MAT 341 HW 2, Carl Liu

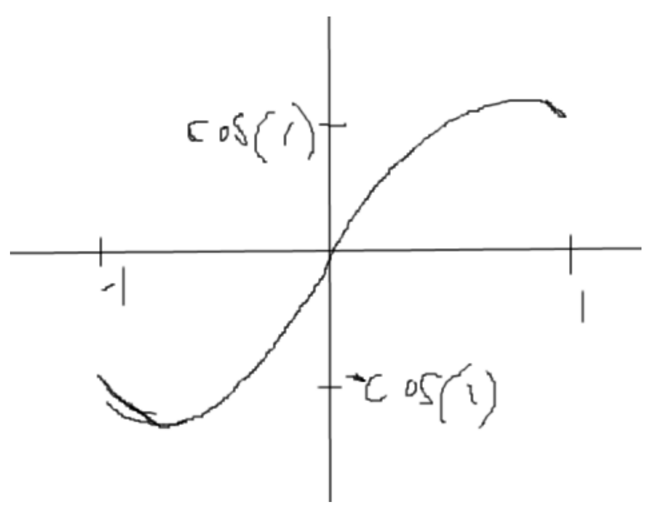
1.

C) Let $-1 < x < 1$. Then $-1cos(x) < xcos(x) < 1cos(x)$. But $-1 \leq cos(x) \leq 1$ thus $-1 \leq -1cos(x) < xcos(x) < 1cos(x) \leq 1$. So $-1 < xcos(x) < 1$ meaning $xcos(x) \in [-1, 1]$. Therefore the function is bounded. Since x is continuous and $cos(x)$ is continuous on the given interval, $xcos(x)$ is also continuous because arithmetic preserves continuity. The derivative of $xcos(x)$ is $-xsin(x) + cos(x)$. Since $sin(x)$ is continuous, $xsin(x)$ is also continuous. Thus $-xsin(x) + cos(x)$ is continuous since it is the difference of two continuous functions. We can therefore conclude that $xcos(x)$ on the interval $-1 < x < 1$ is indeed piece-wise smooth.

Therefore the Fourier series at each point x converges to $(f(x+) + f(x-))/2$. But because $f(x)$ is continuous for $-1 < x < 1$, $f(x+) = f(x-) = f(x)$ for all $-1 < x < 1$. So $(f(x+) + f(x-))/2 = (f(x) + f(x))/2 = 2xcos(x)/2 = xcos(x)$. Meaning the Fourier series converges to $xcos(x)$ on the interval $-1 < x < 1$.

At the end-points of ± 1, the Fourier series converges to $f(-1) = f(1) = (f(1-) + f(-1+))/2$. But $xcos(x)$ is continuous on the entire real number line so $f(1-) = f(1) = 1cos(1) = cos(1)$ and $f(-1+) = f(-1) = -1cos(-1) = -cos(1)$. So $f(-1) = f(1) = (f(1-) + f(-1+))/2 = (cos(1) - cos(1))/2 = 0$.

A graph is shown below

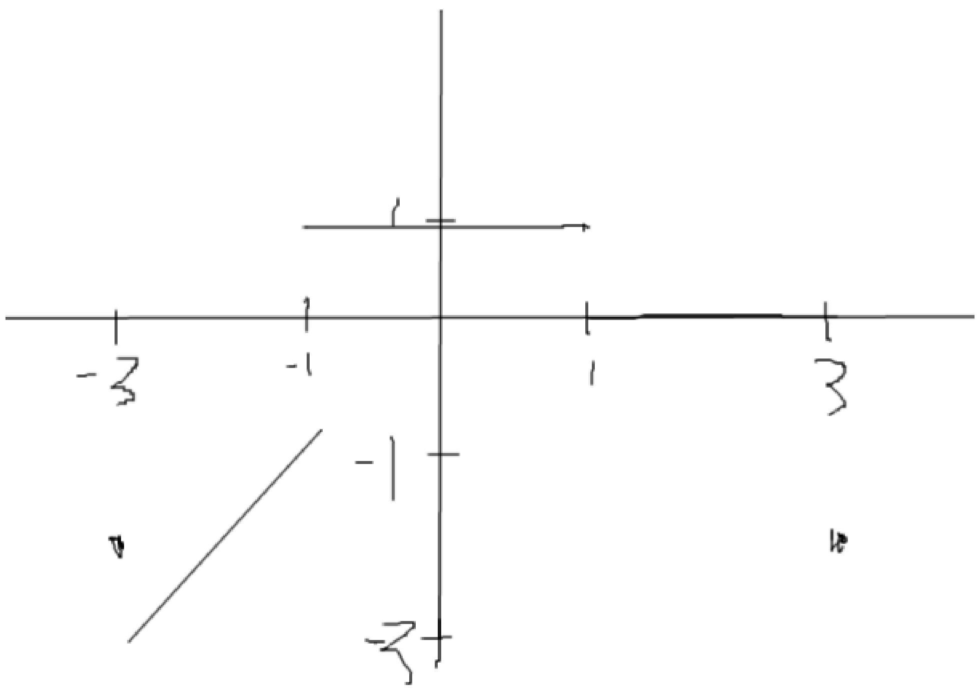


D) Consider the partitions $(-3,-1),(-1,1)$, and $(1,3)$. On $(1,3)$, we have $f(x)=0$ which is continuous on the interval, with derivative 0 on the same interval. On $(-1,1)$, we have $f(x)=1$ which is continuous on the interval, with derivative 0 on the same interval. On $(-3,-1)$, we have $f(x)=x$ which is continuous on the interval, with derivative 1 which is also continuous on the same interval. Both the function and it's derivative are bounded. Since the partitions cover the entire interval, we can thus conclude that the function is piece-wise smooth.

That means the Fourier series of the function converges to $(f(x-)+f(x+))/2$. On the interval $(1,3)$, we have $f(x)=0$ and is continuous on the interval so $f(x-)=f(x+)=f(x)$ thus $(f(x-)+f(x+))/2=(0+0)/2=0$. On the interval $(-1,1)$ we have $f(x)=1$ and is continuous on the interval so $f(x-)=f(x+)=f(x)$ thus $(f(x-)+f(x+))/2=(1+1)/2=1$. On the interval $(-3,-1)$ we have $f(x)=x$ and is continuous on the interval so $f(x-)=f(x+)=f(x)$. Thus $(f(x-)+f(x+))/2=2x/2=x$.

For the edges we have $f(x)=(f(-x+)+f(x-))/2$. So at $f(\pm 1)$, $f(-1+)=1$ and $f(1-)=1$. Thus $f(\pm 1)=1$. At $f(\pm 3)$, $f(-3+)=3$ and $f(3-)=0$. So $f(\pm 3)=3/2=1.5$.

A graph is shown below



2.

B) The even extension of $f(x)$ is defined as

$$f_e(x) = \begin{cases} 1+x & -2 < x < -1 \\ -x & -1 < x < 0 \\ x & 0 < x < 1 \\ 1-x & 1 < x < 2 \end{cases}$$

Clearly the above are all continuous in it's given interval and bounded, It's derivatives are all either -1 or 1 which is also continuous along it's given partition. Thus we can conclude that the even extension is piece-wise smooth. But the periodic extension will have the same properties in intervals that are shifted $4n$ where $n \in \mathbf{Z}$. Thus we have a partition that covers the entire domain such that the function limited to each interval of the partition is continuous and has a continuous derivative on the interval. So we can conclude that the even periodic extension, $\bar{f}_e(x)$ is also piece-wise smooth as required.

C) Since $\bar{f}_e(x)$ is piece-wise smooth and periodic, the Fourier function converges to $(f(x-)+f(x+))/2 = f(x)$. But for edges, $f(\pm x) = (f(-x+) + f(x-))/2$. So at $x = 1$, $\bar{f}_e(1) = (f(-1+)+f(1-))/2 = (-(-1)+1)/2 = 1$.

At $x = 2$, $\bar{f}_e(2) = (f(-2+)+f(2-))/2 = ((1+(-2))+(1-2))/2 = -1$.

At $x = 9.6$, we have $\bar{f}_e(1.6) = 1 - 1.6 = -0.6$ because the period of the even periodic extension is 4. Thus $\bar{f}_e(x+4n) = \bar{f}_e(x)$ where $n \in \mathbf{Z}$. So $\bar{f}_e(9.6) = \bar{f}_e(1.6+4*2) = \bar{f}_e(1.6) = 1-1.6 = -0.6$

At $x = -3.8$, we have $\bar{f}_e(-3.8) = \bar{f}_e(0.2-4) = \bar{f}_e(0.2) = 0.2$

3. We have the function is continuous on the intervals of $(-\pi, 0)$ and $(0, \pi)$ because $\frac{1}{x}$, $\sin(x)$ are both continuous on those intervals resulting in $\frac{\sin(x)}{x}$ being continuous as well.

Since $\sin(0) = 0$, we can take the derivative of the top and bottom then take the limit. So we have $\lim_{x\to 0} sin(x)/x = \lim_{x\to 0} cos(x)/1 = cos(0)/1 = 1$. So we have at x, the function will converge to 1. We also have $\lim_{x\to -\pi+} \frac{\sin(x)}{x} = \lim_{x\to\pi-} \sin(x)x = 0$. So define $g : [-\pi, \pi] \to \mathbf{R}$ as $g(x) = 1$ when $x = 1$ and $g(x) = f(x)$ otherwise. This function is continuous and because the interval is closed, must also be bounded. But we have $g(x)$ restricted to the interval $(-\pi, 0)$ must also be bounded, but such a restriction is the same as $f(x)$ restricted to the same interval, same can be said about the interval $(0, \pi)$. Therefore we can conclude that $f(x)$ on the given two intervals are bounded and so can thus conclude piecewise-continuity

So by using the Theorem in section 1-3, we now have the Fourier series of $f(x)$ converges at edges to $\frac{f(-a+)+f(a-)}{2}$. So as shown above, we have $f(0) = 1$. At $x = \pi$, we consider the averaging property of the Fourier function. So $f(\pi) = (f(\pi-)+f(-\pi)+)/2$. So we have $f(\pi-) = \sin(\pi)/\pi = 0$ and $f(-\pi+) = \sin(-\pi)/-\pi = 0$. So $f(\pi) = 0$.

By defining $f(0) = 1$, we have a continuous function. Since f has $f(-\pi) = f(\pi) = 0$, the periodic extension of f would indeed be continuous. The derivative of the function is $\frac{-\sin(x)}{x^2} + \frac{\cos(x)}{x}$ which is continuous on $(-\pi, 0)$ and $(0, \pi)$, has

$$\lim_{x\to 0} \frac{-\sin(x)}{x^2} + \frac{\cos(x)}{x} = \lim_{x\to 0} \frac{-\sin(x)+\cos(x)x}{x^2} =$$

$$\lim_{x\to 0}\frac{-\sin(x)x}{2x} = \lim_{x\to 0}\frac{-\cos(x)x - sin(x)}{2} = 0$$

Since $\lim_{x\to-\pi+}\frac{-\sin(x)}{x^2}+\frac{\cos(x)}{x} = \lim_{x\to\pi-}\frac{-\sin(x)}{x^2}+\frac{\cos(x)}{x} = -\frac{1}{\pi}$, we can conclude using similar reasoning as above that the derivative is also piece-wise continuous thereby making the function piece-wise smooth. Thus by theorem 2 of section 1-4, we can conclude that the Fourier series of f converges uniformly to f as required.

4.

Since $\sum_{n=1}^{N} a_n\cos(nx) + b_n\sin(nx)$ are continuous, and the limit uniformly converges to $f(x)$, we can conclude that $f(x)$ is continuous. $f(x)$ must also be periodic, meaning that $f(-\pi) = f(\pi) = 0$. The function must also have a piece-wise continuous derivative. We can then do two integrals one from 0 to π which result in 0, and another over $-\pi$ to 0. So we have

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{0} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos(nx)dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin(nx)dx$$