

- Runge-Kutta method of 4th Order :-

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

$$y_1 = y(x_0 + h) = y_0 + \frac{h}{6} (k_1 + 2k_2 + 2k_3 + k_4) = (y_0 + k)$$

$$k_1 = h f(x_0, y_0)$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = h f\left(x_0 + h, y_0 + k_3\right)$$

Q) Apply 4th order Runge-Kutta method to find the approximate solution of at $x=0.1, 0.2$ given that

$$y' + x = y, \quad y(0) = 2.$$

$$\Rightarrow y' + x = y$$

$$y' = y - x, \quad y_0 = 2, \quad x_0 = 0, \quad h = 0.1.$$

Step 1:-

$$k_1 = h f(x_0, y_0)$$

$$= (0.1)(2 - 0)$$

$$k_1 = \underline{0.2}.$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= (0.1) f\left(0 + \frac{0.1}{2}, 2 + \frac{0.2}{2}\right)$$

$$= (0.1) f(0.05, 2.01)$$

$$k_2 = 0.1 \times 2.05$$

$$= \underline{0.205}.$$

$$k_3 = h f \left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2} \right)$$

$$= (0.1) f(0.05, 2.1025)$$

$$= \underline{\underline{0.2053}}$$

$$k_4 = h f \left(x_0 + h, y_0 + k_3 \right)$$

$$= 0.1 f(0.1, 2.2053)$$

$$= \underline{\underline{0.2105}}$$

$$k = \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6} (0.2 + 0.41 + 0.4106 + 0.2105)$$

$$= \underline{\underline{0.2052}}$$

$$y_1(0.1) = y_0 + k$$

$$= 2 + 0.2052$$

$$= \underline{\underline{2.2052}}$$

$$y_1(0.2) = 2.4152$$

Step 2:-

$$y' = y - x, \quad y_1 = 2.2052, \quad x_1 = 0.1, \quad h = 0.1$$

$$k_1 = h f(x_1, y_1)$$

$$= (0.1) f(0.1, 2.2052)$$

$$= \underline{\underline{0.2105}}$$

$$k_2 = h f \left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2} \right)$$

$$= (0.1) f(0.15, 2.3105)$$

$$k_2 = \underline{\underline{0.2161}}$$

$$k_3 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= 0.1 f(0.1 + 0.05, 2.2052 + 0.1081)$$

$$= (0.1) f(0.15, 2.3133)$$

$$k_3 = \underline{\underline{0.2163}}$$

$$k_4 = h f(x_1 + h, y_1 + k_3)$$

$$= (0.1) f(0.1 + 0.1, 2.2052 + 0.2163)$$

$$= (0.1) f(0.2, 2.4215)$$

$$k_4 = \underline{\underline{0.2222}}$$

$$k = \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6} (0.2105 + 0.4322 + 0.4326 + 0.2222)$$

$$= \frac{1}{6} (1.2975)$$

$$k = \underline{\underline{0.2163}}$$

$$y_2^{(0.2)} = y_1 + k$$

$$= 2.2052 + 0.2163$$

$$= \underline{\underline{2.4215}}$$

2) Apply RK method / RKM to find $y(1.2)$, $y(1.4)$
given that $y' = x^2 + y^2$, $y(1) = 1.5$.

$$\Rightarrow y' = x^2 + y^2 ; h = 0.2 ; y_0 = 1.5, x_0 = 1.$$

Step 1:-

$$k_1 = h f(x_0, y_0)$$

$$= (0.2) f(1, 1.5)$$

$$= (0.2) (3.25)$$

$$k_1 = \underline{0.65}$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= (0.2) f(1.1, 1.825)$$

$$= (0.2) (1.21 + 3.306)$$

$$k_2 = \underline{0.9081}$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= (0.2) f(1.1, 1.9541)$$

$$= (0.2) (1.21 + 3.8185)$$

$$k_3 = \underline{1.0057}$$

$$k_4 = h f(x_0 + h, y_0 + k_3)$$

$$= (0.2) f(1.2, 2.5057)$$

$$= (0.2) (1.44 + 6.285)$$

$$= \underline{\underline{1.5437}}$$

$$k = \frac{1}{6} (0.65 + 1.8162 + 2.0114 + 1.5437)$$

$$= \underline{\underline{1.0036}}$$

$$\begin{aligned}
 y_1(0.1.2) &= y_0 + k \\
 &= 1.5 + 8.9614 \cdot 1.0036 \\
 &= \underline{\underline{2.4614}} \\
 &= \underline{\underline{2.5036}}
 \end{aligned}$$

Step 2:

$$y' = x^2 + y^2 ; \quad x_1 = 1.2 ; \quad y_1 = 2.5036 ; \quad h=0.2$$

$$k_1 = h f(x_1, y_1)$$

$$= (0.2) f(1.2, 2.5036)$$

$$= (0.2) (1.44 + 6.2680)$$

$$k_1 = \underline{\underline{1.5416}}$$

$$k_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= (0.2) f\left(1.2 + 0.1, 2.5036 + 0.7708\right)$$

$$= (0.2) f(1.3, 3.2744)$$

$$= (0.2) (1.69 + 10.7217)$$

$$k_2 = \underline{\underline{2.4823}}$$

$$k_3 = (0.2) f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= (0.2) f\left(1.2 + 0.1, 2.5036 + 1.2412\right)$$

$$= (0.2) f(1.3, 3.7448)$$

$$= (0.2) (1.69 + 14.0235)$$

$$= (0.2) (15.7135) \Rightarrow \underline{\underline{3.1427}}$$

$$k_4 = h f(x_1 + h, y_1 + k_3)$$

$$= (0.2) f(1.4, 5.6463)$$

$$= (0.2) (1.96, 31.8807)$$

$$k_4 = ~~6.9844~~ \cdot \underline{\underline{6.7681}}$$

$$k = \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6} (1.5416 + 4.9646 + 6.2854 + 6.7681)$$

$$k = \underline{\underline{3.26}}$$

$$y_2(1.4) = y_1 + k$$

$$= 2.5036 + 3.26$$

$$= \underline{\underline{5.7636}}$$

3) $\frac{dy}{dx} = \frac{y-x}{y+x}$; $y(0) = 1$; $x = 0.1$ taking $h = 0.1$.

$$\rightarrow y' = \frac{y-x}{y+x}$$

$$x_0 = 0$$

$$y_0 = 1$$

$$k_1 = h f(x_0, y_0)$$

$$= (0.1)$$

$$k_1 = \underline{\underline{0.1}}$$

$$k_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$$

$$= (0.1) f(0.05, 1.05)$$

$$= (0.1) \left(\frac{0.9}{1.2} \right) \Rightarrow \underline{\underline{0.075}}$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.0909$$

$$k_4 = h f(x_0 + h, y_0 + k_3)$$

$$= \cancel{0.0832} \Rightarrow \underline{\underline{0.0832}}$$

$$k = \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$\cancel{\frac{1}{6} k} = \underline{\underline{0.0911}}$$

$$y_1'(0.1) = y_0 + k$$

$$= 1 + 0.0911$$

$$= \underline{\underline{1.0911}}$$