PHY 300 HW 7, Carl Liu

1.2

We have
$$\frac{\partial^2 f(z-ut)}{\partial t^2} = u^2 \frac{d^2 f(z-ut)}{d(z-ut)^2}$$
and
$$\frac{\partial^2 f(z-ut)}{\partial z^2} = \frac{d^2 f(z-ut)}{d(z-ut)^2}$$
since a space partial derivative results in a multiplication of factor 1. Thus we have
$$\frac{\partial^2 f(z-ut)}{\partial z^2} = \frac{1}{u^2} \frac{\partial^2 f(z-ut)}{\partial t^2}$$
as required

1.3

We have
$$\nabla^2 f(\hat{\mathbf{n}}\cdot\mathbf{r}-ut) = \frac{\partial^2 f(\hat{\mathbf{n}}\cdot\mathbf{r}-ut)}{\partial x^2} + \frac{\partial^2 f(\hat{\mathbf{n}}\cdot\mathbf{r}-ut)}{\partial y^2} + \frac{\partial^2 f(\hat{\mathbf{n}}\cdot\mathbf{r}-ut)}{\partial z^2} =$$
$$n_1^2 \frac{d^2 f(\hat{\mathbf{n}}\cdot\mathbf{r}-ut)}{d(\hat{\mathbf{n}}\cdot\mathbf{r}-ut)^2} + n_2^2 \frac{d^2 f(\hat{\mathbf{n}}\cdot\mathbf{r}-ut)}{d(\hat{\mathbf{n}}\cdot\mathbf{r}-ut)^2} + n_3^2 \frac{d^2 f(\hat{\mathbf{n}}\cdot\mathbf{r}-ut)}{d(\hat{\mathbf{n}}\cdot\mathbf{r}-ut)^2}$$
But because $\hat{\mathbf{n}}$ is a unit vector, we have $n_1^2+n_2^2+n_3^2=1$ and so
$$n_1^2 \frac{d^2 f(\hat{\mathbf{n}}\cdot\mathbf{r}-ut)}{d(\hat{\mathbf{n}}\cdot\mathbf{r}-ut)^2} + n_2^2 \frac{d^2 f(\hat{\mathbf{n}}\cdot\mathbf{r}-ut)}{d(\hat{\mathbf{n}}\cdot\mathbf{r}-ut)^2} + n_3^2 \frac{d^2 f(\hat{\mathbf{n}}\cdot\mathbf{r}-ut)}{d(\hat{\mathbf{n}}\cdot\mathbf{r}-ut)^2}$$
$$(n_1^2+n_2^2+n_3^2) \frac{d^2 f(\hat{\mathbf{n}}\cdot\mathbf{r}-ut)}{d(\hat{\mathbf{n}}\cdot\mathbf{r}-ut)^2} = \frac{d^2 f(\hat{\mathbf{n}}\cdot\mathbf{r}-ut)}{d(\hat{\mathbf{n}}\cdot\mathbf{r}-ut)^2}$$
We also have
$$\frac{\partial^2 f(\hat{\mathbf{n}}\cdot\mathbf{r}-ut)}{\partial t^2} = u^2 \frac{d^2 f(\hat{\mathbf{n}}\cdot\mathbf{r}-ut)}{d(\hat{\mathbf{n}}\cdot\mathbf{r}-ut)^2}$$
Thus we have
$$\nabla^2 f(\hat{\mathbf{n}}\cdot\mathbf{r}-ut) = \frac{1}{u^2} \frac{\partial^2 f(\hat{\mathbf{n}}\cdot\mathbf{r}-ut)}{\partial t^2}$$

1.5

We have $n = c/u$ meaning $u = c/n$ where u is the speed of propagation. Since $\lambda * f = u$, we then have $\lambda = u/f$. By definition $k = \frac{2\pi}{\lambda}$ meaning $k = \frac{2\pi f}{u}$. We have $f = c/(633 * 10^{-9})$. Thus we have

$$k = \frac{2\pi f}{u} = \frac{2\pi f * n}{c} = \frac{2\pi c * n}{c * 633 * 10^{-9}} = \frac{2\pi * n}{633 * 10^{-9}} = 13,201,637m^{-1}$$

1.6

A)

Since $\omega = k/u$, we have

$$u_g = \frac{d\omega}{dk} = \frac{d}{dk}(ku) = \frac{u * dk + k * du}{dk} = u + k\frac{du}{dk}$$

Since $k = 2\pi/\lambda$, we have

$$\frac{dk}{d\lambda} = \frac{d}{d\lambda}\left(\frac{2\pi}{\lambda}\right) = -\frac{2\pi}{\lambda^2}$$

Then we have

$$k = \frac{2\pi}{\lambda} = -\lambda\left(-\frac{2\pi}{\lambda^2}\right) = -\lambda\frac{dk}{d\lambda}$$

Therefore

$$u_g = u + k\frac{du}{dk} = u - \lambda\frac{dk}{d\lambda}\frac{du}{dk} = u - \lambda\frac{du}{d\lambda}$$

as required.

B)

Since $\omega = 2\pi f = 2\pi c/\lambda_0$, we have

$$u_g = \frac{d\omega}{dk} = -\frac{2\pi c}{\lambda_0^2}\frac{d\lambda_0}{dk}$$

and so

$$\frac{1}{u_g} = -\frac{\lambda_0^2}{2\pi c}\frac{dk}{d\lambda_0}$$

But we also have $\omega = kc/n$ and so $k = 2\pi n/\lambda_0$. This means

$$\frac{dk}{d\lambda_0} = \frac{d}{d\lambda_0}\left(\frac{2\pi n}{\lambda_0}\right) = \frac{2\pi}{\lambda_0}\frac{dn}{d\lambda_0} - \frac{2\pi n}{\lambda_0^2}$$

and so

$$\frac{1}{u_g} = -\frac{\lambda_0^2}{2\pi c}\left(\frac{2\pi}{\lambda_0}\frac{dn}{d\lambda_0} - \frac{2\pi n}{\lambda_0^2}\right) = \frac{n}{c} - \frac{\lambda_0}{c}\frac{dn}{d\lambda_0}$$

though we have $n = c/u$ and so $n/c = 1/u$ therefore we have

$$\frac{1}{u_g} = \frac{1}{u} - \frac{\lambda_0}{c}\frac{dn}{d\lambda_0}$$

as required