MAT 312 HW 3 Carl Liu

1. for $\mathbf{Z}_6$

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

for $\mathbf{Z}_7$

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

2.

i) Since $gcd(7, 11) = 1$, we have $7s + 11r = 1$. Through inspection we see that for $r = -5$ and $s = 8$ we satisfy the equation. Thus we have $7 * 8 = 11r + 1$ and $[7]_{11}^{-1} = [8]_{11}$

ii) Since 2 is a factor of 10 and 26, we can conclude that there is no inverse

iii) Since $gcd(11, 31) = 1$, we have $11s + 31r = 1$. Using the matrix method

$$\left(\begin{array}{cc|c} 1 & 0 & 11 \\ 0 & 1 & 31 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 11 \\ -2 & 1 & 9 \end{array}\right) \to \left(\begin{array}{cc|c} 3 & -1 & 2 \\ -2 & 1 & 9 \end{array}\right) \to \left(\begin{array}{cc|c} 3 & -1 & 2 \\ -14 & 5 & 1 \end{array}\right)$$

and so $s = -14$, $r = 5$. We then have $-14 * 11 = -31 * 5 + 1$. So $[11]_{31}^{-1} = [-14]_{31} = [17]_{31}$

iv) Since $gcd(23, 31) = 1$, we have $23s + 31r = 1$, we have through inspection $r = 3$ and $s = -4$. Thus we have $-4 * 23 = -3 * 31 + 1$. Meaning $[23]_{31}^{-1} = [-4]_{31} = [27]_{31}$

v) Since $gcd(91, 237) = 1$, we have $91s + 237r = 1$. Using the matrix method

$$\left(\begin{array}{cc|c} 1 & 0 & 237 \\ 0 & 1 & 91 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -2 & 55 \\ 0 & 1 & 91 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -2 & 55 \\ -1 & 3 & 36 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & -5 & 19 \\ -1 & 3 & 36 \end{array}\right) \to$$

$$\left(\begin{array}{cc|c} 2 & -5 & 19 \\ -3 & 8 & 17 \end{array}\right) \to \left(\begin{array}{cc|c} 5 & -13 & 2 \\ -3 & 8 & 17 \end{array}\right) \to \left(\begin{array}{cc|c} 5 & -13 & 2 \\ -43 & 112 & 1 \end{array}\right)$$

Thus $-43 * 237 + 112 * 91 = 1$. So $112 * 91 = 43 * 237 + 1$ meaning $[91]_{237}^{-1} = [112]_{237}$

3.

The hw says to do problem 5 but the hint is for problem 6 so i did 6.

We have $x^2 \equiv 1 \mod p \to x^2 - 1 = (x-1)(x+1) \equiv 0 \mod p$. Clearly we see that $[p-1]_p$ and $[1]_p$ are solutions since $p - 1 + 1 = p \equiv 0 \mod p$ and $1 - 1 = 0 \equiv 0 \mod p$. Now suppose for contradiction that there exists a solution which isn't one of the above. Then we would have $(x-1)(x+1) = (pk+r)^2 - 1 = p^2k^2 + 2pkr + r^2 - 1$ but because $r \neq p-1$ and $r \neq 1$, we cannot divide $r^2 - 1$ by p and thus $(pk+r)^2 - 1 \equiv 0 \mod p$ is false. Thus we only have two solutions as required.

4.

a) We shall prove using induction. In the base case $n = 1$, we have $10 \equiv 1 \mod 9$ clearly. Now suppose as inductive hypothesis that $10^n \equiv 1 \mod 9$ for $n \geq 1$. Then $10^{n+1} = 10^n * 10$. Since $10^n * 10 \equiv 10^n * 1 = 10^n \equiv 1 \mod 9$, we can thus close the induction. Therefore we conclude that for all $n \geq 1$, $10^n \equiv 1 \mod 9$.

b) Let x be defined as in the question. We have

$$x = \sum_{n=0}^{k} a_n * 10^n$$

Then

$$x \equiv \left(\sum_{n=0}^{k} a_n * 10^n\right) \mod 9 \equiv \left(\sum_{n=0}^{k} a_n * 1\right) \mod 9$$

due to part a. Thus we are done.

c) Suppose x is divisible by 9. Then $x = \sum_{n=0}^{k} a_n * 10^n \equiv 0 \mod 9$. But we have $\sum_{n=0}^{k} a_n * 10^n \equiv \sum_{n=0}^{k} a_n$ by a. Thus we have $x = \sum_{n=0}^{k} a_n * 10^n \equiv \sum_{n=0}^{k} a_n \equiv 0 \mod 9$ and thus we must have the sum of the digits be divisible by 9 as required. Now suppose $\sum_{n=0}^{k} a_n \equiv 0 \mod 9$. Since $\sum_{n=0}^{k} a_n \equiv \sum_{n=0}^{k} a_n * 10^n$ by a, we conclude $x \equiv 0 \mod 9$ as required. Thus we are done

5.

We will prove that $10^n \equiv -1 \mod 11$ when n is odd and $10^n \equiv 1 \mod 11$ when n is even. In the case that n is even we have $n = 2m$. For the base case $m = 1$, we have $10^2 = 100 \equiv 1 \mod 11$. Now suppose as inductive hypothesis that $10^{2m} \equiv 1$

mod 11 where $m \geq 1$. Then $10^{2(m+1)} = 10^{2m} * 100 \equiv 10^{2m} * 1 \equiv 1 \mod 11$ as required. In the case that n is odd we have $n = 2m+1$. For the base case $m = 0$ we have $10^{2m+1} = 10 \equiv -1 \mod 11$. Now suppose as inductive hypothesis $10^{2m+1} \equiv -1 \mod 11$ where $m \geq 0$. Then $10^{2(m+1)+1} = 10^{2m+1+2} = 10^{2m+1} * 100 \equiv 10^{2m+1} * 1 \equiv -1 \mod 11$ as required. Thus we have finished the proof.

Since we have $x = \sum_{n=0}^{k} a_n * 10^n$. Suppose $x \equiv 0 \mod 11$. Then $x = \sum_{n=0}^{k} a_n * 10^n \equiv \sum_{n=0}^{k} (-1)^n a_n \equiv 0 \mod 11$. The converse is also true.

6.

i) since the $gcd(3,12) = 3$ and $3 \not| 1$ we conclude that there are no solutions for x.

ii) We have $gcd(3,11) = 1$ and so we have one unique solution. Through inspection we find that the inverse of $[3]_{11}^{-1} = [4]_{11}$. Thus $x \equiv 4 \mod 11$ and so $[4]_{11}$ is a solution

iii) $gcd(64,84) = 4$. Thus dividing through we have $16x \equiv 8 \mod 21$.

$$\left(\begin{array}{cc|c} 1 & 0 & 16 \\ 0 & 1 & 21 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 16 \\ -1 & 1 & 5 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 4 & -3 & 1 \\ -1 & 1 & 5 \end{array}\right)$$

Thus making the inverse $[16]_{21}^{-1} = [4]_{21}$. We then have $x \equiv 32 \mod 21 \equiv 11 \mod 21$ and thus we have solution of $[11]_{21}$ which is the same as having $[11 + 21k]_{84}$ for $0 \geq k \geq 3$.

iv) Since $gcd(15,17) = 1$, we have one unique solution.

$$\left(\begin{array}{cc|c} 1 & 0 & 15 \\ 0 & 1 & 17 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 15 \\ -1 & 1 & 2 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 8 & -7 & 1 \\ -1 & 1 & 2 \end{array}\right)$$

Thus we have $[15]_{17}^{[}-1] = [8]_{17}$ and so $x \equiv 8 * 5 \mod 17 \equiv 6 \mod 17$. Therefore the solution for x is $[6]_{17}$.

v) Since we have $gcd(15,18) = 3$ and $3 \not| 5$, we can conclude that there are no solutions.

vi) Since $gcd(15,100) = 5$, we have $3x \equiv 1 \mod 20$ and through inspection we see that $[3]_{20}^{-1} = [7]_{20}$ and so $x \equiv 7 \mod 20$ making $[7]_{20}$ a solution, which also means $[3 + 20k]_{100}$ for $0 \geq k \geq 4$ are also solutions.

vii) Since $gcd(23,107) = 1$, we have one unique solution for x.

$$\left(\begin{array}{cc|c} 1 & 0 & 23 \\ 0 & 1 & 107 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 23 \\ -4 & 1 & 15 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 5 & -1 & 8 \\ -4 & 1 & 15 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 5 & -1 & 8 \\ -14 & 3 & -1 \end{array}\right)$$

So $-14 * 23 + 3 * 107 = -1$ meaning $14 * 23 - 3 * 107 = 1$. Therefore we have $[23]_{107}^{-1} = [14]_{107}$ and so $x \equiv 14 * 16 \mod 107 \equiv 10 \mod 107$ and so $[10]_{107}$ is a solution.

7.

To find the solution to this problem, we need first to find $7x \equiv 13 \mod 30$ to find which week's friday will land on the $13th$. Thus we find through inspection that $7 * 13 - 30 * 3 = 1$. Thus $[7]_{30}^{-1} = [13]_{30}$ and we have $x \equiv 13^2 \mod 30 \equiv 19 \mod 30$. Thus we have solutions $[19]_{30}$. Therefore we conclude that 19 weeks will have to pass, and it will occur every 30 weeks.