PHY 300 HW 3, Carl Liu

4-16

A) Through Kirchhoff's junction rule, we have that the current into a junction is equal to that out of the junction. So we have $I_0 \cos(\omega t) = I_1 + I_2 + I_3$. But $I_1 = V_R/R$. We also have $C * V_C = q$. Since V_C is dependent on the current, which is dependent on time, we have $C * dV_C/dt = I_2$. Finally $V_L = LdI_3/dt$. Thus

$$\frac{1}{L}\int V_L dt = \int dI_3 = I_3$$

So

$$I_0 \cos(\omega t) = \frac{V_R}{R} + C\frac{dV_C}{dt} + \frac{1}{L}\int V_L dt$$

But because $V_C = V_L = V_R$ due to the components being in parallel and $V_L = LdI_3/dt$, we can thus conclude that

$$I_0 \cos(\omega t) = CL\frac{d^2 I_3}{dt^2} + \frac{L}{R}\frac{dI_3}{dt} + I_3$$

So

$$\frac{I_0}{LC}\cos(\omega t) = \frac{d^2 I_3}{dt^2} + \frac{1}{RC}\frac{dI_3}{dt} + \frac{1}{LC}I_3$$

This is the same form as a damped driven harmonic oscillator and so we have

$$\omega_0^2 = \frac{1}{LC} \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

and

$$B) \quad \gamma = \frac{1}{RC}$$

C) Since the resistor is the only draining power source we have $< P > = < V^2/R > = V_0^2/2R$

Since $V_0 = I_0 R$ at resonance, we have $< V^2/R > = I_0^2 R/2$

5-2

Lets consider the right pendulum clamped. We then have

$$m\frac{d^2 x}{dt^2} = -m\frac{g}{l}x - kx$$

$$\frac{d^2x}{dt^2} + (\frac{g}{l} + \frac{k}{m})x = 0$$

for the left pendulum since the right pendulum causes no displacement to the spring. But that means $\omega_0^2 = \frac{g}{l} + \frac{k}{m}$ and so $\omega_0 = \sqrt{\frac{g}{l} + \frac{k}{m}}$. We know that $T = 1.25s$. So $\omega_0 = 2\pi/T = 5.027rad/s$. Then $5.027rad/s = \sqrt{\frac{g}{l} + \frac{k}{m}}$ meaning $0.770729s^{-2} = \frac{k}{m}$

A) The forces on the pendulums allowed to swing freely is governed by the differential equations

$$m\frac{d^2x_1}{dt^2} = -\frac{mg}{l}x_1 - k(x_1 - x_2)$$

$$m\frac{d^2x_2}{dt^2} = -\frac{mg}{l}x_2 - k(x_2 - x_1)$$

Thus

$$\frac{d^2x_1}{dt^2} + \left(\frac{g}{l} + \frac{k}{m}\right)x_1 - \frac{k}{m}x_2 = 0$$
$$\frac{d^2x_2}{dt^2} + \left(\frac{g}{l} + \frac{k}{m}\right)x_2 - \frac{k}{m}x_1 = 0$$

We then have

$$\frac{d^2(x_1 - x_2)}{dt^2} + \left(\frac{g}{l} + 2\frac{k}{m}\right)(x_1 - x_2) = 0$$

$$\frac{d^2(x_1 + x_2)}{dt^2} + \left(\frac{g}{l}\right)(x_1 + x_2) = 0$$

Letting $y = x_1 - x_2$ and $z = x_1 + x_2$, we then have solutions of

$$y = C\cos\left(\sqrt{\frac{g}{l} + 2\frac{k}{m}}t\right)$$

$$z = D\cos\left(\sqrt{\frac{g}{l}}t\right)$$

So for the two normal modes, we have

$$\omega_1 = \sqrt{\frac{g}{l} + 2\frac{k}{m}} = 5.1031rad/s$$

$$\omega_2 = \sqrt{\frac{g}{l}} = 4.95 rad/s$$

and thus their periods are

$$T_1 = \frac{2\pi}{\omega_1} 1.23s$$

$$T_2 = \frac{2\pi}{\omega_2} = 1.27s$$

B) We have

$$x_1(t) = \frac{y+z}{2} = C\cos\left(\sqrt{\frac{g}{l} + 2\frac{k}{m}} t\right) + D\cos\left(\sqrt{\frac{g}{l}} t\right)$$

$$x_2(t) = \frac{z-y}{2} = D\cos\left(\sqrt{\frac{g}{l}} t\right) - C\cos\left(\sqrt{\frac{g}{l} + 2\frac{k}{m}} t\right)$$

These can be considered beats and we thus have successive maximums with a period of $2\pi/(5.1031 - 4.95) = 41.04s$

5-4

The differential equation governing this system is

$$m\frac{d^2x_A}{dt^2} = -k_A x_A - k_C(x_A - x_B)$$

$$m\frac{d^2x_B}{dt^2} = -k_B x_B - k_C(x_B - x_A)$$

Then

$$\frac{d^2x_A}{dt^2} + \frac{k_A + k_C}{m} x_A - \frac{k_C}{m} x_B = 0$$

$$\frac{d^2x_B}{dt^2} + \frac{k_B + k_C}{m} x_B - \frac{k_C}{m} x_A = 0$$

Using the eigenvalue method we have

$$e^{\alpha t}\mathbf{c} = \mathbf{x} \quad Let\ \alpha^2 = \lambda = -\omega^2 \quad Thus\ \alpha = \pm i\omega$$

$$\lambda\mathbf{c} = \begin{bmatrix} -\frac{k_A+k_C}{m} & \frac{k_C}{m} \\ \frac{k_C}{m} & -\frac{k_B+k_C}{m} \end{bmatrix}\mathbf{c}$$

$$0 = \begin{bmatrix} -\frac{k_A+k_C}{m} - \lambda & \frac{k_C}{m} \\ \frac{k_C}{m} & -\frac{k_B+k_C}{m} - \lambda \end{bmatrix} \mathbf{c}$$

So

$$\left(-\frac{k_A+k_C}{m} - \lambda\right)\left(-\frac{k_B+k_C}{m} - \lambda\right) - \frac{k_C^2}{m^2} =$$

$$\lambda^2 + \lambda\left(\frac{k_A+k_C}{m} + \frac{k_B+k_C}{m}\right) + \left(\frac{k_A+k_C}{m}\right)\left(\frac{k_B+k_C}{m}\right) - \frac{k_C^2}{m^2} =$$

$$\lambda^2 + \lambda\left(\frac{k_A+2k_C+k_B}{m}\right) + \left(\frac{k_Ak_B + k_C(k_A+k_B)}{m^2}\right) = 0$$

Meaning

$$\lambda = -\left(\frac{k_A+2k_C+k_B}{2m}\right) \pm \frac{1}{2}\sqrt{\left(\frac{k_A+2k_C+k_B}{m}\right)^2 - 4\left(\frac{k_Ak_B + k_C(k_A+k_B)}{m^2}\right)} =$$

$$-\left(\frac{k_A+2k_C+k_B}{2m}\right) \pm \frac{1}{2}\sqrt{\frac{k_A^2 - 2k_Bk_A + 4k_C^2 + k_B^2}{m^2}}$$

But we have $k_C^2 = k_Ak_B$ and so

$$\lambda = -\left(\frac{k_A+2k_C+k_B}{2m}\right) \pm \frac{1}{2m}\sqrt{k_A^2 + 2k_Bk_A + k_B^2} =$$

$$-\left(\frac{k_A+2k_C+k_B}{2m}\right) \pm \frac{1}{2m}\sqrt{(k_A+k_B)^2} =$$

$$-\left(\frac{k_A+2k_C+k_B}{2m}\right) \pm \frac{1}{2m}(k_A+k_B)$$

So we have

$$\lambda = -\frac{k_C}{m}$$

or

$$\lambda = -\frac{k_A+k_C+k_B}{m}$$

From the earlier definition of α, we have

$$\alpha = \pm i\sqrt{\frac{k_C}{m}}$$

or

$$\alpha = \pm i\sqrt{\frac{k_A + k_C + k_B}{m}}$$

This means we have linearly independent solutions of

$$\exp\left(\pm i\sqrt{\frac{k_C}{m}}t\right)$$

$$\exp\left(\pm i\sqrt{\frac{k_A + k_C + k_B}{m}}t\right)$$

Thus we have

$$\omega' = \sqrt{\frac{k_C}{m}}$$

and

$$\omega'' = \sqrt{\frac{k_A + k_C + k_B}{m}}$$

5-9

A) We have the differential equations below describing the molecules motion

$$m_1\frac{d^2x_1}{dt^2} = -k(x_1 - x_2)$$

$$m_2\frac{d^2x_2}{dt^2} = -k(x_2 - x_3) - k(x_2 - x_1) = -k(2x_2 - x_1 - x_3)$$

$$m_3\frac{d^2x_3}{dt^2} = -k(x_3 - x_2)$$

So

$$\frac{d^2x_1}{dt^2} = -\frac{k}{m_1}(x_1 - x_2)$$

$$\frac{d^2x_2}{dt^2} = -\frac{k}{m_2}(2x_2 - x_1 - x_3)$$

$$\frac{d^2x_3}{dt^2} = -\frac{k}{m_3}(x_3 - x_2)$$

We will solve for the normal modes by using the eigenvalue method

$$e^{\alpha t}\mathbf{c} = \mathbf{x} \quad Let\ \alpha^2 = \lambda = -\omega^2 \quad Thus\ \alpha = \pm i\omega$$

$$\lambda \mathbf{c} = \begin{bmatrix} -\frac{k}{m_1} & \frac{k}{m_1} & 0 \\ \frac{k}{m_2} & -2\frac{k}{m_2} & \frac{k}{m_2} \\ 0 & \frac{k}{m_3} & -\frac{k}{m_3} \end{bmatrix} \mathbf{c}$$

$$0 = \begin{bmatrix} -\frac{k}{m_1} - \lambda & \frac{k}{m_1} & 0 \\ \frac{k}{m_2} & -2\frac{k}{m_2} - \lambda & \frac{k}{m_2} \\ 0 & \frac{k}{m_3} & -\frac{k}{m_3} - \lambda \end{bmatrix} \mathbf{c}$$

So

$$0 = \left(\frac{-k}{m_1} - \lambda\right)\left(\left(\frac{-2k}{m_2} - \lambda\right)\left(\frac{-k}{m_3} - \lambda\right) - \frac{k^2}{m_2 m_3}\right) - \frac{k^2}{m_1 m_2}\left(\frac{-k}{m_3} - \lambda\right) =$$

$$\frac{-k}{m_1}\left(\frac{-2k}{m_2} - \lambda\right)\left(\frac{-k}{m_3} - \lambda\right) + \frac{k^3}{m_1 m_2 m_3} - \lambda\left(\frac{-2k}{m_2} - \lambda\right)\left(\frac{-k}{m_3} - \lambda\right) +$$

$$\frac{k^2\lambda}{m_2 m_3} + \frac{k^3}{m_1 m_2 m_3} + \frac{k^2\lambda}{m_1 m_2} =$$

$$\frac{-2k^3}{m_1 m_2 m_3} - \frac{k^2\lambda}{m_1 m_3} - \frac{2k^2\lambda}{m_1 m_2} - \frac{\lambda^2 k}{m_1} + \frac{k^3}{m_1 m_2 m_3} - \frac{2k^2\lambda}{m_2 m_3} - \frac{k\lambda^2}{m_3} - \frac{2k\lambda^2}{m_2} - \lambda^3 +$$

$$\frac{k^2\lambda}{m_2 m_3} + \frac{k^3}{m_1 m_2 m_3} + \frac{k^2\lambda}{m_1 m_2} =$$

$$-\frac{k^2\lambda}{m_1 m_3} - \frac{k^2\lambda}{m_1 m_2} - \frac{\lambda^2 k}{m_1} - \frac{k^2\lambda}{m_2 m_3} - \frac{k\lambda^2}{m_3} - \frac{2k\lambda^2}{m_2} - \lambda^3 =$$

$$-\lambda\left(\frac{k^2}{m_1 m_3} + \frac{k^2}{m_1 m_2} + \frac{\lambda k}{m_1} + \frac{k^2}{m_2 m_3} + \frac{k\lambda}{m_3} + \frac{2k\lambda}{m_2} + \lambda^2\right) =$$

$$-\lambda\left(\lambda^2 + \left(\frac{k}{m_1} + \frac{k}{m_3} + \frac{2k}{m_2}\right)\lambda + \left(\frac{k^2}{m_1 m_3} + \frac{k^2}{m_1 m_2} + \frac{k^2}{m_2 m_3}\right)\right) = 0$$

So we have $\lambda = 0$ as a solution and so $\alpha = 0$ thus $0 = \omega_0$ as a normal mode. We also have

$$\lambda = -\frac{1}{2}\left(\frac{k}{m_1} + \frac{k}{m_3} + \frac{2k}{m_2}\right) \pm$$

$$\frac{1}{2}\sqrt{\left(\frac{k}{m_1} + \frac{k}{m_3} + \frac{2k}{m_2}\right)^2 - 4\left(\frac{k^2}{m_1 m_3} + \frac{k^2}{m_1 m_2} + \frac{k^2}{m_2 m_3}\right)} =$$

$$-\frac{1}{2}\left(\frac{k}{m_1}+\frac{k}{m_3}+\frac{2k}{m_2}\right)\pm$$

$$\frac{1}{2}\sqrt{\frac{k^2}{m_1^2}+\frac{k^2}{m_3m_1}+\frac{2k^2}{m_2m_1}+\frac{k^2}{m_1m_3}+\frac{k^2}{m_3^2}+\frac{2k^2}{m_3m_2}+\frac{2k^2}{m_1m_2}+\frac{2k^2}{m_3m_2}+\frac{4k^2}{m_2^2}-4\left(\frac{k^2}{m_1m_3}+\frac{k^2}{m_1m_2}+\frac{k^2}{m_2m_3}\right)}=$$

$$-\frac{1}{2}\left(\frac{k}{m_1}+\frac{k}{m_3}+\frac{2k}{m_2}\right)\pm$$

$$\frac{1}{2}\sqrt{\frac{k^2}{m_1^2}+\frac{2k^2}{m_3m_1}+\frac{4k^2}{m_2m_1}+\frac{k^2}{m_3^2}+\frac{4k^2}{m_3m_2}+\frac{4k^2}{m_2^2}-4\left(\frac{k^2}{m_1m_3}+\frac{k^2}{m_1m_2}+\frac{k^2}{m_2m_3}\right)}=$$

$$-\frac{1}{2}\left(\frac{k}{m_1}+\frac{k}{m_3}+\frac{2k}{m_2}\right)\pm\frac{1}{2}\sqrt{\frac{k^2}{m_1^2}-\frac{2k^2}{m_3m_1}+\frac{k^2}{m_3^2}+\frac{4k^2}{m_2^2}}$$

But $m_1 = m_3$ and so we have

$$\lambda=-\frac{1}{2}\left(\frac{k}{m_1}+\frac{k}{m_1}+\frac{2k}{m_2}\right)\pm\frac{1}{2}\sqrt{\frac{k^2}{m_1^2}-\frac{2k^2}{m_1^2}+\frac{k^2}{m_1^2}+\frac{4k^2}{m_2^2}}=$$

$$-\frac{1}{2}\left(\frac{2k}{m_1}+\frac{2k}{m_2}\right)\pm\frac{1}{2}\frac{2k}{m_2}$$

So we have

$$\lambda=-\frac{k}{m_1}$$

or

$$\lambda=-\frac{k}{m_1}-\frac{2k}{m_2}$$

From the earlier definitions for α, we have $\alpha=\pm i\sqrt{\frac{k}{m_1}}$ or $\alpha=\pm i\sqrt{\frac{k}{m_1}+\frac{2k}{m_2}}$
Thus we have

$$\exp\left(\pm i\sqrt{\frac{k}{m_1}}t\right)$$

$$\exp\left(\pm i\sqrt{\frac{k}{m_1}+\frac{2k}{m_2}}t\right)$$

as linearly independent solutions. Thus we have

$$\omega' = \sqrt{\frac{k}{m_1}}, \quad \omega'' = \sqrt{\frac{k}{m_1}+\frac{2k}{m_2}}, \ \textit{and}\ \omega_0 = 0$$

as normal modes (ω_0 was shown earlier). If $m_3 \neq m_1$, we could have taken the square root of the λ's before substituting $m_3 = m_1$ and arrived at the normal modes through the same continuing process.

B) By having $m_1 = m_3 = 16$ and $m_2 = 12$ and taking a ratio

$$\frac{\omega'}{\omega''} = \frac{\sqrt{\frac{k}{m_1}}}{\sqrt{\frac{k}{m_1}+\frac{2k}{m_2}}} = \sqrt{\frac{1}{m_1\left(\frac{1}{m_1}+\frac{2}{m_2}\right)}} = \sqrt{\frac{1}{1+\frac{2m_1}{m_2}}} = \sqrt{\frac{1}{1+\frac{32}{12}}} =$$

$$0.5222$$

Taking the ratio the other way we arrive at

$$\frac{\omega''}{\omega'} = \left(\frac{\omega'}{\omega''}\right)^{-1} = 1.915$$

as the ratio

5-10

Not taking into account the gravitational force, we have

$$m\frac{d^2x_1}{dt} = -kx_1 - k(x_1 - x_2)$$

$$m\frac{d^2x_2}{dt} = -k(x_2 - x_1)$$

So

$$\frac{d^2x_1}{dt} + 2\frac{k}{m}x_1 - \frac{k}{m}x_2 = 0$$

$$\frac{d^2x_2}{dt} + \frac{k}{m}(x_2 - x_1) = 0$$

We shall use the eigenvalue method

$$e^{\alpha t}\mathbf{c} = \mathbf{x} \quad Let\ \alpha^2 = \lambda = -\omega^2 \quad Thus\ \alpha = \pm i\omega$$

$$\lambda\mathbf{c} = \begin{bmatrix} -2\frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{k}{m} \end{bmatrix}\mathbf{c}$$

$$0 = \begin{bmatrix} -2\frac{k}{m} - \lambda & \frac{k}{m} \\ \frac{k}{m} & -\frac{k}{m} - \lambda \end{bmatrix}\mathbf{c}$$

So

$$0 = \left(-2\frac{k}{m} - \lambda\right)\left(-\frac{k}{m} - \lambda\right) - \frac{k^2}{m^2} = 2\frac{k^2}{m^2} + \frac{k\lambda}{m} + 2\frac{k\lambda}{m} + \lambda^2 - \frac{k^2}{m^2} =$$

$$\lambda^2 + 3\lambda\frac{k}{m} + \frac{k^2}{m^2}$$

Thus

$$\lambda = -\frac{3k}{m2} \pm \frac{1}{2}\sqrt{\frac{9k^2}{m^2} - 4\frac{k^2}{m^2}} = -3\frac{k}{2m} \pm \frac{1}{2}\sqrt{5\frac{k^2}{m^2}} = (-3 \pm \sqrt{5})\frac{k}{2m}$$

Therefore we have linearly independent solutions of

$$\exp(\pm i\sqrt{(3 \pm \sqrt{5})\frac{k}{2m}}$$

So

$$\omega = \sqrt{(3 \pm \sqrt{5})\frac{k}{2m}}$$

for the normal modes meaning

$$\omega^2 = (3 \pm \sqrt{5})\frac{k}{2m}$$

as required. The ratio is thus

$$\frac{\omega_+}{\omega_-} = \sqrt{\frac{(3+\sqrt{5})\frac{k}{2m}}{(3-\sqrt{5})\frac{k}{2m}}} = \sqrt{\frac{3+\sqrt{5}}{3-\sqrt{5}}} = \frac{\sqrt{4}}{3-\sqrt{5}} = \sqrt{\frac{3+\sqrt{5}}{3-\sqrt{5}}}$$

$$= \frac{2+2\sqrt{5}}{3-\sqrt{5}+3\sqrt{5}-5} = \frac{2+2\sqrt{5}}{-2+2\sqrt{5}} = \frac{\sqrt{5}+1}{\sqrt{5}-1}$$

as required. Since the solutions have the same frequencies when in normal modes, we have $x_1 = A\cos(\omega t)$ and $x_3 = B\cos(\omega t)$. Consider the first normal mode ω_+, we have from the differential equations

$$-A\omega_+^2 + A2\frac{k}{m} = B\frac{k}{m}$$

$$-A(3+\sqrt{5})\frac{k}{2m} + A2\frac{k}{m} = B\frac{k}{m}$$

$$\frac{B}{A} = 2 - \frac{1}{2}(3+\sqrt{5}) = -0.618$$

Consider the second normal mode ω_-, we have from the differential equations

$$-A\omega_-^2 + A2\frac{k}{m} = B\frac{k}{m}$$

$$-A(3-\sqrt{5})\frac{k}{2m} + A2\frac{k}{m} = B\frac{k}{m}$$

$$\frac{B}{A} = 2 - \frac{1}{2}(3-\sqrt{5}) = 1.618$$