MAT 310 HW 4, Carl Liu

3.A.1

Suppose T is linear. Let $\lambda \in \mathbf{F}$ such that $\lambda \neq 1$. Then we have $\lambda T(x,y,z) = \lambda(2x-4y+3z+b, 6x+cxyz) = (\lambda 2x - \lambda 4y + \lambda 3z + \lambda b, \lambda 6x + \lambda cxyz)$ and $T(\lambda x, \lambda y, \lambda z) = (\lambda 2x - \lambda 4y + \lambda 3z + b, \lambda 6x + \lambda^3 cxyz)$. Since T is linear we must have $(\lambda 2x - \lambda 4y + \lambda 3z + \lambda b, \lambda 6x + \lambda cxyz) = (\lambda 2x - \lambda 4y + \lambda 3z + b, \lambda 6x + \lambda^3 cxyz)$. So $\lambda 2x - \lambda 4y + \lambda 3z + \lambda b = \lambda 2x - \lambda 4y + \lambda 3z + b$ and $\lambda 6x + \lambda cxyz = \lambda 6x + \lambda^3 cxyz$. Then $\lambda b = b$ and $\lambda cxyz = \lambda^3 cxyz$. Suppose for contradiction that b or c are not equal to 0. In the case $b \neq 0$, we would have $\lambda b = b$ meaning $\lambda = 1$ is a must, but we have $\lambda \neq 1$ and so a contradiction. In the case $c \neq 0$, we would have $\lambda cxyz = \lambda^3 cxyz$. Thus $1 = \lambda^2$. Meaning $\lambda = 1$ is a must, which once again is a contradiction. Therefore we can thus conclude that $c = b = 0$

Suppose $b = c = 0$

Let $v, u \in \mathbf{R}^3$. Then $T(v+u) = (2(x_v+x_u) - 4(y_v+y_u) + 3(z_v+z_u) + b, 6(x_v+x_u) + c(x_v+x_u)(y_v+y_u)(z_v+z_u)) = (2(x_v+x_u) - 4(y_v+y_u) + 3(z_v+z_u), 6(x_v+x_u))$ since $b = c = 0$. Also $T(v)+T(u) = (2x_v - 4y_v + 3z_v + b, 6x_v + cx_vy_vz_v) + (2x_u - 4y_u + 3z_u + b, 6x_u + cx_uy_uz_u) = (2(x_v+x_u) - 4(y_v+y_u) + 3(z_v+z_u) + 2b, 6(x_v+x_u) + cx_vy_vz_v + cx_uy_uz_u) = (2(x_v+x_u) - 4(y_v+y_u) + 3(z_v+z_u), 6(x_v+x_u))$ since $b = c = 0$. Thus we have $T(v+u) = T(v)+T(u)$

Let $\lambda \in \mathbf{F}$ and $v \in \mathbf{R}^3$ Then $\lambda T(v) = \lambda(2x-4y+3z+b, 6x+cxyz) = (\lambda 2x - \lambda 4y + \lambda 3z + \lambda b, \lambda 6x + \lambda cxyz) = (\lambda 2x - \lambda 4y + \lambda 3z, \lambda 6x)$ since $b = c = 0$. Also $T(\lambda v) = (\lambda 2x - \lambda 4y + \lambda 3z + b, \lambda 6x + \lambda^3 cxyz) = (\lambda 2x - \lambda 4y + \lambda 3z, \lambda 6x)$ since $b = c = 0$. Thus we have $T(\lambda v) = \lambda T(v)$.

Therefore we can conclude that T has additivity and multiplicativity. That in turn means T is a linear map by definition.

3.A.7

Suppose hypothesis. Let $b \in V$ such that $b \neq 0$. Since $\dim V = 1$ and $0 = 0b$ is the only representation of 0 in the span of b, we can conclude that b is linearly independent and thus is a basis of V. Also by definition of T, $T(b) = u$ for some $u \in V$. But because $u \in V$, we have $u = \lambda_1 b$ for some $\lambda_1 \in \mathbf{F}$. So $T(b) = \lambda_1 b$. Now let $v \in V$. Then $\lambda b = v$ for some $\lambda \in \mathbf{F}$. Since $\lambda b = v$, we have $T(v) = T(\lambda b)$. Due to T being a linear map we then have $T(\lambda b) = \lambda T(b) = \lambda \lambda_1 b$. But we established that $\lambda b = v$. Thus

$T(v) = \lambda\lambda_1 b = \lambda_1 v$. Since $v \in V$ was arbitrary we can conclude that there exists $\lambda_1 \in \mathbf{F}$ such that $T(v) = \lambda_1 v$ for all $v \in V$ as required.

3.A.11

Suppose hypothesis. Since U is a subspace of V and V is finite dimensional, we have $V = U \oplus P$ for some subspace P of V. That means $v \in V$ can be written uniquely as a sum $v = u + p$ where $u \in U$ and $p \in P$. Consider the transformation T defined as $T(v) = S(u) + 0p = S(u)$.

Let $v, x \in V$. Then $T(v) = S(u_1) + 0p_1 = S(u_1)$ and $T(x) = S(u_2) + 0p_2 = S(u_2)$. So $T(v) + T(x) = S(u_1) + S(u_2)$ but since S is a linear map, we have $T(v)+T(x) = S(u_1+u_2)$. Also $T(v+x) = S(u_1+u_2)+0(p_1+p_2) = S(u_1+u_2)$ meaning $T(v + x) = T(v) + T(x)$.

Let $\lambda \in \mathbf{F}$ and $v \in V$. Then $\lambda T(v) = \lambda S(u_1) + \lambda 0 w_1 = \lambda S(u_1)$ and $T(\lambda v) = S(\lambda u_1) + 0(\lambda w_1) = S(\lambda u_1)$ but because S is a linear map we have $T(\lambda v) = \lambda S(u_1)$. Thus $T(\lambda v) = \lambda T(v)$.

Therefore T is additive and multiplicative making it a linear map.

Let $u \in U$, we then have $T(u) = S(u) + 0(0) = S(u)$ since $u \in U$ and $u = 1u + 0w$. Since $u \in U$ was arbitrary we can conclude $T(u) = S(u)$ for all $u \in U$ as required.

3.B.6

Suppose for contradiction that there exists a linear map such that $range(T) = null(T)$. Then we have $\dim(\mathbf{R}^5) = \dim(null(T)) + \dim(range(T))$. But because $null(T) = range(T)$, we have $\dim(null(T)) = \dim(range(T))$. So $\dim(null(T)) + \dim(range(T)) = 2\dim(null(T))$. Since $\dim(\mathbf{R}^5) = 5$ we therefore must have $5 = 2\dim(null(T))$. But because the dimension must be an integer number, we have $odd = even$, a contradiction. Therefore there does not exist such a linear map as required.

3.B.13

We have $range(T) \subseteq \mathbf{F}^2$ by definition of range.

Consider $a_1 = (5, 1, 0, 0), a_2 = (0, 0, 7, 1) \in \mathbf{F}^4$. Let $0 = c_1(5, 1, 0, 0) + c_2(0, 0, 7, 1)$ and $0 = b_1(5, 1, 0, 0) + b_2(0, 0, 7, 1)$. Then $(5b_1, b_1, 7b_2, b_2) = (5c_1, c_1, 7c_2, c_2)$. So $b_1 = c_1$ and $b_2 = c_2$, a unique representation of 0. Thus we can conclude that a_1, a_2 is linearly independent.

Now let $v \in null(T)$, then $v = (5x_2, x_2, 7x_4, x_4)$ for $x_2, x_4 \in \mathbf{F}$. But we have $(5x_2, x_2, 7x_4, x_4) = x_2(5,1,0,0) + x_4(0,0,7,1)$. Thus we have $v \in span(a_1, a_2)$ and so $null(T) \subseteq span(a_1, a_2)$

Let $v \in span(a_1, a_2)$. Then $v = c_1(5,1,0,0) + c_2(0,0,7,1)$ for some $c_1, c_2 \in \mathbf{F}$. But we have $c_1(5,1,0,0) + c_2(0,0,7,1)$ $(5c_1, c_1, 7c_2, c_2)$. But that means $v \in null(T)$ by definition of $null(T)$. Thus $span(a_1, a_2) \subseteq null(T)$ meaning $span(a_1, a_2) = null(T)$. Thus we can conclude that a_1, a_2 is a basis of $null(T)$ and so $\dim(null(T)) = 2$.

Then we have from the fundamental theorem of linear maps, $\dim(\mathbf{F}^4) = \dim(null(T)) + \dim(range(T))$. So $4 = 2 + \dim(range(T))$. Meaning $2 = \dim(range(T))$. and we have a basis (v_1, v_2) of $range(T)$. But $range(T)$ is a subspace of $\mathbf{F}^2$ and so $range(T) \subseteq \mathbf{F}^2$. But that means v_1, v_2 are linearly independent in $\mathbf{F}^2$. Since $\dim(\mathbf{F}^2) = 2$, we have (v_1, v_2) is a basis of $\mathbf{F}^2$. But $range(T) = span(v_1, v_2)$ and $\mathbf{F}^2 = span(v_1, v_2)$. Thus we can conclude $range(T) = \mathbf{F}^2$ meaning T is surjective as required.