MAT 310 HW 3, Carl Liu

2.B.3

A) Consider the vectors $\mathbf{v}_1 = (3,1,0,0,0), \mathbf{v}_2 = (0,0,7,1,0),$ *and* $\mathbf{v}_3 = (0,0,0,0,1)$. $\mathbf{v}_1$ is in U since $(3,1,0,0,0) \in \mathbf{R}^5$ and $x_1 = 3$, $x_2 = 1$ meaning $x_1 = 3x_2$. Also since $x_4 = 0$ we have $x_3 = 0 = 7x_4$. $\mathbf{v}_2$ is in U since $(0,0,7,1,0) \in \mathbf{R}^5$, $x_4 = 1$ so $x_3 = 7x_4$ and $x_2 = 0$ so $x_1 = 3x_2 = 0$. $\mathbf{v}_3$ is in U since $(0,0,0,0,1) \in \mathbf{R}^5$ and $x_1 = x_2 = x_3 = x_4 = 0$ so $x_1 = 3x_2 = 0$ and $x_3 = 7x_4 = 0$.

Let $\mathbf{v} \in U$. Then $\mathbf{v} = (3x_2, x_2, 7x_4, x_4, x_5)$ for $x_2, x_4, x_5 \in \mathbf{R}$. But we have $x_2\mathbf{v}_1 + x_4\mathbf{v}_2 + x_5\mathbf{v}_3 = (3x_2, x_2, 0, 0, 0) + (0, 0, 7x_4, x_4, 0) + (0, 0, 0, 0, x_5) = (3x_2, x_2, 7x_4, x_4, x_5) = \mathbf{v}$. Thus we can conclude that $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ spans U

Since $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ spans U, we must have $a_1, a_2, a_3 \in \mathbf{R}$ such that $0 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$. Now suppose that we also have $c_1, c_2, c_3 \in \mathbf{R}$ such that $0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. We have $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = (3c_1, c_1, 7c_2, c_2, c_3)$ and $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = (3a_1, a_1, 7a_2, a_2, a_3)$. Thus $(3c_1, c_1, 7c_2, c_2, c_3) = (3a_1, a_1, 7a_2, a_2, a_3)$ meaning $c_1 = a_1$, $c_2 = a_2$, and $c_3 = a_3$. Therefore $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ uniquely represents 0 meaning the vectors are linearly independent. Thus it can be concluded that the list $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is indeed a basis of U.

B) Consider the vectors in **5** of $\mathbf{v}_4 = (1,0,0,0,0)$, $\mathbf{v}_5 = (0,0,1,0,0)$, $\mathbf{v}_6 = (0,0,0,1,0)$, $\mathbf{v}_7 = (0,1,0,0,0)$, and $\mathbf{v}_8 = (0,0,0,0,1)$. This spans $\mathbf{R}^5$ and also is linearly independent. So $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7, \mathbf{v}_8)$ spans $\mathbf{R}^5$. Starting from $\mathbf{v}_1$, we have $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent in relation to the vectors before their respective vector. Then let $0 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + a_5\mathbf{v}_5$ and $0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5$. We have $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + a_5\mathbf{v}_5 = (3a_1 + a_4, a_1, 7a_2 + a_5, a_2, a_3)$ and $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5 = (3c_1 + c_4, c_1, 7c_2 + c_5, c_2, c_3)$. We must have $(3c_1 + c_4, c_1, 7c_2 + c_5, c_2, c_3) = (3a_1 + a_4, a_1, 7a_2 + a_5, a_2, a_3)$. So we have $c_1 = a_1$, $a_2 = c_2$, $a_3 = c_3$, $3a_1 + a_4 = 3c_1 + c_4$, and $7a_2 + a_5 = 7c_2 + c_5$. That in turn means $3a_1 + a_4 = 3c_1 + a_4 = 3c_1 + c_4$. So $a_4 = c_4$. Also $7a_2 + a_5 = 7c_2 + a_5 = 7c_2 + c_5$ so $a_5 = c_5$. Thus we can conclude that the coefficients are the same and there is a unique representation of 0. That therefore means the vectors $\mathbf{v}_1 ... \mathbf{v}_5$ are linearly independent. Since $\mathbf{v}_6 = \mathbf{v}_2 - 7\mathbf{v}_5 = (0,0,7-7,1,0) = (0,0,0,1,0)$ is linearly dependent we remove $\mathbf{v}_6$ from the list. Since $\mathbf{v}_7 = \mathbf{v}_1 - 3\mathbf{v}_4 = (3-3,1,0,0,0) = (0,1,0,0,0)$ is linearly dependent, we also remove it from the list. Since $\mathbf{v}_8 = \mathbf{v}_3$ is linearly dependent, it is also removed from the list. Thus we are left with $(\mathbf{v}_1 ... \mathbf{v}_5)$ being a linearly independent list that spans $\mathbf{R}^5$ by 2.33 of section 2.B and is thus a basis

C) Consider the subspace $W = \{(x_1, 0, x_2, 0, 0) \in \mathbf{R}^5 : x_1, x_2 \in \mathbf{R}\}$. Let $0 = W_1 + U_1$ where $W_1 \in W$ and $U_1 \in U$. Suppose $0 = W_2 + U_2$ as well. Since $W_1 + U_1 = (3a_1 + x_1, a_1, 7a_2 + x_2, a_2, a_3)$ and $W_2 + U_2 = (3a_1 + y_1, a_1, 7a_2 + y_2, a_2, a_3)$. We must have $(3a_1 + x_1, a_1, 7a_2 + x_2, a_2, a_3) = (3c_1 + y_1, c_1, 7c_2 + y_2, c_2, c_3)$. But we already established in B that $a_n = c_n$ and $x_n = y_n$. Thus we can conclude that 0 has a unique representation and it is such that all terms must be 0 as required. Thus $U \oplus W$ is a direct sum.

Now let $x \in \mathbf{R}^5$. It is then a linear combination of $(\mathbf{v}_1 ... \mathbf{v}_5)$ due to these vectors being a basis and therefore spanning $\mathbf{R}^5$. So $x = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + a_5\mathbf{v}_5 = (3a_1, a_1, 7a_2, a_2, a_3) + (a_4, 0, a_5, 0, 0)$ and $(3a_1, a_1, 7a_2, a_2, a_3) \in U$ as well as $(a_4, 0, a_5, 0, 0) \in W$, we can conclude that $x = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + a_5\mathbf{v}_5 \in U \oplus W$.

Since $U \oplus W$ can be broken into the components $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + a_5\mathbf{v}_5$, we can also conclude that $U \oplus W$ spans $\mathbf{R}^5$ and thus $\mathbf{R}^5 = U \oplus W$ as required.

2.B.6

Since V is a vector space and $(v_1 ... v_2)$ is a basis of V, we can conclude that $v_1 ... v_2 \in V$ and because vector spaces are closed under addition, $(v_1 + v_2), (v_2 + v_3), (v_3 + v_4)$ are also in V.

Let $v \in V$. Suppose $v = a_1(v_1 + v_2) + a_2(v_2 + v_3) + a_3(v_3 + v_4) + a_4 v_4$ and $v = b_1(v_1 + v_2) + b_2(v_2 + v_3) + b_3(v_3 + v_4) + b_4 v_4$. Then $v = a_1 v_1 + (a_1 + a_2)v_2 + (a_2 + a_3)v_3 + (a_3 + a_4)v_4$ and $v = b_1 v_1 + (b_1 + b_2)v_2 + (b_2 + b_3)v_3 + (b_3 + b_4)v_4$. But because $(v_1 ... v_4)$ is a basis of V, the coefficients of $a_1 v_1 + (a_1 + a_2)v_2 + (a_2 + a_3)v_3 + (a_3 + a_4)v_4 = v = b_1 v_1 + (b_1 + b_2)v_2 + (b_2 + b_3)v_3 + (b_3 + b_4)v_4$ must be unique. Thus we must have $a_1 = b_1$, $(b_1 + b_2) = (a_1 + a_2)$, $(b_2 + b_3) = (a_2 + a_3)$, and $(b_3 + b_4) = (a_3 + a_4)$. Therefore we must have $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$, and $a_4 = b_4$. But these are just the coefficients of v expressed in the $((v_1 + v_2) ... v_4)$ vectors. Therefore we have a unique representation of v and thus $((v_1 + v_2) ... v_4)$ is indeed a basis of V.

2.C.1

Suppose hypothesis. Since $dim(U) = dim(V)$ and is finite dimensional, we have $(u_1...u_n)$ is a basis of U and $(v_1...v_n)$ is a basis of V. Since $u_1...u_n \in V$, due to U being a subspace of V, linearly independent, due to being a basis of U, and having length $dim(V)$, we can use 2.39 of section 2.C to conclude that $(u_1...u_n)$ is a basis of V. So by definition of basis we must also have $span(u_1...u_n) = V$. But we also have $span(u_1...u_n) = U$ by definition of basis and thus $U = V$ as required.

2.C.10

We have $1, z, ..., z^m$ as a basis of $\mathcal{P}_m(\mathbf{F})$. Thus we have $\dim(\mathcal{P}_m(\mathbf{F})) = m+1$. We also have $p_n \in \mathcal{P}_m(\mathbf{F})$ for all $n \leq m$ by definition. Now consider $0 = \sum_{n=0}^{m} a_n p_n$. We shall use strong induction to prove that a_n must be 0 for all $0 \leq n \leq m$

Suppose as base case $k = 0$, then $a_{m-k} = a_m$. We have p_m being a polynomial of degree m, so there is a term $a_m z^m$. But because all the other polynomials in the sum has degrees less then m, we must have $a_m = 0$ since 0 has no terms of degree m.

Suppose as strong inductive hypothesis that for some $m \geq k \geq 0$, $a_{m-l} = 0$ for all $l \leq k$. Then we have $0 = \sum_{n=0}^{k-1} a_n p_n = 0$ since all coefficients above $k-1$ is 0. But consider $a_{k-1}p_{k-1}$. This has a term of $a_{k-1}z^{k-1}$, but 0 has no such terms of such degree and so we must have $a_{k-1} = 0$. This closes the induction. Thus we can conclude that we must have $a_m = 0$ in order for $\sum_{n=0}^{m} a_n p_n = 0$. So we have $(p_0...p_n)$ is linearly independent. But because the list has the same length, $m+1$, as $dim(\mathcal{P}_m(\mathbf{F}))$, we can thus conclude that $(p_0...p_n)$ is a basis of $\mathcal{P}_m(\mathbf{F})$ as required.

2.C.12

Suppose hypothesis. Suppose for contradiction that $U_1 \cap U_2 = \{0\}$. Since U_1 and U_2 are subspaces of a finite dimensional vector space, we have by 2.43 of section 2.C, $dim(U_1 + U_2) = dim(U_1) + dim(U_2) - dim(U_1 \cap U_2)$ But because $U_1 \cap U_2 = \{0\}$, we have $dim(U_1 \cap U_2) = 0$. So $dim(U_1) + dim(U_2) - dim(U_1 \cap U_2) = 5 + 5 = 10$. Thus $dim(U_1 + U_2) = 10$. But $U_1 + U_2$ is a subspace of $\mathbf{R}^9$ since $\mathbf{R}^9$ is a vector space and is thus closed under addition. So $dim(U_1 + U_2) = 10 \leq dim(\mathbf{R}^9) = 9$ a contradiction. Thus we can conclude that $U_1 \cap U_2 \neq \{0\}$