MAT 312 HW 7 Carl Liu

1.

We have

$$G_{14} = \{[1]_{14}, [3]_{14}, [5]_{14}, [9]_{14}, [11]_{14}, [13]_{14}\}$$

Let $S = \{[1]_{14}, [13]_{14}\}$. So the distinct left subsets are

$$[1]_{14}S = \{[1]_{14}, [13]_{14}\} \quad [3]_{14}S = \{[3]_{14}, [11]_{14}\} \quad [5]_{14}S = \{[5]_{14}, [9]_{14}\}$$

2.

By corollary 5.2.4, the order of G_{20} must be a multiple of the order of $g \in G_{20}$ for all $g \in G_{20}$. Since we see $[9]_{20} \in G_{20}$ and the order is 2, we can conclude that the order of G_{20} is $2n$ for some natural number n. We know that the order of G_{20} is $\phi(20) = \phi(2^2 * 5) = \phi(2^2)\phi(5) = (2^2 - 2)(5 - 1) = 8$ which checks out with 5.2.4

3.

Suppose hypothesis. Since $n = dk$. Consider the congruence class $[k]_n \in G$. Then we have

$$\sum_1^d [k]_n = [dk]_n = [n]_n = [0]_n$$

which is the identity element. This means that the order of $[k]_n$ is d and since $[k]_n \in G$ we are done.

4.

a) Suppose h and k are in H. Then $h(n) = n$ and $k(n) = n$. But that means $k^{-1}k(n) = n = k^{-1}(n)$ and so $k^{-1}(n) \in H$. Also $hk^{-1}(n) = h(n) = n$ which means $hk^{-1} \in H$, thus H is indeed a subgroup. We know that S_{n-1} has $(n-1)!$ elements. The amount of permutations in which n is fixed is also $(n-1)!$ which we can see by first fixing n, then $n-1$ has $n-1$ different choices it can map to, $n-2$ has $n-2$ different choices and so on. Thus the total different combinations is $(n-1)!$ as required. Since the two sets have the same cardinality, they are thus isomorphic.

b) Since $s_2 = s_1 h$ and $h \in H$, we have $h(n) = n$. So $s_2(n) = s_1 h(n) = s_1(n)$ as required.

c) Suppose hypothesis. We have $s_1(n) = s_2(n) = k$. So consider $s_2^{-1}s_1 = h$. Since $s_2^{-1}s_1(n) = s_2^{-1}(k) = n$, we have $h \in H$. Since $s_2 h = s_2 s_2^{-1} s_1 = s_1$, we can thus conclude that s_1, s_2 are in the same $H - coset$

d) Let $s \in S_n$ then $s(n) = k$ for some $1 \leq k \leq n$. Then $s \in H_k = s_k H$ where s_k is defined as $s_k(n) = k, s_k(k) = n \textit{ and } s_k(m) = m$ for all other cases. Now suppose there is more than n cosets, we have for $s \in H_{other}$, $s(n) = l$ where $1 \leq l \leq n$. But that means $s \in H_l$ and by part b and c, we conclude that $H_{other} = H_l$ thus we can have at most n unique cosets. We also see that all the sets $s_k H$ are non empty because the permutation s_k is in it since $s_k h(n) = s_k(n)$. Thus we have at least n cosets. Therefore we can conclude that there is exactly n H-cosets as required.

5.

a) Let $g, k \in H_v$, then $k(v) = v$ so $k^{-1}(v) = v$ as well. Then $gk^{-1}(v) = g(v) = g$ meaning that H_v is indeed a subgroup as required

b) Suppose g_1, g_2 are in the same $H_v - coset$, then $g_2 = g_1 h$ for some $h \in H_v$. That means $g_2(v) = g_1 h(v) = g_1(v)$ as required.

Suppose $g_1(v) = g_2(v) = k$. Following in the same vein as part c, we consider $g_2^{-1} g_1 = h$. Since $g_2^{-1} g_1(v) = g_2^{-1}(k) = v$, we thus have $h \in H_v$. Then because $g_2 h = g_2 g_2^{-1} g_1 = g_1$, we can thus conclude that g_2, g_1 are in the same $H_v - coset$.

c) Following the same proof as part d of question 4 we will arrive at exactly 4 $H_v - cosets$ and by considering $g_k(v) = k$ for $1 \leq k \leq 4$ we arrive at cosets of the form

$$g_k H_v = \{g_k g \in R | g(v) = v\} = \{g_k g \in R | g_k(v) = k\}$$