MAT 310 HW 6, Carl Liu

4.4

Suppose m and n are positive integers with $m \leq n$, and suppose $\lambda_1, ..., \lambda_m \in \mathbf{F}$. Consider the polynomial $p(z) = (z - \lambda_1)...(z - \lambda_m)^{n-m+1}$. It can be seen that $p(\lambda_j) = 0$ where $1 \leq j \leq m$ since the factor $(z - \lambda_j) = (\lambda_j - \lambda_j) = 0$. Also $deg(p) = (m - 1) + (n - m + 1) = n$. So suppose for contradiction that there exists a $\lambda \in \mathbf{F}$ such that $p(\lambda) = 0$ and $\lambda \neq \lambda_j$ for all $1 \leq j \leq m$. Then $p(\lambda) = (\lambda - \lambda_1)...(\lambda - \lambda_m)^{n-m+1}$. But clearly no terms are 0 in this factorization. So $p(\lambda) \neq 0$. But that is a contradiction, thus we must have $\lambda = \lambda_j$ for some $i \leq j \leq m$. Therefore p has no other zeroes as required.

4.6

Suppose $p \in \mathcal{P}(\mathbf{C})$ has degree m.

Suppose p has m distinct zeroes. In the base case $m = 1$, we have $p = az + r$ for some constant $a, r \in \mathbf{C}$ and $p' = a$, the two don't have any zeroes in common since p' has no zeroes and the zero of p is $-r/a$. Now suppose as inductive hypothesis that for $m > 1$, all $q \in \mathcal{P}(\mathbf{C})$ that has degree $m-1$ and $m-1$ distinct zeroes has q' such that q and q' have no zeroes in common. Since p has m distinct zeroes and p has a factorization $c(z-\lambda_1)...(z-\lambda_m)$ which is unique, we must have $\lambda_1, ..., \lambda_m$ being distinct since they are all zeroes. But $p(z) = c(z - \lambda_1)...(z - \lambda_{m-1})(z - \lambda_m)$. Setting $k(z) = c(z - \lambda_1)...(z - \lambda_{m-1})$, we have a polynomial of degree $m - 1$ with distinct zeroes. So by inductive hypothesis we have $k'(z)$ also has no zeroes in common with $k(z)$. We then have $p(z) = k(z)(z - \lambda_m)$ and $p'(z) = k'(z)(z - \lambda_m) + k(z)$. Suppose for contradiction that for some $1 \leq j \leq m$, λ_j is a zero of $p'(z)$. Then $p'(\lambda_j) = 0$ and $p'(\lambda_j) = k'(\lambda_j)(\lambda_j - \lambda_m) + k(\lambda_j)$. In the case $1 \leq j \leq m - 1$, we can see that λ_j is a root of k and so can't be a root of k' meaning $k'(\lambda_j) \neq 0$. Also $\lambda_j \neq \lambda_m$ by hypothesis so $\lambda_j - \lambda_m \neq 0$. Thus we have $p'(\lambda_j) = k'(\lambda_j)(\lambda_j - \lambda_m) + k(\lambda_j) = p'(\lambda_j) = k'(\lambda_j)(\lambda_j - \lambda_m) \neq 0$. A contradiction. Thus $p'(z)$ does not have λ_j as zeroes where $1 \leq j \leq m - 1$. In the case $j = m$, we have $p'(\lambda_m) = k'(\lambda_m)(\lambda_m - \lambda_m) + k(\lambda_m) = k(\lambda_m)$. But because λ_m is not a root of k, we must have $p'(\lambda_m) \neq 0$. A contradiction. Thus $p'(z)$ can't have λ_m as a root. Therefore there exists no common roots between p and p'. Thus we close the induction

Suppose p and it's derivative p' have no zeroes in common. When $m = 1$, we have $p = az + r$ for some constant $a, r \in \mathbf{C}$ and $p' = a$. There is only one zero for p specifically $-r/a$ and is thus distinct thereby proving case $m = 1$. For $m > 1$ p has a unique factorization $p(z) = c(z - \lambda_1)...(z - \lambda_m)$. Define $q_j(z)$ as $p(z)$ with the factor $(z - \lambda_j)$ removed. We have by product rule

$$p'(z) = \sum_{n=1}^{m} q_n(z)$$

Suppose for contradiction that λ_j is not distinct for some $1 \leq j \leq m$. Then there exists λ_r such that $\lambda_r = \lambda j$ and $j \neq r$. We have $p'(\lambda_j) = n * q_j(j)$ for some integer $n > 1$. But $q_j(z)$ is just $p(z)$ with the factor $(z - \lambda_j)$ removed. Though it still has the factor $(z - \lambda_r)$ which when evaluated at λ_j results in 0. This means $q_j(\lambda_j) = 0$ and so λ_j is a zero of both $p'(z)$ and $p(z)$. A contradiction. Thus we must have p having distinct zeroes

5.A.8

We have λ is an eigenvalue when $T(w, z) = \lambda(w, z)$. That means we must have $T(w, z) = (z, w) = \lambda(w, z)$. Thus $z = \lambda w$ and $w = \lambda z$ resulting in $z = \lambda^2 z$. Therefore $1 = \lambda^2$, resulting in $\lambda = \pm 1$ as eigenvalues. In the case $\lambda = 1$, we have $z = \lambda w = w$. Therefore we have the eigenvector (z, z). In the case $\lambda = -1$, we have $z = \lambda w = z = -w$ resulting in eigenvector $(w, -w)$.

5.A.9

We have λ is an eigenvalue when $T(z_1, z_2, z_3) = \lambda(z_1, z_2, z_3)$. So $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3) = \lambda(z_1, z_2, z_3)$. Thus we have $2z_2 = \lambda z_1$, $z_2 = \lambda 0$, and $5z_3 = \lambda z_3$. Therefore $5 = \lambda$ and we must have $z_2 = 0$ which in turn means $z_1 = 0$. Thus we only have one eigenvalue for this equation. The eigenvector corresponding to this eigenvalue is then $(0, 0, z)$.

5.A.15 A)

Suppose $T \in \mathcal{L}(V)$ and $S \in \mathcal{L}(V)$ is invertible. Let λ be a eigenvalue of v. Then $T - \lambda I$ is not invertible. Consider now $S^{-1}(T - \lambda I)S$. Let $v \in V$. Then $S^{-1}(T - \lambda I)Sv = S^{-1}(T - \lambda I)(Sv)$. Since S maps to V, we have $Sv = w$ for some $w \in V$. Which then results in $S^{-1}(T - \lambda I)w$. But because $(T - \lambda I)$ is not invertible, $S^{-1}(T - \lambda I)$ is also not invertible and $S^{-1}, (T - \lambda I)$ are both operators, by question $3.D.9$ we must have $S^{-1}(T - \lambda I)$ is not invertible. That in turn means $S^{-1}(T - \lambda I)S$ is not invertible. Since

$S^{-1}(T - \lambda I)S = S^{-1}TS - \lambda S^{-1}IS = S^{-1}TS - \lambda S^{-1}S = S^{-1}TS - \lambda I$, we have by 5.6 that λ is an eigenvalue of $S^{-1}TS$ as required.