MAT 310 HW 7, Carl Liu

5.B.2

Suppose λ is an eigenvalue of T and v is it's corresponding eigenvector. Then $Tv = \lambda v$. Thus we have $(T-2I)(T-3I)(T-4I)v = (\lambda-2)(\lambda-3)(\lambda-4) = 0$. But that means $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$ as required.

5.B.4

Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. If either $null(P) = \{0\}$ or $range(P) = \{0\}$, we have $range(P) = V$, $null(P) = V$ respectively and so $V = null(P) \oplus range(P)$ is satisfied. In the case $null(P) \neq \{0\}$ and $range(P) \neq \{0\}$. Let $v \in null(P)$. Then $Pv = 0v$ meaning 0 is an eigenvalue with v as an eigenvector. Now let $w \in range(P)$. Then $Pu = w$ for some $u \in V$. But we have $P^2(u) = P(u) = w$ and $P^2(u) = P(P(u)) = P(w)$ thus meaning $P(w) = w$. Therefore we have an eigenvalue of 1 with corresponding eigenvector w. But that means v and w are linearly independent since they correspond to unique eigenvalues. Now let $v \in V$. We have $v = v - Pv + Pv$. But $P(v-Pv) = Pv-P^2v = Pv-Pv = 0$ meaning $(v-Pv) \in null(P)$ and clearly $Pv \in range(P)$. Thus $v \in V$ is a linear combination of some $v \in null(P)$ and $w \in range(P)$. Meaning $V = range(P) + null(P)$. But we established earlier that $v \in range(P)$ and $w \in null(P)$ are linearly independent and thus we can conclude $V = range(P) \oplus null(P)$ as required.

5.B.7

Suppose $T \in \mathcal{L}(V)$.

Suppose 9 is an eigenvalue of T^2. Let $v \in V$ be an eigenvector corresponding to eigenvalue 9. Then $T^2v = 9v$. But that in turn means $T^2v - 9v = 0$ so $(T^2 - 9I)v = 0$. But we have $(T^2 - 9I) = (T - \sqrt{9}I)(T + \sqrt{9}I) = (T-3I)(T+3I)v = 0$. Since v is an eigenvector, it can't be 0 and so $(T-3I)$ or $(T+3I)$ is not injective. That in turn means 3 or -3 are eigenvalues of T as required.

Suppose 3 or -3 are eigenvalues of T. In the case 3 is an eigenvalue of T, let $v \in V$ be an eigenvector corresponding to eigenvalue 3 then $Tv = 3v$ and so $T^2v = 3Tv = 9v$. But that means $(T^2 - 9I)v = 0$. Since $v \neq 0$, we must have $(T^2 - 9I)v$ is not injective and thus 9 is a eigenvalue of T^2. In the case -3 is an eigenvalue of T, let $v \in V$ be an eigenvector corresponding to eigenvalue -3 then $Tv = -3v$ and so $T^2v = -3Tv = 9v$. But that means $(T^2 - 9I)v = 0$. Since $v \neq 0$, we must have $(T^2 - 9I)v$ is not injective and thus 9 is a eigenvalue of T^2. In the case both are eigenvalues of T, we have 9 as an eigenvalue of T^2 following from the above cases. Thus we have for all cases 9 being an eigenvalue of T^2 as required.

Thus we have equivalence as required.

5.C.1

Suppose $T \in \mathcal{L}(V)$ is diagonalizable. Since T is diagonalizable we have a diagonal matrix M with respect to some basis $v_1...$ of V. So we have $Tv_j = \lambda_j v_j$ where $\lambda_j = M_{jj}$. Since $v_1...$ is a basis of V we have for $v \in V$, $v = a_1v_1 + ...$ Let $w = \sum_{j \in J} a_j v_j$ where $J = \{j \in \mathbf{N} : \lambda_j = 0\}$. Also let $u = \sum_{l \in L} a_l v_l$ where $L = \{j \in \mathbf{N} : \lambda_j \neq 0\}$. Clearly $v = w + u$ and we also have

$$T(w) = \sum_{j \in J} a_j T v_j = 0$$

whereas

$$u = \sum_{l \in L} a_l v_l = \sum_{l \in L} \frac{\lambda_l a_l}{\lambda_l} v_l = T \sum_{l \in L} \frac{a_l}{\lambda_l} v_l$$

Thus we can conclude $w \in null(T)$ and $u \in range(T)$. Meaning $V = null(T) + range(T)$. But because w and u consist of linearly independent vectors, we have $0 = 0v_1...$ being the only unique representation of 0. That in turn means $0 = w + u$ must have $w = 0$ and $u = 0$. Thus we can conclude $V = null(T) \oplus range(T)$

5.C.9

Suppose $T \in \mathcal{L}(V)$ is invertible. Let $\lambda \in \mathbf{F}$. Then $E(\lambda, T) = null(T - \lambda I)$ and $E(1/\lambda, T^{-1}) = null(T^{-1} - I/\lambda)$. Let $w \in null(T - \lambda I)$. Then $(T - \lambda I)w = 0$. That in turn results in $Tw = \lambda w$. Applying T^{-1} to both sides we obtain $w = \lambda T^{-1} w$. Dividing both sides by λ we obtain $\frac{1}{\lambda} w = T^{-1} w$ which means that $(T^{-1} - I/\lambda)w = 0$. Thus $w \in null(T^{-1} - I/\lambda)$ which means $E(\lambda, T) \subseteq E(1/\lambda, T^{-1})$. Now let $u \in null(T^{-1} - I/\lambda)$. Then $(T^{-1} - I/\lambda)u = 0$ which means $T^{-1}u = u/\lambda$. Applying T and multiplying both sides by λ we obtain $\lambda u = Tu$ which means $(T - \lambda I)u = 0$. Thus $u \in null(T - \lambda I)$ which means $E(1/\lambda, T^{-1}) \subseteq E(\lambda, T)$. Therefore we can conclude $E(1/\lambda, T^{-1}) = E(\lambda, T)$ for all λ as required.