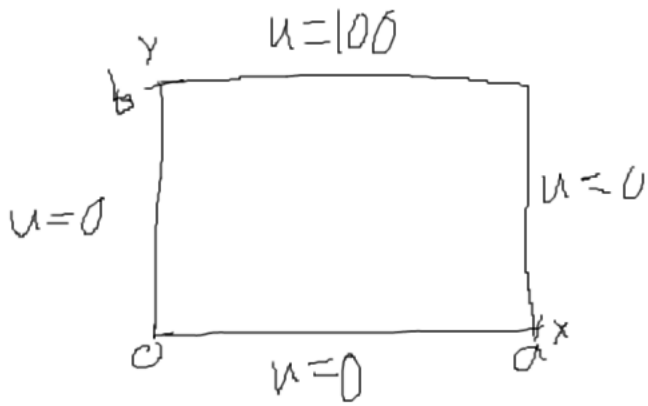
MAT 341 HW10, Carl Liu

1.

A)



We have $u(x, y) = X(x)Y(y)$. Then

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}$$

We must have

$$\frac{X''(x)}{X(x)} = -\lambda^2 = -\frac{Y''(y)}{Y(y)}$$

because a positive value would only fit the given boundary condition if $X(x) = 0$. Therefore we have

$$X''(x) + \lambda^2 X(x) = 0 \quad Y''(y) - \lambda^2 Y(y) = 0$$

This results in

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x) \quad Y(y) = C\cosh(\lambda y) + D\sinh(\lambda y)$$

But since $u(0, y) = X(0)Y(y) = 0$, we must have $A = 0$ resulting in $X(x) = \sin(\lambda x)$. Due to the boundary condition $u(a, y) = X(a)Y(y) = 0$, we must also have $\sin(\lambda a) = 0$. Thus resulting in $\lambda = \frac{n\pi}{a}$. Therefore we have the solution

$$u(x, y) = \sum_{n=1}^{\infty} (a_n \cosh(\lambda_n y) + b_n \sinh(\lambda_n y)) \sin(\lambda_n x)$$

We have

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin(\lambda_n x) = 0$$

$$a_n = \frac{2}{a}\int_0^a 0 * \sin(\lambda_n x)dx = 0$$

We also have

$$u(x,a) = \sum_{n=1}^{\infty} (a_n \cosh(\lambda_n a) + b_n \sinh(\lambda_n b)) \sin(\lambda_n x) = 100$$

$$(a_n \cosh(\lambda_n a) + b_n \sinh(\lambda_n b)) = \frac{200}{a}\int_0^a \sin(\lambda_n x)dx$$

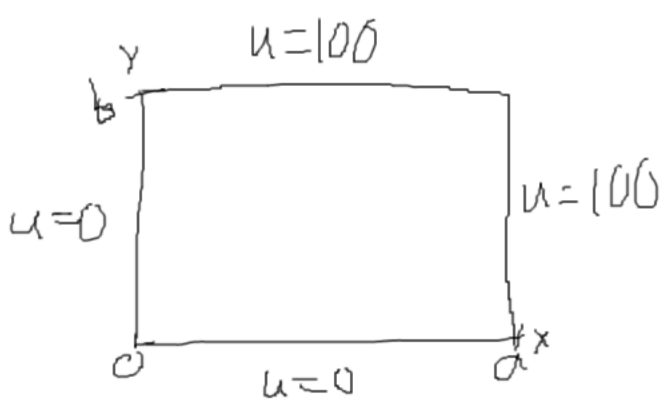
But because $a_n = 0$

$$b_n = \frac{200}{n\pi \sinh(\lambda_n b)}(1 - \cos(\lambda_n a)) = \frac{200}{n\pi \sinh(\lambda_n b)}(1 + (-1)^n)$$

Therefore we have

$$u(x,y) = \sum_{n=1}^{\infty} \frac{200}{n\pi \sinh(\lambda_n b)}(1 + (-1)^n) \sinh(\lambda_n y) \sin(\lambda_n x)$$

B)



Following from part A, we separate $u(x,y) = u_1(x,y) + u_2(x,y)$ where both are separable functions of x and y. We can conclude

$$u_1(x,y) = \sum_{n=1}^{\infty} (a_n \cosh(\lambda_n y) + b_n \sinh(\lambda_n y)) \sin(\lambda_n x)$$

$$u_2(x,y) = \sum_{n=1}^{\infty} (c_n \cosh(\gamma_n x) + d_n \sinh(\gamma_n x)) \sin(\gamma_n y)$$

where

$$\lambda_n = \frac{n\pi}{a} \quad \gamma_n = \frac{n\pi}{b}$$

Also continuing from part A we can conclude that $a_n = 0$ and $c_n = 0$. Resulting in

$$b_n = \frac{200}{n\pi \sinh(\lambda_n b)} (1 + (-1)^n)$$

$$d_n = \frac{200}{n\pi \sinh(\gamma_n a)} (1 + (-1)^n)$$

Meaning

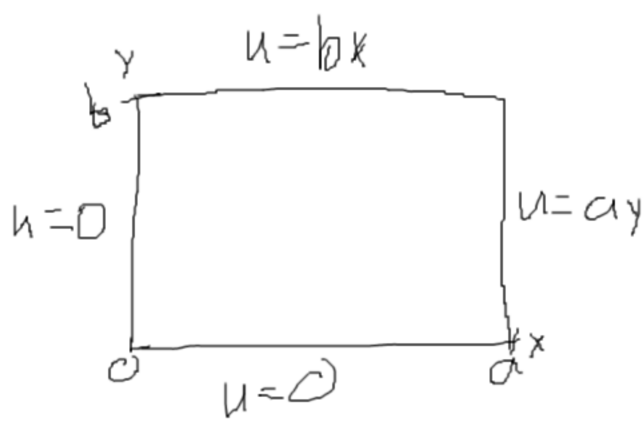
$$u_1(x,y) = \sum_{n=1}^{\infty} \frac{200}{n\pi \sinh(\lambda_n b)} (1 + (-1)^n) \sinh(\lambda_n y) \sin(\lambda_n x)$$

$$u_2(x,y) = \sum_{n=1}^{\infty} \frac{200}{n\pi \sinh(\gamma_n a)} (1 + (-1)^n) \sinh(\gamma_n y) \sin(\gamma_n x)$$

and so

$$u(x,y) = \sum_{n=1}^{\infty} \frac{200(1 + (-1)^n)}{n\pi} \left(\frac{\sinh(\lambda_n y)}{\sinh(\lambda_n b)} \sin(\lambda_n x) + \frac{\sinh(\gamma_n y)}{\sinh(\gamma_n a)} \sin(\gamma_n x) \right)$$

C)



In this case we have

$$b_n = \frac{2}{a} \frac{\int_0^a bx \sin(\lambda_n x) dx}{\sinh(\lambda_n b)} = -\frac{2ba}{a\lambda_n} \frac{\cos(\lambda_n a)}{\sinh(\lambda_n b)} = -\frac{2ba}{n\pi} \frac{(-1)^n}{\sinh(\lambda_n b)}$$

and so

$$u_1(x,y) = \sum_{n=1}^{\infty} \frac{2ba}{n\pi} \frac{(-1)^{n+1} \sinh(\lambda_n y) \sin(\lambda_n x)}{\sinh(\lambda_n b)}$$

We have

$$d_n = \frac{2}{b} \frac{\int_0^b ay \sin(\gamma_n y) dy}{\sinh(\gamma_n a)} = -\frac{2ab}{b\gamma_n} \frac{\cos(\gamma_n b)}{\sinh(\gamma_n a)} = -\frac{2ba}{n\pi} \frac{(-1)^n}{\sinh(\gamma_n a)}$$

and so

$$u_2(x,y) = \sum_{n=1}^{\infty} \frac{2ba}{n\pi} \frac{(-1)^{n+1} \sinh(\gamma_n y) \sin(\gamma_n y)}{\sinh(\gamma_n a)}$$

Thus resulting in

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2ba(-1)^{n+1}}{n\pi} \left(\frac{\sinh(\lambda_n y) \sin(\lambda_n x)}{\sinh(\lambda_n b)} + \frac{\sinh(\gamma_n y) \sin(\gamma_n y)}{\sinh(\gamma_n a)} \right)$$

2.

A) Consider the polynomial $v(x) = 1$. This satisfies the potential equation and the boundary equations

$$\frac{\partial v}{\partial x}(0,y) = 0 \quad v(x,0) = 1 \quad v(x,b) = 1 \quad v(a,y) = 1$$

So we can express $u(x,y) = 1 + w(x,y)$ where we now have

$$\frac{\partial^2 w}{\partial x^2}(x,y) + \frac{\partial^2 w}{\partial y^2}(x,y) = 0 \quad \frac{\partial w}{\partial x}(0,y) = 0 \quad w(x,0) = w(x,b) = w(a,y) = 0$$

We have

$$w(x,y) = X(x)Y(y)$$

resulting in

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}$$

and the eigenvalue problem

$$X'(0) = 0 \quad X''(x) + \lambda^2 X(x) = 0 \quad Y''(y) - \lambda^2 Y(y) = 0$$

This results in $X(x) = A\cos(\lambda x) + B\sin(\lambda x)$. Due to the boundary condition, we must have $X'(0) = 0$ and so $B = 0$ leaving $X(x) = A\cos(\lambda x)$. But because we also have $X(a) = 0$, we must also have $\lambda_n = \frac{(2n-1)\pi}{2a}$. We then have $Y(y) = a_n \cosh(\lambda_n y) + b_n \sin(\lambda_n y)$. Thus there is a solution of the form

$$w(x, y) = \sum_{n=1}^{\infty} (a_n \cosh(\lambda_n y) + b_n \sinh(\lambda_n y)) \cos(\lambda_n x)$$

that in turn means

$$w(x, 0) = \sum_{n=1}^{\infty} a_n \cos(\lambda_n x) = 0$$

$$a_n = 0$$

and

$$w(x, b) = \sum_{n=1}^{\infty} (a_n \cosh(\lambda_n b) + b_n \sinh(\lambda_n b)) \cos(\lambda_n x) = 0$$

$$a_n \cosh(\lambda_n b) + b_n \sinh(\lambda_n b) = 0$$

$$b_n = 0$$

Thus we have

$$w(x, y) = 0$$

and since $u(x, y) = 1 + w(x, y)$, we have

$$u(x, y) = 1$$

B) Let $v(y) = y/b$. This satisfies the potential equation and the boundary equations

$$\frac{\partial v}{\partial x}(0, y) = 0 \quad \frac{\partial v}{\partial x}(a, y) = 0 \quad v(x, 0) = 0 \quad v(x, b) = 1$$

Thus we have $u(x, y) = v(y) + w(x, y)$ where

$$\frac{\partial^2 w}{\partial x^2}(x, y) + \frac{\partial^2 w}{\partial y^2}(x, y) = \frac{\partial w}{\partial x}(0, y) = \frac{\partial w}{\partial x}(a, y) = w(x, 0) = w(x, b) = 0$$

The solution is thus

$$X(x) = 1 \quad X(x) = \cos(\lambda_n x) \quad Y(y) = a_n \cosh(\lambda_n y) + b_n \sinh(\lambda_n y)$$

$$Y(y) = a_0 + b_0 y$$

where $\lambda_n = \frac{n\pi}{a}$. Thus resulting in

$$w(x, y) = a_0 + b_0 y + \sum_{n=1}^{\infty} (a_n \cosh(\lambda_n y) + b_n \sinh(\lambda_n y)) \cos(\lambda_n x)$$

We then have

$$w(x, 0) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\lambda_n x) = 0$$

$$a_0 = 0 \quad a_n = 0$$

and

$$w(x, b) = a_0 + b_0 b + \sum_{n=1}^{\infty} (a_n \cosh(\lambda_n b) + b_n \sinh(\lambda_n b)) \cos(\lambda_n x)$$

$$a_0 + b_0 b = 0 \quad a_n \cosh(\lambda_n b) + b_n \sinh(\lambda_n b) = 0$$

so

$$b_0 = 0 \quad b_n = 0$$

Therefore

$$w(x, y) = 0$$

But that means

$$u(x, y) = y/b$$

C) We have $u(x, y) = X(x)Y(y)$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}$$

Given the boundary conditions, this results in

$$X''(x) - \lambda^2 X(x) = 0 \quad Y''(y) + \lambda^2 Y(y) = 0 \quad Y'(0)X(x) = 0$$

which means $Y'(0) = 0$. Then we have

$$Y(y) = A\cos(\lambda y) + B\sin(\lambda y)$$

Due to $Y'(0) = 0$, we must have $B = 0$ leaving us with $Y(y) = A\cos(\lambda y)$. Since $u(x, b) = 0$, we must have $Y(b) = A\cos(\lambda b) = 0$. So $\lambda_n = (2n-1)\pi/2b$. Then we have $X(x) = a_n \cosh(\lambda_n x) + b_n \sinh(\lambda_n x)$. This means a solution of the form

$$u(x, y) = \sum_{n=1}^{\infty} (a_n \cosh(\lambda_n x) + b_n \sinh(\lambda_n x))\cos(\lambda_n y)$$

We have

$$u(0, y) = \sum_{n=1}^{\infty} a_n \cos(\lambda_n y) = 1$$

$$a_n = \frac{2}{b}\int_0^b \cos(\lambda_n y)dy = \frac{2}{b\lambda_n} = \frac{4}{(2n-1)\pi}$$

We also have

$$u(a, y) = \sum_{n=1}^{\infty} (a_n \cosh(\lambda_n a) + b_n \sinh(\lambda_n a))\cos(\lambda_n y) = 0$$

$$a_n \cosh(\lambda_n a) + b_n \sinh(\lambda_n a) = 0$$

$$b_n = -\frac{a_n \cosh(\lambda_n a)}{\sinh(\lambda_n a)} = -\frac{4}{(2n-1)\pi}\frac{\cosh(\lambda_n a)}{\sinh(\lambda_n a)}$$

Resulting in

$$u(x, y) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi}\left(\cosh(\lambda_n x) - \frac{\cosh(\lambda_n a)}{\sinh(\lambda_n a)\sinh(\lambda_n x)}\right)\cos(\lambda_n y)$$