MAT 310 HW 5, Carl Liu

3.C.2

We can express $p \in \mathcal{P}_3(\mathbf{R})$ as $p = a_1 + a_2x + a_3x^2 + a_4x^3$. Then $Dp = p' = a_2 + 2 * a_3x + 3 * a_4x^2$. So consider the bases $(x^3, x^2, x, 1)$ and $(3x^2, 2x, 1)$. We then have $\mathcal{M}(D)$ defined as $Dx^3 = 1(3x^2) + 0(2x) + 0(1)$, $Dx^2 = 0(3x^3) + 1(2x) + 0(1)$, $Dx = 0(3x^3) + 0(2x) + 1(1)$, and $D1 = 0(3x^3) + 0(2x) + 0(1)$ resulting in

$$\mathcal{M}(D) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

as required.

3.D.9

Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$

Suppose ST is invertible. Since ST is a map from V to V and is invertible, we have $range(ST) = V$ and $null(ST) = \{0\}$. Since T is a linear map we have $T(0) = 0$ and so $0 \in null(T)$. Now consider $v \in null(T)$, we have $T(v) = 0$, then $S(T(v)) = 0$, and since $null(ST) = 0$, we must have $v = 0$ meaning $null(T) = \{0\}$ making T injective. Therefore we have T is invertible due to it also being an operator. Now consider $v \in V$. Then $v \in range(ST)$. That means there is a $w \in V$ such that $ST(w) = v$. But that means $S(T(w)) = v$ and so $v \in range(S)$. But that means $V \subseteq range(S)$ and since $range(S) \subseteq V$ by definition of S, we have $range(S) = V$. Thus S is surjective. Since S is an operator it is therefore invertible as required.

Suppose S and T is invertible. Then we have $range(S) = V$, $null(S) = \{0\}$, $range(T) = V$, and $null(T) = \{0\}$. We have $ST(0) = 0$ by definition of linear maps. Let $v \in null(ST)$. Then $ST(v) = S(T(v)) = 0$. That means $T(v) = 0$ since $null(S) = \{0\}$. But we also have $null(T) = \{0\}$ and so $v = 0$. Thus we have $null(ST) = \{0\}$ and therefore injectivity of ST. But ST is a linear map from $V \to V$, and thus an operator which in turn means ST is invertible as required.

Thus we can conclude ST is invertible if and only if S and T are both invertible

3.D.18

Let r be a basis of $\mathbf{F}$ and $v_1, ..., v_m$ a basis for V. Then we have the matrix map $\mathcal{M}_F : \mathcal{L}(\mathbf{F}, V) \to \mathbf{F}^{m,1}$ as an isomorphism between $\mathcal{L}(\mathbf{F}, V)$ and $\mathbf{F}^{m,1}$. But $\mathbf{F}^{m,1}$ has dimension m and is a vector space over $\mathbf{F}$. Same can be said about $\mathcal{L}(\mathbf{F}, V)$. That means $dim(\mathcal{L}(\mathbf{F}, V)) = dim(\mathbf{F}^{m,1}) = m$. We also have V is a vector space of dimension m over $\mathbf{F}$. Thus $dim(V) = dim(\mathcal{L}(\mathbf{F}, V))$. But that means $\mathcal{L}(\mathbf{F}, V)$ and V are isomorphic as required.

3.E.2

Suppose $V_1, ..., V_m$ are vector spaces such that $V_1 \times ... \times V_m$ is finite dimensional. Let $1 \leq j \leq m$. Consider the map $M_j : V_j \to V_1 \times ... \times V_m$ that takes $v \in V_j$ to $(a_n)_{n=1}^m$ where $a_n = 0$, when $n \neq j$ and $a_n = v$, when $n = j$. We will prove that this map is linear. Let $v, w \in V_j$. Then $M_j(v) + M_j(w) = (0, ..., v, ...0) + (0, ..., w, ...0)$ where w, v are both in the jth position. Thus we have $(0, ..., v, ...0) + (0, ..., w, ...0) = (0, ..., v + w, ...0) = M_j(v + w)$ proving additivity. Let $\lambda \in \mathbf{F}$. Then $\lambda M_j(v) = \lambda(0, ..., v, ...0) = (0, ..., \lambda v, ...0) = M_j(\lambda v)$. Thus we can conclude multiplicativity and thus M_j is a linear map. We will now prove injectivity. We have $M_j(0) = 0$ and so $0 \in nullM_j$. Now let $x \in nullM_j$. We have $M_j(x) = 0$. So $(0, ..., x, ...0) = 0$ meaning $x = 0$ and so we can conclude $nullM_j = \{0\}$. Therefore we have injectivity. But by the fundamental theorem of linear maps we have $dim(V_j) = dim(range(M_j)) + dim(null(M_j))$. So $dim(V_j) = dim(range(M_j))$. But $range(M_j)$ is a subspace of $V_1 \times ... \times V_m$ and so we must have $dim(V_j) = dim(range(M_j)) \leq dim(V_1 \times ... \times V_m)$. Thus we can conclude that V_j is finite dimensional. Since $1 \leq j \leq m$ was arbitrary, we can conclude this for all V_j as required.

3.E.7

Suppose v, x are vectors in V and U, W are subspaces of V such that $v + U = x + W$. Since U, W are subspaces, we have $0 \in U$ and $0 \in W$. Meaning $v + 0 \in v + U$ and $x + 0 \in x + W$. So $v + 0 \in x + W$. Thus $v + 0 = x + w$ for some $w \in W$. This means $v - x = w$ and so $v - x \in W$. Thus $v + W = x + W$. We also have $x + 0 \in v + U$ and so $x + 0 = v + u$ for some $u \in U$. Thus $x - v = u$ resulting in $x - v \in U$ and so $x + U = v + U$. Thus we have $v + U = v + W$. So let $u \in U$. Then $v + u \in v + U$ and so $v + u \in v + W$. Thus $v + u = v + w$ for some $w \in W$. Therefore $u = w$ and so $u \in W$. That means $U \subseteq W$. Let $w \in W$. Then $v + w \in v + W$ and so $v + w \in v + U$.

Thus $v+w=v+u$ for some $u \in U$. Therefore $w=u$ and so $w \in U$ resulting in $W \subseteq U$. This means we can conclude $W=U$ as required.