Game Theory Assignment: AKQ Poker

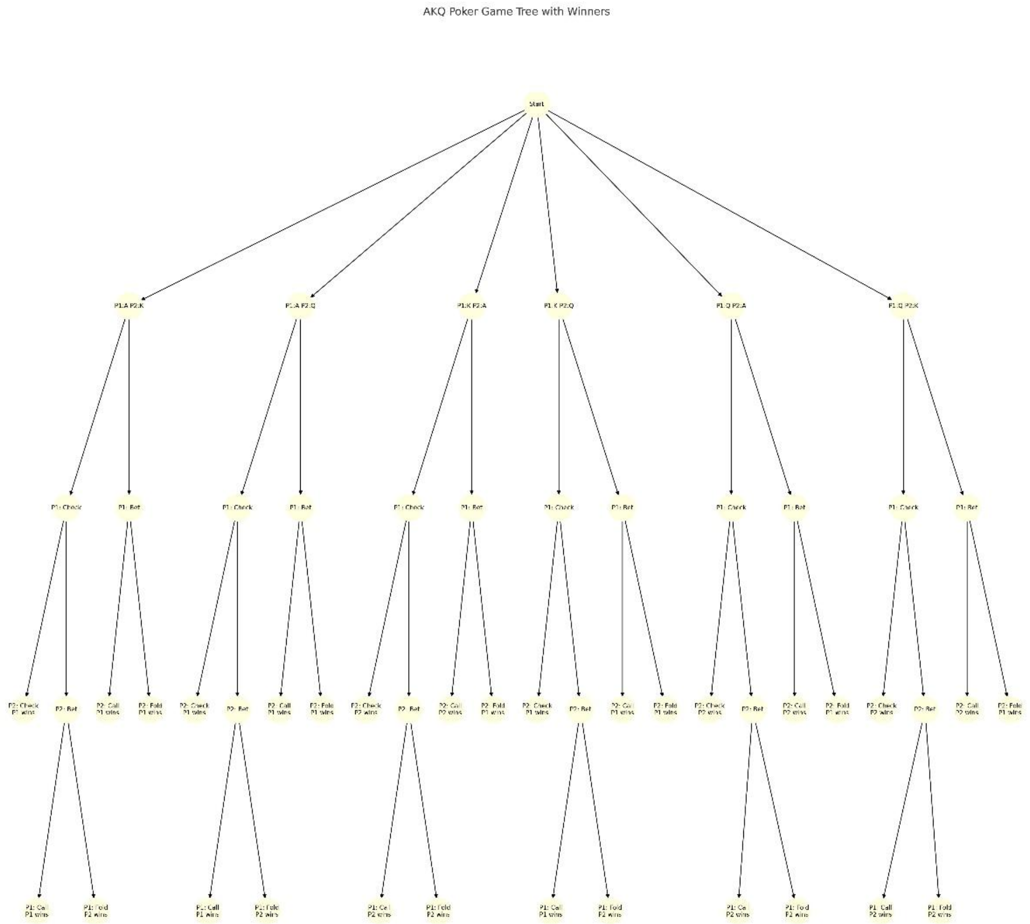
Team members:

1. Amogh Singh Rathore(Team Leader)
2. Ajinkya Kulkarni

Game Rules of AKQ Poker

1. Only three cards are used in the game, Ace, King, Queen with the usual hierarchy order of Ace>King>Queen.
2. Only two players are allowed to play.
3. Only 1 Round is played that means bet can only be placed once without having any option of raising the bet subsequently.
4. Initially, the ante is of 1 dollar i.e., both players have to place 1 dollar into the pot before any rounds occur.
5. Each player is dealt one card with the third card being set aside without being seen by either player.
6. Player Actions:
   i. Player 1 can either check or bet 1 dollar.
   ii. If Player 1 checks: Player 2 can either check or bet 1 dollar.
      a. If Player 2 checks: Game proceeds to showdown.
      b. If Player 2 bets: Player 1 can either call or fold. If Player 1 calls, then the game proceeds to showdown. If Player 1 folds, then immediately Player 2 becomes the winner.
   iii. If Player 1 bets: Player 2 can either call or fold.
      a. If Player 2 calls: Game proceeds to showdown.
      b. If Player 2 folds: immediately Player 1 becomes the winner.

Game Tree



Strategic Analysis

In total, there are 6 cases possible considering all the permutations in AKQ poker as depicted in the game tree. I will delve into each case in detail one by one.

Note: The following analysis is made with the assumption that it is the first ever game being played between them without any previous game history.(If game history is considered then the analysis will be heavily complicated.) I will consider game history later under a separate section specifically for bluffing.

Case-1: P1:A :: P2:K

P1 knows that he has the highest hand possible and will win in showdown.

Let's say P1 checks, now P2 knows there is 50% probability of P1 having an Ace and 50% probability of P1 having a Queen. So, if P2 checks then game proceeds to showdown and P2 stands to gain 1 dollar or lose 1 dollar but if P2 bets then, P1 has two options to either call or fold, and here there is a clear dominant strategy for P1, he will call and win 2 dollars. But continuing from P2's perspective, P1 will fold if he has a Queen and P1 will call if he has an Ace. So, P2 will either win 1 dollar or lose 2 dollars. Theoretically, P2 can win 2 dollars if P1 had a Queen and still decided to call but that is a completely irrational decision since, P1 would have known that he has the lowest hand possible in the game, and he cannot win in any scenario through showdown. Clearly, for P2 the better option is to check since, it has less risk associated with it while still offering the same reward. Hence, if P1 checks then P2 has a clear dominant strategy i.e., to check.

Lets say P1 bets, now P2 again knows there is 50% probability of P1 having an Ace and 50% probability of P1 having a Queen. So, P1 can either be bluffing or have the higher hand from P2's perspective. If P2 decides to call, he will either gain 2 dollars or lose 2 dollars and if he decides to fold, then he will lose 1 dollar. Since, there is no prior game history, there is no info about the player's playing style and it is safer to fold.

Conclusion: There is a clear dominant strategy for P1 to bet since, betting offers practical chances of winning 2 dollars. On the other hand, if P1 checks, then P2 has a clear dominant strategy of checking in response and P1 can only gain 1 dollar logically.

Case-2: P1:A :: P2:Q

P1 knows that he has the highest hand possible and will win in showdown. P2 knows that he has the lowest hand possible and will lose in showdown.

Let's say P1 checks. P1 is now hoping that P2 might try to bluff. From P2's perspective, he must decide whether to check or bet. P2 knows that P1 cannot have a Queen, so P1 must hold either a King or an Ace. If P2 decides to check, the game goes to showdown, and P2 knows he will lose his $1 ante. If P2 decides to bet, he must consider P1's likely reaction. P1, holding either an Ace or a King, would have a clear dominant strategy to call a bet from a player who could only be bluffing. Knowing this, P2 understands that a bluff attempt will be called, resulting in a loss of $2 ($1 ante + $1 bet). Clearly, for P2, the better option is to check and minimize his certain loss to $1. Hence, if P1 checks, P2's best response is to also check.

Let's say P1 bets. P1 is now betting for value, trying to get more money into the pot. P2, holding the lowest possible hand, must decide whether to call or fold. Since P2 knows he cannot win the hand, calling the bet would be a completely irrational decision, as it would mean losing an additional $1 for no reason. P2's only logical move is to fold to the bet to prevent any further loss. By folding, P2 only loses his initial $1 ante.

Conclusion: In this specific scenario, both checking and betting lead to the same logical outcome of P1 winning $1. P1 cannot extract any additional value because P2 holds the worst possible hand and therefore has no incentive to risk any more money.

Case-3: P1:K :: P2:A

P1 knows that he has a medium-strength hand, which loses to an Ace but beats a Queen. P2 knows that he has the highest hand possible and will win in any showdown.

Let's say P1 checks. P1 is playing cautiously, hoping to see a cheap showdown or perhaps induce a bluff from a Queen. From P2's perspective, holding the Ace, he sees the check from P1. P2 knows P1 cannot have an Ace (since P2 has it) and therefore must have a King or a Queen. If P2 checks back, the game proceeds to showdown where P2 is guaranteed to win $1. However, if P2 bets, he can potentially extract more value. A bet forces P1, who might have a King, into a very difficult decision of whether to call or fold. Since P2 has the guaranteed winning hand, there is a strong incentive for P2 to bet to maximize his profit. Hence, if P1 checks, P2's best response is to bet.

Let's say P1 bets. P1 is now attempting a semi-bluff, representing the Ace that he does not have. P2, who actually holds the Ace, sees this bet. For P2, the decision is simple. Holding the unbeatable hand, there is no reason to fold. Calling is the only logical action to take, as it ensures he can win the now larger pot at showdown. Therefore, if P1 bets, P2's only rational move is to call. This results in a showdown where P1 is guaranteed to lose $2.

Conclusion: In this scenario, betting with a King is a disastrous play for P1, as it only gets called by the better hand and guarantees a $2 loss. P1's clear dominant strategy is to check. Checking keeps the pot small and gives P1 the option to fold to a subsequent bet from P2, thereby minimizing his potential losses against a stronger hand.

Case-4: P1:K :: P2:Q

P1 knows that he has a medium-strength hand, but in this specific matchup, it is the superior hand. P2 knows that he holds the weaker hand and can only win the pot if P1 can be convinced to fold.

Let's say P1 checks. P1 is playing passively, perhaps hoping to induce a bluff from P2. From P2's perspective, he holds a Queen and must decide whether to check or bet. P2 knows that P1 must have either an Ace or a King, both of which beat his Queen. If P2 checks, the game proceeds to a showdown that he is guaranteed to lose, resulting in a loss of his $1 ante. If P2 decides to bet, he is attempting a bluff. However, P1, holding a King, knows that P2's bet can only be a bluff with a Queen. P1 will therefore always call the bet, as he holds the winning hand. A rational P2 would understand this and realize that a bluff is very likely to be called, leading to a $2 loss. Clearly, for P2, the better option is to check and accept the smaller $1 loss. Hence, if P1 checks, P2's best response is to also check.

Let's say P1 bets. P1 is now betting for value with his stronger hand. P2, holding the losing Queen, sees this bet. For P2, the decision is straightforward. Knowing his hand is beaten by either a potential King or Ace from P1, calling the bet would be an irrational loss of more money. P2's only logical move is to fold to the bet, thereby limiting his loss to the initial $1 ante. Therefore, if P1 bets, P2's only rational move is to fold.

Conclusion: While both checking and betting result in a $1 win for P1 in this instance, the strategically sound and dominant play is to check. Betting with a King is a flawed strategy overall, as it guarantees a large loss against an Ace, making checking the superior choice.

Case-5: P1:Q :: P2:A

P1 knows that he has the worst possible hand and can only win if P2 can be convinced to fold. P2 knows that he has the best possible hand and cannot lose in a showdown.

Let's say P1 checks. P1 is playing passively, likely hoping to minimize his loss. From P2's perspective, holding the Ace, he sees the check from P1. P2 knows P1's hand can only be a King or a Queen. P2's dominant strategy here is to bet for value, trying to get called by a King. P1, holding the Queen and now facing a bet, has no choice but to fold as calling would be irrational. Hence, if P1 checks, the logical sequence is that P2 will bet and P1 will fold, resulting in a $1 loss for P1.

Let's say P1 bets. P1 is now attempting a pure bluff, representing a strong hand that he does not have. P2, who actually holds the unbeatable Ace, sees this bet. For P2, the decision is simple. Holding the best possible hand, his strategy is to call any bet to get to a showdown. Folding would be completely irrational. Therefore, if P1 bets, P2's only logical move is to call, which exposes the bluff and leads to a showdown that P1 is guaranteed to lose. This results in a $2 loss for P1.

Conclusion: In this scenario, bluffing with a Queen is a costly mistake for P1, as it gets called by P2's Ace and leads to a $2 loss. The strategically superior play is to check, which allows P1 to fold to P2's inevitable bet and minimize the loss to just $1.

Case-6: P1:Q :: P2:K

P1 knows that he has the weaker hand and will lose in a showdown. P1's only path to winning the pot is to make P2 fold. P2 knows that he has the medium strength card in this matchup.

Let's say P1 checks. P1 is playing passively and essentially giving up on winning the pot, aiming only to minimize his loss. From P2's perspective, he holds a King and sees the check. P2 knows P1's hand could be an Ace or a Queen. The optimal strategy for P2 holding a King is to check back to control the pot size against a potential Ace and to take his guaranteed win against a Queen. Hence, if P1 checks, P2 will also check, leading to a showdown where P1 is guaranteed to lose $1.

Let's say P1 bets. P1 is now attempting a pure bluff, representing the Ace that he does not have. P2, holding a King, now faces the most critical decision in the game. He knows P1's bet could be for value with an Ace or a bluff with a Queen. According to the established Nash Equilibrium, P2's strategy is to mix his play: he will call the bet 1/3 of the time and fold 2/3 of the time. This mixed response makes it impossible for P1 to know for certain if his bluff will succeed. If P2 calls, P1 loses $2; if P2 folds, P1 wins $1.

Conclusion: In this scenario, checking with a Queen guarantees a $1 loss for P1. Bluffing, while risky, is the strategically necessary play as it gives P1 a chance to win the pot. The uncertainty created by P2's mixed response is precisely what makes bluffing a viable and essential part of P1's optimal strategy with a losing hand.

Summary of Strategic Analysis

Using Bayesian probability and the Indifference Principle, the exact Nash equilibrium strategies are (note: alpha, beta and gamma have been defined under the section “Calculations for Bluffing Frequencies”.):

- P1
  - Ace: always Bet
  - King: always Check on the first chance; Call any bet two times out of three ($\delta = 2/3$), Fold against any bet one time out of three (1/3)
  - Queen: Bet one time out of three ($\alpha = 1/3$), Check two times out of three (2/3); Fold against any bet
- P2
  - Ace: Call any bet; Bet after check
  - King: Call one time out of three ($\beta = 1/3$) any bet; Check after check
  - Queen: Fold against any bet; Bet after check one time out of three ($\gamma = 1/3$), Check after check two times out of three (2/3)

AKQ Poker Payoff Matrix

Card Combo	Check-Check	Check-Bet-Call	Check-Bet-Fold	Bet-Call	Bet-Fold
P1:A, P2:K	(+1,-1)	(+2,-2)	(-1,+1)	(+2,-2)	(+1,-1)
P1:A, P2:Q	(+1,-1)	(+2,-2)	(-1,+1)	(+2,-2)	(+1,-1)
P1:K, P2:A	(-1,+1)	(-2,+2)	(-1,+1)	(-2,+2)	(+1,-1)
P1:K, P2:Q	(+1,-1)	(+2,-2)	(-1,+1)	(+2,-2)	(+1,-1)
P1:Q, P2:A	(-1,+1)	(-2,+2)	(-1,+1)	(-2,+2)	(+1,-1)
P1:Q, P2:K	(-1,+1)	(-2,+2)	(-1,+1)	(-2,+2)	(+1,-1)

Calculations for Bluffing Frequencies

Notation

- α = P1's probability of betting with Queen
- β = P2's probability of calling with King when facing a bet
- γ = P2's probability of betting with Queen when P1 checks
- δ = P1's probability of calling with King when facing a bet

1. Derivation of $\alpha = 1/3$

When P1 holds Queen and bets:

1. Posterior probabilities after a bet:
   $\Pr(\text{Ace} \mid \text{bet}) = 1/(1+\alpha)$
   $\Pr(\text{Queen} \mid \text{bet}) = \alpha/(1+\alpha)$
2. P2's payoffs if calling:
   –2 if P1=Ace, +2 if P1=Queen
   Now, $EV\_call = [1/(1+\alpha)]\cdot(-2) + [\alpha/(1+\alpha)]\cdot(+2)$
   $= (-2 + 2\alpha)/(1+\alpha)$
3. P2's payoff if folding: –1
4. Indifference requires EV_call =EV_fold:
   $(-2 + 2\alpha)/(1+\alpha) = -1$
5. Solve for alpha:
   $\alpha = 1/3$

2. Derivation of $\beta = 1/3$

We are deriving the value of β, such that Player 1 (P1) becomes indifferent between betting and checking when holding a Queen.

1. P1's Expected Value when Betting with Queen

If P1 holds Queen and chooses to bet:

There are only two cards P2 might have: Ace or King. So, the probability that P2 has Ace is 1/2, and probability P2 has King is 1/2.

Now calculating the expected value:

- If P2 has Ace (probability 1/2), he always calls, and P1 loses 2 dollars.
- If P2 has King (probability 1/2):
  - With probability β, P2 calls → P1 loses 2 dollars.
  - With probability $(1 - \beta)$, P2 folds → P1 wins 1 dollar.

Now,

$EV(\text{bet with Queen}) = (1/2) \times (-2) + (1/2) \times [\beta \times (-2) + (1 - \beta) \times (+1)]$
$= -1 + (1/2) \times (-2\beta + 1 - \beta)$
$= -1 + (1 - 3\beta)/2$
$= (-1 - 3\beta)/2$

2. P1's Expected Value when Checking with Queen

If P1 checks with Queen:

- If P2 has Ace, he will bet → P1 folds → P1 loses 1 dollar.
- If P2 has King, he checks → showdown → Queen loses → P1 loses 1 dollar.

So,

$EV(\text{check with Queen}) = -1$

3. Applying the Indifference Principle

To make P1 indifferent between betting and checking with Queen:

Set both expected values equal:

$(-1 - 3\beta)/2 = -1$

On solving,
$\beta = 1/3$

3.Derivation of $\gamma = 1/3$ and $\delta = 2/3$ (NEXT PAGE)

Note: refer next page for 'p' notation.

Now, $p = \gamma/1+ \gamma$

$= 1/4 = \gamma/1+ \gamma$ (here, 1 is there since if P2 has Ace, he always bets)

On solving,

$= \gamma = 1/3$

Mixed Nash Equilibrium

- Derivation of 'y' and 'g'

$A \rightarrow P1$ has King
$B \rightarrow P1$ has Ace
$C \rightarrow P1$ checks

$P(B/C) = 0$

Now,

$P(A/C) = 1$

let, p = prob. of bluff by P2

Now,

let, x = prob. of call by P1
$1-x$ = prob. of fold by P1

$$EV_{call} = p.(+2) + 1-p.(-2)$$
$$= 2p - 2 + 2p$$
$$= 4p - 2$$

$EV_{fold} = -1$

Using indiff. principle,

$$4p - 2 = -1$$
$$4p = 1$$
$$\boxed{p = 1/4}$$

Now,

$$EV_{bluff} = x.(-2) + 1-x.(+1)$$
$$= -2x + 1 - x = 1 - 3x$$

$EV_{\cancel{fold}\ check} = -1$

Using indiff. principle, $\Rightarrow \boxed{x = 2/3}$

$\Rightarrow 1 - 3x = -1$

$\Rightarrow \boxed{g = 2/3}$

Calculations for Expected Value (EV) in each case

Case 1: P1 has Ace, P2:?

a. EV Calculation: P1 (Ace) Checks

Scenario 1: P2 has a King (Probability: 1/2)

- P1 checks.
- P2's strategy is to check back ("Check after check").
- Outcome: Showdown. P1 wins the $2 ante pot.
- Payoff: +1

Scenario 2: P2 has a Queen (Probability: 1/2)

- P1 checks.
- P2's strategy is to bet with probability $\gamma=1/3$ or check with probability 2/3.
  - If P2 bets (Prob 1/3): P1 holds an Ace and calls. P1 wins the pot plus the bet. Payoff: +2.
  - If P2 checks (Prob 2/3): Showdown. P1 wins the pot. Payoff: +1.
- Blended Payoff vs. Queen: (1/3)*(+2) + (2/3)*(+1) = +4/3

Final Expected Value

The total EV is the average of the two scenarios.

- Total EV = [(1/2) * (Payoff vs. King)] + [(1/2) * (Payoff vs. Queen)]
- Total EV = [(1/2) * (+1)] + [(1/2) * (+4/3)]
- Total EV = 1/2 + 4/6 = 3/6 + 4/6 = +7/6

The final expected value for Player 1 checking with a Ace is +7/6.

b. EV Calculation: P1 (Ace) Bets

Scenario 1: P2 has a King (Probability: 1/2)

- P1 bets.
- P2's strategy is to call with probability

$\beta=1/3$ and fold with probability 2/3.

  - If P2 calls (Prob 1/3): Showdown. P1 wins. Payoff: +2.
  - If P2 folds (Prob 2/3): P1 wins the pot. Payoff: +1.
- Blended Payoff vs. King: (1/3)*(+2) + (2/3)*(+1) = +4/3

Scenario 2: P2 has a Queen (Probability: 1/2)

- P1 bets.
- P2's strategy is to always fold against any bet.
- Outcome: P2 folds. P1 wins the pot.
- Payoff: +1

Final Expected Value

The total EV is the average of the two scenarios.

- Total EV = [(1/2) * (Payoff vs. King)] + [(1/2) * (Payoff vs. Queen)]
- Total EV = [(1/2) * (+4/3)] + [(1/2) * (+1)]
- Total EV = 4/6 + 1/2 = 3/6 + 4/6 = +7/6

The final expected value for Player 1 betting with a Ace is +7/6.

Note: It may seem like P1 is indifferent between betting and checking from the above EVs but that is not the case, there is a clear dominant strategy for P1 of betting. Refer Case 1 under Strategic Analysis for more information.

Case 2: P1 has King, P2:?

a. **EV Calculation: P1 (King) Checks**

Scenario 1: P2 has an Ace (Probability: 1/2)

- P1 checks.
- P2's strategy is to always bet after a check.
- P1 must then react to the bet, calling with probability

δ=2/3 and folding with 1/3.

  - If P1 calls (Prob 2/3): Showdown. P2 wins. Payoff: -2.
  - If P1 folds (Prob 1/3): P2 wins the pot. Payoff: -1.

- Blended Payoff vs. Ace: (2/3)*(-2) + (1/3)*(-1) = -4/3 - 1/3 = -5/3

Scenario 2: P2 has a Queen (Probability: 1/2)

- P1 checks.
- P2's strategy is to bet with probability γ=1/3 or check with probability 2/3.
  - If P2 bets (Prob 1/3): P1 must react, calling with probability δ=2/3 or folding with 1/3. The blended payoff if P2 bets is +1.
  - If P2 checks (Prob 2/3): Showdown. P1 wins. Payoff: +1.
- Blended Payoff vs. Queen: (1/3)(+1) + (2/3)(+1) = +1

Final Expected Value

The total EV is the average of the two scenarios.

- Total EV = [(1/2) * (Payoff vs. Ace)] + [(1/2) * (Payoff vs. Queen)]
- Total EV = [(1/2) * (-5/3)] + [(1/2) * (+1)]
- Total EV = -5/6 + 1/2 = -5/6 + 3/6 = -2/6 = -1/3

The final expected value for Player 1 checking with a King is -1/3.

b. **EV Calculation: P1 (King) Bets**

Scenario 1: P2 has an Ace (Probability: 1/2)

- P1 bets.
- P2's strategy with an Ace is to "Call any bet".
- Outcome: Showdown. P2's Ace wins.
- Payoff: -2

Scenario 2: P2 has a Queen (Probability: 1/2)

- P1 bets.
- P2's strategy with a Queen is to "Fold against any bet".
- Outcome: P2 folds. P1 wins the pot.
- Payoff: +1

Final Expected Value

The total EV is the average of the two scenarios.

- Total EV = [(1/2) * (Payoff vs. Ace)] + [(1/2) * (Payoff vs. Queen)]
- Total EV = [(1/2) * (-2)] + [(1/2) * (+1)]
- Total EV = -1 + 1/2 = -1/2

The final expected value for Player 1 betting with a King is -1/2.

Case 3: P1 has Queen, P2: ?

a. **EV Calculation: P1 (Queen) Checks**

Scenario 1: P2 has an Ace (Probability: 1/2)

- P1 checks.
- P2's strategy is to always bet after a check.

- P1's strategy is to fold against any bet when holding a Queen.
- Outcome: P1 folds to the bet and loses their ante.
- Payoff: -1

Scenario 2: P2 has a King (Probability: 1/2)

- P1 checks.
- P2's strategy is to "Check after check".
- Outcome: Showdown. P2's King wins. P1 loses their ante.
- Payoff: -1

Final Expected Value

The total EV is the average of the two scenarios.

- Total EV = [(1/2) * (Payoff vs. Ace)] + [(1/2) * (Payoff vs. King)]
- Total EV = [(1/2) * (-1)] + [(1/2) * (-1)]
- Total EV = -1/2 - 1/2 = -1

The final expected value for Player 1 checking with a Queen is -1.

b. **EV Calculation: P1 (Queen) Bets**

Scenario 1: P2 has an Ace (Probability: 1/2)

- P1 bets.
- P2's strategy is to "Call any bet".
- Outcome: Showdown. P2's Ace wins.
- Payoff: -2

Scenario 2: P2 has a King (Probability: 1/2)

- P1 bets.
- P2's strategy is to call with probability

β=1/3 and fold with probability 2/3.

  - If P2 calls (Prob 1/3): Showdown. P2's King wins. Payoff: -2.
  - If P2 folds (Prob 2/3): P1's bluff succeeds. P1 wins the pot. Payoff: +1.

- Blended Payoff vs. King: (1/3)*(-2) + (2/3)*(+1) = -2/3 + 2/3 = 0

Final Expected Value

The total EV is the average of the two scenarios.

- Total EV = [(1/2) * (Payoff vs. Ace)] + [(1/2) * (Payoff vs. King)]

- Total EV = [(1/2) * (-2)] + [(1/2) * (0)]
- Total EV = -1 + 0 = -1

The final expected value for Player 1 betting with a Queen is -1.

Note: The expected values for both the sub-cases came out to be same as a direct consequence of the indifference principle.

Case 4: P1 bets, P2 has Ace and he has two options: either to call or fold

a. **EV Calculation: P2 (Ace) Calls P1's Bet**

Reasoning

- P1 has placed a bet. According to P1's strategy, they only bet with an Ace or a Queen (as a bluff).
- P2 holds the Ace. Therefore, P2 knows with certainty that P1 cannot have an Ace.

Calculation

- P2 calls the bet, knowing they have the winning hand.
- Outcome: Showdown. P2's Ace beats P1's Queen. P2 wins the $2 ante plus P1's $1 bet.
- Payoff: +2

Final Expected Value

Since the outcome is certain, the expected value is equal to the payoff.

- Total EV = 1 * (+2) = +2

The final expected value for Player 2 calling with an Ace is +2.

b. **EV Calculation: P2 (Ace) Folds to P1's Bet**

Reasoning

- Both players have placed a $1 ante into the pot.
- P1 has placed a bet.
- P2 chooses to fold. When a player folds, they surrender the hand and lose any money they have already put into the pot.

Calculation

- P2's only contribution to the pot was their initial $1 ante.
- By folding, P2 forfeits this ante.
- Outcome: P1 wins the pot. P2 loses their $1 ante.

- Payoff: -1

Final Expected Value

Since P2's action of folding leads to a definite outcome, the expected value is equal to the payoff.

- Total EV = 1 * (-1) = -1

The final expected value for Player 2 folding with an Ace is -1.

Note: This complete case was trivial but we have shown it for completeness.

Case 5: P1 bets, P2 has King and he has two options: either to call or fold

a. **EV Calculation: P2 (King) Calls P1's Bet**

Reasoning

- Player 1 (P1) has bet. P1's strategy is to bet with an Ace and to bluff with a Queen at a frequency of $\alpha=1/3$.
- From Player 2's (P2) perspective, the probability that P1 holds an Ace is 3/4, and the probability that P1 is bluffing with a Queen is 1/4 (using Bayes and Total Probability Theorem, derived under calculation).
- P2 holds a King and chooses to call. We average the outcomes based on P1's hand.

Calculation

- Derivation for the probability that P1 holds an Ace or Queen

  Probability P1 has an Ace = 3/4:

  P(P1=A | Bet) = [P(Bet | P1=A) * P(P1=A)] / P(Bet)

  P(P1=A | Bet) = [1 * (1/2)] / (2/3)

  P(P1=A | Bet) = (1/2) / (2/3) = 3/4

  Probability P1 has a Queen = 1/4:

  P(P1=Q | Bet) = [P(Bet | P1=Q) * P(P1=Q)] / P(Bet)

  P(P1=Q | Bet) = [(1/3) * (1/2)] / (2/3)

  P(P1=Q | Bet) = (1/6) / (2/3) = 1/4

- Scenario 1: P1 has an Ace (Probability: 3/4)
  - Outcome: Showdown. P1's Ace wins. P2 loses their ante and their call.
  - Payoff: -2
- Scenario 2: P1 has a Queen (Probability: 1/4)

- Outcome: Showdown. P2's King wins. P2 wins the pot and P1's bet.
- Payoff: +2

Final Expected Value

The total EV is the average of the two scenarios.

- Total EV = [(3/4) * (Payoff vs. Ace)] + [(1/4) * (Payoff vs. Queen)]
- Total EV = [(3/4) * (-2)] + [(1/4) * (+2)]
- Total EV = -6/4 + 2/4 = -4/4 = -1

The final expected value for Player 2 calling with a King is -1.

b. EV Calculation: P2 (King) Folds to P1's Bet

- P2's only contribution to the pot was their initial $1 ante.
- By folding, P2 forfeits this ante.
- Outcome: P1 wins the pot. P2 loses their $1 ante.
- Payoff: -1

Final Expected Value

Since P2's action of folding leads to a definite outcome, the expected value is equal to the payoff.

- Total EV = 1 * (-1) = -1

The final expected value for Player 2 folding with a King is -1.

Case 6: P1 bets, P2 has Queen and he has two options: either to call or fold

This is a trivial case. P2 knows he has the lowest hand possible and will lose in showdown.

EV_calling = -2

EV_folding = -1

Even without thinking mathematically, it is fairly obvious that P2 always folds in such a situation.

Case 7: P1 checks, P2 has Ace and he has two options: either to bet or check

a. EV Calculation: P2 (Ace) Bets After P1 Checks

Reasoning

- Player 1 (P1) has checked. P1's strategy is to check with a King or a Queen, but never an Ace.

- From Player 2's (P2) perspective, the probability that P1 holds a King is 3/5, and the probability that P1 holds a Queen is 2/5 (using Bayes Theorem and Total Probability Theorem, derived under calculation).
- P2 holds an Ace and chooses to bet. We average the outcomes based on P1's hand and subsequent reaction.

Calculation

- Derivation for probabilities of King or Queen

  Probability P1 has a King:

  P(P1=K | Check) = [P(Check | P1=K) * P(P1=K)] / P(Check)

  P(P1=K | Check) = [1 * (1/2)] / (5/6)

  P(P1=K | Check) = (1/2) / (5/6) = 3/5

  Probability P1 has a Queen:

  P(P1=Q | Check) = [P(Check | P1=Q) * P(P1=Q)] / P(Check)

  P(P1=Q | Check) = [(2/3) * (1/2)] / (5/6)

  P(P1=Q | Check) = (1/3) / (5/6) = 2/5

- Scenario 1: P1 has a King (Probability: 3/5)
  - P2 bets. P1 will call with probability δ=2/3 or fold with 1/3.
    - If P1 calls, P2's Ace wins. Payoff: +2.
    - If P1 folds, P2 wins the pot. Payoff: +1.
  - Blended Payoff vs. King: (2/3)*(+2) + (1/3)*(+1) = +5/3.
- Scenario 2: P1 has a Queen (Probability: 2/5)
  - P2 bets. P1's strategy is to always fold with a Queen when facing a bet.
  - Outcome: P1 folds. P2 wins the pot.
  - Payoff: +1.

Final Expected Value

The total EV is the average of the two scenarios.

- Total EV = [(3/5) * (Payoff vs. King)] + [(2/5) * (Payoff vs. Queen)]
- Total EV = [(3/5) * (+5/3)] + [(2/5) * (+1)]
- Total EV = 1 + 2/5 = +7/5

The final expected value for Player 2 betting with an Ace is +7/5.

b. EV Calculation: P2 (Ace) Checks After P1 Checks

Reasoning

- Player 1 (P1) has checked. P1's strategy is to check only with a King or a Queen.
- Player 2 (P2) holds an Ace and chooses to check.

Calculation

- At the showdown, P2's Ace will beat either of P1's possible hands (King or Queen).
- Outcome: P2 wins the $2 ante pot.
- Payoff: +1

Final Expected Value

Since this action leads to a definite showdown where P2 always wins, the expected value is equal to the certain payoff.

- Total EV = 1 * (+1) = +1

The final expected value for Player 2 checking with an Ace is +1.

Note: Also, logically without mathematical calculations P2 will always bet since, it provides a chance to win more.

Case 8: P1 checks, P2 has King and he has two options: either to bet or check

a. **EV Calculation: P2 (King) Bets After P1 Checks**

Reasoning

- Player 1 (P1) has checked. P1's strategy is to check only with a King or a Queen.
- Player 2 (P2) holds the King. Therefore, P2 knows with certainty that P1 must have the Queen.
- P2 chooses to bet.
- P1 holds a Queen and faces a bet. P1's strategy is to "Fold against any bet".

Calculation

- P1 folds to P2's bet.
- Outcome: P2 wins the $2 ante pot uncontested. P2's net gain is P1's $1 ante.
- Payoff: +1

Final Expected Value

Since this action leads to a definite outcome, the expected value is equal to the payoff.

- Total EV = 1 * (+1) = +1

The final expected value for Player 2 betting with a King is +1.

b. EV Calculation: P2 (King) Checks After P1 Checks

Reasoning

- Player 1 (P1) has checked. P1's strategy is to check only with a King or a Queen.
- Player 2 (P2) holds the King. Therefore, P2 knows with certainty that P1 must have the Queen.
- P2 chooses to check, which is the optimal strategy ("Check after check").

Calculation

- At the showdown, P2's King beats P1's Queen.
- Outcome: P2 wins the $2 ante pot. P2's net gain is P1's $1 ante.
- Payoff: +1

Final Expected Value

Since this action leads to a definite showdown where P2 always wins, the expected value is equal to the certain payoff.

- Total EV = 1 * (+1) = +1

The final expected value for Player 2 checking with a King is +1.

Note: Again, this is the same as the Case-1 we discussed, even though the expected values of bet and check are same, the dominant strategy according to Nash Equilibrium is to check.

Case 9: P1 checks, P2 has Queen and he has two options: either to bet or check

a. EV Calculation: P2 (Queen) Bets After P1 Checks

Reasoning

- Player 1 (P1) has checked. P1's strategy is to check only with a King or a Queen.
- Player 2 (P2) holds the Queen. Therefore, P2 knows with certainty that P1 must have the King.
- P2 chooses to bet (bluff)..

Calculation

- Scenario 1: P1 Calls (Probability: 2/3)
    - Outcome: Showdown. P1's King wins. P2 loses their ante and their bet.
    - Payoff: -2

- Scenario 2: P1 Folds (Probability: 1/3)
  - Outcome: P2's bluff succeeds. P2 wins the pot.
  - Payoff: +1

Final Expected Value

The total EV is the average of the two possible outcomes.

- Total EV = [(2/3) * (Payoff if P1 Calls)] + [(1/3) * (Payoff if P1 Folds)]
- Total EV = [(2/3) * (-2)] + [(1/3) * (+1)]
- Total EV = -4/3 + 1/3 = -1

The final expected value for Player 2 betting with a Queen is -1.

b. EV Calculation: P2 (Queen) Checks After P1 Checks

Since this action leads to a definite showdown where P2 always loses, the expected value is equal to the certain payoff.

Total EV = 1 * (-1) = -1

The final expected value for Player 2 checking with a Queen is -1.

Case 10: P1 checks, P2 bets and now P1 has Ace; he can either call or fold

This case is trivial since, P1 knows he has the highest possible hand and will always win in showdown. So, he always calls.

EV_call = +2

EV_fold = -1

Case 11: P1 checks, P2 bets and now P1 has King; he can either call or fold

a. EV Calculation: P1 (King) Calls P2's Bet

Reasoning

- Player 1 (P1) has checked with a King and now faces a bet from Player 2 (P2).
- P2's strategy is to bet with an Ace and to bluff with a Queen at a frequency of $\gamma=1/3$.

- From P1's perspective, the probability that P2 holds an Ace is 3/4, and the probability that P2 is bluffing with a Queen is 1/4 (using Bayes Theorem and Total Probability Theorem, derived under calculation).
- P1 holds a King and chooses to call. We average the outcomes based on P2's hand.

Calculation

- Derivation of probabilities of King or Queen:

  Probability P2 has an Ace:

  P(P2=A | Bet) = [P(Bet | P2=A) * P(P2=A)] / P(Bet)

  P(P2=A | Bet) = [1 * (1/2)] / (2/3)

  P(P2=A | Bet) = (1/2) / (2/3) = 3/4

  Probability P2 has a Queen:

  P(P2=Q | Bet) = [P(Bet | P2=Q) * P(P2=Q)] / P(Bet)

  P(P2=Q | Bet) = [(1/3) * (1/2)] / (2/3)

  P(P2=Q | Bet) = (1/6) / (2/3) = 1/4

- Scenario 1: P2 has an Ace (Probability: 3/4)
  - Outcome: Showdown. P2's Ace wins. P1 loses their ante and their call.
  - Payoff: -2
- Scenario 2: P2 has a Queen (Probability: 1/4)
  - Outcome: Showdown. P1's King wins. P1 wins the pot and P2's bet.
  - Payoff: +2

Final Expected Value

The total EV is the average of the two scenarios.

- Total EV = [(3/4) * (Payoff vs. Ace)] + [(1/4) * (Payoff vs. Queen)]
- Total EV = [(3/4) * (-2)] + [(1/4) * (+2)]
- Total EV = -6/4 + 2/4 = -4/4 = -1

The final expected value for Player 1 calling with a King is -1.

b. EV Calculation: P1 (King) Folds to P2's Bet

Since P1's action of folding leads to a definite outcome, the expected value is equal to the payoff.

Total EV = 1 * (-1) = -1

The final expected value for Player 1 folding with a King is -1.

Case 12: P1 checks, P2 bets and now P1 has Queen; he can either call or fold

This is a trivial case since, P1 knows he has the lowest possible hand he will always fold.

EV_call = -2

EV_fold = -1

4. Nash Equilibrium and Mixed Strategies

i) Nash Equilibrium in AKQ Poker

A Nash Equilibrium is a stable state in a game where no player can gain an advantage by unilaterally changing their own strategy, assuming the other players stick to their strategies. It represents a point where all players are playing an optimal response to their opponents' moves.

In the context of AKQ Poker, the set of strategies we have calculated represents this equilibrium. For example, Player 1's strategy to bluff with a Queen exactly 1/3 of the time is a perfect response to Player 2's strategy of calling with a King 1/3 of the time. If Player 2 were to suddenly start calling more often, Player 1's bluffing would become unprofitable, giving P1 an incentive to change their strategy. Conversely, if P2 started calling less often, P1 would be incentivized to bluff more. The equilibrium holds because the strategies are in perfect balance, leaving neither player with a reason to deviate from their plan.

ii) The Use of Mixed Strategies

Players use mixed strategies—randomizing their actions based on set probabilities—to avoid becoming predictable. In a game of incomplete information like this, a predictable player is easily exploited. For example, if P2 knew P1 only bet with an Ace, P2 could simply fold every time P1 bet, minimizing their losses and preventing P1 from profiting on their best hands.

To prevent this, players adopt a mixed strategy. By sometimes betting with a weak hand like a Queen, Player 1 makes their bets ambiguous. P2 can no longer be certain if a bet represents an Ace or a Queen. This uncertainty forces P2 to also adopt a mixed strategy, sometimes calling with a King to catch a bluff and sometimes folding to a potential Ace. The calculated frequencies are not random guesses; they are derived from the Indifference Principle, where one player's bluffing frequency is calculated specifically to make the other player indifferent between two choices (e.g., calling or folding).

iii) Equilibrium Strategies

The Nash Equilibrium strategies for both players, as derived from the analysis, are as follows:

- **Player 1**
  - **With an Ace:** Always bet.
  - **With a King:** Always check initially. If facing a bet, call with a probability of $\delta=2/3$.

  - **With a Queen:** Bet (bluff) with a probability of $\alpha=1/3$ and check with a probability of 2/3.
- **Player 2**
  - **With an Ace:** Always call a bet; always bet after a check.
  - **With a King:** If facing a bet, call with a probability of $\beta=1/3$. If P1 checks, always check back.
  - **With a Queen:** Always fold to a bet. If P1 checks, bet (bluff) with a probability of $\gamma=1/3$.

5. Reflection and Application

1. Why Bluffing is Necessary

In AKQ poker, without the concept of bluff, the game would become very predictable and transparent. For eg- If you only bet when you have the best hand (the Ace), your opponent will figure it out making them simply fold every time you bet, which means you'll never win more than the small starting pot with your best hands.

By sometimes betting with your worst hand (the Queen), you create confusion. Now, your opponent can no longer be sure if you have the Ace or if you're bluffing. This uncertainty forces them to take a chance and call your bet sometimes with their medium hand (the King). As a result, when you actually have the Ace, you get paid off and win a bigger pot. Bluffing isn't just about trying to win with a bad hand; it's about making sure you win more when you have a good hand.

2. Lessons in Strategic Thinking and Decision-Making

This project demonstrates several core principles of advanced strategic thinking:

- **The Power of Mixed Strategies:** In games of incomplete information, any predictable, pure strategy can be exploited. The solution is to randomize your actions according to a precisely calculated mixed strategy. The frequencies we derived (e.g., bluffing 1/3 of the time) are not arbitrary; they are the exact frequencies required to make the opponent indifferent to their own choices, thereby preventing them from finding an easy counterstrategy.

- **Thinking from the Opponent's Perspective:** The entire analysis was a recursive process of putting ourselves in the opponent's shoes. Our best move is only "best" in the context of our opponent's likely reaction. The Nash Equilibrium is achieved not when we have found a way to "win," but when both players have chosen strategies that are optimal responses to each other, leaving neither player with an incentive to unilaterally change their play.

- **Decision-Making Under Uncertainty:** Every decision in this game is based on probabilities and updated beliefs. An action as simple as a "check" or a "bet" fundamentally alters the likely holdings of a player (a concept formalized by Bayesian probability). The exercise teaches that strong decision-making is about making the most logical choice based on the limited information available, while simultaneously managing your own actions to control the information you reveal to your opponent.

3. Real-Life Application: Business Negotiation

The strategic reasoning in AKQ Poker is directly applicable to many real-world scenarios, most notably **business negotiations**.

Consider a company negotiating a supply contract. Each side has incomplete information about the other's true position—their "bottom line," their urgency, and whether they have viable alternative partners (their Best Alternative to a Negotiated Agreement, or BATNA).

- **A strong BATNA** (a great offer from another supplier) is like holding an **Ace**.
- **A weak or non-existent BATNA** is like holding a **Queen**.
- **Making a firm, aggressive "take-it-or-leave-it" offer** is equivalent to making a **bet**.

If a company only makes aggressive offers when it has a strong BATNA (the Ace), its negotiation style becomes predictable. The other party will learn to concede only when faced with an aggressive offer and will hold firm otherwise.

To counteract this, the company must sometimes **bluff**—make a firm stand even when its negotiating position is weak (holding the Queen). This creates uncertainty. The other party can no longer be sure if the aggressive stance is backed by a strong alternative or not. This forces them to take all firm offers more seriously and makes them more likely to concede, thereby making the company's truly strong positions more profitable. Just as in AKQ poker, the goal is to mix one's strategy to remain unpredictable and maximize value across all possible scenarios.