UNIT V – Numerical Methods :-

Interpolation – The process of estimating the (interm) dependent variable (y) for the independent variable (x) is called interpolation.

- The process of estimating y for the outside range of x is called Extrapolation.

- Interpolation techniques depend on 3 types of differences

1) forward differences
2) Backward differences
3) Central differences.

Forward Differences →

Let $y = f(x)$ be defined for the equal intervals of x as

x_0	x_0+h	x_0+2h	x_0+3h	---	x_0+nh
y_0	y_1	y_2	y_3	---	y_n

Step length $= h$.

The differences $y_1 - y_0$, $y_2 - y_1$, $y_n - y_{n-1}$ are called forward differences denoted by Δy_0, Δy_1 -- Δy_{n-1} respectively.

Backward Differences →

Let $y = f(x)$ be defined for equal intervals of x as

x_0	x_0+h	x_0+2h	---	x_0+nh
y_0	y_1	y_2	...	y_n

The diff → $y_1 - y_0$, $y_2 - y_1$, $y_n - y_{n-1}$ are called backward diff denoted by ∇y_1, ∇y_2, ∇y_3 --- ∇y_n $\boxed{\nabla \to \text{nabla / del}}$

① $\Delta f(x) = f(x+h) - f(x)$

$\nabla f(x) = f(x) - f(x-h)$

② $\Delta y_0, \Delta y_1$ --- first order forward diff

$\nabla y_1, \nabla y_2$ ---- " " backward "

③ $\Delta^2 y_0, \Delta^2 y_1$ --- second order forward diff

$\nabla^2 y_1, \nabla^2 y_2$ --- second order backward diff

④ $\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0)$

$= \Delta y_1 - \Delta y_0$

$= y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$

$\Delta^3 y_0 = \Delta(\Delta^2 y_0)$

$= \Delta(y_2 - 2y_1 + y_0)$

$= \Delta y_2 - 2\Delta y_1 + \Delta y_0$

$= (y_3 - y_2) - 2(y_2 - y_1) + (y_1 - y_0)$

$= y_3 - 3y_2 + 3y_1 - y_0$

$\nabla^2 y_2 = y_2 - y_1 - y_1 + y_0$

pascals triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4

$\therefore \boxed{\Delta^r y_k = \nabla^r y_{r+k}}$

$\therefore \boxed{\Delta^n y_0 = y_n - {}^nC_1\, y_{n-1} + \cdots + {}^nC_n\, y_0}$

Q] $\Delta \tan^{-1} x$

Q] $\Delta \log(2x+3)$

Q] $\nabla \sin(x+2)$

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta \tan^{-1} x = \tan^{-1}(x+h) - \tan^{-1} x$$

$$= \tan^{-1}\left(\frac{x+h-x}{1+(x+h)x}\right)$$

$$= \tan^{-1}\left(\frac{h}{1+(x+h)x}\right)$$

Q] Δe^{2x+3}, $h = 1$

$$= e^{2(x+h)+3} - e^{2x+3}$$

$$= e^{2x+\cancel{4}5} - e^{2x+3}$$

$$= (e^2 - 1)\, e^{2x+3}$$

Difference tables →

Let a function $y = f(x)$ be defined for equal intervals of x as

x_0	$x_1 = x_0 + h$	x_2	x_3	x_4
y_0	y_1	y_2	y_3	y_4

Forward difference Tables

x	y	Δ	Δ^2	Δ^3
x_0	y_0			
		$y_1 - y_0 = \Delta y_0$		
x_1	y_1		$\Delta y_1 - \Delta y_0 = \Delta^2 y_0$	
		$y_2 - y_1 = \Delta y_1$		$\Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y$
x_2	y_2		$\Delta y_2 - \Delta y_1 = \Delta^2 y_1$	
		$y_3 - y_2 = \Delta y_2$		
x_3	y_3			

$\Rightarrow y_0, \Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ are called leading entries in forward diff table.

Backward difference tables

x	y	∇	∇^2	∇^3
x_0	y_0			
		$y_1 - y_0 = \nabla y_1$		
			$\nabla y_2 - \nabla y_1 = \nabla^2 y_2$	
x_1	y_1			$\nabla^3 y_3$
		$y_2 - y_1 = \nabla y_2$		
			$\nabla y_3 - \nabla y_2 = \nabla^2 y_3$	
x_2	y_2			$\nabla^3 y_4$
		$y_3 - y_2 = \nabla y_3$		
			$\nabla y_4 - \nabla y_3 = \nabla^2 y_4$	
x_3	y_3			
		$y_4 - y_3 = \nabla y_4$		
x_4	y_4			

$\Rightarrow y_4, \nabla y_4, \nabla^2 y_4, \nabla^3 y_4 \ldots$ are leading entries in backward diff table

Ex $\rightarrow$

x : 1 2 3 4

y : 52 65 72 86

Construct F.D table, B.D table find, $\Delta^2 y_1$, $\nabla^2 y_2$

x	y	Δ	Δ^2	Δ^3
1	52			
		13		
2	65		$-6 \rightarrow \nabla^2 y_2$	
		7		13.
3	72		$7 \rightarrow \Delta^2 y_1$	
4		14		
	86			

Newton's forward Interpolation Formula (NFIF)

Let the function $y = f(x)$ be defined for equal intervals of x as

x_0	x_0+h	x_0+2h	x_0+3h	---
y_0	y_1	y_2	y_3	---

Then, NFIF is →

$$y = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

where, $\boxed{p = \frac{x - x_0}{h}}$

Newton's Backward Interpolation formula (NBIF)

$$y = y_n + p\,\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$

where, $\boxed{p = \frac{x - x_n}{h}}$

Q] In an experiment, the values of o/p y are recorded for I/p x from 1.0 to 3.5 at intervals of 0.5. Estimate the values of y by using appropriate interpolation formulas, for $x = 1.2$, $x = 3.4$, $x = 3.8$

$x = \quad 1 \quad 1.5 \quad 2 \quad 2.5 \quad 3 \quad 3.5$

$y = \quad 277 \quad 166 \quad 146 \quad 130 \quad 115 \quad 102$

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
1	277					
		-111				
			91			
1.5	166			-87		
		-20			84	
			4			
2	146			-3		-80
		-16				
			1		4	
2.5	130					
		-15				
				1		
3	115		2			
		-13				
3.5	102					

$\rightarrow \boxed{x = 1.2} \qquad p = \dfrac{x - x_0}{h}, \quad h = 0.5$

$$p = \frac{1.2 - 1}{0.5} = 0.4$$

$$y = 277 + (0.4)(-111) + \frac{0.4 \times (-0.6)}{2} \times 91 + \frac{0.4 \times (-0.6) \times (-1.6)}{6} \times (-87)$$

$$+ \frac{(0.4)(-0.6)(-1.6)(-2.6)}{24} \times 84 + \frac{(-0.24)(-1.6)(-2.6)(-3.6)}{24 \times 5} \times (-80)$$

$y \Rightarrow 277 - 44.4 - 10.92 - 5.568 - 3.4944 - 2.39616$

$y \Rightarrow \underline{210.22144}$

$\Rightarrow \boxed{x = 3.4}$

$$p = \frac{3.4 - 3.5}{0.5} = \frac{-0.1}{0.5} = -0.2$$

$$y = \cancel{35}\,102 + (-0.2)(-13) + \frac{(-0.2)(0.8)}{2}(2) + \cdots\cdots$$

$\Rightarrow \boxed{x = 3.8}$

$$p = \frac{3.8 - 3.5}{0.5} = 0.6$$

$$y = 102 + (0.6)(-13) + \frac{(0.6)(1.6)}{2}(2) + \frac{(0.6)(1.6)(2.6)}{6}(1)$$
$$+ \frac{(0.6)(1.6)(2.6)(3.6)}{24}(4) + \frac{(0.6)(1.6)(2.6)(3.6)(4.6)}{24 \times 5}(-80)$$

$$= 102 + (-7.8) + 0.96 + 0.416 + 1.4976 + (-27.55584)$$
$$= \underline{\underline{69.51776}}$$

Q) The details below are regarding monthly salaries of 2000 adults in a colony. Using interpolation estimate no of persons who have monthly income b/w 8000 & 10000

$\Rightarrow$

Monthly income	Below 5K	5K-10K	10K-15K	15K-20K	20K-25K
No. of person	535	660	470	270	65

$\Rightarrow$ Cumulative values of y

Monthly income (x)	(y)	Δ	Δ^2	Δ^3	Δ^4
5k	535				
		660			
10k	1195		-190		
		470		-10	
15k	1665		-200		+5
		270		-5	
20k	1935		-205		
		65			
25k	2000				

$\Rightarrow \quad x = 8000$

$$p = \frac{x - x_0}{h} = \frac{8000 - 5000}{5000} = \underline{0.6}$$

$$y = 535 + 0.6 \times \cancel{535}\,660 + \frac{(0.6)(-0.4)}{2}(-190) + (0.\cancel{064}\,056) \times -10$$

$$+ (-0.168) \times \cancel{5}$$

$$= 535 + 19 + (-0.56) + (-0.168)\,4 \Big\} +$$

$$= \cancel{553.8722+386}$$

$$= \underline{953.072}$$

$y(8000) = 953.072$

$y(10000) = 1195$

$\therefore \quad 1195 - 953 = \underline{242}$

Q] Details regarding marks secured by 280 students in an examination are given below. Estimate the number of students who secured marks b/w 45 & 52

Marks	below 30	30 - 40	40 - 50	50 - 60	60 - 70	70 - 80
No. of st	35	49	62	74	40	20

Q] Find an interpolating polynomial for the following data. Hence find, $y(12)$

x	y	Δ	Δ^2	Δ^3	Δ^4
3	1				
		2			
5	3		3		
		5		0	
7	8		3		0
		8		0	
9	16		3		
		11			
11	27				

$$y = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$p = \frac{x-3}{2}$$

$$y = 1 + \left(\frac{x-3}{\cancel{2}}\right)\cancel{2} + \frac{1}{2}\left(\frac{x-3}{2}\right)\left(\frac{x-5}{2}\right)3.$$

$$= 1 + x - 3 + (x-3)(x-5)\frac{3}{8}$$

$$y = \frac{8 + 8x - 24 + (x^2 - 8x + 15)3}{8} = \frac{-16x + 3x^2 + 29}{8}$$

$$\boxed{y = \frac{3}{8}x^2 - 2x + \frac{29}{8}}$$

$y(12) = \underline{33.625}$

Q] apply NBIF find the cubic polynomial satisfying $f(-4) = \cancel{24}\ -25$

$f(-2) = 1$ $f(0) = 3$, $f(2) = 29$ $f(4) = 127$

$\Rightarrow$

x	y	∇	∇^2	∇^3	∇^4
-4	-25				
		26			
-2	1		-24		
		2		48	
0	3		24		0
		26		48	
2	29		72		
		98			
4	127				

~~4~~ ~~24~~

$$y = y_4 + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$$p = \frac{x-4}{2}$$

$$y = 127 + \left(\frac{x-4}{\cancel{2}}\right)\cancel{98}\ 49 + \left(\frac{x-4}{2}\right)_2\left(\frac{x-2}{\cancel{2}}\right)\cancel{72}\ \cancel{36}\ \cancel{18}\ 9 + \left(\frac{x^2-6x+8}{4\times\cancel{6}}\right)\left(\frac{x+2}{\cancel{2}}\right)\cancel{48}\ \cancel{24}\ 6$$

$$y = 127 + 49x - \cancel{98}\ 196 + \cancel{18}\ 9x^2 + (\cancel{18}\ 9\times 6)x + (\cancel{18}\ 9\times 8) + (x^3 + 2x^2 - 6x^2 - 12x + 8x + 16)6$$

$$y = \cancel{173}\ 75 - 59x + 18x^2 + 6x^3 - \cancel{24}\ 4x^2 - \cancel{24}\ 4x + \cancel{96}\ 16$$

$$y = \cancel{x^3} + \cancel{14x^2} - \cancel{63x}\ 3$$

$$y = x^3 + 3x^2 + 3x + 3$$

Numerical Diff.

Let the function $f(x)$ be given for equally spaced values of x as,

x:	x_0	x_0+h	x_0+2h	---	x_0+nh
y:	y_0	~~y_0+h~~ y_1	~~y_0+2h~~ y_2	---	~~y_0+n~~ y_n

The process of estimating the derivatives of ~~a~~ y is called numerical Diff

By Newtons FIF, the derivatives of y for the specified values of x are as follows.

$$y' = f'(x)\Big|_{x=x_i} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots\right]$$

$$y'' = f''(x)\Big|_{x=x_i} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

$$y''' = f'''(x)\Big|_{x=x_i} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \frac{7}{4}\Delta^5 y_0 \cdots\right]$$

By NBIF, the derivatives of y for specified values of x are as follows,

$$y' = f'(x)\Big|_{x=x_i} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \cdots\right]$$

$$y'' = f''(x)\Big|_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n \cdots\right]$$

$$y''' = f'''(x)\Big|_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \frac{7}{4}\nabla^5 y_n + \cdots\right]$$

Q] A rod is rotating in a plane, the following table gives, The following table gives, θ in rad, through which the rod has turned for various values of time t in seconds,

$t=$	0	0.2	0.4	0.6	0.8	1.0	1.2
$\theta=$	0	0.12	0.49	1.12	2.02	3.2	4.67

find ω & α when $t =$ 0.2, 1.0, 1.2, 0.4

t	θ	Δ	Δ^2	Δ^3	Δ^4
0	0				
		0.12			
0.2	0.12		0.25		
		0.37		0.01	
0.4	0.49		0.26		0
		0.63		0.01	
0.6	1.12		0.27		0
		0.9		0.01	
0.8	2.02		0.28		0
		1.18		0.01	
1.0	3.2		0.29		
		1.47			
1.2	4.67				

$h = 0.2$

$\boxed{\text{at } t = 0.2}$

$$\omega = \frac{d\theta}{dt} \Rightarrow \theta' = \frac{1}{h}\left[\Delta\theta_0 - \frac{\Delta^2\theta_0}{2} + \frac{\Delta^3\theta_0}{3} - \frac{\Delta^4\theta_0}{4} - \cdots\right]$$

$$= \frac{1}{0.2}\left[0.37 - \frac{0.26}{2} + \frac{0.01}{3}\right]$$

$$= 1.2167 \text{ rad/s.}$$

$$\alpha = \frac{d^2\theta}{dt^2} \Rightarrow \theta'' = \frac{1}{(0.2)^2}\left[0.26 - 0.01\right]$$

$$= 6.25 \text{ rad/s}^2$$

$\boxed{\text{at } t = 1.0}$

$$\omega = \frac{d\theta}{dt} \Rightarrow \theta' = \frac{1}{h}\left[\nabla\theta_n + \frac{\nabla^2\theta_n}{2} + \frac{\nabla^3\theta_n}{3} + \cdots\right]$$

$$= \frac{1}{0.2}\left[1.18 + \frac{0.28}{2} + \frac{0.01}{3}\right]$$

$$= 6.6167 \text{ rad/s.}$$

$$\alpha = \frac{d^2\theta}{dt^2} \Rightarrow \theta'' = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n\right]$$

$$= \frac{1}{(0.2)^2}\left[0.28 + 0.01\right]$$

$$= 7.25 \text{ rad/s}^2$$

Interpolation with unequal intervals. [Lagrange's Interpolation]

Let $y = f(x)$ be given by (x not necessarily equally spaced)

x : $x_0 \quad x_1 \quad x_2 \, \ldots \, x_n$

y : $y_0 \quad y_1 \quad y_2 \, \ldots \, y_n$.

$$y = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} y_1$$

$$\cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} y_n$$

Inverse Lagrange's Interpolation →

To Estimate x for the given value of y we use I.L.I.

$$x = \frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)} x_0 + \frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)} x_1$$

$$+ \cdots + \frac{(y-y_0)(y-y_1)\cdots(y-y_{n-1})}{(y_n-y_0)\cdots(y_n-y_{n-1})} x_n$$

Q] x : $\overset{x_0}{1}$ $\overset{x_1}{2}$ $\overset{x_2}{5}$ $\overset{x_3}{8}$ $\overset{x_4}{9}$ $y(10) = ?$

y : $\underset{y_0}{5}$ $\underset{y_1}{-4}$ $\underset{y_2}{41}$ $\underset{y_3}{24}$ $\underset{y_4}{1613}$

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} y_1$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} y_3$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} y_4$$

$$y = 1.7857 + 2.8571 + 41 + (-68.5714) + 5184.6428$$

$$= \underline{5161.714257}$$

Q] x : 1 3 4 6

y : 0 12 33 135 $y(7) = ?$

Q] $f(5) = -30$, $f(8) = -12$, $f(11) = 4$, $f(14) = 20$

find x when $f(x) = 0$.

$\Rightarrow$ By inverse interpolation,

$x_0 = 5$ $x_1 = 8$ $x_2 = 11$ $x_3 = 14$

$y_0 = -30$ $y_1 = -12$ $y_2 = 4$ $y_3 = \cancel{(14)}\,20$

$$x = \frac{(y+12)(y-4)(y-20)}{(-30+12)(-30-4)(-30-20)}(5) + \frac{(y+30)(y-4)(y-20)}{(-12+30)(-12-4)(-12-20)} 8 +$$

$$\frac{(y+30)(y+12)(y-20)}{(4+30)(4+12)(4-20)} 11 + \frac{(y+30)(y+12)(y-4)}{(20+30)(20+12)(20-4)} 14$$

$\underline{\underline{y=0}}$

$$x = \underbrace{\frac{48\times 20}{-30600}\times 5}_{\frac{-8}{51}} + \left(\frac{-25}{12}\right) + \left(\frac{2475}{272}\right) + \frac{-9}{160}$$

$\underline{\underline{x = 10.2382}}$

Q] using G G L.I find the polynomial $P(x)$ to the foll data hence find y at $x=6.5$

	x_0	x_1	x_2	x_3
x	6	7	10	12
y	3	10	43	75
	y_0	y_1	y_2	y_3

$$y = \frac{(x-7)(x-10)(x-12)}{(6-7)(6-10)(6-12)} 3 + \frac{(x-6)(x-10)(x-12)}{(7-6)(7-10)(7-12)} 10 +$$

$$\frac{(x-6)(x-7)(x-12)}{(10-6)(10-7)(10-12)} 43 + \frac{(x-6)(x-7)(x-10)}{(12-6)(12-7)(12-10)} 75$$

$$\boxed{(x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc}$$

$$y = \frac{3}{-24}\left[x^3 - 29x^2 + 274x - 840\right] + \frac{10}{15}\left[x^3 - 28x^3 + 252x - 720\right]$$

$$+ \frac{43}{-24}\left[\left(x^3 - 25x^2 + 198x - 504\right)\right] + \frac{75}{60}\left[x^3 - 23x^2 + 172x - 420\right]$$

$$y = 0(x^3) + (1)x^2 - 6x + 3$$

$$\boxed{y = x^2 - 6x + 3}$$

$x = 6.5$, $y = 6.249$

Q] Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1, 5$ for the given data

x	y	Δ	Δ^2	Δ^3	Δ^4
0	6.9944				
		0.4092			
1	7.4036		-0.0313	0.003	
		0.3779			0
2	7.7815		-0.0283		
		0.3496		0.003	
3	8.1311		-0.0253		0
		0.3243		0.003	
4	8.4554		-0.0223		0
		0.302 ~~0.002~~		0.003	
5	8.7574		-0.0193		
		0.2827			
6	9.0401				

$h = 1$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_0 + \frac{\nabla^2 y_0}{2} + \frac{\nabla^3 y_0}{3}\right]$$

$$= \left[0.302 - \frac{0.0223}{2} + \frac{0.003}{3}\right]$$

$$= \underline{0.29185}$$

$\boxed{x = 5}$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n\right]$$

$$= \left[-0.0223 + 0.003\right] = \underline{-0.0193}$$

Q] Find the max^m value of $f(x)$ using the foll given table

x	1	2	3	4	5
$f(x)$	149.1	154.2	157.7	159.6	159.9

Ans → $x = 4.6875$

$f(x) = 159.89$ (Max)

Q] A smooth curve passes thru the pt.

(1, 17) (2, 9), (3, -19) & (5, 45)

find Slope at $x = 2$

→ Lagrange's

Q] Data given below is regarding age of workers.
Estimate no of person in group 55 – 60

Age	below 30	30-40	40-50	50-60
No. of workers	2380	3456	1820	644

Q] The value of $\sqrt{x}$ $p(x)$

x 1.5 (0.02) 1.6

$p(1.51)$, $p(1.61)$, $p(1.55)$

x	1.5	1.52	1.54	1.56	1.58	1.6
$p(x)$	0.8862	0.8870	0.8881	0.8896	0.8914	0.8935