UNIT-V
NUMERICAL METHODS

The study of numerical analysis is aimed at providing convenient methods for obtaining useful solutions to problems of advanced learning in science and technology. Besides this, analytical solutions to a large number of problems are not available. Numerical analysis provides methods for obtaining approximate solutions to such problems also. Numerical methods, in general, are of repeated nature. In each step, better approximation to the exact solution of the problem is obtained and the process is continued till accuracy to the desired degree is arrived at. In this unit some numerical methods to solve algebraic, transcendental and ordinary differential equations are discussed.

Algebraic and Transcendental Equation:
An expression of the form $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$, where a_i(i = 0, 1, 2, 3…) are constants $(a_0 \neq 0)$ and n is a positive integer, is called a polynomial in x of degree n. The polynomial f (x) =0 is called an algebraic equation of degree n.

Examples:

(1) $3x^2 - 6x + 9 = 0$

(2) $(2x-8)(x+5) = 0$

If $f(x)$ contains some other functions such as trigonometric, logarithmic, exponential etc., then $f(x) = 0$ is called a transcendental equation.

Examples:

(1) $2e^x - 4 = 0$

(2) $\cos x - xe^x = 0$

Root of an equation: The value 'a' of x which satisfies $f(x) = 0$ is called a root of $f(x) = 0$. Geometrically, a root of $f(x) = 0$ is that value of x where the graph of $y = f(x)$ crosses the x-axis.

Simple root: A number ξ is a simple root of $f(x) = 0$ if $f(\xi) = 0$, $f^{'}(\xi) \neq 0$.

Multiple root: A number ξ is a multiple root of multiplicity m if

$f(\xi)=f'(\xi)=.....=f^{(m-1)}(\xi) \quad \ni \quad f^{(m)}(\xi)\neq 0.$

i.e. $f(x)=(x-\xi)^m g(x)\,;\, g(\xi)\neq 0.$

The process of finding the roots of an equation is called solution of that equation. This is a problem of great importance in Scientific and Engineering studies. To solve higher order and transcendental equations no analytical methods exist. Such equations are solved by numerical methods.

Basic properties of equations:

1. If $f(x)$ is exactly divisible by x-a, then 'a' is a root of $f(x)=0$.

2. Every polynomial equation of the nth degree has only 'n' roots (real or imaginary).

3. In an equation with real coefficients, imaginary roots occur in conjugate pairs, i e., if $\alpha+i\beta$ is a root of the equation $f(x)=0$, then $\alpha - i\beta$ must also be its root.

4. Every polynomial equation of odd degree has at least one real root.

Descartes' rule of sign:

The number of positive real roots of polynomial equation $P_n(x) = 0$ cannot exceed the number of sign changes in $P_n(x)$ and the number of negative real roots of $Pn(x) = 0$ cannot exceed the number of sign changes in $P_n(-x)$.

Example: Equation $P_5(x) = 8x^5 + 12x^4 – 10x^3 + 17x^2 – 18x + 5 = 0$ has maximum of four positive real roots and maximum of one negative real root since $P_5(x)$ has four times changes in sign and $P_5(-x)$ has one-time changes in sign.

Intermediate value property: If $f(x)$ is continuous in the interval [a, b] and $f(a)$, $f(b)$ have different signs, then the equation $f(x)=0$ has at least one root between x=a and x=b.

Example:

1. Solve the equation $3x^3 - 4x^2 + x + 88 = 0$, one root being $2+\sqrt{7}i$.

Solution:

Since one root is $2+\sqrt{7}i$, the other root must be $2-\sqrt{7}i$

Therefore, the factors corresponding to these roots are $(x-2-\sqrt{7}i)$ and $(x-2+\sqrt{7}i)$

or $(x-2-\sqrt{7}i)\,(x-2+\sqrt{7}i) = (x-2)^2+7 = x^2-4x+11$

$x^2-4x+11$ is a divisor of $3x^3-4x^2+x+88=0$

$\Rightarrow 3x+8 = 0$

$\Rightarrow x = -8/3$

Iterative method:

This method is based on the idea of successive approximation. Starting with one (or) more initial approximation to the root we obtain a sequence of approximates $\{x_K\}$ which converges to the root. Further this method give only one root at a time.

Example: Consider $a_0 x^2 + a_1 x + a_2 = 0$

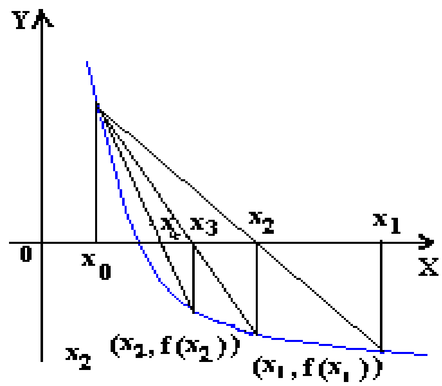
The successive approximations are given by, $x_{K+1} = -\dfrac{(a_0 x_K^2 + a_2)}{a_1}$; K = 0,1,2,.....

Initial approximation:

The common method used to obtain the initial approximation to the root is based upon the intermediate value theorem which states that "If f(x) is a continuous function in the closed interval [a, b] and if f(a) and f(b) are of opposite signs [i.e. f(a).f(b)<0] then the equation f(x) = 0 has atleast one simple root (or odd number of roots) lying between a and b". By using this property we take a (or) b as the initial approximation of the root to start the Iterative process.

Regula-Falsi method(Method of false position):

This is method of finding the real root of an equation $f(x) = 0$. Here we choose two points x_0, x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs i.e., the graph of y=f(x) crosses the x- axis between these points a root lies between x_0 *and* x_1 and consequently $f(x_0)f(x_1) < 0$.
Consider the equation f(x) = 0, where f(x) may be Algebraic (or) Transcendental which is known to have a root between $[x_0, x_1]$ then $f(x_0)f(x_1) < 0$.



Approximate the function f(x) by the straight line joining the points $[x_0, f(x_0)]$ and$[x_1, f(x_1)]$. Let this straight line cut the x-axis at x_2.

The equation of the chord joining the two points $[x_0, f(x_0)]$ and $[x_1, f(x_1)]$ is given by,

$$\frac{y-f(x_0)}{f(x_1)-f(x_0)}=\frac{x-x_0}{x_1-x_0} \quad \Rightarrow y-f_0=\frac{f(x_1)-f(x_0)}{x_1-x_0}(x-x_0) \qquad ---------(1)$$

The method consists in replacing the curve AB by means of the chord AB and taking the point of intersection of the chord with x-axis as an initial approximation to the root. The point of intersection is obtained by putting y = 0

$\therefore$ Equation (1) becomes, $0-f(x_0)=\dfrac{f(x_1)-f(x_0)}{x_1-x_0}(x_2-x_0)$

$$\Rightarrow x_2-x_0=-\frac{f(x_0)(x_1-x_0)}{f(x_1)-f(x_0)}$$

$$\Rightarrow x_2=x_0-\frac{(x_1-x_0)}{f(x_1)-f(x_0)}f(x_0)$$

$$x_2=\frac{x_0f(x_1)-x_1f(x_0)}{f(x_1)-f(x_0)} \qquad ----(2)$$

The formula given (2) is the first approximation to the root.

If $f(x_2)$ and $f(x_0)$ are opposite signs then replace x_1 by x_2 and draw a straight line joining $f(x_2)$ and $f(x_0)$ to find the new intersection point. If $f(x_2)$ and $f(x_0)$ are of same sign then x_0 is replaced by x_2 and repeat the above procedure till the root is found to the desired accuracy. Repeating the above process n times we get the general formula given by

$$x_{n+1}=\frac{x_{n-1}f(x_n)-x_nf(x_{n-1})}{f(x_n)-f(x_{n-1})} \qquad ---------(3) \qquad n=1,2,3,....$$

Note:

1. Regula-Falsi method has linear rate of convergence.
2. Geometrically this method is equivalent to replacing the curve y=f(x) by a chord that passes through the points A [a, f (a)] and B [b, f (b)] and intersection of chord with x-axis is obtained as solution of equation.

Examples:

1. Find a real root of the equation x^3-2x-5=0 by the method of false position correct to three decimal places.

Solution: Let f(x) = x^3 - 2x -5

f(1) = -6(-ve) f(2) = -1(-ve), f(3) = 16(+ve)

A root lies between 2 and 3

therefore $x_0 = 2$ $x_1 = 3$ $f(x_0) = -1$ $f(x_1) = 16$

By the method of false position,

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

We get, $x_2 = 2 + \frac{1}{17} = 2.0588$

Since $f(2.0588) = -0.3908(-ve)$, therefore the root lies between 2.0588 and 3.

Now, $x_0 = 2.0588,\ x_1 = 3,\ f(x_0) = -0.3908,\ f(x_1) = 16$

$$x_3 = 2.0588 - \frac{0.9412}{16.3908}(-0.3908) = 2.0813$$

Repeating this process, the successive approximations are

$x_4 = 2.0862,\ \ x_5 = 2.0915,\ \ x_6 = 2.0934,\ \ x_7 = 2.0941,\ \ x_8 = 2.0943$

Hence, the root is 2.094 correct to 3 decimal places.

2. Find the root of the equation $xe^x = \cos x$ using Regula - falsi method correct to four decimal places.

Solution:

Let $f(x) = \cos x - xe^x = 0$

f(0) = 1(+ve), f(1) = -2.17798(-ve)

root lies between 0 and 1

therefore, $x_0 = 0, x_1 = 1, f(x_0) = 1, f(x_1) = -2.17798$

By the method of false position,

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = 0 + \frac{1}{3.17798} = 0.31467$$

Since f(0.31467) = 0.51987

The root lies between 0.31467 and 1

therefore $x_0 = 0.31467, x_1 = 1,\ f(x_0) = 0.51987, f(x_1) = -2.17798$

$$x_3 = 0.31467 - \frac{0.68533}{2.69785}(0.51987) = 0.44673$$

The successive approximations are

$x_4 = 0.49402$, $x_5 = 0.50995$, $x_6 = 0.51520$, $x_7 = 0.51692$, $x_8 = 0.51748$, $x_9 = 0.51767$, $x_{10} = 0.51775$
Hence the root is 0.5177.

3. Find a real root of the equation $x\log_{10} x = 1.2$ by Regular - Falsi method correct to four decimal places.

Solution: Let $f(x) = x\log_{10} x - 1.2$

$f(1) = -ve,\ f(2) = -ve\ and\ f(3) = +ve$

Therefore a root lies between 2 and 3

Now, $x_0 = 2, x_1 = 3, f(x_0) = -0.59794, f(x_1) = 0.23136$

By Regular-Falsi method

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= 2.72102$$

$f(x_2) = f(2.72102) = -0.01709$

$\Rightarrow$ the root lies between 2.72102 and 3

$$x_3 = 2.72102 + \frac{0.27898}{0.23136 + 0.01709}(0.01709) = 2.74021$$

Repeating this process, the successive approximations are

$x_4 = 2.74024,\ x_5 = 2.74063$

Hence the root is 2.7406 correct to 4 decimal places.

Newton-Raphson method:

Let f(x) = 0 ---------- (1), be the given equation, where f(x) may be Algebraic (or) Transcendental and x_0 be the initial approximation to the root.

Let $x_1 = x_0 + h$ ---------- (2) be the exact root of the equation f(x) = 0, where h is very small quantity.

From (1), we have $f(x_1) = f(x_0 + h) = 0$ ---------- (3)

Expanding $f(x_0 + h)$ by Taylor's series expansion we get,

$$f(x_0+h) = f(x_0) + \frac{h}{1!} f'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \ldots \quad \text{------------(4)}$$

Since $f(x_1) = f(x_0 + h) = 0$ and h is very small, so higher powers of h can be neglected.

$\therefore$ Equation (3) becomes,

$$f(x_0+h) = f(x_0) + hf'(x_0)$$

$$\Rightarrow 0 = f(x_0) + hf'(x_0) \qquad (\because f(x_0+h) = 0)$$

$$\Rightarrow h = -\frac{f(x_0)}{f'(x_0)} \qquad \text{----------------------(5)}$$

Substituting (5) in (2), we get the first approximation to the root given by,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly starting with x_1 and repeating the above procedure we get the second approximation to the root given by,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Proceeding like this, we get the n^{th} approximation to the root given by,

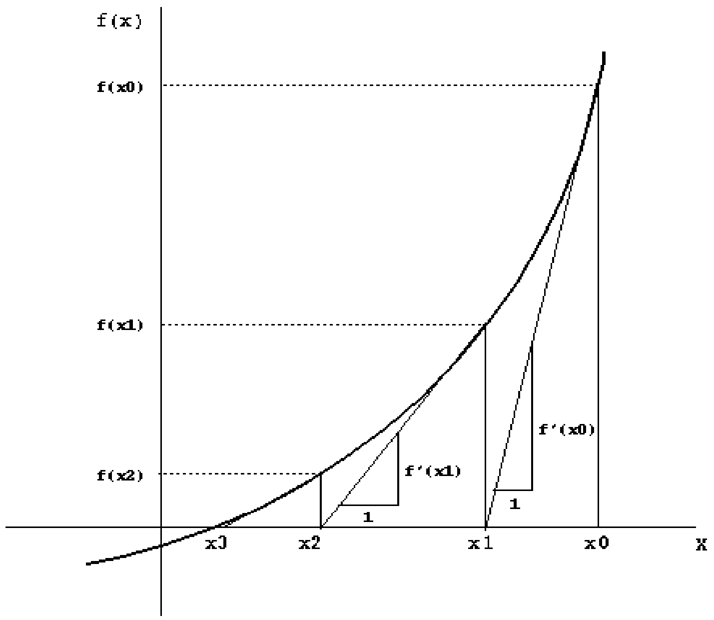
$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{---------(6)}}$$

The formula given in (6) is known as Newton-Raphson Iterative formula to find the root of the equation $f(x) = 0$.

Note: 1 Newton's formula converges provided initial approximation x_0 is chosen sufficiently close to the root.

Note: 2 Newton's method has a quadratic convergence. i.e., the subsequent error at each step is proportional to the square of the previous error.

Geometrical Interpretation:



Let x_0 be a point near the root α of the equation $f(x)=0$

Then, the equation of the tangent at $A_0[x_0,\ f(x_0)]$ is $y - f(x_0) = f'(x_0)(x - x_0)$

It cuts the x-axis at $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ ________ 1st approximation to the root α

If A_1 is a point corresponding to x_1 on the curve, then the tangent at A_1 will cut the x- axis at x_2, which is nearer to α is a second approximation to the root. Hence the method consists in replacing the part of the curve between the point A_0 and the x- axis by means of tangent to the curve at A_0.

Examples:

1. Apply Newton-Raphson's method to obtain the smallest positive root of the equation

$\cos(x) - x\,e^x = 0$.

Solution: Given $\cos(x) - x\,e^x = 0$.

$\Rightarrow f(x) = \cos(x) - x\,e^x$ and $f'(x) = -\sin x - xe^x - e^x$

Since f (0) = 1>0 and f(1) =-2.1780 <0 , the root lies in the interval (0,1)

Let $x_0 = 1$ be the initial approximation to the root.

The approximate root is calculated as follows:

k	x_k	$\Delta x_k = \frac{f(x_k)}{f'(x_k)}$	$x_{k+1} = x_k - \Delta x_k$	$f(x_{k+1})$
0	1.0	0.3469	0.65307940	-0.4606
1	0.65307940	0.1217	0.53134337	-0.4180
3	0.53134337	0.1343	0.51790991	-0.4641
4	0.51790991	0.1525	0.51775738	-0.5926
5	0.51775738	0.1948	0.51775736	-0.2910

Hence, x = 0.51775736 is the approximate root.

2. Find by Newton's method, the real root of the equation $3x = \cos x + 1$.

Solution:

Let $f(x) = 3x - \cos x - 1$

$f(0) = -2 = -ve \quad f(1) = 1.4597 = +ve$

So a root of $f(x) = 0$ lies between 0 and 1.

It is nearer to 1 let us take $x_0 = 0.6$

$$f'(x) = 3 + \sin x$$

Therefore Newton's iteration formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}$$

$$x_{n+1} = \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n}$$

Put n = 0 first approximation x_1 is

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} = 0.6071$$

Put n = 1 second approximation x_2 is

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1} = 0.6071$$

clearly $x_1 = x_2$

Hence the desired root is 0.6071 correct to 4 decimal places.

3. Using Newton's iterative method, find the real root of $x\log_{10}x = 1.2$ correct to five decimal places.

Solution:

$f(x) = x\log_{10}x - 1.2$

$f(1) = -1.2,\ f(2) = 0.59794,\ f(3) = 0.23136$

so a root of $f(x) = 0$ lies between 2 and 3

Take $x_0 = 2$

$$f'(x) = \log_{10}x + x.\frac{1}{x}.\log_{10}e = \log_{10}x + 0.43429$$

The approximate root is calculated as follows

k	$x_{k+1} = x_k - \Delta x_k$
1	2.81
2	2.741
3	2.74064
4	2.74065
5	2.74065

Since $x_4 = x_5$

Hence the required root is 2.74065.

4. Evaluate $\sqrt{12}$ to four decimal places by Newton's iterative method.

Solution:

Let $\sqrt{x} = 12$

$\Rightarrow x^2 - 12 = 0$

Therefore $f(x) = x^2 - 12$

Newton's iterative formula is $x_{n+1} = x_n - \frac{f(x_n)}{f^{'}(x_n)}$

$$x_{n+1} = x_n - \frac{x_n^{\ 2}\text{-}12}{2x_n} = \frac{1}{2}\left(x_n + \frac{12}{x_n}\right)$$

Since f(3) =-3 , f(4) = 4

Root lies between 3 and 4

$x_0 = 3.5$

$$x_1 = \frac{1}{2}\left(x_0 + \frac{12}{x_0}\right) = 3.4643$$

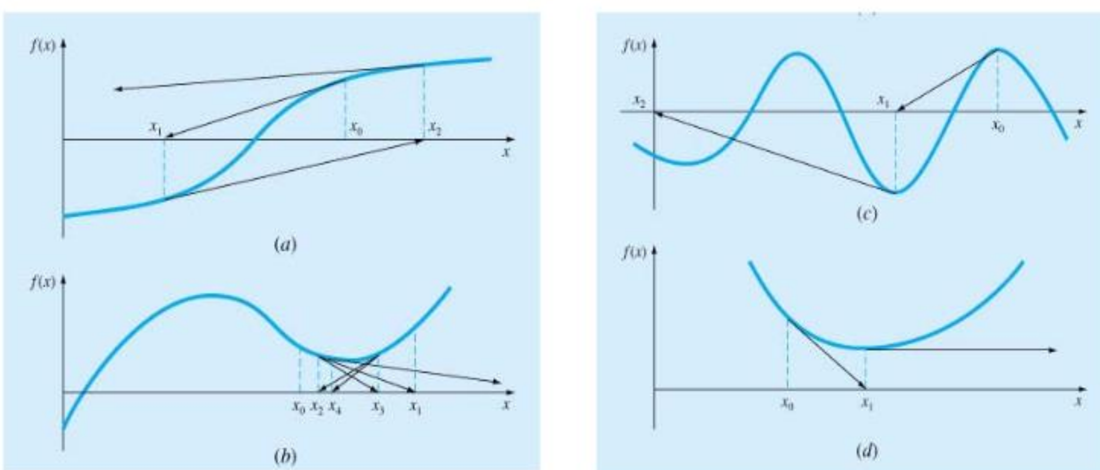
$$x_2 = \frac{1}{2}\left(3.4643 + \frac{12}{3.4643}\right) = 3.4641$$

$x_3 = 3.4641$

since $x_2 = x_3$ up to 4 decimal places, therefore $\sqrt{12} = 3.4641$.

Failure of Newton-Raphson method:

- Inflection point in vicinity of root.
- Oscillate around local maximum or minimum.
- Jump away for several roots.
- Disaster from zero slope.



Numerical solutions of ordinary differential equation:

A general solution of a differential equation of the n th order has n arbitrary constants. In order to compute Numerical solution of such an equation we need n conditions. If all the n conditions are prescribed at the initial point (one point) only, then it is called initial value problem. If the n conditions are specified at two (or) more points then it is called Boundary Value Problem.

Taylors's series:

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{\lfloor 2} f''(a) + \dots + \frac{(x-a)^{n-1}}{\lfloor n-1} f^{(n-1)}(a) + \dots.$$

$$= f(a) + \sum_{n=1}^{\infty} \frac{(x-a)^n}{\lfloor n} f^{(n)}(a) \rightarrow (4)$$

Using (4) we can write a Taylor's series expansion for the given function f(x) in powers of (x-a) or about the point 'a'.

Maclaurin's series:

(Scottish Mathematician Colin Maclaurin 1698-1746)

When a=0, expression (4) reduces to a Maclaurin's expansion given by

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \dots + \frac{x^{n-1}}{\lfloor n-1} f^{(n-1)}(0) + \dots.$$

$$= f(0) + \sum_{n=1}^{\infty} \frac{x^n}{\lfloor n} f^{(n)}(0) \rightarrow (5)$$

Example 1: Obtain a Taylor's expansion for $f(x) = Sin\, x$ in the ascending powers of $\left(x - \frac{\pi}{4}\right)$ up to the fourth degree term.

The Taylor's expansion for $f(x)$ about $\frac{\pi}{4}$ is

$$f(x)=f(\frac{\pi}{4})+(x-\frac{\pi}{4})f'(\frac{\pi}{4})+\frac{(x-\frac{\pi}{4})^2}{\underline{|2}}f''(\frac{\pi}{4})+\frac{(x-\frac{\pi}{4})^3}{\underline{|3}}f'''(\frac{\pi}{4})+\frac{(x-\frac{\pi}{4})^4}{\underline{|4}}f^{(4)}(\frac{\pi}{4})\ldots.\rightarrow(1)$$

$$f(x)=Sin\,x\Rightarrow f\left(\frac{\pi}{4}\right)=Sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}\;;\qquad f'(x)=Cos\,x\Rightarrow f'\left(\frac{\pi}{4}\right)=Cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$$

$$f''(x)=-Sin\,x\Rightarrow f''\left(\frac{\pi}{4}\right)=-Sin\frac{\pi}{4}=-\frac{1}{\sqrt{2}}$$

$$f'''(x)=-Cos\,x\Rightarrow f'''\left(\frac{\pi}{4}\right)=-Cos\frac{\pi}{4}=-\frac{1}{\sqrt{2}}$$

$$f^{(4)}(x)=Sin\,x\Rightarrow f^{(4)}\left(\frac{\pi}{4}\right)=Sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$$

Substituting these in (1) we obtain the required Taylor's series in the form

$$f(x)=\frac{1}{\sqrt{2}}+(x-\frac{\pi}{4})(\frac{1}{\sqrt{2}})+\frac{(x-\frac{\pi}{4})^2}{\underline{|2}}(-\frac{1}{\sqrt{2}})+\frac{(x-\frac{\pi}{4})^3}{\underline{|3}}(-\frac{1}{\sqrt{2}})+\frac{(x-\frac{\pi}{4})^4}{\underline{|4}}(\frac{1}{\sqrt{2}})\ldots.$$

$$f(x)=\frac{1}{\sqrt{2}}\left[1+(x-\frac{\pi}{4})-\frac{(x-\frac{\pi}{4})^2}{\underline{|2}}+\frac{(x-\frac{\pi}{4})^3}{\underline{|3}}-\frac{(x-\frac{\pi}{4})^4}{\underline{|4}}+\ldots\right]$$

Example 2: Obtain a Taylor's expansion for $f(x)=\log_e x$ up to the term containing $(x-1)^4$ and hence find $\log_e(1.1)$.

The Taylor's series for $f(x)$ about the point 1 is

$$f(x)=f(1)+(x-1)f'(1)+\frac{(x-1)^2}{\underline{|2}}f''(1)+\frac{(x-1)^3}{\underline{|3}}f'''(1)+\frac{(x-1)^4}{\underline{|4}}f^{(4)}(1)\ldots.\rightarrow(1)$$

Here $f(x)=\log_e x \Rightarrow f(1)=\log 1=0$; $\quad f'(x)=\dfrac{1}{x} \Rightarrow f'(1)=1$

$f''(x)=-\dfrac{1}{x^2} \Rightarrow f''(1)=-1$; $\quad f'''(x)=\dfrac{2}{x^3} \Rightarrow f'''(1)=2$

$f^{(4)}(x)=-\dfrac{6}{x^4} \Rightarrow f^{(4)}(1)=-6$ etc.,

Using all these values in (1) we get

$$f(x)=\log_e x=0+(x-1)(1)+\frac{(x-1)^2}{\lfloor 2}(-1)+\frac{(x-1)^3}{\lfloor 3}(2)+\frac{(x-1)^4}{\lfloor 4}(-6)\ldots.$$

$$\Rightarrow \log_e x=(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}\ldots.$$

Taking x=1.1 in the above expansion we get

$$\Rightarrow \log_e(1.1)=(0.1)-\frac{(0.1)^2}{2}+\frac{(0.1)^3}{3}-\frac{(0.1)^4}{4}\ldots.=0.0953$$

Example 3: Using Taylor's theorem Show that

$$\log_e(1+x)<x-\frac{x^2}{2}+\frac{x^3}{3} \text{ for } 0<\theta<1, x>0$$

Taking n=3 in the statement of Taylor's theorem, we can write

$$f(a+x)=f(a)+xf'(a)+\frac{x^2}{\lfloor 2}f''(a)+\frac{x^3}{\lfloor 3}f'''(a+\theta x)\to(1)$$

Consider $f(x)=\log_e x \Rightarrow f'(x)=\dfrac{1}{x}$; $f''(x)=-\dfrac{1}{x^2}$ *and* $f'''(x)=\dfrac{2}{x^3}$

Using these in (1), we can write,

$$\log(a+x) = \log a + x\left(\frac{1}{a}\right) + \frac{x^2}{\lfloor 2}\left(-\frac{1}{a^2}\right) + \frac{x^3}{\lfloor 3}\left(\frac{2}{(a+\theta x)^3}\right) \to (2)$$

For a=1 in (2) we write,

$$\log(1+x) = \log 1 + x - \frac{x^2}{2} + \frac{x^3}{3}\frac{1}{(1+\theta x)^3} = x - \frac{x^2}{2} + \frac{x^3}{3}\frac{1}{(1+\theta x)^3}$$

Since $x > 0$ *and* $\theta > 0$, $(1+\theta x)^3 > 1$ *and therefore* $\frac{1}{(1+\theta x)^3} < 1$

$$\therefore \log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$$

Example 4: Obtain a Maclaurin's series for $f(x) = Sin\ x$ up to the term containing x^5.

The Maclaurin's series for f(x) is

$$f(x) = f(0) + x\ f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \frac{x^4}{\lfloor 4} f^{(4)}(0) + \frac{x^5}{\lfloor 5} f^{(5)}(0) \ldots. \to (1)$$

Here $f(x) = Sin\ x \Rightarrow f(0) = Sin\ 0 = 0 \qquad f'(x) = Cos\ x \Rightarrow f'(0) = Cos\ 0 = 1$

$$f''(x) = -Sin\ x \Rightarrow f''(0) = -Sin\ 0 = 0 \qquad f'''(x) = -Cos\ x \Rightarrow f'''(0) = -Cos\ 0 = -1$$

$$f^{(4)}(x) = Sin\ x \Rightarrow f^{(4)}(0) = Sin\ 0 = 0 \qquad f^{(5)}(x) = Cos\ x \Rightarrow f^{(5)}(0) = Cos\ 0 = 1$$

Substituting these values in (1), we get the Maclaurin's series for $f(x) = Sin\ x$ as

$$f(x) = Sin\ x = 0 + x(1) + \frac{x^2}{\lfloor 2}(0) + \frac{x^3}{\lfloor 3}(-1) + \frac{x^4}{\lfloor 4}(0) + \frac{x^5}{\lfloor 5}(1) \ldots. \Rightarrow Sin\ x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} \ldots.$$

Taylor's series Method:

Consider the first order ordinary differential equation,

$$\frac{dy}{dx} = f(x, y), \quad \text{subject to initial conditions} \quad y(x_0) = y_0 \qquad \text{----- (1)}$$

Let $y = y(x)$ be the exact solution of equation (1) expanding y(x) in Taylor series about the point x_0, we get,

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!}y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \dots$$

Since $y(x_0) = y_0$, $y'(x_0) = y'_0$, $y''(x_0) = y''_0$ etc., the above equation becomes

$$y(x) = y_0 + \frac{(x-x_0)}{1!}y'_0 + \frac{(x-x_0)^2}{2!}y''_0 + \frac{(x-x_0)^3}{3!}y'''_0 + \dots$$

Setting h = $x_1 - x_0$ and denoting y(x) = $y(x_1) = y_1$, the above equation becomes,

$$y_1 = y_0 + \frac{h}{1!}y'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + ... \qquad \text{--------------(2)}$$

This is the Taylor series solution at the point $x_1 = x_0 + h$. Starting with this x_1 and repeating the above procedure we get the solution at the point x_2 as,

Proceeding like this we get the Taylor series solution at the n^{th} step given by,

$$y_{n+1} = y_n + \frac{h}{1!}y'_n + \frac{h^2}{2!}y''_n + \frac{h^3}{3!}y'''_n + ... \qquad \text{---------------(3)}$$

The formula given in (3) is known as Taylor's series method.

Note: It is a single step method.

Example:

1. Use Taylor's series method to find y at the points $x_1 = 0.1$ and $x_2 = 0.2$, given that $\frac{dy}{dx} = x^2 + y^2$, y(0) = 1.

Solution: Here $x_0 = 0$, the Taylor's series solution of the given problem is

$$y = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + ... \qquad \text{with } y(0) = 1. \quad \text{(i)}$$

Also, $y' = x^2 + y^2$, $y'' = 2x + 2yy'$, $y''' = 2 + 2\{yy'' + (y')^2\}$, $y^{iv} = 2\{yy''' + 3y'y''\}$ and so on.

Using the condition y(0) = 1 in these we obtain,

$y'(0)=1,\qquad y''(0)=2,\qquad y'''(0)=8,\qquad y^{iv}(0)=28.$

Using these and the condition y(0) = 1 in (i) we get,

$$y=1+x+\left(\frac{x^2}{2!}\times 2\right)+\left(\frac{x^3}{3!}\times 8\right)+\left(\frac{x^4}{4!}\times 28\right)+\ldots\ldots$$

$$=1+x+x^2+\frac{4}{3}x^3+\frac{7}{6}x^4+\ldots\ldots \qquad \text{(ii)}$$

Under the given condition $y(x_0)=y(0)=1$, this is the Taylor's series solution for y at a point x in a neighbourhood of

$x_0=0$.

For $x=x_1=0.1$, expression (ii) yields the solution at $x_1=0.1$ as,

$$y(0.1)=1+(0.1)+(0.1)^2+\frac{4}{3}(0.1)^3+\frac{7}{6}(0.1)^4+\ldots\ldots\approx 1.11145$$

For $x=x_2=0.2$, expression (ii) yields the solution at $x_2=0.2$ as,

$$y(0.2)=1+(0.2)+(0.2)^2+\frac{4}{3}(0.2)^3+\frac{7}{6}(0.2)^4+\ldots\ldots\approx 1.25253.$$

Runge-Kutta fourth order method:

The general formula for finding the solution of the initial value problem $\frac{dy}{dx}=f(x, y), y(x_0)=y_0$ is given by:

$$y_{n+1}=y_n+\frac{1}{6}[K_1+2K_2+2K_3+K_4]$$

Where $K_1=h\,f(x_n,y_n)$, $K_2=h\,f\left(x_n+\frac{h}{2},y_n+\frac{K_1}{2}\right)$

$K_3=h\,f\left(x_n+\frac{h}{2},y_n+\frac{K_2}{2}\right)$ and $K_4=h\,f(x_n+h, y_n+K_3)$

Example:

1. Using fourth – order Runge – Kutta method, solve the problem $y' = x + y^2$, y(0) = 1 at the points x = 0.1 and x = 0.2 in steps of 0.1.

Solution: Here, $f(x,y) = x + y^2, x_0 = 0, y_0 = 1$, h=0.1, so that $x_1 = x_0 + h = 0.1$ and $x_2 = x_1 + h = 0.2$

Step 1: We have to find $y_1 = y(x_1) = y(0.1)$ *and* $y_2 = y(x_2) = y(0.2)$. Hence,

$k_1 = hf(x_0, y_0) = (0.1)\times f(0,1) = (0.1)\times(0+1^2) = 0.1,$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)\times f(0.05, 1.05)$

$= (0.1)\times\{0.05 + (1.05)^2\} = 0.11525$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)\times f(0.05, 1.057625)$

$= (0.1)\times\{0.05 + (1.057625)^2\} = 0.116857$

$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1)\times f(0.1, 1.116857)$

$= (0.1)\times\{0.1 + (1.116857)^2\} = 0.13474$

and, $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}(0.1 + 2\times 0.11525 + 2\times 0.116857 + 0.13474) = 0.11649$

Thus, the solution at $x_1 = 0.1$ is,

$y_1 = y_0 + k = 1 + 0.11649 \approx 1.1165$

Step 2: Again, we compute k_1, k_2, k_3, k_4 and k by changing (x_0, y_0) to (x_1, y_1)

$k_1 = hf(x_1, y_1) = (0.1)\times f(0.1, 1.1165) = 0.134657,$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.1)\times f(0.15, 1.1838) = 0.15514,$

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.1)\times f(0.15, 1.1941) = 0.15759,$

$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1)\times f(0.2, 1.27409) = 0.18233,$

and $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}(0.134657 + 2\times 0.15514 + 2\times 0.15759 + 0.18233) = 0.15708.$

Hence, the solution at $x_2 = 0.2$ is, $y_2 = y_1 + k = 1.1165 + 0.15708 = 1.27358.$

Predictor-Corrector Method:

The methods discussed above to solve a first order differential equation, require information about the solution at a single point say x_n for computing the solution at x_{n+1}. Hence these methods are called single-step methods.

In the Predictor-Corrector Methods, a predictor formula is used to predict the value y_{i+1} of y at x_{i+1} and then a corrector formula is used to improve the value of y_{i+1}.The corrector formula is used repeatedly until two consecutive values of y_{i+1} are almost equal.

Milne's Predictor-Corrector method:

The Predictor-Corrector methods form a large class of general methods for numerical integration of ordinary differential equations. As an illustration, consider Milne's method for the first-order equation $y'(x) = f(y(x), x)$ initial value $y(x_0) = y_0$.

Define

$$y_n = y(x_0 + nh),$$

$$y'_n = y'(x_0 + nh) = f(y_n, x_0 + nh)$$

Then by Simpson's rule

$$y'_{n+1} = f(y_{n+1,} x + (n+1)h)$$

This corrector equation is an implicit equation for y_{n+1}; if h is sufficiently small, and if a first approximation for y_{n+1} can be found, the equation is solved simply by iteration, i.e. by repeated evaluations of the right hand side. To provide the first approximation for y_{n+1}, an explicit predictor formula is needed, e.g. Milne's formula

$$y_{n+1} = y_{n-3} + (4h/3)(2y'_n - y'_{n-1} + 2y'_{n-2}) + O(h^5)$$

Predictor Formula:

$$y_{n+1}^{(p)} = y_{n-3} + \frac{4h}{3}\left[2y_{n-2}^1 - y_{n-1}^1 + 2y_n^1\right]$$

Corrector Formula:

$$y_{n+1}^{(c)} = y_{n-1} + \frac{h}{3}\left[y_{n-1}^1 + 4y_n^1 + y_{n+1}^1\right].$$

The need for a corrector formula arises because the predictor alone is numerically unstable; it gives spurious solutions growing exponentially. Milne's predictor uses four previous values of y, these values can be obtained by using Taylor's or Runge-Kutta method.

Example:

1. Using Milne's Predictor-Corrector method evaluate the integral of $y'-4y=0$ *at* 0.4, *given that* $y(0)=1,\ y(0.1)=1.492,\ y(0.2)=2.226,\ y(0.3)=3.320$

Solution:

Here, $\dfrac{dy}{dx}=4y$ So $f(x,y)=y'=4y$

h=0.1, $x_0=0,\ x_1=0.1,\ x_2=0.2,\ x_3=0.3,\ x_4=0.4$

For predictor value of y(0.4) :

$$y_{n+1}^{(p)}=y_{n-3}+\frac{4h}{3}\left[2y_{n-2}^1-y_{n-1}^1+2y_n^1\right] \text{ with n=3}$$

$$y_4^{(p)}=y_0+\frac{4h}{3}\left[2y_1^1-y_2^1+2y_3^1\right]$$

Here, h=0.1,

$$y_0=1,\ y_1^1=4y_1=4(1.492)=5.968$$

$$y_2^1=4y_2=4(2.226)=8.904$$

$$y_3^1=4y_3=4(3.32)=13.28$$

$$\therefore y^p{}_4=y(x_4)=y(0.4)=1+\frac{4(0.1)}{3}[2(5.968)-8.904+2(13.28)]=4.9456$$

$$So\ y_4^1=4y_4=4(4.9456)=19.7824$$

For corrector value of y(0.4):

$$y_{n+1}^{(c)}=y_{n-1}+\frac{h}{3}\left[y_{n-1}^1+4y_n^1+y_{n+1}^p\right]$$

with n=3, $y_4^{(c)}=y_2+\dfrac{h}{3}\left[y_2^1+4y_3^1+y_4^1\right]$

$$\therefore y_4=y(0.4)=2.226+\frac{0.1}{3}(2.226+4(13.28)+19.7824=4.95288 \text{ (first approximation of } y_4)$$

Second approximation of y_4:

$$y_4^{(C_1)}=y_2+\frac{h}{3}\left[y_2^1+4y_3^1+y_4^{(c)}\right]$$

$$y_4{}^{(1)}=4(4.95288)=19.81152$$

$$\therefore y_4=y(0.4)=2.226+\frac{0.1}{3}(2.226+4(13.28)+19..81152)=4.731251.$$

Similarly third approximation of y_4:

$$\therefore y_4 = y(0.4) = 2.226 + \frac{0.1}{3}(2.226 + 4(13.28) + 4(4.731251)) = 4.701700.$$

Fourth approximation of y_4:

$$\therefore y_4 = y(0.4) = 2.226 + \frac{0.1}{3}(2.226 + 4(13.28) + 4(4.701700)) = 4.69776.$$

Fifth approximation of y_4:

$$\therefore y_4 = y(0.4) = 2.226 + \frac{0.1}{3}(2.226 + 4(13.28) + 4(4.69776)) = 4.697234.$$

$\therefore$ The approximate value of y(0.4) is 4.697.

2. Using the Milne's predictor-corrector method, find an approximate solution of the initial value problem $\frac{dy}{dx} = \frac{2y}{x}, x \neq 0,$ at the point x = 2, given y(1) = 2, y(1.25) = 3.13, y(1.5) = 4.5 and y(1.75) = 6.13. Apply the corrector twice.

Solution: Here $f = f(x,y) = \frac{2y}{x}$, $x_0 = 1$, $y_0 = 2$, $x_1 = 1.25$, $y_1 = 3.13$, $x_2 = 1.5$, $y_2 = 4.5$, $x_0 = 1$, $y_3 = 6.13$
h = 0.25, also $f_1 = f(x_1,y_1) = 5.008$, $f_2 = f(x_2,y_2) = 6.00$, $f_3 = f(x_3,y_3) = 7.0057$

Using Milne predictor yields the value of y_4 as : $y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) = 8.0091$

This give $f_4^{(p)} = f(x_4,y_4) = 8.0091$

The Milne corrector gives the improved value of y_4 as:

$$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^{(p)}) = 8.0027,$$

Further computing $f_4^{(p)} = f(x_4,y_4) = 8.0027$, applying the corrector again $y_4^{(c)} = 8.0021$

The approximate solution for 'y' at the point x =2, is 8.0021.

Exercise:

1. Determine the smallest positive real root of $xe^x - 2 = 0$ to four significant figures using false position method.
(Ans: 0.8526)

2. Solve the equation $x\tan x = -1$ by Regular-Falsi method between 2.5 and 3 correct to three decimal places.
(Ans: 2.7982)

3. Find the real root of the following equations by Newton - Raphson's method

i) $x^3 - x - 11 = 0$ in the interval (2 , 3) (Ans: x = 1.532)

ii) $x^3 - 3x + 11 = 0$ in the interval (1 , 2) (Ans: x = 1.532)

4. Use Modified Newton-Raphson method to evaluate the multiple roots of $f(x) = x^3 - x^2 - x + 1 = 0$. Find the double root of the equation with $x_0 = 0.8$.
(Ans: x=1.0002)

5. The volume 'V' of a liquid in a spherical tank of radius 'a' is related to the depth 'l' of the liquid given by $l^3 - 3l^2 + 0.4775 = 0$, using Newton-Raphson method find a positive real root of the equation which lies between 0 and 1.
(Ans: 0.431137997)

6. If $y' = 1 + xy$ and y = 2 when x = 0, use Taylor's series method to obtain the value of y for x = 0.4.
(Ans: 2.5884139)

7. Solve the differential equation $y' = 2y + 3e^x$ with x(0) = 0 using Taylor's series method and find the value of y for x = 0.1, 0.2. Compare your answer with exact values.
(Ans: Calculated values: y(0.1) = 0.3487, y(0.2) = 0.8112
Exact values: y(0.1) = 0.3486, y(0.2) = 0.8112)

8. Solve the initial value problem $\frac{dy}{dx} = \log(x + y)$ when the initial condition that y =1when x = 0 use Runge-Kutta method to find y(0.2) and y(0.5) in more accurate form.
(Ans: y(0.2) =1.0082, y(0.5) =1.0490)

9. Using Runge-Kutta 4^{th} order method compute y(0.2) and y(0.4) from $y' = x^2 + y^2$, y(0) =1, taking h = 0.1.

(Ans: y(0.2) =1.0207,y(0.4) =1.038).

10. Solve the initial value problem $\frac{dy}{dx} = 3x - 4y$, y(0) =2 at x =0.4 taking h = 0.2 by Runge-Kutta method of order 4.
(Ans: y(0.4) = 0.5543).

11. Solve the initial value problem $\frac{dy}{dx} = 1 + xy^2$, y(0)=1 for x = 0.4 by using Milne's method when it is given that

x:	0.1	0.2	0.3
y:	1.105	1.223	1.335

(Ans: y =1.538 at x = 0.4)

Video link:

https://www.youtube.com/watch?v=QwyjgmqbR9s