MAT 312 HW 2 Carl Liu

1.

We have $a_1 = 1$, $a_2 = 2a_1 + 1 = 3, a_3 = 2a_2 + 1 = 7, a_4 = 2a_3 + 1 = 15, a_5 = 2a_4 + 1 = 31$.

In the base case of $n = 1$, we have $a_1 + 1 = 1 + 1 = 2 = 2^1$, a power of 2 as required. Now as inductive hypothesis suppose $a_k + 1 = 2^n$ where k and n are natural number. Then $a_{k+1} + 1 = 2a_k + 1 + 1 = 2a_k + 2 = 2(a_k + 1) = 2 * 2^n = 2^{n+1}$ which is a power of 2 as required. Thus we can close the induction and conclude that $a_k + 1$ is indeed a power of 2 for all k.

2.

We want $gcd(a_n, a_{n+1}) = 1$. The fibonacci sequence is defined as $a_1 = a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n > 2$. For the base case of $n = 1$, we have $gcd(a_1, a_2) = gcd(1, 1) = 1$ as required. Now suppose as inductive hypothesis that $gcd(a_n, a_{n+1}) = 1$ where n is a natural number. Then $gcd(a_{n+1}, a_{n+2}) = gcd(a_{n+1}, a_n + a_{n+1})$. Since a_n and a_{n+1} are relatively prime, we have integers b, c such that $a_nb + a_{n+1}c = 1$. In the case $b > c$, we would have $b = c + d$, where d is a positive integer. Then $a_nb + a_{n+1}c = a_nb + a_{n+1}(b - d) = (a_n + a_{n+1})b - a_{n+1}d = 1$. Since this is a linear integral combination of $(a_n + a_{n+1})$ and a_{n+1} as well as being the smallest possible positive number, we conclude that $gcd(a_{n+1}, a_{n+2}) = 1$. In the case $c \geq b$, we have $c = b + e$, where e is a natural number. Then $a_nb + a_{n+1}c = a_nb + a_{n+1}(b + e) = (a_n+a_{n+1})b+a_{n+1}e = 1$ which is once again a linear integral combination of (a_n+a_{n+1}) and a_{n+1}. Since 1 is the smallest possible positive number we can once again conclude that $gcd(a_{n+1}, a_{n+2}) = 1$. We can thus close the induction and conclude that a_n and a_{n+1} are relatively prime for all positive n as required.

3.

i) In the base case $n = 1$, we have $1^5 - 1 = 0$ and $5|0$ since $0 = 5 * 0$. Now suppose as inductive hypothesis that $5|(n^5 - n)$ where n is a positive integer. Then $(n+1)^5-(n+1) = n^5+5n^4+10n^3+10n^2+5n+1-n-1 = n^5+5n^4+10n^3+10n^2+5n-n$ Since we have from inductive hypothesis $n^5 - n = 5x$ for some integer x, we then have $(n + 1)^5 - (n + 1) = 5n^4 + 10n^3 + 10n^2 + 5n + 5x = 5(n^4 + 2n^3 + 2n^2 + n + x)$ and so we can conclude $5|((n + 1)^5 - (n + 1))$, thus closing the induction. Therefore for all positive integers n, $n^5 - n$ is divisible by 5.

ii) In the base case $n = 1$, we have $3^2 - 1 = 8$ which is divisible by 8 because $8 = 8 * 1$. Now suppose as inductive hypothesis that $8|(3^{2n} - 1)$ where n is a positive integer. Then $(3^{2n} - 1) = 8x$ where x is an integer. Then $3^{2(n+1)} - 1 = 3^{2n}3^2 - 1 =$

$3^{2n} * 8 + 3^{2n} - 1 = 3^{2n} * 8 + 8x = 8(3^{2n} + x)$. Thus we can conclude that $3^{2(n+1)} - 1$ is divisible by 8 thereby closing the induction. Therefore for all positive integer n, we have $3^{2n} - 1$ is divisible by 8.

4.

In this proof we had assumed that the set of k pens after the first pen is removed is the same color as the set of k pens after the last pen is removed. For example consider a set of 2 pen, $\{a, b\}$. By taking out, a, the first element, we are left with the set $\{b\}$ which is of one color. By taking out the last pen we are left with $\{a\}$ which is of one color. But that does not mean a and b are of the same color and so $\{a\} \cup \{b\}$ is not guaranteed to be of one color. Also the first and last element of a set are not well defined as a set itself has no real ordering.

5.

In the base case where $r = 1$ and we have $a = p_1^{n_1}$ for some prime p_1, we have $1, p_1 ... p_1^{n_1}$ as unique factors. So we have (n_1+1) unique factors as required. Suppose as inductive hypothesis that for all a that has a prime factorization of r unique prime numbers where we use $n_1...n_r$ are the powers of their respective prime factors, (So we have the form $a = p_1^{n_1}...p_r^{n_r}$) there is $(n_1 + 1)...(n_r + 1)$. Then let a be any number with $r + 1$ unique prime number factor so $a = p_1^{n_1}...p_{r+1}^{n_{r+1}}$. We then have $a = (p_1^{n_1}...p_r^{n_r})p_{r+1}^{n_{r+1}}$ But we have from inductive hypothesis $(p_1^{n_1}...p_r^{n_r})$ has $(n_1 + 1)...(n_r + 1)$ unique factor and since $p_{r+1}^{n_{r+1}}$ has $(n_r + 1) + 1$ unique factors, we can form $(n_1 + 1)...(n_r + 1)((n_r + 1) + 1)$ unique factors. Thus we can close this induction and can conclude that there are $(n_1 + 1)...(n_r + 1)$ unique factors as required.

6.

a) We have $2^{3m} + 1 = (2^m)^3 + 1 = (2^m + 1)(2^{2m} - 2^m + 1)$. Therefore $2^{3m} + 1$ is divisible by $2^m + 1$, which is never equal to 1 nor equal to $2^{3m} + 1$ and so cannot be prime.

b) In the case l is odd we have $l = 2n - 1$ where n is a positive integer. Then $2^{lm} + 1 = 2^{(2n-1)m} + 1$. We shall use induction on n for this proof. In the base case of $n = 1$, we have $2^{(2-1)m} + 1 = 2^m + 1$ which is divisible by $2^m + 1$ as required. Now suppose as inductive hypothesis that $2^{(2n-1)m} + 1$ is divisible by $2^m + 1$. Then $2^{(2(n+1)-1)m} + 1 = 2^{(2n+1)m} + 1 = 2^{(2n-1)m+2m} + 1 = 2^{(2n-1)m}2^{2m} + 1 = 2^{(2n-1)m}(2^{2m} - 1) + (2^{(2n-1)m} + 1) = 2^{(2n-1)m}(2^m - 1)(2^m + 1) + (2^{(2n-1)m} + 1)$ by the inductive hypothesis we have $2^{(2n-1)m} + 1 = (2^m + 1)q$ because it is divisible by $2^m + 1$. Therefore

$$2^{(2n-1)m}(2^m - 1)(2^m + 1) + (2^{(2n-1)m} + 1) = (2^m + 1)(2^{(2n-1)m}(2^m - 1) + q)$$

and thus is divisible by 2^{m+1}, closing the induction. Therefore we can conclude that

$2^{lm}+1$ is divisible by 2^m+1 for all odd l.

c) Suppose 2^n+1 is prime. n is either odd or even. Suppose n is odd Suppose for contradiction that n is not a power of 2. Then $n = 2^x(p_1)^{n_1}...(p_r)^{n_r}$ where $r > 0$, $n_k > 0$, and p_k are unique primes that aren't equal to 2. Then $2^n+1 = 2^{((p_1)^{n_1}...(p_r)^{n_r})2^x}$ where $(p_1)^{n_1}...(p_r)^{n_r})$ must be odd because if 2 divides it, then 2 divides one of the primes, a contradiction. So by b, we have 2^n+1 is divisible by $2^{2^x}+1$ and cannot be prime. Therefore we conclude that n must be a power of 2 for 2^n+1 to be prime as required.

7.

Since we have unique prime factorizations $a = \prod_{k=1}^r p_k^{n_k}$ and $b = \prod_{k=1}^r p_k^{m_k}$. Also $gcd(a,b) = \prod_{k=1}^r p_k^{min(n_k,m_k)}$ while $lcm(a,b) = \prod_{k=1}^r p_k^{max(n_k,m_k)}$. Since we have

$$ab = \prod_{k=1}^{r} p_k^{n_k} \prod_{k=1}^{r} p_k^{m_k} = \prod_{k=1}^{r} p_k^{n_k+m_k}$$

and

$$gcd(a,b)lcm(a,b) = \prod_{k=1}^{r} p_k^{min(n_k,m_k)} \prod_{k=1}^{r} p_k^{max(n_k,m_k)} = \prod_{k=1}^{r} p_k^{min(n_k,m_k)+max(n_k,m_k)}$$

We will prove that $min(n_k,m_k)+max(n_k,m_k) = n_k+m_k$. There are two cases. In the case $n_k > m_k$, we have $min(n_k,m_k) = m_k$ and $max(n_k,m_k) = n_k$. Thus $min(n_k,m_k)+max(n_k,m_k) = m_k+n_k$ as required. In the case $n_k \leq m_k$, we have $min(n_k,m_k) = m_k$ and $max(n_k,m_k) = n_k$ making $min(n_k,m_k)+max(n_k,m_k) = n_k+m_k$ as required. Thus we have

$$\prod_{k=1}^{r} p_k^{min(n_k,m_k)+max(n_k,m_k)} = \prod_{k=1}^{r} p_k^{n_k+m_k}$$

and so $gcd(a,b)lcm(a,b) = ab$ as required.