MAT 341 HW4, Carl Liu

1.

Given the initial conditions and a steady state of $v(x) = 0$, we must have $\lim_{t\to\infty} u(x,t) = v(x) = 0$. That means we then have $\lim_{t\to\infty} u(0,t) = \lim_{t\to\infty} T_0$. But $\lim_{t\to\infty} u(0,t) = v(0) = 0$ and $\lim_{t\to\infty} T_0 = T_0$. So we must have $T_0 = 0$. We also have

$$\lim_{t\to\infty} \frac{\partial u}{\partial x}(a,t) = \frac{dv}{dx}(a) = 0$$

So

$$\lim_{t\to\infty} -\kappa\frac{\partial u}{\partial x}(a,t) = -\kappa\frac{dv}{dx}(a) = 0$$

Meaning

$$0 = \lim_{t\to\infty} -\kappa\frac{\partial u}{\partial x}(a,t) = \lim_{t\to\infty} h(u(a,t) - T_1) = h(v(a) - T_1) = h(0 - T_1) = -hT_1$$

Thus we have $0 = -hT_1$ and so $T_1 = 0 = T_0$

2.

Suppose hypothesis. Then we have

$$L(\alpha u_1(x,t) + \beta u_2(x,t)) =$$

$$\frac{\partial^2(\alpha u_1(x,t) + \beta u_2(x,t))}{\partial^2 x} - \frac{1}{k}\frac{\partial(\alpha u_1(x,t) + \beta u_2(x,t))}{\partial x} =$$

$$\frac{\partial^2(\alpha u_1(x,t))}{\partial^2 x} + \frac{\partial^2(\beta u_2(x,t))}{\partial^2 x} - \frac{1}{k}\left(\frac{\partial(\alpha u_1(x,t))}{\partial x} + \frac{\partial(\beta u_2(x,t))}{\partial x}\right) =$$

$$\alpha\left(\frac{\partial^2(u_1(x,t))}{\partial^2 x} - \frac{1}{k}\frac{\partial(u_1(x,t))}{\partial x}\right) + \beta\left(\frac{\partial(u_2(x,t))}{\partial x} - \frac{1}{k}\frac{\partial(u_2(x,t))}{\partial x}\right) =$$

$$\alpha * 0 + \beta * 0 = 0$$

as required

3.

$$g(x) = \begin{cases} \frac{2T_0 x}{a}, & 0 < x < \frac{a}{2} \\ \frac{2T_0(a-x)}{a}, & \frac{a}{2} < x < a \end{cases}$$

We know that

$$w(x,t) = \sum_{n=1}^{\infty} b_n \sin(\lambda_n x) \exp(-\lambda_n^2 kt)$$

where $\lambda_n = n\pi/a$

That means the initial condition must be

$$w(x,0) = \sum_{n=1}^{\infty} b_n \sin(\lambda_n x) = g(x)$$

Consider the odd periodic extension

$$g_o(x) = \begin{cases} \frac{-2T_0(a+x)}{a}, & -a < x < \frac{-a}{2} \\ \frac{2T_0 x}{a}, & \frac{-a}{2} < x < 0 \\ \frac{2T_0 x}{a}, & 0 < x < \frac{a}{2} \\ \frac{2T_0(a-x)}{a}, & \frac{a}{2} < x < a \end{cases}$$

So we have

$$\int_{-a}^{a} g_o(x) \sin(\lambda_n x) dx = \int_{-a}^{a} b_n \sin^2(\lambda_n x) dx$$

$$b_n = \frac{2}{a} \int_0^a g_o(x) \sin(\lambda_n x) dx =$$

$$\frac{2}{a} \left(\int_0^{a/2} \frac{2T_0 x}{a} \sin(\lambda_n x) dx + \int_{a/2}^{a} \frac{2T_0(a-x)}{a} \sin(\lambda_n x) dx \right) =$$

$$\frac{4T_0}{a^2} ([-\frac{x}{\lambda_n} \cos(\lambda_n x)]_0^{a/2} + [\frac{1}{\lambda_n^2} \sin(\lambda_n x)]_0^{a/2} +$$

$$[-\frac{a-x}{\lambda_n} \cos(\lambda_n x)]_{a/2}^{a} - [\frac{1}{\lambda_n^2} \sin(\lambda_n x)]_{a/2}^{a}) =$$

$$\frac{4T_0}{a^2} \left(2 \frac{a^2}{n^2\pi^2} \sin(n\pi/2) \right) = \frac{8T_0}{n^2\pi^2} \sin(n\pi/2) =$$

$$\frac{8T_0}{n^2\pi^2} (-1)^{(n^2+n+2)/2} (1/2 - (-1)^n/2)$$

($(n^2+n+2)/2$ alternates between even and odd every other n allowing for use in alternating -1)

Thus we have

$$w(x,0) = \sum_{n=1}^{\infty} \frac{8T_0}{n^2\pi^2} \sin(n\pi/2)\sin(n\pi x/a) =$$

$$\sum_{n=1}^{\infty} \frac{8T_0}{n^2\pi^2}(-1)^{(n^2+n+2)/2}(1/2-(-1)^n/2)\sin(n\pi x/a)$$

and the final solution is

$$w(x,t) = \sum_{n=1}^{\infty} \frac{8T_0}{n^2\pi^2} \sin(n\pi/2)\sin(n\pi x/a)\exp(-\frac{n^2\pi^2}{a^2}kt)$$

which is the same as

$$w(x,t) = \sum_{n=1}^{\infty} \frac{8T_0}{n^2\pi^2}(-1)^{(n^2+n+2)/2}(1/2-(-1)^n/2)\sin(n\pi x/a)\exp(-\frac{n^2\pi^2}{a^2}kt)$$