Chapter1

Saturday, February 27, 2021 5:59 PM

Exercises

1. Assume A_n is monotone decreasing

$A_1 \supset A_2 \supset \cdots$, $A = \lim_{n\to\infty} A_n = \bigcap_{i=1}^{\infty} A_i$

$P(A_n) = P(\bigcap_{i=1}^{n} A_i)$

Take the complementary series:

$A_1^c, A_2^c, \ldots$

This series is monotone increasing

$\lim_{n\to\infty} P(A_n^c) = P(A^c)$

$\lim_{n\to\infty} P(A_n^c) = 1 - P(A^c)$

$\lim_{n\to\infty} P(A_n) = P(A) \Rightarrow P(A_n) \to P(A)$

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2. a) $P(\emptyset) = 0$; $P(\Omega) = 1$

$P(\Omega \cup \emptyset) = P(\Omega) + P(\emptyset)$

$1 = 1 + P(\emptyset)$

$P(\emptyset) = 0$

b) $A \subset B \Rightarrow P(A) \leqslant P(B)$

$C = B - A$ $\quad P(C \cup A) = P(C) + P(A)$

$C \cup A = B$ $\quad P(B) = P(C) + P(A)$

$P(C) \geqslant 0 \Rightarrow P(A) \leqslant P(B)$

c) Assume $P(A) > 1$

$A \subset \Omega \Rightarrow P(A) \leqslant P(\Omega) \Rightarrow P(\Omega) > 1$ X

d) $(A \cup A^c) = \Omega$ $\quad P(A) + P(A^c) = P(\Omega)$

$P(A^c) = 1 - P(A)$

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3. $B_n = \bigcup_{i=n}^{\infty} A_i$ $\quad C_n = \bigcap_{i=n}^{\infty} A_i$

a) $B_n = B_{n+1} \cup A_n$ $\quad C_n = C_{n+1} \cap A_n$

$w \in B_{n+1} \Rightarrow w \in B_n$ $\quad w \in C_n \Rightarrow w \in C_{n+1}$ $\forall w \in A_n$

$\Rightarrow B_{n+1} \subset B_n$ $\quad \Rightarrow C_n \subset C_{n+1}$

b)

c)

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4. $I = \{1, \ldots, n\}$?

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5. a) T H T H

b) $\frac{1}{2} \times \frac{\binom{k-1}{1}}{2^{k-1}} = \frac{k-1}{2^k}$ ✓

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6. $\Omega = \{0, 1, \ldots\}$ $\quad P(\{0\}) = \varepsilon$ if $p > 0$

$|A| = \lceil \frac{2}{\varepsilon} \rceil \Rightarrow P(A) \geqslant 2$ X

if $p = 0 \Rightarrow P(\Omega) = 0$ X

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7. $B_n = A_n - \bigcup_{i=1}^{n-1} A_i$

$B_1 = A_1$ $\quad B_2 = A_2 - A_1$ $\quad B_3 = A_3 - (A_1 \cup A_2)$

$w \in B_n \Rightarrow w \in A_n$ and $w \notin \bigcup_{i=1}^{n-1} A_i$

$\Rightarrow w \notin A_{n-1} \Rightarrow w \notin B_{n-1}$ ✓

$P(\bigcup_{n=1}^{\infty} A_n) = P(\bigcup_{n=1}^{\infty} B_n) \Rightarrow P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(B_n)$

$P(B_n) \leqslant P(A_n)$ $\forall n \Rightarrow P(\bigcup_{1}^{\infty} A_n) \leqslant \sum_{1}^{\infty} P(A_n)$

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8. $P(A_i) = 1 \Rightarrow P(A_i^c) = 0$

8. $P(A_i)=1 \Rightarrow P(A_i^c)=0$

$P\left(\left(\bigcup_{n=1}^{\infty} A_i\right)^c\right)=0 \Rightarrow P\left(\bigcap_{n=1}^{\infty} A_i\right)=1$

9. $P(B)>0 \quad P(\cdot|B) \quad \frac{P(\cdot \cap B)}{P(B)} \geq 0$

$\frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)}=1$

$P\left(\bigcup_1^{\infty} A_i|B\right) = \frac{P\left(\bigcup_1^{\infty} A_i B\right)}{P(B)} = \frac{\sum_1^{\infty} P(A_i B)}{P(B)} = \sum_1^{\infty} P(A_i|B)$

10. $\Omega = \{(\omega_1, \omega_2) : \omega_i \in \{1,2,3\}\}$

$\Omega = (1,2)$
$(1,3)$
$(2,3)$
$(3,2)$

$P(\omega_1=2, \omega_2=3) = \underbrace{P(\omega_2=3|\omega_1=2)}_{1} \cdot \underbrace{P(\omega_1=2)}_{\frac{1}{3}}$

$P(\omega_1=1, \omega_2=2) = \underbrace{P(\omega_2=2|\omega_1=1)}_{\frac{1}{2}} \cdot \underbrace{P(\omega_1=1)}_{\frac{1}{3}}$

11. $P(AB)=P(A)P(B)$

$P(A^c)=1-P(A) \quad P(B^c)=1-P(B)$

$P(AB)=(1-P(A^c))(1-P(B^c))$

$1-P(AB)=P((AB)^c)=P(A^c)+P(B^c)-P(A^cB^c)$

$\cancel{P(B^c)}+\cancel{P(A^c)}-P(A^c)P(B^c)=\cancel{P(A^c)}+\cancel{P(B^c)}-P(A^cB^c)$

$P(A^cB^c)=P(A^c)P(B^c)$

12. $\Omega = \{(w, c) : w \in \{RR, GG, RG\}, c \in \{R, G\}\}$

12. $\Omega = \{(W, C) : W \in \{RR, GG, RG\}, C \in \{R, G\}\}$

$$P(W=GG \mid C=G) = \frac{P(W=GG, C=G)}{P(C=G)}$$

$$= \frac{P(C=G \mid W=GG)\,P(GG)}{\frac{1}{2}\times\frac{1}{3} + \frac{1}{3}} = \frac{1 \times \frac{1}{3}}{\frac{1}{2}\times\frac{1}{3}+\frac{1}{3}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

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13. $\Omega = \{(w_1, w_2, \dots, w_n) \mid n \geq 2, w_n = H, w_i = T \; \forall i \in \{1, \dots, n-1\}\}$ $\cup$

HHT $\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 = \frac{1}{4}$

TTH

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14. $P(A) = 0 \Rightarrow P(A \cap \cdot) \subset P(A) = P(A \cap \cdot) \leq 0$

$= P(A \cap \cdot) = 0$

$P(A) = 1 \Rightarrow B = (A \cap B) \cup (A^c \cap B) \Rightarrow$

$P(B) = P(A \cap B) + P(A^c \cap B) \Rightarrow P(A)P(B) = P(A$

$P(A)P(A) = P(A \cap A) = P(A)$

$P(A)^2 = P(A) \Rightarrow P(A) = 1$ or 0

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15. a) $\Omega = \{x \mid 0, 1, 2, 3, 4, 5\}$

$$P(x \geq 2 \mid x \geq 1) = \frac{4}{5}$$

b) $\Omega = \{c_1, c_2, c_3, c_4, c_5 \mid c_i \in \{b, n\}\}$

$$1 - \left(\frac{3}{4}\right)^4 =$$

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16. $P(A)P(B) = P(AB)$

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A$$

$P(AB) = P(A|B)P(B)$ also holds the other way

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17. $P(ABC) = P(A|BC)P(B|C)P(C)$

$$P(ABC) = P(A|BC)P(BC)$$

$$P(BC) = P(B|C)P(C)$$

18. $\bigcup_{i=1}^{k} A_i = \Omega \quad P(B) > 0$

if $P(A_1|B) < P(A_1)$ then $P(A_i|B) > P(A_i)$

$P(A_1 B) < P(A_1)P(B) \quad P(B) = \sum P(A_i B$

$\sum P(B|A_i)P(A_i) = \sum P(A_i|B)P(B)$

$\sum \cancel{P(B)}P(A_i) = \sum P(A_i|B)\cancel{P(B)}$

$\sum P(A_i) = \sum P(A_i|B)$ ✓

Insight: if B increases th
of an event A_i it MUST
the chance of some other

19. $P(M) = 0.3 \quad P(W) = 0.5 \quad P(L) =$

$P(V|M) = 0.65 \quad P(V|W) = 0.82 \quad P(V$

$P(W|V) = \frac{P(WV)}{\ldots} = \frac{0.41}{0.17 + 0.\ldots}$

) for some i

B)

e chance
decrease
event(s) A

$= 0.2$

$(|L) = 0.5$

$+ 0.195$

$$P(W|V) = \frac{\quad}{\sum_{os \in \{w, a, m, e\}} P(V|os)P(os)} = 0.417\ 0.$$

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20. $p_1 = 0 \quad p_2 = \frac{1}{4} \quad p_3 = \frac{1}{2} \quad p_4 = \frac{3}{4} \quad p_5$

a) $$P(C_i|H) = \frac{P(HC_i) = P(H|C_i)P(C_i) \frac{1}{5} \times P_i}{P(H)} = \frac{\quad}{\sum_j^5 P_j \cdot \frac{1}{5}}$$

b) $$P(H_2|H_1) = \frac{P(H_2H_1)}{P(H_1)} = \frac{\sum_j P(C_j H}{\sum_j P(C_j}$$

$$\sum_i^k P(C_i|H) \times P_i \checkmark$$

c) $P(C_i|B_4) \quad B_4 = $ first head on

$$P(C_i|B_4) = \frac{\frac{1}{5} \cdot (1-P_i)^3 P_i}{\sum_j^5 (1-P_j)^3 P_j \cdot \frac{1}{5}}$$

$1 + 0.195$

$= 1$

$H_1 H_2)$

$H_1)$

n toss 4

$$\sum_{\theta} (1-p_\theta)\, p_\theta \cdot \frac{1}{5}$$

Chapter 2

Saturday, February 27, 2021 6:00 PM

Random Variables

Exercises

1 $P(X=x)=F(x^+)-F(x^-)$

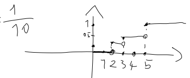
$F(x)=P(X\leqslant x)=\sum f_X(x_i)$

$F(x^+)=\lim_{y\to x, y>x} F(y)=\lim_{y\to x, y>x} P(X\leqslant y)=$

$F(x^+)-F(x^-)=P(X\leqslant x)-P(X<x)=P(X=x)$

2. $P(X=2)=P(X=3)=\frac{1}{10}$

$P(X=5)=\frac{8}{10}$



$P(2<X\leqslant 4.8)=P(X\leqslant 4.8)-P(X<2)$

$=0.2-0=0.2$

$P(2\leqslant X\leqslant 4.8)=0.2-0.1=0.1$

3. a) $P(X=x)=F(x)-F(x^-)$ $F(x^-)=\lim_{y\uparrow x}F(y)$

b) $P(x<X<y)=F(y)-F(x)$

$\int_a^b f_X(x)dx=\int_{-\infty}^b f_X(x)dx-\int_{-\infty}^a f_X(x)dx$

$=F(b)-F(a)$

c) $P(X>x)=1-F(x)$

$P(x<X<\infty)=\int_{-\infty}^{\infty} f_X(x)dx-\int_{-\infty}^{x} f_X(x)dx$

$=1-F(x)$

d) $P(X=x)=0$; continous

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4. $f_X(x)=\begin{cases}\frac{1}{4} & 0<x<1\\ \frac{3}{8} & 3<x<5\\ 0 & \text{o.w}\end{cases}$

a) $F_X(x)=\int_{-\infty}^{x} f_X(t)dt$

$F_X(x)=\begin{cases}\frac{1}{4}x & 0<x<1\\ \frac{1}{4} & 1\leqslant x\leqslant 3\\ \frac{3}{8}(x-3)+\frac{1}{4} & 3<x<5\\ 0 & x<0\\ 1 & 5\leqslant x\end{cases}$

b) $Y=\frac{1}{X}$ $A_y=\{x; r(x)\leqslant y\}=\{x; x\leqslant r^{-1}(y)\}$

REDO

5. $f_{X,Y}(x,y)=f_X(x)f_Y(y)$

$f_X(x)=P(X=x)$

$P(X=x, Y=y)=P(X=x)P(Y=y)$

6. $X\sim F$ $I_A(x)=\begin{cases}1 & x\in A\\ 0 & x\notin A\end{cases}$

$Y=I_A(X)$ $\int_{x\in A} f_X(x)=P(x\in A)$

$f_Y(y)=P(I_A(X)=y)=\begin{cases}1-\int_{x\in A} f_X(x) & y=0\\ \int_{x\in A} f_X(x) & y=1\end{cases}$

$F_Y(y)=\begin{cases}1-\int_{x\in A} f_X(x) & y=0\\ 1 & y=1\end{cases}$

7. $P(x\in X)P(y\in Y)=P(x\in X, y\in Y)$ $Z=\min\{X,Y\}$

$0\leqslant z\leqslant 1$

$f_{X,Y}(x,y)=\begin{cases}1 & 0\leqslant x, y\leqslant 1\\ 0 & \text{o.w}\end{cases}$

$A_{X,Y}=\{(x,y): \min(x,y)\leqslant z\}$

$P(Z>z)=P(\min(x,y)\geqslant z)=P(x\geqslant z, y\geqslant z)$

$=P(x\geqslant z)\cdot P(y\geqslant z)=(1-P(x\leqslant z))(1-P(y\leqslant z))$

$=(1-F_X(x=z))(1-F_Y(y=z))$

$1-P(Z>z)=F_X(x=z)+F_Y(y=z)-F_{X,Y}(x=z, y=z)$

8. $X\sim F$ $X^+=\max\{0,X\}$

$A_y=\{x; \max(0,x)\leqslant y\}$

$P(X<x)=F_X(x)$

$P(\max(x,0)<y)=F_Y(y)$ $y\geqslant 0$

12. $f(x,y)=g(x)h(y)$

$f_X(x)=\int f(x,y)dy=Hg(x)$

$f_Y(y)=\int f(x,y)dx=Gh(y)$

$f(x,y)=\frac{f_X(x)\cdot f_Y(y)}{GH}$

$f(x,y)GH\to$ has to integrate to $1\to GH=1$ ✓

13. $X\sim N(0,1)$ $Y=e^x$ $y\geqslant 0$

$f(x)=\frac{\exp\{-\frac{x^2}{2}\}}{\sqrt{2\pi}}=\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$

$A_y=\{y; r(x)\leqslant y\}\Rightarrow\{y; x\leqslant \ln y\}$

$F_X(x)=\Phi(x)\Rightarrow F_X(\ln y)=\Phi(\ln y)$

$\begin{cases}\frac{1}{y}\cdot\Phi'(\ln y) & y\geqslant 0\\ 0 & \text{o.w}\end{cases}=\begin{cases}\frac{1}{y}z(\ln y) & y\geqslant 0\\ 0 & \text{o.w}\end{cases}$

14. $\{(x,y): x^2+y^2\leqslant 1\}$

$R=\sqrt{X^2+Y^2}$

$A_n=\{(x,y); \sqrt{X^2+Y^2}\leqslant z\}$

$r^2=x^2+y^2$ $\int_0^1 f_R(r)dr=1$

$F_R(r)=\frac{\pi r^2}{\pi 1}=r^2$ $f_R(r)=2r$

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15. $X\sim F_X(x)$ $Y=F(X)$ $0\leqslant y$

$f_X(x)=F'(x)$

$A_y=\{y; F(x)\leqslant y\}$ $x\leqslant F_X^{-1}($

16. $X \sim \text{Poisson}(\lambda)$ $Y \sim \text{Poisson}(\mu)$

$X+Y \sim \text{Poisson}(\mu+\lambda)$

$f(x) = e^{-\lambda} \frac{\lambda^x}{x!}$ $x \geq 0$ Binomial. $f(x) = ($

$$P(X \mid X+Y=n) = \frac{P(X=x, X+Y=n)}{P(X+Y=n)}$$

$$= \frac{P(X=x)\,P(Y=n-x)}{P(X+Y=n)} = \checkmark$$

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17. $f_{X,Y}(x,y) = \begin{cases} c(x+y^2) & 0 \leq x \leq 1 \text{ an} \\ 0 & \text{o.w} \end{cases}$

$$\iint c(x+y^2)\,dx\,dy = \int \frac{c}{2} +$$

$\frac{c}{2} = \frac{2}{3}$ $c = \frac{4}{3}$

$$\frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{c(x+y^2)}{\int_0^1 \frac{4}{3}(x+y^2)\,dx}$$

≤ 1

(y)

$\binom{n}{x} p^{x}(1-p)^{n-x}$

nd $0 \leq y \leq 1$

$y^2 dy = \frac{c}{2} + \frac{1}{3} = 1$

$y = \frac{1}{2}$

$\frac{\frac{4}{3}(x+y^2)}{\frac{2}{3}+\frac{4}{3}y^2} = \frac{x+\frac{1}{4}}{\frac{1}{2}+\frac{1}{4}}$

$P(X<x) = F_X(x)$

$P(\max(X,0) \leq y) = F_X(y) \quad y \geq 0$

9. $X \sim \mathrm{Exp}(\beta) \quad F(x) \text{ and } F^{-1}(q)$

$f_X(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}, \quad x>0$

$F_X(x) = \int_0^x \frac{1}{\beta} e^{-\frac{x}{\beta}} dx = \Big|_0^x - e^{-\frac{x}{\beta}}$

$= -e^{-\frac{x}{\beta}} + 1 = 1 - e^{-\frac{x}{\beta}}$

$F^{-1}(q) = \inf\{x : F(x) \geq q\}$

$1 - e^{-\frac{x}{\beta}} = q \quad e^{-\frac{x}{\beta}} = 1-q$

$-\frac{x}{\beta} = \ln(1-q) \Rightarrow x = \frac{\beta}{\ln(1-q)}$

10. $f_X(x) f_Y(y) = f_{X,Y}(x,y)$ or with sets

$f_G(g(X)=g') f_H(h(Y)=h') = f_G(X=g^{-1}(g')) f_H(Y=h^{-1}(h'))$

$= f_{G,H}(X=g^{-1}(g'), Y=h^{-1}(h')) = f_{G,H}(g(X)=g', h(Y)=h')$

11. a) $f_X(x) = px + (1-p)(1-x)$

$f_{X,Y}$	$Y=0$	$Y=1$
$X=0$	0	$1-p$
$X=1$	p	0

$f_Y(y) = (1-p)y + p(1-y)$

$$f_{X,Y}(x,y) = \begin{cases} 1-p & x=0, y=1 \\ p & x=1, y=0 \end{cases}$$

$= px + p(1-y) + (1-p)y + (1-p)(1-x)$

$f_X(x) f_Y(y) \neq f_{X,Y}(x,y)$

b) $N \sim \mathrm{poisson}(\lambda) \quad f(n) = e^{-\lambda} \frac{\lambda^n}{n!}, n \geq 0$

$f_{X|N}(x|n) = \frac{f_{N,X}(n,x)}{f_X(x)} = \binom{n}{x} p^x (1-p)^{n-x}$

$X + Y = N$

$P(X=x, Y=y) = P(N=x+y, X=x)$

$= e^{-\lambda} \frac{\lambda^n}{n!} \cdot \binom{n}{x} p^x (1-p)^y$

$= e^{-\lambda} \frac{\lambda^{x+y}}{(x+y)!} \cdot \binom{x+y}{x} p^x (1-p)^y$

$= \underbrace{\left(e^{-\lambda} \frac{\lambda^x}{x!} p^x\right)}_{g(x)} \underbrace{\left(\frac{\lambda^y}{y!} (1-p)^y\right)}_{h(y)}$

$A_y = \{y : F(x) \leq y\}$

$x \geq 1$, X, $F(x$

$F_X(F_X^{-1}(y)) = y$

$F_Y(y) = \Pr(Y \leq y)$

$Y \sim \mathrm{Uniform}(0,1)$

$U \sim \mathrm{Uniform}(0,1) \quad X = F^{-1}(U$

$A_x = \{x ; F^{-1}(u) \leq x\} = \{x; u$

$X \sim F(X)$

$f(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}, \quad x > 0$

$F_X(x) = \int_0^x \frac{1}{\beta} e^{-\frac{x}{\beta}}$ x

$= 1 - e^{-\frac{x}{\beta}} = F_X(x)$

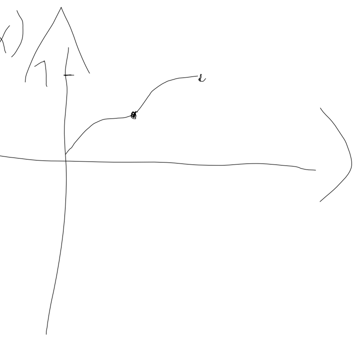
$F_X^{-1}(x) = -\beta \ln(1-x)$

$U \sim \mathrm{Uniform}(0,1)$

$R = -\beta \ln(1-U)$

$R \sim F_X(x)$

(0)



$\leq F(x)\}$

$= -e^{-\frac{x}{\beta}} \Big|_0^x$

$f_Y(y) \quad \int_0 \frac{1}{3} (\ldots)$

$= \frac{4x+1}{3} \quad \int \frac{4x+1}{3}\, dx =$

18. $X \sim N(3, 16)$

$P(X < 7) = P\left(Z < \frac{7-3}{16}\right)$

$P(X > -2) = 1 - P(X < -2) =$

$P(X > x) = 0.05 \Rightarrow P(X < x)$

$P\left(Z < \frac{x-3}{16}\right) = 0.95$

$P(0 \leq X < 4) = P\left(\frac{-3}{16} \leq Z < \frac{1}{16}\right)$

$P(|X| > |x|) = 0.05$

$P(|X| < |x|) = 0.95 \quad P(-x <$

19. $Y = r(x)$ r is strictly mo

$s = r^{-1} \quad f_Y(y) = f(s(y)) \left|\frac{ds(y)}{dy}\right|$

$$\frac{2x^2}{3}+\frac{x}{3}\Big|_0^{\frac{1}{2}}$$ ✓

$$1-P\left(Z<\frac{-2-3}{16}\right)$$

$$=0.95$$

$$)=P\left(Z<\frac{1}{16}\right)-P\left(Z<\frac{-3}{16}\right)$$

$$X<x)=0.95$$

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notone increasing

$x!$

$g(x) \qquad h(y)$

$f(x,y) = g(x)\,h(y) \Rightarrow X, Y$ independent $\quad R \sim F_X(x)$

$s = r^{-1} \quad f_Y(y) = f_X(s(y)) \left| \frac{d\,s(y)}{dy} \right|$

$A_y = \{y : r(x) < y\} = \{y : x <$

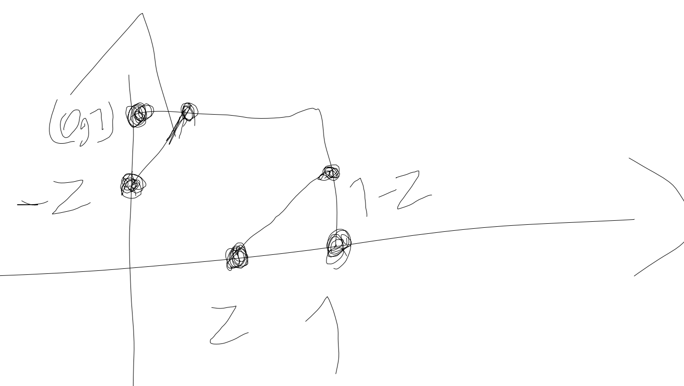
flip in decreasing

$F_Y(y) = F_X(s(y)) \quad f_Y(y) = f_X(s(y)$

20. $X, Y \sim \text{Uniform}(0, 1)$

$Z = X - Y$

$A_Z = \{z : r(X, Y) < z\} = \{z : X - Y$



b) $\frac{X}{Y}$

monotone increasing

$s(y)\}$

$)) \frac{d(s(y))}{dy}$

$$f_{X,Y}(x,y) = \begin{cases} 1 & 0 \leq x, y \leq 1 \\ 0 & o.w \end{cases}$$

$\leq z\}$

$1-z)^2 \quad 0 \leq z \leq 1$

$-1 \leq z < 0$

Y

21. $X_1, \ldots, X_n \sim \mathrm{Exp}$

$Y = \max\{X_1, \ldots$

$A_y = \{y : \max\{X_1, \ldots, X_n\} < y\}$

$\prod_{i=1}^{n} F_{X_i}(y) = F_Y(y) = (\,)^n$

$$\overline{}$$
$$P(B)$$

$$\left.\vphantom{\int}gX_n\right\}$$

chapter 3

Wednesday, March 17, 2021 8:38 PM

1- $E(X)=\sum f_X(x)x$

X_i = money after i trials

$x_0=c$

$P(x_{i+1}|x_i)=\begin{cases}\frac{1}{2} & x_{i+1}=\frac{x}{2}\\ \frac{1}{2} & x_{i+1}=2x\end{cases}$

$E(X_{i+1}|X_i=x)=\frac{x}{4}+x=\frac{5}{4}x$

$E(E(X_{i+1}|X_i))=E(\frac{5}{4}X_i)$

$E(X_{i+1})=E(\frac{5}{4})X_i$

$E(X_n)=(\frac{5}{4})^n c$

$c2^{x_1}\cdot2^{x_2}\cdot2^{x_3}\cdots2^{x_n}=X$

$E(X)=cE[2^{x_1}]E[2^{x_2}]\cdots E[2^{x_n}]=c(\frac{5}{4})^n$

$x_i=\begin{cases}1 & \frac{1}{2}\\ -1 & \frac{1}{2}\end{cases}$

$E[2^{x_i}]=2^1\cdot\frac{1}{2}+2^{-1}\cdot\frac{1}{2}=\frac{5}{4}$

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2. $V(X)=E(X-\mu)^2=0$

$\sum(x-\mu)^2f(x)=0$

$P(X=c)=1\Rightarrow f_X(x)=\begin{cases}1 & x=c\\ 0 & \text{o.w}\end{cases}\Rightarrow E(X)=c$

$E(x-\mu)^2=E(X^2)-E(X)^2=c^2-c^2=0$

---

3. $X_1,\dots,X_n\sim \text{Uniform}(0,1)$ $E[X_i]=\frac{1}{2}$

$Y_n=\max\{X_1,\dots,X_n\}$

$F_{X_i}(x_i)=\int_0^{x} f(x)dx=x_i$ $\quad 0\leq y\leq 1$

$A_n=\{y;\max\{x_1,\dots,x_n\}\leq y\}$

$F_Y(y)=\prod_1^n F_{X_i}(y)=\prod_1^n y=y^n$

$\int y\cdot dF_Y(y)=\int_0^1 ny^n=\frac{n}{n+1}$

4.

X_n = position after n jumps

$-n\leq X_n\leq n$

Y_i = position change $Y=\begin{cases}+1 & 1-p\\ -1 & p\end{cases}$

$X_n=\sum_1^n Y_i\Rightarrow E(X_n)=n-2pn$

$E(Y_i)=1-p-p=1-2p$

$E(Y_i-\mu)^2=\cancel{1-p+p}\ \cancel{-1}-4p^2+4p$

$=4p(1-p)$

$V(X_n)=4np(1-p)$

---

5. X = index of first head

$f(X=x)=(\frac{1}{2})^{x-1}\cdot(\frac{1}{2})=\frac{1}{2^x}$ Geometric distribution $E(X)=\frac{1}{p}$

$E(X)=\sum_1^\infty x\cdot\frac{1}{2^x}=\sum_1^\infty\frac{x}{2^x}$

---

6. $E(Y)=E(r(X))=\sum r(x)f_X(x)$

$Y=r(X)$

$E(Y)=\sum y\,P(Y=y)=\sum r(x)P(X=r^{-1}(y))$

$P(Y=y)=P(X=r^{-1}(y))$

---

7. $P(X>0)=1\Rightarrow P(X\leq 0)=0$ $F(0)=0$

$E(X)=\int_0^\infty P(X>x)dx=\int_0^\infty 1-F_X(x)dx$

$\lim_{x\to\infty} x[1-F(x)]=0$

$E(X)=\int_0^\infty x\,dF_X(x)=\left|xF_X(x)\right|_0^\infty-\int_0^\infty F_X(x)dx$

$=x-\int_0^\infty F_X(x)dx$

better approach: $=\int_0^\infty(1-F_X(x))dx$

$\int_0^\infty F_X(x)dx \dots$

13. $Y=\begin{cases}1 & \frac{1}{2}\\ 0 & \frac{1}{2}\end{cases}$

$E(X|Y=0)=\frac{1}{2}$ $\quad E(E(X|Y))=E(X)=\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{7}{2}=2$

$E(X|Y=1)=\frac{7}{2}$

$f_{XY}(x,y)=\begin{cases}\frac{1}{2}(1) & 0\leq x\leq 1 & y=0\\ \frac{1}{2}(1) & 3\leq x\leq 4 & y=1\end{cases}$ $\quad f_{X|Y}(x|y)=\begin{cases}1 & 0\leq x\leq 1 & y=0\\ 1 & 3\leq x\leq 4 & y=1\end{cases}$

$\int_0^1\frac{1}{2}x\,dx+\int_3^4\frac{1}{2}x\,dx=\frac{1}{4}+\frac{16}{4}-\frac{9}{4}=2$

$V(X)=EV(X|Y)+VE(X|Y)$

$X=(Y-1)\text{Uniform}(0,1)+Y\text{Uniform}(3,4)$

$\int_0^1 x^2\frac{1}{2}dx+\int_3^4 x^2\frac{1}{2}dx=\frac{1}{6}+\frac{4^3}{6}-\frac{3^3}{6}$

$\frac{64-27+1}{6}=\frac{38}{6}=\frac{19}{3}=E(X^2)$

$E(X)=2$ $\quad E(X^2)-E(X)^2=\frac{19}{3}-\frac{12}{3}=\frac{7}{3}$

$V(X|Y=0)=\frac{1}{12}$ $\quad EV(X|Y)=\frac{1}{12}$

$V(X|Y=1)=\frac{1}{12}$

$\frac{1}{2}\cdot(\frac{7}{2}-2)^2+\frac{1}{2}\cdot(\frac{1}{2}-2)^2=(\frac{3}{2})^2=\frac{9}{4}$

$\frac{9}{4}+\frac{1}{12}=\frac{28}{12}=\frac{7}{3}$

---

14. $\sum_{i=1}^m\sum_{j=1}^n a_ib_j\,\text{Cov}(X_i,Y_j)$

$=E\left(\sum_1^m a_iX_i\cdot\sum_j^n b_jY_j\right)-E\left(\sum_1^m a_iX_i\right)E\left(\sum_j^n b_jY_j\right)$

$=E\left(\sum_1^m\sum_{j}^n a_ib_jX_iY_j\right)-E\left(\sum_1^m a_iX_i\right)E\left(\sum_j^n b_jY_j\right)$

$=\sum_1^m\sum_1^n E(a_ib_jX_iY_j)-\sum_1^m E(a_iX_i)\cdot\sum_1^n E(b_jY_j)$

$=\sum_1^m\sum_1^n E(a_ib_jX_iY_j)-\sum_1^m\sum_1^n E(a_iX_i)E(b_jY_j)$

$=\sum_1^m\sum_1^n a_ib_j\cdot\text{Cov}(X_i,Y_j)$

---

15. $f_{X,Y}(x,y)=\begin{cases}\frac{1}{3}(x+y) & 0\leq x\leq 1,\ 0\leq y\leq 2\\ 0 & \text{o.w}\end{cases}$

$V(2X-3Y+8)$

$= x - \int_0 \dots$

better approach: $= \int_0^\infty (1 - F_X(x))dx$

$\int_0^\infty 1 - F_X(x)dx = \underbrace{(1-F_X(x)\,x)\Big|_0^\infty}_{0} + \underbrace{\int_0^\infty x\,dF_X(x)}_{E(X)}$

---

8. $X_1,\dots,X_n$ IID $\mu = E(X_i)$ $\sigma^2 = V(X_i)$

$E(\bar{X}_n) = \mu$, $V(\bar{X}_n) = \frac{\sigma^2}{n}$ $E(S_n^2) = \sigma^2$

$\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ $E(X_i^2) - E(X_i)^2 = \sigma^2$

a) $E(\bar{X}_n) = E\left(\frac{1}{n}\sum_{i=1}^n X_i\right) = \frac{1}{n}(n\mu) = \mu$

b) $V(\bar{X}_n) = V\left(\frac{1}{n}\sum_{i=1}^n X_i\right) = \frac{1}{n^2}V\left(\sum_1^n X_i\right) =$

$\frac{1}{n^2}\sum V(X_i) = \frac{nV(X_i)}{n^2} = \frac{\sigma^2}{n}$

c) $E(S_n^2) = E\left(\frac{1}{n-1}\sum_1^n (X_i - \bar{X}_n)^2\right)$

$\frac{1}{n-1}\sum_1^n (E(X_i - \bar{X}_n)^2) =$

---

10. $X \sim N(0,1)$ $E(X) = 0$ $V(X) = 1$

$Y = e^x$

$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ $A_y = \{y \mid e^x < y$

$F_Y(y) = \Phi(\ln y)$ $f_Y(y) = \frac{1}{y} \cdot N(\ln y)$

$E(Y) = \int y \cdot \frac{1}{y} \cdot N(\ln y) = \int_0^\infty N(\ln y)\,dy =$

$\frac{1}{\sqrt{2\pi}}\int e^{-\frac{1}{2}(\ln y)^2}$ or $\int_{-\infty}^\infty e^x f(x)dx$, $x = x-1$

---

11. from question 4

a) $E(X_n) = 0$ $V(X_n) = n$

b)

---

12. Bernoulli $\to f_X(x) = \begin{cases} 1-p & 0 \\ p & 1 \end{cases}$

$E(X) = p \cdot 1 + (1-p)0 = p$

$V(X) = E(X-p)^2 = (1-p)(0-p)^2 + p(1-p)^2$

$= \cancel{p^2} - \cancel{p^3} + \cancel{p^2} + \cancel{p^3} - \cancel{2p^2} = 0$

Poisson $\to f_X(x) = e^{-\lambda}\frac{\lambda^x}{x!}$

$\sum_1^\infty x \cdot e^{-\lambda}\frac{\lambda^x}{x!} = e^{-\lambda}\sum_1^\infty \lambda \frac{\lambda^{x-1}}{(x-1)!} = \lambda e^{-\lambda}e^{\lambda} = \lambda$

$\sum_1^\infty x(x-1)\cdot e^{-\lambda}\frac{\lambda^x}{x!} = e^{-\lambda}\sum_1^\infty \frac{\lambda^x}{(x-2)!}$

$= \lambda^2 = E(X^2 - X) \Rightarrow E(X^2) = \lambda^2 + \lambda$

$E(X^2) - E(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

Gamma $\to f_X(x) = \frac{1}{\beta^\alpha\Gamma(\alpha)} x^{\alpha-1}e^{-x/\beta}$ $x>0$

$\int \frac{1}{\beta^\alpha\Gamma(\alpha)} x^\alpha e^{-x/\beta}dx = \frac{1}{\beta^\alpha\Gamma(\alpha)}\int x^\alpha e^{-x/\beta}dx$

$\frac{1}{\beta^\alpha\Gamma(\alpha)} \cdot \frac{\beta^{\alpha+1}}{\Gamma(\alpha+1)} \cdot \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}}\int x^\alpha e^{-\frac{x}{\beta}}dx$

$= \frac{\beta^{\alpha+1}\Gamma(\alpha+1)}{\beta^\alpha\Gamma(\alpha)}\underbrace{\int \frac{1}{\beta^{\alpha+1}\Gamma(\alpha+1)} x^\alpha e^{-\frac{x}{\beta}}dx}_{1} = \beta \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = \alpha\beta$

$E(X^2) = \int \frac{1}{\alpha}\, x^{\alpha+1}e^{-\frac{x}{\beta}}dx = \frac{1}{\alpha} \cdot \frac{\beta^{\alpha+2}}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+2)}{\beta^{\alpha+2}}\int x^{\alpha+1}e^{-\frac{x}{\beta}}dx$

$V(2X - 3Y + 8)$

$f_X(x) = \int_0^2 \frac{1}{3}(x+y)dy = \Big|_0^2 \frac{1}{3}xy + \frac{y^2}{6}$

$= \frac{2x}{3} + \frac{4}{6}$ $E(X) = \int_0^1 \frac{2x^2}{3} + \frac{4x}{6}dx$

$= \Big|_0^1 \frac{2x^3}{9} + \frac{4x^2}{12}$

$f_Y(y) = \frac{y}{3} + \frac{1}{6}$ $= \frac{2}{9} + \frac{1}{3} = \frac{5}{9}$

$E(Y) = \int_0^2 \frac{y^2}{3} + \frac{y}{6}dy$

$= \Big|_0^2 \frac{y^3}{9} + \frac{y^2}{12} = \frac{8}{9} + \frac{1}{3} = \frac{11}{9}$

OR $E(r(x,y)) = \iint r(x,y)f(x,y)\,dx\,dy$

---

16. $E(r(X)s(Y) \mid X) = r(X)E(s(Y) \mid X)$

$= \int r(x)s(y)f_{Y|X}(y|x)dy$

$= r(x)\int s(y)f_{Y|X}(y|x)dy = r(x)E(s(Y)|X)$

$E(r(X)|X) = \int r(x)f_{X|X}(x|x)dx$?

$s(y) = 1$ is the case we want

---

17. $V(Y) = EV(Y|X) + VE(X|Y)$

$m = E(Y)$ $b(x) = E(Y|X=x)$ $E(b(x)) = m$

---

18. $E(X|Y=y) = c \to E(X) = c$

$E[E[X|Y]] = E[X] = c$

$E(XY) = EE[XY|Y] = E(Y E[X|Y]) = E(Y$

---

20. X vector μ Σ

$$E(X^2)=\int \frac{1}{\beta^\alpha\Gamma(\alpha)} x^{\alpha+1} e^{-\frac{x}{\beta}}dx=\frac{1}{\beta^\alpha\Gamma(\alpha)}\cdot\frac{\beta^{\alpha+2}}{\Gamma(\alpha+2)}\cdot\frac{\Gamma(\alpha+2)}{\beta^{\alpha+2}}\int x^{\alpha+1}e^{-\frac{x}{\beta}}dx$$

$$=\frac{\beta^{\alpha+2}\cdot\Gamma(\alpha+2)}{\beta^\alpha\cdot\Gamma(\alpha)}\underbrace{\int\frac{1}{\beta^{\alpha+2}\Gamma(\alpha+2)}x^{\alpha+1}e^{-\frac{x}{\beta}}dx}_{1}=\alpha^2\beta^2+\alpha\beta^2$$

$\Gamma(\alpha+2)=(\alpha+1)\Gamma(\alpha+1)=\alpha(\alpha+1)\Gamma(\alpha)$ Very similar to above

$E(X^2)-E(X)^2=\cancel{\alpha^2\beta^2}+\alpha\beta^2-\cancel{\alpha^2\beta^2}=\alpha\beta^2$

Beta $\rightarrow \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$

20. X vector μ Σ

$E(a^TX)=E(a_1X_1+a_2X_2+\cdots+a_nX_n)=a_1E(X_1)+$

$=a^TE(X)$

$V(a^TX)=V(a_1X_1+a_2X_2+\cdots+a_nX_n)=\sum_i a_i^2V(X_i$

$\Sigma=\begin{bmatrix} V(X_1) & COV(X_1,X_2) & \cdots \\ \vdots & V(X_2) & \\ \vdots & \vdots & \ddots \end{bmatrix}$ $=\left[\sum a_i COV(X_i,X_1),\ldots\right.$

a^T

21. $E(Y|X)=X\rightarrow COV(X,Y)=V($

$E(Y)=E(X)$ $COV(X,Y)=E(XY)-$

$E(XY)=EE(XY|X)=EXE(Y|X)=E(X^2$

22. $X\sim \text{Uniform}(0,1)$ $0<a<b<1$

$Y=\begin{cases}1 & 0<x<b\\ 0 & \text{o.w}\end{cases}$ $Z=\begin{cases}1 & a<x<1\\ 0 & \text{o.w}\end{cases}$

$P(X<b)=\int_0^b dx=b$ $f_Y(y)=\begin{cases}1\\ 0 & 1\end{cases}$

$P(X>a)=1-P(X<a)=1-a$ $f_Z(z)=\begin{cases}1\\ 0\end{cases}$

$Pr[Z=1|Y=1]=\frac{Pr(Z=1,Y=1)}{Pr(Y=1)}=$

$Pr[Z=1]=b$ $Pr[Z=1|Y=1]\neq Pr[Z=1]$

b) $E(Y|Z)=\int\frac{b-a}{1-a}(1-a)$ $z=1$

$-a_2E(X_2)+\cdots$

$)+2\sum\sum a_i a_j \mathrm{Cov}(X_i, X_j)$

$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \end{bmatrix}$

$X)$

$E(X)E(Y)$

$)$

b

$-b$

$1-a$

a

$b-a$

b

$)$

b) $E(Y|Z) = \begin{cases} \frac{b-a}{1-a}(1-a) & z=1 \\ a & z=0 \end{cases}$

23. $\psi_X(t) = E(e^{tX}) = \int e^{tx} dF(x)$

$f(x) = e^{-\lambda}\frac{\lambda^x}{x!}$ $\sum_0^\infty e^{tx} \cdot e^{-\lambda}\frac{\lambda^x}{x!} = e^{-\lambda}$

$= e^{-\lambda}\sum_0^\infty \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t-1)}$

$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$

$\frac{1}{\sigma\sqrt{2\pi}}\int \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2 + tx\right\} =$

$\frac{1}{\sigma\sqrt{2\pi}}\int$ Late

24. $X_1 \dots X_n \sim Exp(\beta) = Gamma$

$f_{X_i}(x_i) = \frac{1}{\beta} e^{-\frac{x}{\beta}}$ $\int e^{xt} \frac{1}{\beta} e^{-\frac{x}{\beta}}$

$= \frac{\beta}{\beta(t\beta - 1)} \int_0^\infty e^{x(t-\frac{1}{\beta})} =$

$$\sum_{0}^{\infty} e^{tx} \cdot \frac{\lambda^{x}}{x!}$$

$$(1, \beta)$$

$$= \frac{1}{\beta} \left(e^{x\left(t - \frac{1}{\beta}\right)} \right)$$

chapter 4

Monday, April 12, 2021 6:47 PM

1. $X \sim \text{Exponential}(\beta) \to \frac{1}{\beta} e^{\frac{-x}{\beta}}$ $F(x) = 1 - e^{\frac{-x}{\beta}}$

$P(|X - \mu_X| \geq k\sigma_X)$ for $k > 1$

$1 - P(|X - \mu_X| \leqslant k\sigma_X) = 1 - \left(1 - e^{\frac{-k\sigma_X - \mu_X}{\beta}} - 1 + e^{\frac{k\sigma_X - \mu_X}{\beta}}\right)$

$= 1 - e^{\frac{k\sigma_X - \mu_X}{\beta}} + e^{\frac{-k\sigma_X - \mu_X}{\beta}} = 1 - e^{k-1} + e^{-k-1}$

$\leqslant 1 - 1 + \frac{1}{e^2} \leqslant \frac{1}{e^2}$

chebyshev $\to P(|X - \mu_X| \geq k\sigma_X) \leqslant \frac{1}{k^2}$

---

2. $X \sim \text{Poisson}(\lambda)$

$P(X \geq 2\lambda) = P(X - \lambda \geq \lambda) \leqslant \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$

---

3. $X_1, \ldots, X_n \sim \text{Bernoulli}(p)$ $\bar{X}_n = n^{-1} \sum X_i$

$P(|\bar{X}_n - p| > \varepsilon)$ $E(\bar{X}_n) = E(X_i) = p$

$V(\bar{X}_n) = \frac{V(X_i)}{n} = \frac{p(1-p)}{n}$

Chebyshev $\to P(|\bar{X}_n - p| > \varepsilon) \leqslant \frac{\frac{p(1-p)}{n}}{\varepsilon^2} = \frac{p(1-p)}{n\varepsilon^2} = \frac{1}{n} \cdot \frac{p(1-p)}{\varepsilon^2}$

Hoeffding $\to P(|\bar{X}_n - p| > \varepsilon) \leqslant 2e^{-2n\varepsilon^2} = 2 \cdot \left(\frac{1}{e^{2\varepsilon^2}}\right)^n$

---

4. $X_1, \ldots, X_n \sim \text{Bernoulli}(p)$

(a) $\alpha > 0$ $\varepsilon_n = \sqrt{\frac{1}{2n} \log\left(\frac{2}{\alpha}\right)}$

$\hat{p}_n = n^{-1} \sum_{1}^{n} X_i$ $C_n = (\hat{p}_n - \varepsilon_n, \hat{p}_n + \varepsilon_n)$

$P(C_n \text{ contains } p) \geq 1 - \alpha \to P(\hat{p}_n - \varepsilon_n \leqslant p \leqslant \hat{p}_n + \varepsilon_n)$

$P(|\hat{p}_n - p| > \varepsilon) = 1 - P(|\hat{p}_n - p| < \varepsilon)$

$= 1 - P(C_n \text{ contains } p) \leqslant \alpha$

$\Rightarrow P(C_n \text{ contains } p) \geq 1 - \alpha$

(b)

5. $Z \sim N(0,1)$ $P(|Z|>t) \leq \sqrt{\frac{2}{\pi}} \frac{e^{-t^2/2}}{t}$

$$tP(Z>t) = t\int_t^{\infty} f(x)\,dx = \int_t^{\infty} t f(x)\,dx \leq \int_t^{\infty} x f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_t^{\infty} x e^{-\frac{x^2}{2}}\,dx = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$$

$$P(Z>t) \leq \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{t^2}{2}}}{t} \Rightarrow$$

$$2P(Z>t) = P(|Z|>t) \leq \frac{\sqrt{2}}{\sqrt{\pi}} \frac{e^{-\frac{t^2}{2}}}{t}$$

7. $X_1 \ldots X_n \sim N(0,1)$ $P(|\overline{X}_n|>t)$

$\overline{X}_n = n^{-1}\sum_1^n X_i$ $E(\overline{X}_n) = 0$ $V(\overline{X}_n) = \frac{1}{n}$

chebyshev $P(|\overline{X}_n| \geq t) \leq \frac{1}{nt^2}$

Chapter 5

Wednesday, April 21, 2021 6:47 PM

1. $X_1, \dots, X_n$ I.I.D $\mu = E(X_1)$ $\sigma^2 = V(X_1) = E(X_1 - \mu)^2 = E(X^2) - E(X)^2$

a) $E(S_n^2) = \sigma^2$ $S_n^2 = \frac{1}{n-1} \sum_1^n (X_i - \bar{X}_n)^2$

b) $S_n^2 \xrightarrow{P} \sigma^2$

$P(|S_n^2 - \sigma^2| > \varepsilon) \to 0$

if $X_n \rightsquigarrow X$ and $Y_n \rightsquigarrow c$ then $X_n Y_n \rightsquigarrow cX$

WLLN: if $X_1, \dots, X_n$ IID then $\bar{X}_n \xrightarrow{P} \mu$

$E(X_i^2) = \sigma^2 + \mu^2$

$\frac{1}{n-1} \sum_1^n (X_i - \bar{X}_n)^2 = \frac{1}{n-1} \sum_1^n (X_i^2 + \bar{X}_n^2 - 2X_i\bar{X}_n) = \frac{1}{n-1} \sum_1^n X_i^2 + \frac{2\sum_1^n X_i \bar{X}_n}{n-1} + \frac{n}{n-1} \bar{X}_n^2$

$\xrightarrow{\times \frac{n}{n}} \frac{n}{n-1} \frac{1}{n} \sum_1^n X_i^2 - \frac{n}{n-1} \frac{2}{n} \sum X_i \bar{X}_n + \frac{n}{n-1} \bar{X}_n^2$

$\to \underbrace{\frac{n}{n-1}}_{c_n} n^{-1} \sum_1^n X_i^2 - \underbrace{\frac{n}{n-1}}_{d_n} \bar{X}_n^2$

$S_n^2 = c_n n^{-1} \sum_{i=1}^n X_i^2 - d_n \bar{X}_n^2$ $S_n^2 \to \sigma^2 + \cancel{\mu^2} - \cancel{\mu^2}$

$n^{-1} \sum_{i=1}^n X_i^2 \to \sigma^2 + \mu^2$ $\bar{X}_n \to \mu$

---

2. $X_n \xrightarrow{qm} b$ iff $\lim_{n\to\infty} E(X_n) = b$ and $\lim_{n\to\infty} V(X_n) = 0$

$X_n \xrightarrow{qm} b = E(X_n - b)^2 \to 0$ $E[X_n^2] = V[X_n] + (E[X_n])^2$

$E(X_n - b)^2 = E[X_n^2 - 2X_n b + b^2]$

$= E[X_n^2] - 2E[X_n]b + b^2 = V[X_n] + (E[X_n])^2 - 2E(X_n)b + b^2$

$V[X_n] + (E[X_n])^2 - 2E(X_n)b + b^2 \to 0$

$V[X_n] + (E[X_n] - b)^2 \to 0$

both non-negative

$\lim_{n\to\infty} V[X_n] = 0$ $\lim_{n\to\infty} E[X] - b = 0$

$\lim_{n\to\infty} E(X_n) = b$ and $\lim_{n\to\infty} V(X_n) = 0 \Rightarrow$

$V[X_n] + (E[X_n])^2 - 2E(X_n)b + b^2 \to 0$

---

3. $X_1, \dots, X_n$ I.I.D $\mu = E(X_1)$ $\sigma^2 = V(X_1) < \infty$

$\bar{X}_n \xrightarrow{qm} \mu$ $E(\bar{X}_n - \mu)^2 \to 0$ $\bar{X}_n = \frac{1}{n} \sum X_i$

$E(\bar{X}_n - \mu)^2 = E\left[\left(\frac{1}{n} \sum (X_i - \mu)\right)^2\right] = \frac{1}{n^2} E\left[\left(\sum (X_i - \mu)\right)^2\right]$

$= \frac{1}{n^2}\left[V\left(\sum (X_i - \mu)\right) + E\left(\sum (X_i - \mu)\right)^2\right]$

$= \frac{1}{n^2}\left[n V(X_i) + 0\right] = \frac{1}{n} V(X_i) \to 0$

---

4. $X_1, \dots, X_n$ $P(X_n = \frac{1}{n}) = 1 - \frac{1}{n}$ and $P(X_n = n) = 1$

7. $\lambda_n = \frac{1}{n}$ $X_n \sim \text{Poisson}(\lambda_n)$

a) $X_n \xrightarrow{P} 0$ $P(|X_n - 0| > 3) \longrightarrow 0$

$f(x) = e^{-\lambda} \frac{\lambda^x}{x!}$ $x \geq 0$

$P(X_n > 3) \leq \frac{E(X_n)}{t} \leq \frac{1}{nt}$ ✓

$E(X_n) = \frac{1}{n}$

b) $P(|Y_n^2| \cancel{\geq} t) \leq \frac{E(Y_n^2)}{t}$ $Y_n = n X_n$

$E(X\ ) = \frac{1}{n}$ $V(X_n) = \frac{1}{n}$

$E(Y_n) = 1$ $V(Y_n) = n = \sigma^2$

Theorem 5.5

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8. X_i $E(X_i) = 1$ $Y = \sum_1^n X_i$ $n = 100$

$V(X_i) = 1$ $E(Y) = 100$

4. $X_1, \dots, X_n$ $P(X_n = \frac{1}{n}) = 1 - \frac{1}{n^2}$ and $P(X_n = n) = \frac{1}{n^2}$

$X_n \xrightarrow{P} X$: $P(|X_n| > \varepsilon) \to 0$ ✓

$$E(X_n) = \frac{1}{n} \cdot \left(1 - \frac{1}{n^2}\right) + n \cdot \frac{1}{n^2} = \frac{1}{n} - \frac{1}{n^3} + \frac{1}{n}$$

$$= \frac{2}{n} - \frac{1}{n^3} \quad P(|X_n| > \varepsilon) \leq \left(\frac{2}{n} - \frac{1}{n^3}\right)\frac{1}{\varepsilon}$$

$X_n \xrightarrow{qm} X$: $E(X_n - X)^2 \to 0$ $\quad E(X_n) = \frac{2}{n} - \frac{1}{n^3}$

$$= V(X_n - X) + (E(X_n - X))^2 \quad (E(X_n))^2 = \frac{4}{n^2} + \frac{1}{n^6} - \frac{4}{n^4}$$

$$E(X_n^2) = \frac{1}{n^2} \cdot \left(1 - \frac{1}{n^2}\right) + n^2 \cdot \frac{1}{n^2} = 1 + \frac{1}{n^2} - \frac{1}{n^4}$$

$$\to V(X_n) - V(X) + (E(X_n))^2 + (E(X))^2 - 2E(X_n)E(X)$$

5. $X_1, \dots, X_n \sim \text{Bernoulli}(p) \to f_X(x) = \begin{cases} 1-p & x = 0 \\ p & x = 1 \end{cases}$

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{P} p \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{qm} p$$

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i^2 - p\right| > \varepsilon\right) \to 0$$

$$E\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - p\right) = 0$$

Same techniques as above

$V(X_i) = 1$ $\qquad E(Y) = 100$

$P(Y < 90) = P\left(\frac{Y}{100} < 0{,}9\right)$ $\qquad V(Y) = 100$

$E\left(\frac{Y}{100}\right) = 1$ $\qquad P(Z < (0{,}9-1) \times 100) = P(Z < -10)$

$V\left(\frac{Y}{100}\right) = \frac{1}{100}$

---

9. $P(X=1) = P(X=-1) = \frac{1}{2}$ $\quad X_n = \begin{cases} X & 1- \\ e^n & \end{cases}$

$P(|X_n - X| > \varepsilon) \to 0$ $\quad E(X_n) = \frac{1}{n} e^n + \left(1-\frac{1}{n}\right) X$

$P(|X_n - X| > \varepsilon) \leq \frac{E(X_n - X)}{t} = \frac{\frac{1}{n}e^n}{t}$ $\quad \times$?

$F(X) = P(X \leq x)$

$F_{X_n}(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2}\left(1-\frac{1}{n}\right) & -1 \leq x < 1 \\ 1-\frac{1}{n} & 1 \leq x < e^n \\ 1 & e^n \leq x \end{cases}$ $\qquad F_X(x) = \begin{cases} 0 \\ \frac{1}{2} \\ 1 \end{cases}$

$$- \frac{1}{n}$$

$$\frac{1}{n}$$

$$\times \frac{1}{2} \times 1 + \left(1 - \frac{1}{n}\right) \times \frac{1}{2} \times -1$$

$$n < -1$$

$$-1 < n < 1$$

$$1 < n$$

6. $E(X_i)=68 \quad V(X_i)=2.6^2$ √

$X_1, \ldots, X_{100}$ IID C.L.T $\overline{X_n} \approx N(\mu, \frac{\sigma^2}{n})$

$P(\overline{X_n} \geq 68) = 1 - P(\overline{X_n} \leq 68)$

$= 1 - P\left(\frac{\overline{X_n} - \mu}{\frac{\sigma^2}{n}} \leq \frac{68-\mu}{\frac{\sigma^2}{n}}\right) = 1 - P(Z \leq 0)$

10. $Z \sim N(0,1) \quad P(|Z| > t) \leq \frac{E|Z|^k}{t^k}$

$P(|Z| > t) \leq \frac{E|Z|}{t}$ trivial

markov's inequality

mill's inequality $P(|Z| > t) \leq \sqrt{\frac{2}{\pi}} \frac{e^{-t^2/2}}{t}$

$E|Z|^k = \int_{-\infty}^{\infty} |z|^{k+1} \left(\frac{1}{\sqrt{2\pi}} e^{-z^2/2}\right) dz = \int_0^{\infty} z^{k+1} ($

11. $X_n \sim N(0, \frac{1}{n})$ $F($

$$F_{x_n}(x) \xrightarrow{n\to\infty} \begin{cases} 0 & x < -1 \\ \frac{1}{2} & -1 \leq x < 1 \\ 1 & 1 \leq x \end{cases} \quad F_{X_n}(x) \to F$$

skipped

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \Big) + \int_{-\infty}^{0} (-z)^{k+1} \Big(\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\Big)$$

$$(X) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

$F_X(x)$

$P(|X_n - X| > \varepsilon)$ $f($

$\sqrt{n}\, X_n \sim N(0,1)$

$F_n(t) = P(X_n < t) = P(\sqrt{n}\, X_n <$

$t > 0:\ P(Z < \sqrt{n}\, t) \longrightarrow 1$

$t < 0:\ P(Z < \sqrt{n}\, t) \longrightarrow 0$

$P(X = 0) = 1 \implies X_n \xrightarrow{P} X$

---

12. $X, X_1, \ldots$ $\quad X_n \rightsquigarrow X$ iff

$\lim_{n \to \infty} F_n(t) = F(t) \implies \lim_{n \to \infty} f_n(t) =$

$$1 \quad x > 0$$

$$(x) = \begin{cases} 0 & x < 0 \\ 1 & x = 0 \\ 0 & x > 0 \end{cases}$$

$$< \sqrt{n}\, t) = P(Z < \sqrt{n}\, t)$$

$$F_n(t) \to F(t) \checkmark$$

$$\lim_{n \to \infty} P(X_n = k) = P(X = k)$$

$$f(t) \Rightarrow \lim P(X_n = t) = P(X = k)$$

$n \to \infty$ $n \to \infty$

$\lim_{n\to\infty} P(X_n = k) = P(X = k)$ for eve

$F_n(t) = \sum P(X_n = k)$ $F(t) = \sum$

$n \to \infty \Rightarrow$ they are equal

---

13. $Z_1, Z_2, \ldots$ IID density f

$P(Z_i > 0) = 1$, $\lambda = \lim_{x \downarrow 0} f(x) > 0$, $X_n = n \min\{Z_1, \ldots$

$X_n \rightsquigarrow Z$ $\quad Z \sim \begin{cases} \frac{1}{\lambda} e^{-\frac{1}{\lambda}x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

$F_{X_n}(x) = P(X_n < x) = P(n \min\{Z_1, \ldots, Z_n) < x) = P(\min\{$

$= 1 - P(Z_1, \ldots, Z_n \geq \frac{x}{n}) = 1 - P(Z_1 \geq \frac{x}{n})$

$= 1 - \prod_1^n P(Z_n \geq \frac{x}{n}) = 1 - \prod_1^n (1 - F(\frac{x}{n}$

$= \ldots [\log[1 - F(x/n)]]$

$$\lim_{n\to\infty} P(X_n = k) = P(X = k)$$

ery k

$= P(X = k)$

$\{\ldots, Z_n\}$

$Z_1, \ldots, Z_n \} \left(\frac{x}{n}\right)$

$\ldots P\left(Z_n, \frac{x}{n}\right)$

$)) = 1 - \left(1 - F\left(\frac{x}{n}\right)\right)^n$?

$$= 1 - \exp\left\{ \frac{\log[1 - F(x/n)]}{\frac{1}{n}} \right\} \longrightarrow$$

$$F_n(x) \longrightarrow 1 - e^{-\lambda x}$$

---

14. $X_1, \ldots, X_n \sim \text{Uniform}(0,1)$ $Y_n =$

$$\left(\frac{1}{n}\sum_{1}^{n} X_i\right)^2$$

---

15. $\begin{pmatrix} X_{11} \\ X_{21} \end{pmatrix}, \begin{pmatrix} X_{12} \\ X_{22} \end{pmatrix}, \ldots \begin{pmatrix} X_{1n} \\ X_{2n} \end{pmatrix}$ $\mu = ($

$$\overline{X_1} = \frac{1}{n}\sum_{1}^{n} X_{1i} \quad \overline{X_2} = \frac{1}{n}\sum_{1}^{n} X_{2i} \quad Y_n =$$

$$\sqrt{n}(\overline{X} - \mu) \rightsquigarrow N(0, \Sigma)$$

Delta method: $\sqrt{n}(Y_n - \mu)$

$$\nabla g(y) = \begin{pmatrix} \frac{\partial g}{\partial y_1} \\ \frac{\partial g}{\partial} \end{pmatrix} \quad \nabla_\mu = \nabla g(y = \mu)$$

L'Hopital rule limit

$\to e^{-nf(0)} = e^{-\lambda n}$?

$= \overline{X}_n^2 \quad \overline{X}_n \xrightarrow{P} \frac{1}{2}$

$\overline{Y}_n \xrightarrow{P} \frac{1}{4}$

(μ_1, μ_2) Variance Σ

$\frac{\overline{X}_1}{\overline{X}_2} = \frac{\frac{1}{n}\sum X_{1i}}{\frac{1}{n}\sum X_{2i}} \quad X_i = \begin{pmatrix} X_{1i} \\ X_{2i} \end{pmatrix}$

$\rightsquigarrow N(0, \Sigma)$

)

$$\nabla g(y) = \begin{pmatrix} \frac{\partial g}{\partial y_1} \\ \frac{\partial g}{\partial y_2} \end{pmatrix} \qquad \nabla_\mu = \nabla g(y = \mu)$$

$$\sqrt{n}\,\big(g(Y_n) - g(\mu)\big) \rightsquigarrow N$$

---

16. $X_n \rightsquigarrow X \qquad Y_n \rightsquigarrow$

$$X_n + Y_n \not\rightsquigarrow X + Y$$

$$X_n \sim \frac{n+1}{n} N(0,1) \qquad Y_n \sim$$

$$X_n + Y_n = \frac{2}{n} N(0,1) \rightsquigarrow 0$$

or

$$X_n := X \qquad Y_n := \cancel{X} \qquad X, Y$$

$$\frac{(0, \nabla_\mu^T \Sigma \nabla_\mu}{Y}$$

$$\frac{1-n}{n} \quad N(0, 1)$$

non constant symmetrical

$X_n \rightsquigarrow X \quad Y_n \rightsquigarrow Y \qquad Y = -X$

$X + Y = 0 \qquad X_n + Y_n = 2X$

$X + Y = 0$

Practice Test 1

Sunday, May 16, 2021 6:27 PM

1. $f_X(x)=\begin{cases}\frac{1}{2} & 0<x<1\\ \frac{1}{2} & 3<x<4\\ 0 & o.w\end{cases}$

$F_X(x)=\int_{-\infty}^{x} f_X(x)\,dx \Rightarrow \begin{cases}\int_0^x f_X(x)dx & 0<x<1\\ \frac{1}{2}+\int_3^x f_X(x)dx & 3<x<4\\ 1 & 4<x\\ 0 & o.w\end{cases} = \begin{cases}0 & x<0\\ \frac{1}{2}x & 0<x<1\\ \frac{1}{2} & 1\le x<3\\ \frac{1}{2}+\frac{(x-3)}{2} & 3<x<4\\ 1 & 4\le x\end{cases}$

---

2. $I_A(x)=\begin{cases}1 & x\in A\\ 0 & x\notin A\end{cases}$ $Y=I_A(X)$

?

$F_Y(y)=\begin{cases}0 & y<0\\ 1-P_A & 0\le y<1\\ 1 & y\ge 1\end{cases}$

$f_Y(y)=\begin{cases}P(x\in A) & 1\\ P(x\notin A)=1-P(x\in A) & 0\end{cases}$

$f_Y(y)=\begin{cases}\int_A f_X(x) & 1\\ 1-\int_A f_X(x) & 0\end{cases}$ $F_Y(y)=\begin{cases}1 & 1\\ 1-\int_A f_X(x) & 0\end{cases}$

---

3. (X,Y) uniform unit square $0\le x,y\le 1$

$Z=\log(X)+\log(Y)$ $f_Z(z)=?$

$f_{X,Y}(x,y)\begin{cases}1 & 0<x<1,\ 0<y<1\\ 0 & o.w\end{cases}$ $f_X(x)\begin{cases}1 & 0<x<1\\ 0 & o.w\end{cases}$

$f_Y(y)\begin{cases}1 & 0<y<1\\ 0 & o.w\end{cases}$

$r(x)=\log x$ $s(y)=e^y$

$f_{L_x,L_y}(l_x,l_y)=\begin{cases}e^{l_x+l_y} & l_x,l_y<0\\ 0 & o.w\end{cases}$ independence

$f_{L_x}(l_x)=\begin{cases}e^{l_x} & l_x<0\\ 0 & l_x\ge 0\end{cases}$

$f_{L_y}(l_y)=\begin{cases}e^{l_y} & \cancel{-\infty<l_y<1}\\ 0 & \cancel{o.w}\end{cases}$

---

4. Ω $C\subset S$ $P(C)>0$ $Q(A)=P(A|C)$ ✓

$Q(A)=P(A|C)=\frac{P(A\cap C)}{P(C)}$ $\quad P(C)>0,\ P(A\cap C)\ge 0 \Rightarrow$ Axiom 1

$Q(S)=\frac{P(S\cap C)}{P(C)}=\frac{P(C)}{P(C)}=1 \Rightarrow$ Axiom 2

$A_1,A_2,\dots$ disjoint

$Q\left(\bigcup_1^\infty A_i\right)=\frac{P\left(\bigcup_1^\infty (A_i\cap C)\right)}{P(C)}=\frac{\sum_1^\infty P(A_i\cap C)}{P(C)}=\sum_1^\infty \frac{P(A_i\cap C)}{P(C)}$

$=\sum_1^\infty Q(A_i)=$ Axiom 3

---

5. $f_X(x)=\begin{cases}2x & 0<x<1\\ 0 & o.w\end{cases}$ $F_X(x)=\begin{cases}x^2 & 0<x<1\\ 1 & 1\le x\\ 0 & o.w\end{cases}$

$Y=0$ if $X<\frac{1}{2}$

$Y=2X$ o.w

$A_y=\{x\,|\,r(x)\le y\} \Rightarrow 2x\le y \Rightarrow \frac{1}{2}\le x\le \frac{y}{2}$

$\int_{\frac{1}{2}}^{\frac{y}{2}} f_X(x)\,dx=\int_{\frac{1}{2}}^{\frac{y}{2}} 2x\,dx=x^2\Big|_{\frac{1}{2}}^{\frac{y}{2}}=\frac{y^2-1}{4}$

$\int_0^{\frac{1}{2}} f_X(x)\,dx=\int_0^{\frac{1}{2}} 2x\,dx=\frac{1}{4}$

$F_Y(y)=\begin{cases}\frac{1}{4} & y=0 \quad 0\le y\le 1\\ \frac{y^2-1}{4}+\frac{1}{4} & 1\le y\le 2\\ 1 & 2\le y\\ 0 & o.w\end{cases}$

---

6. $f_X(x)=\begin{cases}2e^{-2x} & x>0\\ 0 & o.w\end{cases}$ ✓

find $Y=r(x) \Rightarrow f_Y(y)=\begin{cases}e^{-y} & y>0\\ 0 & o.w\end{cases}$

$f_Y(y)=f_X(r^{-1}(y))\left|\frac{dr^{-1}(y)}{dy}\right|$

$2e^{-2s(y)}\times\frac{ds(y)}{dy}=e^{-y}$

$s(y)=\frac{1}{2}y=r^{-1}(y)$ $r(x)=2x$

---

7. $f_{X,Y}(x,y)=\begin{cases}c(x+y) & 0<x<1 \text{ and } 0<y<1\\ 0 & o.w\end{cases}$

a) $\int_0^1\int_0^1 c(x+y)\,dx\,dy=c\int_0^1\left(\frac{1}{2}+y\right)dy=c\left(\frac{1}{2}+\frac{1}{2}\right)=1$

$\boxed{c=4}$ $\boxed{c=1}$

b) $f_{Y|X}(y|x)=\frac{f_{X,Y}(x,y)}{\dots}=\frac{x+y}{\dots}=\frac{2x+2y}{\dots}$

? ✓

8. $E(Y|X) = X \quad E(Y) = E(X)$ ✓

b) $f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{x+y}{x+\frac{1}{2}} = \frac{2x+2y}{2x+1}$

$f_X(x) = \int_0^1 4(x+y)\,dy = \cancel{4}\left(x+\frac{1}{2}\right)$

c) $1 - P(Y \leq \frac{1}{2} | X=1) = 1 - \int_0^{\frac{1}{2}} \frac{2+2y}{3}\,dy$

$1 - \left(\frac{1}{3} + \frac{1}{12}\right) = \frac{7}{12}$ ✓

d) $1 - P(Y \leq \frac{1}{2} | X \leq \frac{1}{2}) = \frac{\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} 4(x+y)\,dx\,dy}{\int_0^{\frac{1}{2}} f_X(x)\,dx}$

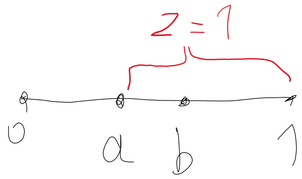
$= \frac{\frac{2}{8}}{4\left(\frac{1}{8} + \frac{1}{4}\right)} = \cancel{\frac{1}{6}} \quad \frac{2}{3}$

9. $X \sim \text{Uniform}(0,1) \quad 0 < a < b < 1$

$Y = \begin{cases} 1 & 0 < x < b \\ 0 & \text{o.w} \end{cases} \qquad Z = \begin{cases} 1 & a < x < 1 \\ 0 & \text{o.w} \end{cases}$

$f_Y(y) = \begin{cases} b & 1 \\ 1-b & 0 \end{cases} \qquad f_Z(z) = \begin{cases} a & 1 \\ 1-a & 0 \end{cases}$

$f_{Y,Z}(y,z) = \begin{cases} 1-b & (0,1) \\ a & (1,0) \\ b-a & (1,1) \end{cases}$



a) not independent ?

b) $E(Y|Z) = \sum y\, f_{Y|Z}(y|z)$

$= 1.$

$E(Y|Z=0) = 1$

$E(Y|Z=1) = \frac{b-a}{b} \times 1 + 0$

Chapter 6

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1. $X_1, \ldots, X_n \sim \text{Poisson}(\lambda)$ $\hat{\lambda} = n^{-1}\sum_1^n X_i$

bias $= E_\theta(\hat{\theta}_n) - \theta \Rightarrow \lambda - \lambda = 0$ unbiased

$E(\hat{\lambda}) = \frac{n\lambda}{n} = \lambda$

$se = se(\hat{\theta}_n) = \sqrt{V(\hat{\theta}_n)} = \sqrt{V(\hat{\lambda})} = \sqrt{\frac{n\lambda}{n^2}} = \sqrt{\frac{\lambda}{n}}$

$MSE = E_\theta(\hat{\theta}_n - \theta)^2 = \text{bias}^2(\hat{\theta}_n) + V_\theta(\hat{\theta}_n)$

$= \frac{\lambda}{n}$

---

2. $X_1, \ldots, X_n \sim \text{Uniform}(0, \theta)$ $\hat{\theta} = \max\{X_1, \ldots, X_n\}$

$E_\theta(\hat{\theta}_n) = \int y f_Y(y)\,dy = \frac{n}{\theta^n}\int y^n\,dy = \Big|_0^\theta \frac{n}{n+1}\frac{y^{n+1}}{\theta^n} = \frac{\theta n}{n+1}$

$P_\theta(\hat{\theta} \leq y) = P_\theta(X_1 \leq y)^n = \left(\frac{y}{\theta}\right)^n = F_Y(y)$ $0 \leq y \leq \theta$

$f_Y(y) = \frac{n y^{n-1}}{\theta^n}$

bias $= \frac{\theta n}{n+1} - \theta = \frac{-\theta}{n+1}$

$V(\hat{\theta}) = E(\hat{\theta}^2) - E(\hat{\theta})^2 = \frac{n\theta^2}{n+2} - \frac{\theta^2 n^2}{(n+1)^2}$

$E(\hat{\theta}^2) = \int y^2 f_Y(y)\,dy = \frac{n}{\theta^n}\int y^{n+1} = \frac{n}{\theta^n}\Big|_0^\theta \frac{y^{n+2}}{n+2} = \frac{n\theta^2}{n+2}$

$= \theta^2\left(\frac{n}{n+2} - \frac{n^2}{(n+1)^2}\right)$ $se = \sqrt{V(\hat{\theta})}$

$MSE = \text{bias}^2 + V(\hat{\theta}) = \frac{\theta^2}{\ldots} + \theta^2\left(\frac{n}{\ldots} - \frac{n^2}{\ldots}\right)$

$$MSE = bias^2 + V_\theta(\hat{\theta}) = \frac{\theta^2}{(n+1)^2} + \theta^2\left(\frac{n}{n+2} - \frac{n^2}{(n+1)^2}\right)$$

---

3. $X_1, \ldots, X_n \sim \text{Uniform}(0, \theta)$ $\hat{\theta} = 2\overline{X_n}$

$$E_\theta(\hat{\theta}_n) = E\left(\frac{2}{n} \cdot \sum_1^n X_i\right) = 2 \cdot \frac{\theta}{2} = \theta$$

bias $= 0$

$$V(X_i) = \frac{\theta^2}{12} \quad V(\hat{\theta}) = \frac{4}{n^2} \cdot n\,\frac{\theta^2}{12} = \frac{\theta^2}{3n}$$

$$se = \frac{\theta}{\sqrt{3n}}$$

$$MSE = 0 + \frac{\theta^2}{3n} = \frac{\theta^2}{3n}$$

Chapter 7

Tuesday, May 18, 2021 5:46 PM

1. a) $E(\hat{F}_n(x)) = F(x)$ $\quad \hat{F}_n(x) = \dfrac{\sum I(X_i \leq x)}{n}$

$E\left(\dfrac{\sum I(X_i \leq x)}{n}\right)$ $\quad I(X_i \leq x) = \begin{cases} 1 & \text{if } X_i \leq x \\ 0 & \text{if } X_i > x \end{cases}$

$= \dfrac{1}{n}\sum_1^n E[I(X_i \leq x)] = \dfrac{1}{n}\sum_1^n \left(1 \cdot F(x) + 0 \cdot (1-F(x))\right)$

$= \dfrac{1}{n}(nF(x)) = F(x)$

b) $V(\hat{F}_n(x)) = \dfrac{V(\sum I(X_i \leq x))}{n^2} = \dfrac{n V(I(X_1 \leq x))}{n^2}$

$= \dfrac{V(I(X_1 \leq x))}{n} = \dfrac{E(I^2(X_1 \leq x)) - E(I(X_1 \leq x))^2}{n}$

$= \dfrac{F(x) - F(x)^2}{n} = \dfrac{F(x)(1-F(x))}{n}$

c) $MSE = bias^2 + V = 0 + V = \dfrac{F(x)(1-F(x))}{n} \to 0$

d) $P(|\hat{F}_n(x) - F(x)| > \varepsilon) \leq \dfrac{\sigma^2}{\varepsilon} = \dfrac{F(x)(1-F(x))}{n^2 \varepsilon} \to 0$

---

2. $X_1, \ldots, X_n \sim Bernoulli(p)$

$Y_1, \ldots, Y_m \sim Bernoulli(q)$

$\hat{p} = \int x \, d\hat{F}_n(x) = \dfrac{1}{n}\sum X_i = \overline{X_n}$

$V(\hat{p}) = V(\overline{X_n}) = \dfrac{1}{n^2}\sum V(X_i) = \dfrac{V(X_i)}{n} = \dfrac{p(1-p)}{n} = \dfrac{\hat{p}(1-\hat{p})}{n}$

90% CI $\quad \overline{X_n} \pm \underbrace{z_{0.05}}_{1.64} \cdot \dfrac{\overline{X_n}(1-\overline{X_n})}{n}$

b) same

---

3. Todo computer

---

4. $X_1 \cdots X_n \sim F$ $\quad \hat{F}_n(x) = \dfrac{\sum I(X_i \leq x)}{n}$

$E(I(X_i \leq x)) = F(x)$ $\quad I(X_i \leq x) = \begin{cases} 1 & \text{if } X_i \leq x \\ 0 & \text{if } X_i > x \end{cases}$

$V(I(X_i \leq x)) = F(x)(1-F(x))$

$I(X_1 \leq x), I(X_2 \leq x) \ldots$ I.I.D mean $F(x)$
variance $F(x)(1-F(x))$

$\hat{F}_n(x) = \overline{I(X_n \leq x)}$

$\hat{F}_n(x) \approx N\left(F(x), \dfrac{F(x)(1-F(x))}{n}\right)$

---

5. $Cov(\hat{F}_n(x), \hat{F}_n(y))$ $\quad \mu_X = F(x)$ $\quad \hat{F}_n(x) = \dfrac{\sum I(X_i \leq x)}{n}$

$\mu_Y = F(y)$

$= E((X - \mu_X)(Y - \mu_Y))$ $\quad I(X_i \leq x) = \begin{cases} 1 & \text{if } X_i \leq x \\ 0 & \text{if } X_i > x \end{cases}$

$= E(XY) - E(X)E(Y)$

$E(XY) = E\left[\dfrac{\sum I(X_i \leq x)}{n} \dfrac{\sum I(X_i \leq y)}{n}\right]$

[illegible]

$= E(XY) - E(X)E(Y)$

w.o Loss of generality $x < y$!

$$E(XY) = E\left[\frac{\sum I(X_i \le x)}{n} \frac{\sum I(X_i \le y)}{n}\right]$$

$$= \frac{1}{n^2}\left[E\left(\sum I(X_i \le x) \sum I(X_j \le y)\right)\right]$$

$$= \frac{1}{n^2}\left\{\sum_i \sum_j E[I(X_i \le x) I(X_j \le y)] + \sum_i E[I(X_i \le x) I(X_i \le y)]\right\}$$

$$= \frac{1}{n^2}\left\{\sum_i \sum_{j \ne i} F(x)F(y) + nF(x)\right\}$$

$$= \frac{1}{n^2}\left[n^2 F(x)F(y) - nF(x)F(y) + nF(x)\right]$$

$$F(x)F(y) - \frac{1}{n}F(x)F(y) + \frac{1}{n}F(x) = E(XY)$$

$$E(x)E(y) = F(x)F(y) \Rightarrow cov = \frac{F(x)(1-F(y))}{n}$$

or use

$$cov\left(\sum_1^m a_i X_i, \sum_1^n b_j Y_j\right) = \sum_1^m \sum_1^n a_i b_j cov(X_i, Y_j)$$

from Exercise 14 chapter 3

---

6. $X_1, \dots, X_n \sim F$ $\quad a < b$

$$\theta = T(F) = F(b) - F(a)$$

$$\hat{\theta} = T(\hat{F}_n) = \hat{F}_n(b) - \hat{F}_n(a)$$

$$V(\hat{F}_n(b) - \hat{F}_n(a)) = \qquad E(\hat{\theta}) = F(b) - F(a)$$

$$V(\hat{F}_n(b)) + V(\hat{F}_n(a)) - 2cov(\hat{F}_n(b), \hat{F}_n(a))$$

$$\frac{F(b)(1-F(b))}{n} + \frac{F(a)(1-F(a))}{n} - \frac{2F(a)(1-F(b))}{n}$$

$$= \frac{F(b) - F^2(b) + F(a) - F^2(a) - 2F(a) + 2F(a)F(b)}{n}$$

$$= \frac{F(b) - F(a) + 2F(a)F(b) - F^2(b) - F^2(a)}{n} = \frac{F(b) - F(a) + (F(b) - F(a))^2}{n}$$

n n

$$\hat{F}_n(b) - \hat{F}_n(a) \pm z_{\frac{\alpha}{2}}\, se$$

---

9. $X_1, \dots X_{100} \sim F(x) = \text{Bernoulli}(p_1)$

$Y_1, \dots, Y_{100} \sim G(y) = \text{Bernoulli}(p_2)$

$$\hat{p}_1 = \frac{1}{100} \sum X_i = \overline{X_{100}} = 0{,}9$$

$$\hat{p}_2 = \frac{1}{100} \sum Y_i = \overline{Y_{100}} = 0{,}85$$

$$\theta = p_1 - p_2 = \int x\, dF(x) - \int y\, dG(y)$$

$$\hat{\theta} = \hat{p}_1 - \hat{p}_2 = \int x\, d\hat{F}(x) - \int y\, d\hat{G}(y) = 0{,}9 - 0{,}85 = 0{,}05$$

$$V(\theta) = V(p_1) - V(p_2) \Rightarrow V(\hat{\theta}) = V(\hat{p}_1) - V(\hat{p}_2)$$

from 2 $\Rightarrow \dfrac{\hat{p}_1(1-\hat{p}_1)}{100} + \dfrac{\hat{p}_2(1-\hat{p}_2)}{100} = \dfrac{0{,}09 + 0{,}1275}{100} = \dfrac{-0{,}0375}{100}$?

$$CI = 0{,}05 \pm z_{0{,}025} \frac{0{,}0375}{100}$$

---

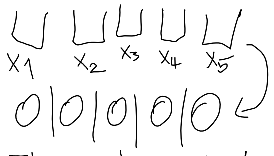
10. $\hat{\theta} = \hat{\mu}_1 - \hat{\mu}_2 = \int x_1\, dF_1(x_1) - \int x_2\, dF_2(x_2)$

Chapter 8

Tuesday, June 1, 2021 6:00 PM

4. $X_1, \dots, X_n$ $\binom{2n-1}{n}$

Each X_i can be seen as a bucket
each ball in a bucket shows how
many times we pick that datapoint



The grouping can be done by looking at
is like a sticks and balls problem
$\binom{2n-1}{n-1} = \binom{2n-1}{n}$ = number of ways to arrange the stick and balls

---

5. $X_1, \dots, X_n$ (no ties) $X_1^*, \dots, X_n^*$ bs

$$\overline{X}_n^* = n^{-1}\sum_1^n X_i^*$$

a) $E(\overline{X}_n^* | X_1, \dots, X_n) = E\left(n^{-1}\sum_1^n X_i^* \middle| X_1, \dots, X_n\right)$

$= \frac{1}{n} E\left(\sum_1^n X_i^* \middle| X_1, \dots, X_n\right) = \frac{1}{n}\sum_1^n E(X_i^* | X_1, \dots, X_n)$

$= \frac{1}{n}\sum_1^n \left(\sum_1^n X_i \cdot \frac{1}{n}\right) = \frac{1}{n^2}\sum_1^n\left(\sum_1^n X_i\right) = \frac{1}{n}\sum_1^n X_i = \overline{X}_n$

b) $E(\overline{X}_n^*) = E\,E(\overline{X}_n^* | X_1, \dots, X_n) = E(\overline{X}_i) = E(X_i) = \mu$

c) $V(\overline{X}_n^* | X_1, \dots, X_n) = \frac{1}{n^2} V\left(\sum_1^n X_i^* \middle| X_1, \dots, X_n\right)$

$= \frac{1}{n} V(X_i^* | X_1, \dots, X_n) = \frac{1}{n} E(X_i^* - \overline{X}_n)^2$

$= \frac{1}{n}\sum^n \frac{1}{n}\cdot(X_i - \overline{X}_n)^2 = \frac{1}{n^2}\sum_i (X_i - \overline{X}_n)^2 = \frac{n-1}{n^2} S^2$

d) $V(\overline{X}_n^*) = E\,V[\overline{X}^* | X_1, \dots, X_n] + V\,E[\overline{X}^* | X_1, \dots, X_n]$

$= E\left(\frac{1}{n^2}\sum (X_i - \overline{X}_n)^2\right) + V(\overline{X}_n)$

$= E\,\frac{n-1}{n^2} S^2 + V(\overline{X}_n) = \frac{n-1}{n^2}\sigma^2 + \frac{\sigma^2}{n}$

$= \frac{\sigma^2}{n}\left[2 - \frac{1}{n}\right) = Var(\overline{X})\left[2 - \frac{1}{n}\right] \to 2Var(\overline{X})$

---

6. a $P(\hat{\theta} \leq t) = P(e^{\overline{X}} \leq t) =$

$P(\overline{X} \leq \log t) = F_{\overline{X}}(\log t)$?

$$P(\bar{X} < \log t) = F_{\bar{X}}(\log t)$$ (?)

$$\bar{X} = \frac{1}{n}\sum_1^n X_i \sim \frac{1}{n} N(n\mu, n) = N\left(\mu, \frac{1}{n}\right)$$

---

7. $X_1, \dots X_n \sim \text{Uniform}(0, \theta)$ $\hat{\theta} = \max\{X_1, \dots, X_n\}$

a)

$$F(\hat{\theta}) = P(\hat{H} < \hat{\theta}) = P(X_1 < \hat{\theta}) \cdots P(X_n < \hat{\theta})$$

$$= \left(\frac{\hat{\theta}}{\theta}\right)^n = \hat{\theta}^n \qquad E(\hat{\theta}) = \frac{\theta n}{n+1}$$

$$V(\hat{\theta}) = \frac{n\theta^2}{n+2} - \frac{\theta^2 n^2}{(n+1)^2}$$

b) $P(\hat{\theta}^* = \hat{\theta}) = 1 -$ probability that none of the bootstraps is the maximum

$$= 1 - \left(\frac{n-1}{n}\right)^n \xrightarrow[n\to\infty]{} 1 - e^{-1} \simeq 0.632$$

---

8. $T_n = \bar{X}_n^2$, $\mu = E(X_1)$, $\alpha_k = \int |x-\mu|^k dF(x)$

$$\hat{\alpha}_k = n^{-1} \sum_1^n |X_i - \bar{X}_n|^k$$

$$V_{boot} = \frac{4\bar{X}_n^2 \hat{\alpha}_2}{n} + \frac{4\bar{X}_n \hat{\alpha}_3}{n^2} + \frac{\hat{\alpha}_4}{n^3}$$?

$$V_{boot} = \frac{1}{B} \sum_{b=1}^B \left(T_{n,b}^* - \frac{1}{B}\sum_{r=1}^B T_{n,r}^*\right)^2$$

[nonparametric - Bootstrap variance of squared sample mean - Cross Validated (stackexchange.com)]()

$$V(\bar{X}_n^2) = E\left[\left(\bar{X}_n^2 - E(\bar{X}_n^2)\right)^2\right] = E\left[\bar{X}_n^2 + E(\bar{X}_n^2)^2 - 2\bar{X}_n^2 E(\bar{X}_n^2)\right]$$

$$= E(\bar{X}_n^2) + E\left[E(\bar{X}_n^2)^2\right] - 2E\left[\bar{X}_n^2 E(\bar{X}_n^2)\right]$$

$$E(\bar{X}^4) - E(\bar{X}_n)^2$$

$$E(\overline{X_n}^4) - E(\overline{X_n})^2$$

Chapter 9

Sunday, June 6, 2021 2:05 PM

1. $X_1,\dots,X_n \sim \text{Gamma}(\alpha,\beta)$

$\alpha_1 = E(X) = \dfrac{\alpha}{\beta}$

$\alpha_2 = E(X^2) = V(X)+E(X)^2 = \dfrac{\alpha}{\beta^2}+\dfrac{\alpha^2}{\beta^2} = \dfrac{\alpha(\alpha+1)}{\beta^2}$

$\dfrac{\hat\alpha}{\hat\beta} = \dfrac{1}{n}\sum_i X_i = \overline{X_n}$ $\quad \dfrac{\hat\alpha(\hat\alpha+1)}{\hat\beta^2} = \dfrac{1}{n}\sum X_i^2$ $\quad i^2$

$\hat\alpha = \dfrac{(E_{\hat F}X)^2}{Var_{\hat F}(X)}$ $\quad \hat\beta = \dfrac{E_{\hat F}(X)}{Var_{\hat F}(X)}$

---

2 $X_1,\dots,X_n \sim \text{Uniform}(a,b)$ $\quad a<b$

a)
$\alpha_1 = E(X_1) = \dfrac{a+b}{2}$ $\quad \alpha_2 = E(X_1^2) = \dfrac{(b-a)^2}{12}$

$\dfrac{\hat a+\hat b}{2} = \overline{X_n}$ $\quad \hat a^2+\hat b^2 = 4\overline{X_n}^2 - 2\hat a\hat b$

$\left(\dfrac{\hat a+\hat b}{2}\right)^2 + \dfrac{(\hat b-\hat a)^2}{12} = \dfrac{1}{n}\sum X_i^2$

$\dfrac{(\hat b-\hat a)^2}{12} = \dfrac{1}{n}(X_i-\overline{X})^2$

$\hat b-\hat a = \sqrt{\dfrac{12}{n}}\,(X_i-\overline{X})$ $\quad \hat a+\hat b = 2\overline{X_n}$

$\hat b = \overline{X_n} + \sqrt{\dfrac{3}{n}}(X_i-\overline{X_n}) = \overline{X_n} + \sqrt{3}\,Std_{\hat F}(X)$

$\hat a = \overline{X_n} - \sqrt{\dfrac{3}{n}}(X_i-\overline{X_n}) = \overline{X_n} - \sqrt{3}\,Std_{\hat F}(X)$

b) $\mathcal{L}(\theta) = \prod f(x;\theta) = \left(\dfrac{1}{b-a}\right)^n \prod_i I(a,b)(X_i)$

$\hat\alpha = \min_i X_i$ $\quad \hat\beta = \max_i X_i$

c) $T = \int x\,dF(x) = \int_a^b x f(x;\theta) = \dfrac{1}{b-a}\int_a^b x$

$= \dfrac{1}{2}\dfrac{1}{b-a}\,b^2-a^2 = \dfrac{b+a}{2} \Rightarrow \text{MLE } \hat T = \dfrac{\max X_i + \min X_i}{2}$

d) $\tilde T = \overline{X}$ $\quad V(\overline{X}) = \dfrac{1}{n^2}\sum V(X_1) = \dfrac{V(X_1)}{n} = \dfrac{4}{120} = \dfrac{1}{30}$

simulated $= 0.032$

---

3. $X_1,\dots,X_n \sim N(\mu,\sigma^2)$

T 0,95 Percentile $P(X<T)=0{,}95$

a) $P(X<T) = P\left(Z<\dfrac{T-\mu}{\sigma}\right) = \phi\left(\dfrac{T-\mu}{\sigma}\right) = 0{,}95$

$\dfrac{T-\mu}{\sigma} = \phi^{-1}(0{,}95) \Rightarrow T = \sigma\phi^{-1}(0{,}95)+\mu$

$\hat\mu_{mle} = \overline{X}$ $\quad \hat\sigma_{mle} = S \Rightarrow \hat T_{mle} = S\phi^{-1}(0{,}95)+\overline{X}$

(b) $\hat T_{mle} \pm 2\hat{se}$

$\ell(\mu,\sigma) = -n\log\sigma - \dfrac{nS^2}{2\sigma^2} - \dfrac{n(\overline{X}-\mu)^2}{2\sigma^2}$

from 9 29 or Exercise 8 $\quad J_n = I_n^{-1}(\mu,\sigma) = \dfrac{1}{n}\begin{bmatrix}\sigma^2 & 0\\ 0 & \frac{\sigma^2}{2}\end{bmatrix}$

$\nabla g = \begin{pmatrix}1\\ \phi^{-1}(0.95)\end{pmatrix}$ using the delta method;

$\hat{se}(\hat T) = \sqrt{\nabla g^T \hat J_n \nabla g} = \sqrt{\begin{pmatrix}1\\ \phi^{-1}(0.95)\end{pmatrix}\dfrac{1}{n}\begin{bmatrix}\sigma^2 & 0\\ 0 & \frac{\sigma^2}{2}\end{bmatrix}\begin{pmatrix}1\\ \phi^{-1}(0.95)\end{pmatrix}}$

$= \sigma\sqrt{\dfrac{1}{n}+\dfrac{1}{2n}(\phi^{-1}(0{,}95))^2}$

c)

---

4. $X_1,\dots,X_n \sim \text{Uniform}(0,\theta)$

$P(|\hat\theta-\theta^*|>\varepsilon)\to 0$ $\quad Y=\max\{X_1,\dots,X_n\}$

$P(|Y-\theta^*|>\varepsilon) \xrightarrow{Y \text{ always} < \theta^*} P(Y-\theta^*>\varepsilon) + P(Y-\theta^*<-\varepsilon)$

$= P(Y<\theta^*-\varepsilon) = P(X_1<\theta^*-\varepsilon)^n = \left(\dfrac{\theta^*-\varepsilon}{\theta^*}\right)^n \to 0$

---

5. $X_1,\dots,X_n \sim \text{Poisson}(\lambda)$

method of moments; $\alpha_1 = E(X_1) = \lambda$

$\hat\lambda = \overline{X_n}$

mle; $f_X(x) = \dfrac{\lambda^x e^{-\lambda}}{x!}$

$L(\theta) = \prod_1^n \dfrac{\theta^{X_i}e^{-\theta}}{X_i!} = e^{-\theta}\dfrac{\theta^{\sum_1^n X_i}}{\prod_1^n X_i!}$

$\ell(\theta) = \log\left(e^{-n\theta}\dfrac{\theta^{\sum_1^n X_i}}{\prod_1^n X_i!}\right) = \log(e^{-n\theta}) + \log\left(\theta^{\sum_1^n X_i}\right) - \log\left(\prod_1^n X_i!\right)$

$= \log(e^{-n\theta}) + \sum_1^n \log(\theta^{X_i}) - \left[\sum_1^n \log(X_i!)\right]$

$= -n\theta\,\cancel{\log(e)}^{1} + \log(\theta)\sum_1^n X_i - \left[\sum_1^n \log(X_i!)\right]$

$\Rightarrow -n\theta + \log(\theta)\sum_1^n X_i - \left[\sum_1^n \log(X_i!)\right]$

6. $X_1,\dots,X_n \sim N(\theta,1)$ $\quad \psi = P(Y_1=1)$

$Y_i = \begin{cases}1 & \text{if } X_i>0\\ 0 & \text{if } X_i\leqslant 0\end{cases}$

a) $\psi = P(Y_1=1) = P(X_1>\theta) = 1-P(X_1\leqslant 0)$

$= 1-P\left(\dfrac{X_1-\theta}{1}\leqslant-\theta\right) = 1-P(Z\leqslant-\theta) = 1-\phi(-\theta)$

$\hat\psi = 1-\phi(-\hat\theta) = 1-\phi(-\overline{X_n}) = \phi(\overline{X_n})$

b) $\hat\psi = \phi(\overline{X_n})$ $\quad \psi = g(\theta) = \phi(\theta)$

using the delta method;

$\hat{se}(\hat\psi) = |g'(\hat\theta)|\,\hat{se}(\hat\theta) = f_Z(\overline{X_n})\dfrac{\sqrt{n}}{n}$

$V(\hat\theta) = V(\overline{X_n}) = \dfrac{1}{n}V(X_1) = \dfrac{1}{n}$

$\Rightarrow \hat{se}(\hat\theta) = \dfrac{\sqrt{n}}{n}f_Z(\overline{X_n})$

$CI = \phi(\overline{X_n}) \pm \dfrac{\sqrt{n}}{n}f_Z(\overline{X_n})$

c) $\tilde\psi = \left(\dfrac{1}{n}\right)\sum_i Y_i \Rightarrow \tilde\psi \xrightarrow{P} \psi$

$P(|\tilde\psi-\psi|>\varepsilon)\to 0$

$E(Y_1) = P(Y_1=1) = \phi(\overline{X_n}) = \psi$ $\quad \tilde\psi = \overline{Y_n}$

By the LLN $\tilde\psi$ converges in probability to ψ

d) $V(\tilde\psi) = V\left(\dfrac{1}{n}\sum_i^n Y_i\right) = \dfrac{1}{n}V(Y_1) = \dfrac{\psi(1-\psi)}{n}$

$\hat{se}(\psi) = \dfrac{\sqrt{n}}{n}f_Z(\overline{X_n})$

$\tilde\psi \sim N\left(\psi, \dfrac{\psi(1-\psi)}{n}\right)$

$\hat\psi \sim N\left(\psi, \dfrac{f_Z(\theta)}{n}\right)$

$ARE(\tilde\psi,\hat\psi) = \dfrac{f_Z^2(\theta)}{\psi(1-\psi)}$

e) $\hat\psi = \phi(\overline{X_n})$

By the law of large numbers $\overline{X_n}\to E(X_1)=\mu$

$\hat\psi = \phi(\overline{X_n}) \to \phi(\mu)$

$\phi(\mu)$

True value of ψ is $1-P(X_1\leqslant 0) = 1-F_X(0)$

$1-F_X(0) \neq \phi(\mu)$ for any arbitrary distribution

So mle is inconsistent.

---

7. $X_1 \sim \text{Binomial}(n_1,p_1)$ $\quad \psi = p_1-p_2$

$X_2 \sim \text{Binomial}(n_2,p_2)$

a) $X_1 = \sum$ Bernoulli $\quad \hat p_{1mle} = \dfrac{\hat X_1}{n_1}$ $\quad \hat p_{2mle} = \dfrac{\hat X_2}{n_2}$

By equivariance of the mle $\hat\psi = \hat p_1 - \hat p_2$

$\ell_n = \log f(x_1;p_1) = \log\left[\binom{n_1}{x_1}p_1^{x_1}(1-p_1)^{n_1-x_1}\right]$

$= \log\binom{n_1}{x_1} + x_1\log p_1 + (n_1-x_1)\log(1-p_1)$

$S(X_1;p_1) = \dfrac{\partial\ell_n}{\partial p_1} = 0 + \dfrac{x_1}{p_1} + (-1)\dfrac{1}{1-p_1}(n_1-x_1)$

$= \dfrac{x_1}{p_1} + \dfrac{(x_1-n_1)}{1-p_1}$

b) X_1 and X_2 are independent

$$=n\theta\,\cancel{\log(e)}+\log(\theta)\sum_1^n x_i-\left[\sum_1^n\log(x_i!)\right]$$

$$\Rightarrow -n\theta+\log(\theta)\sum_1^n x_i-\left[\sum_1^n\log(x_i!)\right]$$

$$\Rightarrow \frac{d\ell(\theta)}{d\theta}=-n+\frac{1}{\theta}\sum_1^n x_i=0\quad \hat\theta_{mle}=\bar X_n$$

FI score function: $s(X;\theta)=\dfrac{\partial\log f(X;\theta)}{\partial\theta}$

$$\log f(X;\theta)=\log\left(\frac{\theta^X e^{-\lambda}}{X!}\right)=\log(\theta^X)+\log(e^{-\theta})-\log(X!)$$

$$=X\log(\theta)-\theta-\log(X!)\quad s(X;\theta)=\frac{X}{\theta}-1$$

$$I_n(\theta)=\sum_1^n V_\theta\left(\frac{X_i}{\theta}-1\right)=\sum_1^n V\left(\frac{X_i}{\theta}\right)=\frac{1}{\theta^2}\sum_1^n V(X_i)$$

$$=\frac{n\theta}{\theta^2}=\frac{n}{\theta}\quad I(\theta)=\frac{1}{n}I_n(\theta)=\frac{1}{\theta}$$

$$=\frac{\cdot}{p_1}+\frac{(x_1-n_1)}{1-p_1}$$

b) X_1 and X_2 are independent

$$L(p_1,p_2)=\binom{n_1}{x_1}p_1^{x_1}(1-p_1)^{n_1-x_1}\binom{n_2}{x_2}p_2^{x_2}(1-p_2)^{n_2-x_2}$$

$$\ell(p_1,p_2)=\log\binom{n_1}{x_1}+x_1\log p_1+(n_1-x_1)\log(1-p_1)+\log\binom{n_2}{x_2}+x_2\log p_2+(n_2-x_2)\log(1-p_2)$$

$$H=\begin{bmatrix}-\frac{x_1}{p_1^2}-\frac{1-x_1}{(1-p_1)^2} & 0\\ 0 & -\frac{x_2}{p_2^2}-\frac{1-x_2}{(1-p_2)^2}\end{bmatrix}\quad I=E(-H)=\begin{bmatrix}\frac{n_1}{p_1(1-p_1)} & 0\\ 0 & \frac{n_2}{p_2(1-p_2)}\end{bmatrix}$$

c) $\psi=p_1-p_2=g(p_1,p_2)$ $\nabla g=\begin{bmatrix}1\\-1\end{bmatrix}$

$$\hat{se}(\hat\psi)=\sqrt{(\hat\nabla g)^T\hat J_n(\hat\nabla g)}$$

$$=\sqrt{[1\ -1]\,I^{-1}(\hat p_1,\hat p_2)\begin{bmatrix}1\\-1\end{bmatrix}}=\sqrt{\frac{\hat p_1(1-\hat p_1)}{n_1}+\frac{\hat p_2(1-\hat p_2)}{n_2}}$$

d) $\hat\psi=\dfrac{160}{200}-\dfrac{148}{200}=\dfrac{12}{200}=0.06\qquad 0.06\pm1.64\hat{se}$

$$\hat{se}=\sqrt{\frac{0.8\cdot0.2}{200}+\frac{0.74\cdot0.26}{200}}=0.04$$

---

9. Continuation of problem 6 from chapter 8:

a) $\hat\mu_{mle}=\bar X$ $\theta=e^\mu$ $\hat\theta=e^{\bar X}$

$$\hat{se}(\hat\tau_n)=|g'(\hat\theta)|\,\hat{se}(\hat\theta_n)$$

$$\Rightarrow e^{\bar X}se(\bar X)=\frac{e^{\bar X}}{\sqrt n}$$

---

10. Continuation of problem 7 from chapter 8:

a)

---

8. $X_1,\ldots,X_n\sim N(\mu,\sigma^2)$

$$\ell(\mu,\sigma)=-n\log\sigma-\frac{nS^2}{2\sigma^2}-\frac{n(\bar X-\mu)^2}{2\sigma^2}$$

$$\frac{\partial^2\ell}{\partial\mu^2}=\frac{\partial}{\partial\mu}\left(\frac{2n(\bar X-\mu)}{2\sigma^2}\right)=\frac{-n}{\sigma^2}=H_{11}$$

$$\frac{\partial^2\ell}{\partial\sigma^2}=\frac{\partial l}{\partial\sigma}\left(\frac{-n}{\sigma}+\frac{nS^2}{\sigma^3}+\frac{n(\bar X-\mu)^2}{\sigma^3}\right)$$

$$=\frac{n}{\sigma^2}-\frac{3nS^2}{\sigma^4}-\frac{3n(\bar X-\mu)^2}{\sigma^4}=H_{22}$$

$$\frac{\partial^2\ell}{\partial\mu\partial\sigma}=\frac{-2n(\bar X-\mu)}{\sigma^3}=H_{12}$$

$$\frac{\partial^2\ell}{\partial\sigma\partial\mu}=\frac{-2n(\bar X-\mu)}{\sigma^3}=H_{21}$$

$$E(H_{12})=E(H_{21})=0$$

$$E(H_{11})=\frac{-n}{\sigma^2}\qquad I_n(\mu,\sigma)=\begin{bmatrix}\frac{n}{\sigma^2} & 0\\ 0 & 2n\end{bmatrix}$$

$$E(H_{22}) = \frac{-2n}{\sigma^2}$$

$$\begin{bmatrix} 0 & \frac{2n}{\sigma^2} \end{bmatrix}$$

Chapter 10 (not done)

Sunday, July 4, 2021 6:24 PM

chapter 11

Sunday, July 4, 2021 6:25 PM

1. $X_1, \dots, X_n \sim N(\theta, \sigma^2)$ $\hat{\mu}_{mle} = \bar{X} = \hat{\theta}_{mle}$

prior: $\theta \sim N(a, b^2)$

$$f(\theta \mid x^n) = \frac{f(x^n \mid \theta) f(\theta)}{\int_\theta f(x^n \mid \theta) f(\theta) d\theta} \propto \mathcal{L}_n(\theta) f(\theta)$$

$$f(x^n \mid \theta) = \prod_{i=1}^n f(x_i \mid \theta) = \prod_i \frac{1}{\sigma} \exp\left\{-\frac{1}{2\sigma^2}(X_i - \theta)^2\right\} \qquad f(\theta) = \frac{1}{b}\exp\left\{-\frac{1}{2b^2}(\theta - a)^2\right\}$$

remember to remove constants

$$f(\theta \mid x^n) \propto \mathcal{L}_n(\theta) f(\theta) = \prod_i \frac{1}{\sigma} \exp\left\{-\frac{1}{2\sigma^2}(X_i - \theta)^2\right\} \cdot \exp\left\{-\frac{1}{2b^2}(\theta - a)^2\right\}$$

$$= \frac{1}{b\sigma^n} \exp\left\{\frac{-ns^2 b^2 + n(\bar{x} - \theta)^2 b^2 + (\theta - a)^2 \sigma^2}{2\sigma^2 b^2}\right\}$$

$$= \frac{1}{b\sigma^n} \exp\left\{\frac{-ns^2b^2 + n\bar{x}^2 b^2 + n\theta^2 b^2 - 2n\bar{x}\theta b^2 - \theta^2\sigma^2 - a^2\sigma^2 + 2\theta a \sigma^2}{2\sigma^2 b^2}\right\}$$

$$= \frac{1}{b\sigma^n} \exp\left\{\frac{\theta^2(nb^2 - \sigma^2) + \theta(2a\sigma^2 - 2n\bar{x}b^2) - ns^2b^2 + n\bar{x}^2 b^2}{2\sigma^2 b^2}\right\}$$

$$= \frac{1}{b\sigma^n} \exp\left\{\theta^2\left(\frac{1}{\sigma^2} \dots\right) + \theta\left(\frac{a}{b^2} - \frac{n\bar{x}}{\sigma^2}\right) - \frac{ns^2}{2\sigma^2} + \frac{n\bar{x}^2}{2\sigma^2}\right\}$$

---

2. $X_1, \dots, X_n \sim \text{Normal}(\mu, 1)$ a) experiment

b) $f(\mu) \propto 1$ $\quad f(\mu \mid X_1, \dots, X_n) \propto \mathcal{L}(\mu) f(\mu)$

$$\mathcal{L}(\mu) \propto \exp\left\{-\frac{ns^2}{2}\right\} \exp\left\{-\frac{n(\bar{X} - \mu)^2}{2}\right\}$$

$$= \exp\left\{-\frac{n(\mu - \bar{X})^2}{2}\right\}$$

d) $\theta = e^{\mu}$ $\quad f(\mu \mid X_1, \dots, X_n) = \frac{n}{\sqrt{2\pi}} \exp\left\{-\frac{n}{2}(\mu - \bar{X})^2\right\}$

$$P(\Theta \leq \theta) = P(e^{\mu} \leq \theta) = P(\mu \leq \ln\theta)$$

$$= \frac{1}{\theta} \frac{n}{\sqrt{2\pi}} \exp\left\{-\frac{n}{2}(\ln\theta - \bar{X})^2\right\}$$

e) experiment $\quad \mu$ posterior $\sim N\left(\bar{X}, \frac{1}{n}\right)$

$$\int_a^b f(\mu) d\mu = 0.9 \quad P(U \leq a) = 0.05 \quad P(Z \leq \sqrt{n}(a - \bar{X})) = 0.05 \Rightarrow a = \frac{z^{-1}(0.05)}{\sqrt{n}} + \bar{X}$$

$$P(U \leq b) = 0.95 \quad P(Z \leq \sqrt{n}(a - \bar{X})) = 0.95 \Rightarrow a = \frac{z^{-1}(0.05)}{\sqrt{n}} + \bar{X}$$

$Y = g(X)$ $\quad P(Y \leq y) = P($ $\quad g^{-1}(y)$ $\quad f_X(g^{-1}$

with experiment

f) experiment $P(\theta \leq a) = P(\mu \leq \underbrace{\ln a}_{a'}) = P(\mu \leq a')$

a' is equal to a from part e)

so $\ln a = a' \Rightarrow \boxed{a = e^{a'}}$

---

3. $X_1, \dots, X_n \sim \text{Uniform}(0, \theta)$ $\quad f(\theta) \propto \frac{1}{\theta}$

$$f(\theta \mid x^n) \propto \mathcal{L}(\theta) f(\theta) = \prod_{i=1} f(x_i \mid \theta) f(\theta)$$

$$= \frac{1}{\theta} \prod_i f(x_i \mid \theta) = \frac{1}{\theta} \prod \frac{I(X_i \leq \theta)}{\theta} \Rightarrow \begin{cases} \frac{1}{\theta^{n+1}} & \theta \geq \max\{X_1, \dots, X_n\} \\ 0 & \text{o.w} \end{cases}$$

$$c = \int \mathcal{L}_n(\theta) f(\theta) d\theta = \int_{\max(x_n)}^{\infty} \frac{1}{\theta^{n+1}} d\theta = \int_{\max(x_n)}^{\infty} \theta^{-n-1} d\theta$$

$$\theta^{-n} \Big|^{\infty} \qquad -(\max(x_n))^{-n}$$

$c = \int \mathcal{L}_n(\theta) f(\theta) d\theta = \int_{max(x_n)} \frac{1}{\theta^{n+1}} d\theta = \int_{max(x_n)} \theta^{-n-1} d\theta$

$= \frac{\theta^{-n}}{-n} \Big|_{max(x_n)}^{\infty} = \frac{(max(x_n))^{-n}}{n}$

---

4. $\tau = p_2 - p_1$ p_2: treatment p_1: placebo

a) By equivariance of MLE $\hat{\tau}_{mle} = \hat{p}_{2mle} - \hat{p}_{1mle} = 0.2$

for $\hat{se}$ it is the same as chapter 9 question 7

$$se(\hat{\tau}) = \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}} = \sqrt{\frac{0.6(0.4)}{50} + \frac{0.2(0.8)}{50}}$$

$$= \sqrt{\frac{0.4}{50}} \qquad CI = \left(0.2 \pm \underbrace{z_{0.05}}_{1.645} \sqrt{\frac{0.4}{50}}\right)$$

b) computer experiment

c) $f(p_1, p_2) = 1$

$$f(p_1, p_2 | x_1, x_2) \propto f(x_n | p_1, p_2) f(p_1, p_2) = \mathcal{L}(p_1, p_2) f(p_1, p_2)$$

$$= \binom{n_1}{s_1} p_1^{s_1} (1-p_1)^{n_1 - s_1} \binom{n_2}{s_2} p_2^{s_2} (1-p_2)^{n_2 - s_2} \cdot 1$$

$$\propto p_1^{s_1} (1-p_1)^{n_1 - s_1} p_2^{s_2} (1-p_2)^{n - s_2}$$

$f(p_1, p_2 | x_1, x_2) = f(p_1 | x_1) f(p_2 | x_2)$ => p_1 and p_2 independent

$f(p_1 | x_1) \xrightarrow{\text{similar to example 11.7}} Beta(s_1 + 1, n_1 - s_1 + 1)$

$f(p_2 | x_2) \longrightarrow Beta(s_2 + 1, n_2 - s_2 + 1)$

now we can simulate p_1, p_2 and compute $\tau = p_1 - p_2$

simulation in code

d) by equivariance $\hat{\psi}_{mle} = \log\left(\left(\frac{\hat{p}_{1mle}}{1-\hat{p}_{1mle}} \div \frac{\hat{p}_{2mle}}{1-\hat{p}_{2mle}}\right)\right) = -0.98$

From chapter 9 question 7b) $I(p_1, p_2) = \begin{bmatrix} \frac{n_1}{p_1(1-p_1)} & 0 \\ 0 & \frac{n_2}{p_2(1-p_2)} \end{bmatrix}$

$$\nabla g = \begin{bmatrix} \frac{1}{\hat{p}_{1mle}} + \frac{1}{1-\hat{p}_{1mle}} \\ \\ \frac{1}{\hat{p}_{2mle}} + \frac{1}{1-\hat{p}_{2mle}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\hat{p}_1(1-\hat{p}_1)} \\ \\ \frac{-1}{\hat{p}_2(1-\hat{p}_2)} \end{bmatrix} \qquad \hat{se}(\hat{\psi}_{mle}) = \sqrt{(\hat{\nabla} g)^T \hat{J}_n (\hat{\nabla} g)} \simeq 0.46$$

$$90\%\ CI = -0.95 \pm \underset{\substack{\downarrow \\ 1.645}}{z_{0.05}}\, 0.46 = [-1.73, 0.23]$$

e) computer experiment

---

5. $f(p | x_1) \propto \mathcal{L}(p) f(p) = \prod_i f(x_i | p) f(p) =$

$$p^s (1-p)^{n-s} f(p) \Rightarrow p^s (1-p)^{n-s} \underset{\substack{\downarrow \\ \text{constant}}}{C}\, p^{\alpha-1} (1-p)^{\beta-1} = p^{s+\alpha-1} (1-p)^{n-s+\beta-1} \Rightarrow$$

$f(p) = Beta(\alpha, \beta)$

$$\Rightarrow f(p | x_1) = Beta(s + \alpha, n - s + \beta)$$

plugin values and computer experiment

$n=10 \quad S=2$

---

6. $X_1, \dots, X_n \sim \text{Poisson}(\lambda)$

a) $\lambda \sim \text{Gamma}(\alpha, \beta)$ prior

$$f(\lambda) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \lambda^{\alpha-1} e^{-\frac{\lambda}{\beta}}$$

constant

$$f(\lambda \mid X_1, \dots, X_n) \propto \mathcal{L}(\lambda) f(\lambda) = \prod_i^n \Big[e^{-\lambda} \frac{\lambda^{x_i}}{\underbrace{x_i!}_{\text{constant}}} \Big] \cdot \frac{\overset{\downarrow}{\beta^\alpha}}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda\beta}$$

$$\propto \prod_i^n \left[e^{-\lambda} \lambda^{x_i} \right] \lambda^{\alpha-1} e^{-\lambda\beta} = e^{-n\lambda} \lambda^{\sum x_i} \lambda^{\alpha-1} e^{-\lambda\beta}$$

$$= e^{-n\lambda - \lambda\beta} \lambda^{\alpha - 1 + \sum x_i} \Rightarrow \text{Gamma}\left(\sum x_i + \alpha, n + \beta\right)$$

$$\bar{\lambda} = \frac{\sum x_i + \alpha}{n + \beta}$$

b) $f(\theta) \propto I(\theta)^{\frac{1}{2}} \qquad I(\lambda) = -E_\theta\left(\frac{\partial^2 \log f(x;\theta)}{\partial \theta^2} \right)$

$$f(x;\lambda) = e^{-\lambda} \frac{\lambda^x}{x!} \quad \log f(x;\theta) = \log e^{-\lambda} + \log \lambda^x - \log x!$$

$$= -\lambda + x \log \lambda - \log x! \quad \frac{\partial^2 \log f(x;\lambda)}{\partial \lambda^2} = \frac{-x}{\lambda^2}$$

$$I(\theta) = -E_\theta\left(\frac{-x}{\lambda^2} \right) = \frac{1}{\lambda} \quad \text{Jeffrey's prior} = \lambda^{-\frac{1}{2}}$$

$$f(\lambda \mid x_n) \propto \prod_i^n \left[e^{-\lambda} \frac{\lambda^{x_i}}{x_i!} \right] \cdot \lambda^{-\frac{1}{2}} = e^{-n\lambda} \lambda^{\sum x_i - \frac{1}{2}} \Rightarrow$$

$$\text{Gamma}\left(\sum x_i + \tfrac{1}{2}, n\right)$$

---

7. $\hat{\psi} = \frac{1}{\;} \sum^n R_i Y_i$

7. $\hat{\psi} = \frac{1}{n}\sum_{i=1}^{n} \frac{R_i Y_i}{\xi_{X_i}}$

$$E\left(\frac{1}{n}\sum_{1}^{n}\frac{R_iY_i}{\xi_{X_i}}\right) = \frac{1}{n}\sum_{1}^{n}\frac{E(R_iY_i)}{\xi_{X_i}} = \frac{n\psi}{n} = \psi$$

$$E(R_iY_i) = P(R_i = 1, Y_i = 1) = \xi_{X_i}\cdot P(Y_i = 1) = \xi_{X_i}\cdot\psi$$

$$V(\hat{\psi}) = V\left(\frac{1}{n}\sum_{1}^{n}\frac{R_iY_i}{\xi_{X_i}}\right) = \frac{1}{n^2}\sum_{1}^{n} V\frac{R_iY_i}{\xi_{X_i}}$$

$$= \frac{1}{n^2}\sum\left[E\left(\left(\frac{R_iY_i}{\xi_{X_i}}\right)^2\right) - \underbrace{\left(E\frac{(R_iY_i)}{\xi_{X_i}}\right)}_{\psi}^2\right] = \frac{1}{n}\left[E\left(\left(\frac{R_1Y_1}{\xi_{X_1}}\right)^2\right) - \psi^2\right]$$

$$E\left(\left(\frac{R_1Y_1}{\xi_{X_1}}\right)^2\right) = \sum_j \frac{1}{\xi_{x_j}^2} E(R_1^2 \mid X_1 = j)\, E(Y_1^2 \mid X_1 = j)\, P(X_1 = j)$$

?

$$= \sum$$

Chapter 12

Tuesday, July 13, 2021 5:15 PM

1. a) $X \sim \text{Binomial}(n, p)$, $p \sim \text{Beta}(\alpha, \beta)$

$L(\hat{p}, p) = (p-\hat{p})^2 \Rightarrow R(\hat{p}, p) = MSE = V(\hat{p}) + \text{bias}^2(\hat{p})$

Bayes risk $= \int R(\hat{p}, p) f(p)\, dp$

$= \int\int L(p, \hat{p}) f(x; p)\, dx \;\; f(p)\, dp$

Bayes estimator $\to E(p \mid X = x)$

$p \mid X \sim \text{Beta}(x+1, n-x+1)$

$E(p \mid X) = \dfrac{x+1}{n-x+1} =$ Bayes estimator

---

2. $X_1, \ldots, X_n \sim N(\theta, \sigma^2)$

$L(\theta, \hat{\theta}) = \dfrac{(\theta - \hat{\theta})^2}{\sigma^2}$ $\quad R(\theta, \hat{\theta}) = \dfrac{Var(\hat{\theta}) + \text{bias}^2(\hat{\theta})}{\sigma^2}$

$\hat{\theta} = \overline{X} \Rightarrow \text{bias}(\hat{\theta}) = 0 \quad Var(\hat{\theta}) = \dfrac{\sigma^2}{n}$

$R(\theta, \hat{\theta}) = \dfrac{1}{n}$

according to theorem 12.20 $\overline{X}$ is admissable under square error. in this case the square error is divided by a constant so the relations won't change.

since $\hat{\theta}$ is admissable and $R(\hat{\theta})$ = constant $(= \frac{1}{n})$

$\Rightarrow \hat{\theta}$ is minimax

$\left\{ \begin{array}{ll} 0 & \theta = \hat{\theta} \end{array} \right.$

3. $\Theta = \{\theta_1, \ldots, \theta_k\}$ $L(\theta, \hat{\theta}) \begin{cases} 0 & \theta = \hat{\theta} \\ 1 & \theta \neq \hat{\theta} \end{cases}$

$r(f, \hat{\theta}) = \int R(\theta, \hat{\theta}) f(\theta) d\theta$

bayes rule $= \inf_{\tilde{\theta}} r(f, \tilde{\theta})$

Posterior risk $r(\hat{\theta} | x) = \int L(\theta, \hat{\theta}) f(\theta | x) d\theta$

the bayes rule minimizes $r(\hat{\theta} | x)$

$= \int L(\theta, \hat{\theta}) f(\theta | x) d\theta = \sum_i I(\hat{\theta} \neq \theta_i) f(\theta_i | x)$

$= 1 - \sum_i I(\hat{\theta} = \theta_i) f(\theta_i | x) = 1 - f(\hat{\theta} | x)$

this is minimized when we have as much o's as possible which happens on the mode

4. $L(\sigma^2, \hat{\sigma}^2) = \frac{\hat{\sigma}^2}{\sigma^2} - 1 - \log\left(\frac{\hat{\sigma}^2}{\sigma^2}\right)$ $S^2 = \frac{1}{1-n} \sum (X_i - \bar{X})^2$

$E\left(\frac{bS^2}{\sigma^2} - 1 - \log\left(\frac{bS^2}{\sigma^2}\right)\right)$

$\frac{b}{\sigma^2} E(S^2) - 1 - \left(\log b + E\log(S^2) - \log(\sigma^2)\right)$

$= b - 1 - \log b + E\log(S^2) - \log(\sigma^2)$

$= b - \log b + \text{const} \Rightarrow \frac{dR}{db} = 0 \Rightarrow 1 - \frac{1}{b} = 0$

$\Rightarrow b = 1$

---

5. $X \sim \text{Binomial}(n, p)$

$L(p, \hat{p}) = \left(1 - \frac{\hat{p}}{p}\right)^2$ where $0 < p < 1$

$\hat{p}(X) = 0 \Rightarrow L(p, \hat{p}) = 1 \Rightarrow R(p, \hat{p}) = 1$

$R(p, \hat{p}) = E\left(1 - \frac{\hat{p}}{p}\right)^2 = E\left(1 + \frac{\hat{p}^2}{p^2} - \frac{2\hat{p}}{p}\right)$

$= 1 + \frac{E(\hat{p}^2)}{p^2} - \frac{2E(\hat{p})}{p}$

$\frac{E(\hat{p}^2)}{p^2} - \frac{2E(\hat{p})}{p} > 0$, $\frac{E(\hat{p}^2)}{p^2} > \frac{2E(\hat{p})}{p}$

$\frac{E(\hat{p}^2)}{2E(\hat{p})} > p$ if we choose p like this the risk will be higher than $\underline{1}$

Practice test 2

Friday, July 23, 2021 4:24 PM

1. $X_1, \dots X_n \sim \text{Bernoulli}(p)$

$\frac{1}{n}\sum_{i=1}^{n} X_i^{8} \xrightarrow{P} p \qquad E(X_i^{8}) = E(X_i) = p$

$V(X_i^{8}) = p - p^2 = p(1-p)$

$X_n \xrightarrow{qm} X : E(X_n - X)^2 \to 0$

2. $\mu = 68 \quad \sigma^2 = 16 \qquad E(\bar{X}) = 68 \quad V(\bar{X}) = \frac{16}{100}$

$\mu_f = 64 \quad \sigma_f^2 = 9 \qquad E(\bar{Y}) = 64 \quad V(\bar{Y}) = \frac{9}{100}$

$X_1, \dots, X_{100} \sim N(68, 16) \qquad W = \bar{X} - \bar{Y}$

$Y_1, \dots, Y_{100} \sim N(64, 9) \qquad E(W) = 4$

$V(W) = \frac{1}{4}$

3. $\lambda_n = \frac{1}{n} \quad X_n \sim \text{Poisson}(\lambda_n)$

$P(X_n > t) < \frac{E(X_n)}{t} = \frac{1}{nt} \quad n \to \infty \Rightarrow \frac{1}{nt} \to 0$

$Y_n = nX_n$

$P(|Y_n| \leq \varepsilon) \geq P(Y_n = 0) = P(X_n = 0) = e^{-\frac{1}{n}} \to 1$

$P(Y_n > \varepsilon) \to 0$

---

4) $X_1, X_2, \dots, X_n \sim \text{Bernoulli}(p)$

a) $f_X(x) = p^x (1-p)^{1-x}$

$\mathcal{L}_n(p) = \prod_i^n f(X_i; p) = \prod p^{x_i}(1-p)^{1-x_i} = p^{\sum x_i}(1-p)^{n-\sum x_i}$

$\ell_n(\theta) = \log p^{\sum x_i}(1-p)^{n-\sum x_i} = \sum X_i \log p + (n - \sum X_i)\log(1-p)$

$\frac{\sum X_i}{p} + \frac{\sum X_i - n}{1-p} = 0 \quad (p-1)\sum X_i = p(\sum X_i - n)$

$\sum X_i = s \quad \cancel{sp} - s = \cancel{sp} - pn \quad \hat{p} = \frac{s}{n} = \bar{X_i}$

$\frac{\partial^2 \log f(X;p)}{\partial p^2} = \frac{\partial^2\, x\log p + (1-x)\log(1-p)}{\partial p^2} = \frac{\partial\ \frac{x}{p} + \frac{x-1}{1-p}}{\partial p}$

$= \frac{-x}{p^2} + \frac{x-1}{(1-p)^2}$

$I(p) = -E_p\left(\frac{-x}{p^2} + \frac{x-1}{(1-p)^2}\right) = -\frac{1}{p} - \frac{1}{1-p} = \frac{-1}{p(1-p)}$

b) $\hat{\psi} = e^{\hat{p}} \quad g(\theta) = e^{\theta} \quad g'(\theta) = e^{\theta}$

$V(\hat{p}) = \frac{V(X_1)}{n} = \frac{p(1-p)}{n} \qquad \hat{se}(\hat{p}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

$\hat{se}(\hat{\psi}) = e^{\hat{p}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

c) we repeatedly sample from Berouli($\hat{p}$)

and calculate $\hat{p}_{bi}$ from the sample

then we apply $e^{\hat{p}_{bi}}$ to get $\hat{\psi}_{bi}$

repeat this B times

then we apply $e^{r_{bi}}$ to get Y_{bi}
repeat this B times

d) $\hat{F}(0) = 1 - \hat{p}$ $\hat{F}(1) = 1$ drawing from $\hat{F}$ is the same as drawing from Bernoulli($\hat{p}$)

---

5)

a) $X \sim \text{Binomial}(n, p)$ $X \to$ flower or not

$Y|X \sim \text{Binomial}(x, q)$ $Y \to$ fruit or not

$$f(x, y) = f(x) f(y|x) = \binom{n}{x} p^x (1-p)^{n-x} \binom{x}{y} q^y (1-q)^{x-y}$$

$$\mathcal{L}(p, q) \propto p^x (1-p)^{n-x} q^y (1-q)^{x-y}$$

b)

$$\ell(p, q) = x \log p + (n-x) \log(1-p) + y \log q + (x-y) \log(1-q)$$

$$\frac{\partial \ell}{\partial p} = 0 \Rightarrow \frac{x}{p} + \frac{x-n}{1-p} = 0 \quad n\cancel{p} - x = x\cancel{p} - np$$

$$p = \frac{x}{n}$$

$$\frac{\partial \ell}{\partial q} = 0 \Rightarrow \frac{y}{q} + \frac{y-x}{1-q} \Rightarrow \hat{q} = \frac{y}{x}$$

$$E(X) = np$$

$$E(Y) = E E(Y|X) = E(q\cancel{X}) = q E(X) = qnp$$

$n\hat{p} = X$ $\hat{p} = \frac{X}{n}$ $\hat{q} = \frac{Y}{X}$

$n\hat{q}\hat{p} = Y$ we basically do mme separetly for each

c)

$nqr = r$ we basically do m... ... for each

c)

---

6) $X_n \sim Normal(\frac{1}{n}, \frac{1}{n})$ $X_n \xrightarrow{P} 0$

$P(|X_n| > t) = P(X_n^2 > t^2) \leq \frac{E(X_n^2)}{t^2}$

$E(X_n^2) = V(X_n) + E(X_n)$

---

7) $X_n \sim Normal(\frac{1}{n}, n)$

$X_n \xrightarrow{P} X \Rightarrow X_n \rightsquigarrow X$

$F_n(c) = P(X_n \leq c)$

$= P\left(\frac{X_n - (\frac{1}{n})}{\sqrt{n}} \leq \frac{-\frac{1}{n}}{\sqrt{n}}\right) = P\left(Z \leq \frac{-\frac{1}{n}}{\sqrt{n}}\right)$

$\rightarrow P(Z \leq 0) = \frac{1}{2}$ this does not create a CDF

---

8) $X_n \sim N(0, \frac{1}{n})$ $F(x) \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$

a) $P(|X_n| > \varepsilon) = P(|X_n|^2 > \varepsilon^2)$

$\leq \frac{E(X_n^2)}{\varepsilon^2} = \frac{\frac{1}{n}}{\varepsilon^2} \rightarrow 0$ $X_n \xrightarrow{P} 0 \Rightarrow$

b) example 5.3 in book

---

9) $X_n \rightsquigarrow X \iff \lim_{n\to\infty} P(X_n=k) = P(X=k)$

for every int k

$$\lim_{n\to\infty} P(X_n=k) = P(X=k)$$

$$F(X_n=t) = \sum_{-\infty}^{t} P(X_n=t) = \sum_{-\infty}^{t} P(X=k) = F(X=t)$$

$$\lim_{n\to\infty} F(X_n=t) = F(X=t) \Rightarrow P(X_n=t) = F(X_n=t) - F(X_n=t-1)$$

$$\rightarrow F(X=t) - F(X=t-1) = P(X=t)$$

---

10) $$P_n(x) = \begin{cases} \frac{1}{2} & x = -(\frac{1}{2})^n \\ \frac{1}{2} & x = (\frac{1}{2})^n \\ 0 & \text{o.w} \end{cases}$$

$$P(x) = \lim_{n\to\infty} P_n(x)$$

a) yes the sum of probabilities will always be 1

b) $F_n(x) = \begin{cases} 1 & x \geq (\frac{1}{2})^n \end{cases}$

b) $F_n(x) = \begin{cases} 1 & x \geq (\frac{1}{2})^n \\ \frac{1}{2} & (\frac{1}{2})^n > x \geq (\frac{1}{2})^n \\ 0 & \text{o.w} \end{cases}$

yes to point mass at $\underline{1}$

---

11) $\psi = \dfrac{E(X^5)\,E(X^4)}{E(X^3)} = \dfrac{1}{n}\dfrac{\sum x_i^5 \sum x_i^4}{\sum x_i^3}$

$\int x^5 \, d\hat{F}_n(x) = \dfrac{1}{n}\sum^n x_i^5$

Sample from $\hat{F}_n(x)$ calculate ψ
repeat B times
calculate se from ψ_{bi}'s

---

12) $X_1, \ldots, X_n \sim \text{Bernoulli}(p)$

$\hat{F}_n(x) = \dfrac{\sum I(x_i \leq x)}{n}$

$\hat{F}_n(x) = \begin{cases} 1 & 1 \\ \dfrac{n - \sum x_i}{n} & 0 \end{cases}$ $\quad F(x) = \begin{cases} 1 & 1 \\ 1-p & 0 \end{cases}$

$$\frac{n-\sum X_i}{n} - 1 + p = p - \frac{\sum X_i}{n}$$

$$P\left(p - \sum X_i > \varepsilon\right) < \frac{E\left(p - \frac{\sum X_i}{n}\right)}{\varepsilon} = \frac{0}{\varepsilon}$$

---

13) $X \sim \text{Multinomial}(n, p)$

$X = (X_1 \ldots X_k) \quad p = (p_1 \ldots p_k)$

Practice test 3

Thursday, July 29, 2021 8:04 PM

3) $X_1, \ldots, X_n \sim \text{Poisson}(\lambda)$

$f(\lambda) = \frac{1}{\sqrt{\lambda}}$ prior

a) $f(\lambda | x^{(n)}) \propto \mathcal{L}(\lambda) f(\lambda)$

$$f_\lambda(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$$\frac{\prod \lambda^{x_i} e^{-\lambda}}{x_i!} \cdot \frac{1}{\sqrt{\lambda}} \propto \lambda^{\sum x_i - \frac{1}{2}} \cdot e^{-n\lambda}$$

$\lambda | x \sim \text{gamma}(\sum x_i + \frac{1}{2}, n)$

b) $E(\lambda | x) = \frac{\alpha}{\beta} = \frac{\sum x_i + \frac{1}{2}}{n}$

---

4) $f(x; \beta) = \beta e^{-\beta x}$

---

10) $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$

a)

Chapter 23

Friday, August 6, 2021 4:27 PM

1. $P = \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.9 & 0.1 & 0.0 \\ 0.1 & 0.8 & 0.1 \end{bmatrix}$

$\mu_0 = (0.3, 0.4, 0.3)$

$P(X_0=0, X_1=1, X_2=2) = P(X_2=2 \mid X_1=1, X_0=0)\, P(X_1=1, X_0=0)$

$= P(X_2=2 \mid X_1=1, X_0=0)\, P(X_1=1 \mid X_0=0)\, P(X_0=0)$

$= P(X_2=2 \mid X_1=1)\, P(X_1=1 \mid X_0=0)\, P(X_0=0)$

$= 0$

$P(X_0=0, X_1=1, X_2=1) = 0.1 \times 0.2 \times 0.3 = 0.006$

---

2. $Y_1, Y_2, Y_3, \ldots$

$P(Y=0)=0.1$

$P(Y=1)=0.3$

$P(Y=2)=0.2$

$P(Y=3)=0.4$

$X_n = \max\{Y_1, \ldots, Y_n\}$

$P(X_1=n) = P(Y=n)$

$\begin{bmatrix} 0.1 & 0.3 & 0.2 & 0.4 \\ 0 & 0.4 & 0.2 & 0.4 \\ 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

---

3. $P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$ $\lim_{n\to\infty} P^n = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$

$\begin{bmatrix} 1-a-\lambda & a \\ b & 1-b-\lambda \end{bmatrix} \Rightarrow 1-a-b-2\lambda + \cancel{ab} + \lambda^2 + a\lambda + b\lambda - \cancel{ab}$

$\lambda^2 + (a+b-2)\lambda + (1-a-b) = \det(P - I\lambda)$

$P^2 = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} = \begin{bmatrix} a^2-2a+1+ab & 2a-a^2-ab \\ 2b-ab-b^2 & b^2-2b+1+ab \end{bmatrix}$

$\begin{matrix} (1-a)^2+ab & -(1-a)^2-(ab+1) \\ (1-b)^2-(ab+1) & -(1-b)^2+ab \end{matrix} = \begin{bmatrix} 1-c & c \\ d & 1-d \end{bmatrix}$

$\lambda_1 + \lambda_2 = 2-a-b$ $\quad a^2+b^2+\cancel{4}+2ab-\cancel{4a}-\cancel{4b}$

$\lambda_1 \lambda_2 = 1-a-b$ $\quad -\cancel{4}+\cancel{4a}+\cancel{4b}$

$(a+b)^2 = (a+b)$ $\quad \dfrac{2-a-b \pm (a+b)}{2} = 1, 1-a-b$

$X_1 = \begin{bmatrix} 1 \\ \; \end{bmatrix}$ $\quad \begin{bmatrix} a \\ \; \end{bmatrix}$

$X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad$ $2 \begin{bmatrix} a \\ -b \end{bmatrix}$

$\left[\frac{b}{a+b} \quad \frac{a}{a+b}\right] = \Pi$ sat. detailed balance $\Rightarrow \Pi$ stationary

so $\Pi = \Pi P \Rightarrow \lim_{n\to\infty} P^n = \begin{bmatrix} \Pi \\ \Pi \end{bmatrix}$

---

4. computer experiment

---

5. $p_k = P(Y=k)$ $\quad \sum_0^\infty p_k = 1 \quad X_0 = 1$

$X_{n+1} = Y_1^{(n)} + \cdots + Y_{X_n}^{(n)}$

a) $m(n+1) = \mu m(n) \qquad E(X_{n+1}) = \mu E(X_n)$

$v(n+1) = \sigma^2 m(n) + \mu^2 v(n)$

$m(n+1) = E(X_{n+1}) = E(Y_1^{(n)} + \cdots + Y_{X_n}^{(n)}) =$

$E(Y_1^{(n)}) + \cdots + E(Y_{X_n}^{(n)}) = E\left[\sum_i^{X_n} Y_i^{(n)}\right] = E\left[\sum_i^{X_n} E(Y_i^{(n)})\right]$

$= E\left[\sum_1^{X_n} \mu\right] = E[X_n \mu] = \mu E[X_n] = \mu m(n)$

$V(n+1) = V(X_{n+1}) = E(X_{n+1}^2) - E^2(X_{n+1})$

b) $m(n) = \mu^n$ by induction

c) $\mu > 1 \to V(n) \to \infty$

$\mu = 1 \to V(n) = n\sigma^2$

$\mu < 1 \to V(n) \to 0$

d) $X_n = 0 \quad N = \min\{n : X_n = 0\}$

$F(n) = P(N \leq n) \qquad F(n) = \sum_0^\infty p_k (F(n-1))^k, \; n = 1, 2, \ldots$

$P(N \leq n) = P(X_n = 0)$

$F(n) = P(X_n = 0) = \sum^\infty P(X_n = 0 \mid X_1 = k) P(X_1 = k)$

$$F(n) = P(X_n = 0) = \sum_{k=0}^{\infty} P(X_n = 0 \mid X_1 = k) P(X_1 = k)$$

$$= \sum_0^{\infty} p_k P(X_n = 0 \mid X_1 = k) = \sum_0^{\infty} p_k F(n-1)^k$$

e) $p_0 = \frac{1}{4} \quad p_1 = \frac{1}{2} \quad p_2 = \frac{1}{4}$

$$F(n+1) = p_0 + p_1 F(n) + p_2 F(n)^2 = \frac{1}{4} + \frac{1}{2} F(n) + \frac{1}{4} F(n)^2$$

$$= \frac{1}{4} (F(n) + 1)^2$$

6.
$$P = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.05 & 0.7 & 0.25 \\ 0.05 & 0.5 & 0.45 \end{bmatrix}$$

$$P^T = \begin{bmatrix} 0.4 & 0.05 & 0.05 \\ 0.5 & 0.7 & 0.5 \\ 0.1 & 0.25 & 0.45 \end{bmatrix} \text{ has } \lambda_1 = 1$$

The eigenvector of $\lambda_1 = 1$ is the stationary distribution

rest computer

---

7. i recurrent and $i \leftrightarrow j \Rightarrow j$ recurrent

$$P(X_n = i \text{ for some } n \geq 1 \mid X_0 = i) = 1$$

$$p_{ij}(m) > 0 \text{ for some } m$$

$$p_{ji}(m') > 0 \text{ for some } m'$$

$$\sum_n p_{ii}(n) = \infty$$

$$P_{jj}(x) = P_{ii}(n') P_{ji}(m) P_{ij}(m')$$

$$m + m' + n' = n$$

$$\Rightarrow \sum_n P_{jj}(n) = \infty$$

---

8. recurrent: 6, 3, 5

transient: 1, 2, 4

---

9. $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \pi P = \pi$

10.

$\pi P = \pi$ solve for π

---

11. $\lambda(t) > 0$

$\Lambda(t) = \int_0^t \lambda(u)\,du \quad Y(s) = X(t) \quad s = \Lambda(t)$

$X(t) \sim \text{Poisson}(\Lambda(t)) \Rightarrow Y(s) \sim \text{Poisson}(s)$

$Y(s) = X(\Lambda^{-1}(s)) \sim \text{Poisson}(\Lambda(\Lambda^{-1}(s))) = \text{Poisson}(s)$

---

12. $X(t) \sim \text{Poisson}(\lambda t)$

$P(X(t) = n \mid X(t+s) = n) = P(X(t) = n$

$P(X(t)=n \mid X(t+s)=n) = P(X(t)=n$

$X(s+t)-X(t) \sim \text{Poisson}(\lambda s)$

---

13. $X(t) \sim \text{Poisson}(\lambda t)$ $\quad \frac{\lambda^k e^{-\lambda}}{k!}$

$P(X(t)=1)+P(X(t)=3)+\cdots$

$\frac{(\lambda t)^1 e^{-\lambda t}}{1} + \frac{(\lambda t)^3 e^{-\lambda t}}{3!} + \cdots$

$e^{-\lambda t}\sum_{k=0}^{\infty}\frac{(\lambda t)^{2k-1}}{k!} = e^{-\lambda t}\sinh(\lambda t) = \frac{1}{2}(1-e^{-2\lambda t})$

---

14. $X(t) \sim \text{Poisson}(\lambda t)$

$P(T<t)=G(t)$ $\quad A(j)=X(j)-X(j-1)$

$Y(t)=\sum_{j=1}^{t}\sum_{i=1}^{A(j)} I(W_i^{(j)} \geq t-j)$

$Y(t)=\sum_{j=1}^{t} B(j)$ $\quad B(j)=\sum_{i=1}^{A(j)} I(W_i^{(j)} \geq t-j)$

$B(j) \mid A(j)=n \sim \text{Binomial}(n, 1-G(t))$

$A(j) \sim \text{Poisson}(\lambda) \Rightarrow$

$A(j) \sim \text{Poisson}(\lambda) \Rightarrow$

$B(j) \sim \text{Poisson}(\lambda(1 - G(t-j)))$

$\Rightarrow Y(t) \sim \text{Poisson}(\lambda_Y(t))$,

$$\lambda_Y(t) = \lambda\left[\sum_{1}^{t}(1 - G(t-j))\right]$$

---

15. $X(t) \sim \text{Poisson}(\lambda t)$

$$E\left(\sum_{1}^{X(t)} f(W_i)\right) = \lambda\int_0^t f(w)\,dw$$

$$W_i \sim \text{Gamma}\left(i, \frac{1}{\lambda}\right)$$

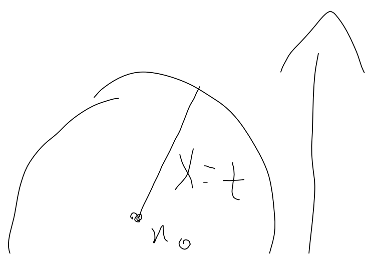
$$\sum_{1}^{X(t)} E\left(f(W_i)\right) = \sum_{1}^{X(t)} \int f(w_i)\frac{1}{\Gamma(i)}\lambda^i w_i^{i-1} e^{-\lambda w_i}\,dw$$

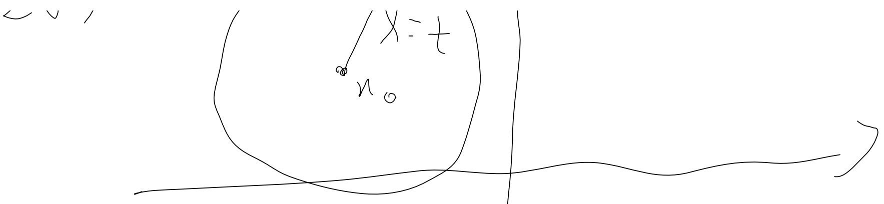
$$= \sum_{1}^{X(t)} \frac{1}{\Gamma(i)}\lambda^i \int f(w_i)\, w_i^{i-1} e^{-\lambda w_i}\,dw$$

---

16. $P(X > t) = e^{-\lambda\pi t^2}$

$$E(X) = \frac{1}{2\sqrt{\lambda}}$$





$$P(\text{point in circle}) = \lambda\mu(A) = \lambda\pi x^2 = P(X<t)$$

$$P(X>t) = P(\#\text{ in circle} = 0)$$

$$= e^{-\lambda\pi t^2} \qquad P(X<t) = 1-e^{-\lambda\pi t^2}$$

$$f(t) = 2\pi\lambda t e^{-\lambda\pi t^2}$$

$$E(X) = \int f(t)\,dt$$

$$= 2\pi\lambda \int t^2 e^{-\lambda\pi t^2}\,dt = \frac{1}{2\sqrt{\lambda}}$$

appendixA.pdf

Wednesday, August 11, 2021 4:33 PM

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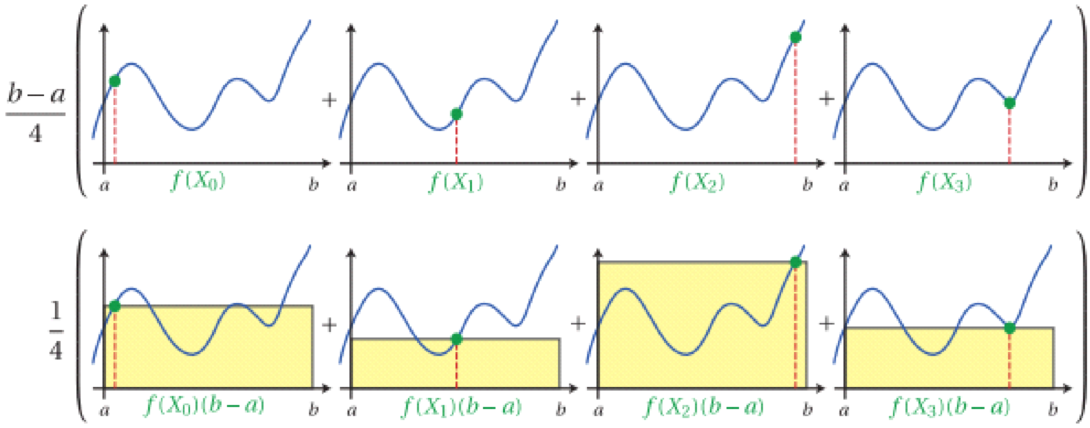

Figure A.1: An illustration of the two interpretations of the basic Monte Carlo estimator in Equation A.12 using four samples: computing the mean value, or height, of the function and multiplying by the interval length (top), or computing the average of several rectangular areas (bottom).

Chapter 24

Thursday, August 12, 2021 6:18 PM

1. $I = \int_1^2 \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = \phi(2) - \phi(1)$

$= \int_1^2 \frac{e^{-x^2/2}}{\sqrt{2\pi}} f(x)\, dx \quad f(x) = \frac{1}{2-1} = 1$

$\hat{I} = \frac{1}{N} \sum_1^n w(X_i)$

b)

$V(\hat{I}) = \frac{1}{N} V(\overset{?}{w}(X_1)) = \frac{1}{n}\left(E(w^2(X_1)) - E^2(w(X_1)) \right)$

$= \frac{1}{N}\left(E(w^2(X_1)) - I^2 \right)$

$\varepsilon_i = Y_i - I$

c) $g \sim N(1.5, v^2)$ $\qquad h(x) = I(1<x<2)$

$\int h(x) f(x) dx = \int \frac{h(x)f(x)}{g(x)} g(x) dx$

$\hat{I} = \frac{1}{N} \sum \frac{h(x)f(x)}{g(x)} = \frac{1}{N} \sum \frac{w(x_i)}{f(x_i)}$ ↳ $\sim Norm(1.5, 1)$

d) $g^*(x) = \frac{|h(x)| f(x)}{\int |h(s)| f(s) ds} = \begin{cases} \frac{f(x)}{\int_1^2 f(s) ds} = \frac{f(x)}{\phi(2) - \phi(1)} & 2 > x > 1 \\ 0 & 0.w \end{cases}$

---

2. a) $\hat{f}_X(x) = \frac{1}{N} \sum_{i=1}^{N} \frac{f_{X,Y}(x, Y_i) w(X_i)}{f_{X,Y}(X_i, Y_i)}$

$I = \int h(x) f(x) dx = \int \frac{h(x)f(x)}{g(x)} g(x) dx$

$\hat{f}_X(x_0) \xrightarrow{P} E\left(\frac{h(x)f(x)}{g(x)} \right) = f_X(x_0)$

$h(x) f(x)$ ... [illegible]

$f_X(x_0) \quad (\quad g(x))$

$$E\left(\frac{h(x)f(x)}{g(x)}\right) = \iint \frac{f_{X,Y}(x_0,y)\,w(x)}{\cancel{f_{X,Y}(x,y)}}\cancel{f_{X,Y}(x,y)}\,dx\,dy$$

$$= \iint f_{X,Y}(x_0,y)\,w(x)\,dx\,dy$$

$$= \int f_{X,Y}(x_0,y)\,dy = f_X(x_0)$$

b) $Y \sim N(0,1) \quad X|Y=y \sim N(y, 1+y^2)$

$$f_{X,Y}(x,y) = f_Y(y)\,f_{X|Y}(x,y)$$

---

3) $f(x) \leq M g(x)$

a) $\dfrac{f(x)}{M g(x)} \leq 1$

$$P(Y \leq y) = P(X_1 \leq y) \cdot P\left(U \leq \frac{f(X)}{M g(x)}\right) = P(X \leq y) \cdot \frac{f(x)}{M g(x)}$$

b) $g(x) = \dfrac{1}{1+x^2}$ $\quad f(x) = \text{normal}(x, 0, 1)$

---

4)