# Pedal Equations (p-r equation) :

Let $r = f(\theta)$ be a polar curve, then the expressions $p = r.\sin(\phi)$ or $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

determines $p$ in terms of '$\theta$'!

If we eliminate '$\theta$' from the relation connecting $p$ and $\theta$, with the equation of the curve, an equation connecting $p$ and $r$ can be obtained. Such a relation is called $p$-$r$ equation of the curve.

Examples :

1. Obtain the pedal equation for the equiangular spiral $r = ae^{\theta.\cot(\alpha)}$, '$\alpha$' is a constant.

Solution: Here $r = ae^{\theta.\cot(\alpha)}$

$$\therefore \log(r) = \log(a) + \log\left[e^{\theta\cot\alpha}\right]$$

$$\log(r) = \log(a) + \theta.\cot(\alpha)$$

Now $\frac{1}{r}\frac{dr}{d\theta} = 0 + 1.\cot(\alpha) = \cot(\alpha)$

$$\therefore \tan(\phi) = r\frac{d\theta}{dr} = \frac{1}{\cot(\alpha)} = \tan(\alpha)$$

$$\therefore \phi = \alpha, \text{ a constant}$$

since $p = r.\sin(\phi) = r.\sin(\alpha)$

In above equation '$p$' is described in terms of '$r$' alone, such a relation is known as $p$-$r$ equation.

2. Show that the $p$-$r$ equation of the cardioid $r = a(1+\cos\theta)$ is given by $2ap^2 = r^3$

Solution: Here $r = a(1+\cos\theta)$ (1)

$$\therefore \frac{dr}{d\theta} = -a\sin\theta$$

$$\therefore \tan(\phi) = r.\frac{d\theta}{dr} = -\frac{(1-\cos\theta)}{\sin(\theta)}$$

$$= -\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = -\tan(\theta/2)$$

$$= \tan(-\theta/2)$$

$$\therefore \phi = -\theta/2$$

Now $p = r\sin(\phi) = r\sin(-\theta/2) = -r\sin(\theta/2)$ ②

from ①, $\frac{2a}{r} = 2\sin^2(\theta/2)$

$$\therefore \sin^2(\theta/2) = \frac{a}{r} \quad \therefore \sin(\theta/2) = \frac{\sqrt{a}}{\sqrt{r}}$$

$$\therefore p = -r.\frac{\sqrt{a}}{\sqrt{r}} = -\sqrt{ar}$$

$$\therefore p^2 = ar$$

4. Find the pedal equation of the curve
$$r^m = a^m\cos(m\theta)$$

Solution: Here $r^m = a^m\cos(m\theta)$

$$\therefore m\log(r) = \log(a^m) + \log[\cos(m\theta)]$$

$$\therefore \frac{m}{r}\frac{dr}{d\theta} = 0 - \frac{m\sin(m\theta)}{\cos(m\theta)} = -m.\tan(m\theta)$$

$$\therefore \tan(\phi) = r\frac{d\theta}{dr} = -\frac{1}{\tan(m\theta)} = -\cot(m\theta)$$

$$= \tan(\pi/2 + m\theta)$$

$$\therefore \phi = (\pi/2 + m\theta)$$

Since $p = r\sin(\phi) = r.\sin(\pi/2 + m\theta) = r.\cos(m\theta)$

and $\cos(m\theta) = \frac{r^m}{a^m}$ (given curve)

$$p = r.r^m/a^m \quad \therefore a^m p = r^{m+1}$$

Find the $p-r$ equation of the curve $\frac{l}{r} = 1 + e\cos(\theta)$.

Solution: Since $\frac{l}{r} = 1 + e\cos(\theta)$, $-\frac{l}{r^2}\frac{dr}{d\theta} = -e\sin(\theta)$

$$\therefore \left(\frac{dr}{d\theta}\right) = \frac{er^2}{l}\sin(\theta)$$

Now $$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$= \frac{1}{r^2} + \frac{1}{r^4} \cdot \frac{e^2 r^4 \sin^2(\theta)}{l^2}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2}{l^2}\sin^2(\theta) = \frac{1}{r^2} + \frac{e^2}{l^2}[1 - \cos^2(\theta)]$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2}{l^2}\left[1 - \frac{1}{e^2}\left(\frac{l}{r} - 1\right)^2\right]$$ since,

$$e\cos(\theta) = \left(\frac{l}{r} - 1\right) \quad \therefore \cos(\theta) = \frac{1}{e}\left(\frac{l}{r} - 1\right)$$

$$\therefore \frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2}{l^2}\left\{1 - \frac{1}{e^2} - \frac{l^2}{e^2 r^2} + \frac{2l}{e^2 r}\right\}$$

$$\therefore \frac{1}{p^2} = \frac{1}{l^2}\left(e^2 + \frac{2l}{r} - 1\right)$$

6. Show that the pedal equation of the curve

$$r^n = a^n \sin(n\theta) + b^n \cos(n\theta)$$ is

$$p^2(a^{2n} + b^{2n}) = r^{2n+2}$$

Solution:

$$r^n = a^n \sin(n\theta) + b^n \cos(n\theta)$$

$$\therefore n r^{n-1}\frac{dr}{d\theta} = n a^n \cos(n\theta) - b^n . n . \sin(n\theta)$$

$$\therefore \frac{dr}{d\theta} = \frac{1}{r^{n-1}}[a^n \cos(n\theta) - b^n \sin(n\theta)]$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$= \frac{1}{r^2} + \frac{1}{r^4} \cdot \frac{[a^n \cos(n\theta) - b^n \sin(n\theta)]^2}{r^{2n-2}}$$

$$= \frac{1}{r^2} + \frac{1}{r^{2n+2}} [a^n \cos(n\theta) - b^n \sin(n\theta)]^2$$

$$= \frac{r^{2n}}{r^{2+2n}} + \frac{1}{r^{2+2n}} [a^n \cos(n\theta) - b^n \sin(n\theta)]^2$$

$$= \frac{\{ r^{2n} + (a^n \cos n\theta - b^n \sin n\theta)^2 \}}{r^{2n+2}}$$

$$\therefore \frac{1}{p^2} = \frac{1}{r^{2n+2}} [ (a^n \sin n\theta + b^n \cos n\theta)^2 + (a^n \cos n\theta - b^n \sin n\theta)^2 ]$$

$$= \frac{1}{r^{2n+2}} \{ a^{2n} (\sin^2 n\theta + \cos^2 n\theta) + b^{2n} (\cos^2 n\theta + \sin^2 n\theta) \}$$

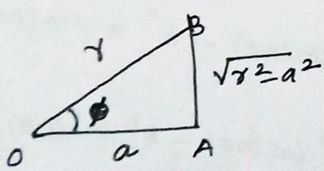
$$\frac{1}{p^2} = \frac{(a^{2n} + b^{2n})}{r^{2n+2}}$$

$$\therefore r^{2n+2} = (a^{2n} + b^{2n}) p^2$$

7. For the curve $\theta = \frac{1}{a}\sqrt{r^2 - a^2} - \cos^{-1}\left(\frac{a}{r}\right)$ show that $p^2 = r^2 - a^2$

Sol: Since, $\theta = \frac{1}{a}\sqrt{r^2 - a^2} - \cos^{-1}\left(\frac{a}{r}\right)$,

$$\tan(\phi) = \frac{\sqrt{r^2 - a^2}}{a}$$




Now $p = r \sin(\phi)$, from $\Delta OAB$

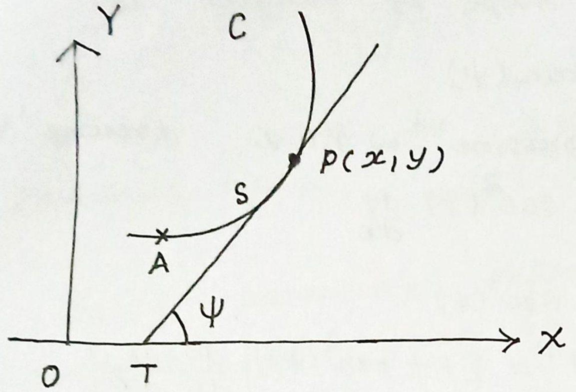
$$\sin(\phi) = \frac{\sqrt{r^2 - a^2}}{r}$$

$$p = \sqrt{r^2 - a^2} \quad \therefore p^2 = r^2 - a^2$$

# Curvature and radius of curvature:

In many practical problems we are concerned with the bending of a curve at different points

The curvature at a point is a numerical measure of the rate of bending of a curve.




Consider a smooth curve $y = f(x)$; A be a fixed point $P(x, y)$ be a variable point on the curve. PT be the tangent to curve at $P(x, y)$. $\widehat{AP} = s$ and $\psi$ be the angle made by tangent with x-axis.

As P varies along the curve the arc length 's' and $\psi$ changes, the ratio $\frac{\Delta\psi}{\Delta s}$ is called as average curvature;

Curvature at a point $P(x, y)$ is defined as

$$\frac{d\psi}{ds} = \lim_{\Delta s \to 0} \left(\frac{\Delta\psi}{\Delta s}\right)$$

it is denoted by greek letter $\kappa$ (kappa)

$$\therefore \quad \kappa = \frac{d\psi}{ds}$$

's' and '$\psi$' are called as intrinsic coordinates and $f(s, \psi) = 0$ is called as the intrinsic equation of the curve.

Curvature is a positive quantity, $\kappa = \left|\frac{d\psi}{ds}\right|$ and if $\kappa \neq 0$, then $\rho = \frac{1}{\kappa} = \frac{ds}{d\psi}$ is called as radius of curvature.

The curvature of a straight line is zero and for a circle it is a constant at any point.

**Radius of curvature for a cartesian curve:**

Let the equation of curve be $y = f(x)$, then at any point $P(x,y)$, slope of tangent is

$$y' = \tan(\psi)$$

Differentiating the last expression w.r.t $x$, treating '$\psi$' as variable;

$$y'' = \sec^2(\psi) \cdot \frac{d\psi}{dx}$$

but, $1 + \tan^2(\psi) = \sec^2(\psi)$

$$\therefore \quad y'' = [1 + \tan^2(\psi)]\left[\frac{d\psi}{ds} \cdot \frac{ds}{dx}\right]$$

(Chain rule)

it can be proved that $\frac{ds}{dx} = \sqrt{1+(y')^2}$

$$\therefore \quad y'' = [1+(y')^2]\, k \cdot \sqrt{1+(y')^2}$$

$$= k \cdot [1+(y')^2]^{1+1/2}$$

$$= k[1+(y')^2]^{3/2}$$

$$\therefore \quad k = \frac{(y'')}{[1+(y')^2]^{3/2}}$$

$$\therefore \quad \rho = \frac{1}{k} = \frac{[1+(y')^2]^{3/2}}{y''}, \quad (y'' \neq 0)$$

$$\therefore \quad \boxed{\rho = \left|\frac{[1+(y')^2]^{3/2}}{y''}\right|}$$

If $y'' > 0$, curve is concave upwards and $\rho > 0$

and $y'' < 0$, curve is concave downwards, $\rho < 0$.

If that tangent drawn to curve is parallel to y-axis, $\psi = \pi/2$, $\frac{dy}{dx}$ does not exist, then radius of curvature is obtained using

$$\rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\left(\frac{d^2x}{dy^2}\right)}$$

Let the curve be $x = x(t)$ and $y = y(t)$, parametric form, and $\dot{x} = \frac{dx}{dt}$ and $\dot{y} = \frac{dy}{dt}$

using parametric differentiation, the radius of curvature is given by:

$$\rho = \frac{\left[(\dot{x})^2 + (\dot{y})^2\right]^{3/2}}{\left[\dot{x}\ddot{y} - \ddot{x}\dot{y}\right]}$$

here $\ddot{x} = \frac{d^2x}{dt^2}$ and $\ddot{y} = \frac{d^2y}{dt^2}$

Examples:

1. Show that the radius of curvature of circle is a constant.

Solution: The general equation of a circle with center $(x_0, y_0)$ is:

$$(x - x_0)^2 + (y - y_0)^2 = a^2 \quad (1)$$

The parametric equations are:

$$x - x_0 = a\cos(t)$$
$$y - y_0 = a\sin(t)$$

$$\therefore \left.\begin{aligned} x &= x_0 + a\cos(t) \\ y &= y_0 + a\sin(t) \end{aligned}\right\} \quad (2)$$

$$\dot{x} = -a\sin(t) \quad , \quad \dot{y} = a\cos(t)$$

$$\ddot{x} = -a\cos(t) \quad , \quad \ddot{y} = -a\sin(t)$$

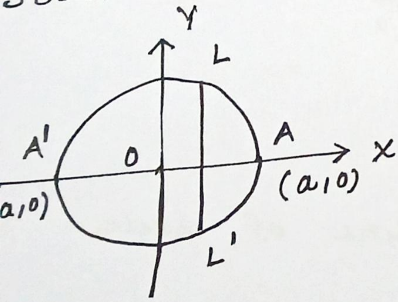
$$\rho = \frac{[(\dot{x})^2 + (\dot{y})^2]^{3/2}}{(\dot{x}\ddot{y} - \dot{y}\ddot{x})} = \frac{[a^2\sin^2(t) + a^2\cos^2(t)]^{3/2}}{[a^2\sin^2(t) + a^2\cos^2(t)]}$$

$$\rho = \frac{(a^2)^{3/2}}{a^2} = \frac{a^3}{a^2} = a \quad , \text{ constant.}$$

also, $\kappa = \frac{1}{a}$.

2. Show that the radius of curvature for an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, at the end of the major axis is equal to semi-latus rectum of the ellipse.

Solution: The parametric equations of an ellipse is $x = a\cos(t)$, $y = b\sin(t)$




$$\therefore \dot{x} = -a\sin(t) \quad , \quad \dot{y} = b\cos(t)$$

$$\ddot{x} = -a\cos(t) \quad , \quad \ddot{y} = -b\sin(t)$$

Since $$\rho = \frac{[(\dot{x})^2 + (\dot{y})^2]^{3/2}}{[\dot{x}\ddot{y} - \dot{y}\ddot{x}]} = \frac{[a^2\sin^2(t) + b^2\cos^2(t)]^{3/2}}{a\cdot b[\sin^2(t) + \cos^2(t)]}$$

x-axis for an ellipse is called as major axis,

$\therefore$ at $A(a,0)$, $x = a$ , $y = 0$

$\Rightarrow a\cos(t) = a$ , $b\sin(t) = 0$

$\Rightarrow \cos(t) = 1$ , $\sin(t) = 0$ as $b \neq 0$

$\therefore t = 0$

$$\therefore \rho = \frac{[0 + b^2]^{3/2}}{ab} = \frac{b^3}{ab} = \frac{b^2}{a}$$

Total length of latus rectum $= \frac{2b^2}{a} = LL'$

5. Find the radius of curvature at any point $P(x, y)$ on the curve $y = c.\cosh\left(\frac{x}{c}\right)$, $c > 0$.

Solution : Here $y = c.\cosh\left(\frac{x}{c}\right)$

$$\therefore \quad y' = c.\sinh\left(\frac{x}{c}\right).\frac{1}{c}$$

$$y' = \sinh\left(\frac{x}{c}\right)$$

and $$y'' = \frac{1}{c}\cosh\left(\frac{x}{c}\right)$$

The radius of curvature at any point $P(x, y)$ is

$$\rho = \frac{\left[1+(y')^2\right]^{3/2}}{y''} = \frac{\left[1+\sinh^2\left(\frac{x}{c}\right)\right]^{3/2}}{\frac{1}{c}\cosh(x/c)}$$

Since, $\cosh^2\left(\frac{x}{c}\right) - \sinh^2\left(\frac{x}{c}\right) = 1$

$$\therefore \quad 1+\sinh^2\left(\frac{x}{c}\right) = \cosh^2\left(\frac{x}{c}\right)$$

$$\rho = \frac{c.\left[\cosh^2(x/c)\right]^{3/2}}{\cosh(x/c)} \qquad \left(\because \frac{1}{1/c} = c\right)$$

$$= c.\frac{\cosh^3\left(\frac{x}{c}\right)}{\cosh(x/c)} = c.\cosh^2\left(\frac{x}{c}\right)$$

Now, $\frac{y}{c} = \cosh\left(\frac{x}{c}\right)$ eqn. of curve

$$\therefore \quad \rho = c.\frac{y^2}{c^2} = \frac{y^2}{c}$$

NOTE ON HYPERBOLIC FUNCTIONS:

1. $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$
2. $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$
3. $\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{(e^x - e^{-x})}{(e^x + e^{-x})}$

4. $\frac{d}{dx}[\sinh(x)] = \cosh(x)$ and $\frac{d}{dx}[\cosh(x)] = \sinh(x)$

5. $\sinh(-x) = -\sinh(x)$ and $\cosh(-x) = \cosh(x)$

6. $\cosh^2(x) - \sinh^2(x) = 1$

4. Find the radius of curvature of the curve $y = xe^{-x}$ at the point where $y$ is maximum.

Solution: Here $y = xe^{-x}$

$$\therefore \quad y' = x(-e^{-x}) + e^{-x}$$

$$y' = e^{-x}(1-x)$$

$$y'' = e^{-x}(-1) + (1-x)(-e^{-x})$$

$$y'' = e^{-x}(x-2)$$

For maximum, $y' = 0 \Rightarrow e^{-x}(1-x) = 0$

but $e^{-x} \neq 0$ for finite '$x$'. $[e^{-\infty} = 0]$

$\therefore \quad 1 - x = 0 \quad \therefore \quad x = 1$

at $x = 1$, $\quad y'' = -\frac{1}{e} < 0$

$\therefore$ $y$ is maximum at $x = 1$.

Hence $\quad \rho = \dfrac{[1 + (y')^2]^{3/2}}{y''}$

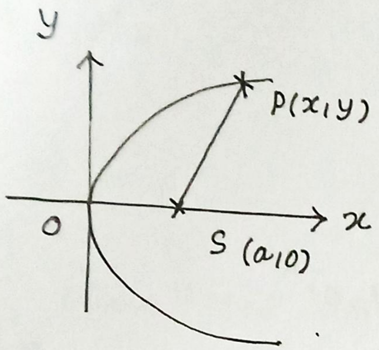
$$\rho = \frac{[1 + (1-x)^2 e^{-2x}]^{3/2}}{e^{-x}\cdot(x-2)}$$

Put $x = 1$, $\quad \rho = \dfrac{[1+0]^{3/2}}{(-1/e)} = -\dfrac{1}{(1/e)} = -e$

$\therefore \quad \rho = e \quad$ (Numerically)

Note : $e \approx 2.7182818$

For the parabola $y^2 = 4ax$, show that the square of the radius of curvature at any point varies as the cube of the focal distance of the point.




Given $y^2 = 4ax$

$$\therefore \quad 2y\frac{dy}{dx} = 4a$$

$$\therefore \quad y' = \frac{2a}{y}$$

and $$y'' = -\frac{2a}{y^2}\cdot\frac{dy}{dx}$$

$$\therefore \quad y'' = -\frac{2a}{4ax}\cdot\left(\frac{2a}{y}\right) = -\frac{a}{xy}$$

The radius of curvature at any point $P(x,y)$ is given by

$$\rho = \frac{[1+(y')^2]^{3/2}}{y''}$$

$$\rho = \frac{\left[1+\left(\frac{4a^2}{y^2}\right)\right]^{3/2}}{\left(-\frac{a}{xy}\right)} = \frac{\left(\frac{1}{y^3}\right)[y^2+4a^2]^{3/2}}{\left(-\frac{a}{xy}\right)}$$

$$= -\frac{xy}{a}\cdot\frac{1}{y^3}[4ax+4a^2]^{3/2}$$

$$= -\frac{x}{ay^2}(4ax+4a^2)^{3/2}$$

$$\rho = -\frac{x}{a(4ax)}[4ax+4a^2]^{3/2}$$

$$\rho = -\frac{4^{3/2}a^{3/2}(x+a)^{3/2}}{4a^2}$$

$$\therefore \quad \rho = -\frac{4^{\frac{1}{2}}}{a^{1/2}}(x+a)^{3/2}$$

$\left[ \text{as } \frac{4^{3/2}}{4} = 4^{\frac{1}{2}} \text{ and } \frac{a^{3/2}}{a^2} = \frac{1}{a^{1/2}} \right]$

Squaring both sides, we get :

$$\therefore \quad \rho^2 = \frac{4}{a}(x+a)^3 \quad (1)$$

Now using distance formula,

$$SP^2 = (x-a)^2 + (y-0)^2$$

$$= (x-a)^2 + y^2 = x^2 - 2ax + a^2 + 4ax$$

$$\therefore \quad SP^2 = x^2 + 2ax + a^2 = (x+a)^2$$

$$\therefore \quad SP = (x+a) \quad = \text{Focal distance of a point.}$$

from (1), $\rho^2 = \frac{4}{a}(SP)^3$

Hence, the required result.

6. Find the radius of curvature of $y^2 = \frac{a^2(a-x)}{x}$, at the point $(a,0)$.

Here $\quad y^2 = \frac{a^2(a-x)}{x}$

$$\therefore \quad 2yy' = \frac{a^2[x(0-1)-(a-x)1]}{x^2} = -\frac{a^3}{x^2}$$

$$\therefore \quad y' = -\frac{a^3}{2x^2y}$$ at $(a,0)$, $y'$ does not exist.

Hence, $\frac{dx}{dy} = -\frac{2x^2y}{a^3}$

$$\frac{d^2x}{dy^2} = -\frac{2}{a^3}\left[x^2(1) + y\,2x\,\frac{dx}{dy}\right]$$

At the point $\frac{dx}{dy} = 0$, and $\frac{d^2x}{dy^2} = -\frac{2}{a}$

$$\therefore \quad \rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\left(\frac{d^2x}{dy^2}\right)} = \frac{(1+0)^{3/2}}{(-2/a)} = -\frac{a}{2}$$

$$\therefore \quad \rho = \frac{a}{2} \quad \text{(numerically)}.$$

7. Show that the radius of curvature for the curve $x = a(t + \sin t)$, $y = a(1 - \cos t)$ is given by $\rho = 4a\cos\left(\frac{t}{2}\right)$

Solution: Here curve is in parametric form:

$$x = a(t + \sin t), \quad y = a(1 - \cos t)$$

$$\dot{x} = a(1 + \cos t), \quad \dot{y} = a\sin(t)$$

$$\ddot{x} = -a\sin(t) \quad \ddot{y} = a\cos(t)$$

$$\rho = \frac{[(\dot{x})^2 + (\dot{y})^2]^{3/2}}{[\dot{x}\ddot{y} - \dot{y}\ddot{x}]} = \frac{[a^2(1+\cos t)^2 + a^2\sin^2(t)]^{3/2}}{a^2(1+\cos t)\cos t + a^2\sin^2(t)}$$

$$= \frac{(a^2)^{3/2}[1 + 2\cos(t) + \cos^2(t) + \sin^2(t)]^{3/2}}{a^2[\cos(t) + \cos^2(t) + \sin^2(t)]}$$

$$= a^3\frac{[2 + 2\cos(t)]^{3/2}}{[1 + \cos(t)]} = \frac{2^{3/2} \cdot a^3[1+\cos(t)]^{3/2}}{[1+\cos(t)]}$$

$$= 2\sqrt{2}\,a[1 + \cos(t)]^{1/2} = 2\sqrt{2}\,a\left[2\cos^2(t/2)\right]^{1/2}$$

$$\therefore \quad \rho = (2\sqrt{2})\,a\sqrt{2}\cos(t/2) = 4a\cos(t/2)$$

## Taylor Series :

Let $f(x)$ be a defined on $[a, a+h]$ such that $f(x)$ has derivatives of all orders in $[a, a+h]$ and the remainder after '$n$' terms

$$R_n = \frac{h^n}{n!} f^{(n)}(a+\theta h) \to 0 \text{ as } n \to \infty \quad (0<\theta<1)$$

then,
$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \cdots$$

the series is known as Taylor series

if $(a+h) = x$ and $h = (x-a)$

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$$

$$f(x) = f(a) + \sum_{r=1}^{\infty} \frac{(x-a)^r}{r!} f^{(r)}(a) \quad (1)$$

The series on RHS gives the expansion of $f(x)$ in ascending powers of $(x-a)$; called as Taylor series of $f(x)$ about $x=a$.

If the point is origin, Taylor series expansion about $x=0$ is given by :

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

$$f(x) = f(0) + \sum_{r=1}^{\infty} \frac{x^r}{r!} f^{(r)}(0) \quad (2)$$

This series is known as Maclaurin series.

Taylor series allow a function to be approximated in terms of polynomials of a specified order. This gives a good indication of how the function behaves locally.

## Some Standard Maclaurin Series:

1. $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots$

2. $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots$

3. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

4 $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - + \cdots$

5. $\sinh(x) = \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

6. $\cosh(x) = \frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

Examples:

1. Expand $f(x) = \cos(x)$ as a Taylor series in ascending powers of $(x - \pi/4)$ up to terms containing $(x - \pi/4)^4$ and hence obtain approximate value of $\cos(46^\circ)$.

Solution: Here $f(x) = \cos(x)$ and $a = \pi/4$

Obtaining successive derivatives of $f(x)$;

$$f'(x) = -\sin(x), \quad f'(\pi/4) = -1/\sqrt{2}$$

$$f''(x) = -\cos(x), \quad f''(\pi/4) = -1/\sqrt{2}$$

$$f'''(x) = \sin(x), \quad f'''(\pi/4) = 1/\sqrt{2}$$

$$f^{(IV)}(x) = \cos(x), \quad f^{(IV)}(\pi/4) = 1/\sqrt{2}$$

$\therefore$ By Taylor series,

$$f(x) = f(\pi/4) + \frac{(x - \pi/4)}{1!} f'(\pi/4) + \frac{\left(x - \frac{\pi}{4}\right)^2}{2!} f''(\pi/4) + \cdots$$

$$f(x) = \frac{1}{\sqrt{2}} - \left(x - \frac{\pi}{4}\right)\frac{1}{\sqrt{2}} - \frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 \frac{1}{\sqrt{2}} + \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{4!}(x - \pi/4)^4\left(\frac{1}{\sqrt{2}}\right) + \cdots$$

$$\therefore \cos(x) \approx \frac{1}{\sqrt{2}}\left[1-\left(x-\frac{\pi}{4}\right)-\frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2+\frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3+\frac{1}{4!}\left(x-\frac{\pi}{4}\right)\right]$$

Now to obtain $\cos(46^\circ)$, we proceed as follows:

put $x = 46^\circ$, but RHS is in radians and

$\cos(x)$ is diffirentiable if $x$ is in radians

$$\left(x-\frac{\pi}{4}\right) = 1^\circ = \frac{\pi}{180} \text{ radians}$$

$$\therefore \cos(46^\circ) \approx \frac{1}{\sqrt{2}}\left[1-\left(\frac{\pi}{180}\right)-\frac{1}{2!}\left(\frac{\pi}{180}\right)^2+\frac{1}{3!}\left(\frac{\pi}{180}\right)^3+\frac{1}{4!}\left(\frac{\pi}{180}\right)^4\right]$$

$$\cos(46^\circ) \approx 0.694658$$

2. Expand $y=\sqrt{x}$ as Taylor Series about the point $x=1$; up to fourth degree terms.

Solution: Here $y = x^{1/2}$ $\qquad y(1) = 1$

$\therefore y' = \frac{1}{2}x^{-1/2}$ $\qquad y'(1) = \frac{1}{2}$

$\therefore y'' = -\frac{1}{4}x^{-3/2}$ $\qquad y''(1) = -\frac{1}{4}$

$\therefore y''' = \frac{3}{8}x^{-5/2}$ $\qquad y'''(1) = 3/8$

$\therefore y^{(IV)} = -\frac{15}{16}x^{-7/2}$ $\qquad y^{(IV)}(1) = -\frac{15}{16}$

$$\therefore f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \cdots$$

$$f(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{4}\frac{(x-1)^2}{2!} + \frac{3}{8}\frac{(x-1)^3}{3!} - \frac{15}{16}\frac{(x-1)^4}{4!} + \cdots$$

$$\sqrt{x} \approx 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{(x-1)^3}{16} - \frac{5}{128}(x-1)^4$$

Expand $y = \log(1+x)$ in ascending powers of '$x$', hence Show that $\log\left(\sqrt{\frac{1+x}{1-x}}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$

Solution:

Let $f(x) = \log(1+x)$ $\qquad f(0) = 0$

$\therefore f'(x) = \frac{1}{(1+x)}$ $\qquad f'(0) = 1$

$\therefore f''(x) = -\frac{1}{(1+x)^2}$ $\qquad f''(0) = -1$

$\therefore f'''(x) = \frac{+2}{(1+x)^3}$ $\qquad f'''(0) = 2$

$f^{(IV)}(x) = -\frac{6}{(1+x)^4}$ $\qquad f^{(IV)}(0) = -6$

By Maclaurin series,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

$$f(x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) - \frac{x^4}{4!}(-6) + \cdots$$

$$\therefore \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - \cdots \quad (1)$$

Replace $x$ by $-x$, we get:

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

$$\therefore -\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots \quad (2)$$

Now $\log\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

$$= \frac{1}{2}\left[\log(1+x) - \log(1-x)\right]$$

$$= \frac{1}{2}\left[2x + \frac{2x^3}{3} + \cdots\right] \text{, using (1) and (2)}$$

$$\therefore \log\sqrt{\frac{1+x}{1-x}} = x + \frac{x^3}{3} + \cdots$$

4. Expand $y = \log[\sec(x)]$ as a series in powers of $x$; hence obtain the series expansion for $\tan(x)$ 5

Solution: Here $y = \log[\sec(x)]$

$$\therefore \quad y' = \frac{1}{\sec(x)} \sec(x).\tan(x)$$

$$y' = \tan(x)$$

Obtaining successive derivatives,

$$y'' = \sec^2(x)$$

$$= 1 + \tan^2(x)$$

$$y'' = 1 + (y')^2$$

$$y''' = 2(y')(y'')$$

$$y^{(IV)} = 2[(y')(y''') + (y'')^2]$$

put $x = 0$, Now $\sec(0) = 1$ $\log(1) = 0$

$$y(0) = 0, \quad y'(0) = \tan(0) = 0$$

$$y''(0) = 1 + (0)^2 = 1$$

$$y'''(0) = 2(0)(1) = 0$$

$$y^{(IV)}(0) = 2[(0)(0) + (1^2)] = 2$$

Now

$$y = y(0) + \frac{x}{1!} \cdot y'(0) + \frac{x^2}{2!} y''(0) + \cdots$$

$$\log[\sec(x)] = 0 + \frac{x}{1!}(0) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(2) + \cdots$$

$$\log[\sec(x)] = \frac{x^2}{2} + \frac{x^4}{12} + \cdots \quad (1)$$

Obtain the series expansion for $\tan(x)$;

get derivative of (1), w.r.t. $x$

i.e. $\frac{d}{dx}[\log\{\sec(x)\}] = \frac{d}{dx}\left[\frac{x^2}{2} + \frac{x^4}{12} + \cdots\right]$

$$\therefore \quad \tan(x) = x + \frac{x^3}{3} + \cdots$$

Expand $f(x) = \tan^{-1}(x)$ as a series in ascending powers of '$x$', hence obtain series for $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Solution: Let $y = \tan^{-1}(x)$

$$\therefore \quad y' = \frac{1}{(1+x^2)}$$

i.e. $\frac{dy}{dx} = (1+x^2)^{-1}$

we have binomial series expansion;

$$(1+t)^{-1} = 1 - t + t^2 - t^3 + - \cdots \quad \text{if } |t| < 1$$

$$\therefore \quad \frac{dy}{dx} = 1 - x^2 + x^4 - x^6 + - \cdots$$

$$dy = [1 - x^2 + x^4 - x^6 + - \cdots]\, dx$$

Integrating, both sides, we get:

$$y = \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + - \cdots\right] + c$$

put $x = 0$, $y(0) = \tan^{-1}(0) = 0 \quad \therefore c = 0$

Hence, $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + - \cdots$

is the required expansion.

Consider $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$,

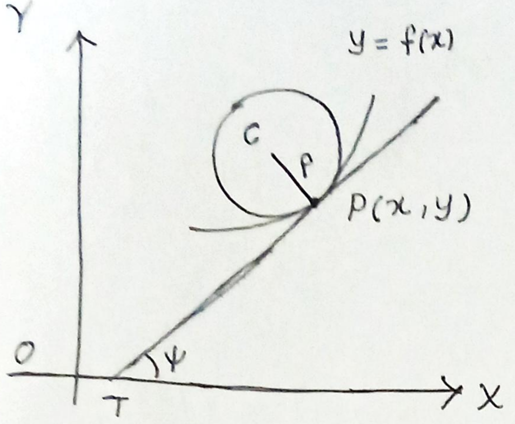
put $x = \tan(\theta/2)$,

then $\frac{2\tan(\theta/2)}{[1 + \tan^2(\theta/2)]} = \sin(\theta)$

$$f(x) = \sin^{-1}[\sin(\theta)] = \theta = 2\tan^{-1}(x)$$

$$\therefore \quad \sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + - \cdots\right]$$

## Centre of Curvature and Circle of Curvature:




Consider a curve $y = f(x)$; and $P(x, y)$ be any point on curve. PT be the tangent to curve at $P(x, y)$, and CP normal at P such that $CP = \rho$ = radius of curvature.

The point C is called as the centre of curvature. A circle can be constructed with C as centre and '$\rho$' as radius known as circle of curvature.

The coordinates of centre of curvature is $C = (\bar{x}, \bar{y})$

Here $\bar{x} = x - \dfrac{y_1(1+y_1^2)}{y_2}$, $\bar{y} = y + \dfrac{(1+y_1^2)}{y_2}$

and circle of curvature is given by:

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2,$$ $y_1 = \dfrac{dy}{dx}$ & $y_2 = \dfrac{d^2y}{dx^2}$

**Examples:**

1. Find the circle of curvature at (3, 4) on $xy = 12$

Solution: Here $y = \dfrac{12}{x}$ ①

$\therefore y_1 = \dfrac{dy}{dx} = \dfrac{-12}{x^2}$ and $y_2 = \dfrac{d^2y}{dx^2} = \dfrac{24}{x^3}$

at (3, 4), $y_1 = \dfrac{-12}{9} = -4/3$, $y_2 = \dfrac{24}{27} = \dfrac{8}{9}$

The coordinates of centre of curvature is given by:

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = 3 - \frac{(-4/3)(1+\frac{16}{9})}{(8/9)}$$

$$\bar{x} = 43/6$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2} = 4 + \frac{\left(1+\frac{16}{9}\right)}{(8/9)} = \frac{57}{8}$$

$\therefore \quad (\bar{x}, \bar{y}) = (43/6,\ 57/8)$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left(\frac{d^2y}{dx^2}\right)} = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1+\frac{16}{9}\right)^{3/2}}{8/9} = \frac{125}{24}$$

$\therefore$ The circle of curvature is

$$\left(x - 43/6\right)^2 + \left(y - 57/8\right)^2 = \left(\frac{125}{24}\right)^2.$$

2. Find the centre of curvature and the circle of curvature on the curve $y = e^x$ at a point where the curve crosses the $y$-axis.

Solution: Here $y = e^x$, $y_1 = e^x$ and $y_2 = e^x$

The equation of $y$-axis is $x = 0$

(on $y$-axis $x=0$) $\therefore$ $y = e^0 = 1$ $\therefore$ $P = (0, 1)$

Now $y_1 = 1$, $y_2 = 1$ at $(0, 1)$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{1} = 2^{3/2} = 2\sqrt{2}$$

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = 0 - \frac{1\,(1+1^2)}{1} = -2$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2} = 1 + \frac{(1+1)}{1} = 3$$

$\therefore$ Centre of curvature is $(\bar{x}, \bar{y}) = (-2, 3)$

Circle of curvature is

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

$$(x+2)^2 + (y-3)^2 = 8$$