MAT 310 HW 9, Carl Liu

7.A.1

Suppose n is a positive integer. Let $z, w \in V$ such that $z = (z_1, ..., z_n)$ and $w = (w_1, ..., w_n)$. Then

$$\langle z, T^*w\rangle = \langle Tz, w\rangle = \langle (0, z_1, ..., z_{n-1}), w\rangle =$$

$$\langle (0, ..., 0) + ... + (0, ..., z_j ..., 0) + ... + (0, ... z_{n-1}), w\rangle =$$

$$\langle (0, ..., 0), w\rangle + ... + \langle (0, ..., z_j ..., 0), w\rangle + ... + \langle (0, ... z_{n-1}), w\rangle =$$

$$\langle (0, z_1, ..., 0), w\rangle + ... + \langle (0, ..., z_j ..., 0), w\rangle + ... + \langle (0, ... z_{n-1}), w\rangle =$$

$$0 + z_1\overline{w_2} + ... + z_{n-1}\overline{w_n} = \langle (z_1 ... z_n), (w_2, ..., w_n, 0)\rangle$$

Thus we can conclude that $T^*(w_1, ..., w_n) = (w_2, ..., w_n, 0)$

7.A.2

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$

Suppose λ is an eigenvalue of T. Then $(T - \lambda I)$ is not invertible and there is a $v \in V$ which is the eigenvector corresponding to λ. Then *null* $(T - \lambda I)^* = (range\ T - \lambda I)^{\perp}$. But because $(T - \lambda I)$ is not invertible, we have $range\ T - \lambda I \neq V$ meaning $dim\ range\ T - \lambda I < dim\ V$. Since $dim\ (range\ T - \lambda I)^{\perp} = dim\ V - dim\ range\ T - \lambda I$, we have $dim\ (range\ T - \lambda I)^{\perp} > 0$ and thus we can conclude that $dim\ null\ (T - \lambda I)^* > 0$. Therefore we can conclude that $null\ (T - \lambda I)^* \neq \{0\}$ and so $(T - \lambda I)^*$ is thus not injective making it not invertible. But we have $(T - \lambda I)^* = (T^* - \overline{\lambda} I)$ is also not invertible. Therefore we can conclude that $\overline{\lambda}$ is an eigenvalue of T^*

Suppose $\overline{\lambda}$ is an eigenvalue of T^*. The proof is the same as above but replace T with T^*, λ with $\overline{\lambda}$, T^* with T and λ with $\overline{\lambda}$.

7.A.5

Let $T \in \mathcal{L}(V, W)$. Then $null\ T^* = (range\ T)^{\perp}$. So $dim\ null\ T^* = dim\ (range\ T)^{\perp} = dim\ W - dim\ range\ T = dim\ W - (dim\ V - dim\ null\ T) = dim\ null\ T + dim\ W - dim\ V$ as required

We have $range\ T^* = (null\ T)^{\perp}$. So $dim\ range\ T^* = dim\ (null\ T)^{\perp} = dim\ V - dim\ null\ T = dim\ V - (dim\ V - dim\ range\ T) = dim\ range\ T$ as required

7.B.2

Suppose that T is a self-adjoint operator on a finite-dimensional inner product space and that 2 and 3 are the only eigenvalues of T. We then have an orthonormal basis that consists of eigenvectors, lets call $(a_1, ...a_n)$ where n is the dimension of V. Let $v \in V$. Then $v = b_1a_1 + ... + b_na_n$ for some $b_1...b_n$. Since $(T - 2I)a_j = 0$ or $(T - 3I)a_j = 0$, and $(T - 3I)(T - 2I) = (T - 2I)(T - 3I) = (T^2 - 5T + 6I)$, we conclude $\langle (T^2 - 5T + 6I)v, v\rangle = \langle 0, v\rangle = 0$ and so $T^2 - 5T + 6I = 0$ since T is self-adjoint.

7.B.6

Suppose T is a normal operator on a complex inner product space.

Suppose T is self-adjoint. Let λ be an eigenvalue of T. Then we have an eigenvector v corresponding to λ. We have $\langle Tv, v\rangle = \langle v, Tv\rangle = \langle v, \lambda v\rangle$. Since $\langle Tv, v\rangle = \langle \lambda v, v\rangle$, we conclude that $\langle \lambda v, v\rangle = \langle v, \lambda v\rangle$ and so $\lambda \langle v, v\rangle = \overline{\lambda} \langle v, v\rangle$. This results in $\overline{\lambda} = \lambda$ which means λ is real. Since λ is an arbitrary eigenvalue, we can conclude this for all eigenvalues as required.

Suppose all eigenvalues are real. V has an orthonormal basis of eigenvectors of T, lets call $e_1, ..., e_n$. Let $v \in V$. Then $v = b_1e_1 + ...b_ne_n$ for $b_1, ..., b_n \in \mathbf{F}$. We have $\langle Tv, v\rangle = \langle T(b_1e_1 + ... + b_ne_n), v\rangle = \langle v, T^*(b_1e_1 + ... + b_ne_n)\rangle$. But because we have $\langle T(b_1e_1 + ... + b_ne_n), v\rangle = \langle T(b_1e_1 + ... + b_ne_n), b_1e_1\rangle + ... + \langle T(b_1e_1 + ... + b_ne_n), b_ne_n\rangle = \langle b_1Te_1, b_1e_1\rangle + ... + \langle b_nTe_n, b_ne_n\rangle = \langle b_1c_1e_1, b_1e_1\rangle + ... + \langle b_nc_ne_n, b_ne_n\rangle = \langle b_1Te_1, b_1e_1\rangle + ... + \langle b_nTe_n, b_ne_n\rangle = c_1 \langle b_1e_1, b_1e_1\rangle + ... + c_n \langle b_ne_n, b_ne_n\rangle = \langle b_1e_1, b_1c_1e_1\rangle + ... + \langle b_ne_n, b_nc_ne_n\rangle = \langle b_1e_1, b_1Te_1\rangle + ... + \langle b_ne_n, b_nTe_n\rangle = \langle b_1e_1 + ... + b_ne_n, T(b_1e_1 + ... + b_ne_n)\rangle = \langle v, Tv\rangle$, where $Te_j = c_je_j$, meaning c_j must be real. Thus we have $\langle v, T^*v\rangle = \langle v, Tv\rangle$ and so $T^* = T$ meaning T is self-adjoint as required.