Ordinal Dependent Variables

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University of Essex

1. Ordinal Data

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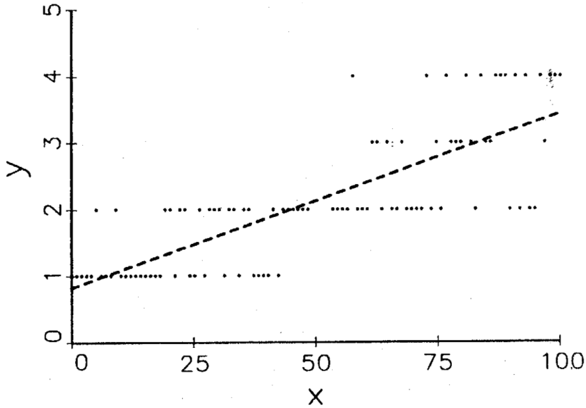
Some people would argue that *all* ordinal data have underlying continuous variables and were measured imperfectly.

Use ordinal logit, ordinal probit, or other ordinal models for these data!

What Happens if we Apply a Linear Model?



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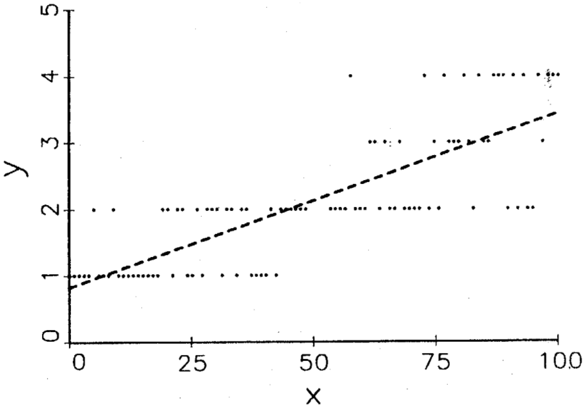

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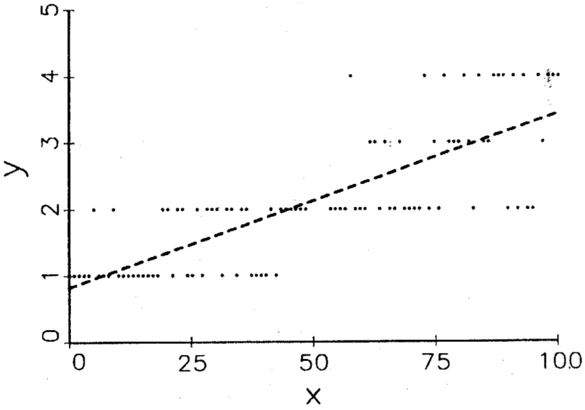

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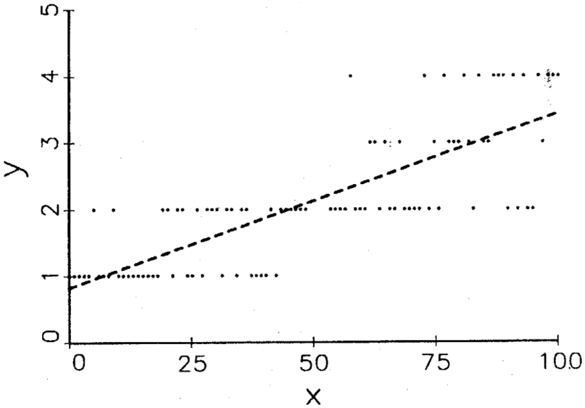
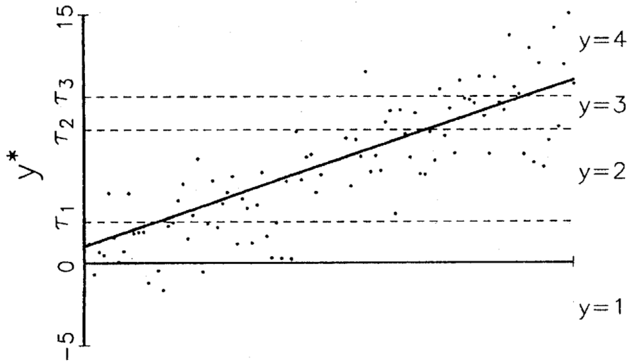


- Heteroskedasticity.
- Predictions outside of the range of the DV.
- Bias because we do not know where the cut points were drawn to discretise the continuous values.

Underlying Continuous Variable: Why the LM is Biased



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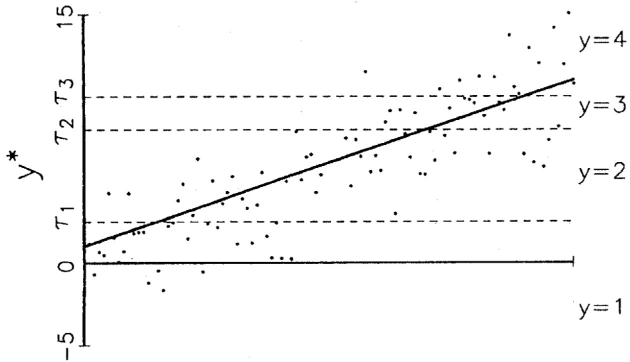

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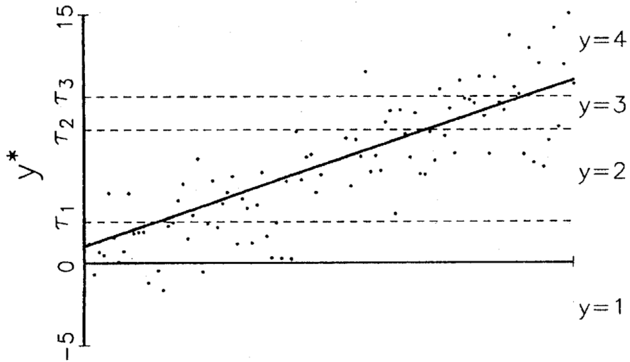

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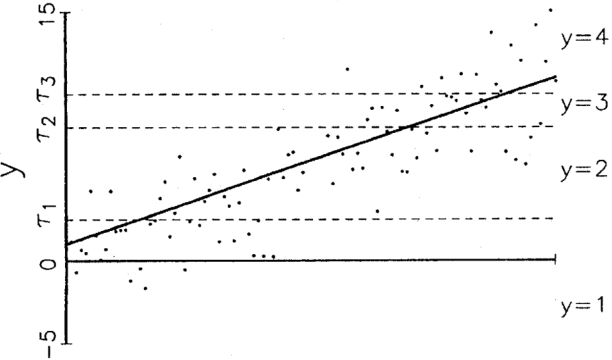

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- The latent underlying continuous variable is unobserved.

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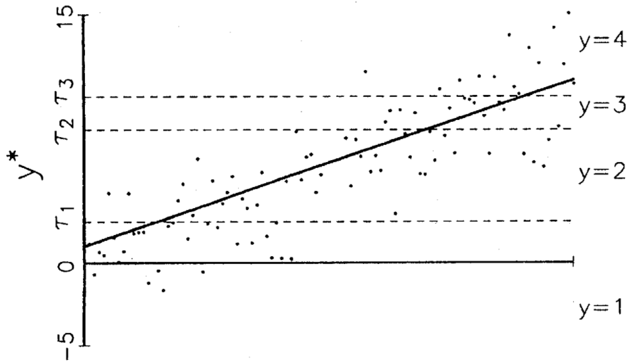
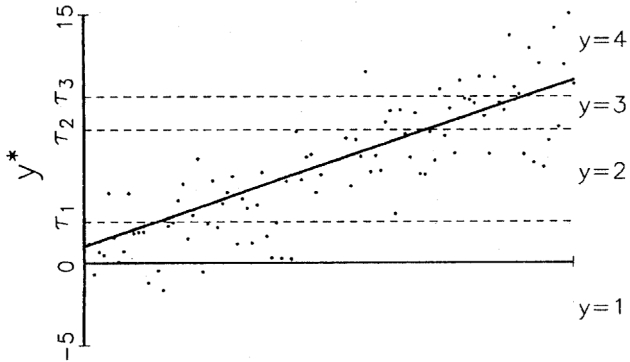


- The observed discrete values correspond to the bands between the cutpoints τ_1, τ_2, and τ_3.
- Four levels: below τ_1; between τ_1 and τ_2; between τ_2 and τ_3; above τ_3.
- The latent underlying continuous variable is unobserved.
- The cut points may not be equidistant.

Vague Intuition: How Ordinal Regression Works



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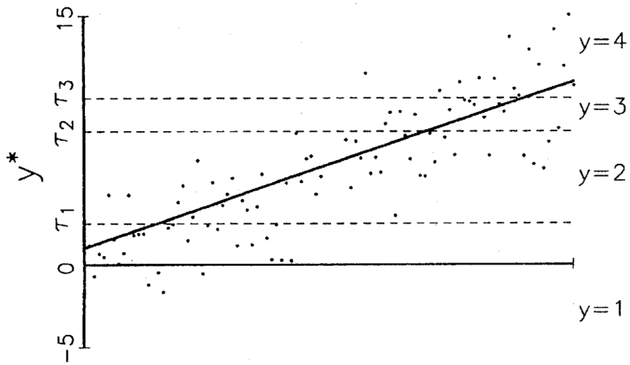

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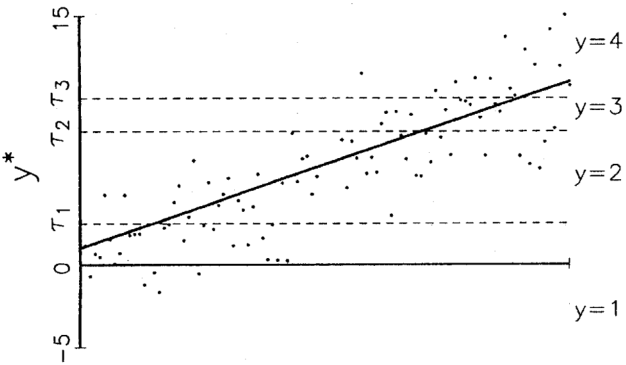

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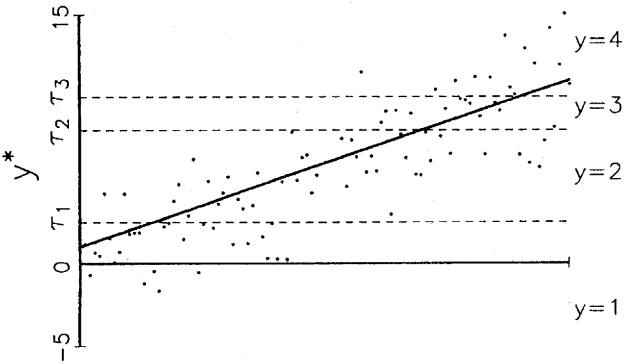

- Logit or probit models for each cut point: are we below or above the cut point?
- The cut points are estimated from the data along with the β coefficients.
- The cut points serve as intercepts for the different levels of the latent variable.

Exercise

Should the Polity IV score of a country be modelled using an ordinal regression model or a linear model?

From the Polity IV website:

> *The "Polity Score" captures [the] regime authority spectrum on a 21-point scale ranging from* -10 *(hereditary monarchy) to* $+10$ *(consolidated democracy). The Polity scores can also be converted into regime categories in a suggested three part categorization of "autocracies"* (-10 *to* -6)*, "anocracies"* (-5 *to* $+5$ *and three special values:* -66, -77 *and* -88)*, and "democracies"* ($+6$ *to* $+10$)*.*

List arguments for and against each choice.

A New Data Type: Ordinal Factors

To model ordinal data, we need a new data type: ordinal factors.

```
v <- c("Negative", "Negative", "Neutral", "Positive", "Neutral")
v
## [1] "Negative" "Negative" "Neutral"  "Positive" "Neutral"

vf <- ordered(v, levels = c("Negative", "Neutral", "Positive"))
vf
## [1] Negative Negative Neutral  Positive Neutral
## Levels: Negative < Neutral < Positive

class(vf)
## [1] "ordered" "factor"

as.numeric(vf) # saved in the right order...
## [1] 1 1 2 3 2
```

The `levels` argument tells `R` the right order of the factors.

World Values Survey Data

```
library("carData")
data(WVS)
head(WVS)
##       poverty religion degree country age gender
## 1  Too Little      yes     no     USA  44   male
## 2 About Right      yes     no     USA  40 female
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- Poverty: "Do you think that what the government is doing for people in poverty in this country is about the right amount, too much, or too little?"
- Ordered factor with three levels.

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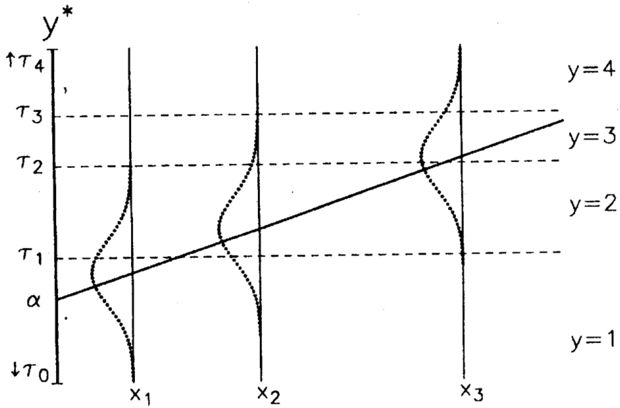
- Ordered factor with three levels.
- Country: Australia; Norway; Sweden; USA.
- 5,381 observations.

2. Ordinal Logit and Probit

Distribution of y^* Given x for Ordered Regression



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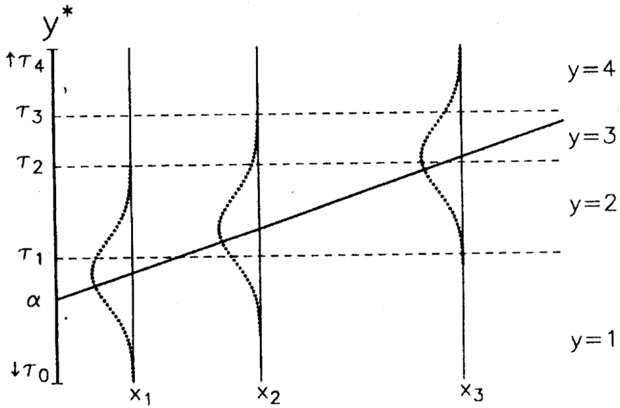

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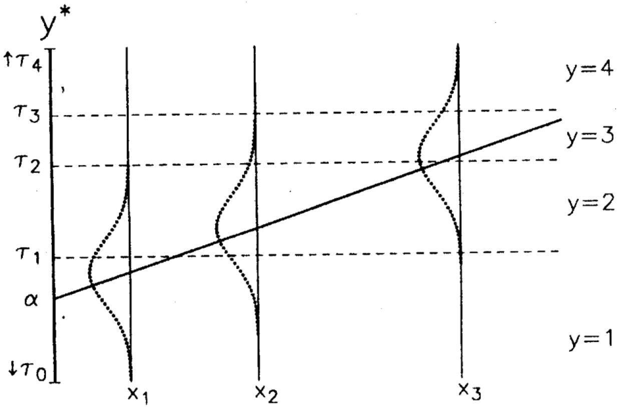

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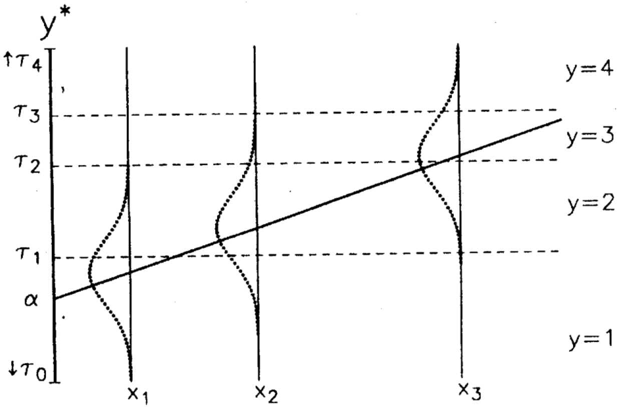

- We imagine a regression line through the underlying latent variable.
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- For each x value, we can derive the probability to be in each level from the cumulative distribution function.

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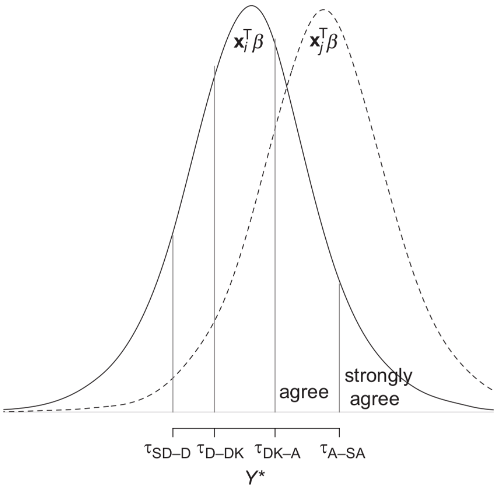

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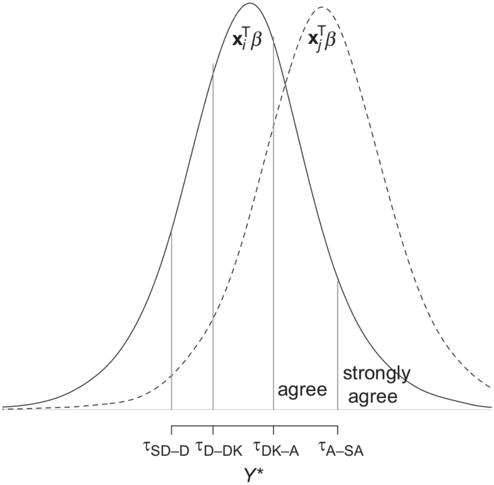

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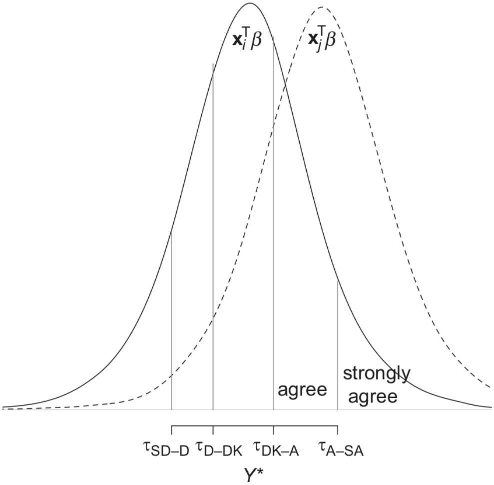

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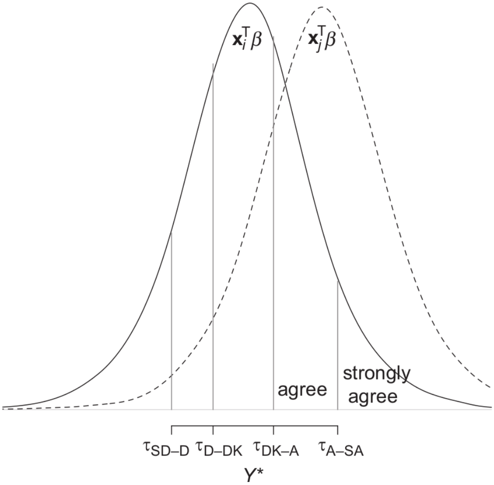

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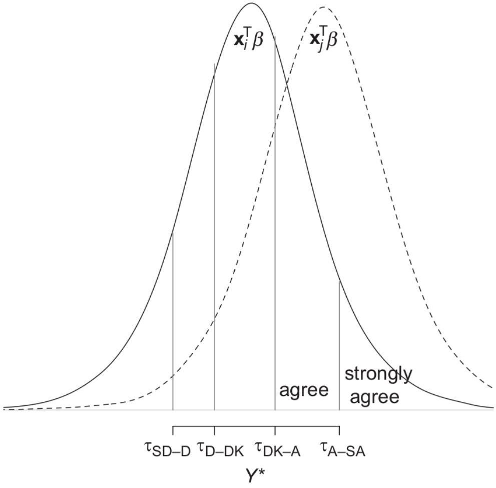

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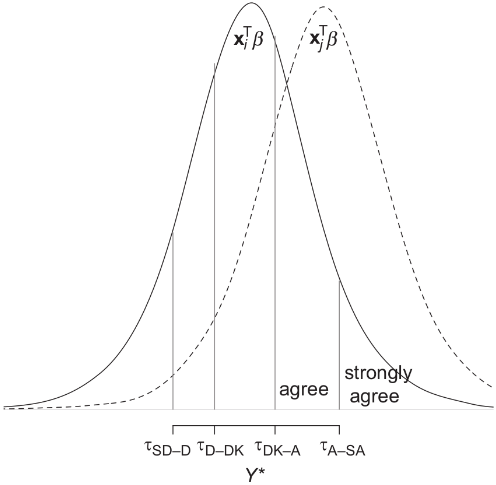

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We use the *cdf* to compute these probabilities!

Example: Calculating Interval Probabilities

Let's say the cut points are $\tau_1 = -3$; $\tau_2 = 1.4$; $\tau_3 = 6.3$.

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Then the probability that the value belongs to the first interval is:

$$P(r \leq \tau_1) = \int_{-\infty}^{\tau_1} f(r)dr = F(\tau_1)$$

where $f(\cdot)$ is the *pdf* and $F(\cdot)$ the *cdf* of the logistic or standard normal distribution.

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We can compute this in `R`:

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That's a very low probability.

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## [1] 0.3631683
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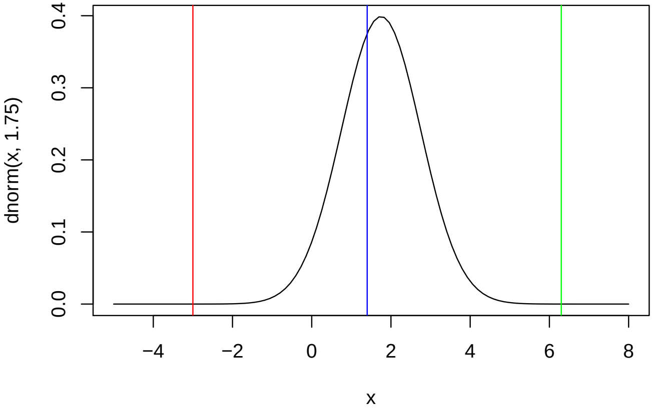
Third and fourth interval:

```
pnorm(6.3, mean = 1.75) - pnorm(1.4, mean = 1.75)
## [1] 0.636828

1 - pnorm(6.3, mean = 1.75)
## [1] 2.682296e-06
```

Visual Representation of these Results

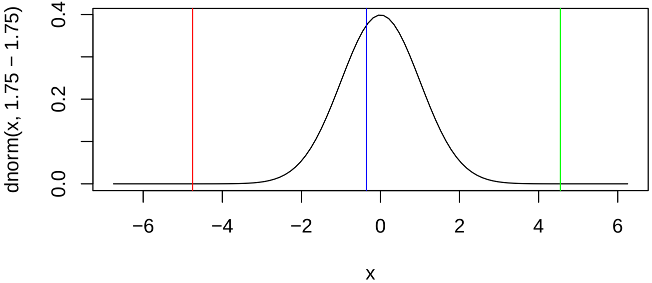
```
x <- seq(-5, 8, length.out = 100)
plot(x, dnorm(x, 1.75), type = "l")
abline(v = -3, col = "red")
abline(v = 1.4, col = "blue")
abline(v = 6.3, col = "green")
```



Since the Φ and Λ distributions are centred around zero, we can obtain the same result by subtracting the predicted value each time and thus shifting the distribution:

$$P(y_i = 1|\mathbf{x}_i) = P(\tau_0 \le \mathbf{x}_i^\top \boldsymbol{\beta} + \epsilon_i < \tau_1|\mathbf{x}_i)$$
$$\Leftrightarrow P(y_i = 1|\mathbf{x}_i) = P(\tau_0 - \mathbf{x}_i^\top \boldsymbol{\beta} \le \epsilon_i < \tau_1 - \mathbf{x}_i^\top \boldsymbol{\beta}|\mathbf{x}_i)$$

```
x <- seq(-5 - 1.75, 8 - 1.75, length.out = 100)
plot(x, dnorm(x, 1.75 - 1.75), type = "l")
abline(v = -3 - 1.75, col = "red")
abline(v = 1.4 - 1.75, col = "blue")
abline(v = 6.3 - 1.75, col = "green")
```



Ordered Probit: Individual Probabilities

Probability of observation i to belong into interval m:

$$P(y_i = m) = \begin{cases} \Phi(\tau_1 - \mathbf{x}_i^\top \boldsymbol{\beta}) & \text{for } m: \text{first level} \\ \Phi(\tau_2 - \mathbf{x}_i^\top \boldsymbol{\beta}) - \Phi(\tau_1 - \mathbf{x}_i^\top \boldsymbol{\beta}) & \text{for } m: \text{second level} \\ \dots & \text{for } m: \text{third etc level} \\ 1 - \Phi(\tau_M - \mathbf{x}_i^\top \boldsymbol{\beta}) & \text{for } m: \text{last level} \end{cases}$$

where Φ denotes the cumulative distribution function of the standard normal distribution:

$$\Phi(\epsilon) = \int_{-\infty}^{\epsilon} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{r^2}{2}\right\} dr$$

Ordered Logit: Individual Probabilities

Probability of observation i to belong into interval m:

$$P(y_i = m) = \begin{cases} \Lambda(\tau_1 - \mathbf{x}_i^\top \boldsymbol{\beta}) & \text{for } m : \text{first level} \\ \Lambda(\tau_2 - \mathbf{x}_i^\top \boldsymbol{\beta}) - \Lambda(\tau_1 - \mathbf{x}_i^\top \boldsymbol{\beta}) & \text{for } m : \text{second level} \\ \dots & \text{for } m : \text{third etc level} \\ 1 - \Lambda(\tau_M - \mathbf{x}_i^\top \boldsymbol{\beta}) & \text{for } m : \text{last level} \end{cases}$$

where Λ denotes the cumulative distribution function of the logistic distribution:

$$\Lambda(\epsilon) = \frac{\exp\{\epsilon\}}{1 + \exp\{\epsilon\}}$$

Likelihood Function

Likelihood Function

So the individual probabilities per observation are:

$$P(y_i = m | \mathbf{x}_i, \boldsymbol{\beta}, \boldsymbol{\tau}) = F(\tau_m - \mathbf{x}_i^\top \boldsymbol{\beta}) - F(\tau_{m-1} - \mathbf{x}_i^\top \boldsymbol{\beta})$$

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To construct the likelihood function, we compute the joint probability for all observations $i \dots n$ and all intervals $m \dots M$:

$$\mathcal{L}(\boldsymbol{\beta}, \boldsymbol{\tau}|\mathbf{y}, \mathbf{X}) = \prod_{m=1}^{M} \prod_{y_i=m} \left[F(\tau_m - \mathbf{x}_i^\top \boldsymbol{\beta}) - F(\tau_{m-1} - \mathbf{x}_i^\top \boldsymbol{\beta})\right]$$

Likelihood Function

So the individual probabilities per observation are:

$$P(y_i = m|\mathbf{x}_i, \boldsymbol{\beta}, \boldsymbol{\tau}) = F(\tau_m - \mathbf{x}_i^\top \boldsymbol{\beta}) - F(\tau_{m-1} - \mathbf{x}_i^\top \boldsymbol{\beta})$$

To construct the likelihood function, we compute the joint probability for all observations $i \dots n$ and all intervals $m \dots M$:

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Corresponding log likelihood:

$$\log \mathcal{L}(\boldsymbol{\beta}, \boldsymbol{\tau}|\mathbf{y}, \mathbf{X}) = \sum_{m=1}^{M} \sum_{y_i=m} \log \left[F(\tau_m - \mathbf{x}_i^\top \boldsymbol{\beta}) - F(\tau_{m-1} - \mathbf{x}_i^\top \boldsymbol{\beta})\right]$$

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- Alternatively, we could drop the first τ estimate and keep the intercept. It does not make a difference for the β estimates.
- Estimation as usual computationally via IWLS, BFGS etc, for example via the `optim` function in `R`.
- There are several ready-to-use implementations in `R`, for example the `polr` function in the `MASS` package.

3. Ordered Logit and Probit in `R`

World Values Survey: Poverty

```
library("MASS")
model1 <- polr(poverty ~ religion + degree + country +
    age + gender, data = WVS, method = "logistic",
    Hess = TRUE)
model2 <- polr(poverty ~ religion + degree + country +
    age + gender, data = WVS, method = "probit",
    Hess = TRUE)

class(model1)
## [1] "polr"

WVS$poverty_ols <- as.numeric(WVS$poverty)
head(WVS$poverty_ols)
## [1] 1 2 1 3 1 2

model3 <- glm(poverty_ols ~ religion + degree +
    country + age + gender, data = WVS)
```

Model Output

```
summary(model1)
## Call:
## polr(formula = poverty ~ religion + degree + country + age +
##     gender, data = WVS, Hess = TRUE, method = "logistic")
##
## Coefficients:
##                  Value Std. Error t value
## religionyes    0.17973   0.077346   2.324
## degreeyes      0.14092   0.066193   2.129
## countryNorway -0.32235   0.073766  -4.370
## countrySweden -0.60330   0.079494  -7.589
## countryUSA     0.61777   0.070665   8.742
## age            0.01114   0.001561   7.139
## gendermale     0.17637   0.052972   3.329
##
## Intercepts:
##                         Value   Std. Error t value
## Too Little|About Right  0.7298  0.1041      7.0128
## About Right|Too Much    2.5325  0.1103     22.9496
##
## Residual Deviance: 10402.59
## AIC: 10420.59
```

Comparison of the Models

	Ordered Logit	Ordered Probit	Linear Model
religionyes	$0.18\ (0.08)^{*}$	$0.11\ (0.05)^{*}$	$0.08\ (0.03)^{**}$
degreeyes	$0.14\ (0.07)^{*}$	$0.08\ (0.04)^{*}$	$0.05\ (0.02)$
countryNorway	$-0.32\ (0.07)^{***}$	$-0.25\ (0.05)^{***}$	$-0.16\ (0.03)^{***}$
countrySweden	$-0.60\ (0.08)^{***}$	$-0.41\ (0.05)^{***}$	$-0.25\ (0.03)^{***}$
countryUSA	$0.62\ (0.07)^{***}$	$0.37\ (0.04)^{***}$	$0.25\ (0.03)^{***}$
age	$0.01\ (0.00)^{***}$	$0.01\ (0.00)^{***}$	$0.00\ (0.00)^{***}$
gendermale	$0.18\ (0.05)^{***}$	$0.10\ (0.03)^{**}$	$0.06\ (0.02)^{**}$
(Intercept)			$1.37\ (0.04)^{***}$
AIC	10420.59	10370.25	11444.39
BIC	10479.91	10429.57	11503.71
Log Likelihood	-5201.30	-5176.13	-5713.20
Deviance	10402.59	10352.25	2633.90
Num. obs.	5381	5381	5381

$^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$

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countryUSA	0.62 (0.07)***	0.37 (0.04)***	0.25 (0.03)***
age	0.01 (0.00)***	0.01 (0.00)***	0.00 (0.00)***
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The linear model is biased, fits worse, and yields different results.

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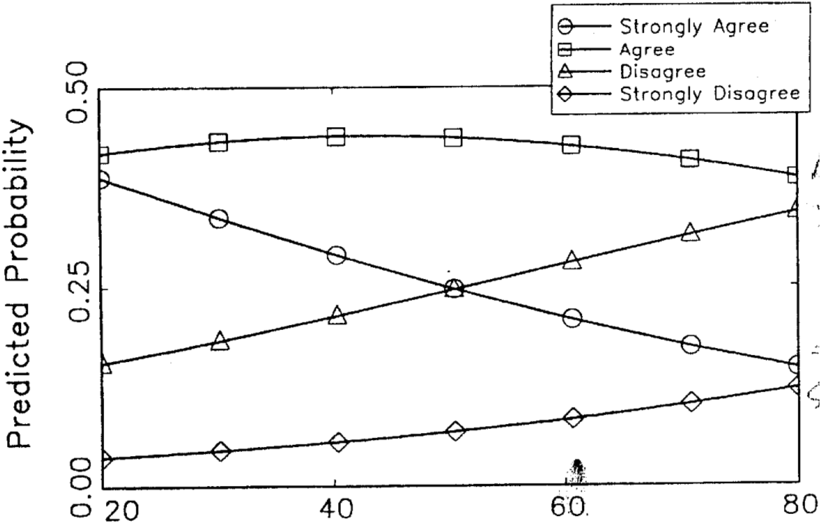
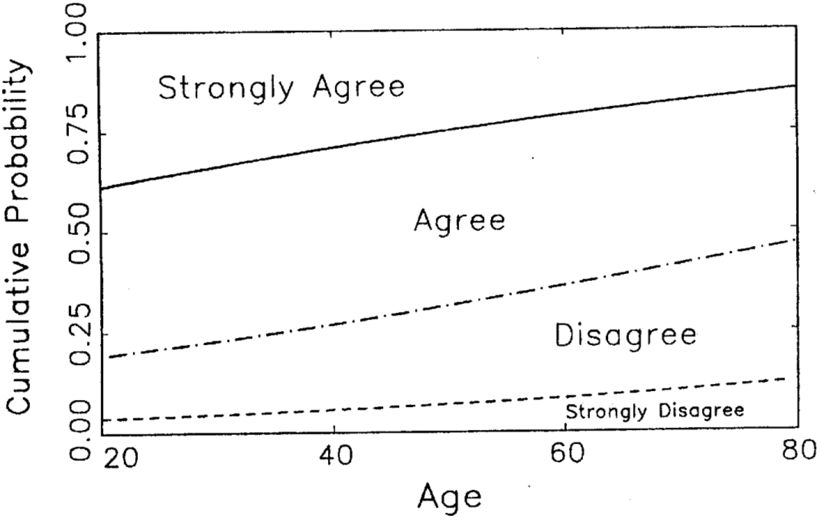
- As this is conditional on the state of all variables, it makes more sense to interpret models by employing prediction and based on scenarios – just like with other GLMs.
- You can also look at marginal effects and other quantities, as detailed in Long (1997), but this is less common in applied work in polisci.

Interpretation: Predicted Probabilities



Age on the x axis. Probabilities can go up and down again.
Otherwise same procedure as with logit and probit models.

Interpretation: Cumulative Probability



Same as before, but stacked up to display cumulative probabilities.

4. The Parallel Regression Assumption

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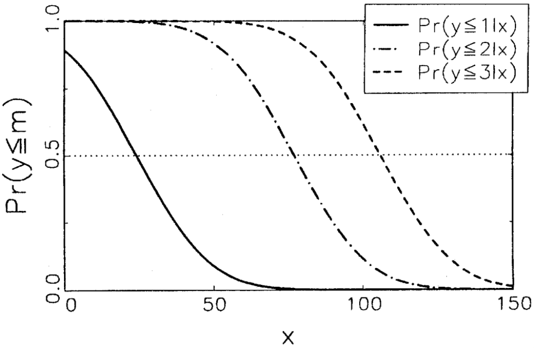

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- The Brant test: a Wald-type test statistic that tests for the whole model and for each model term separately if the assumption holds.
- The assumption is frequently violated in empirical applications. If it is:
  - Use separate binary models to fit the different DGPs.
  - Use a multinomial logit or probit model, which do not assume an ordinal measurement level.

Separate Logit Models

```
WVS_a <- WVS
WVS_a$poverty_ols[WVS_a$poverty_ols == 1] <- 0
WVS_a$poverty_ols[WVS_a$poverty_ols %in% 2:3] <- 1

WVS_b <- WVS
WVS_b$poverty_ols[WVS_b$poverty_ols %in% 1:2] <- 0
WVS_b$poverty_ols[WVS_b$poverty_ols == 3] <- 1

a <- glm(poverty_ols ~ religion + degree + country +
    age + gender, data = WVS_a,
    family = binomial(link = "logit"))

b <- glm(poverty_ols ~ religion + degree + country +
    age + gender, data = WVS_b,
    family = binomial(link = "logit"))
```

Separate Logit Models

```
screenreg(list(a, b), single.row = TRUE)
##
## =========================================================
##                    Model 1                 Model 2
## ---------------------------------------------------------
## (Intercept)           -0.70 (0.11) ***     -2.59 (0.16) ***
## religionyes            0.11 (0.08)          0.37 (0.11) ***
## degreeyes              0.18 (0.07) *        0.02 (0.11)
## countryNorway         -0.13 (0.08)         -1.78 (0.18) ***
## countrySweden         -0.44 (0.08) ***     -2.07 (0.21) ***
## countryUSA             0.36 (0.07) ***      0.90 (0.09) ***
## age                    0.01 (0.00) ***      0.01 (0.00) ***
## gendermale             0.20 (0.06) ***      0.09 (0.08)
## ---------------------------------------------------------
## AIC                 7315.16              3911.40
## BIC                 7367.89              3964.12
## Log Likelihood     -3649.58             -1947.70
## Deviance            7299.16              3895.40
## Num. obs.           5381                 5381
## =========================================================
## *** p < 0.001, ** p < 0.01, * p < 0.05
```

```r
library("brant")
brant(model1)
## --------------------------------------------
## Test for X2 df probability
## --------------------------------------------
## Omnibus 261.22 7 0
## religionyes 6.07 1 0.01
## degreeyes 2.08 1 0.15
## countryNorway 83.64 1 0
## countrySweden 59.58 1 0
## countryUSA 43.85 1 0
## age 0.7 1 0.4
## gendermale 1.69 1 0.19
## --------------------------------------------
##
## H0: Parallel Regression Assumption holds
##                       X2 df  probability
## Omnibus       261.2204841  7 1.130805e-52
## religionyes     6.0674567  1 1.376951e-02
## degreeyes       2.0848479  1 1.487670e-01
## countryNorway  83.6373448  1 5.943954e-20
## countrySweden  59.5786391  1 1.175037e-14
## countryUSA     43.8454840  1 3.553501e-11
## age             0.6951767  1 4.044091e-01
## gendermale      1.6894831  1 1.936691e-01
```

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  4. Use a cumulative link model (CLM)...

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- This means selected slopes are allowed to vary across levels.
- This solves the problem with the parallel regression assumption.

Cumulative Link Models (CLM) in R

Let's first re-estimate Model 1, the ordered logit model:

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```
library("ordinal")
model4 <- clm(poverty ~ religion + degree + country +
    age + gender, data = WVS, link = "logit")
```

(Output not shown here because identical to Model 1).

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`country` was one of the offending variables. We can now re-specify `country` as a variable that has a nominal effect:

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```

(Output not shown here because identical to Model 1).

`country` was one of the offending variables. We can now re-specify `country` as a variable that has a nominal effect:

```
model5 <- clm(poverty ~ religion + degree + age +
    gender, nominal = ~ country, data = WVS,
    link = "logit")
summary(model5) # results on next slide
```

```
## formula: poverty ~ religion + degree + age + gender
## nominal: ~country
## data:    WVS
##
##  link  threshold nobs logLik   AIC      niter max.grad cond.H
##  logit flexible  5381 -5020.12 10064.25 7(0)  9.47e-13 1.7e+05
##
## Coefficients:
##             Estimate Std. Error z value Pr(>|z|)
## religionyes 0.149106   0.076176   1.957  0.05030 .
## degreeyes   0.141428   0.066552   2.125  0.03358 *
## age         0.010605   0.001556   6.816 9.39e-12 ***
## gendermale  0.173844   0.052915   3.285  0.00102 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Threshold coefficients:
##                                      Estimate Std. Error z value
## Too Little|About Right.(Intercept)    0.71784    0.10396   6.905
## About Right|Too Much.(Intercept)      2.36086    0.11461  20.599
## Too Little|About Right.countryNorway  0.12271    0.07779   1.577
## About Right|Too Much.countryNorway    1.78119    0.18170   9.803
## Too Little|About Right.countrySweden  0.44490    0.08240   5.399
## About Right|Too Much.countrySweden    2.06867    0.21338   9.695
## Too Little|About Right.countryUSA    -0.36255    0.07340  -4.939
## About Right|Too Much.countryUSA      -0.87275    0.08666 -10.071
```

Assessing the Parallel Regression Assumption with CLMs

We can now do a likelihood ratio test to see if the parallel regression assumption was actually violated:

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```
anova(model4, model5)
## Likelihood ratio tests of cumulative link models:
##
##        formula:
## model4 poverty ~ religion + degree + country + age + gender
## model5 poverty ~ religion + degree + age + gender
##        nominal: link: threshold:
## model4 ~1       logit flexible
## model5 ~country logit flexible
##
##        no.par   AIC  logLik LR.stat df Pr(>Chisq)
## model4      9 10421 -5201.3
## model5     12 10064 -5020.1  362.35  3  < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Assessing the Parallel Regression Assumption with CLMs

We can now do a likelihood ratio test to see if the parallel regression assumption was actually violated:

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anova(model4, model5)
## Likelihood ratio tests of cumulative link models:
##
##        formula:
## model4 poverty ~ religion + degree + country + age + gender
## model5 poverty ~ religion + degree + age + gender
##        nominal: link: threshold:
## model4 ~1       logit flexible
## model5 ~country logit flexible
##
##        no.par   AIC  logLik LR.stat df Pr(>Chisq)
## model4      9 10421 -5201.3
## model5     12 10064 -5020.1  362.35  3  < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
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This is the best way to test the assumption.

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And it offers a flexible remedy.