MAT 341 HW13, Carl Liu

1.

We have

$$\int_0^b \int_0^a \phi_{mn}(x,y)\phi_{pq}(x,y)dxdy =$$

$$\int_0^b \int_0^a \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi x}{a}\right)\sin\left(\frac{q\pi y}{b}\right)dxdy =$$

$$\int_0^b \sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{q\pi y}{b}\right)\left(\int_0^a \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{p\pi x}{a}\right)dx\right)dy$$

Since

$$\int_0^a \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{p\pi x}{a}\right)dx = \frac{-a}{m\pi}\left[\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{p\pi x}{a}\right)\right]_0^a +$$

$$\frac{p}{m}\left(\frac{a}{m\pi}\left[\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{p\pi x}{a}\right)\right]_0^a + \frac{p}{m}\int_0^a \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{p\pi x}{a}\right)dx\right)$$

We have

$$\left(1-\frac{p^2}{m^2}\right)\int_0^a \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{p\pi x}{a}\right)dx =$$

$$\frac{-a}{m\pi}\left[\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{p\pi x}{a}\right)\right]_0^a + \frac{p}{m}\frac{a}{m\pi}\left[\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{p\pi x}{a}\right)\right]_0^a = 0$$

In the case $1-\frac{p^2}{m^2} \neq 0$, we must have $p \neq m$ and

$$\int_0^a \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{p\pi x}{a}\right)dx = \frac{0}{1-\frac{p^2}{m^2}} = 0$$

resulting in

$$\int_0^b \int_0^a \phi_{mn}(x,y)\phi_{pq}(x,y)dxdy = 0$$

The same process can be went through to arrive at the same result when $q \neq n$. Now for the case $m = p$ and $q = n$, we have

$$\int_0^b \int_0^a \phi_{mn}(x,y)\phi_{pq}(x,y)dxdy =$$

$$\int_0^b \sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{n\pi y}{b}\right)\left(\int_0^a \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx\right)dy =$$

$$\int_0^b \sin^2\left(\frac{n\pi y}{b}\right)\left(\int_0^a \sin^2\left(\frac{m\pi x}{a}\right)dx\right)dy = \frac{a}{2}\int_0^b \sin^2\left(\frac{n\pi y}{b}\right)dy = \frac{ab}{4}$$

as required

2.

We assume $f(x,y) = g(x)h(y)$. We have from the boundary conditions

$$g(x) = \sum_{m=1}^{\infty} a_m \sin\left(\frac{m\pi x}{a}\right) \quad h(y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{b}\right)$$

Then

$$a_m = \frac{2}{a}\int_0^a g(x)\sin\left(\frac{m\pi x}{a}\right)dx \quad b_n = \frac{2}{b}\int_0^b h(y)\sin\left(\frac{n\pi y}{b}\right)dy$$

So

$$f(x,y) = g(x)h(y) = \sum_{m=1}^{\infty} a_m \sin\left(\frac{m\pi x}{a}\right)\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{b}\right) =$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

because the integrals of a_n and b_n do not share the same variable, we have

$$a_{mn} = a_m b_n = \frac{2}{a}\int_0^a g(x)\sin\left(\frac{m\pi x}{a}\right)dx\frac{2}{b}\int_0^b h(y)\sin\left(\frac{n\pi y}{b}\right)dy =$$

$$\frac{4}{ab}\int_0^b\int_0^a g(x)h(y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dxdy =$$

$$\frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dxdy$$

Thus we have the form of equation 15 and 17 as required.

3.

Let $f(x,y)$ denote a real valued function defined on R^2 which is periodic of period 2a in the variable x and is periodic of period 2b in the variable y. The fourier series of $f(x,y)$ is defined as

$$a_0 + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) + b_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) +$$

$$c_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) + d_{mn} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

where

$$a_0 = \int_0^b \int_0^a f(x,y)dxdy$$

$$a_{mn} = \frac{4}{ab} \int_0^b \int_0^a f(x,y) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dxdy$$

$$b_{mn} = \frac{4}{ab} \int_0^b \int_0^a f(x,y) \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dxdy$$

$$c_{mn} = \frac{4}{ab} \int_0^b \int_0^a f(x,y) \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) dxdy$$

$$d_{mn} = \frac{4}{ab} \int_0^b \int_0^a f(x,y) \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) dxdy$$