MAT 312 HW 6 Carl Liu

1.

i) No because there is no inverse for 0.

ii) Let $a, b \in G$. Since both are complex we have $a = x + iy \neq 0$ and $b = d + iz \neq 0$ for real numbers x, y, d, z Then $a * b = xd + idy + izx - yz = (xd - yz) + i(dy + zx)$ which is complex and non zero. Thus G has closure. We also have identity element 1 as for $a = x + iy \in G$, $1 * a = x * 1 + i * 1 * y = x + iy$ as required. Let $a \in G$. Then $a = x + iy$ for real x, y. Since $x + iy \neq 0$, we have $\frac{1}{x+iy} = (x/(x^2 + y^2)) - i(y/(x^2 + y^2) \in G$ and $(x + iy)/(x + iy) = 1$. So the inverse is in G as required. Now suppose $a, b, c \in G$. Then $(ab)c = a(bc)$ by property of complex numbers. Thus we conclude that it is a group

iii) Let $a, b \in G$. Then $a * b$ is also an integer which is not zero since neither a, b are zero. So $a * b \in G$ making it closed. There is the identity element $1 + 0i = 1$ and also for every $a \in G$, we have $1/a \in G$ since $a \neq 0$ and $a/a = 1$. So there are inverse elements. $(ab)c = a(bc)$ by properties of integers.

iv) Not all functions have an inverse. Consider $a(1) = 1, a(2) = 1, a(3) = 1$. By definition of a function, we cannot have a function $b(1) = 1, 2, 3.$ and thus there is no inverse. Thus G not a group

v) Let $x, y \in G$. Then $x = a + b\sqrt{2}, y = c + d\sqrt{2}$ for integers a, b, c, d. Then $x + y = (a + c) + (b + d)\sqrt{2}$ so $x + y \in G$. Thus G has closure. It has the identity element $0 + 0\sqrt{2} = 0 \in G$ since $x \in G$ and $0 + x = 0 + a + b\sqrt{2} = x$. It has the inverse element $-x = -a + -b\sqrt{2} \in G$ where $x - x = a + b\sqrt{2} - a + -b\sqrt{2} = 0$. Also $z = e + f\sqrt{2}$ and the rest is defined as above, we have $(x + y) + z = a + b\sqrt{2} + c + d\sqrt{2} + e + f\sqrt{2} = (a + c + e) + (b + d + f)\sqrt{2} = a + (b + c)$ as required.

vi) Let $A, B \in G$ then we have

$$AB = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & d + a & e + af + b \\ 0 & 1 & f + c \\ 0 & 0 & 1 \end{pmatrix}$$

thus $AB \in G$. We have

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is in G and is the identity element as $IA = A$. We also have inverse of A being

$$\begin{pmatrix} 1 & -a & ca-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}$$

which can be seen from the AB composition above. Associativity is obvious due to the associativity of matrix multiplication.

vii) Yes because it is essentially the set of integers under addition except with a $-$ factor.

viii) Let $x, y \in G$. Then $x * y = x + y + 2$ which is real and is thus in G There is also the identity element -2 since $(-2) * b = -2 + b + 2 = b$. We also have for a, the inverse, $(-a-4)$ as $(-a-4) * a = -a - 4 + a + 2 = -2$ which is the identity element. Now we have $(x * y) * z = (x + y + 2) * z = x + y + 2 + z + 2 = x + y + z + 4$ and $x * (y * z) = x * (y + z + 2) = x + y + z + 2 + 2 = x + y + z + 4$. Therefore we have associativity. Thus G is indeed a group.

2.

First we see that for any $a \in G$, we have $aa^{-1} = e = a^2$. Then we have $a^{-1}aa^{-1} = a^{-1}aa \to a^{-1} = a$. Also let $a, b \in G$, a group. Then we have $(ab)^{-1} = e(ab)^{-1} = b^{-1}a^{-1}ab(ab)^{-1} = e(ab)^{-1} = b^{-1}a^{-1}$.

Let $a, b \in G$. Then $ab = a^{-1}b^{-1} = (ba)^{-1} = ba$. The last equality comes from $(ba)^2 = e$ and what we showed above. Thus we are done.

3.

Let $a, b \in A_n$. Since $A_n \subseteq S(n)$ and $S(n)$ is a group, suppose $h, k \in A_n$. Then $k^{-1} \in A_n$ because $1 = sgn(k) = sgn(k^{-1})$ thus k^{-1} is even. The composition hk^{-1} will then have sign $sgn(hk^{-1}) = sgn(h)sgn(k^{-1}) = 1 * 1 = 1$ as required. Thus we can conclude that A_n is indeed a group. Now consider the odd permutations F_n. This does not form a group because for $h, k \in F_n$ we have $sgn(hk) = sgn(h)sgn(k) = (-1)(-1) = 1$ which is even and is thus not in F_n meaning closure is not satisfied.

4.

a) Suppose $xy = yx$. Since $x, y \in G$ where G is a group there are inverses $xyx^{-1} = yxx^{-1} \to xyx^{-1} = y$ then $xyx^{-1}y^{-1} = yy^{-1} = e$. Thus $[x, y] = e$ as required. Suppose $[x, y] = e$. Then $xyx^{-1}y^{-1} = e \to xyx^{-1}y^{-1}y = y \to xyx^{-1} = y \to xyx^{-1}x = yx \to xy = yx$ as required.

b) we have $x^{-1} = (15432)$ and $y^{-1} = (59876)$. That means the commutator is

$[x, y] = (12345)(56789)(15432)(59876) = (165)$

5.

a) consider an axis which goes through one of the vertices and makes the tetrahedron symmetric about that axis. Then a rotation ρ of $2\pi/3$ around this axis preserves. Since there are 4 vertices, we can have 4 such rotations that preserves. Lets name them $\rho_1, \rho_2, \rho_3, \rho_4$ corresponding to rotations about their respective axis. There are also F which are π rotations about an axis that goes through the midpoint of an edge and goes through the midpoint of a perpendicular edge, we will name them F_1, F_2, F_3 for the three edge pairs. Any combination of these rotations preserve and we also have the properties $\rho_k^3 = e$, $F_k^2 = e$. Then we have the rotations $e, F_1, F_2, F_3, \rho_1, \rho_1^2, \rho_2, \rho_2^2, \rho_3, \rho_3^2, \rho_4, \rho_4^2$ which is 12 elements in R.

c) We can map each rotation to a permutation of $S(4)$. Since F_k for any k can be considered a disjoint cycle of $(x_1, x_2)(x_3, x_4)$ where $x_k \in \{1, 2, 3, 4\}$, we thus have $sgn(F_k) = sgn((x_1, x_2))sgn((x_3, x_4)) = (-1)(-1) = 1$ making F_k even. ρ_k can be considered as a cycle (x_1, x_2, x_3) since it is rotating 3 points about an axis. Since $sgn((x_1, x_2, x_3)) = (-1)^{length((x_1,x_2,x_3))-1} = 1$ and is even. Since the composition of even permutations are even, we thus conclude that all the rotations are therefore in A_4 and since both have 12 distinct elements, we have a bijection. Therefore R can now also be considered a group as A_4 is a group.

b) R is a group follows from A_4 being a group.