MAT 341 HW3, Carl Liu

1.

A)

$$f(x) = \int f'(x)dx = \int \sum_{n=1}^{\infty} \frac{n}{3^n} \sin(nx)dx = \sum_{n=1}^{\infty} \int \frac{n}{3^n} \sin(nx)dx =$$

$$-\sum_{n=1}^{\infty} \frac{1}{3^n} \cos(nx) + C$$

So we have

$$2 = f(0) = -\sum_{n=1}^{\infty} \frac{1}{3^n} \cos(n * 0) + C = -\sum_{n=1}^{\infty} \frac{1}{3^n} + C$$

But we have by geometric series

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=0}^{\infty} \frac{1}{3^n} - 1 = \frac{1}{1-3^{-1}} - 1 = \frac{3}{2} - 1 = \frac{1}{2}$$

Thus $2 = -(1/2) + C$. Resulting in $C = 5/2$ and

$$f(x) = -\sum_{n=1}^{\infty} \frac{1}{3^n} \cos(nx) + 5/2$$

B) Using Theorem 7 of section 1-5, we have continuous derivatives $f^{(k)}$ for all k when

$$\sum_{n=1}^{\infty} |n^k a_n| + |n_k b_n|$$

converges for all k. Since $b_n = 0$, we only need convergences of

$$\sum_{n=1}^{\infty} |\frac{n^k}{3^n}|$$

for all k. So let $k \geq 1$. Using the ratio test we have

$$\frac{x_{n+1}}{x_n} = \frac{(n+1)^k/3^{n+1}}{n^k/3^n} = \frac{(n+1)^k}{3n^k} = \left(\frac{n+1}{3n}\right)^k$$

But for $n \geq 1$, we have $2n \geq n+1$. So $2/3 \geq (n+1)/3n$ for all $n \geq 1$. Meaning $(2/3)^k \geq ((n+1)/3n)^k$ for all $n \geq 1$. Since $2/3 < 1$, we have $(2/3)^k < 1$.So by the squeeze test $1 > \lim_{n\to\infty} 2/3^k = 2/3^k \geq \lim_{n\to\infty}((n+1)/3n)^k \geq \lim_{n\to\infty} 0 = 0$. Thus

$$1 > \lim_{n\to\infty} \frac{(n+1)^k}{3n^k} = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}$$

So by the ratio test we have convergence of the series

$$\sum_{n=1}^{\infty} |\frac{n^k}{3^n}|$$

and since $k \geq 1$ was arbitrary, this is true for all $k \geq 1$ and thus we can conclude that f has derivative $f^{(k)}$ for all $k \geq 1$ as required.

2.

A) We have

$$\frac{\partial u}{\partial t} = -\lambda^2 k e^{-\lambda^2 kt} \cos(\lambda x)$$

and

$$\frac{\partial u}{\partial x} = -\lambda e^{-\lambda^2 kt} sin(\lambda t) \quad \frac{\partial^2 u}{\partial x^2} = -\lambda^2 e^{-\lambda^2 kt cos(\lambda t)}$$

So

$$\frac{\partial^2 u}{\partial x^2} = -\lambda^2 e^{-\lambda^2 kt cos(\lambda t)} = \frac{-\lambda^2 k e^{-\lambda^2 kt} \cos(\lambda x)}{k} = \frac{1}{k}\frac{\partial u}{\partial t}$$

as required.

B) We have

$$\frac{\partial u}{\partial t} = -\lambda^2 k e^{-\lambda^2 kt} \sin(\lambda x) \quad \frac{\partial^2 u}{\partial x^2} = -\lambda^2 e^{-\lambda^2 kt} \sin(\lambda x)$$

So

$$\frac{\partial^2 u}{\partial x^2} = -\lambda^2 e^{-\lambda^2 kt} \sin(\lambda x) = \frac{-\lambda^2 k e^{-\lambda^2 kt} \sin(\lambda x)}{k} = \frac{1}{k}\frac{\partial u}{\partial t}$$

as required.

3.

A) We have at the steady state

$$\frac{d^2v}{dx^2} = 0 \quad v(a) = T_0 \quad \frac{dv}{dx}(0) = 0$$

Integrating we then get

$$\frac{dv}{dx}(x) = A \quad v(x) = Ax + B$$

So

$$\frac{dv}{dx}(0) = 0 = A$$

making $v(x) = B$. Since $v(a) = T_0$, we then have $T_0 = B$ making $v(x) = T_0$ the steady state solution

B) We have at the steady state

$$\frac{d^2v}{dx^2}(x) = 0 \quad v(0) = T_0 + \frac{dv}{dx}(0) \quad \frac{dv}{dx}(a) = 0$$

Integrating results us in

$$\frac{dv}{dx}(x) = A \quad v(x) = Ax + B$$

Since $v'(a) = 0$, we must have $A = 0$ meaning $v'(x) = 0$. Thus $v'(0) = 0$ and $v(x) = B$. Since $v(0) = T_0 + v'(0) = T_0 + 0 = T_0$, we have $B = T_0$ and so the steady state solution is $v(x) = T_0$

C) We have at the steady state

$$\frac{d^2v}{dx^2}(x) = 0 \quad v(0) = T_0 + \frac{dv}{dx}(0) \quad v(a) = T_1 - \frac{dv}{dx}(a)$$

Integrating results us in

$$\frac{dv}{dx}(x) = A \quad v(x) = Ax + B$$

We then have $v'(0) = A = v'(a)$. Thus $v(0) = T_0 + A = A * 0 + B$. Meaning $T_0 + A = B$. Then $v(a) = T_1 - A = Aa + B$. So $T_1 - A = Aa + T_0 + A$. Then

$T_1 = A(2+a)+T_0$. Thus $A = (T_1-T_0)/(2+a)$ and $B = T_0+(T_1-T_0)/(2+a)$. So

$$v(x) = \frac{T_1 - T_0}{2+a}x + T_0 + \frac{T_1 - T_0}{2+a} = \frac{T_1 - T_0}{2+a}(x+1) + T_0$$

4.

The amount of heat going into the rod over a period of time is $q(0,t)A\Delta t = -C(u(0,t+\Delta t) - u(0,t))$ where q is the heat flux. This is because the water and rod has the same temperature at 0 meaning that the change in temperature of the rod is equivalent to the change in temperature of the water. But the water holds C heat per unit of temperature and this must go into the rod when the temperature decreases. So a decreasing temperature means an increasing flux. Then by dividing through with Δt we obtain

$$Aq(0,t) = \frac{-C(u(0,t+\Delta t) - u(0,t))}{\Delta t}$$

Taking the limit as $\Delta t \to 0$, we obtain

$$Aq(0,t) = -C\frac{\partial u}{\partial t}(0,t)$$

But by Fourier's law, we have $q(0,t) = -\kappa \frac{\partial u}{\partial x}$ and so we can conclude

$$\kappa \frac{\partial u}{\partial x} = C\frac{\partial u}{\partial t}(0,t)$$

There is no negative because the flux into the rod increases at $x = 0$ when the temperature decreases at that point, the reason was mentioned above. This gets carried throughout the derivation resulting in the above equation.