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Department of Mathematics
22MA11A
Fundamentals of Linear Algebra, Calculus & Numerical Methods

UNIT II

DIFFERENTIAL CALCULUS

TUTORIAL SHEET - 1

1. If $(-1, -\sqrt{3})$ are Cartesian coordinates of a point in plane, the corresponding polar coordinates are ________ Ans: $(2, 4\pi/3)$
2. If $(\sqrt{2}, 5\pi/4)$ are the polar coordinates of a point in plane, the corresponding Cartesian Coordinates are ________ Ans: $(-1, -1)$
3. The circle $x^2+y^2-2ax=0$ in polar form is ________ Ans: $(r = 2a\cos(\theta))$
4. The polar equation $\theta - k = 0$, geometrically represents ________ Ans: (straight line)
5. If two polar curves C_1 and C_2 are orthogonal, then value of $\cot(\phi_1)\cot(\phi_2)$ = ________ Ans: -1
6. Find the angle of intersection between the polar curves $r = \frac{k\theta}{1+\theta}$ and $r = \frac{k}{1+\theta^2}$ Ans: $\tan^{-1}(3)$
7. Show that the angle made by the tangent and the normal at any point $P(r,\theta)$ on the curve Lemniscate $r^2 = a^2\cos(2\theta)$ with the initial line is '3θ'.

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8. Show that the tangents to the cardioid $r = a(1+\cos\theta)$ at $\theta = \pi/3$ and $\theta = 2\pi/3$ are respectively parallel and perpendicular to the initial line.
9. Show that the circle $r = b$ intersects the curve $r^2 = a^2\cos(2\theta) + b^2$, at an angle given by $\tan^{-1}\left(\frac{a^2}{b^2}\right)$
10. Find the angle of intersection between the curves $r = a(1+\sin\theta)$ and $r = a(1-\sin\theta)$. Ans: $\pi/2$

(1)
$x = -1$
$y = -\sqrt{3}$.

$r = \sqrt{x^2+y^2}$
$= \sqrt{1+3}$
$\boxed{r = 2}$
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$
$= \tan^{-1}\sqrt{3}$
$\theta = \frac{\pi}{3}$ $(60°)$
$\theta' = \pi + \frac{\pi}{3} = \boxed{\frac{4\pi}{3}}$
(III quad)

(2) $r = \sqrt{2}$
$\theta = 5\pi/4 \to$ (III)
$\tan^{-1}\frac{y}{x} = \frac{5\pi}{4}$
$\frac{y}{x} = \tan\left(\pi + \frac{\pi}{4}\right)$
$\frac{y}{x} = 1$
$y = x$
$r^2 = x^2+y^2$
$2 = x^2+y^2$
$2 = 2x^2$
$x^2 = 1$
$x = \pm 1$
$\boxed{x = -1 \text{ (III quad)} \Rightarrow y = -1}$

(3) $x^2+y^2-2ax = 0$
$x^2+y^2 = 2ax$
$(r\cos\theta)^2 + (r\sin\theta)^2 = 2a r\cos\theta$
$r^{\cancel{2}}(\cos^2\theta + \sin^2\theta) = 2a\cancel{r}\cos\theta$
$r = 2a\cos\theta$

(6) $r = \frac{k\theta}{1+\theta}$
$\frac{dr}{d\theta} = \frac{k(1(1+\theta) - \theta(1))}{(1+\theta)^2}$
$\frac{dr}{d\theta} = \frac{k}{(1+\theta)^2}$
$\tan\phi_1 = r\frac{d\theta}{dr} = \frac{\cancel{k}\theta}{\cancel{1+\theta}} \times \frac{(1+\theta)^{\cancel{2}}}{\cancel{k}}$
$= \theta(1+\theta) = \theta + \theta^2$

$r = \frac{k}{1+\theta^2}$
$\frac{dr}{d\theta} = \frac{-k}{(1+\theta^2)^2} \times 2\theta$
$= \frac{-2k\theta}{(1+\theta^2)^2}$
$\tan\phi_2 = r\frac{d\theta}{dr} = \frac{\cancel{k}}{\cancel{1+\theta^2}} \times \frac{(1+\theta^2)^{\cancel{2}}}{-2\cancel{k}\theta}$
$= -\frac{(1+\theta^2)}{2\theta}$

$\to \frac{\cancel{k}\theta}{1+\theta} = \frac{\cancel{k}}{1+\theta^2}$
$\cancel{\theta} + \theta^3 = 1 + \cancel{\theta}$
$\theta^3 = 1$
$\theta = 1$
$\tan\phi_1 = 1 + 1^2 = 2$
$\tan\phi_2 = -\frac{(1+1^2)}{2} = -1$

$\to \tan\alpha = \left|\frac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1\tan\phi_2}\right|$
$\tan\alpha = \left|\frac{2+1}{1+2(-1)}\right| = |-3|$
$\tan\alpha = 3$
$\boxed{\alpha = \tan^{-1}3}$

(9) $r^2 = a^2\cos(2\theta) + b^2$
$\cancel{2}r\frac{dr}{d\theta} = a^2(-\sin2\theta)\times\cancel{2}$
$\tan\phi_1 = r\frac{d\theta}{dr}$
$= r \times \frac{r}{-a\sin2\theta}$
$\tan\phi_1 = \frac{a^2\cos2\theta + b^2}{-a^2\sin2\theta}$
$= \frac{a^2(0)+b^2}{-a^2(1)} = \frac{-b^2}{a^2}$
$\phi_1 = \tan^{-1}(-b^2/a^2)$
$\to \alpha = |\phi_1 - \phi_2|$
$= \left|\tan^{-1}(-b^2/a^2) - \frac{\pi}{2}\right|$
$= |\cot^{-1}(-b^2/a^2)|$
$= \tan^{-1}(a^2/b^2)$

$r = b$
$\frac{dr}{d\theta} = 0$
$\tan\phi_2 = \frac{1}{r}\frac{d\theta}{dr} = \infty$
$\phi_2 = \pi/2$
$\to r^2 = b^2$
$r^2 = a^2\cos2\theta + b^2$
$\cancel{b^2} = a^2\cos2\theta + \cancel{b^2}$
$a^2\cos2\theta = 0$
$\cos2\theta = 0$
$\sqrt{1-\sin^2 2\theta} = 0$
$\sin^2 2\theta = 1$
$\sin2\theta = 1$

(7) $r^2 = a^2\cos(2\theta)$
$\cancel{2}r\frac{dr}{d\theta} = a^2(-\sin2\theta)\times\cancel{2}$
$\frac{dr}{d\theta} = \frac{-a^2\sin2\theta}{r}$
$\tan\phi = r\frac{d\theta}{dr} = r\cdot\frac{r}{-a^2\sin2\theta}$
$= \frac{-a^2\cos2\theta}{a^2\sin2\theta}$
$\tan\phi = -\cot2\theta$
$\tan\phi = \tan(\pi/2 + 2\theta)$
$\phi = \frac{\pi}{2} + 2\theta$
given normal at P, $\phi = 2\theta$
$\psi = \theta + \phi$
$= \theta + 2\theta$
$\psi = 3\theta$

(8) $r = a(1+\cos\theta)$ at $\theta = \frac{\pi}{3}, \frac{2\pi}{3}$
$\frac{dr}{d\theta} = a(-\sin\theta)$
$\tan\phi = r\frac{d\theta}{dr} = \frac{\cancel{a}(1+\cos\theta)}{-\cancel{a}\sin\theta}$
$= \frac{\cancel{2}\cos^2\theta/2}{-\cancel{2}\sin\theta/2\cos\theta/2}$
$= -\cot\theta/2$
$= \tan(\pi/2 + \theta/2)$
$\phi = \frac{\pi}{2} + \frac{\theta}{2}$
$\psi = \phi + \theta = \frac{\pi}{2} + \frac{\theta}{2} + \theta = \frac{\pi}{2} + \frac{3\theta}{2}$
at $\theta = \frac{\pi}{3}$, $\psi = \frac{\pi}{2} + \frac{3}{2}\times\frac{\pi}{3} = \pi$
$\boxed{\tan\psi = \tan\pi = 0 \Rightarrow \text{tangent } \parallel^{l}}$
at $\theta = \frac{2\pi}{3}$, $\psi = \frac{\pi}{2} + \frac{3}{2}\times\frac{2\pi}{3} = \frac{3\pi}{2}$
$\boxed{\tan\psi = \tan\frac{3\pi}{2} = \infty \Rightarrow \text{tangent} \perp\text{er}}$

(10) $r = a(1+\sin\theta)$
$\frac{dr}{d\theta} = a\cos\theta$
$\tan\phi_1 = r\frac{d\theta}{dr} = \frac{\cancel{a}(1+\sin\theta)}{\cancel{a}\cos\theta}$

$r = a(1-\sin\theta)$
$\frac{dr}{d\theta} = -a\cos\theta$
$\tan\phi_2 = r\frac{d\theta}{dr} = -\frac{\cancel{a}(1-\sin\theta)}{\cancel{a}\cos\theta}$

$\to \cancel{a}(1+\sin\theta) = \cancel{a}(1-\sin\theta)$
$2\sin\theta = 0$
$\sin\theta = 0$
$\theta = 0$
$\cos\theta = 1$
$\tan\phi_1 = \frac{1+0}{1} = 1$
$\tan\phi_2 = -\frac{(1-0)}{1} = -1$
$\to \tan\alpha = \frac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1\tan\phi_2}$
$= \frac{1-(-1)}{1+(1)(-1)}$
$\tan\alpha = \frac{1}{0}$
$\boxed{\alpha = \pi/2}$

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TUTORIAL SHEET - 2

1. The curvature of a circle $s = a\psi$ at any point is ________ Ans: $(\kappa = 1/a)$
2. The radius of curvature for straight line $y = mx + c$ is ________ Ans: ($\rho = \infty$, not defined)
3. The curvature of the curve $y = e^x$ at the point where it crosses the y-axis is ________ Ans: $(\kappa = \frac{1}{2^{3/2}})$
4. The Taylor series expansion of $\log(x)$ about $x = 1$ up to second degree term is ________ Ans: $\log(x) = (x-1) - \frac{(x-1)^2}{2} + \cdots \infty$
5. The Maclaurin series expansion of $\cos(x)$ is ________ Ans: $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \infty$
6. Show that the radius of curvature of the Folium $x^3 + y^3 = 3axy$ at the point $(3a/2, 3a/2)$ is given by $\frac{-3a}{8\sqrt{2}}$.
7. Find the radius of curvature of the curve $y^2 = \frac{4a^2(2a-x)}{x}$ where the curve meets the x-axis.
8. For the curve $y = \frac{ax}{a+x}$, show that $\left(\frac{2\rho}{a}\right)^{\frac{2}{3}} = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2$
9. Find the radius of curvature of the $x = a\log(\sec t + \tan t)$, $y = a\sec t$. Ans: $\rho = a\sec^2 t$
10. Show that the curvature of the tractrix $x = a[\cos t + \log\tan(\frac{t}{2})]$, $y = a\sin t$ at any point is given by $\kappa = \frac{\tan t}{a}$.

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11. Find the coordinates of the centre of curvature of $(at^2, 2at)$ on the parabola $y^2 = 4ax$. Ans: $((\bar{x}, \bar{y}) = ((2+3t)at^2, -4\sqrt{2}at^{3/2}))$
12. Find the circle of curvature at the point $(a/4, a/4)$ for the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$. Ans: $\left(x - \frac{3a}{4}\right)^2 + \left(y - \frac{3a}{4}\right)^2 = \frac{a^2}{2}$
13. Find the radius of curvature of the curve $r^n = a^n\cos(n\theta)$. Ans: $\frac{a^n r^{1-n}}{n+1}$
14. Show that the radius of curvature at any point (r, θ) on the Cardioid $r = a(1-\cos\theta)$ varies as $\sqrt{r}$
15. Find the radius of curvature for the parabola $\frac{2a}{r} = 1 - \cos\theta$ at any point (r, θ). Ans: $2\sqrt{\frac{r^3}{a}}$

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TUTORIAL SHEET - 3

1. Match the following:

(1) $s = a\psi$
$\frac{ds}{d\psi} = a$
$\kappa = \frac{d\psi}{ds} = \boxed{\frac{1}{a}}$

(2) $y = mx + c$
$\hookrightarrow$ straight line.
$\kappa = 0$, $\rho = \infty$ (ND)

(3) $y = e^x$ $(y, 0)$ $\langle \kappa ? \rangle$
$y_1 = e^x = 1$ at $x = 0$, $y = e^0 = 1$
$y_2 = e^x = 1$
$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1^2)^{3/2}}{1} = 2^{3/2}$
$\kappa = \frac{1}{\rho} = \boxed{\frac{1}{2^{3/2}}}$

(6) $x^3 + y^3 = 3axy$ $(3a/2, 3a/2)$
ST: ROC $= \rho = \frac{-3a}{8\sqrt{2}}$
$\cancel{3}x^2 + \cancel{3}y^2 y_1 = \cancel{3}a(y + xy_1)$
$x^2 + y^2 y_1 = ay + axy_1$
$(y^2 - ax)y_1 = ay - x^2$
$y_1 = \frac{ay - x^2}{y^2 - ax}$
at $(3a/2, 3a/2)$; $y_1 = \frac{a\cdot\frac{3a}{2} - \left(\frac{3a}{2}\right)^2}{\left(\frac{3a}{2}\right)^2 - a\frac{3a}{2}}$
$y_1 = \frac{\frac{3a^2}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - \frac{3a^2}{2}}$
$= \frac{6-9}{9-6} = \frac{-3}{3}$
$y_1 = -1$
$y_2 = \frac{(ay_1 - 2x)(y^2 - ax) - (ay - x^2)(2yy_1 - a)}{(y^2 - ax)^2}$
at $(3a/2, 3a/2)$,
$y = \left[\left(a\cdot(-1) - 2\cdot\frac{3a}{2}\right)\left(\left(\frac{3a}{2}\right)^2 - a\cdot\frac{3a}{2}\right)\right]$
$- \left[\left(a\cdot\frac{3a}{2} - \left(\frac{3a}{2}\right)^2\right)\left(2\cdot\frac{3a}{2}(-1) - a\right)\right]$

(4) $f(x) = \log x$ $(a = 1)$ $f(a) = \log 1 = 0$
$f'(x) = \frac{1}{x}$ $f''(x) = \frac{-1}{x^2}$ $f'''(x) = \frac{2}{x^3}$
$f'(a) = 1$ $f''(a) = -1$ $f'''(a) = 2$
Taylor's $\to$
$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a)$
$\log x = 0 + (x-1)(1) + \frac{(x-1)^2}{2}(-1) \ldots$
$\log x = (x-1) - \frac{(x-1)^2}{2} + \ldots \infty$

(5) $f(x) = \cos x$ $f(0) = \cos 0 = 1$
$f'(x) = -\sin x$ $f'(0) = -\sin 0 = 0$
$f''(x) = -\cos x$ $f''(0) = -\cos 0 = -1$
$f'''(x) = \sin x$ $f'''(0) = \sin 0 = 0$
$f^{IV}(x) = \cos x$ $f^{IV}(0) = \cos 0 = 1$
Mclaurins $\to$
$f(x) = f(0) + x\cdot f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) \ldots$
$\cos x = 1 + \cancel{x(0)} + \frac{x^2}{2!}(-1) + \cancel{\frac{x^3}{3!}(0)} + \frac{x^4}{4!}(+1) \ldots$
$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \infty$

(7) $y^2 = \frac{4a^2(2a-x)}{x}$ $(x, 0)$.
ROC? $\Rightarrow \rho$?
at $y = 0$, $x = 2a$
$\to 2yy_1 = 4a^2\left(\frac{(-1)(x) - (2a-x)(1)}{x^2}\right)$
$\cancel{2}yy_1 = \cancel{4}^{2}a^2\left(\frac{-\cancel{x} - 2a + \cancel{x}}{x^2}\right)$
$yy_1 = \frac{-4a^3}{x^2}$
$y_1 = \frac{-4a^3}{x^2 y}$
at $(2a, 0)$, $y_1 = \frac{-4a^3}{(2a)^2(0)} = \infty$?
$(y_1)^2 + yy_2 = -4a^3\left(\frac{-2}{x^3}\right)$
$y_1^2 + yy_2 = \frac{8a^3}{x^3} \to$

(9) $x = a\log(\sec t + \tan t)$
$\frac{dx}{dt} = \frac{a}{\sec t + \tan t} \times (\sec t\tan t + \sec^2 t)$
$= a\sec t$
$y = a\sec t$
$\frac{dy}{dt} = a\sec t\tan t$
$\frac{dy}{dx} = \frac{\cancel{a\sec t}\tan t}{\cancel{a\sec t}}$
$y_1 = \tan t$
$y_2 = \sec^2 t \times \frac{dt}{dx}$
$= \sec^2 t \times \frac{1}{a\sec t}$
$y_2 = \frac{\sec t}{a}$
$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$
$= \frac{(1+\tan^2 t)^{3/2}}{\frac{\sec t}{a}}$
$= \frac{a(\sec^2 t)^{3/2}}{\sec t}$
$\rho = a\sec^2 t$

(10) $x = a[\cos t + \log(\tan t/2)]$
$y = a\sin t$
curvature? $\Rightarrow \kappa$?
$\frac{dx}{dt} = a\left[-\sin t + \frac{\sec^2 t/2}{\tan(t/2)} \times \frac{1}{2}\right]$
$= a\left[-\sin t + \frac{1}{\cos^2 t/2}\cdot\frac{\cancel{\cos t/2}}{\sin t/2} \times \frac{1}{2}\right]$
$= a\left[-\sin t + \frac{1}{\sin t}\right]$
$= a\left[\frac{1-\sin^2 t}{\sin t}\right]$
$\frac{dx}{dt} = \frac{a\cos^2 t}{\sin t}$

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TUTORIAL SHEET -3

1. Match the following:

i)	The angle between radius vector and tangent for the polar curve at any point $P(r,\theta)$ is	a)	$\rho \propto y^2$
ii)	The angle between radius vector and tangent for the Cartesian curve at any point $P(x,y)$ is	b)	$\rho \propto \frac{1}{y^2}$
iii)	The radius of curvature at any point $P(x,y)$ on the catenary $y = c\cosh\left(\frac{x}{c}\right)$ is	c)	$\cot(\phi) = \frac{1}{r}\frac{dr}{d\theta}$
		d)	$\tan(\phi) = r\frac{d\theta}{dr}$
		e)	$\tan(\phi) = \frac{xy' - y}{x + yy'}$
		h)	$\tan(\phi) = \frac{xy' + y}{x - yy'}$

Ans: (i) - (c) (ii) - (e) (iii) - (a)

2. Find the Taylor series expansion of the function $y = \log(\cos x)$ about the point $x = \pi/3$.
Ans: $\log(\cos x) = -\log 2 - \sqrt{3}\left(x-\frac{\pi}{3}\right) - 2\left(x-\frac{\pi}{3}\right)^2 - \dots$

3. Obtain the expansion of the function $e^{\sin(x)}$ in ascending powers of x up to terms containing x^4.
Ans: $e^{\sin(x)} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} \dots$

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4. Obtain the Maclaurin series expansion for the function $f(x) = \tan^{-1}(x)$ and hence deduce that $\pi = 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right]$
Ans: $\tan^{-1}(x) = \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \dots\right]$

5. Using Maclaurin's series, prove that $\sqrt{1+\sin(2x)} = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \dots$

6. Show that $\left(\frac{x}{\sin x}\right) = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \dots$

at $(3a/2, 3a/2)$,

$$y = \frac{\left[\left(a\cdot(-1) - 2\cdot\frac{3a}{2}\right)\left(\left(\frac{3a}{2}\right)^2 - a\cdot\frac{3a}{2}\right)\right] - \left[\left(a\cdot\frac{3a}{2} - \left(\frac{3a}{2}\right)^2\right)\left(2\cdot\frac{3a}{2}(-1) - a\right)\right]}{\left(\left(\frac{3a}{2}\right)^2 - a\cdot\frac{3a}{2}\right)^2}$$

$$= \frac{\left[(-a-3a)\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)\right] - \left[\left(\frac{3a^2}{2} - \frac{9a^2}{4}\right)(-3a-a)\right]}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)^2}$$

$$= \frac{\left[(-4a)\left(\frac{3a^2}{4}\right)\right] - \left[\left(+\frac{3a^2}{4}\right)(+4a)\right]}{\left(\frac{3a^2}{4}\right)^2}$$

$$y_2 = \frac{-4a - 4a}{\frac{3a^2}{4}} = \frac{-8a}{3a^2} \times 4 = \frac{-32}{3a}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+(-1)^2)^{3/2}}{y_2}$$

$$= \frac{2^{3/2}}{-32} \times 3a$$

$$= \frac{-2^{3/2}}{2^5} \times 3a$$

$$= -\frac{2\sqrt{2}}{2^5} \times 3a \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$\rho = \frac{-3a}{8\sqrt{2}} \quad \therefore \text{proved}$$

$$\frac{(2a)^2(0)}{}$$

$$(y_1)^2 + yy_2 = -4a^3\left(\frac{-2}{x^3}\right)$$

$$y_1^2 + yy_2 = \frac{8a^3}{x^3} \rightarrow$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

(8) $y = \frac{ax}{a+x} \Rightarrow \frac{y}{x} = \frac{a}{a+x}$

ST: $\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2$

$$y_1 = \frac{a(a+x) - ax(1)}{(a+x)^2}$$

$$y_1 = \frac{a^2}{(a+x)^2}$$

$$y_2 = a^2\left(\frac{-2}{(a+x)^3}\right) = \frac{-2a^2}{(a+x)^3}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\frac{2\rho}{a} = \frac{\left[1 + \left(\frac{a^2}{(a+x)^2}\right)^2\right]^{3/2}}{\frac{-\cancel{2}a^2}{(a+x)^3}} \times \frac{\cancel{2}}{a}$$

$$\left(\frac{2\rho}{a}\right)^{2/3} = \left[\frac{-\left[1 + \left(\frac{a}{a+x}\right)^4\right]^{3/2}(a+x)^3}{a^3}\right]^{2/3}$$

$$= \frac{\left[1 + \left(\frac{a}{a+x}\right)^4\right](a+x)^2}{a^2}$$

$$= \frac{\left[1 + \left(\frac{y}{x}\right)^4\right]\left(\frac{xa}{y}\right)^2}{a^2}$$

$$= \left[1 + \left(\frac{y}{x}\right)^4\right]\left(\frac{x}{y}\right)^2$$

$$= \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2 \quad \therefore \text{proved}$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right]$$

$$= a\left[\frac{1 - \sin^2 t}{\sin t}\right]$$

$$\frac{dx}{dt} = \frac{a\cos^2 t}{\sin t}$$

$$\frac{dy}{dt} = a\cos t$$

$$\frac{dy}{dx} = \frac{\cancel{a\cos t}}{\cancel{a}\cos^{\cancel{2}} t} \times \sin t$$

$$y_1 = \tan t$$

$$y_2 = \sec^2 t \times \frac{dt}{dx}$$

$$= \frac{1}{\cos^2 t} \times \frac{\sin t}{a\cos^2 t}$$

$$y_2 = \frac{\sin t}{a\cos^4 t}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$K = \frac{y_2}{(1+y_1^2)^{3/2}} = \frac{\frac{\sin t}{a\cos^4 t}}{(1+\tan^2 t)^{3/2}}$$

$$= \frac{\sin t \sec^4 t}{a\sec^3 t}$$

$$K = \frac{\sin t}{a\cos t} = \frac{\tan t}{a}$$

(11) $(at^2, 2at)$

$$y^2 = 4ax$$

$$2yy_1 = 4a$$

$$y_1 = \frac{2a}{y} = \frac{1}{t}$$

$$y_2 = -\frac{2a}{y^2} = \frac{-2a}{(2at)^2} = \frac{-1}{2at^2}$$

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$$

$$= at^2 - \frac{1/\cancel{t}}{-1/2at^{\cancel{2}}}\left(1 + \frac{1}{t^2}\right)$$

$$= at^2 + 2at\left(\frac{t^2+1}{t^2}\right)$$

$$= at^2 + \frac{2a}{t}(1+t^2)$$

$$= at^2 + \frac{2a}{t} + 2at$$

$$\bar{y} = +\frac{1}{y_2}(1+y_1^2)$$

$$= 2at - 2at^2\left(1 + \frac{1}{t^2}\right)$$

$$= 2at - 2a(1+t^2)$$

$$= 2at - 2a - 2at^2$$

$$= 2a(t - t^2 - 1)$$

(12) circle of curvature? $\sqrt{x} + \sqrt{y} = \sqrt{a}$ $(a/4, a/4)$

$$x^{1/2} + y^{1/2} = a^{1/2}$$

$\frac{1}{2} - 1$

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y_1 = 0$$

$$y_1 = -\left(\frac{x}{y}\right)^{-1/2} = -\frac{\sqrt{y}}{\sqrt{x}}$$

$\frac{-1}{2} - 1$

at $a/4, a/4 \Rightarrow y_1 = -1$

$$y_1 = \frac{\sqrt{y}}{\sqrt{x}}$$

$$y_2 = \frac{-\frac{1}{2\sqrt{y}}\cdot y_1\sqrt{x} - \left(-\sqrt{y}\cdot\frac{1}{2\sqrt{x}}\right)}{x}$$

$$= \frac{\frac{\sqrt{x}}{2\sqrt{y}} + \frac{\sqrt{y}}{2\sqrt{x}}}{x}$$

$$y_2 = \frac{4}{a}$$

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$$

$$= \frac{a}{4} - \frac{-1}{\frac{4}{a}}(1+(-1)^2)$$

$$\bar{x} = \frac{a}{4} + \frac{a}{4}(2) = \frac{3a}{4}$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$= \frac{a}{4} + \frac{1}{\frac{4}{a}}(1+(-1)^2)$$

$$\bar{y} = \frac{a}{4} + \frac{a}{4}(2) = \frac{3a}{4}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{\left[1 + (-1)^2\right]^{3/2}}{\frac{4}{a}}$$

$$= \frac{2^{3/2}a}{2^2}$$

$$\rho = 2^{-1/2}a$$

$$\rho^2 = \frac{a^2}{2}$$

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

$$\left(x - \frac{3a}{4}\right)^2 + \left(y - \frac{3a}{4}\right)^2 = \frac{a^2}{2}$$

(13) Circle of curvature? $r^n = a^n \cos n\theta$

$$n\log r = n\log a + \log(\cos n\theta)$$

$$\frac{\cancel{n}}{r}\frac{dr}{d\theta} = \frac{1}{\cos n\theta} \times -\sin n\theta \times \cancel{n}$$

$$r_1 = -r\tan n\theta$$

$$r_2 = -r_1\tan n\theta - rn\sec^2 n\theta$$

$$= r\tan^2 n\theta - rn\sec^2 n\theta$$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

$$= \frac{(r^2 + r^2\tan^2 n\theta)^{3/2}}{r^2 + 2r^2\tan^2 n\theta - r^2(\tan^2 n\theta - n\sec^2 n\theta)}$$

$$= \frac{(r^2\sec^2 n\theta)^{3/2}}{r^2(1 + 2\tan^2 n\theta - \tan^2 n\theta + n\sec^2 n\theta)}$$

$$= \frac{r^3\sec^3 n\theta}{r^2(1 + \tan^2 n\theta + n\sec^2 n\theta)}$$

$$= \frac{r\sec^3 n\theta}{\sec^2 n\theta(1+n)}$$

$$= \frac{r\sec n\theta}{1+n}$$

$$\left\{\begin{array}{l} r^n = a^n\cos n\theta \\ r^n = \frac{a^n}{\sec n\theta} \\ \sec n\theta = \frac{a^n}{r^n} \end{array}\right\}$$

$$= r \times \frac{a^n}{r^n} \times \frac{1}{1+n}$$

$$\rho = \frac{a^n r^{1-n}}{1+n}$$

(14) ρ? $P(r,\theta)$

$r = a(1-\cos\theta)$ varies a $\sqrt{r}$

$$\frac{dr}{d\theta} = a\sin\theta = r_1$$

$$\frac{d^2r}{d\theta^2} = a\cos\theta = r_2$$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

$$= \frac{(a^2(1-\cos\theta)^2 + a^2\sin^2\theta)^{3/2}}{a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a^2\cos\theta(1-\cos\theta)}$$

$$= \frac{\left[a^2(1 + \cos^2\theta - 2\cos\theta + \sin^2\theta)\right]^{3/2}}{a^2(1 + \cos^2\theta - 2\cos\theta + 2\sin^2\theta - \cos\theta + \cos^2\theta)}$$

$$= \frac{a^3(2(1-\cos\theta))^{3/2}}{a^2(3(1-\cos\theta))}$$

$$= \frac{a\,2^{3/2}(1-\cos\theta)^{3/2}}{3(1-\cos\theta)}$$

$$r^{1/2} = a^{1/2}(1-\cos\theta)^{1/2}$$

$$= \frac{a\,2^{3/2}}{3}(1-\cos\theta)^{1/2} \times \frac{a^{1/2}}{a^{1/2}}$$

$$= \frac{2\sqrt{2}\sqrt{a}\sqrt{r}}{3}$$

ρ varies as $\sqrt{r}$.