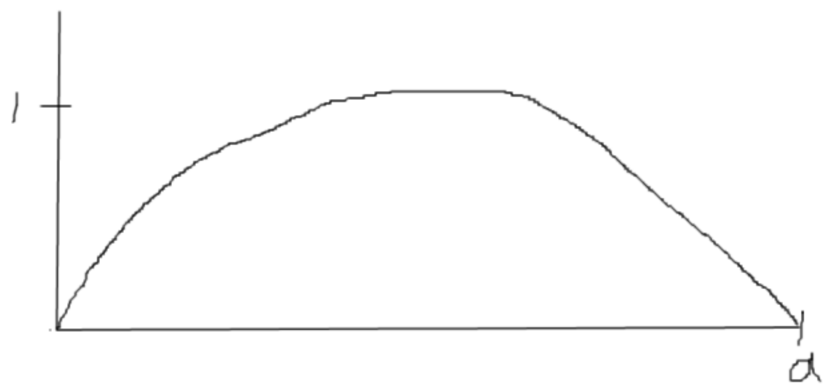
MAT 341 HW8, Carl Liu

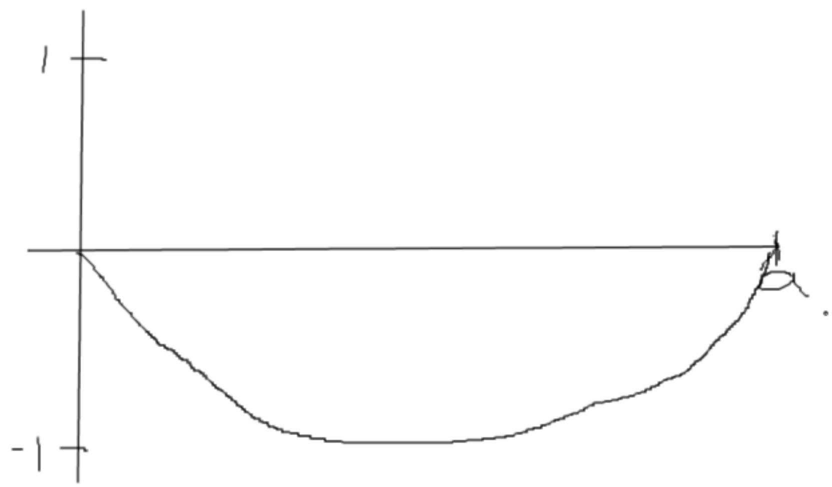
1.

We have $u_1(x,t) = \sin(\lambda_1 x)\cos(\lambda_1 ct) = \sin(\frac{\pi}{a}x)\cos(\frac{\pi}{a}ct)$ and $u_2(x,t) = \sin(\frac{2\pi}{a}x)\cos(\frac{2\pi}{a}ct)$ at $t = 0$ we would have $u_1(x,0) = \sin(\frac{\pi}{a}x)$ and so a graph of the form

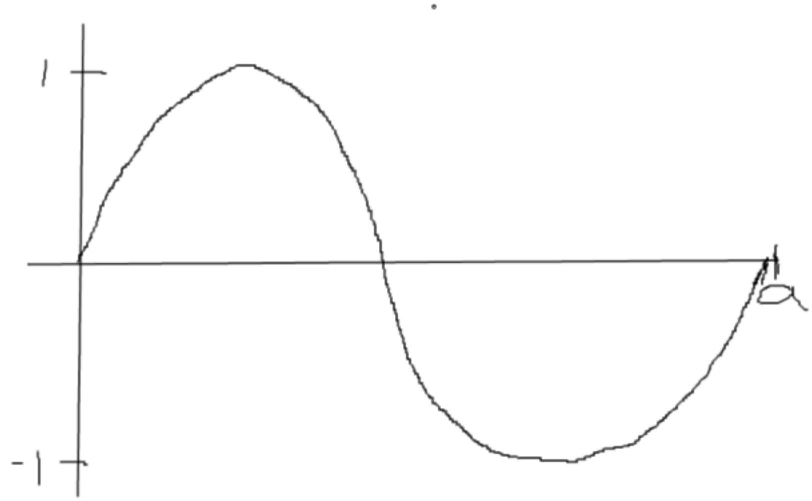


at $t = a/2c$ we would have as our string $u_1(x, a/2c) = \sin(\frac{\pi}{a}x)\cos(\frac{\pi a}{a2c}c) = \sin(\frac{\pi}{a}x)\cos(\frac{\pi}{2}) = 0$. Resulting in a graph

finally at $t = a/c$ we have $u_1(x, a/c) = \sin(\frac{\pi}{a}x)\cos(\frac{\pi a}{ac}c) = -\sin(\frac{\pi}{a})$ resulting in a graph

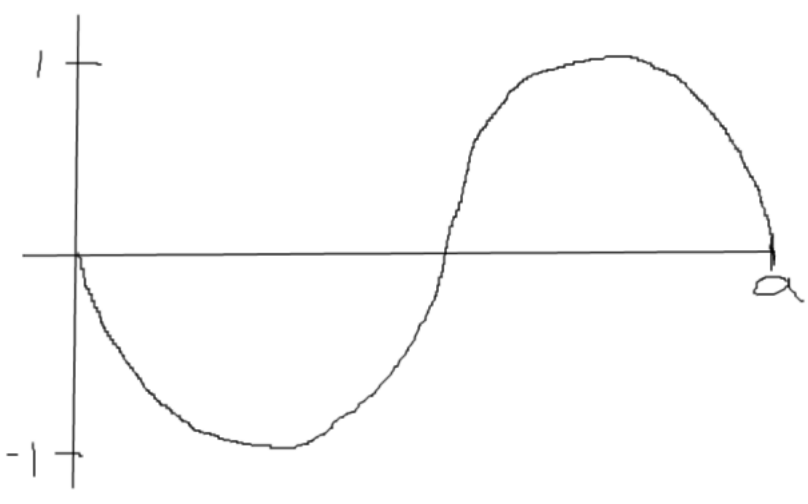


For u_2 at time $t = 0$, we have $u_2(x, 0) = \sin(\frac{2\pi}{a}x)$ resulting in a graph



at time $t = a/4c$ we have as our position, $u_2(x, a/4c) = \sin(\frac{2\pi}{a}x)\cos(\frac{2\pi a}{4ac}c) = \sin(\frac{2\pi}{a}x)\cos(\frac{\pi}{2}) = 0$ and so a graph

at time $t = a/c$ we have $u_2(x, a/c) = \sin(\frac{2\pi}{a}x)\cos(\frac{2\pi a}{ac}c) = -\sin(\frac{2\pi}{a}x)$ and so a graph



2.

We have $u(x,0) = f(x) = \sin(\pi x/a)$ and so

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{a}\right) = \sin(\pi x/a)$$

thus resulting in

$$a_n = \frac{1}{a}\int_{-a}^{a} \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right) dx =$$

where we are considering $\sin\left(\frac{\pi x}{a}\right)$ as an odd extension. But we have by orthogonality that $a_n = 0$ for all $n \neq 1$ and so we have $a_1 = a/a = 1$. Since $g(x) = 0$, we have $b_n = 0$ for all n. Therefore we have

$$u(x,t) = \sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{a}ct\right)$$

For the initial condition of $f(x) = 0$ and $g(x) = 1$, we have

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n \frac{n\pi}{a} c \sin\left(\frac{n\pi x}{a}\right) = 1$$

This results in

$$b_n = \frac{2}{n\pi c}\int_{0}^{a} \sin\left(\frac{n\pi x}{a}\right) dx = -\frac{2a}{n^2\pi^2 c}\cos(n\pi) + \frac{2a}{n^2\pi^2 c} = \frac{2a}{n^2\pi^2 c}(1-(-1)^n)$$

and since $f(x) = 0$ we have $a_n = 0$. Thus we have

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2a}{n^2\pi^2 c}(1-(-1)^n)\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n\pi}{a}ct\right)$$

3.

We have $u(x,t) = \psi(x+ct) + \phi(x-ct)$ and so

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2\psi}{d(x+ct)^2}(x+ct) + \frac{d^2\phi}{d(x-ct)^2}(x-ct)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{d^2\psi}{d(x+ct)^2}(x+ct) + c^2 \frac{d^2\phi}{d(x-ct)^2}(x-ct)$$

where the first equality comes from the fact that the derivative of $x+ct$ and $x-ct$ with respect to x is 1 and the second equality due to the derivative of $x+ct$ and $x-ct$ with respect to t being c and $-c$ respectively. Thus we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2\psi}{d(x+ct)^2}(x+ct) + \frac{d^2\phi}{d(x-ct)^2}(x-ct) =$$

$$\frac{c^2}{c^2}\frac{d^2\psi}{d(x+ct)^2}(x+ct) + \frac{c^2}{c^2}\frac{d^2\phi}{d(x-ct)^2}(x-ct)) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

and so

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

as required thereby satisfying the vibrating string equation.

4.

We have

$$u(x,t) = \psi(x+ct) + \phi(x-ct) =$$

$$\frac{1}{2}(\overline{f}_o(x+ct) + \overline{G}_e(x+ct) + \overline{f}_o(x-ct) - \overline{G}_e(x-ct))$$

but because $\overline{G}_e$ and $\overline{f}_o$ both have their third derivative existing at every point on the real line, we can conclude that ψ and ϕ also have their third derivative existing at every point on the real line. Thus we can conclude through problem 3 that $u(x,t)$ as defined as so satisfies

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Now for $u(0,t)$, we have

$$u(0,t) = \frac{1}{2}(\overline{f}_o(ct) + \overline{G}_e(ct) + \overline{f}_o(-ct) - \overline{G}_e(-ct))$$

Since $\overline{G}_e$ is even, we have $\overline{G}_e(ct) = \overline{G}_e(-ct)$. Since $\overline{f}_o$ is odd we have $\overline{f}_o(ct) = -\overline{f}_o(-ct)$. Thus we have

$$u(0,t) = 0$$

as required. Now

$$u(x,0) = \frac{1}{2}(\overline{f}_o(x) + \overline{G}_e(x) + \overline{f}_o(x) - \overline{G}_e(x)) = \overline{f}_o(x)$$

which is just $f(x)$ over $0 < x < a$ and so we have

$$u(x,0) = f(x)$$

as required. Finally

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{2}\left(c\frac{d\overline{f}_o}{d(x+ct)}(x+ct) + c\frac{d\overline{G}_e}{d(x+ct)}(x+ct) - \right.$$
$$\left. c\frac{d\overline{f}_o}{d(x-ct)}(x-ct) + c\frac{d\overline{G}_e}{d(x-ct)}(x-ct)\right)$$

Thus we have

$$\frac{\partial u}{\partial t}(x,0) = \frac{1}{2}\left(c\frac{d\overline{f}_o}{d(x)}(x) + c\frac{d\overline{G}_e}{d(x)}(x) - c\frac{d\overline{f}_o}{d(x)}(x) + c\frac{d\overline{G}_e}{d(x)}(x)\right) =$$
$$c\frac{d\overline{G}_e}{d(x)}(x)$$

Since it has been established that $\frac{d\overline{G}_e}{d(x)}(x)$ has a derivative that exists for all x, we must have

$$\frac{d\overline{G}_e}{d(x)}(x) = \frac{dG}{d(x)}(x) = \frac{1}{c}\frac{d}{dx}\int_0^x g(y)dy = \frac{1}{c}g(x)$$

for $0 < x < a$ and thus

$$\frac{\partial u}{\partial t}(x,0) = c\frac{d\overline{G}_e}{d(x)}(x) = \frac{c}{c}g(x) = g(x)$$

as required.