MAT 310 HW 2, Carl Liu

1.C.20

Consider the set $W = \{(0, x, y, 0) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}$. This is clearly a subset of $\mathbf{F}^4$. We also have $(0, 0, 0, 0) \in W$. Since $(0, x, y, 0) = (0, 0, 0, 0)$ when $x = 0, y = 0 \in \mathbf{F}$. Consider two vectors $u, v \in W$. Then $u = (0, a, b, 0)$ and $v = (0, c, d, 0)$ for $a, b, c, d \in \mathbf{F}$. So $u+v = (0, a+c, b+d, 0)$, but $a+c, b+d \in \mathbf{F}$ and so have $u+v \in W$ making W closed under addition. Consider $\lambda \in \mathbf{F}$ and $v \in W$. Then $v = (0, a, b, 0)$ where $a, b \in \mathbf{F}$. Then $\lambda v = (0\lambda, \lambda a, \lambda b, 0\lambda) = (0, \lambda a, \lambda b, 0)$ but since $\lambda a, \lambda b \in \mathbf{F}$ we can thus conclude that $\lambda v \in W$ and W is closed under scalar multiplication. So W is a subspace of $\mathbf{F}^4$ as required.

Consider the set $U + W = \{u + w : u \in U, w \in W\} = \{(x, a, b, y) \in \mathbf{F}^4 : x, y, a, b \in \mathbf{F}\}$. Since $0 = (0, 0, 0, 0) \in U + W$, we have $0 = u + w = (x, x, y, y) + (0, a, b, 0)(x, x + a, y + b, y)$ and $0 = u' + w' = (x', x', y', y') + (0, a', b', 0)$. So $x = x' = 0$, $x + a = x' + a' = 0$, $y + b = y' + b' = 0$, and $y = y' = 0$. But because $x = x' = 0$ and $y = y' = 0$, we have $0 = x' + a' = 0 + a' = a'$, $0 = x + a = 0 + a = a$, $0 = y + b = 0 + b = b$, and $0 = y' + b' = 0 + b' = b'$. Thus we have $a = a'$, $b = b'$, $c = c'$, and $d = d'$. Therefore $u' = u$ and $w = w'$. So we have a unique representation of 0. Thus we can conclude that $U + W$ is a direct sum and can express it as $U \oplus W$.

We have $U \oplus W \subseteq \mathbf{F}^4$ since addition is closed in vector spaces. Let $x \in \mathbf{F}^4$. Then $x = (a, b, c, d)$ where $a, b, c, d \in \mathbf{F}$. Then $x \in U \oplus W$. Thus $\mathbf{F}^4 \subseteq U \oplus W$. Therefore $U \oplus W = \mathbf{F}^4$ as required.

2.A.6

Let $0 = a(v_1 - v_2) + b(v_2 - v_3) + c(v_3 - v_4) + dv_4$ where $a, b, c, d \in \mathbf{F}$. Then $a(v_1 - v_2) + b(v_2 - v_3) + c(v_3 - v_4) + dv_4 = a(v_1) + (b-a)v_2 + (c-b)v_3 + (d-c)v_4 = 0$. But we have v_1, v_2, v_3, v_4 is linearly independent and so $a = 0$, $b - a = 0$, $c - b = 0$, and $d - c = 0$. That means $0 = b - a = b - 0 = b$, $0 = c - b = c - 0 = c$, and $0 = d - c = d - 0 = d$. So we must have $0 = a, b, c, d$ and thus we can conclude that $(v_1 - v_2), (v_2 - v_3), (v_3 - v_4), v_4$ is indeed linearly independent.

2.A.10

Since $v_1 + w ... v_m + w$ is linearly dependent, there exists a $1 \leq j \leq m$ such that $a_j \in \mathbf{F}$ and $a_j \neq 0$ and $0 = a_1(v_1 + w)...a_m(v_m + w)$. Then $\sum_{n=1}^{m} a_n(v_n + w) = \sum_{n=1}^{m} a_n v_n + w \sum_{n=1}^{m} a_n = 0$. That means $\sum_{n=1}^{m} a_n v_n = -w \sum_{n=1}^{m} a_n$. We then have two cases, either $w = 0$(vector) or $w \neq 0$. In the case $w = 0$, we have $\sum_{n=1}^{m} 0 * v_n = 0 = w$ and thus we have $w \in span(v_1...v_m)$ as required. In the case $w \neq 0$, suppose for contradiction that $\sum_{n=1}^{m} a_n = 0$. Then $-w \sum_{n=1}^{m} a_n = -w * 0 = 0$. But $\sum_{n=1}^{m} a_n v_n = -w \sum_{n=1}^{m} a_n = 0$ and because $v_1...v_m$ is linearly independent we must have $a_n = 0$ for all n. This is a contradiction since we have $a_j \neq 0$. Thus we can conclude $\sum_{n=1}^{m} a_n \neq 0$. That means we have $\sum_{n=1}^{m}((-a_n/\sum_{n=1}^{m} a_n)(v_n + w)) = w$ and since the left hand side is a sum of $v_1...v_m$, we can thus conclude $w \in span(v_1...v_m)$ as required.

2.A.14

Suppose V is infinite-dimensional. Then $span(v_1...v_m) \neq V$ for all $m \in \mathbf{N}$. Let the sequence w_n be defined recursively as $w_1 \in V$ such that $w_1 \neq 0$, $w_{n+1} \in V$ and $w_{n+1} \notin span(w_1...w_n)$. Such a sequence exists because $span(w_1...w_n) \neq V$ for all n and thus we have $w \in V$ such that $w \notin span(w_1...w_n)$.

We will prove using induction that $w_1...w_n$ is linearly independent for all n. Suppose as base case $n = 1$. Then we have w_1 is linearly independent since it is a list of one vector that is not equal to 0. Suppose as inductive hypothesis that $w_1...w_n$ is linearly independent for some $n \geq 1$. Then $w_{n+1} \neq \sum_{i=1}^{n} a_i w_i$ for all $a_n \in \mathbf{F}$. This is due to how we defined the sequence earlier. Let $0 = \sum_{i=1}^{n} a_i w_i + b w_{n+1}$. Then $-b w_{n+1} = \sum_{i=1}^{n} a_i w_i$. Suppose for contradiction that $b \neq 0$. Then $\sum_{i=1}^{n}(-a_i/b) w_i = w_{n+1}$. But that's a contradiction since w_{n+1} cannot be a linear combination of $w_1...w_n$. Thus $b = 0$. So we have $0 = 0 w_{n+1} = \sum_{i=1}^{i} a_i w_i$. But because $w_1...w_n$ is linearly independent, we must have $a_i = 0$ thus we must have $0 w_i n + 1 = \sum_{i=1}^{n} 0 w_i$ and so we must have $0 = \sum_{i=1}^{n} 0 w_i + 0 w_{n+1}$ meaning $w_1...w_{n+1}$ is linearly independent. Thus we can close the induction and we have $w_1...w_n$ is linearly independent for all $n \geq 1$ as required.

Suppose the converse. Suppose for contradiction that V is finite dimensional. Then there exists an $m \geq 1$ such that a list, $(w_1...w_m)$ of vectors in V spans V. But we have a list $v_1...v_{m+1}$ that is linearly independent in V. By 2.23, a list that spans V must have greater than $m+1$ elements. But $w_1...w_m$ spans V and has only m elements, a contradiction. Thus we can conclude that V is infinite dimensional as required.

We can then conclude the equivalence true.

2.A.15

Consider the sequence $(v_n)_{n=1}^{\infty}$ defined as $v_n = (a_j)_{j=1}^{\infty}$ where $a_j = 1$ when $j = n$ and $a_j = 0$ otherwise. v_n is clearly in $\mathbf{F}^{\infty}$. Let $n \geq 1$ and $0 = \sum_{i=1}^{n} b_i v_i$. Since v_i is 0 except at its ith element, we have $\sum_{i=1}^{n} b_i v_i = (c_i)_{i=1}^{\infty}$ where $c_i = b_i$ when $i \leq n$ and $c_i = 0$ when $i > n$. So $(c_i)_{i=1}^{\infty} = 0$ and so $b_i = 0$. This means $v_1 ... v_n$ is linearly independent. Since $n \geq 1$ was arbitrary we can conclude this for all $n \in \mathbf{N}$ and thus by the exercise above, we have $\mathbf{F}^{\infty}$ is infinite dimensional as required.