MAT 341 HW7, Carl Liu

1.

$f(x) = T_0 e^{-\alpha x}, x > 0$

We will solve

$$B(\lambda) = \frac{2}{\pi}\int_0^\infty f(x)\sin(\lambda x)dx = \frac{2T_0}{\pi}\int_0^\infty e^{-\alpha x}\sin(\lambda x)dx =$$

$$\frac{2T_0}{\pi}\left(-\frac{1}{\lambda}\left[e^{-\alpha x}\cos(\lambda x)\right]_0^\infty - \frac{\alpha}{\lambda}\int_0^\infty e^{-\alpha x}\cos(\lambda x)dx\right) =$$

$$\frac{2T_0}{\pi}\left(-\frac{1}{\lambda}\left[e^{-\alpha x}\cos(\lambda x)\right]_0^\infty - \frac{\alpha}{\lambda^2}\left[e^{-\alpha x}\sin(\lambda x)\right]_0^\infty - \frac{\alpha^2}{\lambda^2}\int_0^\infty e^{-\alpha x}\sin(\lambda x)dx\right)$$

But that means

$$B(\lambda) = \frac{2T_0}{\pi(1+\frac{\alpha^2}{\lambda^2})}\left(-\frac{1}{\lambda}\left[e^{-\alpha x}\cos(\lambda x)\right]_0^\infty - \frac{\alpha}{\lambda^2}\left[e^{-\alpha x}\sin(\lambda x)\right]_0^\infty\right) =$$

$$B(\lambda) = \frac{2T_0}{\pi(1+\frac{\alpha^2}{\lambda^2})}\frac{1}{\lambda} = \frac{2T_0}{\pi(\lambda+\frac{\alpha}{\lambda})}$$

and so our solution is

$$u(x,t) = \int_0^\infty \frac{2T_0}{\pi(\lambda+\frac{\alpha}{\lambda})}\sin(\lambda x)\exp(-\lambda^2 kt)d\lambda$$

2.

A)

Assuming $u(x,t) = \phi(x)T(t)$, we have from equation 1

$$\phi''(x)T(t) = \frac{1}{k}\phi(x)T'(t)$$

Separating we then have

$$\frac{\phi''(x)}{\phi(x)} = \frac{1}{k}\frac{T'(t)}{T(t)}$$

So the two must be equivalent to a constant, $-\lambda^2$ and thus resulting in

$$\phi''(x) + \lambda^2\phi(x) = 0 \quad T'(t) + \lambda^2 kT(t) = 0$$

with solutions of the form

$$\phi(x) = c_1\cos(\lambda x) + c_2\sin(\lambda x) \quad T(t) = \exp(-\lambda^2 kt)$$

Since

$$\frac{\partial u}{\partial x}(0,t) = 0$$

we must have

$$\phi'(0) = -\lambda c_1\sin(\lambda 0) + \lambda c_2\cos(\lambda 0) = \lambda c_2 = 0$$

resulting in $c_2 = 0$ and therefore

$$\phi(x) = c_1\cos(\lambda x)$$

But λ can be any positive number thus we have

$$u(x,t) = \int_0^\infty A(\lambda)\cos(\lambda x)\exp(-\lambda^2 kt)d\lambda$$

where

$$u(x,0) = \int_0^\infty A(\lambda)\cos(\lambda x)d\lambda = f(x)$$

and so being of the form of a Fourier integral we have

$$A(\lambda) = \frac{2}{\pi}\int_0^\infty f(x)\cos(\lambda x)dx$$

B)

Solving with

$$f(x) = \begin{cases} 0 & 0 < x < a \\ T & a < x < b \\ 0 & b < x \end{cases}$$

we have

$$A(\lambda) = \frac{2}{\pi} \int_a^b T \cos(\lambda x) dx = \frac{2T}{\pi\lambda} (\sin(\lambda b) - \sin(\lambda a))$$

meaning

$$u(x,t) = \frac{2T}{\pi} \int_0^\infty \frac{1}{\lambda} (\sin(\lambda b) - \sin(\lambda a)) \cos(\lambda x) \exp(-\lambda^2 kt) d\lambda$$

3.

We have

$$u(x,t) = \phi(x)T(t) \quad \frac{\phi''(x)}{\phi(x)} = \frac{T'(t)}{T(t)} = p$$

where p is a constant. Now suppose for contradiction that p is positive. Then we have

$$\phi''(x) - p\phi(x) = 0 \quad T'(t) - pT(t) = 0$$

with solutions

$$\phi(x) = c_1 e^{\sqrt{p}x} + c_2 e^{-\sqrt{p}x} \quad T(t) = \exp(pt)$$

Thus we will have

$$u(x,t) = (c_1 e^{\sqrt{p}x} + c_2 e^{-\sqrt{p}x}) \exp(pt)$$

But that would mean

$$u(x,0) = c_1 e^{\sqrt{p}x} + c_2 e^{-\sqrt{p}x} = f(x)$$

but we can't fit $c_1 e^{\sqrt{p}x} + c_2 e^{-\sqrt{p}x}$ to any arbitrary function. There is also the condition that $|u(x,t)|$ be bounded as $x \to \pm\infty$. This is not satisfied by the $c_1 e^{\sqrt{p}x}$ term which would cause $u(x,t)$ to explode as x goes to ∞. Therefore we must have p as non-positive

4.

We have

$$f(x) = \begin{cases} T_0(a - |x|) & -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

We have a solution of the form

$$u(x,t) = \int_0^\infty (A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x))\exp(-\lambda^2 kt)d\lambda$$

Where

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\lambda x)dx \quad B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\lambda x)dx$$

But since $f(x)$ is an even function, we have $B(\lambda) = 0$ and

$$A(\lambda) = \frac{2}{\pi}\int_0^a (a-x)\cos(\lambda x)dx = \frac{2}{\pi}\left(\frac{1}{\lambda}\left[(a-x)\sin(\lambda x)\right]_0^a - \frac{1}{\lambda^2}[\cos(\lambda x)]_0^a\right) =$$

$$\frac{2}{\pi\lambda^2}(1 - \cos(\lambda a))$$

Thus we have

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{1}{\lambda^2}(1 - \cos(\lambda a))\cos(\lambda x)\exp(-\lambda^2 kt)d\lambda$$