**MAT 341 HW 1, Carl Liu**

**1.**

We have $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n cos(nx) + b_n sin(nx))$ By integrating everything with respect to x on the interval $(-\pi, \pi)$ we get

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} 3x^2 dx = \frac{1}{2\pi}[x^3]_{-\pi}^{\pi} = \frac{1}{2\pi}(\pi^3 + \pi^3) = \frac{(\pi^2 + \pi^2)}{2} = \pi^2$$

By multiplying through by $cos(mx)$ we get

$$f(x)cos(mx) = a_0 cos(mx) + \sum_{n=1}^{\infty}(a_n cos(nx)cos(mx) + b_n sin(nx)cos(mx))$$

Integrating all with respect to x and on the interval $(-\pi, \pi)$ will get us

$$\int_{-\pi}^{\pi} f(x)cos(mx)dx = a_m \pi$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} 3x^2 cos(mx)dx$$

Since $d(fg)/dx = (d(f)/dx)g + f(d(g)/dx)$ so $d(fg)/dx - (d(f)/dx)g = (d(g)/dx)f$. Then $\int(d(fg)/dx)dx - \int((d(f)/dx)g)dx = \int(f(d(g)/dx))dx$. Then $fg - \int((d(f)/dx)g)dx = \int((d(g)/dx)f)dx$. So we can continue as

$$a_m = \frac{1}{\pi}([\frac{1}{m}3x^2 sin(mx)]_{-\pi}^{\pi} - \frac{1}{m}\int 6x sin(mx)dx) =$$

$$\frac{1}{\pi}([\frac{1}{m}3x^2 sin(mx)]_{-\pi}^{\pi} - \frac{1}{m}(-[\frac{1}{m}6x cos(mx)]_{-\pi}^{\pi} + \frac{6}{m}\int cos(mx)dx) =$$

$$\frac{1}{\pi}([\frac{1}{m}3x^2 sin(mx)]_{-\pi}^{\pi} - \frac{1}{m}(-[\frac{6x}{m}cos(mx)]_{-\pi}^{\pi} + [\frac{6}{m^2}sin(mx)]_{-\pi}^{\pi}) =$$

$$\frac{1}{\pi}(-\frac{1}{m}(-[\frac{6x}{m}cos(mx)]_{-\pi}^{\pi}) = \frac{1}{\pi}(-\frac{1}{m}(-(\frac{6\pi}{m}cos(m\pi) + \frac{6\pi}{m}cos(-m\pi))) =$$

$$\frac{12}{m^2}cos(m\pi)$$

By multiplying through by $sin(mx)$ we get

$$f(x)sin(mx) = a_0 sin(mx) + \sum_{n=1}^{\infty}(a_n cos(nx)sin(mx) + b_n sin(nx)sin(mx))$$

Then integrating through we then get

$$\int_{-\pi}^{\pi} f(x)sin(mx)dx = b_m \pi$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(mx)dx = b_m$$

Since $f(x) = 3x^2$ is even and $sin(mx)$ is odd we know that $f(x)sin(mx)$ must be odd. Since this is true on the interval $(-\pi, \pi)$, we must therefore have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(mx)dx = 0 = b_m$$

So

$$f(x) = \pi^2 + \sum_{n=1}^{\infty}\frac{12}{n^2}\cos n\pi \cos nx = \pi^2 + \sum_{n=1}^{\infty}\frac{12}{n^2}(-1)^n \cos nx$$

**2.**

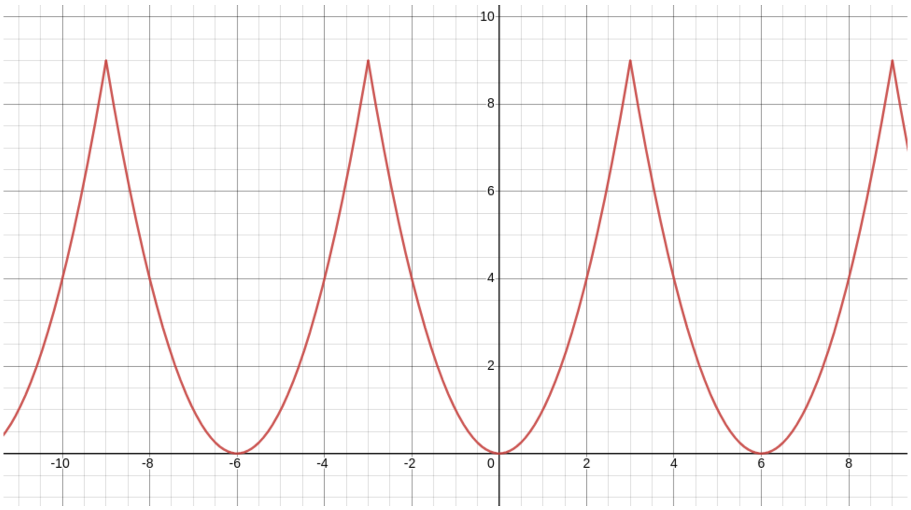
A) We have $cos(2x) = cos^2(x) + sin^2(x) = 2cos^2(x) - 1$. So $cos(2x)/2 + 1/2 = cos^2(x)$ and is thus the Fourier series of $cos^2(x)$ as required.

B) We have $sin(x - \pi/6) = sin(x)cos(\pi/6) - cos(x)sin(\pi/6) = sin(x)\sqrt{3}/2 - cos(x)/2$ which is the Fourier series of $sin(x - \pi/6)$ as required.

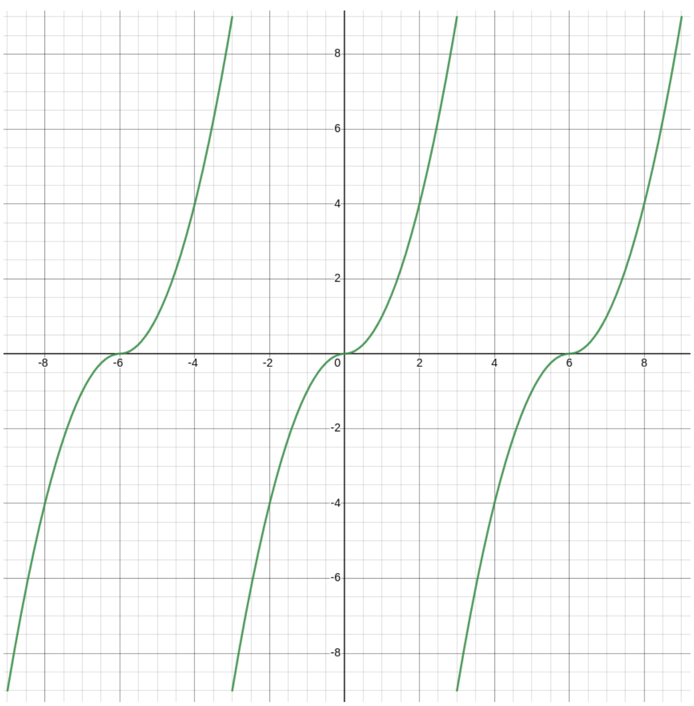
C) Since $cos(2x) = 2cos^2(x) - 1$, we have $sin(x)cos(2x) = 2cos^2(x)sin(x) - sin(x)$. Since $cos^2(x) = 1 - sin^2(x)$, we have $2cos^2(x)sin(x) - sin(x) = 2(1 - sin^2(x))sin(x) - sin(x) = 2sin(x) - 2sin^3(x) - sin(x) = sin(x) - 2sin^3(x)$. Then $2sin(x)cos(2x) = 2sin(x) - 4sin^3(x) = 2sin(x) - 4sin^3(x) - sin(x) + sin(x) = 3sin(x) - 4sin^3(x) - sin(x) = sin(3x) - sin(x)$. So $sin(x)cos(2x) = (sin(3x)/2) - (sin(x)/2)$ which is a Fourier series as required.

**3.**

A) For the even periodic extension we have




For the odd periodic extension we have




B) **Even Period Extension**

Since the even periodic extension of $x^2$ is defined as

$$f_e(x) = \begin{cases} f(x) = x^2, & 0 < x < 3 \\ f(-x) = (-x)^2 = x^2, & -3 < x < 0 \end{cases}$$

Since $f_e$ is even we have

$$f_e(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{a}\right)$$

So

$$\int_{-a}^{a} f_e(x)dx = \int_{-a}^{a} a_0 dx = 2aa_0$$

$$\frac{1}{2a}\int_{-a}^{a} f_e(x)dx = \frac{1}{2a}\int_{-a}^{a} x^2 dx = \frac{1}{6a}(a^3 + a^3) = \frac{a^2}{3} = a_0$$

and

$$\frac{1}{a}\int_{-a}^{a} f_e(x)cos(\frac{n\pi x}{a})dx = \frac{1}{a}\int_{-a}^{a} x^2 cos(\frac{n\pi x}{a})dx =$$

$$\frac{2}{a}\int_{0}^{a} x^2 cos(\frac{n\pi x}{a})dx = a_n$$

We can adapt the integral we had in question 1 to then obtain

$$a_n = \frac{4}{(\frac{n\pi}{a})^2} cos(\frac{n\pi}{a}a) = \frac{4a^2}{n^2\pi^2} cos(n\pi)$$

and so

$$f_e(x) = \frac{a^2}{3} + \sum_{n=1}^{\infty} \frac{4a^2}{n^2\pi^2} cos(n\pi) cos(\frac{n\pi x}{a}) =$$

$$3 + \sum_{n=1}^{\infty} \frac{4 * 3^2}{n^2\pi^2}(-1)^n cos(\frac{n\pi x}{3})$$

**Odd Period Extension**

For the odd period extension we have

$$f_o(x) = \begin{cases} f(x) = x^2, & 0 < x < 3 \\ -f(-x) = -(-x)^2 = -x^2, & -3 < x < 0 \end{cases}$$

We then have

$$0 = a_0$$

$$b_n = \frac{1}{a}\int\limits_{-a}^{a} f_o(x) sin(\frac{n\pi x}{a}) = \frac{2}{a}\int\limits_{o}^{a} f_0(x) sin(\frac{n\pi x}{a}) = \frac{2}{a}\int\limits_{0}^{a} x^2 sin(\frac{n\pi x}{a}) =$$

$$\frac{2}{a}(\frac{a}{n\pi}[-x^2 cos(\frac{n\pi x}{a})]_0^a + \frac{2a}{n\pi}\int\limits_{0}^{a} x cos(\frac{n\pi x}{a})) =$$

$$\frac{2}{a}(\frac{a}{n\pi}[-x^2 cos(\frac{n\pi x}{a})]_0^a + \frac{2a}{n\pi}(\frac{a}{n\pi}[x sin(\frac{n\pi x}{a})]_0^a + \frac{a^2}{n^2\pi^2}[cos(\frac{n\pi x}{a})]_0^a)) =$$

$$\frac{-2a^2 cos(n\pi)}{n\pi} + \frac{4a^2 sin(n\pi)}{n^2\pi^2} + \frac{4a^2 cos(n\pi)}{n^3\pi^3} - \frac{4a^2}{n^3\pi^3} =$$

$$\frac{-2a^2 cos(n\pi)}{n\pi} + \frac{4a^2 cos(n\pi)}{n^3\pi^3} - \frac{4a^2}{n^3\pi^3}$$

So

$$f_o(x) = \sum_{n=1}^{\infty}(\frac{-2a^2 cos(n\pi)}{n\pi} + \frac{4a^2 cos(n\pi)}{n^3\pi^3} - \frac{4a^2}{n^3\pi^3}) sin(\frac{n\pi x}{a})) =$$

$$\sum_{n=1}^{\infty} \frac{(-18n^2\pi^2 + 36)(-1)^n - 36}{n^3\pi^3} sin(\frac{n\pi x}{3})$$