MAT 310 HW 10, Carl Liu

8.A.2

Let $T \in \mathcal{L}(\mathbf{C}^2)$ be defined as $T(w.z) = (-z, w)$. To find the eigenvalues, we have $T(w, z) = \lambda(w, z)$ and so $(-z, w) = (\lambda w, \lambda z)$. This means $-z = \lambda w$ and $w = \lambda z$. This results in $-z = \lambda^2 z$ and so $-1 = \lambda^2$. We then have $\lambda = \pm i$ as eigenvalues. We then have for $\lambda = i$, $(T - iI)^2(x, y) = (T^2 - 2iT - I)(x, y) = (-x, -y) + (2iy, -2ix) - (x, y) = (-2x + 2iy, -2y - 2ix)$. We want the null space and so $(-2x + 2iy, -2y - 2ix) = (0.0)$ resulting in $iy = x$. Thus we have $G(i, T) = \{(iy, y) : y \in \mathbf{C}\}$. For $\lambda = -i$, $(T + iI)^2(x, y) = (T^2 + 2iT - I)(x, y) = (-2x - 2iy, -2y + 2ix)$. Finding the null space we have $(-2x - 2iy, -2y + 2ix) = (0, 0)$ and so $ix = y$. Thus we have $G(-i, T) = \{(x, ix) : x \in \mathbf{C}\}$

8.A.3

Suppose $T \in \mathcal{L}(V)$ is invertible. Let $\lambda \in \mathbf{F}$ such that $\lambda \neq 0$. Using induction on n we will prove that $null(T - \lambda I)^n = null(T^{-1} - \frac{1}{\lambda}I)^n$ for all n. In the base case of $n = 1$, let $v \in null(T - \lambda I)$. Then $(T - \lambda I)v = 0$ so $Tv = \lambda v$. Then $v = \lambda T^{-1}v$. So $0 = (T^{-1} - \frac{1}{\lambda}I)v$ proving that $v \in null(T^{-1} - \lambda I)$ and so $null(T - \lambda I) \subseteq null(T^{-1} - \frac{1}{\lambda}I)$. Now let $v \in null(T^{-1} - \frac{1}{\lambda}I)$. Then $(T^{-1} - \frac{1}{\lambda}I)v = 0$ resulting in $T^{-1}v = \frac{1}{\lambda}v$. Then $v = \frac{1}{\lambda}Tv$ resulting in $(T - \lambda I)v = 0$. Meaning $null(T - \lambda I) \subseteq null(T^{-1} - \frac{1}{\lambda}I)$. Thus we prove the base case. Now suppose as inductive hypothesis that for some $n > 0$, we have $null(T - \lambda I)^m = null(T^{-1} - \frac{1}{\lambda}I)^m$ for all $m \leq n$. Let $v \in null(T - \lambda I)^{n+1}$ then $(T - \lambda I)^{n+1}v = (T - \lambda I)^n((T - \lambda I)v) = 0$. So $(T - \lambda I)v \in null(T - \lambda I)^n$. But that also means $(T - \lambda I)v \in null\ (T^{-1} - \frac{1}{\lambda}I)^n$. This results in $(T^{-1} - \frac{1}{\lambda}I)^n(T - \lambda I)v = (T - \lambda I)((T^{-1} - \frac{1}{\lambda}I)^n v) = 0$. Thus $(T^{-1} - \frac{1}{\lambda}I)^n v \in null(T - \lambda I) = null(T^{-1} - \frac{1}{\lambda}I)$. Therefore $(T^{-1} - \frac{1}{\lambda}I)(T^{-1} - \frac{1}{\lambda}I)^n v = (T^{-1} - \frac{1}{\lambda}I)^{n+1}v = 0$ meaning $null(T - \lambda I)^{n+1} \subseteq null(T^{-1} - \frac{1}{\lambda}I)^{n+1}$. Now let $v \in null(T^{-1} - \frac{1}{\lambda}I)^{n+1}$. So $(T^{-1} - \frac{1}{\lambda}I)^{n+1}v = (T^{-1} - \frac{1}{\lambda}I)^n((T^{-1} - \frac{1}{\lambda}I)v) = 0$. Thus $(T^{-1} - \frac{1}{\lambda}I)v \in null(T^{-1} - \frac{1}{\lambda}I)^n = null(T - \lambda I)^n$. This results in $(T - \lambda I)^n(T^{-1} - \frac{1}{\lambda}I)v = (T^{-1} - \frac{1}{\lambda}I)(T - \lambda I)^n v = 0$. We then have $(T - \lambda I)^n v \in null(T^{-1} - \frac{1}{\lambda}I) = (T - \lambda I)$. So $(T - \lambda I)(T - \lambda I)^n v = (T - \lambda I)^{n+1}v = 0$. Therefore $null(T^{-1} - \frac{1}{\lambda}I)^{n+1} \subseteq null(T - \lambda I)^{n+1}$ and so we can conclude $null(T^{-1} - \frac{1}{\lambda}I)^{n+1} = null(T - \lambda I)^{n+1}$ closing the induction. Since we have $G(\lambda, T) = null\ (T - \frac{1}{\lambda}I)^n$ where $n = dim(V)$ and $G(\frac{1}{\lambda}, T^{-1}) = null\ (T^{-1} - \frac{1}{\lambda}I)^n$, we can thus conclude that $G(\frac{1}{\lambda}, T^{-1}) = G(\lambda, T)$ as required.

8.A.8

Consider the nilpotent operators

$$N(x, y) = (y, 0)$$

and

$$K(x, y) = (0, x)$$

on $dimV = 2$. These are indeed nilpotent because $N^2(x, y) = N(y, 0) = 0$ and $K^2(x, y) = K(0, x) = 0$. But we have

$$(N + K)^2(x, y) = (N + K)((y, 0) + (0, x)) = (N + K)(y, x) = (x, 0) + (0, y) = (x, y)$$

Thus $(N + K)^2 \neq 0$ and so $N + K$ is not nilpotent. Since the sum of two nilpotent operators on V is not nilpotent in this case, we can conclude that it is false that the set of nilpotent operators on V is a subspace of $\mathcal{L}(V)$ for any V.

8.B.1

Suppose V is a complex vector space, $N \in \mathcal{L}(V)$, and 0 is the only eigenvalue of N. Then $V = G(0, N)$ by 8.21. But $G(0, N) = null(N - 0I)^n = null(N^n)$ where $n = dim(V)$. So we have $V = null(N^n)$. Now let $v \in V$. Then $N^n v = 0$ for all v and thus we can conclude that $N^n = 0$ meaning N is nilpotent as required

8.B.2

Consider $T(x, y, z) = (-y, x, 0)$. We look for eigenvalues, $T(x, y, z) = \lambda(x, y, z)$. This results in $(-y, x, 0) = (\lambda x, \lambda y, \lambda z)$. So $-y = \lambda x$, $x = \lambda y$, and $0 = \lambda z$. This results in $x = -\lambda^2 x$ and $\lambda = 0$. So $-1 = \lambda^2$ but that means $\lambda = \pm i$. These cannot be eigenvalues given a real vector space. Thus we only have $\lambda = 0$. Now suppose N is nilpotent. Then $T^3 = 0$. But we have $T^3(x, y, z) = T^2(-y, x, 0) = T(-x, -y, 0) = (y, -x, 0)$ implying $T^3 \neq 0$, a contradiction. Therefore T is not nilpotent.