MAT 341 HW5, Carl Liu

1.

Since both ends are insulated our boundaries and conditions are

$$\frac{\partial u}{\partial x}(0,t) = 0 \quad \frac{\partial u}{\partial x}(a,t) = 0 \quad \frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

$$u(x,0) = \begin{cases} \frac{2T_0 x}{a} & 0 \le x \le \frac{a}{2} \\ \frac{2T_0(a-x)}{a} & \frac{a}{2} < x \le a \end{cases}$$

assume $u(x,t) = \phi(x)T(t)$. Then

$$\phi''(x)T(t) = \frac{1}{k}\phi(x)'T(t)$$

Dividing through by $\phi(x)T(t)$ we obtain

$$\frac{\phi(x)''}{\phi(x)} = \frac{1}{k}\frac{T(t)'}{T(t)}$$

and

$$\frac{\partial u}{\partial x}(0,t) = \phi'(0)T(t) = 0$$
$$\frac{\partial u}{\partial x}(a,t) = \phi'(a)T(t) = 0$$

The only nontrivial solution to the above is $\phi'(0) = \phi'(a) = 0$ and we must also have

$$\frac{\phi''(x)}{\phi(x)} = -\lambda^2 = \frac{1}{k}\frac{T'(t)}{T(t)}$$

We then have $\phi''(x) + \lambda^2\phi(x) = 0$ and $T'(t) + kT(t) = 0$. The solution for the first equation is $\phi(x) = c_1\cos(\lambda x) + c_2\sin(\lambda x)$. But we have $\phi'(0) = \lambda c_2 = 0$. In the case $c_2 = 0$ we have $\phi(x) = c_1\cos(\lambda x)$. But because $\phi'(a) = -c_1\sin(\lambda a) = 0$, we must have $\lambda_n = n\pi/a$. Thus $\phi(x) = c_1\cos(\lambda_n x)$. Then we have

$$T'(t) + \lambda_n^2 kT(t) = 0$$

The solution to this is $T(t) = \exp(-\lambda_n^2 kt)$. So we have solutions of

$$u(x,t) = a_n\cos(\lambda_n x)\exp(-\lambda_n^2 kt)$$

Since superposition of solutions to a homogeneous differential equation are also solutions, we thus have

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\lambda_n x) \exp(-\lambda_n^2 kt)$$

The a_0 comes from $\lambda = 0$. We then have

$$u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\lambda_n x)$$

So we have

$$a_0 = \frac{1}{a}\left(\int_0^{a/2} \frac{2T_0 x}{a} dx + \int_{a/2}^{a} \frac{2T_0(a-x)}{a} dx\right) = \frac{T_0}{4} + \left[\frac{-T_0(a-x)^2}{a^2}\right]_{a/2}^{a} =$$

$$\frac{T_0}{4} + \frac{T_0(a/2)^2}{a^2} = \frac{T_0}{4} + \frac{T_0}{4} = \frac{T_0}{2}$$

and we also have

$$a_n = \frac{4T_0}{a^2}\left(\int_0^{a/2} x\cos(\lambda_n x)dx + \int_{a/2}^{a} (a-x)\cos(\lambda_n x)dx\right) =$$

$$\tfrac{4T_0}{a^2}\left(\left[\tfrac{x}{\lambda_n}\sin(\lambda_n x)\right]_0^{a/2} + \left[\tfrac{1}{\lambda_n^2}\cos(\lambda_n x)\right]_0^{a/2} + \left[\tfrac{(a-x)}{\lambda_n}\sin(\lambda_n x)\right]_{a/2}^{a} - \left[\tfrac{1}{\lambda_n^2}\cos(\lambda_n x)\right]_{a/2}^{a}\right) =$$

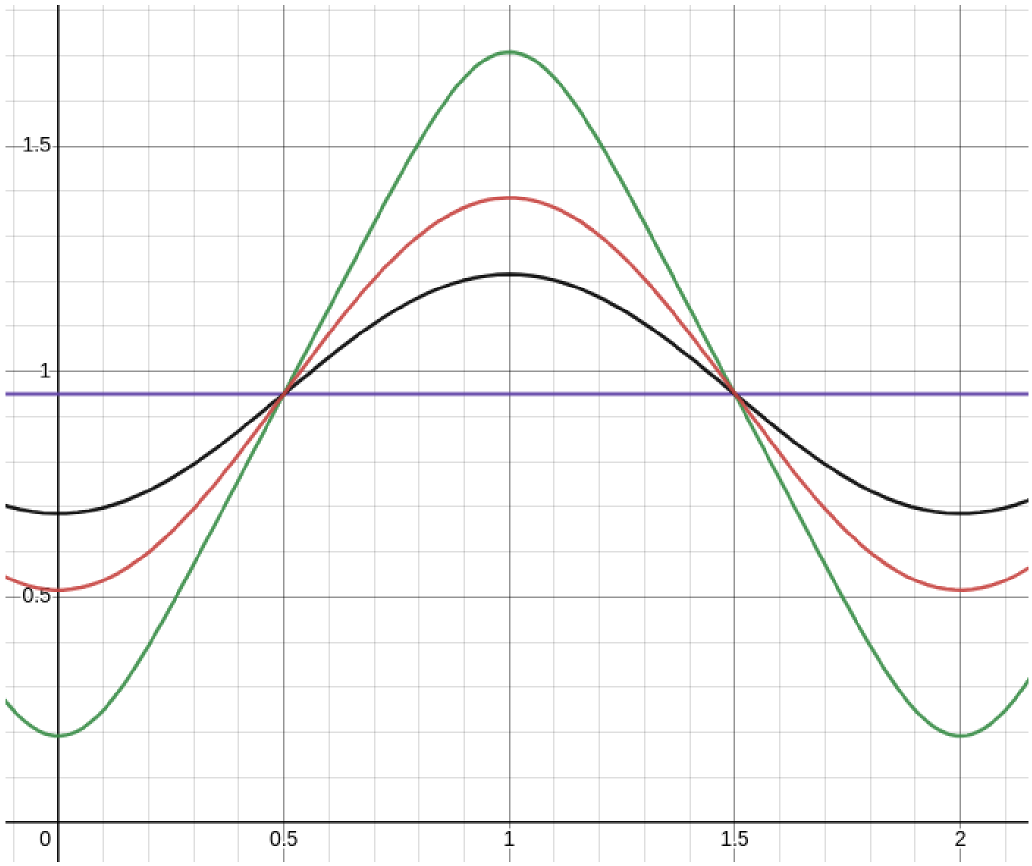
$$\frac{4T_0}{a^2}\left(\frac{2}{\lambda_n^2}\cos\left(\lambda_n a/2\right) - \frac{1}{\lambda_n^2} - \frac{1}{\lambda_n^2}\cos\left(\lambda_n a\right)\right) =$$

$$\frac{4T_0}{a^2}\left(\frac{2a^2}{n^2\pi^2}\cos\left(n\pi/2\right) - \frac{a^2}{n^2\pi^2} - \frac{a^2}{n^2\pi^2}(-1)^n\right)$$

Thus we can conclude that

$$u(x,t) =$$

$$\frac{T_0}{2} + \sum_{n=1}^{\infty} \frac{4T_0}{n^2\pi^2}\left(2\cos\left(n\pi/2\right) - 1 - (-1)^n\right)\cos\left(\frac{n\pi}{a}x\right)\exp\left(-\frac{n^2\pi^2}{a^2}kt\right)$$


1.5
1
0.5
0
0.5
1
1.5
2

2.

We have the initial conditions being

$$u(x,0) = \begin{cases} T_1 & 0 \leq x \leq \frac{a}{2} \\ T_2 & \frac{a}{2} < x \leq a \end{cases}$$

So

$$a_0 = \frac{1}{a}\left(\int_0^{a/2} T_1 dx + \int_{a/2}^{a} T_2 dx\right) = \frac{1}{2}(T_1 + T_2)$$

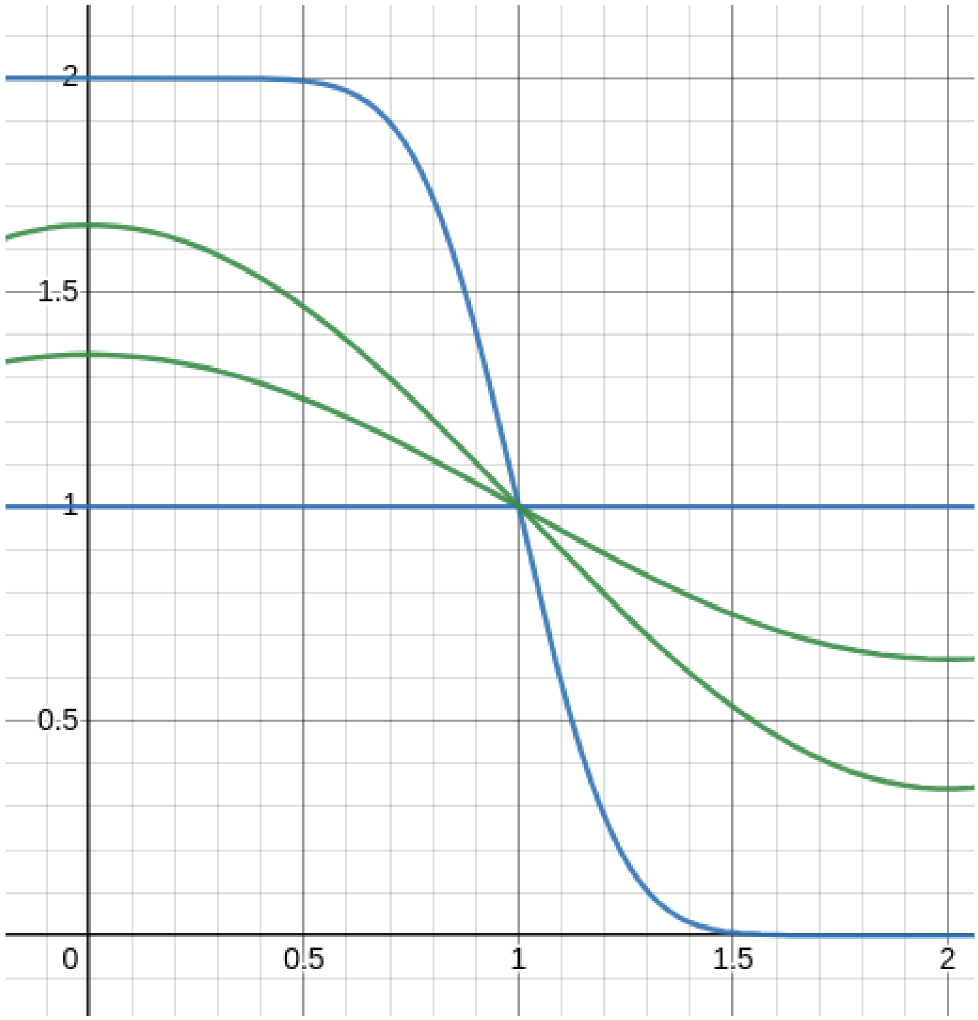
and

$$a_n = \frac{2}{a}\left(\int_0^{a/2} T_1 \cos\left(\frac{n\pi}{a}x\right) dx + \int_{a/2}^{a} T_2 \cos\left(\frac{n\pi}{a}x\right)\right) =$$

$$\frac{T_1 2}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{T_2 2}{n\pi}\sin\left(\frac{n\pi}{2}\right) = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)(T_1 - T_2)$$

resulting in

$$u(x,t) = \frac{1}{2}(T_1 + T_2) + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)(T_1 - T_2)\cos\left(\frac{n\pi}{a}x\right)\exp\left(-\frac{n^2\pi^2}{a^2}kt\right)$$


2
1.5
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0.5
0
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1.5
2

3.

Since one end is insulated and the other held constant, our boundaries and conditions are

$$u(0,t) = T_0 \quad \frac{\partial u}{\partial x}(a,t) = 0 \quad \frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} \quad u(x,0) = \frac{Tx}{a}$$

In this case we don't have a homogeneous equation. So lets consider the transient, we have

$$w(x,t) = u(x,t) - T_0 \quad w(0,t) = 0 \quad \frac{\partial w}{\partial x}(a,t) = 0$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{k}\frac{\partial w}{\partial t} \quad w(x,0) = \frac{Tx}{a} - T_0$$

We then have $\lambda_n = (2n-1)\pi/2a$

$$w(x,t) = \sum_{n=1}^{\infty} b_n \sin(\lambda_n x)\exp(-\lambda_n^2 kt)$$

$$b_n = \frac{2}{a}\int_0^a \left(\frac{Tx}{a} - T_0\right)\sin\left(\frac{(2n-1)\pi x}{2a}\right)dx =$$

$$\tfrac{2}{a}\left(\left[-\tfrac{2a}{(2n-1)\pi}\left(\tfrac{Tx}{a} - T_0\right)\cos\left(\tfrac{(2n-1)\pi x}{2a}\right)\right]_0^a + \left[\tfrac{4a^2T}{(2n-1)^2\pi^2 a}\sin\left(\tfrac{(2n-1)\pi x}{2a}\right)\right]_0^a\right) =$$

$$-\frac{2aT_0}{(2n-1)\pi} + \frac{4a^2T}{(2n-1)^2\pi^2 a}(-1)^{n-1}$$

Thus we have

$$w(x,t) =$$

$$\textstyle\sum_{n=1}^{\infty}\left(\tfrac{4a^2T}{(2n-1)^2\pi^2 a}(-1)^{n-1} - \tfrac{2aT_0}{(2n-1)\pi}\right)\sin\left(\tfrac{(2n-1)\pi}{2a}x\right)\exp\left(\tfrac{-(2n-1)^2\pi^2}{4a^2}kt\right)$$

and so

$$u(x,t) = T_0 + \sum_{n=1}^{\infty} \left(\frac{4a^2 T}{(2n-1)^2 \pi^2 a} (-1)^{n-1} - \frac{2aT_0}{(2n-1)\pi} \right)$$
$$* \sin\left(\frac{(2n-1)\pi}{2a} x \right) * \exp\left(\frac{-(2n-1)^2 \pi^2}{4a^2} kt \right)$$