PHY 300 HW 2, Carl Liu

3-1

Since $F = ma = -kx$. That means $-ma/x = k$. Thus

$$k = \frac{-(1*10^{-3}kg(-1.7514*10^{-2})\frac{m}{s^2})}{43.785*10^{-2}m} = 4*10^{-5}\frac{kg}{s^2} = 4*10^{-5}\frac{N}{m}$$

3-2

A) We have the period of oscillation is $T = 2\pi/\omega$. But $\omega = \sqrt{k/m}$. So the period of period of oscillation is $T = 2\pi/\sqrt{k/m} = 2\pi\sqrt{m/k}$

B)

1) The force applied by the springs to the mass would be $-kx + (-kx)$. This is essentially doubling the spring constant and so $\omega = \sqrt{2k/m}$ making $T = 2\pi\sqrt{m/2k}$

2) The spring constant stays the same for the whole spring because it is a longer version of the same spring. But because x describes only the displacement of one of the springs, we have $F = -kx/2$. So $\omega = \sqrt{k/2m}$. Meaning $T = 2\pi\sqrt{2m/k}$.

3-3

A) It will leave the platform when $x'' < -g = -9.8m/s$. This is because the force on the block is that of the platform and gravity where the platform will enact a force of $g + x''$ as long as $x'' > -g$. So we have $x'' = -g = -5*10^{-2}\text{m}\omega^2\cos(\omega t+\pi)$, where $\omega = 2\pi(10/\pi) = 20rad/s$. So $\cos(\omega t+\pi) = 9.8/(5*10^{-2}*20^2) = 9.8/(5*10^{-2}*20^2) = 0.49$ Then $x = 5*10^{-2}\text{m}\cos(\omega t+\pi) = 5*10^{-2}\text{m}*0.49 = 0.0245\text{m}$. This is the point where the acceleration of the platform is g and when the block will stop touching the platform.

B) The height of the block will be dependent on the energy imparted onto it by the spring. Since $K = mv^2/2$, and $v = x' = -A\omega\sin(\omega t+\pi)$ we have $K = mA^2\omega^2\sin^2(\omega t+\pi)/2 = mA^2\omega^2(1-\cos(\omega t+\pi))/2$. But $K = mgh$ thus $h = A^2\omega^2(1-\cos(\omega t+\pi))/g = 5*10^{-2}*20^2\text{m}(1-0.49)/9.8 = 0.0388m$. Since the initial position is $0.0245m$, we have a final height of $0.0245m+0.0388m = 0.0633m$ above displacement. Since the platform has an amplitude of $0.05m$, we thus conclude that the block goes $0.0633m - 0.05m = 0.0133m$ above the max height of the platform.

3-13

We consider first a solution of the form $Ce^{irt} = x$. Then

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = -Cr^2e^{rt} + iC\gamma re^{rt} + C\omega_0^2 e^{irt} = 0$$

So for Ce^{rt} to be a solution we must have $r^2 - i\gamma r - \omega_0^2 = 0$. Meaning

$$r = \frac{i\gamma}{2} \pm \frac{\sqrt{-\gamma^2 + 4\omega_0^2}}{2}$$

We therefore have a solution of the form

$$C_1 e^{(\gamma t/2)+(i\sqrt{-\gamma^2+4\omega_0^2}t/2)} + C_2 e^{(\gamma t/2)-(i\sqrt{-\gamma^2+4\omega_0^2}t/2)} =$$

$$(C_1 + C_2)e^{\gamma t/2}(e^{i\sqrt{-\gamma^2+4\omega_0^2}t/2} + e^{-i\sqrt{-\gamma^2+4\omega_0^2}t/2}) = x$$

But we know that

$$(e^{i\sqrt{-\gamma^2+4\omega_0^2}t/2} + e^{-i\sqrt{-\gamma^2+4\omega_0^2}t/2}) = 2\cos(\frac{\sqrt{-\gamma^2 + 4\omega_0^2}}{2}t)$$

So we have

$$(C_1 + C_2)2e^{\gamma t/2}\cos(\frac{\sqrt{-\gamma^2 + 4\omega_0^2}}{2}t) = x$$

But we have $(C_1 + C_2)2$ is an arbitrary constant so lets have $A = (C_1 + C_2)2$. Lets define $\alpha = \gamma/2$ and $\omega = \frac{\sqrt{-\gamma^2+4\omega_0^2}}{2} = \sqrt{\omega_0^2 - \gamma^2/4}$. Thus

$$x = (C_1 + C_2)2e^{\gamma t/2}\cos(\frac{\sqrt{-\gamma^2 + 4\omega_0^2}}{2}t) = Ae^{\alpha t}cos(\omega t)$$

is a solution where α and ω has been defined as above.

3-18

A) The increase in potential energy can be seen as moving water up a distance of y. Since the top plane is tilted but flat, we have $y = \frac{y_0}{L/2}x = 2y_0x/L$. Now consider a slice of water in the pond of that height that was also moved up that height. It would have a potential energy increase of $(dm)gy$. But we have $dm = p(dv) = pby(dx)$. So

$$(dm)gy = pby^2g(dx) = \frac{4pgby_0^2}{L^2}x^2(dx)$$

Integrating this from 0 to $L/2$ results in

$$U = \int_0^{L/2} \frac{4pgby_0^2}{L^2}x^2(dx) = \frac{4pgby_0^2}{L^2}\int_0^{L/2} x^2(dx) = \frac{4pgby_0^2L^3}{24L^2} = \frac{pgby_0^2L}{6}$$

as required.

B) Since $vhb - (v + dv)hb = b(dx)(dy/dt)$, we have $-h(dv) = (dx)(dy/dt)$. But from earlier, we have $y = 2y_0x/L$ So $dy = 2dy_0x/L$. Thus $(dv) = -(dx)2dy_0(dx)/((dt)hL)$. Then integrating both sides we obtain

$$v(x) = \int \frac{-2dy_0x}{((dt)hL)}dx = \frac{-x^2}{hL}\frac{dy_0}{dt} + C$$

Since $v(L/2) = 0$, we have

$$0 = \frac{-L^2}{4hL}\frac{dy_0}{dt} + C$$

So

$$\frac{L}{4h}\frac{dy_0}{dt} = C$$

Thus we have

$$v(x) = \frac{L}{4h}\frac{dy_0}{dt} - \frac{x^2}{hL}\frac{dy_0}{dt}$$

But

$$v(0) = \frac{L}{4h}\frac{dy_0}{dt}$$

Thus

$$v(x) = v(0) - \frac{x^2}{hL}\frac{dy_0}{dt}$$

C)

The kinetic energy of a slice of water lying between x and $x + dx$ is $mv^2/2$. But $m = p(dv) = pbh(dx)$ and $v(x) = v(0) - \frac{x^2}{hL}\frac{dy_0}{dt}$. So

$$dK = pbh(dx)(v(0) - \frac{x^2}{hL}\frac{dy_0}{dt})^2/2$$

Integrating between $\pm L/2$ results in

$$\int_{-L/2}^{L/2} pbh(v(0) - \frac{x^2}{hL}\frac{dy_0}{dt})^2/2(dx) = \frac{pbh}{2}\int\limits_{-L/2}^{L/2}(v(0) - \frac{x^2}{hL}\frac{dy_0}{dt})^2(dx) =$$

$$\frac{pbh}{2}(v(0)^2L + \int\limits_{-L/2}^{L/2}(-x^2 2v(0)\frac{1}{hL}\frac{dy_0}{dt} + x^4(\frac{1}{hL}\frac{dy_0}{dt})^2)dx) =$$

$$\frac{pbh}{2}(v(0)^2L - \frac{2}{3}(\frac{L}{2})^3 2v(0)\frac{1}{hL}\frac{dy_0}{dt} + \frac{2}{5}(\frac{L}{2})^5(\frac{1}{hL}\frac{dy_0}{dt})^2) =$$

$$\frac{pbhL^3}{2}((\frac{1}{4h}\frac{dy_0}{dt})^2 - \frac{2}{3}(\frac{1}{2})^3 2\frac{1}{4h}\frac{dy_0}{dt}\frac{1}{h}\frac{dy_0}{dt} + \frac{2}{5}(\frac{1}{2})^5(\frac{1}{h}\frac{dy_0}{dt})^2) =$$

$$\frac{pbL^3}{2h}\left(\frac{dy_0}{dt}\right)^2(\frac{1}{16} - \frac{1}{24} + \frac{1}{80}) = \frac{pbL^3}{2h}\left(\frac{dy_0}{dt}\right)^2(\frac{15 - 10 + 3}{240}) =$$

$$\frac{pbL^3}{60h}\left(\frac{dy_0}{dt}\right)^2$$

as required

D) So $\frac{pbL^3}{60h}\left(\frac{dy_0}{dt}\right)^2 + \frac{pgby_0^2L}{6} = E$. Where $\frac{pbL^3}{60h}\left(\frac{dy_0}{dt}\right)^2 = A$ and $\frac{pgby_0^2L}{6} = B$. Since E is a constant, we can conclude $\omega = \sqrt{B/A}$. Since $T = 2\pi/\omega$, we get

$$T = 2\pi\sqrt{A/B} = 2\pi\sqrt{\frac{pbL^3 6}{60hpgbL}} = 2\pi\sqrt{\frac{L^2}{10hg}}$$

E)

$$T = 2\pi\sqrt{\frac{L^2}{10hg}} = 2\pi{*}70{*}10^3 m\sqrt{\frac{1}{10 * 150m * 9.8m/s}} = 3627.6s = 60.46mins$$

This is about an 18% error in calculations.

4-3

A) We have $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$ and $\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = 17.321 rad/s$. So the period of oscillation is $T = 2\pi/\omega = 0.363s$

B) We have the amplitude of the steady state being

$$A = \frac{F_0/m}{[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]^{1/2}}$$

But

$$\omega_0 = \sqrt{k/m} = 20s^{-1}$$

and

$$\gamma = b/m = 20s^{-1}$$

So

$$A = \frac{2N/0.2kg}{[((20s^{-1})^2 - (30s^{-1})^2)^2 + (20s^{-1}30s^{-1})^2]^{1/2}} = 0.0128m$$

4-5

A) The forces on a pendulum driven as so would depend on the angle between a vertical line going through the pivot of the pendulum and the bob. Specifically we have a force on the bob of $mg\sin(\theta) = -mg(x-\xi)/l$ for small angles. But $m(d^2x)/(dt^2) = -mg(x-\xi)/l - b(dx)/(dt)$. So $(d^2x)/(dt^2) + \gamma(dx)/(dt) = g(\xi - x)/l$ meaning $(d^2x)/(dt^2) + \gamma(dx)/(dt) + gx/l = g\xi/l$ as required. The steady state solution of such an equation is of the form $Acos(\omega t + \alpha)$. Where $A = \xi_0/[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]^{1/2}$ and $\tan(\alpha) = \gamma\omega/(\omega_0^2 - \omega^2)$

B) We have $A = \xi_0/[(\omega_0^2 - \omega_0^2)^2 + (\gamma\omega_0)^2]^{1/2} = \xi_0/(\gamma\omega_0)$ at resonance. We know that $\omega_0^2 = g/l = 9.8s^{-2}$ and we have $e^{-1} = e^{-\gamma 50T/2}$ So $2/50T = \gamma$. So $Q = \omega_0/\gamma = 50T\omega_0/2$. But $T = 2\pi/\omega_0$ so $Q = 50\pi$. Thus at resonance we have $A = \xi_0/(\gamma\omega_0) = \xi_0 Q/\omega_0^2 = 1 * 10^{-3}m * 50\pi/9.8 = 0.016m$

4-10

A) Since $W = \int F dx$, we have $dW = F dx$. Because $P = dW/dt$ we then have $P = F dx/dt$, but we have $F = bv$ and $dx/dt = v$. Thus $P = bv * v = bv^2$ as required,

B) Since $x = A\cos(\omega t - \delta)$, we have $v = -A\omega\sin(\omega t - \delta)$. So $P = bA^2\omega^2\sin^2(\omega t - \delta)$. The average of $\sin^2(\omega t - \delta) = 1/2$ and thus we have $\overline{P} = bA^2\omega^2/2$ as required.

C) We have

$$A = \frac{F_0}{k}\frac{\omega_0/\omega}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{1/2}}$$

Substituting it in results in

$$\overline{P} = b\frac{F_0^2}{2k^2}\frac{\omega_0^2}{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}} =$$

$$\frac{F_0^2}{2kQ}\frac{\omega_0}{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}}$$

as required.

4-13

A) ω_0 is when P is at max power so we have $\omega_0 = 40s^{-1}$. Since $Q = \omega_0/\gamma$ and we have $\gamma = 41 - 39 = 2s^{-1}$, $Q = 20$

B) Since $E = E_0e^{-\gamma t}$, we set $E = E_0/e^5$. Then $E_0/e^5 = E_0e^{-\gamma t}$ and thus $e^{-5} = e^{-\gamma t}$. So $-5 = -\gamma t$ and we must therefore have $t = 5/\gamma = 2.5s$. Since a period is $2\pi/\omega_0 = 0.157s$, we have $2.5s/0.157s = 15.9$ cycles.