PHY 300 HW 5, Carl Liu

6-12

We know that $F * d = W$ and the total work done to deform the string is the energy in the string. We have $F * dy = dW$. In this case $F = 2T * \sin(\theta)$ since the string is pulling it down from both sides in the same direction. θ is the angle between the horizontal and the string. But we have

$$\sin(\theta) = \frac{y}{\sqrt{y^2 + \frac{L^2}{4}}}$$

and so

$$A) \quad \int_0^h F * dy = 2T \int_0^h \frac{y}{\sqrt{y^2 + \frac{L^2}{4}}} * dy = 2T \left[\sqrt{y^2 + \frac{L^2}{4}}\right]_0^h =$$

$$2T \left(\sqrt{h^2 + \frac{L^2}{4}} + \frac{L}{2}\right) = E$$

Since we know that the movement of the spring at a location x can be written as

$$y(t) = \sum_{n=1}^{\infty} a_n \cos\left(\omega_n t - \delta_n\right)$$

where ω_n is the frequencies of the normal modes of the string, meaning

$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}}$$

So in order for $y(t) = y(t + \Delta t)$ to be true, we must have

$$\Delta t \frac{\pi}{L}\sqrt{\frac{T}{\mu}} = 2\pi$$

Since that results in $\cos\left(\omega_n t - \delta_n\right) = \cos\left(\omega_n (t + \Delta t) - \delta_n\right)$ due to $\omega_n = n\omega_1$. This results in

$$\Delta t = 2L\sqrt{\frac{\mu}{T}}$$

and means we have a repeat every

$$B) \quad 2L\sqrt{\frac{\mu}{T}} \text{ seconds}$$

μ is kg/m, aka mass per unit length of the string.

6-14

A) $y(x) = Ax(L-x)$

We first set

$$y(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right)$$

By multiplying through with $\sin(\frac{m\pi}{L}x)$ and integrating from 0 to L, we obtain

$$a_n = \frac{2}{L}\int_0^L y(x)\sin\left(\frac{n\pi}{L}x\right)dx = \frac{2}{L}\int_0^L Ax(L-x)\sin\left(\frac{n\pi}{L}x\right)dx =$$

$$\frac{2A}{L}\left(\int_0^L Lx\sin\left(\frac{n\pi}{L}x\right)dx - \int_0^L x^2\sin\left(\frac{n\pi}{L}x\right)\right) =$$

$$\frac{2A}{L}\left(-\frac{L^3}{n\pi}\cos(n\pi) + \frac{L^3}{n\pi}\cos(n\pi) - \frac{L^3 2}{n^3\pi^3}\cos(n\pi) + \frac{L^3 2}{n^3\pi^3}\right) =$$

$$\frac{2A}{L}\left(-\frac{L^3 2}{n^3\pi^3}\cos(n\pi) + \frac{L^3 2}{n^3\pi^3}\right) = \frac{2A}{L}\left(-\frac{L^3 2}{n^3\pi^3}(-1)^n + \frac{L^3 2}{n^3\pi^3}\right) =$$

$$\frac{2A}{L}\frac{L^3 2}{n^3\pi^3}\left(1-(-1)^n\right) = \frac{4AL^2}{n^3\pi^3}\left(1-(-1)^n\right)$$

Since $(1-(-1)^n) = 0$ when n is even and $(1-(-1)^n) = 2$ when n is odd, we have

$$y(x) = \sum_{n=1}^{\infty}\frac{4AL^2}{n^3\pi^3}\left(1-(-1)^n\right)\sin\left(\frac{n\pi}{L}x\right) =$$

$$\sum_{n=1}^{\infty}\frac{8AL^2}{(2n-1)^3\pi^3}\sin\left(\frac{(2n-1)\pi}{L}x\right)$$

on the interval $0 \le x \le L$

B) The Fourier series is just

$$A \sin\left(\frac{\pi x}{L}\right)$$

Since this is just the first term of the sine series where all other terms other than it's first is 0.

C) We have

$$y(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right)$$

$$a_n = \frac{2}{L}\int_0^L y(x)\sin\left(\frac{n\pi}{L}x\right) dx$$

But because $y(x)$ is piecewise, we will split it into

$$\frac{2}{L}\int_0^L y(x)\sin\left(\frac{n\pi}{L}x\right) dx =$$

$$\frac{2}{L}\left(\int_0^{L/2} y(x)\sin\left(\frac{n\pi}{L}x\right) dx + \int_{L/2}^{L} y(x)\sin\left(\frac{n\pi}{L}x\right) dx\right) =$$

$$\frac{2}{L}\left(\int_0^{L/2} A\sin\left(\frac{2\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right) dx + \int_{L/2}^{L} 0 * \sin\left(\frac{n\pi}{L}x\right) dx\right) =$$

$$\frac{2A}{L}\left(\int_0^{L/2} \sin\left(\frac{2\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right) dx\right) =$$

$$\frac{2A}{L}\left(\int\limits_0^{L/2} \cos\left(\frac{2\pi - n\pi}{L}x\right) + \cos\left(\frac{2\pi + n\pi}{L}x\right) - \sin\left(\frac{2\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right) dx\right)$$

But that means

$$\frac{2A}{L}\int\limits_0^{L/2} \sin\left(\frac{n\pi}{L}x\right) dx = \frac{A}{L}\int\limits_0^{L/2} \cos\left(\frac{2\pi - n\pi}{L}x\right) - \cos\left(\frac{2\pi + n\pi}{L}x\right) =$$

$$\frac{A}{L}\left[\frac{L}{2\pi - n\pi}\sin\left(\frac{2\pi - n\pi}{L}x\right) - \frac{L}{2\pi + n\pi}\sin\left(\frac{2\pi + n\pi}{L}x\right)\right]_0^{L/2} =$$

$$\frac{A}{2\pi - n\pi}\sin\left(\frac{2\pi - n\pi}{2}\right) - \frac{A}{2\pi + n\pi}\sin\left(\frac{2\pi + n\pi}{2}\right) =$$

$$\frac{A}{(n-2)\pi}\sin\left(\frac{\pi}{2}(n-2)\right) - \frac{A}{(2+n)\pi}\sin\left(\frac{\pi}{2}(2+n)\right) = a_n$$

Thus we have

$$y(x) = \sum_{n=1}^{\infty}\left(\frac{A}{2\pi - n\pi}\sin\left(\frac{\pi}{2}(2-n)\right) - \frac{A}{2\pi + n\pi}\sin\left(\frac{\pi}{2}(2+n)\right)\right)\sin\left(\frac{n\pi}{L}x\right)$$

Since $\sin(\frac{\pi}{2}(2-n)) = 0$ and $\sin(\frac{\pi}{2}(2+n))$, when n is even, and alternates when odd. We have

$$y(x) = \sum_{n=1}^{\infty}\left(\frac{A(-1)^{n-1}}{2\pi - (2n-1)\pi} - \frac{A(-1)^{n}}{2\pi + (2n-1)\pi}\right)\sin\left(\frac{(2n-1)\pi}{L}x\right)$$

7-1 We have $\frac{1}{\lambda} = k$ and $\nu = v/\lambda$. Then

$$y = A\sin(2\pi(x - vt)/\lambda) = A\sin(2\pi(\frac{1}{\lambda}x - \frac{v}{\lambda}t)) = A\sin(2\pi(kx - \nu t))$$

Since $v = \lambda/T$ is the velocity of the wave, we have

$$A\sin(2\pi(x - vt)/\lambda) = A\sin(2\pi(\frac{1}{\lambda}x - \frac{v}{\lambda}t)) = A\sin(2\pi(\frac{1}{\lambda}x - \frac{\lambda}{\lambda T}t)) =$$

$$A\sin(2\pi(\frac{x}{\lambda} - \frac{t}{T}))$$

Since $\omega = 2\pi\nu$ due to ν being frequency, we have

$$A\sin(2\pi(x - vt)/\lambda) = A\sin(2\pi(x - \nu\lambda t)/\lambda) = A\sin(2\pi\nu(\frac{x}{\lambda\nu} - t)) =$$

$$-A\sin(\omega(t - \frac{x}{v}))$$

We have

$$A * Im\{\exp[j2\pi(kx - \nu t)]\} = A * Im\{\cos(2\pi(kx - \nu t)) + i\sin(2\pi(kx - \nu t))]\} =$$

$$A\sin(2\pi(kx-\nu t))$$

which we established earlier to be equal to

$$A\sin(2\pi(x-vt)/\lambda)$$

as required.

7-2

A) The amplitude of the wave is just $A = 0.3cm$. We can express the equation as so

$$0.3\sin(\pi(0.5x-50t)) = 0.3\sin(2\pi(0.25x-25t))$$

Thus we have wave number $k = 1/\lambda = 0.25cm^{-1}$, wavelength $\lambda = 4cm$, and frequency $\nu = 25Hz$. The period is then $T = 1/f = 1/25 = 0.04s$. Then we have velocity as $v = \lambda\nu = 100cm/s$

B) Since the transverse wave will pass through every point, we can consider the equation at $x = 0$. We thus have $y(0,t) = 0.3\sin(2\pi(-25t))$. Taking the derivative with respect to time, we obtain

$$\frac{dy}{dt} = -2\pi * 0.3 * 25\cos(2\pi(-25t))$$

for velocity. This is at a maximum when $\cos(2\pi(-25)t) = 1$, meaning $2\pi(-25)t = 0$, resulting in $t = 0$. This in turn results in $2\pi * 0.3 * 25 = 47.124cm/s$ as the maximum speed.

7-3

A longitudinal wave can be expressed with the same equation as a transverse wave and would thus be of the form $A\sin(2\pi(x-vt)/\lambda)$. Since we know $\nu = v/\lambda$, we have $\lambda = v/\nu = 3000/5 = 600m$. Thus we have

$$y = 0.003\sin(2\pi(x-3000t)/600)$$

7-4

A) The wave can be described as $A\sin(2\pi(x-vt)/\lambda)$, where we have $\lambda = v/\nu = 80/20 = 4m$. So $y = A\sin(2\pi(x-vt)/4) = A\sin(2\pi(\frac{1}{4}x-20t))$. Lets consider the time $t = 0$. We then have $y = A\sin(2\pi\frac{1}{4}x)$. A phase displacement of 30^o from $x = 0$ would just mean $2\pi\frac{1}{4}x = \pi/6$ and so $x = \frac{1}{3}$. Thus two points 30^o in phase apart are $\frac{1}{3}m = 0.33m$ apart.

B)

Let us consider the point $x = 0$. We have $A\sin(-2\pi 20t)$. The difference in phase between $t = 0$ and $t = 0.01s$ is then $-2\pi * 20 * 0.01s/\lambda - 0 = -1.256$ which means -72^o