PHY 300 HW 4, Carl Liu

6-1

A) The angular frequencies of the normal modes are

$$\omega = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}}$$

where for the fundamental frequency means we would need $n = 1$. We have $\mu = m/L = 0.004kg/m$. This results in

$$f_1 = \frac{\pi}{2\pi L}\sqrt{\frac{T}{\mu}} = \frac{1}{2 * 2.5m}\sqrt{\frac{10N}{0.004kg/m}} = 10Hz$$

B) If the string is touched at $0.5m$ from one end, the only allowed frequencies would be those that have a node at $0.5m$. This means that we must have $\lambda_l = \frac{2*0.5m}{n} = \frac{1m}{l}$ where l is an integer. But we also have $\lambda_n = \frac{2*2.5m}{n}$. Since we want $\lambda_l = \lambda_n$, we must have $\frac{1m}{l} = \frac{5m}{n}$. This means $n = 5l$ meaning n has to be a multiple of 5 in order for a node to be at $0.5m$. Thus we plug this in for

$$f_l = \frac{5l}{2L}\sqrt{\frac{T}{\mu}} = \frac{5l}{2L}\sqrt{\frac{T}{\mu}} = 5f_1 = l * 50Hz$$

where l is an integer.

6-2

The frequencies of its three lowest normal mode is defined by

$$f_{sn} = \frac{n}{2L}\sqrt{\frac{TL}{M}}$$

where $n = 1, n = 2, n = 3$. In the case of 3 separate masses we have

$$\omega_0 = \sqrt{\frac{T}{ml}}$$

where m is the mass of the individual masses and l the length of the string between each mass. In this case we have $m = M/3$ and $l = L/4$. The normal modes are

$$f_n = \frac{\omega_0}{\pi}\sin\left(\frac{n\pi}{2(N+1)}\right)$$

So we have

$$f_1 = \sqrt{\frac{T12}{ML}}\frac{1}{\pi}\sin\left(\frac{\pi}{8}\right) = \frac{4\sqrt{3}}{2L\pi}\sqrt{\frac{TL}{M}}\sin\left(\frac{\pi}{8}\right) = \frac{4\sqrt{3}\sin(\frac{\pi}{8})}{\pi}f_{s1} = 0.844 f_{s1}$$

$$f_2 = \sqrt{\frac{T12}{ML}}\frac{1}{\pi}\sin\left(\frac{2\pi}{8}\right) = \sqrt{\frac{T12}{ML}}\frac{1}{\pi\sqrt{2}} = \frac{4\sqrt{3}}{2L\sqrt{2}\pi}\sqrt{\frac{TL}{M}} = \frac{4\sqrt{3}}{\pi\sqrt{2}}f_{s1} = 1.56 f_{s1}$$

$$f_3 = \sqrt{\frac{T12}{ML}}\frac{1}{\pi}\sin\left(\frac{3\pi}{8}\right) = \frac{4\sqrt{3}}{2L\pi}\sqrt{\frac{TL}{M}}\sin\left(\frac{3\pi}{8}\right) = \frac{4\sqrt{3}\sin(\frac{3\pi}{8})}{\pi}f_{s1} = 2.04 f_{s1}$$

6-6

A) We have $\frac{\partial^2\xi}{\partial x^2} = \frac{\rho}{Y}\frac{\partial^2\xi}{\partial t^2} = \frac{\partial^2\xi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\xi}{\partial t^2}$ and $\xi(x,t) = f(x)\cos(\omega t)$. We then have

$$\frac{\partial^2\xi}{\partial x^2} = f''(x)\cos(\omega t) = -\frac{\omega^2}{v^2}f(x)\cos(\omega t)$$

$$f''(x) - \frac{\omega^2}{v^2}f(x) = 0$$

Thus $f(x) = A\cos(\frac{\omega}{v}x)$. So

$$\xi(x,t) = A\cos(\frac{\omega}{v}x)\cos(\omega t)$$

We also have the boundary conditions

$$\alpha Y\frac{\partial\xi}{\partial x}(0,t) = \alpha Y\frac{\partial\xi}{\partial x}(L,t) = 0 \quad \xi(L/2,t) = 0$$

due to both ends being free and the middle being clamped. So we must have

$$-A\frac{\omega}{v}\sin\left(\frac{\omega}{v}*0\right)\cos(\omega t) = 0$$

$$-A\frac{\omega}{v}\sin\left(\frac{\omega}{v}*L\right)\cos(\omega t) = 0$$

$$A\cos\left(\frac{\omega L}{2v}\right)\cos(\omega t) = 0$$

For this to be true for all t, we must have

$$\sin\left(\frac{\omega}{v}*L\right) = 0 \quad \cos\left(\frac{\omega L}{2v}\right) = 0$$

meaning

$$\frac{\omega L}{v} = n\pi \quad \frac{\omega L}{2v} = \frac{(2m-1)\pi}{2}$$

where m, n are integers. Thus

$$\omega = \frac{n\pi v}{L} \quad and \quad \omega = \frac{(2m-1)v\pi}{L}$$

For the two to be equivalent we have

$$\frac{(2m-1)\pi v}{L} = \frac{n\pi v}{L}$$

$$2m - 1 = n$$

So

$$A) \quad \omega_n = \frac{(2n-1)\pi v}{L}$$

are the natural frequencies for such a rod. The wavelength of λ_n depends on $\cos(\frac{\omega_n}{v}x)$. Since the angular frequency is $\frac{\omega_n}{v}$, we have

$$\frac{\omega_n}{v} = \frac{2\pi}{\lambda_n}$$

$$B) \quad \lambda_n = \frac{2\pi v}{\omega_n} = \frac{2L}{2n-1}$$

The nodes are then at areas where $\cos(\frac{\omega_n}{v}x) = 0$ where $0 \leq x \leq L$. So

$$\frac{\omega_n}{v}x = \frac{(2m-1)\pi}{2}$$

$$x = \frac{(2m-1)L}{2(2n-1)}$$

But we must have

$$0 \leq \frac{(2m-1)L}{2(2n-1)} \leq L$$

$$\leq 2m - 1 \leq 2(2n-1)$$

$$\frac{1}{2} \leq m \leq \frac{4n-1}{2} = 2n - \frac{1}{2}$$

because m is an integer we can conclude

$$1 \leq m \leq 2n - 1$$

and so the maximum amount of nodes for the nth normal mode is $2n - 1$. We thus have the nodes for the nth normal mode at

$$C) \quad x = \frac{(2m-1)L}{2(2n-1)}$$

where $1 \leq m \leq 2n - 1$ is an integer.

6-11

Since

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}$$

and we assume that all parts of the string vibrate at the same frequency so $y(x,t) = f(x)\cos(\omega t)$ we then have $f''(x) = -\frac{\mu\omega^2}{T}f'(x)$. That means we have $f(x) = \sin(\omega\sqrt{\frac{\mu}{T}}x)$. So $y(x,t) = A\sin(\omega\sqrt{\frac{\mu}{T}}x)\cos(\omega t)$. But both ends of the string are stationary so we have the condition

$$\sin\left(\omega\sqrt{\frac{\mu}{T}}L\right) = 0$$

$$\omega\sqrt{\frac{\mu}{T}}L = n\pi$$

$$\omega_n = n\frac{\pi}{L}\sqrt{\frac{T}{\mu}}$$

We know that

$$v(x,t) = \frac{\partial y}{\partial t} = -\omega\sin(\omega\sqrt{\frac{\mu}{T}}x)\sin(\omega t)$$

The kinetic energy of a small section of the string would be

$$\frac{1}{2}v^2\mu * dx$$

Integrating across the entire length of the string will give us the kinetic energy of the string at a certain time t. So we have

$$KE = \frac{A^2\mu\omega^2}{2}\sin^2(\omega t)\int_0^L \sin^2\left(\omega\sqrt{\frac{\mu}{T}}x\right)dx =$$

$$\frac{A^2\mu\omega^2}{4}\sin^2(\omega t)\int_0^L 1-\cos\left(2\omega\sqrt{\frac{\mu}{T}}x\right)dx =$$

$$\frac{A^2\mu\omega^2}{4}\sin^2(\omega t)\left(L-\sqrt{\frac{T}{\mu}}\frac{1}{2\omega}\sin\left(2\omega\sqrt{\frac{\mu}{T}}L\right)\right) = \frac{A^2\mu\omega^2 L}{4}\sin^2(\omega t)$$

But we must have $E = KE + PE$ where PE is potential energy. In the case the string is not extended, we would have $PE = 0$ meaning $KE = E$. This happens when

$$\omega_n t = \frac{(2n-1)\pi}{2}$$

So

$$t = \frac{(2n-1)L}{n}\sqrt{\frac{\mu}{T}}$$

Thus the energy of the nth normal mode is

$$A)\quad \frac{A_n^2\mu\omega_n^2 L}{4}\sin^2\left(\omega_n\frac{L}{n}\sqrt{\frac{\mu}{T}}\right) = \frac{A_n^2\mu\omega_n^2 L}{4} = \frac{A_n^2 n^2\pi^2 T}{4L}$$

For the superposition given we know that normal modes are independent so the superposition of two normal modes must have an energy that is equivalent to the sum of the energy of the two normal modes separate. Thus we have

$$E = \frac{A_1^2\pi^2 T}{4L} + \frac{A_3^2 9\pi^2 T}{4L} = \pi^2 T\frac{(A_1^2 + 9A_3^2)}{4L}$$