MAT 310 HW 8, Carl Liu

6.A.8

Suppose $u, v \in V$ and $||u|| = ||v|| = 1$ and $\langle u, v\rangle = 1$. Then $\sqrt{\langle u, u\rangle} = ||u|| = 1$ and $\sqrt{\langle v, v\rangle} = ||v|| = 1$. Also $\langle u, v\rangle = \overline{\langle v, u\rangle} = 1$ and so when applying another conjugate we have $\overline{\overline{\langle v, u\rangle}} = \overline{1}$. Since 1 is real and the conjugate of a conjugate is just the original, we end up with $\langle v, u\rangle = 1$ So $\langle v, v\rangle = 1$ and $\langle u, u\rangle = 1$. That means $\langle u, u\rangle = \langle u, v\rangle = \langle v, u\rangle = \langle v, v\rangle$. Thus $\langle u, u\rangle - \langle u, v\rangle = 0$ resulting in $\langle u, u - v\rangle = 0$. Also we have $\langle v, u\rangle - \langle v, v\rangle = 0$ resulting in $\langle v, u - v\rangle = 0$. Thus $\langle u, u - v\rangle - \langle v, u - v\rangle = 0$ which means $\langle u - v, u - v\rangle = 0$. Therefore $u - v = 0$ resulting in $u = v$ as required.

6.A.10

A vector orthogonal to $(1, 3)$ will have the property $v_1 + 3v_2 = 0$. Thus resulting in $v_1 = -3v_2$. Since we also have $(1, 2) = u + v$ where $u = \lambda(1, 3)$, we must have $1 = \lambda - 3v_2$ and $2 = 3\lambda + v_2$. Thus we have $7 = 10\lambda$, so $\lambda = 7/10$. That means $v_2 = -1/10$ resulting in vectors

$$(1, 2) = \frac{7}{10}(1, 3) + \frac{1}{10}(3, -1)$$

6.B.2

Suppose $e_1 ... e_m$ is an orthonormal list of vectors in V. Let $v \in V$. We can extend $e_1 ... e_m$ to an orthonormal basis of V which we will call $e_1 ... e_m, f_1 ... f_n$. Then $||v||^2 = |\langle v, e_1\rangle|^2 + ... + |\langle v, e_m\rangle|^2 + |\langle v, f_1\rangle|^2 + ... + |\langle v, f_n\rangle|^2$.

Suppose $||v||^2 = |\langle v, e_1\rangle|^2 + ... + |\langle v, e_m\rangle|^2$. But from hypothesis we have $0 = |\langle v, f_1\rangle|^2 - ||v||^2 = |\langle v, f_1\rangle|^2 + ... + |\langle v, f_n\rangle|^2$ and so $\langle v, f_n\rangle = 0$. Since $v = \langle v, e_1\rangle e_1 + ... + \langle v, e_m\rangle e_m + \langle v, f_1\rangle f_1 + ... \langle v, f_n\rangle f_n$. We would then have $v = \langle v, e_1\rangle e_1 + ... + \langle v, e_m\rangle e_m$. But this is a linear combination of $e_1 ... e_m$ meaning $v \in span(e_1 ... e_m)$ as required.

Suppose $v \in span(e_1 ... e_m)$. Since $span(e_1 ... e_m)$ is clearly a vector space and $e_1 ... e_m$ is an orthonormal basis of $span(e_1 ... e_m)$, we have $||v||^2 = |\langle v, e_1\rangle|^2 + ... + |\langle v, e_m\rangle|^2$ as required.

6.B.5

We have as our first orthonormal vector

$$e_1 = \frac{\langle 1,1\rangle 1}{\sqrt{\langle 1,1\rangle}} = \frac{\int_0^1 1^2 dx}{\sqrt{\int_0^1 1^2 dx}} = 1$$

Then we have

$$e_2 = \frac{x - \langle x,1\rangle 1}{||x - \langle x,1\rangle 1||} = \frac{x - \int_0^1 x dx}{||x - \langle x,1\rangle 1||} = \frac{x-0.5}{||x-0.5||} = \frac{x-0.5}{\sqrt{\int_0^1 (x-0.5)^2}} =$$

$$\frac{x-0.5}{\sqrt{\int_0^1 (x-0.5)^2}} = (x-0.5)\sqrt{12}$$

Finally we have

$$e_3 = \frac{x^2 - \sqrt{12}\langle x^2, x-0.5\rangle (x-0.5) - \langle x^2,1\rangle 1}{||x^2 - \sqrt{12}\langle x^2, x-0.5\rangle (x-0.5) - \langle x^2,1\rangle 1||} =$$

$$\frac{x^2 - \sqrt{12}\int_0^1 (x^3 - 0.5x^2)dx(x-0.5) - \int_0^1 x^2 dx}{||x^2 - \sqrt{12}\int_0^1 (x^3 - 0.5x^2)dx(x-0.5) - \int_0^1 x^2 dx||} =$$

$$\frac{x^2 - \frac{\sqrt{12}}{12}(x-0.5) - \frac{1}{3}}{||x^2 - \frac{\sqrt{12}}{12}(x-0.5) - \frac{1}{3}||} = \frac{x^2 - \frac{\sqrt{12}}{12}x + \frac{\sqrt{12}-8}{24}}{||x^2 - \frac{\sqrt{12}}{12}x + \frac{\sqrt{12}-8}{24}||} =$$

$$\frac{x^2 - \frac{\sqrt{12}}{12}x + \frac{\sqrt{12}-8}{24}}{\int_0^1 (x^2 - \frac{\sqrt{12}}{12}x + \frac{\sqrt{12}-8}{24})^2 dx} =$$

$$\frac{x^2 - \frac{\sqrt{12}}{12}x + \frac{\sqrt{12}-8}{24}}{\int_0^1 (x^4 - 2\frac{\sqrt{12}}{12}x^3 + 2\frac{\sqrt{12}-8}{24}x^2 + \left(\frac{\sqrt{12}}{12}\right)^2 x^2 - 2\frac{\sqrt{12}-8}{24}\frac{\sqrt{12}}{12}x + \left(\frac{\sqrt{12}-8}{24}\right)^2)dx} =$$

$$\frac{x^2 - \frac{\sqrt{12}}{12}x + \frac{\sqrt{12}-8}{24}}{\frac{1}{5} - \frac{\sqrt{12}}{24} + 2\frac{\sqrt{12}-8}{24*3} + \frac{1}{3}\left(\frac{\sqrt{12}}{12}\right)^2 - \frac{\sqrt{12}-8}{24}\frac{\sqrt{12}}{12} + \left(\frac{\sqrt{12}-8}{24}\right)^2} =$$

$$\frac{x^2 - \frac{\sqrt{12}}{12}x + \frac{\sqrt{12}-8}{24}}{0.0477}$$

6.C.2

Suppose U is a finite dimensional subspace of V.

Suppose $U^{\perp} = \{0\}$. Let $v \in V$. Then $\langle v, 0 \rangle = 0$. Meaning $v \in (U^{\perp})^{\perp}$. Since $(U^{\perp})^{\perp} = U$. We can conclude $v \in U$ and so $V \subseteq U$. Now let $u \in U$. Then $u \in V$ by definition of subspace. Thus $U \subseteq V$. Therefore we can conclude $U = V$ as required.

Suppose $U = V$. Let $u \in U$. Then $\langle U, 0 \rangle = 0$ so $0 \in U^{\perp}$. Thus $\{0\} \subseteq U^{\perp}$. Now let $u \in U^{\perp}$. Since we have $U^{\perp} \subseteq V$ and $V = U$, we have $u \in U$ thus. $\langle u, u \rangle = 0$. But that means $u = 0$ is a must. Therefore $U \subseteq \{0\}$ and so we have $U = \{0\}$ as required