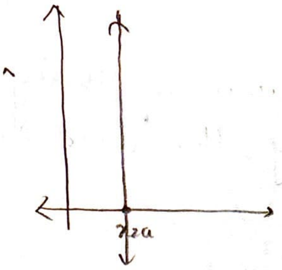
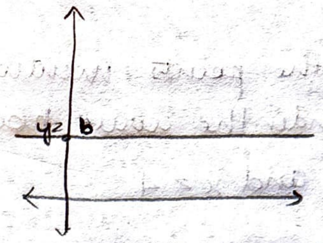
# MULTIPLE INTEGRALS:

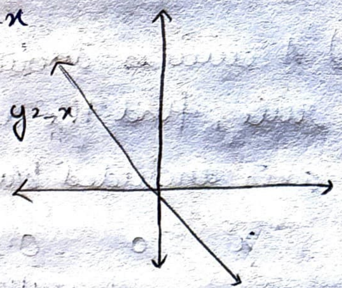
$x = a$



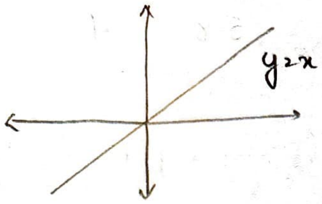

$y = b$



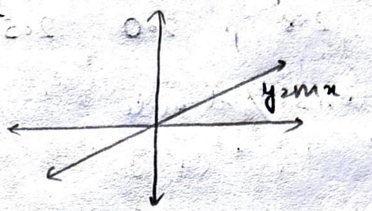

$y = -x$



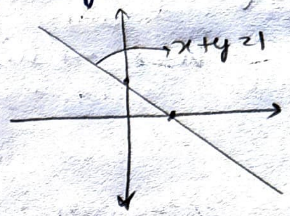

$y = x$



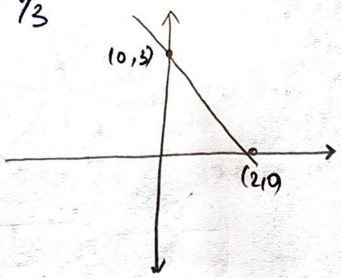

$y = mx$



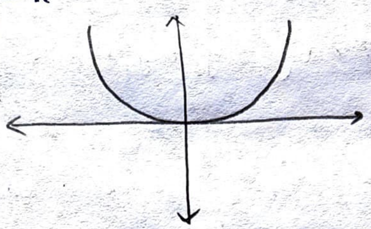

$x+y=1$, $y = 1-x$



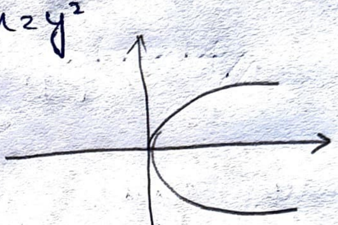

$x/2 + y/3 = 1$



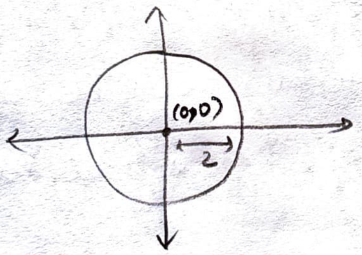

$y = x^2$



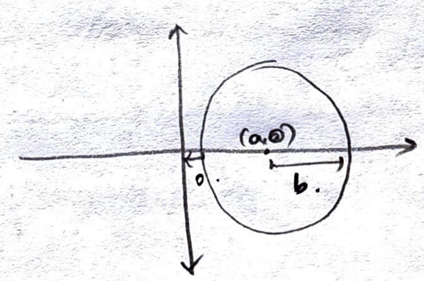
$x = y^2$



$x^2+y^2 = 4$




$(x-a)^2 + y^2 = b^2$




Double Integral

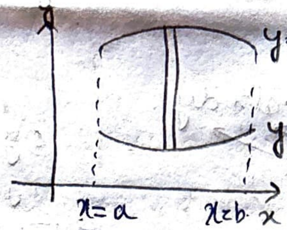
The double $f(x,y)$ over the region $x = x_1$, $x = x_2$, $y = y_1$, $y = y_2$ is denoted by integral of $\rightarrow$

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)\cdot dx\cdot dy$$

Evaluation of double integral:

(1) Let the limits of the region be $x = a$, $x = b$, $y = f(x)$; $y = g(x)$, then

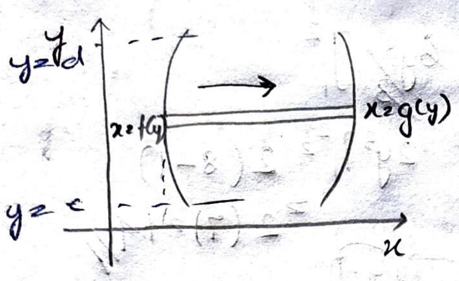
$$\int_{a}^{b}\int_{y=f(x)}^{g(x)} f(x,y)\cdot dx\cdot dy = \boxed{\int_{x=a}^{b}\boxed{\int_{y=f(x)}^{y=g(x)} f(x,y)\cdot dy}\cdot dx}$$




y has variable limits and x has constant limits.
First we integrate wrt y treating x as constant b/w the y limits then we integrate wrt x which has constant limits b/w x limits

② $y = c, y = d; x = f(y); x = g(y)$

$$\int_{x=f(y)}^{g(y)}\int_{y=c}^{d} f(x,y)\cdot dx\cdot dy = \int_{y=c}^{d}\boxed{\int_{x=f(y)}^{g(y)} f(x,y)\cdot dx}\cdot dy$$




If x has functional limits y has constant limits, first we integrate wrt x keeping y constant b/w the x limits, then we integrate wrt y.

③ If the limits are $x=a; x=b; y=c; y=d$ (all are constants), the order does not matter.

$$\int_{x=a}^{x=b}\int_{y=c}^{y=d} f(x,y)\cdot dx\cdot dy$$

(Q) Evaluate the following integrals.

A $\displaystyle\int_{x=2}^{3}\int_{1=y}^{2} x^2 y\cdot dy\cdot dx$

$\displaystyle\int_2^3 x^2\cdot dx \int_1^2 y\cdot dy$ → other way of writing

$$= \int_2^3 \left[x^2\cdot y^2/2\right]_1^2 \cdot dx$$

$$= \int_2^3 x^2\cdot 3/2\cdot dx$$

$$= \left.\frac{x^3\cdot \cancel{3}}{\cancel{3}\cdot 2}\right]_2^3 = \frac{27}{2} - \frac{\cancel{8}\,4}{\cancel{2}}$$

$$19/2$$

(Q) $\displaystyle\int_1^2\int_1^3 (x+y^3)\cdot dy\cdot dx$

$$= \left. x+\frac{y^4}{4}\right]_1^3 = x + 81/4 - 1/4$$

$$\left. x+20\right]_1^2 = \left.\frac{x^2}{2}+20x\right]_1^2 \quad 4/2 - 1/2 + 20[1]$$

$$3/2 + 20 = 43/2$$

(Q) $\int_0^1 \int_0^1 \frac{dx \cdot dy}{1+x+y}$

(Q) $\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} xy \cdot dy \cdot dx$

$\Rightarrow \int_0^{2a} x \, \frac{y^2}{2} \Big]_0^{\sqrt{2ax-x^2}}$

$= \int_0^{2a} \frac{x(2ax-x^2)}{2} \cdot dx$

$= \int_0^{2a} \frac{2ax^2 - x^3}{2} \cdot dx$

$= \frac{1}{2} \left( \frac{2ax^3}{3} - \frac{x^4}{4} \right]_0^{2a}$

$= \frac{1}{\cancel{2}} \left[ \frac{2a \, 8a^3}{3} - \frac{16a^4}{4} \right]$

$= \frac{8a^4}{3} - \cancel{4}2a^4 = \cancel{\frac{-4a^4}{3}} \;\; \frac{2a^4}{3}$

(Q) Evaluate

$\int_1^2 \int_y^{3y} (x+y) \cdot dx \, dy.$

$\Rightarrow \int_1^2 \int_y^{3y} \frac{x^2}{2} + xy$

$= \int_1^2 (4y^2 + 2xy^2) \cdot dy$

$= \frac{4y^3}{3} + \frac{2y^3}{3}$

$\cancel{6}\frac{y^3}{\cancel{3}} \Big]_1^2$

$2y^3 \Big]_1^2 = 2(8-1)$

$= 2(7) = 14$ //

$3y^2 - y^2$

$3y \cdot y -$

$y \cdot y.$

$\frac{9y^2}{2} + x(3y) -$

$\frac{y^2}{2} - x(y)$

$\frac{8y^2}{2}$

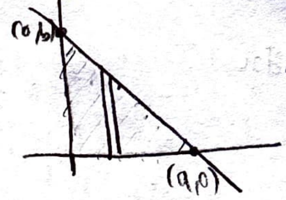
$4y^2 + 2xy$

(Q) $\int_0^{2a} \int_{y^2/4a}^{a} 1 \cdot x^2 \cdot dx \cdot dy.$ ($y^2$)

(Q) $\int_0^4 \int_0^{x^2} e^{y/x} \cdot dx \cdot dy$

(Q) $\int_0^1 \int_0^{\sqrt{4-x^2}} xy \cdot e^{x^2} \cdot dy \cdot dx$

(Q) * $\iint_R xy \cdot dx \cdot dy$ where R is the triangular region bounded by the coordinate axes and $x/a + y/b = 1$




$R = \frac{1}{2} \times a \times b$

$ab/2$

(A) $\iint x \cdot dx \cdot y \cdot dy$   y limits $= \int_0^{b(1-x/a)}$

x limits $= \int_0^a$

$=$ Integral: $\int_0^a \int_0^{b(1-x/a)} y \cdot dy \cdot x \cdot dx$

$\Rightarrow \frac{y^2}{2} \Big]_0^{b(1-x/a)} = \frac{b^2 (1-x/a)^2}{2} \cdot \int x \cdot dx \Big]_0^a$

$b^2 \left(1 + \frac{x^2}{a^2} - \frac{2x}{a}\right) \cdot x \cdot dx$

$= \frac{1}{2} \left[ b^2 x + \frac{b^2 x^3}{a^2} - \frac{2b^2 x^2}{a} \right]$

$= \frac{1}{2} \left[ \frac{b^2 x^2}{2} + \frac{b^2 x^4}{4a^2} - \frac{2b^2 x^3}{3a} \right]_0^a$

$\frac{1}{2} \left[ \frac{a^2 b^2}{2} + \frac{a^4 b^2}{4a^2} - \frac{2b^2 a^3}{3a} \right]$

$\frac{1}{2} \left[ \frac{a^2b^2}{2} + \frac{a^2b^2}{4} - \frac{2a^2b^2}{3} \right)$

$\frac{6a^2b^2 + 3a^2b^2 - 8a^2b^2}{12}$

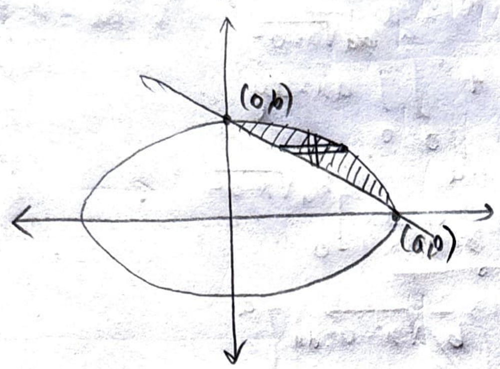
$\left(\frac{a^2b^2}{12}\right)\left(\frac{1}{2}\right) = \frac{a^2b^2}{24}$

(Q) Evaluate $\iint xy \cdot dA$ over the area in the 1st quadrant of the circle and ellipse

~~Hwk~~ $x^2/\cancel{a} + y^2 = a^2$, $x^2/a^2 + y^2/b^2 = 1$

(Q) Evaluate $\iint_R xy \cdot dx \cdot dy$ when R is the region in the 1st quadrant included b/w $x^2/a^2 + y^2/b^2 = 1$ ; $x/a + y/b = 1$

$$y \text{ limits} = \int \int_{b(1-x/a)}^{b\sqrt{1-x^2/a^2}} y \cdot dy \qquad \int_0^a \int_{b(1-x/a)}^{b\sqrt{1-x^2/a^2}} \cdot y \cdot dy$$




$$x^2/a^2 + y^2/b^2 = 1$$

$$1 - x^2/a^2 = y^2/b^2$$

$$b^2(1 - x^2/a^2) = y^2$$

$$b\sqrt{(1 - x^2/a^2)} = y$$

$$\left.\frac{y^2}{2}\right]_{b(1-x/a)}^{b(\sqrt{1-x^2/a^2})} = \frac{b^2(1 - x^2/a^2) - b^2(1 - x/a)^2}{2}$$

$$= \frac{b^2 - \frac{b^2x^2}{a^2} - b^2(1 + x^2/a^2 - \cancel{2a}\,2x/a)}{2}$$

$$= \frac{\cancel{b^2} - b^2x^2/a^2 - \cancel{b^2} - b^2x^2/a^2 + 2xb^2/a}{2}$$

$$= \frac{1}{2}\left(-\frac{2b^2x^2}{a^2} + \frac{2b^2x}{a}\right)\Bigg]_0^a$$

$$= \frac{1}{2}\left(-\frac{2b^2x^3}{3a^2} + \frac{\cancel{2}b^2x^2}{\cancel{2}a}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{-2b^2a^3}{3\cancel{a^2}} + \frac{b^2a^2}{a}\right]$$

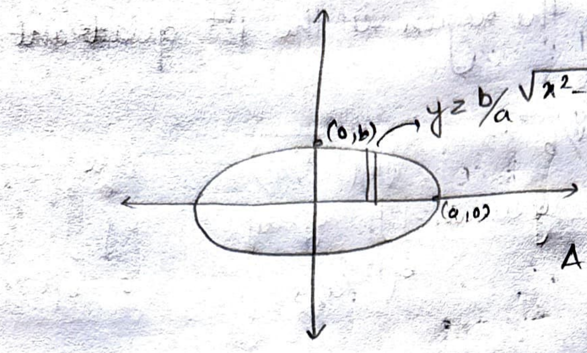
$$= \frac{1}{2}\left[\frac{-2b^2a}{3} + b^2a\right]$$

$$= \frac{1}{2}\left[\frac{-2b^2a + 3b^2a}{3}\right] = \frac{ab^2}{6}.$$

Area $= \iint_R dx \cdot dy$

NOTE: 1

(Q) Find the area of the elipse $x^2/a^2 + y^2/b^2 = 1$ using double integral.




Area of elipse $= 4 \times$ Area of $1^{st}$ quadrant

y limits $y = 0$ to $y = b/a \sqrt{x^2 - a^2}$

x limits $x = 0$ to $x = a$

$$A = 4 \int_0^a \int_0^{b/a\sqrt{x^2-a^2}} dx \cdot dy$$

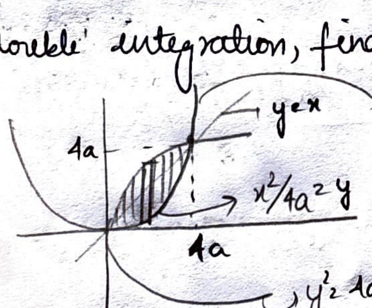
$$A = 4 \int_0^a y \cdot dx \Big]_0^{b/a\sqrt{x^2-a^2}}$$

$$4\left(b/a \sqrt{x^2-a^2} \cdot dx\right)\Big]_0^a$$

$$\Rightarrow 4b/a \left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}(x/a)\right]_0^a$$

$$= 4b/a\left(\frac{a^2}{2} \cdot \frac{\pi}{2}\right) = \pi ab$$

(Q) Using double integration, find the area b/w the curves. $y^2 = 4ax$ and $x^2 = 4ay$




y limits $= y = x^2/4a$ $y = 2\sqrt{ax}$

x limits $= x = 0$ to $x = 4a$

$y^2/4a = \sqrt{4ay}$

$y^2/4a = 2\sqrt{ay}$

$y^4/16a^2 = 4ay$

$y^3 = 64a^3$

$\boxed{y = 4a}$

$4a \times 4a = 4a \times x$.

$$A = \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dx \cdot dy$$

$$= \int_0^{4a} y \Big]_{x^2/4a}^{2\sqrt{ax}}$$

$$2\sqrt{ax} - x^2/4a \Big]_0^{4a}$$

$$\frac{2a^{1/2}x^{3/2} \cdot 2}{3} - \frac{x^3}{3 \times 4a}\Big]_0^{4a}$$

$$\Rightarrow \frac{4a^{1/2} \times (4a)^{3/2}}{3} - \frac{(4a)^{3}}{3(4a)}$$

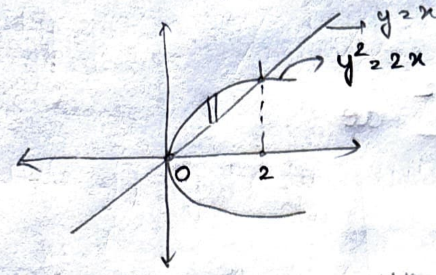
$4 \quad 2^{2 \times 3/2}$

$$\Rightarrow \frac{4a^2 \cdot (64)^{1/4}}{3} - \frac{16a^2}{3} = \frac{4a^2 \cdot 8}{3} - \frac{16a^2}{3}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}$$

(Q) using double integration, area b/w $y^2 = 2x$, $y = x$.

(A)




y limits $y = x$ to $y = \sqrt{2x}$

x limits $x = 0$ to $x = 2$

$y^2 = 2y$

$y^2 - 2y$

$$\int_0^2 \int_x^{\sqrt{2x}} dx \cdot dy$$

$$\int_0^2 y \Big]_x^{\sqrt{2x}} \quad \sqrt{2x} - x \Big]_0^2$$

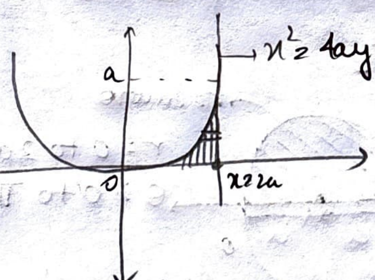
$$\frac{\sqrt{2} \cdot x^{3/2} \cdot 2}{3} - \frac{x^2}{2} \Big]_0^2$$

$$\frac{\sqrt{2} \cdot (2)^{3/2} \cdot 2}{3} - \cancel{4}\, 2$$

$$\frac{2^2 \cdot 2}{3} - 2 = \frac{8}{3} - 2 = 2/3$$

(Q) Evaluate double integral $\iint xy \cdot dA$ over the area bounded by the x axis $x = 2a$, $x^2 = 4ay$.

(A)




$\sqrt{4ay} = 2a$

$\cancel{4}ay = \cancel{4}a^2$

$\boxed{y = a}$

$x^2 = 4a^2$

$\boxed{x = 2a}$

x limits = $\cancel{2a}$ 0 to $\sqrt{4ay}$

y limits = 0 to a

$$= \int_0^a \int_0^{\sqrt{4ay}} xy \cdot dx \cdot dy$$

$$= \int_0^a y\frac{x^2}{2} \Big]_0^{\sqrt{4ay}} = y\frac{(4ay)}{2} \Big]_0^a$$

$$= \frac{4ay^2}{2} \Big]_0^a = \frac{4ay^3}{6} \Big]_0^a$$

$$\frac{4ay^3}{2 \times 3} \Big]_0^a$$

$$\frac{\cancel{4}a^4}{\cancel{6}} = \frac{2a^4}{3}$$

$$\cancel{= \frac{4a(8)}{6} = \frac{32a}{6}}$$

$$\cancel{= \frac{16a}{3}}$$

In polar form, area $= \iint r \cdot dr \cdot d\theta$

(Q) using double integral, find the area enclosed by the curve $r = a(1+\cos\theta)$ and lying above the initial line


r limits at $r = a(1+\cos\theta)$

$\theta$ limits $\theta = 0$ to $\theta = \pi$

$$A = \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= \int_0^{\pi}\left(\frac{r^2}{2}\right)\Big]_0^{a(1+\cos\theta)} = \int_0^{\pi}\frac{a^2(1+\cos\theta)^2}{2}\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} \cancel{4}\,4\cos^4\theta/_2\,d\theta$$

$$= \frac{a^2 4}{2}\int_0^{\pi/2}\cos^4\theta/_2\,d\theta$$

$$\int_0 \cos^n\theta\,d\theta = \frac{(n-1)(n-3)(n-5)\ldots}{n(n-2)\ldots}$$

$4\ n = \pi/_2$ — even

$1 \rightarrow$ odd

$$= \frac{4a^2}{2}\int_0^{\pi/2}$$

$$= 4a^2\left[\frac{3\times1}{4\times2}\times\frac{\pi}{2}\right]$$

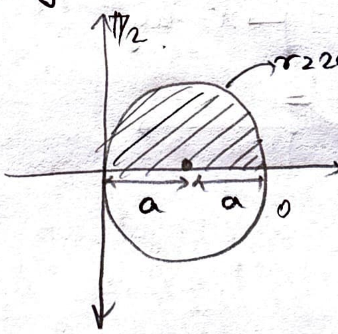
$$= \frac{\cancel{4}a^2\times3\times\pi}{\cancel{8}\times\cancel{2}} = \frac{3\pi a^2}{4}$$

(Q) Evaluate double integral over $R$. $\iint r^2\sin\theta\,dr\,d\theta$ where $R$ is the region bounded by the semicircle $r = 2a\cos\theta$ lying above the initial line.

(A) $r = 2a\cos\theta$

$r^2 = 4a^2\cos^2\theta$

$x^2+y^2 = 2a\cos\theta \qquad (x-a)^2+y^2 = a^2$




$r = 0$ to $2a\cos\theta$

$\theta = 0$ to $\pi/_2$

$$\int_0^{\pi}\int_0^{2a\cos\theta} r^2\sin\theta\,dr\,d\theta$$

$$= r^2 - \cos\theta\Big]_0^{2a\cos\theta}\Big]_0^{\pi}$$

$$= -r^2\cos(2a\cos\theta) + r^2\Big]_0^{\pi}$$

$=$

$$\int_0^{\pi/2}\int_0^{2a\cos\theta}\frac{r^3}{3}\sin\theta$$

$$\frac{(2a\cos\theta)^3}{3}\,d\theta\Big]_0^{\pi}$$

$$= \frac{8a^3}{3}\cdot\cos^3\theta\sin\theta\,d\theta\Big]_0^{\pi/2}$$

~~sin~~ $\cos\theta = t$

$\sin\theta\,d\theta = dt$

$$\frac{8a^3}{3}\,-t^3\,dt\Big]_0^{\pi/2}$$

$$\frac{8a^3}{3}\frac{t^4}{4}\Big]_0^{\pi/2}$$

$$\frac{\cos^4\theta}{4}$$

$$\frac{8a^3}{3(4)} + \frac{8a^3}{3(4)} = 0$$

$$= \frac{16a^3}{12} = \frac{4a^3}{3}$$

Evaluate double integral $r^3 \cdot dr \cdot d\theta$ over the area b/w two circles $r = 2\cos\theta$ and $r = 4\cos\theta$.

(Q) Triple integral:

$$\iiint f(x,y,z)\, dx \cdot dy \cdot dz \quad (\text{or}) \quad \iiint_V f(x,y,z) \cdot dV$$

$$\Rightarrow \boxed{\int_{z=a}^{b} \boxed{\int_{y=k(z)}^{\phi(z)} \boxed{\int_{x=f(y,z)}^{x=g(y,z)} f(x,y,z) \cdot dx} \, dy} \cdot dz}$$

(Q) Evaluate $\int_0^a \int_0^b \int_0^c (x^2+y^2+z^2) \cdot dz \cdot dy \cdot dx$

$$\Rightarrow x^2 z + y^2 z + z^3/3 \Big]_0^c$$

$$\Rightarrow x^2 c + y^2 c + c^3/3 \cdot dy \Big]_0^b$$

$$\Rightarrow x^2 yc + \frac{y^3 c}{3} + \frac{c^3 \cdot y}{3} \Big]_0^b$$

$$\Rightarrow x^2 \cdot bc + \frac{b^3 c}{3} + \frac{bc^3}{3} \cdot dx \Big]_0^a$$

$$\Rightarrow \frac{x^3}{3} \cdot bc + \frac{b^3 c \cdot x}{3} + \frac{bc^3 \cdot x}{3} \Big]_0^a$$

$$\Rightarrow \frac{a^3 \cdot bc}{3} + \frac{ab^3 c}{3} + \frac{abc^3}{3}$$

(Q) Evaluate $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dy \cdot dx \cdot dz$

$$\Rightarrow y \Big]_0^{\sqrt{4z-x^2}} \Rightarrow \sqrt{4z-x^2} \cdot dx \Big]_0^{2\sqrt{z}}$$

$$\Rightarrow \sqrt{(2\sqrt{z})^2 - x^2} \cdot dx \cdot dz \Big]_0^{2\sqrt{z}}$$

$$\Rightarrow \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) \Big]_0^{2\sqrt{z}}$$

$$\Rightarrow \cancel{\frac{2\sqrt{z}\sqrt{(2\sqrt{z})^2 - (2\sqrt{z})^2}}{2}} + \frac{(2\sqrt{z})^2}{2} \sin^{-1}\left(\frac{2\sqrt{z}}{2\sqrt{z}}\right) \Big]_0^4 \cdot dz$$

$$\Rightarrow \frac{\cancel{4}z \cdot \pi}{\cancel{2} \cdot 2} \Big]_0^4$$

$$\Rightarrow \frac{\pi z^2}{2} \Big]_0^4 \Rightarrow \frac{\pi\, \cancel{16}}{\cancel{2}} = \pi 8$$

(Q) Evaluate $\int_{-1}^{1}\int_{0}^{2}\int_{x-z}^{x+z}(x+y+z)\cdot dy\cdot dx\cdot dz$

$\Rightarrow xy+y^2/2+zy\Big]_{x-z}^{x+z}$

$\Rightarrow \left(x(x+z)+\frac{(x+z)^2}{2}+z(x+z)\right)-\left(x(x-z)+\frac{(x-z)^2}{2}+z(x-z)\right)$

$= \cancel{x^2}+xz+\cancel{\frac{x^2+z^2}{2}}+xz+xz+z^2-\cancel{x^2}+xz-\cancel{\frac{x^2-z^2}{2}}+\cancel{xz}-\cancel{xz}+z^2$

$= (4xz+2z^2)\cdot dx\Big]_0^2$

$\Rightarrow \frac{4x^2\cdot z}{2}+2z^2\cdot x\Big]_0^2$

$\Rightarrow \frac{\overset{2}{\cancel{4}}\times 4\cdot z}{\cancel{2}}+2z^2(2)$

$= 8z+4z^2\Big]_{-1}^{1}$

$= 8z^2/2+4z^3/3\Big]_{-1}^{1}$

$= \cancel{8/2}+4/3-\cancel{8/2}+4/3$

$= 8/3$

**NOTE:** Volume of the solid $V = \iiint dV = \iiint dx\cdot dy\cdot dz$

centre of gravity $(\bar{x},\bar{y})$ of a region

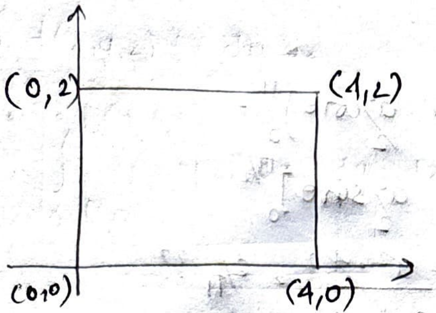
$$\bar{x} = \frac{\iint_R \rho x\cdot dy\cdot dx}{\iint_R \rho\, dy\cdot dx} \quad ; \quad \bar{y} = \frac{\iint_R \rho y\cdot dy\cdot dx}{\iint_R \rho\cdot dy\cdot dx}$$

where mass, $M = \iint_R \rho\cdot dy\cdot dx$ ; $\rho$ - density

**NOTE:** If density is not mentioned in the problem, we can take $\rho$ as a constant either 1 (or) K, you cannot leave density blank.

(Q) Find the coordinates of COG relating to ~~x axis~~ the region bounded by x axis, y axis and the rectangle $0 \le x \le 4$, $0 \le y \le 2$ having mass density $xy$.

(A)




$\rho = xy$.

Mass $M = \iint \rho \cdot dy \cdot dx$

$= \int_0^4 \int_0^2 xy \cdot dy \cdot dx$

$= \int_0^4 \left[ \frac{xy^2}{2} \right]_0^2$

$= x\cancel{4}/\cancel{2} \quad \frac{2x^2}{2}\Big]_0^4 = 16.$

$\bar{x} = \dfrac{\iint \rho \cdot x \cdot dy \cdot dx}{\iint \rho \cdot dy \cdot dx} = \int_0^4 \int_0^2 \cancel{xy \cdot x}\ x^2y \cdot dy \cdot dx$

$\frac{x^2y^2}{2}\Big]_0^2 = x^2 2\Big]_0^4$

$= \frac{2x^3}{3}\Big]_0^4 = \frac{2 \times 64}{3} = \frac{128}{3}$

$= \dfrac{128/3}{16} = \dfrac{128}{3 \times 16} = \dfrac{\cancel{128}}{\cancel{48}} = 8/3$

$\bar{y} = \dfrac{\iint \rho \cdot y \cdot dy \cdot dx}{16} = \int_0^4 \int_0^2 xy^2 \cdot dy \cdot dx$

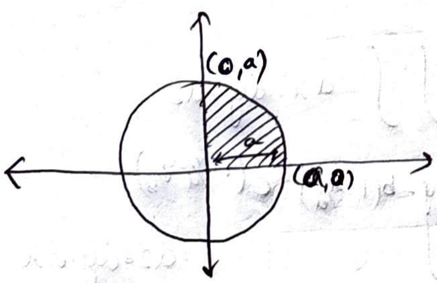
$xy^3/3\Big]_0^2 = \frac{x 8}{3}\Big]_0^4$

$= \frac{8x^2}{6}\Big]_0^4 = \frac{8 \times \cancel{16}}{6 \times \cancel{16}} = 8/6 = 4/3$

$(\bar{x}, \bar{y}) = (8/3, 4/3)$

(Q) Find the COG of the area of ~~semicircle~~ circle $x^2+y^2=a^2$ in the 1st quadrant.

(A)




for circle problems solve in polar form, it will be easier.

$M = \iint \rho \cdot dy \cdot dx = \iint \rho \cdot r dr \cdot d\theta$

$\bar{x} = \dfrac{\iint \rho \cdot r\cos\theta \cdot dr \cdot d\theta}{M}$

$\bar{y} = \dfrac{\iint \rho \cdot r\sin\theta \cdot dr d\theta}{M}$

$$M = \int_0^{\pi/2}\int_{r=0}^{a} 1\cdot r\cdot dr\cdot d\theta = \int_0^{\pi/2} \frac{r^2}{2}\Big]_0^a$$

$$= \frac{a^2}{2}\cdot\frac{\pi}{2} = \frac{\pi a^2}{4}$$

$$\bar{x} = \frac{\int_0^{\pi/2}\int_0^a 1\cdot r\cos\theta\cdot dr\cdot d\theta}{\pi a^2}\cdot 4 = \frac{r^2}{2}\cdot\cos\theta\Big]_0^a = \frac{a^2}{2}\cos\theta\Big]_0^{\pi/2}$$

$$= \frac{a^2}{2}\sin\theta\Big]_0^{\pi/2}$$

$$= \frac{a^2\cdot\cancel{4}\,2}{\cancel{2}\cdot\pi\cancel{a^2}} = 2/\pi \;??$$

$$\bar{y} = \frac{\int_0^{\pi/2}\int_0^a r\sin\theta\cdot dr\cdot d\theta}{\pi a^2}\cdot 4 = \frac{r^2}{2}\sin\theta\Big]_0^a = \frac{a^2}{2}\sin\theta\Big]_0^{\pi/2} = \frac{-\frac{a^2}{2}\cos\theta\Big]_0^{\pi/2}\cdot 4}{\pi a^2}$$

$$= \frac{\frac{a^2}{2}\cdot 4}{\pi a^2} = \frac{\cancel{a^2}\cdot\cancel{4}\,2}{\cancel{2}\times\pi\cancel{a^2}} = 2/\pi.$$

$$(\bar{x},\bar{y}) = (2/\pi, 2/\pi)$$

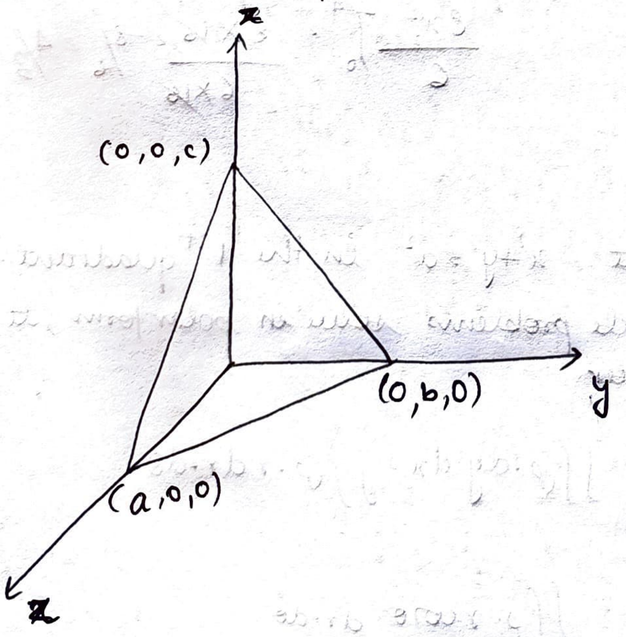
→ p 21

(Q) Find the COG of the tetrahedron bounded by the coordinate plane and the plane $x/a + y/b + z/c = 1$.

(*) → ANS: $abc/6$.

(Q) Find the volume of the tetrahedron $x \geq 0; y \geq 0; z \geq 0, x/a + y/b + z/c = 1$

(A)




$z = 0 \quad z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$

$y = 0 \quad y = b\left(1 - x/a\right)$

$x = 0 \quad x = a$

$$\iiint dx\cdot dy\cdot dz$$

$$= \int_{x=0}^{a}\int_{y=0}^{y=b(1-x/a)}\int_{z=0}^{c(1-x/a-y/b)} \cdot dz\cdot dy\cdot dx$$

$$= z\cdot dy\, dx\Big]_0^{c(1-x/a-y/b)}$$

$$= c\left(1 - x/a - y/b\right).$$

(Q) Find the volume of the Ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

(A) $z = 0$ to $c\sqrt{1 - x^2/a^2 - y^2/b^2}$

$y = 0$ to $b\sqrt{1 - x^2/a^2}$

$x = 0$ to $a$

$V = 8$ (First Octant)

$$V = \int\int\int dz \cdot dy \cdot dx$$

$$V = \int_0^a \int_{y=0}^{b\sqrt{1-x^2/a^2}} \int_{z=0}^{c\sqrt{1-x^2/a^2-y^2/b^2}} dz \cdot dy \cdot dx$$

ANS. = $4abc\pi/3$.

(Q) Using triple integral, find the volume of the sphere $x^2+y^2+z^2 = 4$.

(A) $z = 0$ to $\sqrt{4-x^2-y^2}$

$y = 0$ to $\sqrt{4-x^2}$

$x = 0$ to $2$.

$V = 8 \times$ (First Octant)

~~Ans~~: $\frac{4\pi a^3}{3}$.

total volume = $\frac{32\pi a^3}{3}$.

Change of order of integrals:

NOTE: change of order of integration is changing the limits of integration.

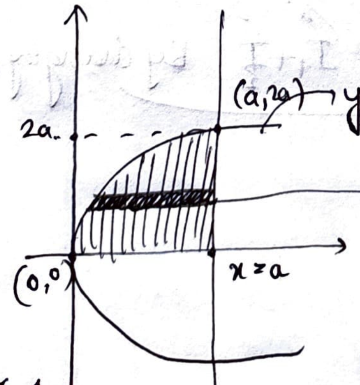
(Q) Change the order of integration and evaluate

$$I = \int_0^a \int_0^{2\sqrt{ax}} x^2 \cdot dy \cdot dx$$

Given $y = 0$ to $y = 2\sqrt{ax}$ = $y^2 = 4ax$

$x = 0$ to $x = a$

=




$y^2 = 4a^2$

$\boxed{y = 2a}$

change the limits of x and y

$x = y^2/4a$ to $x = a$

$y = 0$ to $y = 2a$

By change of order: $\int_0^{2a}\int_{y^2/4a}^{a} x^2 \cdot dx \cdot dy$

$$= \left[\frac{x^3}{3}\right]_{y^2/4a}^{a} = \cancel{\frac{y^6}{(4a)^3 3}}$$

$$\left[\frac{a^3}{3} - \frac{y^6}{(4a)^3 \cdot 3}\right]_0^{2a} = \left[\frac{a^3 y}{3} - \frac{y^7}{21(4a)^3}\right]_0^{2a}$$

$$= a^3(2a)/3 - (2a)^7/21(4a)^3$$

$$\frac{2a^4}{3} - \frac{2^7 a^4}{4^3 \cdot 21}$$

$4a^4/7$.

$$\frac{2a^4}{3} - \frac{2a^4}{21}$$

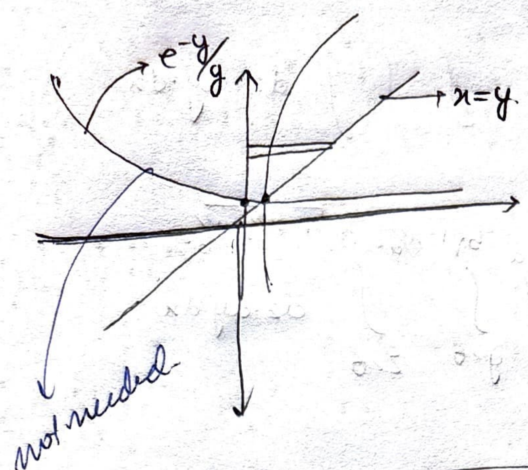
$$\frac{14a^4 - 2a^4}{21}$$

$$\frac{12a^4}{21} = \frac{4a^4}{7}$$

(Q) Evaluate by changing the order $\int_0^{\infty}\int_{y=x}^{\infty} \frac{e^{-y}}{y}\, dx\cdot dy$, given limits $x=0$ to $x=y$

$y=0$ to $y=\infty$

Ans: 1




changing the order.

$$\int_0^{\infty}\int_0^{x} e^{-y}/y \cdot dx\cdot dy$$

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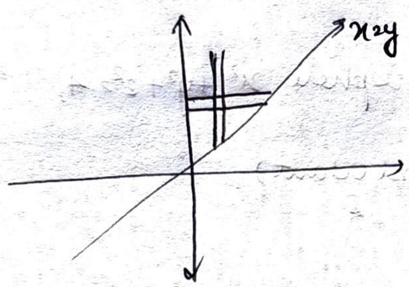



changing the order.

~~y=0~~ to ~~y=∞~~

$x=0$ to $x=y$

$y=0$ to $y=\infty$

$$\int_0^{\infty}\int_0^{y} e^{-y}/y \cdot dx\cdot dy$$

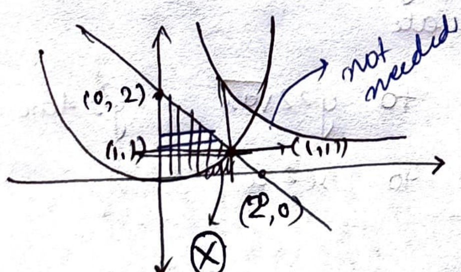
SEE exam

(Q) Evaluate $\int_0^1\int_{x^2}^{2-x} xy\cdot dy\cdot dx$

ANS = 3/8

given limits : $y=x^2$ to $y=2-x$

$x=0$ to $x=1$




split the region into 2 parts

$x-2=x$

$(x-2=x^2$

$\cancel{x^2-x+2=0}$

given integral $= I_1+I_2$ By changing order

$R_1 \Rightarrow x=0$ to $x=\sqrt{y}$.

$y=0$ to $y=1$

$R_2 \Rightarrow x=0$ to $x=2-y$

$y=1$ to $y=2$

$$\int_0^1\int_0^{\sqrt{y}} xy\cdot dy\cdot dx + \int_1^{2}\int_0^{2-y} xy\cdot dy\cdot dx$$

@ Change of variables:

To change the cartesian variable $x, y$ to $r, \theta$ in a double integral, we take

$x = r\cos\theta$; $y = r\sin\theta$

$dx \cdot dy = |J| \cdot dr \cdot d\theta = r \cdot dr \cdot d\theta$.

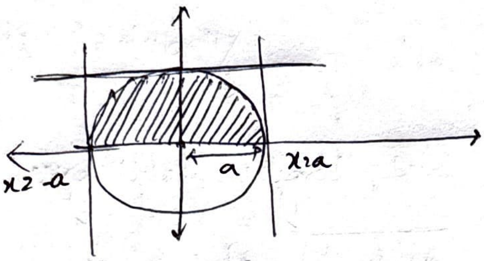
(Q) Transform the following integral into polar coordinates -

$$\int_{-a}^{a} \int_{0}^{y=\sqrt{a^2-x^2}} \sqrt{x^2+y^2} \cdot dy \cdot dx$$

given limits = $r = 0$ to $r = a$

$\theta = 0$ to $\theta = \pi$

(A)




$\Rightarrow x^2+y^2 = a^2$

$r^2 = a^2$

$r = a$

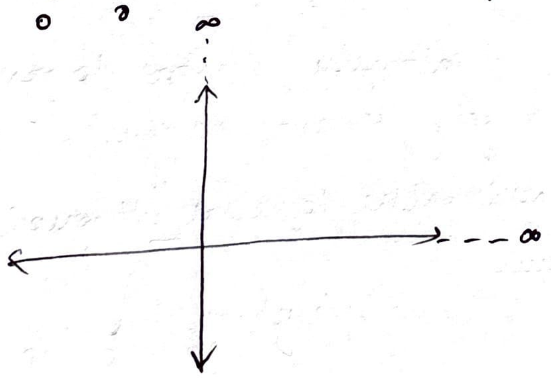
$$\int_{0}^{\pi} \int_{0}^{a} r^2 dr \cdot d\theta$$

Homework

(Q) ① $\int_{0}^{a} \int_{0}^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2} \cdot dx \cdot dy$.

② $\int_{0}^{\infty} \int_{0}^{\infty} e^{-(x^2+y^2)} \cdot dx \cdot dy$.

②.




$x^2+y^2 = a^2$

$r^2 = a^2$

$\boxed{r = a}$

$$= \int_{0}^{\infty} \int_{0}^{\infty} r \cdot r \cdot dr \cdot d\theta$$

$$= \int_{0}^{\infty} \int_{0}^{\infty} r^2 \cdot dr \cdot d\theta$$