi \qquad \partial y/\partial\phi = r\sin\theta\cos\phi \qquad \partial z/\partial\phi = 0$

$$J\left(\frac{x,y,z}{r,\theta,\phi}\right) = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{vmatrix}$$

$$= \sin\theta\cos\phi(r^2\sin^2\theta\cos\phi) - \sin\theta\sin\phi(-r^2\sin^2\theta\sin\phi)$$
$$= r^2\sin^3\theta\cos^2\phi + r^2\sin^3\theta\sin^2\phi$$
$$= r^2\sin^2\theta(1) = r^2\sin^2\theta$$

(Q) If $u = x^2 - 2y^2;\ v = 2x^2 - y^2;\ x = r\cos\theta;\ y = r\sin\theta$; find $\frac{\partial(u,v)}{\partial(r,\theta)}$

$$\begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} \qquad \begin{vmatrix} \cos\theta & -r \\ \sin\theta & r\cos \end{vmatrix}$$

(A) u and v are composite f^n of r and θ.

$$\frac{\partial(u,v)}{\partial(\cancel{x,y}\ r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)} \qquad J = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} 2x & -4y \\ 4x & -2y \end{vmatrix}$$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -4y \\ 4x & -2y \end{vmatrix}\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = (-4xy + 16xy)(r)$$
$$= (12xy\cdot r) = 6r^3\sin\cos$$

(Q) If $u = x^2+3y^2-z^2$; $v = \cancel{2x^2-y^2}\ 4x^2yz$; $w = 2z^2-xy$, find Jacobian of $J\left(\frac{u,v,w}{x,y,z}\right)$ at $(1,-1,0)$

(A) $\frac{\partial u}{\partial x} = 2x\,(1) = 2$ $\quad \frac{\partial v}{\partial x} = 8xyz\,(8\times -1,0) = 0$ $\quad \frac{\partial w}{\partial x} = -y\,(1)$

$\frac{\partial u}{\partial y} = 6y\,(1) = -6$ $\quad \frac{\partial v}{\partial y} = 4xz\,(4\times 1\times 0) = 0$ $\quad \frac{\partial w}{\partial y} = -x\,(-1)$

$\frac{\partial u}{\partial z} = -2z\,(1) = -2$ $\quad \frac{\partial v}{\partial z} = 4x^2y\,(4\times 1\times -1) = -4$ $\quad \frac{\partial w}{\partial z} = 4z\,(0)$

$$\begin{vmatrix} 2 & -6 & -2 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix}$$

$= 2(-4)+6(4)-2\cancel{(0)}$

$= -8+24 = \cancel{32}\ 16$

Homework

(Q) ① $x = r\cos\theta$; $y = r\sin\theta$; ST $JJ' = 1$

② $u+v = e^x\cos y$; $u-v = e^x\sin y$; $J\left(\frac{u,v}{x,y}\right)$

③ $u = \frac{x}{y-z}$; $v = \frac{y}{z-x}$; $w = \frac{z}{x-y}$, find $J\left(\frac{u,v,w}{x,y,z}\right)$

④ $u = \frac{x}{\sqrt{1-r^2}}$; $v = \frac{y}{\sqrt{1-r^2}}$; $r^2 = x^2+y^2$; ST $J\left(\frac{u,v}{x,y}\right) = \frac{1}{(1-r^2)^2}$

5. $u = x+y$, $v = \frac{x}{x+y}$; $J\frac{(u,v)}{(x,y)} = ?$

(6) $u = \frac{yz}{x}$; $v = \frac{zx}{y}$; $w = \frac{xy}{z}$; $J\left(\frac{u,v,w}{x,y,z}\right) = ??$

(7) $x = a(u+v)$; $y = b(u-v)$; $u = r^2\cos 2\theta$; $v = r^2\sin 2\theta$. $\frac{\partial(x,y)}{\partial(r,\theta)} = ??$

(A) $\frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(r,\theta)}$

⇒ **Maxima and Minima:**

$y = f(x) \rightarrow f'(x) = 0$ critical/stationary points

$f''(x) < 0 \rightarrow$ maxima

$f''(x) > 0 \rightarrow$ minima

Maxima: $z = f(x,y)$ be a f^n of two independent variables x and y, then we use the following notation for the derivatives of z

$$p = \frac{\partial z}{\partial x};\ q = \frac{\partial z}{\partial y};\ r = \frac{\partial^2 z}{\partial x^2};\ s = \frac{\partial^2 z}{\partial x \partial y};\ t = \frac{\partial^2 z}{\partial y^2}$$

The necessary and sufficient condition for the f^n $z = f(x,y)$ to have extremum is $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

Working rule:

To find the extremum of $z = f(x,y)$.

1. Find the stationary points $\left(\frac{\partial f}{\partial x} = 0;\ \frac{\partial f}{\partial y} = 0\right)$
2. Find the derivatives $r = \frac{\partial^2 z}{\partial x^2}$; $s = \frac{\partial^2 z}{\partial x \partial y}$; $t = \frac{\partial^2 z}{\partial y^2}$ at the stationary points
3. If at the stationary points
   (i) $rt - s^2 > 0$, $r < 0$, f^n is ~~minimum~~, maximum
   (ii) $rt - s^2 > 0$; $r > 0$, f^n is minimum
   (iii) $rt - s^2 = 0$, is the ~~point of inflection~~ saddle point.
   (iv) $rt - s^2 < 0$; further study is required

(Q) Find the extreme values of $u = x^3y^2(1-x-y)$

(A) $u = x^3y^2 - x^4y^2 - x^3y^3$

$\partial f/\partial x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0.$

$\Rightarrow y^2 x^2 [3 - 4x - 3y] = 0$

$\hookrightarrow 4x + 3y = 3$ — ①

$\partial u/\partial y = 2x^3y - 2x^4y - 3y^2x^3$

$= yx^3 [2 - 2x - 3y] = 0$

$2x + 3y = 2$ — ②

$2x + 3y = 2$
$(-)\ 4x + 3y = 3$
$-2x = -1$

$\boxed{x = 1/2}$; $\boxed{y = 1/3}$

$2 \times 1/2 + 3y = 2$
$3y = 1$

Stationary points
$(0,0), (0,1), (3/4, 0), (0, 2/3),$
$(1,0), (1/2, 1/3)$

$r = \frac{\partial^2 z}{\partial x^2} \Rightarrow \frac{\partial}{\partial x}\left(3x^2y^2 - 4x^3y^2 - 3x^2y^3\right)$

$\Rightarrow 6xy^2 - 12x^2y^2 - 6xy^3 \quad (1/2, 1/3)$

$\Rightarrow \cancel{6}\times 1/2 \times 1/9 - 12 \times 1/4 \times 1/9 - 6 \times 1/2 \times 1/27$

$\Rightarrow 1/3 - 1/3 - 1/9 = -1/9$

$s = \frac{\partial^2 z}{\partial x \cdot \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(2x^3y - 2x^4y - 3y^2x^3\right) \quad (1/2, 1/3)$

$\Rightarrow 6x^2y - 8x^3y - \cancel{6}\,9x^2y^2$

$= 6 \times 1/4 \times 1/3 - 8 \times 1/8 \times 1/3 - 9 \times 1/4 \times 1/9$

$= 1/2 - 1/3 - 1/4$

$= \frac{6-4-3}{12} = \frac{6-7}{12} = -1/12$

$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(2x^3y - 2x^4y - 3y^2x^3\right)$

$= 2x^3 - 2x^4 - 6yx^3 \quad (1/2, 1/3)$

$= 2 \times 1/8 - 2 \times 1/16 - 6 \times 1/3 \times 1/8$

$= -1/8$

$rs - t^2 = -1/9 \cdot -1/12 - 1/64$

$= 1/108 - 1/64 = > 0 ; r < 0 \longrightarrow$ maximum

$= \cancel{\frac{16}{1728}} \Rightarrow$

2 | 64, 108
2 | 32, 54
16 | 16, 27

$27 \times 32 \times 54$

32
54
0128
1690
1728
~~108~~

(Q) $u = xy(a-x-y)$; $f(x,y) = \sin x + \sin y + \sin(x+y)$

(A)

Lagrange's method ~~of~~ of undetermined multipliers:

~~By using~~ Let $f(x,y,z)$ be a function of three independent variables, x,y,z. subject to the constraint $\phi(x,y,z)=c$.

Working rule:

$F(x,y,z) = f(x,y,z) + \lambda(\phi(x,y,z)-c)$ where λ is a parameter.

Eliminating λ from $\frac{\partial F}{\partial x}=0$; $\frac{\partial F}{\partial y}=0$, $\frac{\partial F}{\partial z}=0$ and $\phi(x,y,z)=c$, we get a stationary points.

(Q) Temperature at any point in space is given by $T = 400xyz^2$, find the highest temperature on the unisphere $x^2+y^2+z^2=1$.

(A) $F(x,y,z) = 400xyz^2 + \lambda(x^2+y^2+z^2-1)$

$\partial F/\partial x = 400yz^2 + \lambda(2x) = 0 \Rightarrow \boxed{\lambda = \frac{-200yz^2}{x}}$

$\partial F/\partial y = 400xz^2 + \lambda(2y) = 0 \rightarrow \boxed{\lambda = \frac{-200xz^2}{y}}$

$\partial F/\partial z = 0 \Rightarrow 800xyz + \lambda(2z) = 0$

$\boxed{\lambda = -400xy.}$

$\Rightarrow \frac{-200yz^2}{x} = \frac{-200xz^2}{y} = \boxed{y^2=x^2} \rightarrow \boxed{y = \pm x}$

$\Rightarrow \frac{-\cancel{200}xz^2}{y} = -\overset{2}{\cancel{400}}xy$

$\boxed{z^2 = 2y^2} \longrightarrow \boxed{z = \pm\sqrt{2}y}$

$x^2+y^2+z^2=1$

$x^2+x^2+2x^2=1$

$4x^2=1$

$\boxed{x = \pm\frac{1}{2}} = y$; $\boxed{z = \frac{\sqrt{2}}{\cancel{2}} = \pm\frac{1}{\sqrt{2}}}$

$x = \frac{1}{2}, y = \frac{1}{2}; z = \frac{1}{\sqrt{2}}$

highest temperature $= 400xyz^2$

$= \overset{50}{\cancel{400}} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

$= 50.$

(Q) Find the dimensions of a rectangular parallelopiped that can be enclosed in the ellipse $x^2/a^2+y^2/b^2+z^2/c^2=1$

(A) Let $x,y,z \to$ length, breadth, height of the parallelopiped

Volume of parallelopiped $= 2x \cdot 2y \cdot 2z$
$= 8xyz$

$x^2/a^2+y^2/b^2+z^2/c^2=1$

$F(x,y,z) = 8xyz+\lambda\left(x^2/a^2+y^2/b^2+z^2/c^2-1\right)$

$\partial F/\partial x = 8yz+\lambda \cdot \dfrac{2x}{a^2} = 0$

$\lambda = -\dfrac{8yz \cdot a^2}{2x} = -\dfrac{4a^2yz}{x}$

$\partial F/\partial y = 8xz+\lambda\left(2y/b^2\right) = 0$

$\lambda = -\dfrac{8xz \cdot b^2}{2y} = -\dfrac{4xzb^2}{y}$

$\partial F/\partial z = 8xy+\lambda\left(\dfrac{2z}{c^2}\right) = 0$

$\lambda = -\dfrac{8xy \cdot c^2}{2z} = -\dfrac{4c^2xy}{z}$

$-\dfrac{4a^2y\cancel{z}}{x} = -\dfrac{4x\cancel{z}b^2}{y} \Rightarrow a^2y^2 = b^2x^2 \Rightarrow bx = ay$, $y = bx/a$; $x/y = a/b$

$-\dfrac{4xzb^2}{y} = -\dfrac{4c^2xy}{z} \Rightarrow z^2b^2 = y^2c^2 \Rightarrow bz = cy$; $z/y = c/b \Rightarrow y/z = b/c$

$x/y \cdot y/z = a/b \cdot b/c$

$x/z = a/c$

$cx = az$

$x^2/a^2+y^2/b^2+z^2/c^2=1$

$= x^2/a^2+b^2x^2/a^2+c^2x^2/a^2=1$

$\dfrac{x^2(1+b^2+c^2)}{a^2} = 1 \Rightarrow x^2(1+b^2+c^2) = a^2$

$x = \dfrac{a}{\sqrt{1+b^2+c^2}}$; $y = \dfrac{b}{\sqrt{1+b^2+c^2}}$; $z = \dfrac{c}{\sqrt{1+b^2+c^2}}$

$8xyz = \dfrac{8abc}{\left(\sqrt{1+b^2+c^2}\right)^3}$

$3x^2/a^2 = 1$

$3x^2 = a^2$

$x^2 = a^2/3$

$x = \pm a/\sqrt{3}$

$y = b/\sqrt{3}$

$z = c/\sqrt{3}$

$\dfrac{8 \times abc}{3\sqrt{3}}$

(Q) Find the point on the plane $2x+3y-z=5$ which is the nearest to the origin.
(plane)

(A) Let the point $P(x,y,z)$ on the plane, then the distance from the origin (OP)

$OP = \sqrt{x^2+y^2+z^2}$

$OP^2 = x^2+y^2+z^2$

$F(x,y,z) = x^2+y^2+z^2+\lambda(2x+3y-z-5)$

$\partial F/\partial x = 2x+\lambda(2) = 0$

$2\lambda = -2x$

$\lambda = -\cancel{2}x/\cancel{2} = -x$ — ①

$\partial F/\partial y = 2y+3(\lambda) = 0$

$3\lambda = -2y$

$\lambda = -2y/3$ — ②

$\partial F/\partial z = 2z+\lambda(-1) = 0$

$\lambda = 2z$ — ③

$-x = 2z = -2y/3$ $\quad -x/2$

$2x+3\left(\frac{-3x}{-2}\right)-\left(\frac{-2y}{3(2)}\right)=5$

$\cancel{2x+\frac{9x}{2}+\frac{y}{3}}$ $\quad 2x+\frac{9x}{2}+\frac{x}{2}=5$

$2x+5x=5$

$7x=5$

$x=5/7$

$z = -5/14$

$y = \frac{3\times5}{7\times2} = 15/14$

$P = (5/7, 15/14, -5/14)$

$\frac{49}{\times 4}$

$OP = \sqrt{25/49+225/196+25/196}$

$\sqrt{\frac{100+25+225}{196}}$

$\sqrt{\frac{350}{196}}$

225
250
100
350

$OP = \frac{\sqrt{350}}{14}$

$= \frac{5\sqrt{14}}{14}$

$= \frac{5}{\sqrt{14}}$ //

(Q) A rectangular box open at the top is to have the volume of $32\,ft^3$, find the dimension of the box such that the total surface area is minimum.

(A) $V = 32\,ft^3$; let $P(x,y,z)$

$A = \cancel{2(xy+yz+zx)}$ $2xy+2yz+2x$

$V = xyz$

$F(x,y,z) = 2xy + 2yz + zx + \lambda(xyz - 32)$

$\partial F/\partial x = 2y + z + \lambda yz = 0$

$\lambda(yz) = -2y - z$

$\lambda = \dfrac{-2y-z}{yz}$

$\partial F/\partial y = 2x + 2z + \lambda(xz) = 0$

$\lambda(xz) = -2x - 2z$

$\lambda = \dfrac{-2x-2z}{xz}$

$\partial F/\partial z = 2y + x + \lambda(xy) = 0$

$\lambda(xy) = -2y - x$

$\lambda = \dfrac{-2y-x}{xy}$

$\dfrac{-2y-z}{\cancel{yz}} = \dfrac{-2y-x}{\cancel{xy}} = -2\cancel{xy} - \cancel{xz} = -\cancel{2yz} - \cancel{xz}$

$\boxed{x = z}$

$\dfrac{-2x-2z}{\cancel{x}z} = \dfrac{-2y-x}{\cancel{x}y} = -2\cancel{xy} - \cancel{2yz} = -\cancel{2yz} - \cancel{xz}$

$+2y = z$

$\boxed{y = z/2}$

$xyz = 32$

$z\cdot\left(\dfrac{z}{2}\right)\cdot z = 32$

$z^3/2 = 32$

$z^3 = 64$

$\boxed{z = 4ft}$

$\boxed{x = 4ft}$

$\boxed{y = 2ft}$

$SA = 8 + 16 + 16 + 8$

$= 40$

(Q) Represent 24 as the sum of 3 parts such that the product of the first part, square of the second part and cube of the third part is max^{m}.

(A) $x \cdot y^2 \cdot z^3 = f(x,y,z)$

$F(x,y,z) = xy^2z^3 + \lambda(x+y+z-24)$

$\partial F/\partial x = y^2z^3 + \lambda \Rightarrow \lambda = -y^2z^3$

$\partial F/\partial y = x2y \cdot z^3 + \lambda \Rightarrow \lambda = -2xyz^3$

$\partial F/\partial z = 3z^2 \cdot xy^2 + \lambda \Rightarrow \lambda = -3xy^2z^2$

$\cancel{-}y^{\cancel{2}}\cancel{z^3} = \cancel{-}2x\cancel{y}\cancel{z^3}$

$y = 2x \rightarrow x = y/2$

$-2\cancel{xy}z^3 = -3\cancel{xy}^2z^2$

$2z = 3y$

$\cancel{y} \cdot z = 3y/2$

$y/2 + y + 3y/2 = 24$

$3y = 24$

$\boxed{y = 8}$

$\boxed{x = 4}$; $\boxed{z = \frac{3 \times \cancel{8}^{4}}{\cancel{2}} = 12}$

$xy^2z^3 = 4 \times 64 \times 1728$

2

Unit 5: Numerical Methods:

Interpolation:
It is the technique of finding the dependent variable y for the intermediate values of independent variable (x) of the given data.

Extrapolation:
Estimating the dependent variable for the independent variable outside the range is called extrapolation.

Interpolation techniques depend on 3 types of differences.
(1) Forward difference,
(2) Backward differences } → syllabus
(3) Central difference ⊗

Forward difference:
Let $y=f(x)$ be defined for <u>equal spaced</u> ~~of~~ values of x as

x_0 x_0+h $x_0+2h \dots x_0+nh$ ---
y y_1 $y_2 \dots y_n$ ---

then the differences $y_1-y_0, y_2-y_1, y_3-y_2$ --- y_n-y_{n-1} are called first order forward differences dended by $\Delta y_0, \Delta y_1, \Delta y_2, \Delta y_{n-1}$

Second order forward differences: $\Delta^2 y_0 = \Delta(\Delta y_0)$

$= \Delta(y_1-y_0) = \Delta y_1 - \Delta y_0$
$= y_2-y_1-(y_1-y_0)$
$= y_2-2y_1+y_0.$

binomial expression

$\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0.$

$\Delta^4 y_1 = y_5 - 4y_4 + 6y_3 - 4y_2 + y_1$

$\Delta^n y_0 = y_n\,{}^nC_0 - y_{n-1}\,{}^nC_1 + - {}^nC_2 --- (-1)^n \cdot {}^nC_n y_0$

Backward difference:
Let the f^n $y=f(x)$ given by

x_0 x_0+h x_0+2h · --- x_0+nh
y_0 y_1 y_2 · · · y_n

then the differences y_1-y_0, y_2-y_1, y_n-y_{n-1} are called first order backward differences denoted by ∇y_1,
nabla (or) del.

NOTE: $\Delta f(x) = f(x+h) - f(x)$.

$\nabla f(x) = f(x) - f(x-h)$

$\nabla^2 y_2 = \nabla(\nabla y_2)$

$= \nabla(y_2 - y_1)$

$= \nabla y_2 - \nabla y_1$

$= (y_2 - y_1) - (y_1 - y_0)$

$= y_2 - 2y_1 + y_0 = \Delta^2 y_0$

$$\boxed{\Delta^r y_k = \nabla^r y_{r+k}}$$

(Q) $\Delta \tan^{-1} x$

$\Delta f(x) = f(x+h) - f(x)$

$\Delta \tan^{-1} x = \tan^{-1}(x+h) - \tan^{-1}(x)$

$= \tan^{-1}\left(\frac{\cancel{x}+h-\cancel{x}}{1+(x+h\cdot x)}\right)$

$= \tan^{-1}\left(\frac{h}{1+x^2+xh}\right)$

(Q) $\Delta \log(2x+3)$

$\Delta f(x) = f(x+h) - f(x)$

$\Delta \log(2x+3) = \log(2x+3+2h) - \log(2x+3)$

$= \log\left(\frac{2x+3+2h}{2x+3}\right)$

$= \log\left(\frac{2h}{2x+3}\right)$

(Q) ∇e^{3x+2} with $h=1$

$\nabla f(x) = f(x) - f(x-h)$

$= e^{3x+2} - e^{3(x-h)+2}$

$= e^{3x+2} - e^{3x+2-3h}$

$= e^{3x+2} - \frac{e^{3x+2}}{e^{3h}}$

$= \frac{e^{3x+2+3h} - e^{3x+2}}{e^{3h}} = \frac{e^{3x+5} - e^{3x+2}}{e^3}$

Forward difference table:

let the fn $y = f(x)$ be given by

x_0	$x_0+h=x_1$	x_2	x_3	x_4
y_0	y_1	y_2	y_3	y_4

x	y	Δ	Δ^2	Δ^3	Δ^4
x_0	y_0				
		$y_1-y_0=\Delta y_0$			
x_1	y_1		$\Delta^2 y_0$		
		$y_2-y_1=\Delta y_1$		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		$y_3-y_2=\Delta y_2$		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		$y_4-y_3=\Delta y_3$			
x_4	y_4				

The differences $y_0, \Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0$ are called leading entities in forward difference table

Backward difference table:

x_0	$x_0+h=x_1$	x_2	x_3	x_4
y_0	y_1	y_2	y_3	y_4

x	y	∇	∇^2	∇^3	∇^4
x_0	y_0				
		$y_1-y_0=\nabla y_1$			
x_1	y_1		$\nabla^2 y_2$		
		$y_2-y_1=\nabla y_2$		$\nabla^3 y_3$	
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$
		$y_3-y_2=\nabla y_3$		$\nabla^3 y_4$	
x_3	y_3		$\nabla^2 y_4$		
		$y_4-y_3=\nabla y_4$			
x_4	y_4				

leading entities in backward difference table

x: 10 20 30 40

y: 66 42 85 90

$\Delta^{\cancel{2}}\cancel{85}, \nabla^2 \cancel{42}\ 85.$
85

$\Delta'_{85} \to 5$

$\nabla^2 85 \to 67$

x	y	Δ	Δ^2	Δ^3
10	66			
		-24		
20	42		67.	
				-105.
		43		
30	85			
			-38	
		5		
40	90			

Newton's Forward ~~difference~~ interpolation Formula (NFIF)

To compute y for starting table values of x, we use NFIF, formula is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$\boxed{p = \frac{x - x_0}{h}}$$ $h \to$ step length

Newton's Backward interpolation Formula (NBIF)

To ~~find~~ estimate y at the end values of x, we use ~~we~~ NBIF, formula

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$$\boxed{p = \frac{x - x_n}{h}}$$

When the value of x is in the middle → either forward/backward

(Q) In an experiment, output 5 are regarded for input x from 1.0 to 3.5 at 0.5 intervals, estimate the values of y by using appropriate interpolation formula for the inputs $x=1.2$; $x=3.4$, $x=3.8$; x is 1, 1.5, 2, 2.5, 3.

x	1	1.5	2.0	2.5	3.0	3.5
y	277	166	146	130	115	102

x	y	Δy	Δ^2	Δ^3	Δ^4	Δ^5
1	277					
		-111				
1.5	166		91			
		-20		-87		
2.0	146		4		84	
		-16		-3		-80
2.5	130		1		4	
		-15		1		
3.0	115		2			
		-13				
3.5	102					

NFIF = $y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$

① $x = 1.2$

$x_0 = 1$

$y_0 = 277$

$p = \frac{x - x_0}{h} = \frac{1.2-1}{0.5} = \frac{0.2}{0.5} = 0.4$

$= 277 + 0.4 \times -111 + \frac{0.4\times(0.4-1)}{2!} 91 + \frac{0.4(0.4-1)(0.4-2)}{3!} -87 + \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{4!} 84 + \frac{0.4(0.4-1)(0.4-2)(0.4-3)(0.4-4)(-80)}{5!}$

$= 277 - 44.4 - 10.92 - 5.568 - 3.494 -$ ~~...~~

$= 210.2221$

2.396

② $x = 3.4$

$x_n = 3.5$

$y_n = 102$

$p = \frac{3.4-3.5}{0.5} = \frac{-0.1}{0.5}$

③ $x = 3.8$

$p = \frac{3.8-3.5}{0.5} = \frac{0.3}{0.5} = 0.6$

$y = 102 + 0.6 \times -13 + \frac{0.6\times(0.6+1)\times 2}{2!} + \frac{0.6\times(0.6+1)(0.6+2)\times 1}{3!} + \frac{0.6\times(0.6+1)\times(0.6+2)\times(0.6+3)\times 4}{4!} + \frac{0.6\times(0.6+1)\times(0.6+2)\times(0.6+3)\times(0.6+4)\times(-80)}{5!}$

$= 102 - 7.8 + 0.96 + 0.416 + 1.4975 - 27.55$

$= 69.52$

(Q) The details given below are monthly income of 2000 adults in a colony using interpolation estimate the no. of person who have monthly income b/w 8000 - 10000.

(A)

Monthly income	Below 5K	5K-10K	10K-15K	15K-20K	20K-25K
no. of persons.	535	660	470	270	65

Monthly income (x)	no. of persons (y)	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
5000	535				
		660			
10000	1195		-190		
		470		-10	
15000	1665		-200		+5
		270		-5	
20000	1935		-205		
		65			
25000	2000				

$$y(8000) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$p = \frac{x - x_0}{h} = \frac{8000-5000}{5000}$$

$$= \frac{3\cancel{000}}{5\cancel{000}} = 0.6.$$

$$= 535 + 0.6 \times 660 + \frac{0.6 \times (0.6-1) \times -190}{2} + \frac{(0.6)\times(0.6-1)(0.6-2) \times -10}{3!} + \frac{0.6 \times (0.6-1) \times (0.6-2) \times (0.6-3) \times 5}{4!}$$

$$= 535 + 396 + 22.8 - 0.56 - 0.168$$

$$= 953.072.$$

Diff: $1195 - 953$ income b/w 8000 & 10000

$= 242$

$$\begin{array}{r} 1195 \\ -953 \\ \hline 0242 \end{array}$$

Hmwk

(Q) The details regarding marks secured by 280 candidates in an examination are given by the following table, using interpolation estimate the no. of people having marks between 45 and 50.

Marks	below 30.	30-40	40-50	50-60	60-70	~~80-90~~ 70-80
no. of candidates	35	49	62	74	40	20

(a) Find the interpolated polynomial for the following data, hence find y(12).

x (values) → 3   5   7   9   11

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4$

y (values) → 1   3   8   16   27

$y_0 \quad y_1 \quad y_2 \quad y_3 \quad y_4$

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
3	1				
		+2			
5	3		3		
		5		0	
7	8		3		0
		8		0	
9	16		3		
		11			
11	27				

NFIF : $y = y_0 + \Delta y_0 \cdot p + \frac{\Delta^2 y_0 \, p(p-1)}{2!} + \frac{p(p-1)(p-2)\Delta^3 y_0}{3!} + \dots$

$p = \frac{x - x_0}{h} = \frac{x-3}{2}$

$y_0 = 1$

$y = 1 + \cancel{2}\frac{(x-3)}{\cancel{2}} + \frac{3\frac{(x-3)}{2} \times \left(\frac{x-3}{2} - 1\right)}{2!} \quad \frac{x-5}{2}$

$= 1 + x - 3 + \frac{3(x-3)(x-5)}{8}$ → collect x^2 coefficient

$y(x) = \cancel{8/4} \; \cancel{1 + x - 3 + 3} \rightarrow \frac{3}{8}x^2 - 2x + \frac{29}{8}$

$y(12) = \frac{3}{8} \times 144 - 2 \times 12 + \frac{29}{8}$

$= 54 - 24 + 3.625$

$= 30 + 3.625 = 33.625$

(Q) Using NBIF, find the cubic polynomial satisfying $f(-4) = -2.5$; $f(-2) = 1$; $f(0) = 3$; $f(2) = 29$; $f(4) = 127$; Hence find $f(3)$ and $f(5)$

Hedd

(A)

x	y	∇y_0	$\nabla^2 y_0$	$\nabla^3 y_0$	$\nabla^4 y_0$
-4	-2.5				
		3.5			
-2	1		-1.5		
		2		25.5	
0	3		24		$\boxed{22.5}$
		26		$\boxed{48}$	
2	29		$\boxed{72}$		
		$\boxed{98}$			
4	$\boxed{127}$				

$p = \frac{x_n - x}{h} = \frac{4-x}{2}$; $y_n = 127$

$$\left(\frac{x-4}{2}\right)\left(\frac{x-2}{2}\right)\left(\frac{x}{2}\right) \qquad \left(\frac{x-4}{2}\right)\left(\frac{x-2}{2}\right) \times 72$$

$$\left(\frac{x-4}{2}\right)\left(\frac{x-2}{2}\right)\left(\frac{x}{2}\right)\left(\frac{x+2}{2}\right) \times \frac{22.5}{24} \qquad x-4+2$$

$$y = y_n + p\nabla y_0 + \frac{p(p+1)\nabla^2 y_0}{2!} + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_0 + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_0$$

$$= 127 + \left(\frac{\cancel{4-x}\ x-4}{2}\right) \times 98 + \frac{\left(\frac{\cancel{4-x}\ x-4}{2}\right) \times \left(\cancel{4-x}\ \frac{x-4+1}{2}\right)\left(\frac{x-4+2}{2}\right) \times 48}{3!} + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-4+1}{-2}\right) \times 72}{2!}$$

$$+ \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-4+1}{2}\right)\left(\frac{x-4+2}{2}\right)\left(\frac{x-4+3}{2}\right) \times 22.5}{4!}$$

$x^2 - 6x + 8$

$x(x-2)(x-4)$; $x^3 - 6x^2$

$x[x^2 - 6x + 8]$

$$= 127 + 49(x-4) + (x-4)(x-2)(x) + 9(x-4)(x-2) + 0.058(x-4)(x-2)(x)(x+2)$$

$$= 127 + 49x - 196 + 9x^2 - 54x + 72 + x^3 - 6x^2 + 8x +$$

$$0.058x^4 - 0.348x^3 + 0.464x^2 +$$

$$0.116x^3 - 0.696x^2 + 0.928x.$$

$(x+2)(x^3 - 6x^2 + 8x)$

$x^4 - 6x^3 + 8x^2 +$

$2x^3 - 12x^2 + 16x$

$$= 0.768x^3 + 0.058x^4 + 2.768x^2 + 3.928x + 3.$$

$3x^2$

$-5x+$

$3x$

$f(3) = 65.13$

$f(5) = 224.09$

(Q) n^{th} order differences of a polynomial of degree n are constant, all $(n+1)$ order differences are 0.

Numerical differentiation

Let the f^n $y=f(x)$ be defined

x	x_0	x_0+h	x_0+2h	---	x_0+nh
y	y_0	y_1	y_2	---	y_n

The process of finding the derivatives of y is called numerical differentiation

Newton's forward interpolation formula to find the derivatives for the specified values of x are

$$y' = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} \cdots\right]$$

$$y'' = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

$$y''' = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \frac{7}{4}\Delta^5 y_0 \cdots\right]$$

Newton's backward interpolation formula are:

$$y' = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \cdots\right]$$

$$y'' = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \cdots\right]$$

$$y''' = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \frac{7}{4}\nabla^5 y_n + \cdots\right]$$

(Q) A rod is rotating in a plane, the following table ~~gives~~ gives the angle θ in radian through which the rod has turned for various values of time t in seconds.

t	0	0.2	0.4	0.6	0.8	1.0	1.2
θ	0	0.12	0.49	0.12	2.02	3.2	4.67

Find angular velocity and angular acceleration at $t = 0.2$, 1.0 and 1.2.

Homework

t	θ	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$	$\Delta^5 y_0$	
0	0						
		0.12					
0.2	0.12		0.25				
		+0.37		−0.99			
0.4	0.49		−0.74		4		
		−0.31		3.01		−10	
0.6	0.12		2.27		6		20
		1.9		−2.99		10	
0.8	2.02		−0.72		4		
		1.18		1.01			
1.0	3.2		0.29				
		1.47					
1.2	4.67						

$t = 0.2.$ $d\theta/dt = 1/h = 0.2.$

$$y' = \frac{1}{0.2}\left[\Delta\theta_0 - \frac{\Delta^2\theta_0}{2} + \frac{\Delta^3\theta_0}{3} - \frac{\Delta^4\theta_0}{4} + \frac{\Delta^5\theta_0}{5}\right]$$

$$= \frac{1}{0.2}\left[0.37 + \frac{0.74}{2} + \frac{3.01}{3} + \frac{6}{4} + 10/5\right]$$

$$= 1/0.2\,[0.74 + 1.000 + 1.5 + 2]$$

$$= 1/0.2\,[5.24] = 26.2.$$

$$y'' = \frac{1}{(0.2)^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + 11/12\,\Delta^4 y_0 - 5/6\,\Delta^5 y_0 + \ldots\right]$$

$$= \frac{1}{(0.2)^2}\left[-0.74 - 3.01 + 11/12 \times -6 - 5/6 \times 10\right]$$

$$= \frac{1}{0.04}\left[-0.74 - 3.01 - 5.5 - 8.3\right]$$

$$= \frac{1}{0.04} \times -17.55 = -438.75.$$

$t = 1.0.$

$$y' = \frac{1}{0.2}\left[\Delta\theta_n + \frac{\Delta\theta_n^2}{2} + \frac{\Delta\theta_n^3}{3} + \ldots\right]$$

$$= \frac{1}{0.2}\left[1.18 - \frac{0.72}{2} - \frac{2.99}{3} - 6/4 - 10/5\right]$$

$$= \frac{1}{0.2}\left[1.18 - 0.36 - 0.99 - 1.5 - 2\right]$$

$$= \frac{1}{0.2}\left[-3.67\right] = -18.35$$

Interpolation with unequal intervals Lagrange's interpolation

Let a fn of $f(x)$ be defined as

x	x_0	x_1	x_2	----	x_n
y	y_0	y_1	y_2	----	y_n

(x need not be equally spaced)

$$y = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}\, y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}\, y_1 + \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}\, y_n$$

Inverse Lagrange's interpolation formula:

$$x = \frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)}\, x_0 + \frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)}\, x_1 + \cdots + \frac{(y-y_0)(y-y_1)\cdots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\cdots(y_n-y_{n-1})}\, x_n$$

(Q) Find $y(10)$ for the given data:

	x_0	x_1	x_2	x_3	x_4
x	1	2	5	8	9
y	5	-4	91	24	1613
	y_0	y_1	y_2	y_3	y_4

(A)
$$y = \frac{(x-2)(x-5)(x-8)(x-9)}{(1-2)(1-5)(1-8)(1-9)} \times 5 + \frac{(x-1)(x-2)(x-8)(x-9)}{(2-1)(2-5)(2-8)(2-9)} \times -4 +$$

$$\frac{(x-1)(x-2)(x-8)(x-9)}{(5-1)(5-2)(5-8)(5-9)} \times 91 + \frac{(x-1)(x-2)(x-5)(x-9)}{(8-1)(8-2)(8-5)(8-9)} \times 24 +$$

$$\frac{(x-1)(x-2)(x-5)(x-8)}{(9-1)(9-2)(9-5)(9-8)} \times 1613 \longrightarrow \text{Put } \boxed{x=10}$$

$$y = \cancel{0.357}\ \overset{10.78}{\cancel{8.92}} + \overset{2.85}{\cancel{4.097}} + 91 \cancel{\times}\ 68.57 + 5184.64$$

$$= \cancel{5350.56}\ 5211.7142$$

Numerical Integration: evaluate special f^n and improper integrals

Let the $f^n = f(x)$

$$\left.\begin{array}{cccccc} x & x_0 & x_0+h & x_0+2h & \cdots & x_0+nh \\ y & y_0 & y_1 & y_2 & \cdots & y_n \end{array}\right] y = f(x).$$

The process of evaluating the integral $\int_a^b f(x).dx$ (definite integral) is called numerical integration.

Simpson's $1/3$rd rule:

$$\int_a^b f(x).dx = h/3\,[y_0 + y_n + 2(y_2+y_4+y_6+\cdots) + 4(y_1+y_3+y_5+\cdots)]$$

Simpson's $3/8$th rule:

$$\int_a^b f(x).dx = 3h/8\,[y_0+y_n+2(y_3+y_6+y_9+\cdots)+3(y_1+y_2+y_4+y_5+\cdots)]$$

For eg: $y_n = y_6$ here y_6 should not be added.

Weddle's rule:

$$\int_a^b f(x).dx = 3h/10\,[y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6+$$
$$y_6+5y_7+y_8+6y_9+y_{10}+5y_{11}+y_{12}+$$
$$\cdots\cdots]$$

NOTE: For Simpson's $1/3$rd rule, the number of sub intervals must be even.

For Simpson's $3/8$th rule, the number of subintervals must be multiples of 3.

For Weddle's rule, the number of subintervals must be multiples of 6.

$h = \frac{b-a}{n}$.

For trignometric functions, always keep calculator in radian mode

(Q) Evaluate $\int_0^{\pi/2} \sqrt{\sin x}.dx$ by using (1) Simpson's $1/3$rd rule (2) Simpson's $3/8$ rule (3) Weddle's rule. by taking 7 ordinates $n=6$

(A) interval: $x_0 \; 1 \; x_1 \; 2 \; x_2 \; 3 \; x_3$ — ordinates

$$h = \frac{b-a}{n} = \frac{\pi/2 - 0}{6} = \pi/12$$

x	0	$\pi/12$	$2\pi/12$	$3\pi/12$	$4\pi/12$	$5\pi/12$	$6\pi/12 = \pi/2$
$y = \sqrt{\sin x}$	0	0.5087	0.7071	0.8408	0.9306	0.9828	1
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(1) $\int_a^b f(x)\cdot dx = \frac{\pi}{36}\Big[0+1+2[0.7071+0.9306]+4[0.5087+0.8408+0.9828]\Big]$

$= \pi/36\,[1+3.2754+9.3292]$

$= \pi/36\,[13.6046]$

$= 1.1866$

(2) $\int_a^b f(x)\cdot dx = \frac{3\times\pi}{\cancel{36}^{12}\times 8}[0+1+2[0.8408]+3[0.5087+0.7071+0.9306+0.9828]]$

$= \frac{3\times\pi}{\cancel{36}^{12}\times 8}[0+1+1.6816+9.3876]$

$= \frac{3\times\pi}{\cancel{36}^{12}\times 8}[12.0692] = \cancel{0.3947}\ 1.1848$

(3) $\int_a^b f(x)\cdot dx = \frac{3\times\pi}{12\times10}[0+5(0.5087)+0.7071+6(0.8408)+0.9306+5(0.9828)+1]$

$= \frac{3\times\pi}{12\times10}\times 15.14 = 1.18\cancel{49}849$

Homework
(Q) ① $\int_0^{1.2} \frac{x^3}{y}\cdot dx$ by taking 9 ordinates → 8 intervals, simpson's $\frac{1}{3}$rd rule

② $\int_0^{\pi} \frac{\sin x}{x}\cdot dx$ by taking 12 intervals; ③ $\int_0^1 \frac{dx}{1+x^2}$ by taking 6 intervals and compare with exact result.

simpson's $\frac{1}{3}$rd rule, weddle, simpson's $\frac{3}{8}$th rule

(Q) $\int_0^{0.6} e^{-x^2}\cdot dx$ by using weddle's rule.

(Q) The velocity of a train which starts from rest from a station and reaches the other station in 12 minutes is recorded. at the intervals of 2 minutes.

t(min)	~~0~~ 2	4	6	8	10
V (kmp/h)	20	36	60	64	42

Find the distance b/w the stations. $\boxed{426}$

t (hr)	0	2/60	4/60	6/60	8/60	10/60	12/60
v (kmph)	0	20	36	60	64	42	0

(A) Apply the 3 rules

(Q) A curve is described by the points mentioned below by using Simpson's rule, find the area under the curve bounded by the curve and the x axis included b/w $x = 0$ and $x = 4$

x	0	0.5	1.0	1.5	2.0	2.5	3	3.5	4
y	4.6	3.8	2.8	2.0	2.5	3.2	5.8	4	4

(A) $n = 8$ → simpson's $\frac{1}{3}$rd rule