MAT 312 HW 4 Carl Liu

1.

i) We have $24s + 11r = 1$ and since

$$\left(\begin{array}{cc|c} 1 & 0 & 24 \\ 0 & 1 & 11 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -2 & 2 \\ -5 & 11 & 1 \end{array}\right)$$

we thus have $24(-5) + 11 * 11 = 1$. Then $7 * 24 * (-5) + 4 * 11 * 11 = -356$ is a solution. Thus we have solutions $[-356]_{24*11} = [172]_{264}$

ii) We have $gcd(3, 5) = 1$ and so there is a solution for x. Since $[3]_5^{-1} = [2]_5$ we thus have solution $x = 2 \mod 5$ and since we have $gcd(2, 8) = 2$ then $x \equiv 3 \mod 4$. Then we have $5r + 4s = 1$ where by inspection we see $r = 1$ and $s = -1$. So there is a solution $5 * 3 - 4 * 2 = 7$. Thus we have solutions of $[7]_{20}$.

iii) We have $[2]_7^{-1} = [4]_7$ and so $x \equiv 4 \mod 7$. Then $5r + 7s = 1$ which through inspection we can see that $5*3-7*2 = 1$ which would result in $4*5*3-3*7*2 = 18$ as a solution and so $x \equiv 18 \mod 35$. Since $x \equiv 3 \mod 8$. We must then have $35r+8s = 1$ where we see through inspection that $35 * 3 - 8 * 13 = 1$. Then we have a solution $3 * 35 * 3 - 18 * 8 * 13 = -1557$. Thus we have solution $[-1557]_{35*8} = [123]_{280}$

2.

We are trying to find a number that satisfies $x \equiv 8 \mod 11$, $x \equiv 4 \mod 10$, and $x \equiv 0 \mod 27$. We then have $11 - 10 = 1$ and so $x = 11 * 4 - 10 * 8 = -36$. Thus we have $x \equiv -36 \mod 110 \equiv 74 \mod 110$. Now $27r + 110s = 1$ and since

$$\left(\begin{array}{cc|c} 1 & 0 & 27 \\ 0 & 1 & 110 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 27 \\ -4 & 1 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 53 & -13 & 1 \\ -4 & 1 & 2 \end{array}\right)$$

we have $27*53+110*(-13) = 1$ then we have a solution $74*27*53+0*110*(-13) = 105894$ and so we have $x \equiv [105894]_{110*27} \equiv [1944]_{2970}$ making 1944 the smallest possible integer satisfying the properties given.

3.

The situation can be considered as such $x \equiv 3 \mod 15$, $x \equiv 2 \mod 7$, and $x \equiv 0 \mod 4$. We then have $15r + 7s = 1$ where we see from inspection $15 - 7 * 2 = 1$, Then there is a solution of $15*2-3*7*2 = -12$ and so $x \equiv -12 \mod 105 \equiv 93 \mod 105$. Then we have $105 - 4 * 26 = 1$ and so we have a solution $105 * 0 - 4 * 26 * 93 = -9672$. Thus

$$x \equiv [-9672]_{105*4} \equiv [-9672]_{420} \equiv [408]_{420}$$

Thus the smallest amount of coins they could have is 408.

4.

We have $\phi(32) = \phi(2^5) = 2^5 - 2^4 = 32 - 16 = 16$, $\phi(21) = \phi(7)\phi(3) = (7-1)(3-1) = 6*2 = 12$, $\phi(120) = \phi(8)\phi(15) = \phi(2^3)\phi(5)\phi(3) = (2^3-2^2)(5-1)(3-1) = 4*4*2 = 32$, and $\phi(384) = \phi(2*192) = \phi(2^2*96) = \phi(2^3*48) = \phi(2^4*24) = \phi(2^5*12) = \phi(2^6*6) = \phi(2^7*3) = \phi(2^7)\phi(3) = (2^7-2^6)(3-1) = 128$.

5.

a) We know that $\phi(7) = 6$ and so $5^6 \equiv 1 \mod 7$. Since $2023 = 6k+1$ we have $5^{2023} = 5^{6k} * 5 \equiv 5 \mod 7$.

b) We know that $\phi(11) = 10$ and so $5^{10} \equiv 1 \mod 7$. Since $2023 = 10m + 3$, we then have $5^{2023} = 5^{10m}5^3 \equiv 5^3 \mod 11 \equiv 25*5 \mod 11 \equiv 15 \mod 11 \equiv 4 \mod 11$. Since 5^{2023} satisfies the above and $7, 11$ being relatively prime, we have $7r + 11s = 1$ where $2*11 - 7*3 = 1$. Then we have solutions $2*11*5 - 7*3*4 = 26$ and so $5^{2023} \equiv 26 \mod 77$

6.

We saw from above $5^{2023} \equiv 4 \mod 11$. Now $\phi(13) = 12$ and since $2023 = 12k+7$, we have $5^{2023} = 5^{12k}5^7 \equiv 5^7 \mod 13 \equiv (5^3)^2 * 5 \mod 13 \equiv 8^2 * 5 \equiv 12 * 5 \equiv 8 \mod 13$. We then have $13s + 11r = 1$ where we see that $11*6 - 13*5 = 1$. Then there is a solution $11*6*8 - 13*5*4 = 268$ and so $5^{2023} \equiv 268 \mod 143 \equiv 125 \mod 143$.

7.

a) We solve $x \equiv 1 \mod 20$ and $x \equiv 1 \mod 13$. We have $20*2 + 13*(-3) = 1$ and so a solution is $20*2*1 + 13*(-3)*1 = 1$. Thus $x \equiv 1 \mod 260$ and 1 Imix returns every 260 days.

b) We solve $x \equiv 8 \mod 13$ and $x \equiv 12 \mod 20$ where we see that $20*2*8 + 13*(-3)*12 = -148 \equiv 112 \mod 260$. Since the first day is $1 \mod 260$, we then have 111 days between the two days