MAT 341 HW6, Carl Liu

1.

We have

$$b_m = \frac{\int_0^a g(x)\sin(\lambda_m)dx}{\int_0^a \sin^2(\lambda_m)dx} = \frac{\int_0^a \sin(\lambda_m x)dx}{\int_0^a \sin^2(\lambda_m x)dx} = \frac{\left[-\frac{1}{\lambda_m}\cos(\lambda_m x)\right]_0^a}{\frac{a}{2} + \frac{\kappa}{h}\frac{\cos^2(\lambda_m a)}{2}} =$$

$$\frac{2\left(-\frac{1}{\lambda_m}\cos(\lambda_m a) + \frac{1}{\lambda_m}\right)}{a + \frac{\kappa}{h}\cos^2(\lambda_m a)} = \frac{2 - 2\cos(\lambda_m a)}{\lambda_m\left(a + \frac{\kappa}{h}\cos^2(\lambda_m a)\right)}$$

2.

having $g(x) = T$ will result in

$$b_m = T\frac{2 - 2\cos(\lambda_m a)}{\lambda_m\left(a + \frac{\kappa}{h}\cos^2(\lambda_m a)\right)}$$

Since $\tan(\lambda_m a) = -\frac{\kappa}{h}\lambda_m$, we have the first 3 terms of the solution as

$$T_0 + \frac{xh(T_1 - T_0)}{\kappa + ha} +$$

$$T\left(\frac{(2 - 2\cos(\lambda_1 a))e^{-\lambda_1^2 kt}}{\lambda_1\left(a + \frac{\kappa}{h}\cos^2(\lambda_1 a)\right)} + \frac{(2 - 2\cos(\lambda_2 a))e^{-\lambda_2^2 kt}}{\lambda_2\left(a + \frac{\kappa}{h}\cos^2(\lambda_2 a)\right)} + \frac{(2 - 2\cos(\lambda_3 a))e^{-\lambda_3^2 kt}}{\lambda_3\left(a + \frac{\kappa}{h}\cos^2(\lambda_3 a)\right)}\right)$$

where $\lambda_1, \lambda_2, \lambda_3$ are the first, second, and third positive terms that satisfy

$$\tan(\lambda_m a) = -\frac{\kappa}{h}\lambda_m$$

3.

We have through integration by parts

$$\int_0^a \sin(\lambda_n x)\sin(\lambda_m x)dx =$$

$$\left[-\frac{1}{\lambda_m}\sin(\lambda_n x)\cos(\lambda_m x)\right]_0^a + \frac{\lambda_n}{\lambda_m}\int_0^a \cos(\lambda_n x)\cos(\lambda_m x)dx =$$

$$-\left[\frac{1}{\lambda_m}\sin(\lambda_n x)\cos(\lambda_m x)\right]_0^a + \left[\frac{1}{\lambda_m}\sin(\lambda_n x)\cos(\lambda_m x)\right]_0^a$$

$$-\int_0^a \sin(\lambda_n x)\sin(\lambda_m x)dx =$$

$$\frac{\sin(\lambda_n a)\cos(\lambda_m a) - \sin(\lambda_n a)\cos(\lambda_m a)}{\lambda_m} - \int_0^a \sin(\lambda_n x)\sin(\lambda_m x)dx$$

Therefore we have

$$\int_0^a \sin(\lambda_n x)\sin(\lambda_m x)dx = \frac{\sin(\lambda_n a)\cos(\lambda_m a) - \sin(\lambda_n a)\cos(\lambda_m a)}{2\lambda_m} = 0$$

when $m \neq n$. But when $m = n$ we have

$$\int_0^a \sin(\lambda_n x)\sin(\lambda_m x)dx = \int_0^a \sin^2(\lambda_n x)dx > 0$$

and so we have orthogonality