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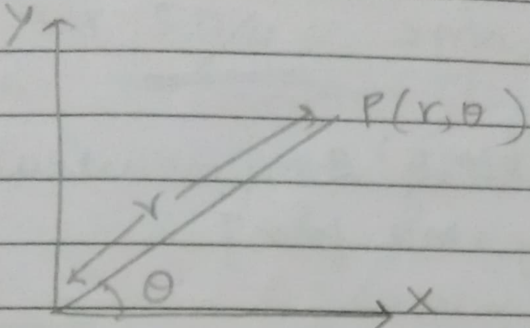
UNIT-2

DIFFERENTIAL CALCULUS

Polar Coordinates:



Let the distance between the fixed point O (origin or pole) to any point P(r, θ) be

$OP = r =$ Radius vector

$OX =$ Initial line.

Let θ be the angle between the radius vector OP and the initial line OX.

$\theta \longrightarrow$ Radial angle.

The relation b/w the cartesian coordinate (x, y) to polar coordinate (r, θ) is one-to-one. It is given by,

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

$$\frac{y}{x} = \tan\theta$$

$$\boxed{\theta = \tan^{-1}\frac{y}{x}}$$

Where r is radial distance, $r \geq 0$, & θ has $0 \leq \theta \leq 2\pi$

→ $r = a$, constant represents the circle which has its centre at origin and radius a.

→ $\theta = c$, constant implies that it is a radial line passing through origin with slope = $\tan c$.

$\theta = c$

$\tan^{-1}\frac{y}{x} = c$

$\frac{y}{x} = \tan c$

$y = (\tan c)\,x$

[$y = mx$ form]

$r = a$

$r^2 = a^2$

$x^2 + y^2 = a^2$ → Circle of radius a & centre at origin.

Ex:-

When $(1,1)$

$r = \sqrt{x^2+y^2}$

$= \sqrt{1+1}$

$= \sqrt{2}$.

$\theta = \tan^{-1}\frac{y}{x}$.

$= \tan^{-1} 1$.

$= \pi/4$

∴ Polar form $= (r, \theta)$

$= (\sqrt{2}, \pi/4)$.

When $(-1, 1)$

$r = \sqrt{x^2+y^2}$

$= \sqrt{2}$.

$\theta = \tan^{-1}(-1)$ ⇒ $\pi - \pi/4$.

$= \cancel{\pi/4}$ ⇒ $\frac{3\pi}{4}$.

Polar form :- $(\sqrt{2}, 3\pi/4)$.

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(iv) When $(1, -1)$

$r = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{-1}{1}\right) \Rightarrow \tan^{-1} -1$ (IIIrd quadrant)

$\therefore \pi + \pi/4$ $\therefore$ Polar form :- $(\sqrt{2}, 5\pi/4)$

$\Rightarrow 5\pi/4$

1) Write the Cartesian point for the polar form given.

i) $(2, 2\pi/3)$

$~~2 = \sqrt{x^2+y}~~$ $[x = r\cos\theta$ & $y = r\sin\theta]$

$x = 2\cos 2\pi/3 = 2 \cdot -1/2 = -1$

$y = 2\sin\frac{2\pi}{3} = 2 \times \sqrt{3}/2 = \sqrt{3}$

$\therefore (2, 2\pi/3) = (-1, \sqrt{3})$

ii) $(3, 5\pi/6)$

$x = r\cos\theta$

$= 3\cos\frac{5\pi}{6} \Rightarrow -3 \cdot \frac{\sqrt{3}}{2} \Rightarrow \frac{-3\sqrt{3}}{2}$

$y = r\sin\theta$

$= 3\sin\frac{5\pi}{6} \Rightarrow 3 \cdot \frac{1}{2} \Rightarrow 3/2$

$\therefore (3, 5\pi/6) = (-3\sqrt{3}/2, 3/2)$

$\rightarrow$ • $r\cos\theta = 3 \Rightarrow x = 3$, line $\parallel$ to y axis

• $r\sin\theta = b \Rightarrow y = b$

$\rightarrow$ $r = 2a\cos\theta$ - represents a circle with centre at $(a, 0)$ & Radius a.

$r^2 = 2ar\cos\theta \Rightarrow x^2 + y^2 - 2ax = 0$

$x^2 + y^2 = 2ax$

It is a circle with centre at $(a, 0)$

$\rightarrow x^2 + (y-b)^2 = b^2$

$x^2 + y^2 - 2yb + \cancel{b^2} = \cancel{b^2}$

$r^2 - 2br\sin\theta = 0$

$r^2 = 2br\sin\theta$

$\boxed{r = 2b\sin\theta}$

$\rightarrow \sin\theta + \cos\theta = \frac{1}{r}$

$r(\sin\theta + \cos\theta) = \frac{r}{r}$

$\boxed{r\sin\theta + r\cos\theta = 1}$

$\Rightarrow y + x = 1 \longrightarrow y = 1 - x$

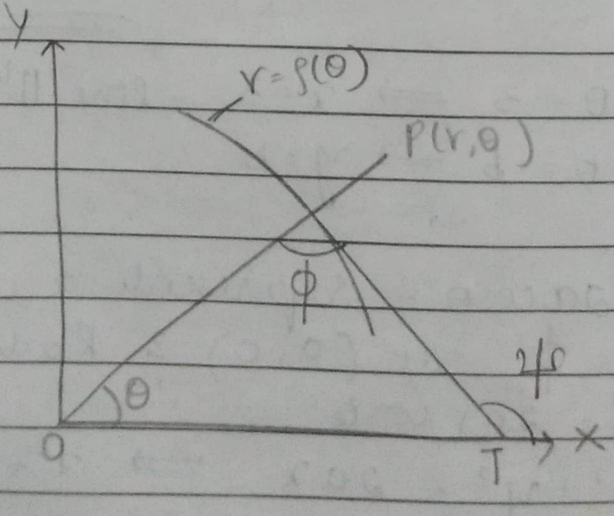
$\downarrow$ st. line

$\rightarrow \boxed{r = 4\operatorname{cosec}\theta}$

$\cancel{\left[r = \frac{4}{\cos\theta} \Rightarrow x = 4\right]}$

$r = \frac{4}{\sin\theta} \Rightarrow r\sin\theta = 4 \Rightarrow y = 4$

ANGLE BETWEEN RADIUS VECTOR AND TANGENT.



Let ϕ be the angle between the radius vector OP and the tangent PT. At the point $P(r,\theta)$ of the curve $r=f(\theta)$. Then

$$\boxed{\tan\phi = r\frac{d\theta}{dr}}$$

(Or)

$$\boxed{\cot\phi = \frac{1}{r}\frac{dr}{d\theta}}$$

If ψ is the angle b/w tangent PT & x axis, then,

$\psi = \theta + \phi$ (Ext angle theorem)

Slope of the tangent $= \tan\psi = \tan(\theta+\phi)$

$$= \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi}$$

1] Show that at any point on the Equiangular spiral $r = ae^{\theta\cot\alpha}$ where a, α are constant, the tangent is inclined at a constant angle to the radius vector

$\rightarrow$ $\frac{dr}{d\theta} = \frac{d\,ae^{\theta\cot\alpha}}{d\theta}$

$= (ae^{\theta\cot\alpha})(\cot\alpha)$

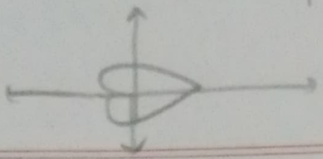
$\cot\phi = \frac{1}{r}\frac{dr}{d\theta}$

$\Rightarrow$ $\cancel{\frac{dr}{d\theta} = r\cot\phi}$

$\Rightarrow$ $\cot\phi = \frac{1}{ae^{\theta\cot\alpha}} \cdot ae^{\theta\cot\alpha}(\cot\alpha)$

$$\boxed{\cot\phi = \cot\alpha}$$

→ Cardioid has ♡ shape



$\Rightarrow \phi = \alpha$.

$\Rightarrow$ Since α is a constant,
ϕ is also a const.

2] Find the angle between the radius vector and tangent to the curve, $r = ae^{b\theta}$ where a, b are constants:-

→ $$\cot\phi = \frac{1}{r}\frac{dr}{d\theta}$$

$$\cancel{\frac{dr}{d\theta} = r\cot\phi}$$

$$\frac{dr}{d\theta} = \frac{d\,ae^{b\theta}}{d\theta}$$

$$= ae^{b\theta}(b)$$

$$\therefore \cot\phi = \frac{1}{r}ae^{b\theta}(b).$$

$$= \frac{1}{ae^{b\theta}} \cdot ae^{b\theta}(b)$$

$$\cot\phi = b$$

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3] Find the angle between the tangent & Radius vector for the cardiod $r = a(1+\cos\theta)$:-

→ $$\frac{dr}{d\theta} = \frac{d(a(1+\cos\theta))}{d\theta}$$

$$= a(-\sin\theta).$$

If $r = f(\theta)$
$\log r = \log f(\theta)$
$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{f(\theta)} f'(\theta) = \cot \phi$

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$$\frac{dr}{d\theta} = -a\sin\theta$$

We know, $\cot\phi = \frac{1}{r}\frac{dr}{d\theta}$

$$\Rightarrow \cot\phi = \frac{-1}{a(1+\cos\theta)} \cdot a\sin\theta$$

$$\Rightarrow \cot\phi = \frac{-\sin\theta}{1+\cos\theta}$$

$$= \frac{-\sin\theta}{2\cos^2\theta/2}$$

$$\Rightarrow -\frac{\cancel{2}\sin\theta/2\cos\theta/2}{\cancel{2}\cos^2\theta/2}$$

$$\cot\phi = -\tan\theta/2 \rightarrow \text{In 2nd quadrant}$$

$$\cot\phi = \cot\left(\frac{\pi}{2} + \theta/2\right)$$

$$\therefore \boxed{\phi = \pi/2 + \theta/2}$$

$\because \cot(\pi/2 + \theta) = -\tan\theta$

5m)

(i) Find the angle between the radius vector & tangent to the curve, $\theta = \frac{1}{a}\sqrt{r^2-a^2} - \cos^{-1}\left(\frac{a}{r}\right)$, $r \neq 0 \neq a$.

$$\rightarrow \frac{d\theta}{dr} = \frac{d}{dr}\left\{ 1/a \sqrt{r^2-a^2} - \cos^{-1}(a/r)\right\}$$

$\frac{d}{dx}(\cos^{-1}x) = \left(\frac{-1}{\sqrt{1-x^2}}\right)$

$$\Rightarrow \frac{1}{a}\left\{\frac{1}{2}(r^2-a^2)^{-1/2}(2r) - \left(-\frac{1}{\sqrt{1-\frac{a^2}{r^2}}}\right)\left(\frac{-a}{r^2}\right)\right.$$

$$\rightarrow \frac{r}{a\sqrt{r^2-a^2}} - \frac{a}{r\sqrt{r^2-a^2}}$$

$$\rightarrow \frac{1}{\sqrt{r^2-a^2}}\left(\frac{r}{a}-\frac{a}{r}\right) \Rightarrow \frac{1}{\sqrt{r^2-a^2}}\left(\frac{r^2-a^2}{ar}\right)$$

$$\frac{d\theta}{dr} \rightarrow \frac{\sqrt{r^2-a^2}}{ar}$$

$$\tan\phi = r\cdot\frac{d\theta}{dr}$$

$$= \cancel{r}\cdot\frac{\sqrt{r^2-a^2}}{a\cancel{r}}$$

$$\tan\phi = \frac{\sqrt{r^2-a^2}}{a}$$

$$\boxed{\phi = \tan^{-1}\frac{\sqrt{r^2-a^2}}{a}}$$

5) Show that the tangent to the curve $r = a(1+\sin\theta)$ at any point with $\theta = \pi/2$ is parallel to the initial line:-

$\longrightarrow$ Tangent is parallel to initial line $\Rightarrow$ Slope of tangent is zero.

$$r = a(1+\sin\theta).$$

$$\log r = \log a + \log(1+\sin\theta)$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta}{1+\sin\theta}$$

$$\tan\phi = r\frac{d\theta}{dr} = \frac{1+\sin\theta}{\cos\theta}$$

$$= \cancel{\sin}\ \cancel{\sin^2\theta/2 + \cos^2\theta/2}$$

$$\Rightarrow \frac{\sin^2\theta/2 + \cos^2\theta/2 + 2\sin\theta/2\cos\theta/2}{\cos^2\theta/2 - \sin^2\theta/2}$$

$$\Rightarrow \frac{(\sin\theta/2 + \cos\theta/2)^{\cancel{2}}}{(\cos\theta/2 - \sin\theta/2)\cancel{(\cos\theta/2 + \sin\theta/2)}}$$

$$\Rightarrow \frac{\sin\theta/2 + \cos\theta/2}{\cos\theta/2 - \sin\theta/2}$$

Divide by $\cos\theta/2$.

$$\Rightarrow \frac{1+\tan\theta/2}{1-\tan\theta/2}$$

$$\Rightarrow \tan(\pi/4 + \theta/2)$$

$$\therefore \tan\phi = \tan(\pi/4 + \theta/2)$$

$$\therefore \phi = \pi/4 + \theta/2$$

Here $\theta = \pi/2$

$$\phi = \pi/4 + \pi/4$$

$$= \frac{2\pi}{4}$$

$$\phi = \pi/2$$

$$\psi = \phi + \theta$$

$$= \pi/2 + \pi/2$$

$$= \pi$$

Slope $= \tan\psi \rightarrow \tan\pi$

$= 0$ $\therefore$ Tangent is $\parallel^{el}$ to initial line.

$\Rightarrow r = \frac{2a}{1-\cos\theta}$.

$\log r = \log 2a - \log(1-\cos\theta)$.

$\frac{1}{r}\frac{dr}{d\theta} = -\frac{1}{1-\cos\theta}(\sin\theta)$,

$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1-\cos\theta}$

$r.\frac{d\theta}{dr} = \frac{1-\cos\theta}{-\sin\theta}$

$\tan\phi = \frac{1-\cos\theta}{-\sin\theta}$

$= \frac{2\sin^2\theta/2}{-2\sin\theta/2\cos\theta/2}$

$\tan\phi = -\tan\theta/2$.

$\cancel{\phi = -\theta/2}$

$\cancel{\tan\phi = \cot(\pi/2 + \theta/2)}$

$\tan\phi = \tan(\pi - \theta/2)$

$\phi = \pi - \theta/2$.

$\theta = 2\pi/3$

$\Rightarrow \phi = \pi - \frac{2\pi}{3\times 2}$

$\phi = 2\pi/3$

$\therefore \psi = \theta + \phi \Rightarrow \frac{2\pi}{3} + \frac{2\pi}{3} \Rightarrow 4\pi/3$

$\therefore$ Slope $= \tan\phi$

$= \tan\frac{4\pi}{3}$

$\Rightarrow \tan(\pi + \pi/3)$

Slope $\Rightarrow \sqrt{3}$

7) Find the angle between radius vector & tangent to the curve, $r^2 \sin 2\theta = a^2$

$\longrightarrow$ $\log(r^2 \sin 2\theta) = \log a^2$

$2\log r + \log \sin 2\theta = 0$

$$\frac{2}{r}\frac{dr}{d\theta} + \frac{2\cos 2\theta}{\sin 2\theta} = 0$$

$$\frac{2}{r}\frac{dr}{d\theta} + 2\cot 2\theta = 0$$

$$\cancel{\frac{1}{2}\tan\phi + \frac{\tan\theta}{2} = 0}$$

$$\cancel{2}\cot\phi = -\cancel{2}\cot 2\theta \qquad \cot(180-\theta) = -\cot\theta$$

$$\cot\phi = -\cot 2\theta$$

$$\boxed{\phi = 180 - 2\theta}$$

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8) Show that for the curve $\log(x^2+y^2) = k\tan^{-1}\frac{y}{x}$ the angle between the radius vector & the tangent is the same at all points of the curve :-

$\rightarrow$ $x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2$

& $\theta = \tan^{-1} y/x$

$\Rightarrow$ $\log r^2 = k\theta$

$(2\cancel{r})\frac{1}{r^{\cancel{2}}}\frac{dr}{d\theta} = k$

$2\cot\phi = k$

$\cot\phi = \frac{k}{2}$

$\boxed{\phi = \cot^{-1}\frac{k}{2}}$ = const.

$\therefore$ Angle b/w radius vector & tangent is same at all points of the curve.

9) Find ϕ if $r^m = a^m(\cos m\theta + \sin m\theta)$

$\rightarrow$ $r^m = a^m(\cos m\theta + \sin m\theta)$.

$m\log r = m\log a + \log(\cos m\theta + \sin m\theta)$.

$m \cdot \frac{1}{r}\frac{dr}{d\theta} = \frac{m\cos m\theta - m\sin\theta}{\cos m\theta + \sin m\theta}$.

$\Rightarrow$ $\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos m\theta - \sin m\theta}{\cos m\theta + \sin m\theta}$

$\cot\phi = \tan(\pi/4 - m\theta)$

$$\Rightarrow \cot\phi = \tan\left(\pi/2 - \pi/4 + m\theta\right)$$

$$\cot\phi = \cot\left(\pi/4 + m\theta\right)$$

$$\therefore \phi = \pi/4 + m\theta$$

10) Find the slope of the curve $r = a\sec^2\theta/2$ at $\theta = 2\pi/3$:-

$\rightarrow$ $r = a\sec^2\theta/2$

$$\log r = \log a + \log \sec^2\theta/2$$

$$\frac{1}{r}\frac{dr}{d\theta} = \cancel{2}\times \frac{\cancel{2}\cancel{\sec^2}\theta/2 \tan\theta/2 \cdot \frac{1}{\cancel{2}}}{\cancel{\sec^2\theta/2}}$$

$$\cot\phi = \tan\theta/2$$

$$\cot\phi = \cot\left(\pi/2 - \theta/2\right)$$

$$\phi = \pi/2 - \theta/2$$

When $\theta = 2\pi/3$

$$\therefore \phi = \pi/2 - \frac{\cancel{2}\pi}{3\times\cancel{2}}$$

$$= \frac{3\pi - 2\pi}{6}$$

$$\phi = \pi/6$$

$$\psi = \phi + \theta \Rightarrow \pi/6 + \frac{2\pi}{3} \rightarrow \frac{\pi + 4\pi}{6} = \frac{5\pi}{6}$$

$\therefore$ Slope :- $\tan\psi \Rightarrow \tan\left(\pi - \pi/6\right)$

$$\Rightarrow \frac{-1}{\sqrt{3}}$$