MAT 312 HW 1 Carl Liu

1.i

Using the Euclidean algorithm, we have

$$\left(\begin{array}{cc|c} 1 & 0 & 11 \\ 0 & 1 & 7 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 4 \\ 0 & 1 & 7 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 4 \\ -1 & 2 & 3 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & -3 & 1 \\ -1 & 2 & 3 \end{array}\right) \to$$

$$\left(\begin{array}{cc|c} 2 & -3 & 1 \\ -7 & 11 & 0 \end{array}\right)$$

meaning $(11, 7) = 1 = 11 * 2 + 7 * (-3)$

1.ii

$$\left(\begin{array}{cc|c} 1 & 0 & -28 \\ 0 & 1 & -63 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 35 \\ 0 & 1 & -63 \end{array}\right) \to \left(\begin{array}{cc|c} -9 & 4 & 0 \\ 2 & -1 & 7 \end{array}\right)$$

Thus $-28 * 2 - 63 * (-1) = 7$ which is the *gcd* of $(-28, -63)$.

1.iii

$$\left(\begin{array}{cc|c} 1 & 0 & 126 \\ 0 & 1 & 91 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 35 \\ 0 & 1 & 91 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 35 \\ -2 & 3 & 21 \end{array}\right) \to \left(\begin{array}{cc|c} 3 & -4 & 14 \\ -2 & 3 & 21 \end{array}\right) \to$$

$$\left(\begin{array}{cc|c} 3 & -4 & 14 \\ -5 & 7 & 7 \end{array}\right) \to \left(\begin{array}{cc|c} 13 & -18 & 0 \\ -5 & 7 & 7 \end{array}\right)$$

Thus we have $(126, 91) = 7 = 126 * (-5) + 91 * 7$ as the gcd

iv

$$\left(\begin{array}{cc|c} 1 & 0 & 630 \\ 0 & 1 & 132 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -4 & 102 \\ 0 & 1 & 132 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -4 & 102 \\ -1 & 5 & 30 \end{array}\right) \to$$

$$\left(\begin{array}{cc|c} 4 & -19 & 12 \\ -1 & 5 & 30 \end{array}\right) \to \left(\begin{array}{cc|c} 4 & -19 & 12 \\ -9 & 43 & 6 \end{array}\right) \to \left(\begin{array}{cc|c} 22 & -105 & 0 \\ -9 & 43 & 6 \end{array}\right)$$

Thus $(630, 132) = 6 = 630 * (-9) + 132 * 43$

v

$$\left(\begin{array}{cc|c} 1 & 0 & 7245 \\ 0 & 1 & 4784 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 2461 \\ -1 & 2 & 2323 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & -3 & 138 \\ -1 & 2 & 2323 \end{array}\right) \to$$

$$\left(\begin{array}{cc|c} 2 & -3 & 138 \\ -33 & 50 & 115 \end{array}\right) \to \left(\begin{array}{cc|c} 35 & -53 & 23 \\ -33 & 50 & 115 \end{array}\right) \to \left(\begin{array}{cc|c} 35 & -53 & 23 \\ -208 & 315 & 0 \end{array}\right)$$

Thus $(7245, 4784) = 23 = 7245 * 35 + 4784 * (-53)$ as the gcd

vi

$$\left(\begin{array}{cc|c} 1 & 0 & 6499 \\ 0 & 1 & 4288 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 2211 \\ 0 & 1 & 4288 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 2211 \\ -1 & 2 & 2077 \end{array}\right) \to$$
$$\left(\begin{array}{cc|c} 2 & -3 & 134 \\ -1 & 2 & 2077 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & -3 & 134 \\ -31 & 47 & 67 \end{array}\right) \to \left(\begin{array}{cc|c} 64 & -97 & 0 \\ -31 & 47 & 67 \end{array}\right)$$

Thus we have $(6499, 4288) = 67 = 6499 * (-31) + 4288 * 47$

2

Using the euclidean algorithm and that $(6, 14, 21) = ((6, 14), 21)$, we get

$$\left(\begin{array}{cc|c} 1 & 0 & 14 \\ 0 & 1 & 6 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & 6 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -2 & 2 \\ -3 & 7 & 0 \end{array}\right) \to$$

Thus $(6, 14) = 2 = 14 - 2 * 6$. Then we have

$$\left(\begin{array}{cc|c} 1 & 0 & 21 \\ 0 & 1 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -10 & 1 \\ 0 & 1 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -10 & 1 \\ -2 & 21 & 0 \end{array}\right)$$

Therefore we have $((6, 14), 21) = 1 = 21-2*10 = 21-10(14-2*6) = 21-14*10+2*60$

3

We have $1 = as + bt$ is the smallest linear combination of the two. Then we have the smallest positive integral linear combination of $d = (a + bk)p + bn$ for some integer p and n is the gcd of $a+bk$ and b. But $d = (a+bk)p+bn = ap+bkp+bn = ap+b(kp+n)$ and we take $p = s$, $n = -kp + t$, we end up with $d = ap + b(kp + n) = as + b(kp - kp + t) = as + bt$ which we established as equal to 1, so we have $d = 1$. This is the smallest possible positive number and thus we can conclude that 1 is the gcd of $a+bk$ and b, making the two relatively prime as required.

4

consider the integers $a = 6$, $b = 3$, and $c = 2$. Clearly $6 \nmid 3$ and $6 \nmid 2$. But $bc = 3 * 2 = 6$, thus $a|bc$ as required.

5

consider the integers $a = 4$, $b = 6$, $c = 12$. Clearly $4|12$ and $6|12$. But we have $ab = 4 * 6 = 24$ meaning $ab \nmid c$ as required.

6

Suppose $(a, c) = 1 = (b, c)$. Then we have the smallest positive integral linear combination as $1 = as + ct$ and $1 = bu + cv$ for integers s, t, u, v. Then the gcd of (ab, c) is the smallest positive integral linear combination, $d = abw + cx$. Since we have $1 = (as + ct)(bu + cv) = absu + bctu + acsv + cctv$. We take $w = su$ and $x = btu + asv + ctv$. This results in $d = abw + cx = ab(su) + c(btu + asv + ctv) = absu + bctu + acsv + cctv = 1$. Since this is the smallest positive number, we can conclude that the gcd of $(ab, c) = 1$ as required.

7

Since the gcd between 12 and 17 is 1, we have $1 = 12s + 17t$ for some integer s and t. Thus $8 = 12 * 8s + 17 * 8t$. We will solve for s and t using the euclidean algorithm.

$$\left(\begin{array}{cc|c} 1 & 0 & 17 \\ 0 & 1 & 12 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 1 & 12 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & -1 & 5 \\ -2 & 3 & 2 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 5 & -7 & 1 \\ -2 & 3 & 2 \end{array}\right) \rightarrow$$

$$\left(\begin{array}{cc|c} 5 & -7 & 1 \\ -12 & 17 & 0 \end{array}\right)$$

resulting in $17{*}5 - 12{*}7 = 1$. Thus we have $8 = 17{*}(5{*}8) - 12{*}(7{*}8) = 17{*}40 - 12{*}56$. Thus we can measure out 8 ounces by filling the 17 unit jug and pouring it into the tub 40 times. Then using the 12 unit jug, take out water from the tub 56 times.