MAT 341 HW11, Carl Liu

1.

We will split the problem into $u(x,y) = u_1(x,y) + u_2(x,y)$ where

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0 \quad u_1(x,b) = 100 \quad \frac{\partial u_1}{\partial y}(x,0) = 0 \quad u_1(0,y) = 0 \quad u_1(a,y) = 0$$

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0 \quad u_2(x,b) = 0 \quad \frac{\partial u_2}{\partial y}(x,0) = 0 \quad u_2(0,y) = 0 \quad u_2(a,y) = 100$$

For $u_1(x,y) = X(x)Y(y)$ we have $X(x) = \sin(\lambda_n x)$ where $\lambda_n = \frac{n\pi}{a}$. Also $Y(y) = A\cosh(\lambda_n y) + B\sinh(\lambda_n y)$. Since

$$Y'(y) = A\lambda_n \sinh(\lambda_n y) + B\lambda_n \cosh(\lambda_n y)$$

and $\frac{\partial u_1}{\partial y}(x,0) = 0$, we must have $Y'(0) = 0$ meaning $B = 0$ leaving us with $Y(y) = a_n \cosh(\lambda_n y)$. This leaves us with a solution of the form

$$u_1(x,y) = \sum_{n=1}^{\infty} a_n \cosh(\lambda_n y)\sin(\lambda_n x)$$

Then

$$u_1(x.b) = \sum_{n=1}^{\infty} a_n \cosh(\lambda_n b)\sin(\lambda_n x) = 100$$

$$a_n \cosh(\lambda_n b) = \sin(\lambda_n)\frac{2}{a}\int_0^a 100\sin(\lambda_n x)dx = \frac{200(1-\cos(\lambda_n a))}{a\lambda_n}$$

$$a_n = \frac{200(1-(-1)^n)}{n\pi\cosh(\lambda_n b)}$$

resulting in

$$u_1(x,y) = \sum_{n=1}^{\infty} \frac{200(1-(-1)^n)\cosh(\lambda_n y)}{n\pi\cosh(\lambda_n b)}\sin(\lambda_n x)$$

For $u_2(x,y) = X(x)Y(y)$ we have

$$Y(y) = A\cos(\gamma y) + B\sin(\gamma y)$$

But since $Y'(y) = 0$, we must have $Y(y) = A\cos(\gamma y)$ Since $Y(b) = 0$, we must have $Y(y) = \cos(\gamma_n y)$ where $\gamma_n = \frac{(2n-1)\pi}{2a}$. Since $X(x) = a_n \cosh(\gamma_n x) + b_n \sinh(\gamma_n x)$, we have a solution of the form

$$u_2(x,y) = \sum_{n=1}^{\infty} (a_n \cosh(\gamma_n x) + b_n \sinh(\gamma_n x)) \cos(\gamma_n y)$$

We then have

$$u_2(0,y) = \sum_{n=1}^{\infty} a_n \cos(\gamma_n y) = 0$$

$$a_n = 0$$

and also

$$u_2(b,y) = \sum_{n=1}^{\infty} b_n \sinh(\gamma_n b) \cos(\gamma_n y) = 100$$

$$b_n = \frac{2}{\sinh(\gamma_n b)a} \int_0^a 100 \cos(\gamma_n y) dx = \frac{200 \sin(\frac{(2n-1)\pi}{2})}{\sinh(\gamma_n b)\gamma_n a} = \frac{(-1)^{n+1} 200}{\sinh(\gamma_n b)(2n-1)\pi}$$

Thus we have

$$u_2(x,y) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 200 \sinh(\gamma_n x)}{\sinh(\gamma_n b)(2n-1)\pi} \cos(\gamma_n y)$$

This results in

$$u(x,y) = 200 \sum_{n=1}^{\infty} \frac{(1-(-1)^n)\cosh(\lambda_n y)}{n\pi \cosh(\lambda_n b)} \sin(\lambda_n x) + \frac{(-1)^{n+1} \sinh(\gamma_n x)}{\sinh(\gamma_n b)(2n-1)\pi} \cos(\gamma_n y)$$

2.

We have a solution of the form

$$v(r,\theta) = a_0 + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} b_n r^n \sin(n\theta)$$

Then

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta d\theta = 0$$

$$a_n = \frac{1}{c^n\pi}\int_{-\pi}^{\pi}\theta\cos(n\theta)d\theta = 0$$

$$b_n = \frac{1}{c^n\pi}\int_{-\pi}^{\pi}\theta\sin(n\theta)d\theta = -\frac{2}{c^n}\cos(n\pi) = (-1)^{n+1}\frac{2}{c^n n}$$

This results in

$$v(r,\theta) = \sum_{n=1}^{\infty}(-1)^{n+1}r^n\frac{2}{c^n n}\sin(n\theta)$$

The boundary condition is satisfied at $\pm\pi$ because $v(r,\pm\pi) = 0$ and so $v(r,\pi+2\pi) = v(r,3\pi) = 0$ as well as $v(r,-\pi+2\pi) = v(r,\pi) = 0$. Also $v(0,\theta) = 0$ for all θ and is thus bounded as required.

3.

We have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) = \sum_{n=1}^{\infty}n^2r^{-n-2}(a_n\cos(n\theta)+b_n\sin(n\theta))$$

$$\frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2} = -\sum_{n=1}^{\infty}n^2r^{-n-2}(a_n\cos(n\theta)+b_n\sin(n\theta))$$

and so

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2} = 0$$

as required. We also have

$$|a_0| + \sum_{n=1}^{\infty}|r^{-n}(a_n\cos(n\theta)+b_n\sin(n\theta))| \geq |v(r,\theta)| \geq 0$$

by triangle inequality. Since $r > c$ and $c \geq 0$, we have $r^{-n} > 0$ for all n and so

$$\lim_{r\to\infty}\left(|a_0| + \sum_{n=1}^{\infty}r^{-n}|(a_n\cos(n\theta)+b_n\sin(n\theta))|\right) =$$

$$|a_0| + \sum_{n=1}^{\infty}\lim_{r\to\infty}r^{-n}|(a_n\cos(n\theta)+b_n\sin(n\theta))| =$$

$$|a_0| + \sum_{n=1}^{\infty}0 * |(a_n\cos(n\theta)+b_n\sin(n\theta))| = |a_0|$$

$$\geq \lim_{r\to\infty} |v(r,\theta)| \geq 0$$

Since a_0 is a constant, we can thus conclude that $|v(r,\theta)|$ is indeed bounded as $r \to \infty$.

4.

Define $\overline{u}(x,y) = u(x+x_0, y+y_0)$. We have

$$\frac{\partial^2 \overline{u}}{\partial x^2} + \frac{\partial^2 \overline{u}}{\partial y^2} = \frac{\partial^2 u}{\partial x^2}(x+x_0, y+y_0) + \frac{\partial^2 u}{\partial y^2}(x+x_0, y+y_0) = 0$$

since

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

on the whole (x,y) plane. It is thus a conformal map. So we have the average value $\overline{u}(x,y)$ on the circle $x^2+y^2=c^2$ is $\overline{u}(0,0)$. But $\overline{u}(0,0) = u(x_0,y_0)$ means that $u(x_0,y_0)$ is also the average value. We have on the circle $(x-x_0)^2+(y-y_0)^2=c^2$, $u(x,y) = u(\sqrt{c^2-y^2}+x_0, \sqrt{c^2-x^2}+y_0)$. This is equivalent to on the circle $x^2+y^2=c^2$, $\overline{u}(x,y) = u(x+x_0,y+y_0) = u(\sqrt{c^2-y^2}+x_0, \sqrt{c^2-x^2}+y_0)$. Thus the average value of $\overline{u}(x,y)$ on the circle $x^2+y^2=c^2$ is equivalent to the average value of $u(x,y)$ on the circle $(x-x_0)^2+(y-y_0)^2=c^2$. We can then conclude that the average value of $u(x,y)$ on the circle $(x-x_0)^2+(y-y_0)^2=c^2$ is $u(x_0,y_0)$ as required. Since $c>0$ was arbitrary, this is thus true for any c, same can be said for (x_0,y_0).