The Linear Regression Model – Model Specification, Interpretation, and Large-Sample Properties

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GV903: Advanced Research Methods, Week 6



1. Prediction and Uncertainty

A Toy Dataset on Weight Loss and Exercise

Data from https://stats.idre.ucla.edu/r/seminars/interactions-r/

```
u <- paste0("https://stats.idre.ucla.edu/wp-content/",
          "uploads/2019/03/exercise.csv")
dat <- read.csv(u)
head(dat)
    id loss hours effort gender prog
##
## 1 1 18.02226 1.836704 37.71218
## 2 2 10.18642 2.389360 26.72401 1
## 3 3 19.74728 2.362117 36.31657 1
## 4 4 1.88360 2.520866 20.70048 1
## 5 5 14.24259 1.889828 24.72712 1
## 6 6 19.69473 2.367162 33.66948
```

900 observations on weight loss, training hours, training effort, gender, and training programme.

A Linear Model of Weight Loss

```
model <- lm(loss ~ hours + effort, data = dat)
library("texreg")
screenreg(model, single.row = TRUE)
##
             Model 1
##
## (Intercept) -15.60 (3.20) ***
## hours 2.35 (0.92) *
## effort 0.71 (0.09) ***
## R^2 0.07
## Adj. R^2 0.07
## Num. obs. 900
## ============
## *** p < 0.001; ** p < 0.01; * p < 0.05
```

Method 1: plug the desired hypothetical values directly into the estimated regression equation:

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Method 2:

- 1. Subtract desired hypothetical values from observed values.
- 2. Re-estimate linear model.

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 (X_1 - c_1) + \ldots + \hat{\beta}_k (X_k - c_k)$$

3. Extract intercept coefficient $(\hat{\beta}_0)$ as the predicted value.

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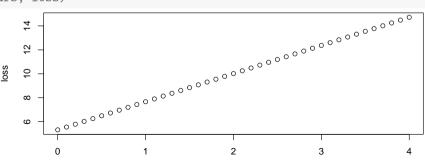
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Method 1 involves less computation, but Method 2 generates a standard error $se(\hat{\beta}_0)$ for constructing confidence intervals.

Simulating Predicted Values: Method 1 in R

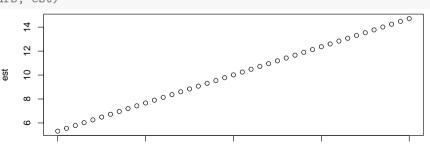
```
hours <- seq(0, 4, 0.1)
loss <- numeric(length(hours))
coefs <- coef(model)
for (i in 1:length(hours)) {
   loss[i] <- coefs[1] + coefs[2] * hours[i] + coefs[3] * mean(dat$effort)
}
plot(hours, loss)</pre>
```



hours

Simulating Predicted Values: Method 2 in R

```
est <- numeric(length(hours))
for (i in 1:length(hours)) {
  dat2 <- dat
  dat2$hours <- dat2$hours - hours[i]
  dat2$effort <- dat2$effort - mean(dat2$effort)
  model2 <- lm(loss ~ hours + effort, data = dat2)
  est[i] <- summary(model2)$coefficients[1, 1]
}
plot(hours, est)</pre>
```



Confidence Intervals for Simulated Values

$$t_{crit} = F^{-1}(1 - \alpha_{\frac{1}{2}}; df = n - k - 1)$$

$$CI(\hat{y}_{|c_1,...,c_k}) = \hat{\beta}_0 \pm t_{crit} \cdot \operatorname{se}(\hat{\beta}_0)$$

Intuition: 95 out of 100 observed values will be in this interval, given the hypothetical values.

Use this if you are interested in mean predictions/expectations.

Prediction Intervals for Simulated Values

$$PI(\hat{y}_{|c_1,...,c_k}) = \hat{\beta}_0 \pm t_{crit} \cdot \sqrt{\text{se}(\hat{\beta}_0)^2 + \sigma^2}$$
$$= \hat{\beta}_0 \pm t_{crit} \cdot \sqrt{\text{se}(\hat{\beta}_0)^2 + \frac{\sum_{i=1}^{n} (\hat{y}_i - y_i)^2}{n - k - 1}}$$

 σ^2 is the squared residual standard error (reported in 1m output).

Intuition: for a specific observation, the probability is 95 % that this interval includes the population value.

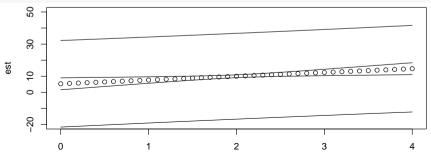
Use this if you need to factor in sample variability because you are interested in a specific observation from the sample.

Adding Intervals in R – Manually

```
est <- numeric(length(hours))</pre>
se <- numeric(length(hours))</pre>
for (i in 1:length(hours)) {
  dat2 <- dat
  dat2$hours <- dat2$hours - hours[i]</pre>
  dat2$effort <- dat2$effort - mean(dat2$effort)</pre>
  model2 \leftarrow lm(loss \sim hours + effort, data = dat2)
  est[i] <- summary(model2)$coefficients[1, 1]</pre>
  se[i] <- summary(model2)$coefficients[1, 2]</pre>
```

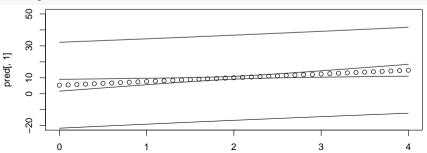
Now adding intercept standard errors ($se(\hat{\beta}_0)$) to the for loop. . .

Adding Intervals in R – Manually



Adding Intervals in R - predict Function

```
newdat <- data.frame(hours = hours, effort = mean(dat$effort))
pred <- predict(model, newdata = newdat, interval = "confidence")
plot(hours, pred[, 1], ylim = c(-20, 50))
lines(hours, pred[, 2])
lines(hours, pred[, 3])
pred2 <- predict(model, newdata = newdat, interval = "prediction")
lines(hours, pred2[, 2])
lines(hours, pred2[, 3])</pre>
```



Other Methods for Generating Predictions with Uncertainty

Method 3: The Delta Method.

- Analytical procedure involving calculus. Taylor series approximation.
- ► This is what the *predict* function does.

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Method 4: Simulation.

- Use $\hat{\sigma}$ to draw points from a multivariate normal distribution and compute CIs based on all the simulated predicted values.
- ► See King, Tomz and Wittenberg (2000), *AJPS*.

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Method 5: Bootstrapping.

- ► Resample from the original data and re-estimate the model, then predict and compute uncertainty over the different predictions.
- Will be discussed in the second semester.

Exercise

You regress election turnout (in per cent) on rain (in millimetres on election day) and expected winning margin (percentage of winning parties minus second-ranked party) according to polls. Your estimates are $\beta_0 = 82.78$, $\beta_1 = -0.34$, and $\beta_3 = -5.03$.

- 1. Interpret these results substantively.
- 2. Discuss ways to improve this model choice and specification.
- 3. You want to predict turnout at the UK election in December 2019. Based on historical data, you expect 0.4 mm of rain, and the Tories lead with a margin of 9 percentage points shortly before the election. Predict the election turnout using two different methods.
- 4. Which kind of interval would you choose for this prediction? What additional details do you need to compute for this?
- 5. Discuss the usefulness of point- and interval prediction in the social sciences. What usage scenarios can you imagine?

Solution

- 1. The baseline turnout is 82.78 per cent when there is no rain and both leading parties are head to head. With each mm of rain, the turnout goes down by 0.34 per cent. With each percentage point difference between the two leading parties, turnout drops by about 5 per cent.
- 2. A difference of > 20 per cent would predict negative turnout; a beta regression model might be a better choice. The effects of rain and margin might be non-linear; check visually if a quadratic term would account for levelling off. Also always check for any omitted variables.
- 3. $\hat{y} = 82.78 0.34(0.4) 5.03(9) = 37.374$ and extract \hat{y} or estimate $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1(X_1 0.4) + \hat{\beta}_k(X_k 9)$ and extract $\hat{\beta}_0$.
- 4. Prediction interval because point prediction. Need $se(\hat{\beta}_0)$ from vcov and squared residual standard error.
- 5. Point prediction has a large uncertainty and is often not sufficiently accurate for policy advice. But out-of-sample prediction is useful to assess model fit.

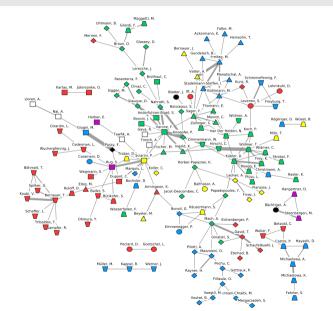
Prediction: Case Study

Based on:

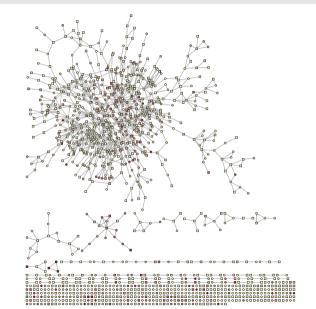
Leifeld, Philip (2018): Polarization in the Social Sciences: Assortative Mixing in Social Science Collaboration Networks is Resilient to Interventions. *Physica A: Statistical Mechanics and its Applications* 507: 510–523.

- Statistical modelling (ERGM) of the German and Swiss political science co-authorship networks.
- ▶ Why do any two researchers (not) have a co-authorship tie?
- Hypothesis: Segregation into two philosophical camps.

Political Science Co-Authorship Network in Switzerland



Political Science Co-Authorship Network in Germany

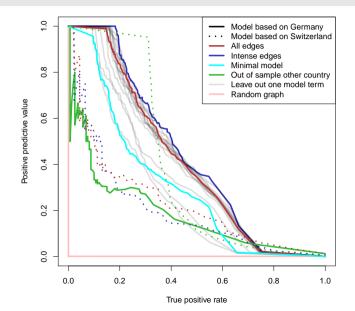


Co-authorship ERGM: Results for Germany

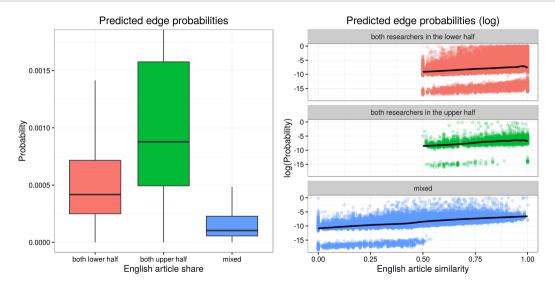
	All ties	Intense collaboration
Endogenous model terms		
Edges	$-11.60 (0.27)^{***}$	$-12.47 (0.32)^{***}$
Edge-wise shared partners	1.94 (0.06)***	2.11 (0.07)***
Degree distribution	0.42 (0.08)***	0.31 (0.09)***
Exogenous covariates		
Publication frequency	0.00 (0.00)***	0.00 (0.00)***
Professor	0.18 (0.04)***	0.17 (0.04)***
Gender: male	0.01 (0.04)	-0.07(0.05)
Gender homophily	0.23 (0.06)***	0.27 (0.07)***
Geographic distance	-0.13 (0.02)***	$-0.14 (0.02)^{***}$
Same affiliation	1.40 (0.07)***	1.42 (0.08)***
Same chair or team	1.29 (0.11)***	1.32 (0.12)***
Supervision	0.35 (0.17)*	0.27 (0.18)
Topic similarity	23.10 (0.74)***	23.81 (0.84)***
Share of English articles	0.49 (0.10)***	0.61 (0.11)***
English article similarity	3.03 (0.26)***	3.64 (0.31)***

^{***}p < 0.001, **p < 0.01, *p < 0.05

Out-of-Sample Predictive Fit: Precision-Recall Curve



Predicted Probabilities: English Article Share Similarity



The effect holds within both camps.

Some Useful Concepts Related to Prediction

- ▶ Within-sample vs. out-of-sample prediction.
- Cross-validation.
- Leave-one-out prediction.
- Point vs. interval prediction.
- Precision–recall curve.
- Predicted probabilities.

2. Interaction Effects

▶ Research often assumes additive causes.

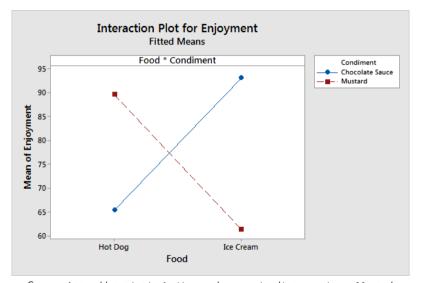
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- ► For example, civil war is explained by low economic development, autocracy, or ethnic fragmentation, independently of each other.
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- ► However, sometimes causes are more complex.
- Example: (autocracy AND ethnic fragmentation) OR intervention from neighbouring state.

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- ► The presence of any of the three factors can increase civil war probability.
- ► However, sometimes causes are more complex.
- Example: (autocracy AND ethnic fragmentation) OR intervention from neighbouring state.
- ▶ Interaction effects can be employed to model some of this complexity.

Example: Interaction between Categorical Variables



Source: http://statisticsbyjim.com/regression/interaction-effects/

Multiplicative Interactions: Centering

You can either multiply the variables manually (entry-wise) or use the multiplication operator inside a formula, like here:

```
model1 <- lm(loss ~ hours*effort + prog, data = dat)</pre>
```

Let's center the variables before using them in an interaction:

```
dat$hrs <- dat$hours - mean(dat$hours)
dat$eff <- dat$effort - mean(dat$effort)
model2 <- lm(loss ~ hrs*eff + prog, data = dat)</pre>
```

Show the resulting table:

```
screenreg(list(model1, model2), single.row = TRUE,
    custom.coef.names = c("cons", "hrs", "eff",
    "prog", "hrs:eff", "hrs", "eff", "hrs:eff"))
```

Multiplicative Interactions: Centering

```
##
##
          Model 1 Model 2
## cons 18.37 (10.95) 21.40 (1.13) ***
## hrs -8.13 (5.32) 2.65 (0.86) **
## eff -0.08 (0.36) 0.65 (0.08) ***
## prog -5.70 (0.52) *** -5.70 (0.52) ***
## hrs:eff 0.36 (0.18) * 0.36 (0.18) *
## R<sup>2</sup> 0.19
                0.19
## Adj. R^2 0.18
                       0.18
## Num. obs. 900
                          900
## *** p < 0.001; ** p < 0.01; * p < 0.05
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                    0.19
## Adj. R^2 0.18
                         0.18
## Num. obs. 900
                         900
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```

prog and hrs:eff stay the same.

```
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## eff -0.08 (0.36) 0.65 (0.08) ***
## prog -5.70 (0.52) *** -5.70 (0.52) ***
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                    0.19
## Adj. R^2 0.18
                         0.18
## Num. obs. 900
                         900
## *** p < 0.001; ** p < 0.01; * p < 0.05
```

cons, hrs, and eff change.

```
##
##
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## eff -0.08 (0.36) 0.65 (0.08) ***
## prog -5.70 (0.52) *** -5.70 (0.52) ***
## hrs:eff 0.36 (0.18) * 0.36 (0.18) *
## R^2 0.19
                    0.19
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```

hrs: slope when eff has a value of zero.

```
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- At an average effort level, an additional hour of training leads to 2.65 units in weight loss.
- ► At an average training time, an additional unit of effort leads to a 0.65 increase in weight loss.
- ► For each additional training hour, effort leads to another 0.36 units in weight loss, in addition to the main effect of effort.

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- ▶ Insignificant in Model 1 because extreme (and rare) value on var2 (i. e., 0).
- ► At an average effort level, an additional hour of training leads to 2.65 units in weight loss.
- At an average training time, an additional unit of effort leads to a 0.65 increase in weight loss.
- ► For each additional training hour, effort leads to another 0.36 units in weight loss, in addition to the main effect of effort.
- Or conversely, for each additional effort unit, an additional training hour leads to another 0.36 units in weight loss, in addition to the main effect of hours.

Significance of interaction can only be evaluated graphically because it spans the whole range of the main variables.

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Marginal effects plots can do this.

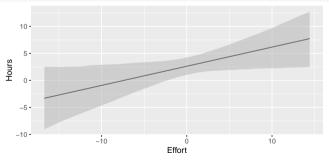
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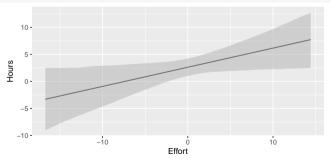
```
library("interplot")
interplot(m = model2, var1 = "hrs", var2 = "eff") +
    xlab("Effort") + ylab("Hours")
```



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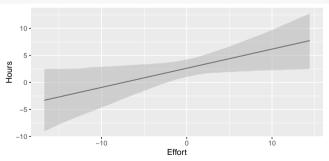


Effort on x-axis, hours on y-axis. Line = conditional slope.

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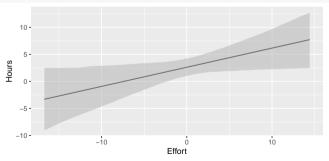


Slope CI excludes zero only for positive values of effort!

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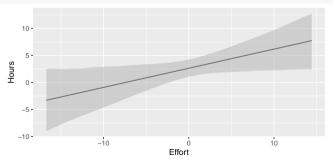


Only above-average effort shows an interaction with hours.

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```
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interplot(m = model2, var1 = "hrs", var2 = "eff") +
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```

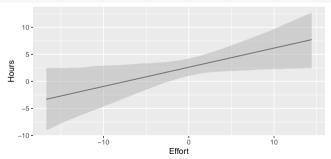


For positive effort: more effort means hours are more effective.

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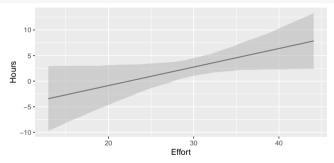
```
library("interplot")
interplot(m = model2, var1 = "hrs", var2 = "eff") +
    xlab("Effort") + ylab("Hours")
```



The CI barely rules out a straight line. See also the SE.

What Happens if we Use the Uncentered Variables?

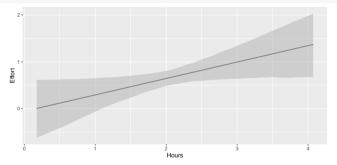
```
interplot(m = model1, var1 = "hours",
  var2 = "effort") + xlab("Effort") + ylab("Hours")
```



- ► Same functional form and CI.
- ▶ Notice how the x-axis is now uncentered.
- ightharpoonup The interaction effect is different from 0 for effort > 27.
- ► Effect at effort = 0 not interpretable anymore.

What Happens if we Switch the Variables?

```
interplot(m = model1, var1 = "effort",
  var2 = "hours") + xlab("Hours") + ylab("Effort")
```



- ▶ Still the same functional form and CI.
- ▶ Notice how the axes are swapped and interpretation changes.
- ightharpoonup Positive interaction for centered hours > 2.

Multiplicative Interactions: Some Advice

- 1. Always include the constitutive terms (= main effects).
- 2. Do not interpret constitutive terms as unconditional marginal effects.
- 3. Consider centering the constitutive terms before inclusion if you want to interpret the main effects at meaningful levels.
- 4. Or (much) better: provide marginal effects plot to evaluate the interaction effect.

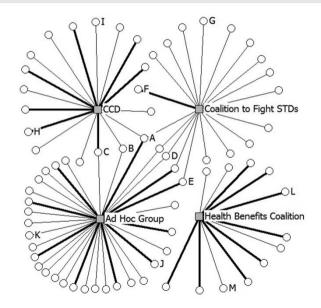
Interaction Effects: Case Study

Based on:

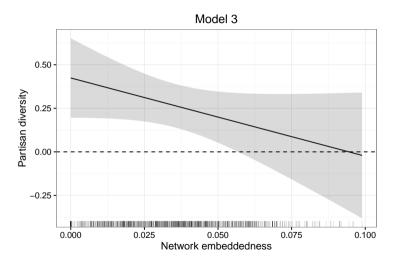
Heaney, Michael T. and Philip Leifeld (2018): Contributions by Interest Groups to Lobbying Coalitions. *The Journal of Politics* 80(2): 494–509.

- Interest groups are members in lobbying coalitions.
- ▶ What determines if a group provides leadership to a coalition?
- ► Two hypotheses: partisan diversity and network embeddedness.
- ► Interaction effect: do they undercut each other?

Interest Groups, Lobbying Coalitions, and Leadership

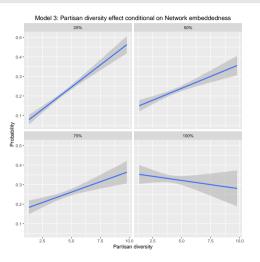


Marginal Effect on Leadership Provision



They undercut each other: the more network embeddedness, the less partisan diversity and vice-versa!

Corresponding Predicted Probabilities



You can also see this by predicting partisan diversity \rightarrow leadership for the quartiles of network embeddedness.

Exercise

- 1. Can you come up with examples of positive and negative interactions in political science? What do they mean substantively?
- 2. Should you center variables before computing interaction effects?
- 3. How could one create a marginal effects plot for an interaction that includes a dummy variable?

Solution

- 1. See previous slides.
- 2. Centering can be helpful in interpreting a main effect at typical levels of the other main effect. But a marginal effects plot is often easier to interpret in absolute terms.
- 3. Show two points on the horizontal axis, one for each group (0 and 1). The vertical axis shows the coefficient of the other, quantitative variable. Confidence intervals can be added as error bars.