UNIT-V

NUMERICAL METHODS

Introduction:

Numerical analysis plays a great role in engineering and in the quantitative parts of pure and applied science. Interpolation, the computing of values for a tabulated function at points not in the table, is historically a most important task. Many famous mathematicians have their names associated with procedures for interpolation: Gauss, Newton, Bessel, Stirling. The need to interpolate began with the early studies of astronomy when the motion of heavenly bodies was to be determined from periodic observations. Interpolation methods demonstrate some important theory about polynomials and the accuracy of numerical methods. Interpolating with polynomials serves as an excellent introduction to some techniques for drawing smooth curves. These methods are the basis of many other procedures. Among these procedures, we will focus on numerical differentiation and integration in this unit.

Interpolation and Extrapolation:

Interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable while the process of computing the value of the function outside the given range is called extrapolation.

Finite Differences:

The finite difference deals with the changes in the value of the function (dependent variable) due to the changes in the values of independent variable. The values of the independent variable x are called Arguments and the corresponding values of dependent variables y are called Entries.

Forward difference:

If $y_0, y_1, y_2, \ldots\ldots\ y_n$ denote the set of values of the function y=f(x).Then Δy_0=y_1- y_2, $\Delta\ y_1$=y_2-y_1........., Δy_{n-1}=y_n-$y_{n-1}$are called the First Forward Differences of the function y=f(x) where Δ is called the forward difference operator. The differences are called first forward differences denoted by Δy_0, $\Delta\ y_1 \ldots\ldots\ldots \Delta\ y_n$, second forward differences and are denoted by $\Delta^2\ y_0$, $\Delta^2\ y_1$,......, $\Delta^2 y_{n-1}$. i.e. $\Delta^2 y_0$= Δy_1- Δy_0, $\Delta^2 y_1$= Δy_2 - Δy_1 , ..., $\Delta^2 y_{n-1}$= Δy_n - $\Delta\ y_{n-1}$.

Similarly we can define third, fourth forward differences etc.

In general, the nth forward differences is defined by the equation, $\Delta^n y_i = \Delta^{n-1}\, y_{i+1} - \Delta^{n-1} y_i$

Forward Difference Table:

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
$x_1 = x_0 + h$	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
$x_2 = x_0 + 2h$	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
$x_3 = x_0 + 3h$	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
$x_4 = x_0 + 4h$	y_4		$\Delta^2 y_3$			
		Δy_4				
$x_5 = x_0 + 5h$	y_5					

Here y_0 is called the First Entry and Δy_0 ,$\Delta^2 y_0$, $\Delta^3 y_0$,.........., $\Delta^5 y_0$ are called the Leading Differences.

Backward Difference:

Consider the function y = f(x). If y_0 , y_1 , y_2 ,...., y_n denote the set of values of y. Then $\nabla y_1 = y_1 - y_0$, $\nabla y_2 = y_2 - y_1$, ...,$\nabla y_n = y_n - y_{n-1}$ are called the First Backward Differences and ∇ is called the backward difference operator.

The second backward difference of the function is given by,

$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$, $\nabla^2 y_3 = \nabla y_3 - \nabla y_2$, …….., $\nabla^n y_n = \nabla y_n - \nabla y_{n-1}$

Similarly we can define higher order differences.

In general the nth backward difference is given by $\nabla^n y_i = \nabla^{n-1} y_i - \nabla^{n-1} y_{i-1}$

Backward Difference Table:

x	$y = f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$

x_0	y_0					
$x_1 = x_0 + h$	y_1	∇y_1	$\nabla^2 y_2$	$\nabla^2 y_3$		
$x_2 = x_0 + 2h$	y_2	∇y_2	$\nabla^2 y_3$	$\nabla^3 y_4$	$\nabla^4 y_4$	$\nabla^5 y_5$
$x_3 = x_0 + 3h$	y_3	∇y_3	$\nabla^2 y_4$	$\nabla^3 y_5$	$\nabla^4 y_5$	
$x_4 = x_0 + 4h$	y_4	∇y_4	$\nabla^2 y_5$			
$x_5 = x_0 + 5h$	y_5	∇y_5				

NOTE: Only the notations are changed not the differences.

(i) Relation between forward and backward operators: $\Delta^n y_r = \nabla^n y_{n+r}$

(ii) $f(x) = f(x + h) - f(x)$

(iii) $\nabla f(x) = f(x) - f(x - h)$

Problems:

1. Evaluate $\Delta \tan^{-1} x$

Solution: $\Delta \tan^{-1} x = \tan^{-1}(x + h) - \tan^{-1} x$

$$= \tan^{-1}\left\{\frac{x+h-x}{1+(x+h)x}\right\} = \tan^{-1}\left\{\frac{h}{1+hx+x^2}\right\}$$

2. Evaluate $\Delta^2 \cos 2x$

Solution: $\Delta^2 \cos 2x = \Delta\{\cos 2(x+h) - \cos 2x\}$

$= \Delta \cos 2(x + h) - \Delta \cos 2x$

$= [\cos 2(x + 2h) - \cos 2(x + h)] - [\cos 2(x + h) - \cos 2x]$

$= -2 \sin(2x + 3h) \sin h + 2 \sin(2x + h) \sin h$

$= -2 \sin h [\sin(2x + 3h) - \sin(2x + h)]$

$= -2 \sin h [2 \cos(2x + 2h) \sin h]$

$= -4 \sin^2 h \cos(2x + 2h)$

Differences of a Polynomial:

The n^{th} differences of a polynomial of the n^{th} degree are constant and all higher order differences are zero.

Let $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$, then

$\Delta^n f(x) = a_0 n(n-1)(n-2)\ldots 1.\, h^n = a_0\, n!\, h^n$ ....(1)

and then for higher orders $\Delta^{n+1}f(x) = \Delta^{n+2}f(x) = \cdots = 0$. ....(2)

Problems:

1. Evaluate $\Delta^{10}[(1 - ax)(1 - bx^2)(1 - cx^3)(1 - dx^4)]$ with h=1, h=2.

Solution: Given,

$\Delta^{10}[(1 - ax)(1 - bx^2)(1 - cx^3)(1 - dx^4)] = \Delta^{10}[abcd\, x^{10} + (\,)\, x^9 + (\,)\, x^8 + \ldots\ldots + 1]$

$= abcd\ \Delta^{10}\ (x^{10})$ $\qquad [\Delta^{10}\ (x^n) = 0 \text{ or } n < 10]$

$= abcd\ (10\ !)$ $\qquad$ (for $h = 1$)

Similarly for h =2, $\qquad = abcd\ (10!)\ (2)^{10}$

2. Construct the forward difference table, given that

x	5	10	15	20	25	30
y	9962	9848	9659	9397	9063	8660

and find the values of $\Delta^2 y_{10}$,$\Delta^3 y_5$, $\Delta^4 y_5$.

Solution:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
5	9962				
		− 114			
10	9848		− 75		
		− 189		2	
15	9659		**− 73**		−1
		− 262		1	
20	9397		− 72		2
		− 334		3	
25	9063		− 69		
		− 403			
30	8660				

Now $\Delta^2 y_{10}$ = - 73 which is second element for the column $\Delta^2 y$.
$\Delta^3 y_5$ = 2 which is the first element of the column $\Delta^3 y$.Similarly $\Delta^4 y_5$= - 1.

3. Construct the finite difference table for the function $f(x) = x^3 + x + 1$ where x takes the values 0,1,2,3,4,5,6. Identify the leading forward and backward differences**.** Hence find $\Delta^2 y_{1,}$ $\nabla^3 y_5$.

Solution:

x	y	First difference	Second difference	Third difference	Fourth difference
0	1				
		2			
1	3		**6**		
		8		**6**	
2	11		12		0
		20		6	
3	31		18		0
		38		6	
4	69		24		0
		62		**6**	
5	131		**30**		
		92			
6	223				

The leading forward differences are 2, 6, 6 and leading backward differences are 92, 30, 6. $\Delta^2 y_1 = 12, \nabla^3 y_5 = 6.$

Note: Third differences are constants and higher order differences are zero as $f(x)$ is a polynomial of third degree.

Interpolation with equal intervals:

Newton's Forward Interpolation Formula:

Let the function $y = f(x)$ takes the values $y_{0,}\ y_1$......., y_n at the points x_0, x_1......., x_n where $x_i = x_0 + ih$. Then Newton's Forward interpolation polynomial is given by,

$y_p = f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ \frac{p(p-1)\ldots\ldots(p-n-1)}{n!}\Delta^n y_0,$

where x= x_0+ ph

Newton's Backward Interpolation Formula:

Let the function y = f(x) takes the values y_0, $y_{1,}$......., y_n at the points x_0, x_1......., x_n,

where $x_i = x_0 + ih$. The Newton's Backward Interpolation polynomial is given by,

$y_p = f(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots\ldots\ldots\ \frac{p(p+1)\ldots\ldots(p+n-1)}{n!}\nabla^n y_n,$

where $x = x_n + ph$

Remark:

Newton's Forward Interpolation Formula is used to interpolate the values of y near the beginning of the set of tabulated values or for extrapolating values of y to the left of the beginning. Newton's Backward Interpolation Formula is used to interpolate the values of y near the end of the set of tabulated values or for extrapolating values of y to the right of the last tabulated value y.

Examples:

1. Find a cubic polynomial which takes the following data

x	0	1	2	3
$f(x)$	1	2	1	10

Solution: The forward difference table is given by,

x	y = f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

Here, x_0= 0, h = 1, $p = \frac{x-x_0}{h} = \frac{x-0}{1} = x$, Δy_0=1, $\Delta^2 y_0$= -2, $\Delta^3 y_0$=12,

By Newton-Gregory forward interpolation formula we have

$f(x) = y_0 + p\Delta y_0 + + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$

$$f(x) = 1 + x(1) + + \frac{x(x-1)}{2!}(-2) + \frac{x(x-1)(x-2)}{3!}(12)$$

$f(x) = 2x^3 - 7x^2 + 6x + 1$ is the required polynomial.

2. The table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface.

x =height:	150	200	250	300	350	400
y=distance:	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of y when $x = 160ft$ and $x = 410ft$.

Solution:

The difference table is

x	y	First difference	Second difference	Third difference	Fourth difference	Fifth difference
150	**13.03**					
		2.01				
200	15.04		**-0.24**			
		1.77		**0.08**		
250	16.81		-0.16		**-0.05**	
		1.61		0.03		**0.04**
300	18.42		-0.13		**-0.01**	
		1.48		**0.02**		
350	19.90		**-0.11**			
		1.37				
400	**21.27**					

(i) x_0=160, y_0=13.03, Δy_0= 2.01, $\Delta^2 y_0$ = -0.24, $\Delta^3 y_0$ =0.08, $\Delta^4 y_0$=-0.05, $h = 50$

$p = \frac{x-x_0}{h} = \frac{10}{50} = 0.2$

Using Newton's forward interpolation formula, we get

$f(160) = y_0 + \text{p}\Delta y_0 + + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots$

$f(160) = 13.03 + 0.402 + 0.0192 + 0.00384 + 0.00168 + 0.001$

$f(160) = 13.46$ nautical miles.

(ii) x =410, x_n=400, y_n=21.27, ∇y_n=1.37,$\nabla^2 y_n$=-0.11, $\nabla^3 y_n$=0.02, $h = 50$

$p = \frac{x-x_n}{h} = \frac{10}{50} = 0.2$

Using Newton's backward interpolation formula we get,

$f(410) = y_n + \text{p}\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n \ldots\ldots$

$$f(410) = 21.27 + 0.2(1.37) + \frac{0.2(1.2)}{2}(-0.11) + \cdots$$
$$f(410) = 21.27 + 0.274 + 0.0132 + 0.0018 + 0 + 0.0024$$
$f(410) = 21.53$ nautical miles.

3. From the following table, estimate the number of students who obtained marks between 40 and 45:

Marks (x):	30-40	40-50	50-60	60-70	70-80
No. of students(y):	31	42	51	35	31

Solution: we prepare cumulative frequency table as follows:

Marks less than(x):	40	50	60	70	80
No. of students(y):	31	73	124	159	190

Now the difference table is:

x	y	Δy	Δ^2y	Δ^3 y	Δ^4 y
40	31				
		42			
50	73		9		
		51		−25	
60	124		−16		37
		35		12	
70	159		−4		
		31			
80	190				

To find $y(45)$ i.e. number of students with marks less than 45.

Taking $x_0 = 40$, $x = 45, h = 10$, $p = \frac{x-x_0}{h} = \frac{5}{10} = 0.5$

Using Newton's Forward Interpolation formula we get,

$$y(45) = y_{40} + p\Delta y_{40} + \frac{p(p-1)}{2}\Delta^2 y_{40} + \frac{p(p-1)(p-2)}{6}\Delta^3 y_{40} + \cdots$$

$$=31 + 0.5(42) + \frac{0.5(-0.5)}{2}(9) + \frac{0.5(0.5)(-1.5)}{6}(-25) + \frac{0.5(-0.5)(-1.5)(-2.5)}{24}(37)$$

$= 47.87$

The number of students with marks less than 45 is 47.87 ≅48.

But the number of students with marks less than 40 is 31.

Hence the number of students getting marks between 40 and 45 =48-31=17.

Exercise:

1. Fit a cubic polynomial to the following data using suitable interpolation formula.

x	0	1	2	3
$f(x)$	-2	2	12	34

2. Using Newton-Gregory Interpolation formula, estimate $f\ (0.12)$ from the following data.

x	0.10	0.15	0.20	0.25	0.30
$f(x)$	0.1003	0.1511	0.2027	0.2553	0.3093

3. Apply Newton's backward difference interpolation formula to find $f(7.5)$ from the following table:

x	1	2	3	4	5	6	7	8
$f(x)$	1	8	27	64	125	216	343	512

4. Using Newton -Gregory Interpolation formulae, find $f(17)$ from the following data.

x	0	4	8	12	16	20	24
$f(x)$	0	0.0699	0.1405	0.2126	0.2867	0.3640	0.4452

Interpolation with unequal intervals

Lagrange's formula for unequal intervals:

Let $y = f(x)$ be a function whose values are $y_0,\ y_1, y_2, \ldots\ldots, y_n$ corresponding to $x_0,\ x_1, x_2, \ldots\ldots, x_n$ not necessarily equally spaced.

$$f\ (x) = \frac{(x-x_1)\ (x-x_2) \ldots\ldots\ldots (x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots\ldots\ldots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\ldots\ldots\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots\ldots\ldots(x_1-x_n)} y_1$$
$$+ \cdots + \frac{(x-x_0)(x-x_1)\ldots\ldots\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots\ldots\ldots(x_n-x_{n-1})}\ y_n$$

This formula is known as Lagrange's Interpolation formula.

Inverse interpolation:

The process of estimating the value of x for a given value of y is called Inverse interpolation. So far given a table of values of x and y, using one of the interpolation formulae we find the value of y corresponding to some value of x which is not in the table. On the other hand the process of estimating the value of x for some value of y which is not in the table is called inverse interpolation.

This method is used when the values of x are not necessarily equally spaced. Lagrange's interpolation formula can be simply viewed as a relation between two variables and any one of the variable can be taken as an independent variable. Therefore, inverse interpolation formula can be obtained by interchanging the variables x and y in Lagrange's formula, we get.

$$x = \frac{(y-y_1)(y-y_2)\dots\dots\dots\dots(y-y_n)}{(y_0-y_1)(y_0-y_2)\dots\dots\dots(y_0-y_n)}\, x_0\, \frac{(y-y_0)(y-y_2)\dots\dots(y-y_n)}{(y_1-y_0)(y_1-y_2)\dots\dots\dots(y_1-y_n)} x_1 + \cdots + \frac{(y-y_0)(y-y_1)\dots\dots\dots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\dots\dots(y_n-y_{n-1})}\, x_n$$

Problems:

1. By using the Lagrange's interpolation formula to fit a polynomial to the data given

x	0	1	3	4
y	-12	0	6	12

Hence, find y when $x = 2$.

Solution:

$x_0 = 0 \quad x_1 = 1 \quad x_2 = 3 \quad x_3 = 4$

$y_0 = -12 \quad y_1 = 0 \quad y_2 = 6 \quad y_2 = 12$

By Lagrange's formula

$$y = \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)}(-12) + \frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-4)}(0) + \frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)}(6) + \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)}(12)$$

$y = (x-1)(x-3)(x-4) - (x)(x-1)(x-4) + x(x-1)(x-3)$

$= (x-1)[x^2 - 7x + 12 - x^2 + 4x + x^2 - 3x]$

$= (x-1)(x^2 - 6x + 12)$

$y = x^3 - 7x^2 + 18x - 12$

for $x = 2$, we get $y = 4$.

2. Apply Lagrange's formula to find f(5) and f(6) given that f(1) =2, f(2)=4,f(3)=8,f(7)=128 and explain why the results differ from those obtained by f(x) = 2^x.

Solution: $x_0 = 1 \quad x_1 = 2 \quad x_2 = 2 \quad x_3 = 7$

By Lagrange's formula, we have

$$f(x) = \frac{(x-2)(x-3)(x-7)}{(1-2)(1-3)(1-7)} f(x_0) + \frac{(x-1)(x-3)(x-7)}{(2-1)(2-3)(2-7)} f(x_1) +$$
$$\frac{(x-1)(x-3)(x-17)}{(3-1)(3-2)(1-7)} f(x_2) + \frac{(x-1)(x-2)(x-3)}{(7-1)(7-2)(7-3)} f(x_3)$$

$$f(x) = \frac{(x-2)(x-3)(x-7)}{-12}(2) + \frac{(x-1)(x-3)(x-7)}{5}(4) +$$
$$\frac{(x-1)(x-2)(x-17)}{-8}(8) + \frac{(x-1)(x-2)(x-3)}{120}(128)$$

$$f(5) = \frac{(3)(2)(-2)}{-12}(2) + \frac{(4)(2)(-2)}{5}(4) + \frac{(4)(3)(-2)}{-8}(8) + \frac{(4)(3)(2)}{120}(128)$$

$$f(5) = 2 - 12.8 + 24 + 25.6 = 38.8$$

$$f(6) = \frac{(4)(3)(-1)}{-12}(2) + \frac{(5)(3)(-1)}{5}(4) + \frac{(5)(4)(-1)}{-8}(8) + \frac{(5)(4)(3)}{120}(128)$$

$$f(6) = 2 - 12 + 20 + 64 = 74$$

But actual values of f(5) and f(6) are f(5) = 2^5= 32 and f(6) = 2^6 = 64.

The difference in values of f(5) and f(6) are due to the assumption of f(x) as a polynomial, when it is an exponential function of the form 2^x.

3. Find x for $y = 7$

x	1	3	4
y	4	12	19

Solution:

x_0=1 x_1=3 x_2=4

y_0= 4 y_1 = 12 y_2 = 14

$$x = \frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)} x_0 + \frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)} x_1 + \frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)} x_2$$

$$= \frac{(y-12)(y-19)}{(4-12)(4-19)}(1) + \frac{(y-4)(y-19)}{(12-4)(12-19)}(3) + \frac{(y-4)(y-12)}{(19-4)(19-12)}(4)$$

at $y = 7$

$$x = \frac{60}{120} + \frac{108}{56} - \frac{60}{105} = 1.85714$$

Numerical Differentiation:

Let the function y = f(x) is given by a table of values (x , y) then the process of computing the derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ etc. for some particular value of x is called Numerical Differentiation.

Derivatives using Newton's forward interpolation formula:

By Newton's Interpolation Formula, we have

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots \quad (1)$$

where, $x = x_0 + Ph$ (or) $P = \frac{x - x_0}{h}$ .... (2)

Differentiating (1) w.r.t. P, we get

$$\frac{dy}{dP} = \Delta y_0 + \frac{(2P-1)}{2!}\Delta^2 y_0 + \frac{3P^2+6P+2}{3!}\Delta^3 y_0 + \frac{(4P^3-18P^2+22P-6)}{4!}\Delta^4 y_0 + \ldots\ldots \quad (3)$$

Differentiating (2) w.r.t. x, we get

$\frac{dP}{dx} = \frac{1}{h}$.......... (4) But, $\frac{dy}{dx} = \frac{dy}{dP} \cdot \frac{dP}{dx}$

Using (3) and (4) the above equation becomes,

$$\left(\frac{dy}{dx}\right)_{x=x_0+Ph} = \frac{1}{h}\left[\Delta y_0 + \frac{(2P-1)}{2\,!}\Delta^2 y_0 + \frac{(3P^2-6P+2)}{3\,!}\Delta^3 y_0 + \frac{(4P^3-18P^2+22P-6)}{4\,!}\Delta^4 y_0 + \cdots\right] \ldots. (5)$$

At $x = x_0$, P= 0, the above equation becomes,

$$\left(\frac{dx}{dy}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right] \ldots. (6)$$

Formula (5) is used to compute y'at any point $x = x_0 + Ph$, whereas formula (6) is used to compute y'at any of the value of x when y is specified.

Similarly,

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0+Ph} = \frac{1}{h^2}\left[\Delta y_0 + (P-1)\Delta^3 y_0 + \frac{(6P^2-18P+11)}{12}\Delta^4 y_0 + \cdots\right] \ldots. (7)$$

and $\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right] \ldots. (8)$

The formula (7) is used to compute y" at any point $x = x_0 + Ph$ whereas formula (8) is used to compute y" at any value of x where y is specified.

Derivatives using Newton's backward interpolation formula:

By Newton's Backward Interpolation Formula, we have

$$y = y_n + \frac{P}{1!}\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n + \ldots\ldots \qquad \ldots\ldots\ldots (1)$$

Where, $x = x_n + Ph$ (or) $P = \frac{x-x_n}{h}$ ……… (2)

Differentiating (1) w.r.t. P, we get

$$\frac{dy}{dP} = \nabla y_n + \frac{(2P+1)}{2!}\nabla^2 y_n + \frac{(3P^2+6P+2)}{3!}\nabla^3 y_n + \ldots \qquad \ldots\ldots\ldots (3)$$

Differentiating (2) w.r.t. x, we get

$\frac{dP}{dx} = \frac{1}{h}$…. (4) But, $\frac{dy}{dx} = \frac{dy}{dP} \cdot \frac{dP}{dx}$

Using (3) and (4) the above equation becomes,

$$\left(\frac{dy}{dx}\right)_{x=x_n+Ph} = \frac{1}{h}\left[\nabla y_n + \frac{(2P+1)}{2!}\nabla^2 y_n + \frac{(3P^2+6P+2)}{3!}\nabla^3 y_n + \ldots\right] \qquad \ldots. (5)$$

At x $= x_n$, P = 0, the above equation becomes,

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right] \qquad \ldots. (6)$$

Formula (5) is used to compute y ′ at any point x $= x_n$ + Ph whereas formula (6) is used to compute y'at any of the values of x when y is specified.

Similarly

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n+Ph} = \frac{1}{h^2}\left[\nabla^2 y_n + (P+1)\nabla^3 y_n + \frac{(6P^2+18P+11)}{12}\nabla^4 y_n + \cdots\right] \qquad \ldots. (7)$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \cdots\right] \qquad \ldots. (8)$$

The formula (7) is used to compute y" at any point $x = x_n + P\,h$ whereas formula (8) is used to compute y" at any of the values of x when y is specified.

Problems:

1. Given

x	1.0	1.2	1.4	1.6	1.8	2.0
y	2.72	3.32	4.06	4.96	6.05	7.39

Find y' and y'' at x = 1.2.

Solution: Here, the step-length is $h = 0.2$. We first form the following difference table.

x	y	first differences	Second differences	Third differences	Fourth differences
1.0	2.27	0.60	0.14	0.02	0.01

1.2	3.32	0.74	0.16	0.03	0.03
1.4	4.06	0.90	0.19	0.06	
1.6	4.96	1.09	0.25		
1.8	6.05	1.34			
2.0	7.39				

We have to compute y' and y''at x = 1.2, which is a specified value of x, for this purpose, we take x_0 = 1.2. Then we find from the table that,

Δy_0= 0.74, $\Delta^2 y_0$= 0.16, $\Delta^3 y_0$= 0.03, $\Delta^4 y_0$= 0.03

Using the formula we have,

$$\left(\frac{dx}{dy}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

Then, $\left(\frac{dy}{dx}\right)_{(1.2)} = \frac{1}{(0.2)}\left[(0.74) - \frac{1}{2}(0.16) + \frac{1}{3}(0.03) - \frac{1}{4}(0.03)\right]$ = 3.3125

Using the formula we have,

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

And $\left(\frac{d^2y}{dx^2}\right)_{1.2} = \frac{1}{(0.2)^2}\left[(0.16) - (0.03) + \frac{11}{12}(0.03)\right]$ = 3.39375

2. Using appropriate interpolation formulas, find the values of $y'and\ y''$when x = 4 using the following table.

x	1	2	3	4
y	4	12	20	36

Solution: Here $x = 4$ is a specified value of x which is at the end of the given table.
For this purpose we take x_n = 4.
The difference table is given by,

x	y	First differences	Second differences	Third differences
1	4			
		8		
2	12		0	
		8		**8**
3	20		**8**	
		16		
4	36			

Here x_n= 4, ∇y_n = 16, $\nabla^2 y_n$= 8, $\nabla^3 y_n$= 8 and h=1

Then, $y'(4) = \frac{1}{h}\left[(\nabla y_n) + \frac{1}{2}(\nabla^2 y_n) + \frac{1}{3}(\nabla^3 y_n)\right] = \frac{1}{1}\left[16 + \frac{1}{2}(8) + \frac{1}{3}(8)\right]$

$$y'(4) = 16 + 4 + \frac{8}{3} = 22.667$$

And $y''(4) = \frac{1}{h^2}\left[\frac{1}{2}(\nabla^2 y_n) + (\nabla^3 y_n)\right] = \frac{1}{1^2}[8 + 8] = 16$

3. A rod is rotating in a plane, the following table gives the angle θ (in radians) through which the rod is turned for various values of time t (in seconds): Find the angular velocity and angular acceleration at t = 0.4 sec.

Solution: Given

t	0	0.2	0.4	0.6	0.8	1.0	1.2
θ	0	0.12	0.49	1.12	2.02	3.20	4.67

To compute velocity $= \frac{d\theta}{dt}$ and Acceleration $= \frac{d^2\theta}{dt^2}$ at $t = 0.4 sec$
The finite difference table is given by

x	θ	First differences	second differences	Third differences	Fourth differences
0	0				
		0.12			
0.2	0.12		0.25		
		0.37		0.01	
0.4	**0.49**		0.26		0.00
		0.63		0.01	
0.6	1.12		**0.27**		0.00
		0.90		**0.01**	
0.8	2.02		0.28		**0.00**
		1.18		0.01	
1.0	3.20		0.29		
		1.47			
1.2	4.67				

To compute $\frac{d\theta}{dt}$ and $\frac{d^2\theta}{dt^2}$ at $t = 0.4\ sec$, which is specified value of t. We take $t_0 = 0.4$

$\Delta\theta_0 = 0.63$, $\Delta^2\theta_0 = 0.21$, $\Delta^3\theta_0 = 0.01$, $\Delta^4\theta_0 = 0.00$

Using the formula,

$$\left(\frac{d\theta}{dt}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$\left(\frac{d\theta}{dt}\right)_{(0.4)} = \frac{1}{0.2}\left[(0.63) - \frac{1}{2}(0.27) + \frac{1}{3}(0.01)\right] = 2.49$$

And using the formula,

$$\left(\frac{d^2\theta}{dt^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

$$\left(\frac{d^2\theta}{dt^2}\right)_{(0.4)} = \frac{1}{(0.2)^2}[(0.27) - 0.01] = 6.5$$

4. The following table gives the temperature θ (in degree Celsius) of a cooling body at different instants of time t (in seconds)

T	1	3	5	7	9
θ	85.3	74.5	67.0	60.5	54.3

Find approximately the rate of cooling at t = 8 seconds.

Solution: Rate of cooling = $\frac{d\theta}{dt}$

The backward difference table is given by,

T	θ	$\nabla\theta$	$\nabla^2\theta$	$\nabla^3\theta$	$\nabla^4\theta$
1	85.3				
		-10.8			
3	74.5		3.3		
		-7.5		-2.3	
5	67.0		1.0		**1.6**
		-6.5		**-0.7**	
7	60.5		**0.3**		
		-6.2			
9	**54.3**				

Here, to compute $\frac{d\theta}{dt}$ at $t = 8\ sec.$, which is not the specified value of t

We have, $t_n = t_4 = 9$, $\theta_n = \theta_4 = 54.3$

$\nabla\theta_4 = -6.2$, $\nabla^2\theta_4 = 0.3$, $\nabla^3\theta_4 = -0.7$, $\nabla^4\theta_4 = 1.6$

$P = \frac{t - t_4}{h} = -0.5$

Using the formula, we have

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+2}{3!}\nabla^3 y_n + \frac{4p^3+18p^2+22p+6}{4!}\nabla^4 y_n + \cdots\right]$$

$$\left(\frac{d\theta}{dt}\right)_{(t=8)} = \frac{1}{2}\left[-6.2 + \frac{2(-.05)+1}{2}(0.3) + \frac{3(-0.5)^2+6(.5)+2}{6}(-0.7) + \frac{4(-0.5)^3+18(-0.5)^2+22(-.05)+6}{24}(1.6)\right]$$

$$= \frac{1}{2}[-6.2 + 0 + 0.029166 - 0.066666] = -3.11875$$

Thus, the body cools at the rate of 3.11875 degree/second.

Exercise:

1. Given that,

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Find $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ at $x = 1.1$ and $x = 1.6$.

2. A slider in a machine moves along a fixed straight rod. Its distance x cm along the rod is given below for various values of the time 't' seconds. Find the velocity of the slider and its acceleration when t = 0.1 second.

t	0	1.0	1.2		1.4	1.6	1.8	2.0
x	30.13	31.62	32.87		33.64	33.95	33.81	33.24

3. Find $\frac{dx}{dy}$ and $\frac{d^2y}{dx^2}$ at $x = 1.1$ of the function tabulated below:

x	1.0	1.2	1.4	1.6	1.8	2.0
$f(x)$	0	0.128	$-.544$	1.296	2.432	4.00

4. Compute y' (0) and y'' (0) from the following table:

x	0	1	2	3	4	5
y	4	8	15	7	6	2

5. Find $f'(1.5)$ using the differentiation formula based on Newton's interpolation for the following data:

x	1	2	3	4	5	6
$f(x)$	4	9	16	25	36	49

Numerical Integration:

Numerical integration is a process of evaluating a definite integral from a set of tabulated values of the integrand f(x). If the integrand is a function of a single-valued, then the numerical integration is known as Quadrature.

Newton – Cote's Quadrature formula:

Let

$$I = \int_a^b f(x)\, dx,$$

where f (x) takes the value $y_{0,}y_1, y_2$……..,y_n for x = x_0,x_1,x_2….,x_n.Divide the interval (a, b) into n sub-intervals of width h, so that

x_0= a, x_1=$x_0 + h$, $x_2 = x_0 + 2h$,….., x_n= $x_0 + nh = b$

Now, I = $\int_{x_0}^{x_0+nh} f(x)$ dx.

Let x=$x_0 + rh$ then $dx = h\, dr$. Also when x = x_0, r = 0 and when $x = x_0 + nh, r = n,$

I=$\int_0^n f(x_0 + rh)h\, dr.$

Now by Newton's forward interpolation formula, we have

I = $h\int_0^n [y_0 + r\,\Delta\, y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots]\, dr$

Integrating terms by terms, we get

I = $h[y_0 r + \frac{r^2}{2}\Delta y_0 + \frac{1}{2!}(\frac{r^3}{3} - \frac{r^2}{2})\Delta^2 y_0 + \frac{1}{3!}(\frac{r^4}{4} - r^3 + r^2)\Delta^3 y_0 + \ldots\ldots]_0^n$

I = $nh\,[y_0 + \frac{n}{2}(\Delta y_0) + \frac{1}{2!}\left(\frac{n^2}{3} - \frac{n}{2}\right)(\Delta^2 y_0) + \frac{1}{3!}\left(\frac{n^2}{4} - n^2 + n\right)(\Delta^3 y_0) + \cdots .]$

This formula is known as Newton-Cote's Quadrature formula or a general Quadrature formula.

Taking n =1, 2, 3, 6 in Quadrature formula, we have

n = 1 ⟶ Trapezoidal rule

n = 2 ⟶ Simpson's 1/3rd rule

n = 3 ⟶ Simpson's 3/8th rule

n = 6 ⟶ Weddle's rule

1. Simpson's 1/3rd rule:

By taking n = 2 in the general Quadrature formula and neglecting terms containing $\Delta^3 y_0, \Delta^4 y_0$, … then we get

$$I = \frac{h}{3}[(y_0 + y_n] + 4\,(y_1 + y_3 + \cdots + y_{n-1}) + 2\,(y_2 + y_4 + \cdots + y_{n-2})]$$

This formula is known as Simpson's one-third rule

Note: This formula is applicable only when n is a multiple of 2.

2. Simpson's 3/8th rule:

By taking n = 3 in the general Quadrature formula and neglecting terms containing $\Delta^4 y_0, \Delta^5 y_0$,……we get

$$I = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2\,(y_3 + y_6 + \cdots + y_{n-3})]$$

This formula is known as Simpson's three-eight rule

Note: This formula is applicable only when n is a multiple of 3

3. Weddle's rule:

By taking n = 6 in the general Quadrature formula

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{10}[(y_0+y_n)+5(y_1+y_5+\cdots+y_{n-5}+y_{n-1})+(y_2+y_4+\cdots+y_{n-4}+y_{n-2})+2(y_6+y_{12}+\cdots+y_{n-6})+6(y_3+y_9+\cdots+y_{n-3})]$$

This formula is known as Weddle's rule.
In particular, taking n = 6 in Weddle's rule

$$I = \frac{3h}{10}(y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6)$$

Note: This formula is applicable only when n is a multiple of 6.

Problems:

1. A solid of revolution is formed by rotating about the x-axis, the area between the x-axis, the line $x = 0$ and $x = 1$ and a curve through the points with the following coordinates.

x	0.00	0.25	0.50	0.75	1.00
y	1.0000	0.9896	0.9589	0.9089	0.8415

Solution:

The volume of the solid generated is given by

$$\int_0^1 \pi y^2\,dx$$

By Simpson's $\frac{1}{3}$ rule, we have (here h = 0.25)

$$\int_0^1 \pi y^2\,dx = \pi\frac{h}{3}[(y_0{}^2+y_4{}^2)+4(y_1+y_3{}^2)+2(y_2{}^2)]$$

$$= \frac{0.25}{3}\pi\{1+(0.8415)^2+4[(0.9896)^2+(0.9089)^2]+2(0.9589)^2\}$$

$$= \frac{(0.25)(3.1416)}{3}[1.7081+4(0.9793+0.8261)+1.8389]$$

= (0.2618) [1.7081 + 7.2216 + 1.8389]

= (0.2618) (10.7686) = 2.8192.

2. Calculate $\int_4^{5.2} \log x\,dx$ using (a) Simpson's 1/3rd rule (b) Simpson's 3/8th rule (c) Weddle's rule.

Solution: $h = \frac{5.2-4}{6} = \frac{1.2}{6} = 0.2$

x	4	4.2	4.4	4.6	4.8	5.0	5.2
y	1.386	1.43508	1.48160	1.52605	1.5686	1.6094	1.6486

(a) By Simpson's 1/3rd rule

$$I = \frac{h}{3}[[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)]]$$

$$= \frac{0.2}{3}[(1.38629+1.6486)+4(1.43508+1.52605+1.60943)+2(1.48160+1.56861)]$$

$$= \frac{0.2}{3}[3.034953+18.28224+6.10042]$$

$= \frac{0.2}{3}(27.41761)$
$= 1.82784$

(b) Simpson's 3/8th Rule

$$I = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3)]$$

$$I = \frac{3(0.2)}{8}[(1.38629 + 1.64865) + 3(1.43508 + 1.48160 + 1.56861 + 1.60943) + 2(1.52605)]$$

$= \frac{0.6}{8}[3.034953 + 18.28416 + 3.052]$

$$= \frac{0.6}{8}[24.371213]$$

$=1.82784$

(c) Weddle's Rule

$I = \frac{3h}{10}[(y_0 + y_6) + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5]$

$I = \frac{3(0.2)}{10}[(1.38629 + 5(1.43508) + 1.48160 + 6(1.52605) + 1.56861 + 5(1.60943) + 1.64865]$

$= \frac{0.6}{10}[1.38629 + 7.1754 + 1.48160 + 9.1563 + 1.5686 + 8.04715 + 1.64865]$

$= \frac{0.6}{10}[30.464]$

$= 1.82784$

3. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ using (a) Simpsons 1/3rd rule (b) Simpsons 3/8th rule(c) Weddle's rule

Solution:

Dividing the interval (0, 6) into six equal parts of width $h = \frac{6-0}{6} = 1$

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.2	0.1	0.0588	0.0385	0.027

(a) Simpson's 1/3rd rule

$$\int_0^6 \frac{1}{1+x^2}dx = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{3}[(1 + 0.027) + 4(0.5 + 0.1 + 0.0385) + 2(0.2 + 0.0588)]$$

$= \frac{1}{3}[1.027 + 2.554 + 0.5176]$

$=1.3662$

(b) Simpson's 3/8th Rule

$$\int_0^6 \frac{1}{1+x^2}dx = \frac{3h}{8}[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3)]$$

$$=\frac{3(1)}{8}[(1+0.027)+3(0.5+0.2+0.0588+0.0385)+2(0.1))]$$

$$= \frac{3}{8}[1.027+2.3919+0.2)]$$

=1.3571

(c) Weddle's Rule

$$\int_0^6 \frac{1}{1+x^2}dx = \frac{3h}{10}[(y_0 \ +5y_1+y_2+6y_3+y_4 \ +5y_5+y_6))]$$

$$=\frac{3(1)}{10}[(1 \ +5(0.5)+0.2+6(0.1)+0.0588+5(0.385)+6.027]$$

= 1.3735

Exercise:

1. By using Simpson's 1/3rd rule evaluate $\int_0^5 \frac{dx}{4x+5}$ by dividing range into 10 equal parts. Hence find an approximate value of $\log_e 5$.

2. By Simpson's 1/3 rule evaluate $\int_0^2 f(x)dx$, given

x	0.0	0.5	1.0	1.5	2.0
$f(x)$	0.399	0.352	0.242	0.129	0.054

3. The following tables gives six values of an independent variable x and the corresponding values of y.

x	0.0	1	2	3	4	5	6
$f(x)$	0.46	0.161	0.176	0.190	0.204	0.217	0.230

Evaluate using Simpson's 3/8th rule.

4. Evaluate $\int_0^1 \frac{x\,dx}{1+x^2}$ using Simpson's 3/8th rule dividing the interval into three equal parts. Hence find an approximate value of $\log\sqrt{2}$.

5. Evaluate $\int_3^6 f(x)dx$ using Weddle's rule from the following data;

x	3.0	3.5	4.0	4.5	5.0	5.5	6.0
$f(x)$	0.4771	0.5440	0.6020	0.6532	0.6996	0.7404	0.7782