MAT 341 HW9, Carl Liu

1.

A) We have $u(x,t) = \psi(x+ct) + \phi(x-ct)$. We then have

$$\frac{\partial u}{\partial t}(x,t) = c\frac{d\psi}{d(x+ct)}(x+ct) - c\frac{d\phi}{d(x-ct)}(x-ct)$$

Using the boundary conditions, we have

$$\psi(x) + \phi(x) = f(x)$$

and

$$c\psi'(x) - c\phi'(x) = 0$$

dividing through by c and integrating both sides then results in

$$\psi(x) - \phi(x) = A$$

where A is some constant. This then results in

$$\psi(x) = \frac{f(x)+A}{2} \quad \phi(x) = \frac{f(x)-A}{2}$$

Now consider the extension of $f(x)$, $\overline{f}(x)$ and now let

$$\psi(x) = \frac{\overline{f}(x)+A}{2} \quad \phi(x) = \frac{\overline{f}(x)-A}{2}$$

These must satisfy

$$u(0,t) = \psi(ct) + \phi(-ct) = 0$$

and so we have

$$\overline{f}(ct) + A + \overline{f}(-ct) - A = 0$$

meaning $\overline{f}(ct) = -\overline{f}(-ct)$ and so must be odd meaning $\overline{f} = \overline{f}_o$, the odd extension of f. Therefore we have

$$u(x,t) = \psi(x+ct) + \phi(x-ct) = \frac{1}{2}(\overline{f}(x+ct) + A + \overline{f}(x-ct) - A) =$$

$$\frac{1}{2}(\overline{f}_o(x+ct) + \overline{f}_o(x-ct))$$

B) Separating variables we have $u(x,t) = \phi(x)T(t)$. This results in by equation (1), $\phi''(x)T(t) = \frac{1}{c^2}T''(t)\phi(x)$. That in turn means

$$\frac{\phi''(x)}{\phi(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)}$$

We must have

$$\frac{\phi''(x)}{\phi(x)} = -\lambda^2 = \frac{1}{c^2}\frac{T''(t)}{T(t)}$$

and so

$$\phi''(x) + \lambda^2\phi(x) = 0$$
$$T''(x) + \lambda^2 c^2 T(x) = 0$$

The solutions of these equations are

$$T(t) = A\sin(\lambda ct) + B\cos(\lambda ct)$$

$$\phi(x) = C\sin(\lambda x) + D\cos(\lambda x)$$

But because $u(0,t) = 0$, we must have $\phi(0)T(t) = 0$ to avoid the trivial solution this means $\phi(0) = 0$ which means $D = 0$ leaving us with $\phi(x) = C\sin(\lambda x)$. Also because

$$\frac{\partial u}{\partial t}(x,t) = \phi(x)T'(t) = C\lambda c\sin(\lambda x)(A\cos(\lambda ct) - B\sin(\lambda ct))$$

and we have the condition

$$\frac{\partial u}{\partial t}(x,0) = 0$$

we must have $A = 0$ for a nontrivial solution. Thus resulting in $T(t) = B\cos(\lambda ct)$ and since λ can take on any value, we have a Fourier integral

$$u(x,t) = \int_0^\infty (F(\lambda)\cos(\lambda ct)\sin(\lambda x))d\lambda$$

with initial conditions

$$u(x,0) = f(x) = \int_0^\infty F(\lambda)\sin(\lambda x)d\lambda$$

which being a fourier integral means

$$F(\lambda) = \frac{2}{\pi}\int_0^\infty f(x)\sin(\lambda x)dx$$

2.

A) We have $u(x,t) = \psi(x+ct) + \phi(x-ct)$. We then have

$$\frac{\partial u}{\partial t}(x,t) = c\frac{d\psi}{d(x+ct)}(x+ct) - c\frac{d\phi}{d(x-ct)}(x+ct)$$

Using the boundary equations we then have

$$u(x,0) = \psi(x) + \phi(x) = f(x)$$

$$\frac{\partial u}{\partial t}(x,0) = c\psi'(x) - c\phi'(x) = g(x)$$

which when integrating and dividing both sides of the second equation results in

$$\psi(x) - \phi(x) = G(x) + A$$

where

$$G(x) = \frac{1}{c}\int_0^x g(y)dy$$

This means

$$\psi(x) = \frac{1}{2}(f(x) + G(x) + A)$$

$$\phi(x) = \frac{1}{2}(f(x) - G(x) - A)$$

Since $g(x)$ and $f(x)$ has been defined for all $-\infty < x < \infty$, we have

$$u(x,t) = \psi(x+ct) - \phi(x-ct) =$$

$$\frac{1}{2}(f(x+ct) + G(x+ct) + A + f(x-ct) - G(x-ct) - A) =$$

$$\frac{1}{2}(f(x+ct) + G(x+ct) + f(x-ct) - G(x-ct)) =$$

$$\frac{1}{2}\left(f(x+ct) + \frac{1}{c}\int_0^{x+ct} g(y)dy + f(x-ct) - \frac{1}{c}\int_0^{x-ct} g(y)dy\right) =$$

$$\frac{1}{2}\left(f(x+ct) + f(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} g(y)dy\right)$$

B) Separating variables we have $u(x,t) = \phi(x)T(t)$. This results in by equation (1), $\phi''(x)T(t) = \frac{1}{c^2}T''(t)\phi(x)$. That in turn means

$$\frac{\phi''(x)}{\phi(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)}$$

We must have

$$\frac{\phi''(x)}{\phi(x)} = -\lambda^2 = \frac{1}{c^2}\frac{T''(t)}{T(t)}$$

and so

$$\phi''(x) + \lambda^2\phi(x) = 0$$

$$T''(x) + \lambda^2 c^2 T(x) = 0$$

The solutions of these equations are

$$T(t) = A\sin(\lambda ct) + B\cos(\lambda ct)$$

$$\phi(x) = C\sin(\lambda x) + D\cos(\lambda x)$$

Since

$$\frac{\partial u}{\partial t}(x,t) = \phi(x)T'(t) = \lambda c(C\sin(\lambda x) + D\cos(\lambda x))(A\cos(\lambda ct) - B\sin(\lambda ct))$$

we have

$$\frac{\partial u}{\partial t}(x,0) = \phi(x)T'(0) = A\lambda c(C\sin(\lambda x) + D\cos(\lambda x)) = g(x)$$

$$u(x,0) = \phi(x)T(0) = B(C\sin(\lambda x) + D\cos(\lambda x)) = f(x)$$

Thus resulting in

$$\phi(x) = \frac{f(x)}{B} \quad \phi(x) = \frac{g(x)}{A\lambda c}$$

and so

$$B = A\lambda c f(x)/g(x)$$

Then we have

$$u(x,t) = \int_0^\infty \left(A\sin(\lambda ct) + A\lambda c\frac{f(x)}{g(x)}\cos(\lambda ct)\right)\frac{g(x)}{A\lambda c}d\lambda =$$

$$\int_0^\infty \left(\frac{g(x)}{\lambda c}\sin(\lambda ct) + f(x)\cos(\lambda ct)\right)d\lambda$$

3.

Suppose for contradiction $X(x) \neq 0$ and $p \geq 0$. Then we have $X" - pX = 0$ and so $X(x) = Ae^{\sqrt{p}x} + Be^{-\sqrt{p}x}$ which at $0 = x$ results in $X(0) = A + B$ and at $x = a$ we must have $X(a) = Ae^{\sqrt{p}a} + Be^{-\sqrt{p}a} = 0$, we must have $A = B = 0$ which results in $X(x) = 0$, a contradiction. Thus either $X(x) = 0$ or $p < 0$.