Terence Tao Analysis 2 Exercise

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1 Metric Spaces

1.1.1

Suppose $(x_n)_{n=m}^{\infty}$ converges to x. Then $|x_n - x| \leq \varepsilon$ for all $n \geq N$ for some $N \geq m$. That means $|x_n - x| = |x_n - x| - 0 = ||x_n - x| - 0| \leq \varepsilon$ for all $n \geq N$ for some $N \geq m$. So by definition we have $\lim_{n \to \infty} |x_n - x| = 0$. But $d(x_n, x) = |x_n - x|$ and so $\lim_{n \to \infty} |x_n - x| = \lim_{n \to \infty} d(x_n, x) = 0$ as required.

Suppose $\lim_{n \to \infty} d(x_n, x) = 0$. Then $|d(x_n, x) - 0| \leq \varepsilon$ for all $n \geq N$ for some $N \geq m$. But we have $|d(x_n, x) - 0| = |d(x_n, x)| = ||x_n - x|| = |x_n - x|$ and so $|x_n - x| \leq \varepsilon$ for all $n \geq N$ for some $N \geq m$. Thus by definition $(x_n)_{n=m}^{\infty}$ converges to 0 as required.

1.1.2

A) for $x \in \mathbf{R}$ we have $d(x, x) = |x - x| = 0$ as required.

B) for $x, y \in \mathbf{R}$ where $x \neq y$, we have $d(x, y) = |x - y|$. Since $|x - y| \geq 0$ and $x - y \neq 0$, we have $|x - y| \neq 0$, therefore $|x - y| > 0$. So $d(x, y) > 0$ as required.

C) for $x, y \in \mathbf{R}$, we have $d(x, y) = |x - y| = |-(x - y)| = |y - x| = d(y, x)$ as required.

D) for $x, y, z \in \mathbf{R}$, we have $d(x, y) + d(y, z) = |x - y| + |y - z|$ and so by the triangle inequality, $d(x, z) = |x - z| \leq |x - y| + |y - z| = d(x, y) + d(y, z)$ as required.

1.1.3

A) consider an arbitrary set X and the function d_{disc} defined by $d(x, y) = 0.1$ when $x = y$ and $d(x, y) = 1$ when $x \neq y$. We then have for $x, y, z \in X$, $d(x, x) = 0.1 \neq 0$ and so a is false, $d(x, y) = 0.1 = d(y, x)$ when $x = y$ and $d(x, y) = 1 = d(y, x)$ when $y \neq x$ and so c is true, finally when $x = y = z, d(x, y) = 0.1$, $d(y, z) = 0.1$, $d(x, z) = 0.1$ and so $d(x, z) = 0.1 \leq d(x, y) + d(y, z) = 0.2$. When $x = y \neq z$ we have $d(x, y) = 0.1$, $d(y, z) = 1$, and $d(x, z) = 1$, so $d(x, z) = 1 \leq d(x, y) + d(y, z) = 1.1$, the same can be said in the case of $x \neq y = z$. Now in the case $x \neq y \neq z$, we have $d(x, y) = 1$, $d(y, z) = 1$, $and d(x, z)$ is equal to either 1 or 0.1. But in either case we have $d(x, z) \leq d(x, y) + d(y, z) = 2$. Since in all cases we have $d(x, z) \leq d(x, y) + d(y, z)$ we can conclude that d is true. For $x, y \in X$ such that $x \neq y$, we have $d(x, y) = 1 > 0$ and so b is true. Therefore (X, d_{disc}) obeys bcd but not a as required.

B) consider the arbitrary set X and the function $d : X \times X \rightarrow [0, \infty)$ defined as $d(x, y) = 0$. We then have, for $x, y, z \in X$, $d(x, x) = 0$ so a is true, $d(x, y) = 0 = d(y, x)$ so c is true, and $d(x, z) = 0 \leq d(x, y) + d(y, z) = 0 + 0 = 0$ so d is true. But for $x, y \in X$ such that $x \neq y$, we have $d(x, y) = 0$ which isn't greater than 0 and therefore b is false as required.

C) consider the set of real numbers $\mathbf{R}$ and the function $d : \mathbf{R} \times \mathbf{R} \rightarrow [0, \infty)$ defined as $d(x, x) = 0$ when $x = x$, $d(x, y) = 1$ when $x > y$ and $d(x, y) = 2$ when $x < y$. We then have for any $x, y, z \in \mathbf{R}$, $d(x, x) = 0$, satisfying a. In the case $x \neq y$, we have $d(x, y)$ is either equal to 2 or 1, in either case it is greater than 0 thereby satisfying b. In the case $x > y$, we have $d(x, y) = 1$, but $d(y, x) = 2$ and so c cannot be true. Now we have $d(x, z)$ is either 0, 1, or 2. In the case $d(x, z) = 0$, we have $d(x, z) = 0 \leq d(x, y) + d(y, z)$ since $d(x, y), d(y, z) \geq 0$. In the case $d(x, z) = 1$, we have $x > z$, then $d(x, y)$ is also either 0, 1, or 2. In the case of 1 and 2, we have $d(x, z) \leq d(x, y) + d(y, z)$. In the case it's 0 we have $x = y$ and so $x = y > z$ meaning $d(y, z) = 1$, we therefore have $d(x, z) = 1 \leq d(x, y) + d(y, z) = 0 + 1 = 1$. In the final case of $d(x, z) = 2$, we have $x < z$. We then have $d(x, y)$ is either 0, 1, or 2. In the case $d(x, y) = 2$, we have $d(x, z) = 2 \leq d(x, y) + d(y, z)$. In the case $d(x, y) = 1$, we have $x > y$ and so $y < x < z$ meaning $y < z$ and therefore

$d(y, z) = 2$. This results in $d(x, z) = 2 \leq d(x, y) + d(y, z) = 1 + 2 = 3$. In the case $d(x, y) = 0$, $x = y$ so $x = y < z$ and $d(y, z) = 2$. This means $d(x, z) = 2 \leq d(x, y) + d(y, z) = 0 + 2 = 2$. So in all cases we have $d(x, z) \leq d(x, y) + d(y, z)$ and so d is true. We therefore have abd true while c is false as required.

D) consider the set $X = \{0, 1, 3\}$ and the function $d : \{0, 1, 3\} \times \{0, 1, 3\} \rightarrow [0, \infty)$ defined as $d(x, y) = 10$. when $x = 0$ and $y = 3$, or when $x = 3$ and $y = 0$. In all other cases $d(x, y) = |x - y|$. We then have for any $x, y, z \in X$, $d(x, x) = |x - x| = 0$, satisfying a. When $x \neq y$ we have either $d(x, y) = 10$ or $d(x, y) = |x - y|$. Since $x - y \neq 0$ we have $|x - y| > 0$. we also have $10 > 0$ so in both cases b is satisfied. Now consider any $x, y \in X$. We have either $d(x, y) = 10$ or $d(x, y) = |x - y|$ when $d(x, y) = 10$ we must have either $x = 3$ and $y = 0$, or $x = 0$ and $y = 3$. In either case $d(y, x) = 10$ therefore $d(x, y) = d(y, x)$. Now in the case $d(x, y) = |x - y|$ we have $|x - y| = |-(x - y)| = |y - x| = d(y, x)$. So in all cases c is satisfied. Now consider $x = 0$, $y = 1$, and $z = 3$. We then have $d(x, z) = 10$, $d(x, y) = 1$, and $d(y, z) = 2$. But that means $d(x, z) = 10 > d(x, y) + d(y, z) = 3$ So d is not satisfied and so we have abc is true while d is false as required.

1.1.4

We have for any $x, y, z \in Y \subseteq X$, $d|_{Y \times Y}(x, x) = d(x, x) = 0$ satisfying a. When $x \neq y$ we have $d|_{Y \times Y}(x, x) = d(x, y) > 0$ because $x, y \in X$ and $x \neq y$. Therefore b is true. Now consider any $x, y \in Y$, then $d|_{Y \times Y}(x, y) = d(x, y)$, but we also have $d|_{Y \times Y}(y, x) = d(y, x)$ and by symmetry of the metric function d on X, we have $d(x, y) = d(y, x)$ and so $d|_{Y \times Y}(y, x) = d|_{Y \times Y}(x, y) = d(x, y)$. Therefore symmetry is true for $d|_{Y \times Y}$ on Y. Now consider $d|_{Y \times Y}(x, y) = d(x, y)$, $d|_{Y \times Y}(y, z) = d(y, z)$, and $d|_{Y \times Y}(x, z) = d(x, z)$. Due to the triangle inequality of function d on X we have $d(x, z) = d|_{Y \times Y}(x, z) \leq d(x, y) + d(y, z) = d|_{Y \times Y}(x, y) + d|_{Y \times Y}(y, z)$ and so the triangle inequality is also satisfied for $d|_{Y \times Y}$ on Y. Therefore we can conclude that $(Y, d|_{Y \times Y})$ is indeed a metric space.

1.1.5

We have $\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{j=1}^n b_j^2\right) = \sum_{i=1}^n \left(a_i^2 \sum_{j=1}^n b_j^2\right) = \sum_{i=1}^n \sum_{j=1}^n a_i^2 b_j^2$

We have $\left(\sum_{i=1}^n a_i b_i\right)^2 = \left(\sum_{i=1}^n a_i b_i\right)\left(\sum_{j=1}^n a_j b_j\right)$ (because index doesn't matter) $= \sum_{i=1}^n \left(a_i b_i \sum_{j=1}^n a_j b_j\right) = \sum_{i=1}^n \sum_{j=1}^n a_j b_j a_i b_i$

We have $\sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 = \sum_{i=1}^n \sum_{j=1}^n (a_i^2 b_j^2 - 2a_i b_j a_j b_i + a_j^2 b_i^2)$.

We then have $\left(\sum_{i=1}^n a_i b_i\right)^2 + (1/2) \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 =$
$\sum_{i=1}^n \sum_{j=1}^n (a_j b_j a_i b_i) + (1/2) \sum_{i=1}^n \sum_{j=1}^n (a_i^2 b_j^2 - 2a_i b_j a_j b_i + a_j^2 b_i^2) =$
$\sum_{i=1}^n \sum_{j=1}^n (a_j b_j a_i b_i - a_i b_j a_j b_i + (1/2)a_i^2 b_j^2 + (1/2)a_j^2 b_i^2) =$

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$\sum_{i=1}^{n}\sum_{j=1}^{n}((1/2)a_i^2b_j^2 + (1/2)a_j^2b_i^2) =$
$\sum_{i=1}^{n}\sum_{j=1}^{n}((1/2)a_i^2b_j^2) + \sum_{i=1}^{n}\sum_{j=1}^{n}((1/2)a_j^2b_i^2) =$
$\sum_{i=1}^{n}\sum_{j=1}^{n}((1/2)a_i^2b_j^2) + \sum_{j=1}^{n}\sum_{i=1}^{n}((1/2)a_j^2b_i^2) = \sum_{i=1}^{n}\sum_{j=1}^{n}a_i^2b_j^2$ (because index doesn't matter). Therefore $(\sum_{i=1}^{n}a_ib_i)^2 + (1/2)\sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_j - a_jb_i)^2 = (\sum_{i=1}^{n}a_i^2)(\sum_{j=1}^{n}b_j^2)$ as required.

Since $(\sum_{i=1}^{n}a_i^2)(\sum_{j=1}^{n}b_j^2)$ is all positive terms we have $0 \le (\sum_{i=1}^{n}a_i^2)(\sum_{j=1}^{n}b_j^2)$. Also we have $0 \le (1/2)\sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_j - a_jb_i)^2$ and $0 \le (\sum_{i=1}^{n}a_ib_i)^2$. So $0 \le (\sum_{i=1}^{n}a_ib_i)^2 = (\sum_{i=1}^{n}a_i^2)(\sum_{j=1}^{n}b_j^2) - (1/2)\sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_j - a_jb_i)^2$. But we have $-(1/2)\sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_j - a_jb_i)^2 \le 0$ and so $(\sum_{i=1}^{n}a_i^2)(\sum_{j=1}^{n}b_j^2) - (1/2)\sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_j - a_jb_i)^2 \le (\sum_{i=1}^{n}a_i^2)(\sum_{j=1}^{n}b_j^2)$. Therefore $0 \le (\sum_{i=1}^{n}a_ib_i)^2 \le (\sum_{i=1}^{n}a_i^2)(\sum_{j=1}^{n}b_j^2)$.

We then have $((\sum_{i=1}^{n}a_ib_i)^2)^{1/2} = |\sum_{i=1}^{n}a_ib_i| \le ((\sum_{i=1}^{n}a_i^2)(\sum_{j=1}^{n}b_j^2))^{1/2} = (\sum_{i=1}^{n}a_i^2)^{1/2}(\sum_{j=1}^{n}b_j^2)^{1/2}$ as required.

Triangle Inequality

We have

$$\left((\sum_{i=1}^{n}a_i^2)^{1/2} + (\sum_{j=1}^{n}b_j^2)^{1/2}\right)^2 = (\sum_{i=1}^{n}a_i^2) + 2(\sum_{j=1}^{n}b_j^2)^{1/2}(\sum_{i=1}^{n}a_i^2)^{1/2} + (\sum_{j=1}^{n}b_j^2)$$

We also have

$$\sum_{i=1}^{n}(a_i + b_i)^2 = \sum_{i=1}^{n}(a_i^2 + 2a_ib_i + b_i^2) = \sum_{i=1}^{n}a_i^2 + 2\sum_{i=1}^{n}a_ib_i + \sum_{i=1}^{n}b_i^2$$

But

$$-(\sum_{j=1}^{n}b_j^2)^{1/2}(\sum_{i=1}^{n}a_i^2)^{1/2} \le \sum_{i=1}^{n}a_ib_i \le (\sum_{j=1}^{n}b_j^2)^{1/2}(\sum_{i=1}^{n}a_i^2)^{1/2}$$

and so

$$\sum_{i=1}^{n}(a_i + b_i)^2 = \sum_{i=1}^{n}a_i^2 + 2\sum_{i=1}^{n}a_ib_i + \sum_{i=1}^{n}b_i^2 \le$$

$$(\sum_{i=1}^{n}a_i^2) + 2(\sum_{j=1}^{n}b_j^2)^{1/2}(\sum_{i=1}^{n}a_i^2)^{1/2} + (\sum_{j=1}^{n}b_j^2) = ((\sum_{i=1}^{n}a_i^2)^{1/2} + (\sum_{j=1}^{n}b_j^2)^{1/2})^2$$

Since both sides of the inequalities are positive we can take the square root and get

$$(\sum_{i=1}^{n}(a_i + b_i)^2)^{1/2} \le (\sum_{i=1}^{n}a_i^2)^{1/2} + (\sum_{j=1}^{n}b_j^2)^{1/2}$$

as required.

1.1.8

Suppose $x, y \in \mathbf{R}^n$, then $d_{l^2}(x, y)^2 = \sum_{i=1}^n (x_i - y_i)^2$ and $d_{l^1}(x, y)^2 = (\sum_{i=1}^n |x_i - y_i|)^2 = \sum_{i=1}^n (|x_i - y_i| \sum_{j=1}^n |x_j - y_j|)$. But $((x_i - y_i)^2)^{1/2} = |x_i - y_i|$ so $(((x_i - y_i)^2)^{1/2})^2 = (x_i - y_i)^2 = |x_i - y_i|^2$ and $|x_i - y_i| \leq \sum_{j=1}^n |x_j - y_j|$. We then have $|x_i - y_i|^2 = (x_i - y_i)^2 \leq |x_i - y_i| \sum_{j=1}^n |x_j - y_j|$. Therefore by the comparison test $\sum_{i=1}^n (x_i - y_i)^2 \leq (\sum_{i=1}^n |x_i - y_i|)^2$. Since both sides are positive, we can take the square root to then obtain $(\sum_{i=1}^n (x_i - y_i)^2)^{1/2} = d_{l^2}(x, y) \leq \sum_{i=1}^n |x_i - y_i| = d_{l^1}(x, y)$ as required.

We also have $\sqrt{n} d_{l^2}(x, y) = \sqrt{n \sum_{i=1}^n (x_i - y_i)^2} = \sqrt{(\sum_{n=1}^n 1)(\sum_{i=1}^n |x_i - y_i|^2)}$. So by exercise 1.1.5, we have $d_{l^1}(x, y) = \sum_{i=1}^n |x_i - y_i| \leq \sqrt{(\sum_{n=1}^n 1)(\sum_{i=1}^n |x_i - y_i|^2)} = \sqrt{n} d_{l^2}(x, y)$ as required.

1.1.9

Suppose $x \in \mathbf{R}^n$, then $d_{l^\infty}(x, x) = sup\{|x_i - x_i| : 1 \leq i \leq n\}$, but $|x_i - x_i| = 0$. So $\{|x_i - x_i| : 1 \leq i \leq n\} = \{0\}$. We then have $d_{l^\infty}(x, x) = sup\{0\} = 0$ as required.

Suppose $x, y \in \mathbf{R}^n$ such that $x \neq y$. Then there exists an $1 \leq i \leq n$ such that $x_i \neq y_i$. We then have $x_i - y_i \neq 0$ and so $|x_i - y_i| > 0$. Since $|x_i - y_i| \in \{|x_i - y_i| : 1 \leq i \leq n\}$, we have $d_{l^\infty}(x, y) = sup\{|x_i - y_i| : 1 \leq i \leq n\} \geq |x_i - y_i| > 0$ as required.

Suppose $x, y \in \mathbf{R}^n$. We then have $d_{l^\infty}(x, y) = sup\{|x_i - y_i| : 1 \leq i \leq n\}$, but $|x_i - y_i| = |y_i - x_i|$ and so $d_{l^\infty}(x, y) = sup\{|x_i - y_i| : 1 \leq i \leq n\} = sup\{|y_i - x_i| : 1 \leq i \leq n\} = d_{l^\infty}(y, x)$ as required.

Suppose $x, y, z \in \mathbf{R}^n$. Then $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$ by the triangle inequality. We then have $sup\{|x_i - z_i| : 1 \leq i \leq n\} \leq sup\{|x_i - y_i| + |y_i - z_i| : 1 \leq i \leq n\}$. But $|x_i - y_i| \leq sup\{|x_i - y_i| : 1 \leq i \leq n\}$ and $|y_i - z_i| \leq sup\{|y_i - z_i| : 1 \leq i \leq n\}$ for all $1 \leq i \leq n$. We then have $|x_i - y_i| + |y_i - z_i| \leq sup\{|x_i - y_i| : 1 \leq i \leq n\} + sup\{|y_i - z_i| : 1 \leq i \leq n\}$ and so $sup\{|x_i - y_i| + |y_i - z_i| : 1 \leq i \leq n\} \leq sup\{|x_i - y_i| : 1 \leq i \leq n\} + sup\{|y_i - z_i| : 1 \leq i \leq n\}$. Therefore $sup\{|x_i - z_i| : 1 \leq i \leq n\} = d_{l^\infty}(x, z) \leq sup\{|x_i - y_i| : 1 \leq i \leq n\} + sup\{|y_i - z_i| : 1 \leq i \leq n\} = d_{l^\infty}(x, y) + d_{l^\infty}(y, z)$ as required.

1.1.10

We have $(1/\sqrt{n}) d_{l^2}(x, y) = \sqrt{(1/n) \sum_{i=1}^n (x_i - y_i)^2}$. But $(x_i - y_i) \leq sup\{|x_i - y_i| : 1 \leq i \leq n\}$ and so $(x_i - y_i)^2 \leq (sup\{|x_i - y_i| : 1 \leq i \leq n\})^2$. Then $(1/n) \sum_{i=1}^n (x_i - y_i)^2 \leq (1/n) \sum_{i=1}^n (sup\{|x_i - y_i| : 1 \leq i \leq n\})^2$. But $\sum_{i=1}^n (sup\{|x_i - y_i| : 1 \leq i \leq n\})^2 = n(sup\{|x_i - y_i| : 1 \leq i \leq n\})^2$. So we have $(1/n) \sum_{i=1}^n (x_i - y_i)^2 \leq (sup\{|x_i - y_i| : 1 \leq i \leq n\})^2$ and so $\sqrt{(1/n) \sum_{i=1}^n (x_i - y_i)^2} = (1/\sqrt{n}) d_{l^2}(x, y) \leq sup\{|x_i - y_i| : 1 \leq i \leq n\} =$

$d_{l\infty}(x, y)$ as required.

Suppose for contradiction that $sup\{|x_i - y_i| : 1 \le i \le n\} > (\sum_{i=1}^{n}(x_i - y_i)^2)^{1/2}$. We then have, there is an $1 \le i_0 \le n$ such that $|x_{i_0} - y_{i_0}| > (\sum_{i=1}^{n}(x_i - y_i)^2)^{1/2}$ and so $|x_{i_0} - y_{i_0}|^2 > \sum_{i=1}^{n}(x_i - y_i)^2 = \sum_{i=1}^{n}|x_i - y_i|^2$, But we have $\sum_{i=1}^{n}|x_i - y_i|^2 = \sum_{i=1}^{i_0}|x_i - y_i|^2 + \sum_{i_0+1=1}^{n}|x_i - y_i|^2 = \sum_{i=1}^{i_0-1}|x_i - y_i|^2 + |x_{i_0} - y_{i_0}|^2 + \sum_{i_0+1=1}^{n}|x_i - y_i|^2$. But since all terms are greater than or equal to 0, we have $\sum_{i=1}^{i_0-1}|x_i - y_i|^2 + |x_{i_0} - y_{i_0}|^2 + \sum_{i_0+1=1}^{n}|x_i - y_i|^2 = \sum_{i=1}^{n}(x_i - y_i)^2 \ge |x_{i_0} - y_{i_0}|^2$, a contradiction, therefore $d_{l\infty}(x, y) = sup\{|x_i - y_i| : 1 \le i \le n\} \le (\sum_{i=1}^{n}(x_i - y_i)^2)^{1/2} = d_{l^2}(x, y)$. as required.

1.1.11

Suppose $x \in \mathbf{R}^n$. Then $x = x$, $therefore$ $d_{disc}(x, x) = 0$ as required.

Suppose $x, y \in \mathbf{R}^n$ such that $x \ne y$. Then $d_{disc}(x, y) = 1 > 0$ as required.

Suppose $x, y \in \mathbf{R}^n$. We have either $x = y$ or $x \ne y$. In the case $x = y$, $d_{disc}(x, y) = 0$. But we also have $y = x$, and so $d_{disc}(y, x) = 0 = d_{disc}(x, y)$. In the case $x \ne y$, $d_{disc}(x, y) = 1$. But we also have $x \ne y$, and so $d_{disc}(y, x) = 1 = d_{disc}(x, y)$. In all cases we have $d_{disc}(x, y) = d_{disc}(y, x)$ as required.

Suppose $x, y, z \in \mathbf{R}^n$. We have either $x \ne z$ or $x = z$. In the case $x \ne z$, we have either $x \ne y$ or $x = y$, In the case $x \ne y$, we then have $d_{disc}(x, y) = d_{disc}(x, z) = 1$, and since $0 \le d_{disc}(y, z)$, we have $d_{disc}(x, z) = 1 \le d_{disc}(x, y) = 1 \le d_{disc}(x, y) + d_{disc}(y, z)$. In the case $x = y$, $y \ne z$ and so $d_{disc}(y, z) = 1$. We then have $d_{disc}(x, z) = 1 \le d_{disc}(x, y) + d_{disc}(y, z)$. Now in the other case of $x = z$, we have $d_{disc}(x, z) = 0$. but since, $0 \le d_{disc}(x, y), d_{disc}(y, z)$, we have $d_{disc}(x, z) \le d_{disc}(x, y) + d_{disc}(y, z)$. In all cases we have $d_{disc}(x, z) \le d_{disc}(x, y) + d_{disc}(y, z)$ as required.

We can then conclude that $(\mathbf{R}^n, d_{disc}(x, y)$ is indeed a metric space.

1.1.13

Suppose $(x^{(n)})_{n=m}^{\infty}$ converges to x with respect to the discrete metric d_{disc}. Then for all $\varepsilon > 0$, $d_{disc}(x^{(n)}, x) \le \varepsilon$ for all $n \ge N$ for some $N \ge m$. Since $0.5 > 0$, we have $d_{disc}(x^{(n)}, x) \le 0.5$. Suppose for contradiction that $x^{(n)} \ne x$. Then $d_{disc}(x^{(n)}, x) = 1$, but $d_{disc}(x^{(n)}, x) \le 0.5$, so $d_{disc}(x^{(n)}, x) = 1 \le 0.5$, a contradiction. Therefore $x^{(n)} = x$ for all $n \ge N$ for some $N \ge m$ as required.

Suppose there exists an $N \ge m$ such that $x^{(n)} = x$ for all $n \ge N$. Then $d_{disc}(x^{(n)}, x) = 0$, and so we have for all $\varepsilon > 0$, $d_{disc}(x^{(n)}, x) = 0 \le \varepsilon$ for all $n \ge N$. So by definition we have $(x^{(n)})_{n=m}^{\infty}$ converges to x with respect to the discrete metric d_{disc} as required.

1.1.14

Suppose for contradiction that $x \ne x'$. Then $d(x, x') > 0$. So $d(x, x') >$

$\varepsilon > 0$ for some real number $\varepsilon > 0$. We have $d(x^n, x) \leq \varepsilon/3$ for all $n \geq N_0$ for some $N_0 \geq m$ and $d(x^n, x') \leq \varepsilon/3$ for all $n \geq N_1$ for some $N_1 \geq 0$. Let $N = max(N_0, N_1)$, Then we have $d(x^n, x), d(x^n, x') \leq \varepsilon/3$ for all $n \geq N$. That means $d(x, x^{(n)}) + d(x^n, x') = d(x^n, x) + d(x^n, x') \leq 2\varepsilon/3$, and by the triangle inequality, we have $d(x, x') \leq 2\varepsilon/3$, a contradiction. Therefore $x = x'$ as required.

1.1.15

For the d_{l^1} metric

Suppose $x \in X$, then $x = (a_n)_{n=0}^{\infty}$, such that $\sum_{n=0}^{\infty} |a_n| < \infty$, We then have $d_{l^1}(x, x) = \sum_{n=0}^{\infty} |a_n| - \sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{\infty} |a_n| - |a_n| = \sum_{n=0}^{\infty} 0 = 0$ as required.

Suppose $x, y \in X$, such that $x \neq y$. Then for some $n \geq 0$, $a_n \neq b_n$, and so $a_n - b_n \neq 0$, meaning $|a_n - b_n| > 0$. But that means $d_{l^1}(x, y) = \sum_{n=0}^{\infty} |a_n - b_n| > 0$ as required.

Suppose $x, y \in X$, we then have $d_{l^1}(x, y) = \sum_{n=0}^{\infty} |a_n - b_n|$, but $|a_n - b_n| = |b_n - a_n|$. So $d_{l^1}(x, y) = \sum_{n=0}^{\infty} |a_n - b_n| = \sum_{n=0}^{\infty} |b_n - a_n| = d_{l^1}(y, x)$ as required.

Suppose $x, y, z \in X$. We have $|a_n - c_n| \leq |a_n - b_n| + |b_n - c_n|$. So $\sum_{n=0}^{\infty} |a_n - c_n| \leq \sum_{n=0}^{\infty} (|a_n - b_n| + |b_n - c_n|) = \sum_{n=0}^{\infty} |a_n - b_n| + \sum_{n=0}^{\infty} |b_n - c_n|$. So $d_{l^1}(x, z) \leq d_{l^1}(x, y) + d_{l^1}(y, z)$ as required.

We can thus conclude that (X, d_{l^1}) is a metric space.

For the d_{l^∞} metric

Suppose $x \in X$, then $|a_n - a_n| = 0$, so $d_{l^\infty}(x, x) = sup_{n \in \mathbf{N}} |a_n - a_n| = sup_{n \in \mathbf{N}} 0 = 0$ as required.

Suppose $x, y \in X$ such that $x \neq y$, then there exists an $n_0 \geq 0$ such that $a_{n_0} \neq b_{n_0}$ and so $|a_{n_0} - b_{n_0}| > 0$. We have $sup_{n \in \mathbf{N}} |a_n - b_n| \geq |a_{n_0} - b_{n_0}| > 0$ and so $d_{l^\infty}(x, y) > 0$ as required.

Suppose $x, y \in X$, then $|a_n - b_n| = |b_n - a_n|$, so $d_{l^\infty}(x, y) = sup_{n \in \mathbf{N}} |a_n - b_n| = sup_{n \in \mathbf{N}} |b_n - a_n| = d_{l^\infty}(y, x)$ as required.

Suppose $x, y, z \in X$. We have $|a_n - c_n| \leq |a_n - b_n| + |b_n - c_n|$. So $sup_{n \in \mathbf{N}} |a_n - c_n| \leq sup_{n \in \mathbf{N}} (|a_n - b_n| + |b_n - c_n|)$. But $|a_n - b_n| \leq sup_{n \in \mathbf{N}} |a_n - b_n|$ and $|b_n - c_n| \leq sup_{n \in \mathbf{N}} |b_n - c_n|$. So $|a_n - b_n| + |b_n - c_n| \leq sup_{n \in \mathbf{N}} |a_n - b_n| + sup_{n \in \mathbf{N}} |b_n - c_n|$. Therefore $sup_{n \in \mathbf{N}} |a_n - c_n| \leq sup_{n \in \mathbf{N}} (|a_n - b_n| + |b_n - c_n|) \leq sup_{n \in \mathbf{N}} |a_n - b_n| + sup_{n \in \mathbf{N}} |b_n - c_n|$. Thus $d_{l^\infty}(x, z) \leq d_{l^\infty}(x, y) + d_{l^\infty}(y, z)$ as required.

We can then conclude that (X, d_{l^∞}) is indeed a metric space.

1.1.16 We have $d(x_n, x) \leq \varepsilon/2$ for all $n \geq N_0$ for some $N_0 \geq 1$, we also have $d(y_n, y) \leq \varepsilon/2$ for all $n \geq N_1$ for some $N_1 \geq 1$. Let $N = max(N_0, N_1)$. Then $d(x_n, x) \leq \varepsilon/2$ and $d(y_n, y) \leq \varepsilon/2$ for all $n \geq N$. By the triangle

inequality we also have $d(x_n, y) \leq d(x_n, x) + d(x, y)$. But $d(x_n, x) + d(x, y) \leq \varepsilon/2 + d(x, y)$ and $d(y_n, y) = d(y, y_n)$. So $d(x_n, y) + d(y, y_n) \leq \varepsilon + d(x, y)$. Then by triangle inequality we have $d(x_n, y_n) \leq \varepsilon + d(x, y)$. Therefore $d(x_n, y_n) - d(x, y) \leq \varepsilon$. We also have $d(x, y_n) \leq d(x, x_n) + d(x_n, y_n) \leq \varepsilon/2 + d(x_n, y_n)$. Then $d(x, y_n) + d(y_n + y) \leq \varepsilon/2 + d(x_n, y_n) + d(y_n + y) \leq \varepsilon + d(x_n, y_n)$. By the triangle inequality we have $d(x, y) \leq \varepsilon + d(x_n, y_n)$ and so $d(x, y) - \varepsilon \leq d(x_n, y_n)$. That means $d(x, y) - \varepsilon \leq d(x_n, y_n) \leq \varepsilon + d(x, y)$. Therefore $|d(x_n, y_n) - d(x, y)| \leq \varepsilon$ for all $n \geq N$ so $\lim_{n \to \infty} d(x_n, y_n) = d(x, y)$ as required.

1.2.1

Suppose $x \in E$, then we have $B(x, 0.5) = \{y \in X : d(y, x) < 0.5\}$. But by the definition of discrete method, $d(y, x) = 1$ if $x \neq y$, and $d(y, x) = 0$ when $x = y$. So $d(y, x) = 0 < 0.5$ when $x = y$. Therefore $B(x, 0.5) = \{y \in X : d(y, x) < 0.5\} = \{x\}$ and since $\{x\} \subseteq E$, we have x is an interior point for all $x \in E$

Suppose $x \notin E$, then we have $B(x, 0.5) = \{y \in X : d(y, x) < 0.5\} = \{x\}$. Then $\{x\} \cap E = \emptyset$. Since $0.5 > 0$, we do have an $r > 0$ such that $B(x, r) \cap E = \emptyset$ and so x is an exterior point.

1.2.2

Suppose x_0 is an adherent point of E. We have either $x_0 \in E$, or $x_0 \notin E$. In the case $x_0 \notin E$, suppose x_0 is an exterior point for contradiction. Then there exists an $r > 0$ such that $B(x_0, r) \cap E = \emptyset$, but we have $B(x_0, r) \cap E \neq \emptyset$ by definition of adherent point, therefore a contradiction and so x_0 must be a boundary point since x_0 is not in E and cannot be an interior point. In the case $x_0 \in E$, suppose for contradiction that x_0 is an exterior point. Then for some $r > 0$, $B(x_0, r) \cap E = \emptyset$, but $x_0 \in B(x_0, r)$ since $d(x_0, x_0) = 0 < r$. We then have $x_0 \in B(x_0, r) \cap E \neq \emptyset$, a contradiction. So x_0 is not an exterior point. We then have x_0 is either an interior point or not. In the case it is not, x_0 is a boundary point by definition. So in all cases we have either x_0 is an interior or boundary point as required.

1.2.3

A) Suppose E is open. Suppose for contradiction that there exists an $x \in E$ such that for all $r > 0$, $B(x, r) \not\subseteq E$. Since $x \in E$, x is an adherent point because the sequence $(x)_{n=1}^{\infty}$ obviously converges to x. We then have x is either an interior point or boundary point. Suppose for contradiction that x is an interior point, then we have, there exists an $r_0 > 0$ such that $B(x, r_0) \subseteq X$. But we have $B(x, r_0) \not\subseteq E$ by hypothesis. So we must have x is a boundary point. But by definition of E is open, $\partial E \cap E = \emptyset$. So we

have $x \in \partial E \cap E \neq \emptyset$, a contradiction. Since all cases lead to contradictions, we must have for all $x \in E$, there exists an $r > 0$ such that $B(x, r) \subseteq E$ as required.

Suppose for all $x \in E$, there exists an $r > 0$ such that $B(x, r) \subseteq E$. Suppose for contradiction that E contains its boundary points. So $E \cap \partial E \neq \emptyset$. We then have for $x \in \partial E \cap E$, there exists an $r > 0$ such that $B(x, r) \subseteq E$. So x is an interior point. But this is a contradiction since x cannot be both a boundary and interior point. Therefore E does not contain its boundary points and is therefore open.

B) Suppose E is closed. Let x be an adherent point of E. Then x is either an interior point or a boundary point. In the case x is a boundary point, we have $x \in \partial E$, but $\partial E \subseteq E$ and so $x \in E$. In the case x is an interior point, we must have $x \in E$ and so in all cases we have $x \in E$. Therefore for all adherent points of E, $x \in E$ as required.

Suppose E contains all it's adherent points. Suppose $x \in \partial E$, then x is a boundary point, and therefore an adherent point of E. So $x \in E$. That means $\partial E \subseteq E$ and E is therefore a closed set.

C) Suppose $x_0 \in X$ and $r > 0$. Let $x \in B(x_0, r)$. We then have $d(x_0, x) < r$ but either $x = x_0$ or $x \neq x_0$. In the case $x = x_0$ we have $B(x, r) \subseteq B(x_0, r)$. In the case $x \neq x_0$ we have $0 < d(x_0, x) < r$. Consider $B(x, r - d(x_0, x))$. Let $y \in B(x, r - d(x_0, x))$. We then have $d(x, y) < r - d(x_0, x)$ and so $d(x, y) + d(x_0, x) < r$. But by the triangle inequality we have $d(x_0, y) < r$ and so $y \in B(x_0, r)$. Therefore $B(x, r - d(x_0, x)) \subseteq B(x_0, r)$ meaning x is an interior point of $B(x_0, r)$. But since x was chosen arbitrarily in $B(x_0, r)$, we have for all $x \in B(x_0, r)$, x is an interior point of $B(x_0, r)$, but that also means x is not a boundary point. So we have $B(x_0, r) \cap \partial(B(x_0, r)) = \emptyset$ and therefore $B(x_0, r)$ is open as required.

Now consider the set $\{x \in X : d(x, x_0) \leq r\}$. Let y be a boundary point of this set. Suppose for contradiction that $y \notin \{x \in X : d(x, x_0) \leq r\}$. Since y is a boundary point, it is an adherent point and so there exists a sequence consisting of elements in the set such that $\lim_{n \to \infty} d(x_n, y) = 0$. But we have $d(y, x_0) > r$ and $d(x_n, x_0) \leq r$. So $d(y, x_0) > q > r$ meaning $d(y, x_0) - d(x_n, x_0) > q - r > 0$ Since $d(x_0, y) \leq d(x_n, y) + d(x_n, x_0)$, we have $q - r < d(x_0, y) - d(x_n, x_0) \leq d(x_n, y)$. By the hypothesis we have $d(x_n, y) \leq (q - r)/2$ for all $n \geq N$ for some $N \geq 1$ and so we have a contradiction. Therefore $\{x \in X : d(x, x_0) \leq r\}$ is indeed a closed set.

D) Suppose $\{x_0\}$ is a singleton set where $x_0 \in X$. Let $(x_n)_{n=m}^{\infty}$ be a convergent sequence such that $x_n \in \{x_0\}$. Then $x_n = x_0$ so $\lim_{n \to \infty} x_n =$

$\lim_{n\to\infty} x_0$, but we have $d(x_0, x_0) = 0$ and so $\lim_{n\to\infty} x_0 = x_0$. So $\{x_0\}$ contains all it's adherent points. Therefore $\{x_0\}$ is indeed a closed set.

E) Suppose E is open then $E = int(E)$ and so $X - E = X - int(E)$. Suppose x is a boundary point of $X - E$. Suppose for contradiction that $x \in E$. Then x is an interior point of E and so there exists an $r > 0$ such that $B(x, r) \subseteq E$. But that means $B(x, r) \cap X - E = \emptyset$ and therefore x is an exterior point of $X - E$. But this is a contradiction since x is a boundary point of $X - E$ and cannot be an exterior point. Therefore $x \in X - E$ and so $X - E$ is closed as required.

Suppose $X - E$ is closed. Let x be a boundary point of E. Suppose for contradiction that $x \in E$. Then x is neither an interior or a boundary point of $X - E$. This means that it must be an exterior point of $X - E$ and so there exists an $r > 0$ such that $B(x, r) \cap (X - E) = \emptyset$. Let $x \in B(x, r)$. Then $x \in X$, but $x \notin X - E$ and so $x \in E$, meaning $B(x, r)$ is a subset of E. Therefore x is an interior point of E, but this is a contradiction since x is a boundary point of E by hypothesis. We therefore have $x \notin E$ and so $\partial E \cap E = \emptyset$ meaning E is indeed an open set.

F) First we shall prove that the intersection of two open sets is open. Let E_1 and E_2 be open sets. Let x be a boundary point of $Z = E_1 \cap E_2$. Suppose for contradiction that $x \in Z$, then x is an interior point of both E_1 and E_2. That means there exists an $r_1 > 0$ such that $B(x, r_1) \subseteq E_1$ and there exists an $r_2 > 0$ such that $B(x, r_2) \subseteq E_2$. Suppose $r = min(r_1, r_2)$, then $B(x, r) \subseteq B(x, r_1)$ and $B(x, r) \subseteq B(x, r_2)$. This is because for an $x_0 \in B(x, r)$, we have $d(x, x_0) < r \leq r_1, r_2$. Since $B(x, r_1)$ and $B(x, r_2)$ are subsets of E_1 and E_2 respectively. We have for an $x \in B(x, r)$, $x \in E_1 \cap E_2$ meaning $B(x, r)$ is a subset of $E_1 \cap E_2$ and so x is an interior point of $E_1 \cap E_2$. But x is a boundary point and so we have a contradiction. Therefore $x \notin E_1 \cap E_2$ and so $E_1 \cap E_2$ is open as required.

We will prove using induction. In the case $n = 1$, we have a collection E_1 which is open and since E_1 is open, we are done. Now suppose for inductive hypothesis that for some $n \geq 1$, if $E_1 ... E_n$ are open sets in X then $Z = E_1 \cap ... \cap E_n$ is an open set. Then let $E_1 ... E_{n+1}$ be a collection of open sets in X then $E_1 \cap ... \cap E_n \cap E_{n+1} = Z \cap E_{n+1}$ where Z and E_{n+1} are both open sets. This means that $Z \cap E_{n+1} = E_1 \cap ... \cap E_n \cap E_{n+1}$ is open thereby closing the induction.

1.4.1

Suppose hypothesis. Let $(x^{(n_j)})_{j=1}^{\infty}$ be a subsequence of $(x^{(n)})_{n=m}^{\infty}$. Since

$(x^{(n)})_{n=m}^{\infty}$ converges to x_0, we have $d(x^{(n)}, x_0) \leq \varepsilon$ for all $n \geq N$ for some $N \geq m$. But we have $n_j \geq N$ for all $j \geq N$, this means that $d(x^{(n_j)}, x_0) \leq \varepsilon$ for all $j \geq N$. Since ε was arbitrary we can conclude that the subsequence does converge to x_0 and since the subsequence was also arbitrary, we can then conclude that all subsequences of the convergent sequence is also convergent to x_0 as required.

1.4.2

Suppose L is a limit point of $(x^{(n)})_{n=m}^{\infty}$. Let n_j be defined inductively as $n_1 = n$ such that $n \geq m$ where $d(x^{(n)}, L) \leq 1$ and $n_{j+1} = n$ such that $n \geq n_j + 1$ where $d(x^{(n)}, L) \leq 1/(j+1)$. The sequence $(x^{(n_j)})_{j=1}^{\infty}$ is then a subsequence of $(x^{(n)})_{n=m}^{\infty}$. Let $\varepsilon > 0$ we then have some $N \geq 1$ such that $\varepsilon > 1/N > 0$, but $d(x^{(n_j)}, L) \leq 1/N$ for all $j \geq N$. We can therefore conclude that $(x^{(n_j)})_{j=1}^{\infty}$ converges to L and so there exists a subsequence of $(x^{(n)})_{n=m}^{\infty}$ that converges to L as required.

Suppose there exists a subsequence $(x^{(n_j)})_{j=1}^{\infty}$ of $(x^{(n)})_{n=m}^{\infty}$ which converges to L. Then $d(x^{(n_j)}, L) \leq \varepsilon$ for all $j \geq N$ for some $N \geq 1$. Let $M \geq m$. Then either $M \geq N$ or $M < N$. In the case $M \geq N$, we have $M + 1 \geq N$ and so $d(x^{(n_{M+1})}, L) \leq \varepsilon$, where $n_{M+1} \geq M + 1$. In the case $M < N$, we have $N \geq N$ and so $d(x^{(n_N)}, L) \leq \varepsilon$, where $n_N \geq N > M$. In both cases we have, for $M \geq m$, there exists an $n \geq M$ such that $d(x^{(n)}, L) \leq \varepsilon$. We can therefore conclude that L is indeed a limit point of $(x^{(n)})_{n=m}^{\infty}$ as required.

1.4.3

Suppose hypothesis. Then $d(x^{(n)}, x_0) \leq \varepsilon/3$ for all $n \geq N$ for some $N \geq m$. Let $j \geq N$, we then have $d(x^{(j)}, x_0) \leq \varepsilon/3$ and so $d(x^{(n)}, x^{(j)}) \leq 2\varepsilon/3$. Since j was chosen arbitrary we can say $d(x^{(n)}, x^{(j)}) \leq 2\varepsilon/3 < \varepsilon$ for all $n, j \geq N$. Since $\varepsilon > 0$ was chosen arbitrary we are done and therefore $(x^{(n)})_{n=m}^{\infty}$ is indeed cauchy as required.

1.4.4

Suppose hypothesis. Then x_0 is a limit point of $(x^{(n)})_{n=m}^{\infty}$. Since the sequence is Cauchy, we have $d(x^{(j)}, x^{(n)}) < \varepsilon/2$ for all $j, n > N$ for some $N \geq m$. But since x_0 is a limit point of the sequence we also have $d(x^{(l)}, x_0) \leq \varepsilon/2$ for some $l \geq N$. But that also means $d(x^{(l)}, x^{(n)}) \leq \varepsilon$. We then have by triangle inequality, $d(x^{(n)}, x_0) < \varepsilon$ for all $n \geq N$. We can therefore conclude that $(x^{(n)})_{n=m}^{\infty}$ converges to x_0 as required.

1.4.5

Suppose L is a limit point of the sequence $(x^{(n)})_{n=m}^{\infty}$, then there exists a subsequence of the original sequence such that $(x^{(n_j)})_{j=1}^{\infty}$ converges to L. We

know that $n_j \geq m$ for all j so $x^{n_j} \in \{x^{(n)} : n \geq m\}$. The sequence $(x^{(n_j)})_{j=1}^{\infty}$ is then contained in $\{x^{(n)} : n \geq m\}$ and converges to L, thereby making L an adherent point of the set as required.

We know $x^{(m)}$ is an adherent point of $\{x^{(n)} : n \geq m\}$. Suppose for contradiction that the converse of the above is true. Then $x^{(m)}$ is a limit point of $(x^{(n)})_{n=m}^{\infty}$. But since $(x^{(n)})_{n=m}^{\infty}$ is any sequence of any set X, we can define $(x^{(n)})_{n=m}^{\infty}$ as $x^{(n)} = x^{(m)}$ for $n = m$ and $d(x^{(n)}, x^{(m)}) \geq 1$ otherwise for some set X containing $x^{(m)}$ and $x^{(n)}$ such that $d(x^{(n)}, x^{(m)}) \geq 1$. This obviously leads to a contradiction because for all $n \geq m+1$, $d(x^{(n)}, x^{(m)}) \geq 1$ but by hypothesis we have an $n \geq m+1$ such that $d(x^{(n)}, x^{(m)}) \leq 0.5$ and so the converse is false.

1.4.6

Let $(x^{(n)})_{n=m}^{\infty}$ be a Cauchy sequence of points in the metric space (X, d). Suppose for contradiction that $(x^{(n)})_{n=m}^{\infty}$ has more than one limit point $L > 1$. Then there exists L subsequences that converge to different numbers. Consider two of these subsequences, $(x^{(n_j)})_{j=1}^{\infty}$ and $(x^{(n_k)})_{k=1}^{\infty}$ that converges to Q and P respectively. $Q \neq P$. The original sequence $(x^{(n)})_{n=m}^{\infty}$ then converges to both Q and P. But a sequence cannot converge to different points and so we have a contradiction. Therefore the sequence can have at most 2 limit points.

1.4.7

A) Suppose $(Y, d|_{Y \times Y})$ is complete. Let x_0 be a point of X that is an adherent point of Y. So there is a sequence $(y^{(n)})_{n=m}^{\infty}$, consisting of elements in Y that converges to x_0. But we know that all convergent sequences are Cauchy, so $(y^{(n)})_{n=m}^{\infty}$ is a Cauchy sequence in Y. But by hypothesis, $(y^{(n)})_{n=m}^{\infty}$ converges in Y and therefore we have, all adherent points of Y is in Y. Therefore Y is closed in X

B) Suppose (X, d) is a complete metric space and Y is a closed subset of X. Since Y is a closed subset of X, all adherent points of Y is in Y. Since $Y \subseteq X$ a cauchy sequence in Y is a cauchy sequence in X. So every cauchy sequence in Y converges in X. But that means every cauchy sequence in Y is convergent and by definition of closed, must converge in Y. We therefore have $(Y, d|_{Y \times Y})$ is indeed complete.

1.5.1. Let $(\mathbf{R}, d)$ be a metric space with the standard metric. Let X be a subset of $\mathbf{R}$. Suppose X is bounded by definition 1.5.3, we then have $Y \subseteq B(x, r)$ for some $x \in \mathbf{R}$ and some $r > 0$. Then $d(y, x) < r$ and so $|y - x| < r$ for all y. Then $-r + x \leq y \leq r + x$. So for $M = r + x + 1$, we

have $y \in [-M, M]$ and since $M \neq y$ for all y we have $Y \subset [-M, M]$. But that means Y is bounded by definition 9.1.22 as required.

Let X be a subset of the real line that is bounded by definition 9.1.22. We know that $(\mathbf{R}, d)$ is a metric space and by definition $X \subset [-M, M]$ for some $M > 0$. Consider the ball $B(0, M + 1)$. Let $x \in X$, we then have $|x| \leq M$, but that also means $|x| < M + 1$, and since $d(0, x) = |x|$, we have $x \in B(0, M+1)$. This means $X \subseteq B(0, M+1)$ and so we have X is bounded by definition 1.5.3

1.5.2

Suppose (X, d) is a compact metric space. Let $(x_n)_{n=m}^{\infty}$ be a Cauchy sequence in X. Since $(x_n)_{n=m}^{\infty}$ is a sequence in X, it also has the property that there exists a subsequence of it that converges. But that means it must converge in X. That in turn means $(x_n)_{n=m}^{\infty}$ must also converge in X thereby making (X, d) complete. Now suppose for contradiction that for all $B(x, r)$ in X, there exists a $y \in X$ such that $y \notin B(x, r)$. Fix $x \in X$ and consider the set $B_n = \{y \in X : y \notin B(x, n)\}$. This is always non empty. Now using the axiom of choice define the sequence $(x_n)_{n=1}^{\infty}$ recursively as $x_n = y$ for some $y \in B_n$. This sequence is obviously in X and so we must have a subsequence that converges to some $x_0 \in X$. We have $2d(x_0, x) \leq N$ for some natural number N. But there exists an $n_0 \geq 2N$ such that $d(x_0, x_{n_0}) \leq d(x_0, x)$. That means $d(x, x_{n_0}) \leq 2d(x_0, x) \leq N$ by the triangle inequality. But we have $d(x, x_{n_0}) \geq n_0$ and so $d(x, x_{n_0}) > 2N$, a contradiction. Therefore X is indeed bounded.

1.5.4

Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined as $f(x) = x$ for $x \in (-1, 1)$ and $f(x) = 1$ otherwise. This is continuous. Now consider the open set $V = (-2, 2) \subseteq \mathbf{R}$ which is open. The image of this set would be $f(V) = [-1, 1]$. This set is not open since the boundary points $-1, 1$ are both in the set.

1.5.6

Suppose for contradiction that $\bigcap_{n=1}^{\infty} K_n = \emptyset$. Consider the compact metric space $(K_1, d|_{K_1, K_1})$ and the sets $V_n = K_1 - K_n$. These sets are open in K_1. We have $\bigcup_{n=1}^{\infty}(K_1 - K_n) = K_1 - \bigcap_{n=1}^{\infty}(K_n)$. But since $\bigcap_{n=1}^{\infty}(K_n) = \emptyset$, $\bigcup_{n=1}^{\infty}(K_1 - K_n) = K_1$. We know that K_1 is compact and is also a subset of $\bigcup_{n=1}^{\infty}(K_1 - K_n)$ by definition of equality. So by theorem 1.5.8, we have a finite subset $F \subseteq \mathbf{N}$ such that $K_1 \subseteq \bigcup_{n \in F}(K_1 - K_n)$. But because F is a finite subset of $\mathbf{N}$, F is bounded meaning that there exists an $M \in \mathbf{N}$ such that for all $x \in F$, $x \leq M$. Consider the set $K_1 - K_M$. Let $n \in F$. Let $x \in K_1 - K_n$. Then $x \in K_1$ and $x \notin K_n$. But because $K_M \subseteq K_n$, $x \notin K_M$. Therefore

$x \in K_1 - K_M$. That means $K_1 - K_n \subseteq K_1 - K_M$ for all $n \in F$. Now let $x \in K_1$, then $x \in \bigcup_{n \in F}(K_1 - K_n)$ and so $x \in (K_1 - K_n)$ for some $n \in F$. This means that $x \in K_1 - K_M$. We can then conclude that $K_1 \subseteq K_1 - K_M$. But because $K_1 - K_M \subseteq K_1$, we have $K_1 = K_1 - K_M$. But K_M subseteq K_1 so for an $x \in K_M$, $x \in K_1 - K_M$. Since K_M is non empty, there is an $x \in K_M$ and it must also $x \in K_1 - K_M$. So $x \in K_1$ and $x \notin K_M$, which is a contradiction. Therefore $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$ as required.

1.5.7

A) Let Y be a compact subset of X, and $Z \subseteq Y$

Suppose Z is compact. Let x be an adherent point of Z. Then there exists a sequence $(x_n)_{n=m}^{\infty}$ in Z that converges to x. Since this sequence is in Z and Z is compact, there exists a subsequence of it that converges in Z. But because $(x_n)_{n=m}^{\infty}$ converges to x, all of it's subsequences must also converge to x, therefore x convereges in Z. So all adherent points of Z are in Z meaning that Z is closed as required.

Suppose Z is closed. Let $(x_n)_{n=m}^{\infty}$ be a sequence in Z. This sequence is also in Y since $Z \subseteq Y$. This in turn means that $(x_n)_{n=m}^{\infty}$ has a subsequence that converges in Y since Y is compact. But this subsequence is a convergent sequence in Z. Since Z is closed, this subseqence must converge in Z. Therefore we can conclude that Z is compact as required.

B) Suppose $Y_1, ..., Y_n$ are a finite collection of compact subsets of X. Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of the set $\bigcup_{i=1}^{n} Y_i$. Consider the sets $D_i = \{n \in \mathbf{N} : x^{(n)} \in Y_i\}$ for $1 \leq i \leq n$. Suppose for contradiction that all D_i are finite. The union $\bigcup_{i=1}^{n} D_i$ is then finite. But we have for $n \in \mathbf{N}$, $x^{(n)} \in Y_i$ for some $1 \leq i \leq n$ and so $n \in D_i$ for some $1 \leq i \leq n$ which in turn means $n \in \bigcup_{i=1}^{n} D_i$. This means that $\mathbf{N} \subseteq \bigcup_{i=1}^{n} D_i$, and we also have $\bigcup_{i=1}^{n} D_i \subseteq \mathbf{N}$. So $\bigcup_{i=1}^{n} D_i = \mathbf{N}$. But because $\bigcup_{i=1}^{n} D_i$ is finite, it is bounded by some M, and so we have $M + 1 \in \mathbf{N}$ while $M + 1 \notin \bigcup_{i=1}^{n} D_i$ a contradiction. Therefore there exists a D_i such that D_i is infinite. Let D_{i_0} be this infinite set. Consider the sequence $(x^{(n_j)})_{j=1}^{\infty}$ defined as so, $n_1 = min(D_{i_0})$ and $n_{j+1} = min\{x \in D_{i_0} : x > n_j\}$. This sequence is a subsequence of $(x^{(n)})_{n=m}^{\infty}$ and is in Y_{i_0} so this subsequence converges in Y_{i_0} since Y_{i_0} is compact. This in turn means that the subsequence converges in $\bigcup_{i=1}^{n} Y_i$. Therefore we can conclude that all sequences of $\bigcup_{i=1}^{n} Y_i$ has a subsequence which converges in $\bigcup_{i=1}^{n} Y_i$ and so $\bigcup_{i=1}^{n} Y_i$ is compact as required.

C) First we will prove that all singleton subsets of X is compact. Suppose $\{x\} \subseteq X$. Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence in $\{x\}$. Since $x_n \in \{x\}$, it must equal

x, and so we have $(x^{(n)})_{n=m}^{\infty} = (x)_{n=m}^{\infty}$. This sequence has the subsequence $(x)_{n=m}^{\infty}$, which converges to $x \in \{x\}$. Therefore we can conclude that all singleton subsets of X are indeed compact.

Suppose Y is a finite subset of X. Y then has some cardinality $N \in \mathbf{N}$ and a bijection $f : Y \to \{n \in \mathbf{N} : 1 \le n \le N\}$. Consider the set $L_i = \{y \in Y : f(y) = i\}$ for $1 \le i \le N$. This is a singleton and is therefore finite. Now consider the union $\bigcup_{n=1}^{N} L_i$. Let $x \in Y$, we then have $f(y) = i$ for some $1 \le i \le N$. So $x \in L_i$. This in turn means $x \in \bigcup_{n=1}^{N} L_i$. So $Y \subseteq \bigcup_{n=1}^{N} L_i$ and since $\bigcup_{n=1}^{N} L_i \subseteq Y$, we have $Y = \bigcup_{n=1}^{N} L_i$, but because $\bigcup_{n=1}^{N} L_i$ is a union of finite compact subsets of X, we have $\bigcup_{n=1}^{N} L_i$ is compact and so Y is compact as required.

1.5.8

Suppose $x \in \{e^{(n)} : n \in \mathbf{N}\}$. Then $x = e^{(n)} = (e_j^{(n)})_{j=0}^{\infty}$ and we have $\sum_{j=0}^{\infty} e_j^{(n)} = \sum_{j=0}^{n-1} e_j^{(n)} + e_n^{(n)} + \sum_{j=n+1}^{\infty} e_j^{(n)} = 0 + 1 + 0 = 1$. But since $e_j^{(n)} \ge 0$ for all j, we have $|e_j^{(n)}| = e_j^{(n)}$ and so $\sum_{j=0}^{\infty} |e_j^{(n)}| = \sum_{j=0}^{\infty} e_j^{(n)} = 1$. This makes it so that the sequence is absolutely convergent and so we can conclude that $x \in X$, making $\{e^{(n)} : n \in \mathbf{N}\} \subseteq X$.

Let $(x^{(m)})_{m=p}^{\infty}$ be a convergent sequence in $\{e^{(n)} : n \in \mathbf{N}\}$. Since the sequence is convergent, it must also be Cauchy. We then have $d(x^{(m)}, x^{(z)}) \le 1/2$ for all $m, z \ge N$ for some $N \ge p$. So $d(x^{(m)}, x^{(z)}) = \sum_{j=0}^{\infty} |e_j^{(l)} - e_j^{(k)}| \le 1/2$ where $x^{(m)} = (e_j^{(l)})_{j=0}^{\infty}$ and $x^{(z)} = (e_j^{(k)})_{j=0}^{\infty}$. Suppose for contradiction that $l \ne k$, we then have $e_l^{(l)} = 1$, but $e_l^{(k)} = 0$. This means $e_l^{(l)} - e_l^{(k)} = 1$ and so $|e_l^{(l)} - e_l^{(k)}| = 1$. But $\sum_{j=0}^{\infty} |e_j^{(l)} - e_j^{(k)}| = \sum_{j=0}^{l-1} |e_j^{(l)} - e_j^{(k)}| + |e_l^{(l)} - e_l^{(k)}| + \sum_{j=l+1}^{l-1} |e_j^{(l)} - e_j^{(k)}|$, and since all elements are positive, $0 + |e_l^{(l)} - e_l^{(k)}| = 1 \le \sum_{j=0}^{\infty} |e_j^{(l)} - e_j^{(k)}|$, a contradiction. Therefore $l = k$ and so $(e_j^{(l)})_{j=0}^{\infty} = (e_j^{(k)})_{j=0}^{\infty}$. This then means $e_j^{(l)} = e_j^{(k)}$ and so $e_j^{(l)} - e_j^{(k)} = 0 = |e_j^{(l)} - e_j^{(k)}|$. We then have $\sum_{j=0}^{\infty} |e_j^{(l)} - e_j^{(k)}| = 0$. We then have for $d(x^{(N)}, x^{(m)}) = 0 \le \varepsilon$ for all $m \ge N$. So we can conclude that for any convergent sequence $(x^{(m)})_{m=p}^{\infty}$ in $\{e^{(n)} : n \in \mathbf{N}\}$, $(x^{(m)})_{m=p}^{\infty}$ converges to $x^{(N)}$ for some $N \ge p$. But $x^{(N)} \in \{e^{(n)} : n \in \mathbf{N}\}$ and so we can conclude that $\{e^{(n)} : n \in \mathbf{N}\}$ is closed.

Now consider the ball $B(e^{(0)}, 3)$. Suppose $x \in \{e^{(n)} : n \in \mathbf{N}\}$, we then have $d(e^{(0)}, x) = \sum_{j=0}^{\infty} |e_j^{(0)} - e_j^{(n)}|$. We have either $n = 0$ or $n \ne 1$, in the case $n = 0$, we have $\sum_{j=0}^{\infty} |e_j^{(0)} - e_j^{(n)}| = \sum_{j=0}^{\infty} |e_j^{(0)} - e_j^{(0)}| = \sum_{j=0}^{\infty} 0 = 0$ and so $d(e^{(1)}, x) = 0 < 3$ therefore $x \in B(e^{(0)}, 3)$ In the case $n \ne 0$, we have $d(e^{(0)}, x) = \sum_{j=0}^{\infty} |e_j^{(0)} - e_j^{(n)}| = |e_0^{(0)} - e_0^{(n)}| + \sum_{j=1}^{n-1} |e_j^{(0)} - e_j^{(n)}| + |e_n^{(0)} - e_n^{(n)}| +$

$\sum_{j=n+1}^{\infty} |e_j^{(0)} - e_j^{(n)}| = 1 + 0 + 1 + 0 = 2$. So $d(e^{(0)}, x) = 2 < 3$ therefore $x \in B(e^{(0)}, 3)$. We can therefore conclude that $\{e^{(n)} : n \in \mathbf{N}\} \subseteq B(e^{(0)}, 3)$ and therefore is bounded.

Consider the sequence $(e^{(n)})_{n=0}^{\infty}$, this sequence is in $\{e^{(n)} : n \in \mathbf{N}\}$. Let $(e^{(n_j)})_{j=0}^{\infty}$ be a subsequence of it. Let $N \geq 0$ then for $k > j \geq N$, $d(e^{(n_j)}, e^{(n_k)}) = 2$ and so for all $N \geq 0$, we have $d(e^{(n_j)}, e^{(n_k)}) > 1$ for $j \geq N$ and $k = j + 1$. This means that $(e^{(n_j)})_{j=0}^{\infty}$ isn't Cauchy and therefore can't be convergent. Since $(e^{(n_j)})_{j=0}^{\infty}$ is an arbitrary subsequence of $(e^{(n)})_{n=0}^{\infty}$, we can conclude that all subsequences of $(e^{(n)})_{n=0}^{\infty}$ does not converge in $\{e^{(n)} : n \in \mathbf{N}\}$. This in turn means we can conclude that $\{e^{(n)} : n \in \mathbf{N}\}$ is not compact as required.

1.5.9

Suppose (X, d) is compact. Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence in X. Since X is compact, this sequence must have a subsequence $(x^{(n_j)})_{j=0}^{\infty}$ that converges to some $x_0 \in X$. That in turn means $(x^{(n)})_{n=m}^{\infty}$ has a limit point x_0. So we can conclude that all sequences in X has at least 1 limit point.

Let (X, d) be a metric space. Suppose every sequence in X has at least one limit point. Let $(x^{(n)})_{n=m}^{\infty}$. This sequence has a limit point L and so there must be a subsequence which converges to L. L must be in X because the metric d is not defined outside of X. Therefore for all sequences in X, there exists a subsequence that converges in X and so we can conclude that (X, d) is indeed compact as required.

1.5.10

A) Suppose (X, d) is totally bounded. Suppose for contradiction that X is not bounded. Then for all balls $B(x, r)$ in X, X is not a subset of $B(x, r)$. Since (X, d) is totally bounded, we have, for some positive integer n, and finite number of balls $B(x^{(1)}, \varepsilon), ..., B(x^{(n)}, \varepsilon)$, $X = \bigcup_{i=1}^{n} B(x^{(i)}, \varepsilon)$. But X is not a subset of $B(x^{(i)}, \varepsilon)$ for all $1 \leq i \leq n$. Define $z = \sum_{i=1}^{n} d(x^{(1)}, x^{(i)})$. Consider the ball $B(x^{(1)}, z + 2\varepsilon)$. Let $x \in X$, then $x \in B(x^{(i)}, \varepsilon)$ for some $1 \leq i \leq n$. That then means $d(x^{(i)}, x) \leq \varepsilon$. So $d(x^{(1)}, x) \leq d(x^{(i)}, x) + d(x^{(1)}, x^{(i)}) \leq \varepsilon + d(x^1, x^{(i)})$. But $\varepsilon + d(x^1, x^{(i)}) \leq z + 2\varepsilon$. So we have $d(x^{(1)}, x) \leq z + 2\varepsilon$ and therefore $x \in B(x^{(1)}, z + 2\varepsilon)$. We therefore have $X \subseteq B(x^{(1)}, z + 2\varepsilon)$. But this contradicts the hypothesis that X is not bounded and so we must have X is bounded.

B) Suppose (X, d) is compact. Then from P1.5.5, it is complete and bounded. Suppose for contradiction that (X, d) is not totally bounded. We then have for some $\varepsilon > 0$, X cannot be covered by finitely many $B(x, \varepsilon)$ balls.

Consider the set $\bigcup_{x \in X} B(x, \varepsilon)$. Suppose $x \in X$, we then have $x \in B(x, \varepsilon)$ since $d(x, x) = 0 \leq \varepsilon$. This means $X = \bigcup_{x \in X} B(x, \varepsilon)$. But since we can't have a finite collection of $B(x, \varepsilon)$ that covers X, we must have X is infinite. Now consider the set of all balls $V = \{B(x, \varepsilon/2) : x \in X\}$. This set is infinite since X is infinite. It is also $V \subset 2^X$. We then have a subset of $Y \subseteq V$ such that for all $y \in Y$ and all $z \in Y$, $z \cap y = \emptyset$. Also for all $l \in V$, there exists a $h \in Y$ such that $l \cap h \neq \emptyset$.(Where to go?)

Alternate proof. Suppose (X, d) is compact. Suppose for contradiction that (X, d) is not totally bounded. We then have for some $\varepsilon > 0$, We then have for some $\varepsilon > 0$, X cannot be covered by finitely many $B(x, \varepsilon)$ balls. Consider the set $\bigcup_{x \in X} B(x, \varepsilon)$. Suppose $x \in X$, we then have $x \in B(x, \varepsilon)$ since $d(x, x) = 0 \leq \varepsilon$. This means $X = \bigcup_{x \in X} B(x, \varepsilon)$. But since we can't have a finite collection of $B(x, \varepsilon)$ that covers X, we must have X is infinite. Since balls are open sets we also must have a finite subset of $F \subseteq X$ such that $X \subseteq \bigcup_{x \in F} B(x, \varepsilon)$. This is a contradiction since we can't have a finite set of $\varepsilon - balls$ that cover X and so we must have X is totally bounded.

2 Continuous Functions on Metric Spaces

2.1.3

A) Suppose hypothesis. We then have for every open set that contains $g(f(x_0))$ there is an open subset $U \subset Y$ containing $f(x_0)$ such that $g(U) \subseteq Z$. But since U is an open proper subset of Y that contains $f(x_0)$, there then exists an open subset $P \subset X$ containing x_0 such that $f(P) \subseteq Y$. But since $f(P) \subseteq Y$, we have $g(f(P)) \subseteq Z$. So we have, for every open set that contains $g(f(x_0))$ there is an open subset $P \subset X$ containing x_0 such that $g(f(P)) \subseteq Z$. This in turn means that $g \circ f$ is indeed continuous at x_0 as required.

B) Suppose hypothesis. Let V be an open set in Z we then have $g^{-1}(V)$ is an open set. But $g^{-1}(V) \subseteq Y$ and so we also have $f^{-1}(g^{-1}(V))$ is an open set. But $f^{-1}(g^{-1}(V)) \subseteq X$. So we have for any open set V in Z, the set $f^{-1}(g^{-1}(V))$ is an open subset of X. Therefore $g \circ f$ is continuous as required.

2.1.4

A) consider the function $f : \mathbf{R} \to \mathbf{R}$, defined as $f(x) = x$ for all $x < 0$ and $f(x) = x + 1$ for all $x >= 0$. There is a discontinuity at $x = 0$. Let $g : \mathbf{R} \to \mathbf{R}$ be defined as $g(x) = 0$. This is continuous. The composition

$g \circ f : \mathbf{R} \to \mathbf{R}$ is defined as $g(f(x))$ and so $g(f(x)) = 0$ for all $x \in \mathbf{R}$. Therefore it is continuous.

B) consider the function $g : \mathbf{R} \to \mathbf{R}$ defined as $g(x) = 1$ for all $x \geq 0$, and $g(x) = 0$ for all $x < 0$. Now consider the function $f : \mathbf{R} \to \mathbf{R}$ defined as $f(x) = 1$. We then have $g(f(x)) = g(1) = 1$ for all x. So $g \circ f$ is continuous

C) consider the function $g : \mathbf{R} \to \mathbf{R}$ defined as $g(x) = 1$ for all $x \geq 0$ and $g(x) = 0$ for all $x < 0$. Now consider the function $f : \mathbf{R} \to \mathbf{R}$ defined as $f(x) = 1$ for all $x \geq 0$ and $f(x) = 2$ for all $x < 0$. We then have $g(f(x)) = g(1) = 1$ for all $x \geq 0$ and $g(f(x)) = g(2) = 1$ for all $x < 0$. Therefore $g \circ f$ is continuous as required.

2.1.5

Suppose hypothesis. Let $x_0 \in E$. Let $\varepsilon > 0$. Now consider $\delta = \varepsilon > 0$. Consider the ball $B(x_0, \delta)$ in E. For $x \in B(x_0, \delta)$, $d_{E \times E}(x, x_0) = d(x, x_0) < \delta$. We then have $\iota_{E \to X}(x) = x$ and $\iota_{E \to X}(x_0) = x_0$ so $d(\iota_{E \to X}(x), \iota_{E \to X}(x_0)) = d(x, x_0) < \delta = \varepsilon$. Since ε is arbitrary we can then conclude that for all $\varepsilon > 0$, there exist a $\delta > 0$ such that $d(\iota_{E \to X}(x), \iota_{E \to X}(x_0)) < \varepsilon$ for all x such that $d_{E \times E}(x, x_0) < \delta$. Therefore $\iota_{E \to X}$ is continuous at x_0. But since x_0 is arbitrary in E we have $\iota_{E \to X}$ is continuous at every point in E and so is continous as required.

2.1.6

Suppose hypothesis. Consider the function $f \circ \iota_{E \to X} : E \to Y$. Let $x \in E$. Then $f(\iota_{E \to X}(x)) = f(x)$, but we also have $f|_{E \times E}(x) = f(x)$ and so we have $f \circ \iota_{E \to X} = f|_{E \times E}(x)$. But since $\iota_{E \to X}$ is continuous, it is continuous at x_0. Since the composition of two functions continuous at x_0 results in a function that is also continuous at x_0, we have $f \circ \iota_{E \to X}$ is also continuous at x_0. This in turn means $f|_{E \times E}$ is continuous at x_0 as required.

The converse is not true because suppose for contradiction that it is. Now consider the function $g : \mathbf{R} \to \mathbf{R}$ defined as $g(x) = 1$ for all $x \neq 0$ and $g(0) = 0$. Now consider the restriction $g|_{\{0\}}$. This is continuous at 0 because for all ε, there exists $\delta = 1$, such that $d(f(x), f(0)) < \varepsilon$ When $d|_{\{0\}}(x, 0) < 1$. This is since $x = 0$ is the only $x \in \{0\}$. But the original function g is obviously not continuous, a contradiction. So the converse cannot be true.

2.2.1

A) Suppose $x_0 \in X$

Suppose f and g are both continuous at x_0. Let $((x_n))_{n=1}^{\infty}$ be a sequence in X that converges to x_0 in accordance with the d_X metric. We have $(f(x_n))_{n=1}^{\infty}$ and $(g(x_n))_{n=1}^{\infty}$ converging to $f(x_0)$ and $g(x_0)$ respectively. This is using the standard metric on $\mathbf{R}$. Now consider the sequence $(f \oplus g(x_n))_{n=1}^{\infty}$ which

is equivalent to $((f(x_n), g(x_n)))_{n=1}^{\infty}$. Let $\varepsilon > 0$. We have for all $n \geq N_1$, $|f(x_n) - f(x_0)| \leq \varepsilon/2$ for some $N_1 \geq 1$. Also for all $n \geq N_2$, $|g(x_n) - g(x_0)| \leq \varepsilon/2$ for some $N_1 \geq 1$. Let $N = max(N_1, N_2)$. Then for all $n \geq N$, $|f(x_n) - f(x_0)|, |g(x_n) - g(x_0)| \leq \varepsilon/2$. We then have $|f(x_n) - f(x_0)| + |g(x_n) - g(x_0)| \leq \varepsilon$ for all $n \geq N$. But this is just the taxicab metric on $\mathbf{R}^2$. So we can conclude that $((f(x_n), g(x_n)))_{n=1}^{\infty}$ converges to $(f(x_0), g(x_0))$ with respect to the taxi cab metric. But because of the equivalence between the taxicab and Euclidean metric, we also have $((f(x_n), g(x_n)))_{n=1}^{\infty}$ converges to $(f(x_0), g(x_0))$ with respect to the Euclidean metric. This means that $(f \oplus g(x_n))_{n=1}^{\infty}$ converges to $(f \oplus g(x_0))$ with respect to the Euclidean metric. We can then conclude that $f \oplus g$ is indeed continuous at x_0.

Suppose $f \oplus g$ is continuous at x_0. Let $(x_n)_{n=1}^{\infty}$ be a sequence in X that converges to x_0. Then $((f(x_n), g(x_n)))_{n=1}^{\infty}$ converges to $(f(x_0), g(x_0))$ with respect to the Euclidean metric. But because of the equivalence between the Euclidean and taxi-cab metric, we have that the sequence also converges to $(f(x_0), g(x_0))$ with respect to the taxi-cab metric. Let $\varepsilon > 0$, we then have for all $n \geq N$, $|f(x_n) - f(x_0)| + |g(x_n) - g(x_0)| \leq \varepsilon$ for some $N \geq 1$. But since both $|f(x_n) - f(x_0)|, |g(x_n) - g(x_0)| > 0$ we have $|f(x_n) - f(x_0)|, |g(x_n) - g(x_0)| \leq \varepsilon$. That in turn means $(f(x_n))_{n=1}^{\infty}$ and $(g(x_n))_{n=1}^{\infty}$ converges to $f(x_0)$ and $g(x_0)$ respectively. So we can conclude that f and g are both continuous at x_0.

B) Suppose f and g are both continuous. Let $x_0 \in X$, we then have f and g are both continuous at x_0. But by A, this means $f \oplus g$ is also continuous at x_0. But since x_0 was arbitrary in X, we have $f \oplus g$ is continuous for all $x \in X$ and so we have $f \oplus g$ is continuous as required.

Suppose $f \oplus g$ is continuous. Let $x_0 \in X$. Then $f \oplus g$ is continuous at x_0. But by A, this means f and g are also continuous at x_0. Since x_0 was arbitrary in X, We have f and g are continuous for all $x \in X$. So f and g are both continuous as required.

2.2.2

Let $(x_0, y_0) \in \mathbf{R}^2$. Let $((x^{(n)}, y^{(n)}))_{n=1}^{\infty}$ be a sequence in $\mathbf{R}^2$ that converges to (x_0, y_0). Due to the equivalence of the taxi-cab and Euclidean metric, we have $|x_n - x_0| + |y_n - y_0| \leq \varepsilon$ for all $n \geq N$ for some $N \geq 1$. This means $|x_n - x_0|, |y_n - y_0| \leq \varepsilon$ and so $(x^{(n)})_{n=1}^{\infty}$ and $(y^{(n)})_{n=1}^{\infty}$ converge to x_0 and y_0 respectively. But by limit law we then have $(x^{(n)} + y^{(n)})_{n=1}^{\infty}$ converge to $x_0 + y_0$. So we have that addition is continuous at (x_0, y_0), but since (x_0, y_0) was arbitrary in $\mathbf{R}^2$. We have addition is continuous as required. The rest can be proven similarly

2.3.1

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Suppose hypothesis. Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence in $f(K)$. Define the set $K^{(n)} = \{x \in X : f(x) = f^{(n)}\}$. These are not empty because $f^{(n)} \in f(K)$ and so there exists an $x \in K$ such that $f(x) = f^{(n)}$. By the axiom of choice we then have a sequence $(x^{(n)})_{n=1}^{\infty}$ where $x^{(n)} \in K^{(n)}$. This sequence is in K and so has a subsequence $(x^{(n_j)})_{j=1}^{\infty}$ which converges to some x_0 in K. But f is continuous so we have $(f(x^{(n_j)}))_{j=1}^{\infty}$ converges to $f(x_0)$. Since $x_0 \in K$, we have $f(x_0) \in f(K)$ and since $f(x^{(n_j)}) = f^{(n_j)}$, $(f(x^{(n_j)}))_{j=1}^{\infty}$ is a subsequence of $(f^{(n)})_{n=1}^{\infty}$. So there exists a subsequence of $(f^{(n)})_{n=1}^{\infty}$ that converges in $f(K)$. Since the sequence was arbitrary in $f(K)$, we have $f(K)$ is compact as required.

2.3.2

Suppose hypothesis. Since $X \subseteq X$ and X is compact, we have $f(X)$ is compact. Therefore it is bounded and there exists a $y \in f(X)$ such that $f(X) \subseteq B(y, r)$ for some $r > 0$. Now let $x \in X$. Then $f(x) \in f(X)$ and so $f(x) \in B(y, r)$ This means $|f(x) - y| \le r$. Which in turn means $-r + y \le f(x) \le r + y$. Since $-|y| \le y \le |y|$, we can conclude $-r - |y| \le f(x) \le r + |y|$, but that means $|f(x)| \le r + |y|$ a real number. Since x was arbitrary we have, for all $x \in X$, $|f(x)| \le r + |y|$ and so the function is indeed bounded. Since $f(X)$ is bounded by some $M > 0$, we have $sup(f(X)) \le M$. Since $f(X)$ is compact, it is also closed so all adherent points of $f(X)$ are in $f(X)$. But we know $sup(f(X))$ is an adherent point. We then have $sup(f(X)) \in f(X)$ and for all $f(x) \in f(X)$, $f(x) \le sup(f(X))$. Also there exists an $x_0 \in X$ such that $f(x_0) = sup(f(X))$ and so $f(x) \le f(x_0)$. We therefore have f attains a maximum at some point $x_0 \in X$ as required. Minimum can be proven similarly.

2.3.3

Let $f : X \to Y$ be a map from (X, d_X) to (Y, d_Y), two metric spaces. Suppose f is uniformly continuous. Let $x_0 \in X$, We then have for all $\varepsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x_0, x)) < \varepsilon$ when $x \in X$ such that $d_X(x_0, x) < \delta$. This means that the function is continuous at x_0. But since x_0 is arbitrary in X, we have f is continuous as required.

2.3.4

Suppose hypothesis. Let $\varepsilon > 0$. Then there exists a $\delta_1 > 0$ such that $d_Z(g(y), g(y')) < \varepsilon$ whenever $y, y' \in Y$ are such that $d_Y(y, y') < \delta_1$. We also have a $\delta_2 > 0$ such that $d_Y(f(x), f(x')) < \delta_1$ whenever $x, x' \in X$ are such that $d_X(x, x') < \delta_2$. But since $f(x), f(x') \in Y$, we have $d_Z(g(f(x)), g(f(x'))) < \varepsilon$ whenever $x, x' \in X$ such that $d_X(x, x') < \delta_2$. So we can conclude that $g \circ f$

is indeed uniformly continuous.

2.4.1

Suppose hypothesis. Let $x_0 \in E$. Consider the set $F = \{x_0\}$ and the set $G = \{x \in E : x \neq x_0\}$. These are both subsets of E and non-empty since E has at least 2 elements. Now let $f \in F$. We then have $f = x_0$ and the ball $B(f, 0.5)$ consists of only x_0. We thus have $B(f, 0.5) \subseteq F$ therefore f is an interior point of F. And since $f \in F$ was chosen arbitrarily, we can thus conclude that F is indeed open. Now let $g \in G$. We then have the ball $B(g, 0.5)$ consists of only e and therefore we have $B(g, 0.5) \subseteq G$ and so g is an interior point of G. Since g was arbitrarily chosen in G, we have G is open. Suppose $x \in F$. Then $x = x_0$. This means $x \notin G$. Suppose $x \in G$. Then $x \neq x_0$ and so $x \notin F$. Thereby the two sets are disjoint. We also have for $x \in X$, x is either $x \neq x_0$ or $x = x_0$. In the case $x \neq x_0$, we have $x \in F$. In the case $x = x_0$, we have $x \in G$. So $X \subseteq G \cup F$. We therefore have two disjoint open sets that cover X and so E is disconnected.

2.4.2

Suppose f is continuous. We know that Y is disconnected. So it is covered by two disjoint open sets lets call Z and L. Consider the set $f^{-1}(Z)$ and $f^{-1}(L)$. These sets are open since f is continuous. We either have the two sets are disjoint or they are not. In the case they are disjoint, $f^{-1}(Z) \cup f^{-1}(L)$ does not cover X because X is connected. But we have, for $x \in X$, $f(x) \in Y$. So $f(x) \in Z \cup L$. Meaning $x \in f^{-1}(Z \cup L)$ and so $X \subseteq f^{-1}(Z \cup L)$, a contradiction. In the case $f^{-1}(Z) \cap f^{-1}(L) \neq \emptyset$. Let $x, x_0 \in X$ such that $x \neq x_0$. Suppose for contradiction that $f(x) \neq f(x_0)$. Consider the sets $Z = \{y \in Y : y = f(x)\}$ and $L = \{y \in Y : y \neq f(x)\}$. These two sets are open disjoint and non-empty as shown in exercise 2.4.1 and because $f(x_0) \in L$. So $f^{-1}(Z)$ and $f^{-1}(L)$ are both open. Therefore $f^{-1}(Z) \cap f^{-1}(L) \neq \emptyset$ since X is connected. But that means there exists an $x_1 \in f^{-1}(Z) \cap f^{-1}(L)$. Since $x_1 \in f^{-1}(Z)$, we have $f(x_1) = f(x)$. But we also have $x_1 \in f^{-1}(L)$ and so $f(x_1) \neq f(x)$ a contradiction. Therefore $f(x) = f(x_0)$. Since x and x_0 are arbitrary in X, we can thus conclude that for all $x, x_0 \in X$, $f(x) = f(x_0)$. Therefore f is indeed a constant function.

Suppose f is constant. We then have for all $x, x' \in X$, $f(x) = f(x')$. So given an $x_0 \in X$, let $\varepsilon > 0$. We then have $d(f(x), f(x_0)) = 0 < \varepsilon$ for all $x \in X$. The same can be said about any subset of X. So for any $\delta > 0$, $d(f(x), f(x_0)) = 0 < \varepsilon$ for all $x \in X$ such that $d(x, x_0) < \delta$. Since $x_0 \in X$ was arbitrary, we have f is continuous as required.

2.4.4

Suppose hypothesis. Let F, G be two disjoint open non-empty subsets of $f(E)$. Consider the set $f|_E^{-1}(F)$ and $f|_E^{-1}(G)$. Suppose for contradiction that they aren't disjoint. We then have an $x \in f|_E^{-1}(F) \cap f|_E^{-1}(G)$ such that $f|_E(x) \in F$ and $f(x)|_E \in G$. but F and G are disjoint, a contradiction. Since they are in E, we then have $f|_E^{-1}(F) \cup f|_E^{-1}(G)$ cannot cover E. This in turn means that there is an $x \in E$ such that $x \notin f|_E^{-1}(F) \cup f|_E^{-1}(G)$. So $f(x) \in f(E)$ but $f(x) \notin F \cup G$ and therefore $F \cup G$ does not cover $f(E)$. This is true for all two disjoint open non-empty subsets of $f(E)$ and so we can say $f(E)$ is connected as required.

2.4.5

Suppose hypothesis. We consider the case $f(a) \leq y \leq f(b)$. We have $f(E)$ is connected by T2.4.6. $f(a) \leq f(b)$ so either $f(a) = f(b)$ or $f(a) < f(b)$. In the case $f(a) = f(b)$, we have $y = f(a)$. This in turn means for $a \in E$, we have $f(a) = y$ as required. In the case $f(a) < f(b)$, the interval $[f(a), f(b)] \subseteq f(E)$ since $f(E)$ is connected. That means $y \in f(E)$ and so we have an $x \in E$ such that $f(x) = y$ as required.

2.4.6

Suppose hypothesis. Suppose for contradiction that there exists two disjoint open non-empty subsets of $\bigcup_{\alpha \in I} E_\alpha$, F and L, such that $F \cup L = \bigcup_{\alpha \in I} E_\alpha$. We have $f \in F$ and $l \in L$ since the two sets are non-empty. We also have $f \neq l$ since the two sets are disjoint. Since $f \in \bigcup_{\alpha \in I} E_\alpha$. We have $f \in E_\alpha$ for some $\alpha \in I$. Then either $l \in E_\alpha$ or $l \notin E_\alpha$. In the case $l \in E_\alpha$, we have $E_\alpha \cap F$ is relatively open with respect to E_α since F is open in, and a subset of $\bigcup_{\alpha \in I} E_\alpha$ and $E_\alpha \cap F = F \cap E_\alpha$. Same can be said for $E_\alpha \cap L$. The two sets are also non empty and disjoint so $E_\alpha \cap F \cup E_\alpha \cap L$ does not cover E_α. But $E_\alpha \cap F \cup E_\alpha \cap L = E_\alpha \cap (F \cup L)$ and $E_\alpha \subseteq F \cup L$. So $E_\alpha = E_\alpha \cap F \cup E_\alpha \cap L = E_\alpha$. A contradiction. In the case $l \notin E_\alpha$. We have either $E_\alpha \cap L = \emptyset$ or $E_\alpha \cap L \neq \emptyset$. In the case $E_\alpha \cap L = \emptyset$, we have $E_\alpha \subseteq F$. In the case $E_\alpha \cap L \neq \emptyset$, we have the $l \in E_\alpha$ case again which leads to a contradiction. So we only have $E_\alpha \subseteq F$, when $f \in E_\alpha$. The same can be said for L. Now consider $g \in \bigcap_{\alpha \in I} E_\alpha$. We have either $g \in F$ or $g \in L$. In the case $g \in F$, we have $E_\alpha \subseteq F$ for all $\alpha \in I$. But that means $L = \emptyset$. A contradiction. Same can be said for $g \in L$. So we must have that $\bigcup_{\alpha \in I} E_\alpha$ is connected as required.

2.4.7

Suppose E is path-connected. Let F and L be two non-empty disjoint open subsets of E. Let $x \in F$ and $y \in L$. This can be done since both sets are non-empty. Also $x \neq y$. We have a continuous function $f : [0, 1] \to E$

such that $f(0) = x$ and $f(1) = y$. Since F and L are open, we have $f^{-1}(F)$ and $f^{-1}(L)$ are both open sets in $[0,1]$. Where $0 \in f^{-1}(F)$ and $1 \in f^{-1}(L)$. They are disjoint as well. Since $[0,1]$ is an interval. We know that it must be connected and so $f^{-1}(F) \cup f^{-1}(L)$ does not cover $[0,1]$. So there exists an $x \in [0,1]$ such that $x \notin f^{-1}(F) \cup f^{-1}(L)$. We then have $f(x) \in E$. Suppose for contradiction $f(x) \in F \cup L$. We then have $x \in f^{-1}(F) \cup f^{-1}(L)$ a contradiction. Thus E is connected as required.

2.5.1

It has the property of $\emptyset \in \mathcal{F}$, and $X \in \mathcal{F}$ as required. Also the only intersection of elements in $\mathcal{F}$ is $\emptyset \cap X = X$ which is open. So all finite intersections of elements in X are indeed open as required. Now the only family of sets in $\mathcal{F}$ consists of only $\emptyset$ and X. The intersection of the two is $\emptyset$ which is in $\mathcal{F}$ as required. So it can be concluded that $(X, \mathcal{F})$ is indeed a topology.

Suppose X has more than one element. Consider the metric space (X, d). Since x has more than one element we have $x \in X$ and $y \in X$ such that $x \neq y$. That means $d(x, y) > 0$ so there exists some r such that $d(x, y) > r > 0$. Now consider the ball $B(x, r)$. $y \notin B(x, r)$ because $d(x, y) > r$. That in turn means $B(x, r) \neq X$. But $B(x, r)$ is open. So $B(x, r) \in \mathcal{F}$. Thus a trivial topology on X cannot be constructed by placing a metric d on X when X has more than one element. There are only two open covers of X, it is X and $\emptyset \cup X$. Both are finite and so a finite subset that covers X, would be themselves. Therefore the space is compact. It is also connected since there aren't two disjoint open, non-empty subsets thereby making it vacuously true.

2.5.2

Suppose the sequence $(x^{(n)})_{n=1}^{\infty}$ converges to $x_0 \in X$ by definition 2.5.4. Let $\varepsilon > 0$. Since the ball $B(x_0, \varepsilon)$ contains x_0 and is open, it is a neighborhood of x_0. That means for all $n \geq N$, $x^{(n)} \in B(x_0, \varepsilon)$ for some $N \geq 1$. That means $d(x^{(n)}, x_0) < \varepsilon$ for all $n \geq N$. Since $\varepsilon > 0$ was arbitrary, we have convergence by definition 1.1.14 as required.

Suppose the sequence $(x^{(n)})_{n=1}^{\infty}$ converges to $x_0 \in X$ by definition 1.1.14. Let V be a neighborhood of x_0. Since V is open and we have $x_0 \in V$, there exists a ball $B(x_0, r)$ such that $B(x_0, r) \subseteq V$ and $r > 0$. But we have $d(x^{(n)}, x_0) < r$ for all $n \geq N$ for some $N \geq 1$. So $x^{(n)} \in B(x_0, r)$ for all $n \geq N$, but that in turn means $x^{(n)} \in V$ for all $n \geq N$ and therefore converges by definition 2.5.4 as required.

2.5.4

Consider the metric space (X, d). Let $x, y \in X$ be two distinct points. Then $d(x, y) > 0$. So there is an $r > 0$ such that $d(x, y) > r > 0$. Consider the ball $B(x, r/2)$ and $B(y, r/2)$. Let $x_0 \in B(x, r/2)$. Suppose for contradiction that $x_0 \in B(y, r/2)$. Then $d(x_0, x) < r/2$ and $d(x_0, y) < r/2$. That means $d(x, y) < d(x_0, x) + d(x_0, y) < r$ a contradiction. So $x_0 \notin B(y, r/2)$. We also have for $y_0 \in B(y, r/2)$, $y_0 \notin B(x, r/2)$. So the two sets are disjoint. Therefore we have a neighbor of y and a neighbor of x that are disjoint to each other as required. So (X, d) is indeed hausdorff. This can be extended to all metric spaces since (X, d) is an arbitrary metric space. It is vacuously true only if X has one element or less. When X has more than one element, we have $x, y \in X$ such that $x \neq y$. The only neighborhood of x and y are X. But $x \in X$ and $y \in X$. So there does not exist a neighborhood of x and a neighborhood of y that are disjoint. Therefore the trivial topology is not hausdorff.

3 Uniform Convergence

3.1.1

Suppose for some $L \in Y$, $lim_{x \to x_0; x \in E} f(x) = L$. Now consider $lim_{x \to x_0; x \in E - \{x_0\}} f(x)$. Let $\varepsilon > 0$. There exists a $\delta > 0$ such that $d_Y(f(x), L)$ for all $x \in E$ such that $d_X(x, x_0) < \delta$. But $E - \{x_0\}$ is a subset of E and so $d_Y(f(x), L) < \varepsilon$ for all $x \in E - \{x_0\}$ such that $d_X(x, x_0) < \delta$. So we have $lim_{x \to x_0; x \in E - \{x_0\}} f(x) = L$. Suppose for contradiction that $L \neq f(x_0)$. Then $d_Y(L, f(x_0)) > 0$ and so we have an r such that $d_Y(L, f(x_0)) > r > 0$. But there exists a $\delta > 0$ such that $d_Y(f(x), L) < r$ for all $x \in E$ such that $d_X(x, x_0) < \delta$. $x_0 \in E$ and $d(x_0, x_0) = 0 < \delta$ so we have $d_Y(f(x_0), L) < r$, a contradiction. So $L = f(x_0)$ as required.

Suppose the limit $lim_{x \to x_0; x \in E - \{x_0\}} f(x) = f(x_0)$. Let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \varepsilon$ for all $x \in E - \{x_0\}$ such that $d_X(x, x_0) < \delta$. We know that $x_0 \in E$ and $d_X(x_0, x_0) = 0 < \delta$ as well as $d_Y(f(x_0), f(x_0)) = 0 < \varepsilon$. So for all $x \in E - \{x_0\} \cup \{x_0\} = E$ such that $d_X(x, x_0) < \delta$, $d_Y(f(x), f(x_0)) < \varepsilon$. So we can conclude that $lim_{x \to x_0; x \in E} f(x) = f(x_0)$ and therefore exists as required.

3.2.1

A) Suppose f is continuous. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers that converge to 0. Consider the sequence of functions $(f_{a_n})_{n=1}^{\infty}$. Let $x_0 \in \mathbf{R}$. Let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that $d(f(x), f(x_0)) < \varepsilon$ for all

$x \in \mathbf{R}$ such that $d(x, x_0) < \delta$. Since $(a_n)_{n=0}^{\infty}$ converges to 0, $|a_n| \leq \delta/2$ for all $n \geq N$ for some $N \geq 1$. We then have $d(x_0 - a_n, x_0) = |x_0 - a_n - x_0| = |-a_n| \leq \delta/2 < \delta$. So $d(f(x_0 - a_n), f(x_0)) < \varepsilon$. But $f(x_0 - a_n) = f_{a_n}(x_0)$. So $d(f_{a_n}(x_0), f(x_0)) < \varepsilon$ for all $n \geq N$. Therefore $f_{a_n}(x_0)$ converges to $f(x_0)$. Since $x_0 \in X$ was arbitrary we have $f_{a_n}(x_0)$ converges to $f(x_0)$ for all $x_0 \in X$. Therefore we have $(f_{a_n})_{n=1}^{\infty}$ converges point-wise to f.

Suppose hypothesis of the converse. Let $x_0 \in \mathbf{R}$. Let $(x^{(n)})_{n=1}^{\infty}$ be a sequence that converges to x_0. Let $(a_n)_{n=1}^{\infty}$ be defined as $a_n = x_0 - x^{(n)}$. Let $\varepsilon > 0$. Then $|x^{(n)} - x_0| = |x_0 - x^{(n)}| \leq \varepsilon$ for all $n \geq N$ for some $N \geq 1$. But that means $|a_n| \leq \varepsilon$ for all $n \geq N$. So we have $(a_n)_{n=1}^{\infty}$ converges to 0. That in turn means $(f_{a_n})_{n=1}^{\infty}$ converges point-wise to f. So $(f_{a_n}(x_0))_{n=1}^{\infty}$ converges to $f(x_0)$. But $f_{a_n}(x_0) = f(x_0 - x_0 + x^{(n)}) = f(x^{(n)})$. So we have $(f(x_n))_{n=1}^{\infty}$ converges to $f(x_0)$. Since $(x^{(n)})_{n=1}^{\infty}$ was an arbitrary sequence that converges to x_0 we can conclude that f is continuous at x_0. Since $x_0 \in \mathbf{R}$ was arbitrary as well we can conclude that f is continuous as required.

B) Suppose f is uniformly continuous. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers that converge to 0. Let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that $d(f(x), f(x')) < \varepsilon$ for all $x, x' \in \mathbf{R}$ such that $d(x, x') < \delta$. We have $|a_n| \leq \delta/2$ for all $n \geq N$ for some $N \geq 1$. So $|x - a_n - x| = |-a_n| = |a_n| \leq \delta/2 < \delta$. Since $x - a_n, x \in \mathbf{R}$ we have $d(f(x - a_n), f(x)) < \varepsilon$. But $f(x - a_n) = f_{a_n}(x)$ and so $d(f(x - a_n), f(x)) = d(f_{a_n}(x), f(x)) < \varepsilon$ for all $n \geq N$, for all $x \in \mathbf{R}$. So we have uniform convergence as required.

Suppose hypothesis of converse. Suppose for contradiction that f is not uniformly continuous. Then there exists an $\varepsilon > 0$ such that for all $\delta > 0$, $d(f(x), f(x')) > \varepsilon$ for some $x, x' \in \mathbf{R}$ such that $d(x, x') < \delta$. Consider the sequence defined as such $a_n = x - x'$ where x, x' has the property $d(f(x), f(x')) > \varepsilon$ and $-1/n < x - x' < 1/n$. This sequence exists by hypothesis. This sequence also converges to 0. So we have $d(f_{a_n}(x), f(x)) < \varepsilon$ for all $n \geq N$ for some $N \geq 1$. But for $n = N$, we have $d(f_{a_N}(x), f(x)) = d(f(x - x + x'), f(x)) = d(f(x'), f(x)) < \varepsilon$. This is a contradiction. Therefore f must be uniformly continuous.

3.2.2

A) Suppose hypothesis. Let $x_0 \in X$. Let $\varepsilon > 0$. We then have $d_Y(f^{(n)}(x_0), f(x_0)) < \varepsilon$ for all $n \geq N$ for some $N \geq 1$. Since ε was arbitrary, $(f^{(n)}(x_0))_{n=1}^{\infty}$ converges to $f(x_0)$. Since $x_0 \in X$ was arbitrary, we can conclude this for the whole domain X and so we have point-wise convergence as required.

B) Let $x_0 \in (-1, 1)$ and $g : (-1, 1) \to \mathbf{R}$ be defined as $g(x) = 0$. We

then have $f^{(n)}(x_0) = x_0^n$. But since $-1 < x_0 < 1$, we have $|x_0| < 1$ and the sequence $(|x_0|^n)_{n=1}^{\infty}$ converges to 0. But we have $|x_0|^n = |x_0^n|$ and so $-|x_0^n| \leq x_0^n \leq |x_0^n|$. Therefore $f^{(n)}(x_0)_{n=1}^{\infty}$ converges to $0 = g(x_0)$. Since $x_0 \in (-1, 1)$ was arbitrary we can conclude that $(f^{(n)})_{n=1}^{\infty}$ does indeed converge point-wise to g, the zero function.

Suppose for contradiction that there exists a function $f : (-1, 1) \to \mathbf{R}$ such that $(f^{(n)})_{n=1}^{\infty}$ converges uniformly to f. Then it also converges point-wise to f. But we have $(f^{(n)})_{n=1}^{\infty}$ converges point-wise to g and so $f = g$. We then have $d(f^{(n)}(x), 0) < 1/2$ for all $n \geq N$ for some $N \geq 1$ for all $x \in (-1, 1)$. So $|(1/2)^n| = (1/2)^n < 1/2$. But we have $(1/2)^{1/n} \in (-1, 1)$ and so $f^{(n)}((1/2)^{1/n}) = 1/2$. But $d(f^{(n)}((1/2)^{1/n}), 0) = |1/2| = 1/2 < 1/2$, a contradiction. So we must have for all functions $f : (-1, 1) \to \mathbf{R}$, $(f^{(n)})_{n=1}^{\infty}$ does not converge uniformly to f as required.

C) Let $x_0 \in (-1, 1)$. We have $\sum_{n=1}^{N} f^{(n)}(x_0) = \sum_{n=1}^{N} x_0^n$. Since $|x_0| < 1$, we have $(\sum_{n=1}^{N} f^{(n)}(x_0))_{n=1}^{\infty}$ converges to $g(x_0) = x_0/(1 - x_0)$ by theorem 7.3.3. Since $x_0 \in (-1, 1)$ is arbitrary, we can conclude that $(\sum_{n=1}^{N} f^{(n)})_{n=1}^{\infty}$ does indeed converge point-wise to g as required.

We need to find an $M \geq N_1$ such that $|\sum_{n=1}^{M} f^{(n)} - f| > \varepsilon$ for some $x \in (-1, 1)$. We know that $\sum_{n=1}^{M} f^{(n)} = (1 - x^{M+1})/(1 - x) - 1$. So we have $|(1 - x^{M+1})/(1 - x) - 1 - x/(1 - x)| = |-x^{M+1}/(1 - x)| = x^{M+1}/(1 - x) > \varepsilon$. So $x^{M+1} > \varepsilon(1 - x)$. Then $x^{M+1} + \varepsilon x = (x^M + \varepsilon)x > \varepsilon$. Let $x = 0.5^{1/M}$ and $\varepsilon = 1$. We then have $(0.5 + 1)0.5^{1/M} > 1$ and so $0.5^{1/M} > 2/3$. But we have $0.5^{1/M}$ converges to 1 as $M \to \infty$. So for all $n \geq N_2$, we have $|1 - 0.5^n| < 0.1$ for some $N_2 > 1$. So $0.9 < 0.5^n < 1.1$. Now let $N = max(N_1, N_2)$ and $M = N$. We then have $M \geq N_1$ such that $2/3 < 0.9 < 0.5^M$. Since N_1 was arbitrary, we have this being true for all $N_1 > 0$ and thus can conclude that $\sum_{n=1}^{M} f^{(n)}$ does not uniformly converge to f.

3.2.3

Let $x_0 \in X$. Let $\varepsilon > 0$. Since $f(x_0) \in \mathbf{R}$ we have $h(f(x_0))$ exists. Since h is continuous there exists a $\delta > 0$ such that $d(h(f(x_0)), h(x)) < \varepsilon$ for all $d(f(x_0), x) < \delta$. But because we have point-wise convergence of $f^{(n)}$ to f, for all $n \geq N$, $d(f^{(n)}(x_0), f(x_0)) < \delta$ for some $N \geq 1$. So we then have $d(h(f^{(n)}(x_0)), h(f(x_0))) < \varepsilon$. So $h(f^{(n)}(x_0))$ converges to $h(f(x_0))$. Since x_0 is arbitrary we can conclude that $h \circ f^{(n)}$ converges point-wise to $h \circ f$ as required.

3.3.1

Suppose hypothesis. Let $x_0 \in X$. Then we have $d(f(x), f^{(n)}(x)) < \varepsilon/3$ for all $n \geq N_1$ and $x \in X$, for some $N_1 \geq 1$ and $d(f^{(n)}(x_0), f(x_0)) < \varepsilon/3$ for all

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$n \geq N_2$ for some $N_2 \geq 1$. Let $N = max(N_1, N_2)$. Then $d(f^{(n)}(x_0), f(x_0)) < \varepsilon/3$ and $d(f(x_0), f^{(n)}(x_0)) < \varepsilon/3$ for all $n \geq N$. There also exists a $\delta > 0$ such that $d_Y(f^{(n)}(x_0), f^{(n)}(x)) < \varepsilon/3$ for all $x \in X$ such that $d_X(x_0, x) < \delta$. By letting $n = N$, we have $d(f(x_0), f(x)) < \varepsilon$, by triangle inequality, for all $x \in X$ such that $d(x, x_0) < \delta$. Since $\varepsilon > 0$ was arbitrary we can conclude that f is continuous as required.

3.3.2

Suppose hypothesis. Consider the sequence $(lim_{x \to x_0; x \in E} f^{(n)}(x))_{n=1}^{\infty}$. Then $d_Y(f^{(n)}(x), f(x)) < \varepsilon/4$ for all $n \geq N$ for some $N \geq 1$. Let $m \geq N$. Then $d_Y(f^{(m)}(x), f(x)) < \varepsilon/4$. There also exists a $\delta^{(l)} > 0$ such that $d_Y(f^{(l)}(x), lim_{x \to x_0; x \in E}, f^{(l)}(x)) < \varepsilon/4$ for all $x \in E$ such that $d_X(x_0, x) < \delta^{(l)}$ for all $l > 0$. So we have $d_Y(lim_{x \to x_0; x \in E} f^{(n)}(x), lim_{x \to x_0; x \in E} f^{(n)}(x)) \leq d_Y(f^{(n)}(x), lim_{x \to x_0; x \in E} f^{(n)}(x)) + d_Y(f^{(m)}(x), f(x))$
$+ d_Y(f^{(n)}(x), lim_{x \to x_0; x \in E}, f^{(n)}(x)) + d_Y(f^{(m)}(x), lim_{x \to x_0; x \in E} f^{(m)}(x)) < \varepsilon$ for all $n, m \geq N$. We thus have $(lim_{x \to x_0; x \in E} f^{(n)}(x))_{n=1}^{\infty}$ is Cauchy and since Y is complete, it also converges to some $L \in Y$. But we have for $\varepsilon > 0$, $d_Y(f^{(n)}, f(x)) < \varepsilon/3$ for all $n \geq N_1$ for some $N_1 \geq 1$ and $d_Y(lim_{x \to x_0; x \in E} f^{(n)}(x), L) < \varepsilon$ for all $n \geq N_2$ for some $N_2 \geq 1$. Also there exist a $\delta^{(n)} > 0$ such that $d_Y(f^{(n)}(x), lim_{x \to x_0; x \in E} f^{(n)}(x)) < \varepsilon/3$ for all $x \in E$ such that $d_X(x, x_0) < \delta^{(n)}$. So we have $d_Y(f(x), L) < \varepsilon$ for all $n \geq max(N_1, N_2)$ and for all $x \in E$ such that $d_X(x, x_0) < \delta^{(n)}$. So we can conclude that $lim_{x \to x_0; x \in E} f(x) = L$ as required.

3.3.3

Since the function $x^{(n)}$ does not uniformly converge to any function f, we cannot apply proposition 3.3.3. The statement is false as well because the $lim_{n \to \infty} x^n$ for $x < 1$ is 0. So $lim_{x \to 1^-} lim_{n \to \infty} x^n = 0$. Yet $lim_{x \to 1^-} x^n = 1$ and so $lim_{n \to \infty} lim_{x \to 1^-} x^n = 1$ a contradiction.

3.3.4

Suppose hypothesis. Let $\varepsilon > 0$. Since $f^{(n)}$ converges uniformly to f, we have $d_Y(f^{(n)}(x), f(x)) < \varepsilon/2$ for all $n \geq N_1$ for some $N \geq 1$. Since $f^{(n)}$ is continuous we have for some $\delta > 0$, $d_Y(f^{(n)}(y), f^{(n)}(x)) < \varepsilon/2$ for all $y \in X$ such that $d_X(y, x) < \delta$. But we have $(x_n)_{n=1}^{\infty}$ converges to x and so $d(x_n, x) < \delta/2$ for all $n \geq N_2$ for some $N_2 \geq 1$. So $d_Y(f^{(n)}(x_n), f(x)) \leq d_Y(f^{(n)}(x_n), f^{(n)}(x)) + d_Y(f^{(n)}(x), f(x)) < \varepsilon$ for all $n \geq max(N_1, N_2)$. Therefore we can conclude that $(f^{(n)}(x_n))_{n=1}^{\infty}$ converges to $f(x)$ as required.

3.3.5 Consider the functions defined as $f^{(n)}(x) = x^n$ defined on the interval $(-1, 1]$. This does not uniformly converge to any f but it converges point-wise to the function lets call f defined as $f(x) = 0$ for all $x \in (-1, 1)$

and $f(x) = 1$ for $x = 1$. Consider the sequence $(0.2^{(1/n)})_{n=1}^{\infty}$ that converges to 1 and $(f^{(n)}(0.2^{1/n}))_{n=1}^{\infty}$. Let $\varepsilon = 0.5$, we then have $d(f^{(n)}(0.2^{1/n}).f(1)) = |(0.2^{1/n})^n - 1| = 0.8 > 0.5$ for all $n \geq 1$. Therefore we can conclude that for every $N \geq 1$, we have $n = N$ such that $d(f^{(n)}(0.21/n), f((0.2^{1/n}))) = 0.8 > 0.5$. So $(f^{(n)}(0.2^{(1/n)}))_{n=1}^{\infty}$ does not converge to $f(1)$. as required.

3.4.1

Let $f \in B(X \to Y)$. Then $d_Y(f(x), f(x)) = 0$ for all $x \in X$. So $sup\{d_Y(f(x), f(x)) : x \in X\} = 0$ since for all $x \in \{d_Y(f(x), f(x)) : x \in X\}$, $x \geq 0$ as required.

Let $f, g \in B(X \to Y)$ where $f \neq g$. Then there exists an $x \in X$ such that $f(x) \neq g(x)$. Then $d_Y(f(x), g(x)) > 0$. Since $d_Y(f(x), g(x)) \in \{d_Y(f(x), g(x)) : x \in X\}$, we must have $sup\{d_Y(f(x), g(x)) : x \in X\} > 0$ as required.

Let $f, g \in B(X \to Y)$. We have $d_Y(f(x), g(x)) = d_Y(g(x), f(x))$ for all $x \in X$. So $\{d_Y(f(x), g(x)), x \in X\} = \{d_Y(g(x), f(x)), x \in X\}$ meaning $sup\{d_Y(f(x), g(x)), x \in X\} = sup\{d_Y(g(x), f(x)), x \in X\}$ as required.

Let $f, g, h \in B(X \to Y)$. Then $d_Y(f(x), h(x)) \leq d_Y(f(x), g(x)) + d_Y(g(x), h(x))$ for all $x \in X$. Suppose for contradiction that $sup\{d_Y(f(x), h(x)) : x \in X\} > sup\{d_Y(f(x), g(x)) : x \in X\} + sup\{d_Y(g(x), h(x)) : x \in X\}$. Then there exists an $x_0 \in X$ such that $d_Y(f(x_0), h(x_0)) > sup\{d_Y(f(x), g(x)), x \in X\} + sup\{d_Y(g(x), h(x)) : x \in X\} \geq d_Y(f(x), g(x)) + d_Y(g(y), h(y))$ for all $x, y \in X$. But that means $d_Y(f(x_0), h(x_0)) > d_Y(f(x_0), g(x_0)) + d_Y(g(x_0), h(x_0))$, a contradiction. So the triangle inequality holds as required.

3.4.2

Suppose $(f^{(n)})_{n=1}^{\infty}$ converges to f in the metric $d_{B(X \to Y)}$. Then there exists an $N \geq 1$ such that $d_{B(X \to Y)}(f^{(n)}, f) \leq \varepsilon/2$ for all $n \geq N$. So $d_{B(X \to Y)}(f^{(n)}, f) = sup\{d_Y(f^{(n)}(x), f(x)) : x \in X\} \leq \varepsilon/2$. But that means $d_Y(f^{(n)}(x), f(x)) \leq \varepsilon/2 < \varepsilon$ for all $n \geq N$ for all $x \in \mathbf{X}$. Therefore we have uniform convergence as required

Suppose $(f^{(n)})_{n=1}^{\infty}$ converges uniformly to f. Then there exists an $N \geq 1$ such that $d_Y(f^{(n)}(x), f(x)) < \varepsilon/2$ for all $n \geq N$ and all $x \in X$. So for all $n \geq N$ $d_{B(X \to Y)}(f^{(n)}, f) = sup\{d_Y(f^{(n)}(x), f(x)) : x \in X\} \leq \varepsilon$ because ε is an upper bound of the set. Therefore we can conclude that $(f^{(n)})_{n=1}^{\infty}$ converges to f with respect to the $d_{B(X \to Y)}$ metric as required.

3.4.3

Suppose hypothesis. Let $(f^{(n)})_{n=1}^{\infty}$ be a Cauchy series in $C(X \to Y)$. Then $sup\{d_Y(f^{(n)}(x), f^{(m)}(x)) : x \in X\} \leq \varepsilon$ for all $n, m \geq N_0$ for some

$N_0 \geq 1$. Then $d_Y(f^{(n)}(x), f^{(m)}(x)) \leq \varepsilon$ for all $x \in X$. But Y is a complete metric space. So we can conclude that the sequence $(f^{(n)}(x))_{n=1}^{\infty}$ converges to some $y \in Y$ for all $x \in X$. Define a function as $\lim_{n \to \infty} f^{(n)}(x) = f(x)$ for all $x \in X$. Clearly we have $(f^{(n)})_{n=1}^{\infty}$ converges point-wise to f.

Let $\varepsilon > 0$. Then for some $N_1 \geq 1$, $\sup\{d_Y(f^{(n)}(x), f^{(m)}(x)) : x \in X\} \leq \varepsilon/2$ for all $n, m \geq N_1$. So $d_Y(f^{(n)}(x), f^{(m)}(x)) \leq \varepsilon/2$ for all $n, m \geq N_1$ for all $x \in X$. Let x be fixed. We have for some $N_2 \geq 1$, $d_Y(f^{(l)}(x), f(x)) \leq \varepsilon/2$ for all $l \geq N_2$. Let $N = max(N_1, N_2)$. Then $d_Y(f^{(l)}(x), f(x)) \leq \varepsilon/2$ and $d_Y(f^{(n)}(x), f^{(l)}(x)) \leq \varepsilon/2$ for all $l \geq N$ and $n \geq N_1$. So $d_Y(f^{(n)}(x), f(x)) \leq d_Y(f^{(l)}(x), f(x)) + d_Y(f^{(n)}(x), f^{(l)}(x)) \leq \varepsilon$. Since x was arbitrary, we can conclude this for all x and thus we have uniform convergence. But because the sequence $f^{(n)}$ is continuous and bounded we must have f is also continuous and bounded. By P3.4.4, we must have $f^{(n)}$ converges in the metric $B(X \to Y)$. Since f is also continuous, it converges to a function in $C(X \to Y)$ as required.

3.5.1

Suppose hypothesis. For the base case of $N = 1$, we have $f^{(1)}$ is a bounded function. Then $\sum_{n=1}^{1} f^{(1)} = f^{(1)}$ which is bounded as required. Suppose for inductive hypothesis $f^{(1)}, ..., f^{(N)}$ was a sequence of bounded functions and $\sum_{n=1}^{N} f^{(i)}$ is bounded for $N \geq 1$. Then $\sum_{n=1}^{N} f^{(i)} + f^{(N+1)} = \sum_{n=1}^{N+1} f^{(i)}$ where $f^{(N+1)}$ is a bounded function. But since $\sum_{n=1}^{N+1} f^{(i)}$ is bounded, we have $\sum_{n=1}^{N+1} f^{(i)}(x) \in [-M, M]$ for some $M > 1$ for all $x \in X$. We also have $f^{(N+1)}(x) \in [-R, R]$ for some $R > 0$ for all $x \in X$. Consider the interval $[-(R+M), R+M]$. Since $-R \leq f^{N+1}(x) \leq R$ and $-M \leq \sum_{n=1}^{N} f^{(i)}(x) \leq M$, we have $-M - R \leq \sum_{n=1}^{N} f^{(i)}(x) + f^{N+1}(x) = \sum_{n=1}^{N+1} f^{(i)}(x) \leq M + R$. Thus we have $\sum_{n=1}^{N+1} f^{(i)}(x) \in [-(R+M), R+M]$ for all $x \in X$. Therefore $\sum_{n=1}^{N+1} f^{(i)}$ is bounded and we can close the induction.

Replace bounded with continuous for the hypothesis. The induction is similar, we know for the base case of 1 that the sum is continuous. Suppose hypothesis for some $N \geq 1$, then the function for $N+1$ is still continuous because the sum of continuous functions are continuous. Therefore we can close the induction and say all finite sums of continuous functions are continuous.

Same proof as for continuous function because addition of uniformly continuous functions are also uniformly continuous.

3.5.2 (Unfinished)

Suppose hypothesis. Then $|\sum_{n=q}^{p} ||f^{(n)}||_{\infty}| = |\sum_{n=1}^{p} ||f^{(n)}||_{\infty} - \sum_{n=1}^{q-1} ||f^{(n)}||_{\infty}| \leq \varepsilon$ for all $q, p \geq N$ for some $N \geq 1$. Since $|f^{(n)}|| = \sup\{|f^{(n)}(x)| : x \in X\}$,

we have $|f^{(n)}(x)| \leq ||f^{(n)}||_\infty$ for all $x \in X$. That in turn means $-||f^{(n)}|| \leq f^{(n)}(x) \leq ||f^{(n)}||$. So $-\sum_{n=1}^{N} ||f^{(n)}|| \leq \sum_{n=1}^{N} f^{(n)}(x) \leq \sum_{n=1}^{N} ||f^{(n)}||$

3.6.1

First we will prove that a finite sum of Riemann integrable functions is also Riemann integrable. Let $N = 1$ be the base case, we have $f^{(1)}$ is Riemann integrable on the interval $[a, b]$. Then $\sum_{n=1}^{1} f^{(n)} = f^{(1)}$ which is Riemann integrable as required. Suppose as inductive hypothesis that for some $N \geq 1$, $f(1), ..., f^{(n)}$ is a sequence of Riemann integrable functions on the interval $[a, b]$ and $\sum_{n=1}^{N} f^{(n)}$ is Riemann integrable. Then, let $f^{(N+1)}$ be a Riemann integrable fun ction, $\sum_{n=1}^{N} f^{(n)} + f^{(N+1)} = \sum_{n=1}^{N+1} f^{(n)}$. But the sum of Riemann integrable functions are also Riemann integrable so $\sum_{n=1}^{N+1} f^{(n)}$ is Riemann integrable and so we close the induction.

Now suppose hypothesis. Since $(f^{(n)})_{n=1}^{\infty}$ is a sequence of Riemann integrable functions, we have $\sum_{n=1}^{N} f^{(n)}$ is Riemann integrable for any $N \geq 1$. So $(\sum_{n=1}^{N} f^{(n)})_{N=1}^{\infty}$ is a sequence of Riemann integrable functions on the interval $[a, b]$ that uniformly converge. Thus by Theorem 3.6.1 we then have $\lim_{N\to\infty} \int_{[a,b]} \sum_{n=1}^{N} f^{(n)} = \lim_{N\to\infty} \sum_{n=1}^{N} \int_{[a,b]} f^{(n)} = \int_{[a,b]} \sum_{n=1}^{\infty} f^{(n)}$ as required.

3.7.3

Suppose hypothesis. Since the series

$$\sum_{n=1}^{\infty} ||f_n'||$$

is absolutely convergent, it is also conditionally convergent. That means we can use the Weierstrass M-test and conclude that

$$\sum_{n=1}^{\infty} f_n'$$

converges uniformly to some function f. But

$$\sum_{n=1}^{\infty} f_n' = \lim_{n\to\infty} \sum_{n=1}^{N} f_n'$$

Since $\sum_{n=1}^{N} f_n'$ is a finite sum of continuous functions and is the derivative of $\sum_{n=1}^{N} f_n$, we can apply Theorem 3.7.1 to obtain $\frac{d}{dx} \lim_{n\to\infty} \sum_{n=1}^{N} f_n = \lim_{n\to\infty} \sum_{n=1}^{N} f_n'$ which is equivalent to saying

$$\frac{d}{dx} \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \frac{d}{dx} f_n$$

as required

3.8.1

Suppose hypothesis. Suppose for contradiction that $f(x) \neq 0$ for $x \notin [a,b] \cap [c,d]$. Then $x \notin [a,b]$ or $x \notin [c,d]$. In the case $x \notin [a,b]$ we must have $f(x) = 0$ by definition of supported. In the case $x \notin [c,d]$ we must have $f(x) = 0$. Thus this means

$$\int_{[a,b]} f = \int_{[a,b] \cap [c,d]} f = \int_{[c,d]} f$$

as required.

3.8.2

a) In the base case of $n = 0$ we have

$$(1-y)^0 = 1 \geq 1 - 0 * y = 1$$

as required. Now suppose as inductive hypothesis that

$$(1-y)^n \geq 1 - ny$$

for some $n \geq 0$. Then we have

$$(1-y)^{n+1} = (1-y)^n(1-y) \geq (1-ny)(1-y) = 1 - ny - y + ny^2 =$$

$$1 - (n+1)y + ny^2 \geq 1 - (n+1)y$$

as required thereby closing the induction

b) We have

$$\int_{-1}^{1}(1-x^2)^n dx = \int_{-1/\sqrt{n}}^{1/\sqrt{n}}(1-x^2)^n dx + \int_{1/\sqrt{n}}^{1}(1-x^2)^n dx + \int_{-1}^{-1/\sqrt{n}}(1-x^2)^n dx$$

When $|x| \leq \frac{1}{\sqrt{n}}$, because $x \in [-1,1]$ we have $0 \leq x^2 \leq 1$ and so by part a we see that $(1-x^2)^n \geq 1 - nx^2$. That means

$$\int_{-1}^{1}(1-x^2)^n dx \geq \int_{-1/\sqrt{n}}^{1/\sqrt{n}}(1-nx^2)dx + \int_{1/\sqrt{n}}^{1}(1-x^2)^n dx + \int_{-1}^{-1/\sqrt{n}}(1-x^2)^n dx =$$

$$\frac{2}{\sqrt{n}} - \frac{2}{3\sqrt{n}} + \int_{1/\sqrt{n}}^{1}(1-x^2)^n dx + \int_{-1}^{-1/\sqrt{n}}(1-x^2)^n dx \geq$$

$$\frac{4}{3\sqrt{n}} \geq \frac{1}{\sqrt{n}}$$

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where the second to last inequality comes from $1 - x^2$ being positive over it's interval and thus making it's integral greater than 0.

c) Let $\varepsilon > 0$ and $0 < \delta < 1$. We then have for some N, $1 < \delta N$ and so $\frac{1}{N} \leq \delta$. Then $1 - \frac{1}{N^2} = \frac{N^2 - 1}{N^2} < 1$. Then we have

$$\sqrt{n} \left(\frac{N^2 - 1}{N^2} \right)^n \leq n \left(1 - \frac{1}{N^2} \right)^n$$

Using the ratio test we have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1) \left(\frac{N^2-1}{N^2} \right)^n}{n \left(\frac{N^2-1}{N^2} \right)^n} = \frac{(n+1) \left(\frac{N^2-1}{N^2} \right)}{n}$$

by letting $n > N^2 + 1$ we have $n = N^2 - 1 + k$ and so

$$\frac{(n+1) \left(\frac{N^2-1}{N^2} \right)}{n} = \frac{(N^2 - +k) \left(\frac{N^2-1}{N^2} \right)}{N^2 - 1 + k} = \frac{(N^2 + k)}{N^2 - 1 + k} \frac{N^2 - 1}{N^2}$$

where we now see that

$$N^4 + N^2 k - N^2 - k < N^4 - N^2 + N^2 k$$

and thus we have

$$\frac{(N^2 + k)}{N^2 - 1 + k} \frac{N^2 - 1}{N^2} < 1$$

Meaning with n starting at N^2, we have

$$\limsup_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} < 1$$

thus proving the ratio test and we can therefore conclude that the sequence converges to 0 since the sum converges. That means there exists an $n \geq 1$ such that

$$n \left(1 - \frac{1}{N^2} \right)^n < \varepsilon$$

Since we also have

$$\int_{-1}^{1} (1 - x^2)^n dx \geq \frac{1}{\sqrt{n}} \geq \frac{1}{n} \to \int_{-1}^{1} n(1 - x^2)^n dx = c \geq 1 \to$$

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$$\int_{-1}^{1} \frac{n}{c}(1-x^2)^n dx = 1$$

we also know that for all $\frac{1}{N} < \delta \leq |x| \leq 1$ we have

$$1 - \frac{1}{N^2} \geq 1 - x^2$$

and so

$$n(1-x^2)^n \leq n\left(1 - \frac{1}{N^2}\right)^n < \varepsilon$$

and because

$$\frac{1}{c} \leq 1$$

we see that

$$\frac{1}{c}n(1-x^2)^n < \varepsilon$$

So taking $f(x) = \frac{1}{c}n(1-x^2)^n$ on the interval $[-1,1]$ and 0 otherwise, we see that $f(-1) = 0 = f(1)$ and is continuous on its interval as well as boundary. The function is a polynomial on the given interval, and also the integral is given as above due to the function being supported on $[-1,1]$ and finally for all $\delta \leq |x| \leq 1$ we have shown above that $f(x) < \varepsilon$. Since $f(x)$ is a polynomial we can thus conclude that there indeed exists a polynomial which is an $(\varepsilon, \delta) - approximation$ to the identity as required.

3.8.3

Since $f(x)$ is compactly supported, it is supported on some interval $[a,b]$ which means that for all $x \notin [a,b]$, $f(x) = 0$. Consider the function $f_{[a,b]}$ which is f restricted to the domain $[a,b]$. Since $[a,b]$ is compact, by proposition 2.3.2, we see that for some $x_{max} \in [a,b]$, $f_{[a,b]}(x_{max}) \geq f_{[a,b]}(x)$ and for some $x_{min} \in [a,b]$, $f_{[a,b]}(x_{min}) \leq f_{[a,b]}(x)$ for all $x \in [a,b]$. Then we have

$$|f_{[a,b]}(x)| \leq \max(|f_{[a,b]}(x_{min})|, |f_{[a,b]}(x_{max})|) = M$$

and is thus bounded. We can also use theorem 2.3.5 to conclude that $f_{[a,b]}$ is also uniformly continuous. Now because f is continuous, it is also continuous at $f(a)$ and $f(b)$. Also because f is supported on $[a,b]$, we have for $x < a$ or $x > b$, $f(x) = 0$ and so $|f(x)| \leq M$ for all x. Thus we conclude that $f(x)$ is bounded. Now let $\varepsilon > 0$. Due to the uniform continuity of $f_{[a,b]}$, we know that $|f(x) - f(x')|\varepsilon$ for all x, x' such that $|x - x'| < \delta$ for some δ on the interval $[a,b]$. But this can be extended. Let $x, x' \in \mathbf{R}$ such that

$|x - x'| < \delta$. Then either $x \in [a, b]$ and $x' \in [a, b]$ or $x \in [a, b]$ and $x' \notin [a, b]$ (opposite is true by symmetry) or $x \notin [a, b]$ and $x' \notin [a, b]$. In the first case we have $|f(x) - f(x')| < \varepsilon$ by uniform continuity of $f_{[a,b]}$. In the second case, we must have $|f(x) - f(x')| = |f(x) - 0|$. Since either $|x - a| < \delta$ or $|x - b| < \delta$ in order for x' to satisfy the conditions, we must therefore have either $|f(x) - f(a)| < \varepsilon$ or $|f(x) - f(b)| < \varepsilon$ in both cases we have $|f(x) - 0| < \varepsilon$ and so $|f(x) - f(x')| = |f(x) - 0| < \varepsilon$ as required. Now in the final case we would have $|f(x) - f(x')| = 0 < \varepsilon$ as required. Therefore we can conclude uniform continuity of the entire function and we are done.

3.8.4

Suppose hypothesis

a) Since the functions are continuous and compactly supported, we have by the above exercise that they are also bounded and uniformly continuous. That means we also have $f(y)g(x - y)$ as a function of y being continuous and compactly supported. So it is also bounded and uniformly continuous. Also we have $g(x) = 0$ for $x \notin [a, b]$ and $f(x) = 0$ for $x \notin [c, d]$ for some interval $[a, b]$ and $[c, d]$. Thus we see that $f(y)g(x - y) = 0$ when $y \notin [a, b]$ or $x - y \notin [c, d]$. This means that $f(y)g(x - y)$ as a function of y is supported on $I = [a, b]$ and so

$$\int_{-\infty}^{\infty} f(y)g(x - y)dy = \int_I f(y)g(x - y)dy$$

The second case is equivalent to

$$y \notin [x - d, x - c]$$

But we see that when $x < c + a$, we have $x - c < a$ meaning $x - c = a - s$ where s is positive. So the function is 0 for $y \notin [x - d, a - s]$ which is disjoint from $[a, b]$ meaning for all y, we have $f(y)g(x - y) = 0$. Now when $x > d + b$, we have $x - d > b$ and so $x - d = b + r$ where r is a positive number. This results in the function being 0 for $y \notin [b + r, x - c]$ which is disjoint from $[a, b]$ and thus meaning its 0 for all y. Thus for $x \notin [c + a, d + b]$ we have

$$\int_{-\infty}^{\infty} f(y)g(x - y)dy = \int_I 0dy = 0$$

Thus as a function of x, we have $\int_{-\infty}^{\infty} f(y)g(x - y)dy$ is supported on $[c + a, d + b]$ and therefore compactly supported.

Now we shall prove that the function is also continuous. Let $\varepsilon > 0$. $f(y)$ is bounded by some M and $g(x - y)$ is uniformly continuous on y. Then

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fixing x we have $|g(x - y) - g(x - y')| < \varepsilon/(M|I|)$ for all y, y' such that $|y - y'| < \delta$ for some δ where we have $|I|$ being the length of the I interval. Now consider $|x - x'| < \delta$ we see that $x + \delta > x' > x - \delta$. So $x' - s = x - \delta$ where $s > 0$ meaning $s = x' - x + \delta < 2\delta$ and so we have $|s - \delta| < \delta$. That means $x' = x + s - \delta$ which results in $g(x' - y) = g(x - (-s + \delta + y))$ but $|y - (-s + \delta/2 + y)| = |s - \delta| < \delta$. So we have

$$|g(x - y) - g(x' - y)| = |g(x - y) - g(x - (-s + \delta + y))| < \varepsilon/(M|I|)$$

for all y. This then means that by letting $|x - x'| < \delta$ we have

$$\left| \int_I f(y)g(x - y)dy - \int_I f(y)g(x' - y)dy \right| =$$

$$\left| \int_I f(y)(g(x - y) - g(x' - y))dy \right| \leq$$

$$\int_I |f(y)||(g(x - y) - g(x' - y))|dy < \int_I M\varepsilon/(M|I|)dy = \varepsilon$$

as required.

b) We have

$$f * g(x) = \int_{-\infty}^{\infty} f(y)g(x - y)dy$$

Now consider

$$g * f(x) = \int_{-\infty}^{\infty} g(z)f(x - z)dz$$

if we let $z = x - y$, we have $dz = -dy$ and since $z = x - (-\infty) \to z = \infty + x$ as well as $z = x - \infty$ the new integral will thus be from $\infty \to -\infty$. Thus

$$g * f(x) = -\int_{\infty}^{-\infty} g(x - y)f(x - (x - y))dy = \int_{-\infty}^{\infty} g(x - y)f(y)dy$$

as required

c) We have

$$f * (g + h)(x) = \int_{-\infty}^{\infty} f(y)(g + h)(x - y)dy = \int_{-\infty}^{\infty} f(y)(g(x - y) + h(x - y))dy =$$

$$\int_{-\infty}^{\infty} f(y)(g(x - y) + h(x - y))dy =$$

$$\int_{-\infty}^{\infty} f(y)g(x - y)dy + \int_{-\infty}^{\infty} f(y)h(x - y)dy = f * g(x) + f * h(x)$$

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as required. Also

$$f * (cg)(x) = \int_{-\infty}^{\infty} f(y)cg(x-y)dy = \int_{-\infty}^{\infty} cf(y)g(x-y)dy = (cf) * g(x) =$$

$$c \int_{-\infty}^{\infty} f(y)g(x-y)dy = c(f * g(x))$$

as required.

3.8.5

Suppose hypothesis. Since $f(y)$ is supported on $[0, 1]$ we see that the function $f(y)g(x-y)$ is also supported on the same interval since since for all $y \notin [0, 1]$, $f(y)g(x-y) = 0g(x-y) = 0$. Thus we have

$$f * g(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy = \int_{[0,1]} f(y)g(x-y)dy$$

Now let $x \in [1, 2]$. Since $y \in [0, 1]$ we see that $-1 \leq -y \leq 0 \rightarrow 0 \leq x-y \leq 2$ and since g is constant on $[0, 2]$, we then have

$$g(x-y) = c$$

for $y \in [0, 1]$. This means that

$$\int_{[0,1]} f(y)g(x-y)dy = c \int_{[0,1]} f(y)dy = cM$$

and since x was arbitrary in $[1, 2]$ we can thus conclude that

$$f * g(x) = cM$$

on $[1, 2]$ and is therefore constant.

3.8.6

a) Suppose hypothesis. Then

$$\int_{-\infty}^{\infty} g(x)dx = \int_{[-1,1]} g(x)dx = \int_{[-1,-\delta]} g(x)dx + \int_{[-\delta,\delta]} g(x)dx + \int_{[\delta,1]} g(x)dx = 1$$

we see that

$$\int_{[-\delta,\delta]} g(x)dx = 1 - \int_{[-1,-\delta]} g(x)dx - \int_{[\delta,1]} g(x)dx$$

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but because $0 \leq g(x) \leq \varepsilon$ for $\delta \leq |x| \leq 1$, we then have

$$0 \leq \int_{[-1,-\delta]} g(x)dx \leq \int_{[-1,-\delta]} \varepsilon dx \rightarrow$$

$$0 \leq \int_{[-1,-\delta]} g(x)dx \leq \varepsilon(-\delta + 1)$$

and also

$$0 \leq \int_{[\delta,1]} g(x)dx \leq \int_{[\delta,1]} \varepsilon dx \rightarrow$$

$$0 < \int_{[\delta,1]} g(x)dx < \varepsilon(1 - \delta)$$

which means

$$0 \leq \int_{[-1,-\delta]} g(x)dx + \int_{[\delta,1]} g(x)dx \leq \varepsilon(1 - \delta) + \varepsilon(-\delta + 1) \rightarrow$$

$$0 \leq \int_{[-1,-\delta]} g(x)dx + \int_{[\delta,1]} g(x)dx \leq 2\varepsilon(1 - \delta)$$

and so we have

$$1 - 2\varepsilon(1 - \delta) \leq 1 - \int_{[-1,-\delta]} g(x)dx - \int_{[\delta,1]} g(x)dx \leq 1 \rightarrow$$

$$1 - 2\varepsilon \leq \int_{[-\delta,\delta]} g(x)dx \leq 1$$

as required. where the last line comes from $0 < \delta < 1 \rightarrow 0 < 1 - \delta < 1$ meaning $1 - 2\varepsilon > 1 - 2\varepsilon(1 - \delta)$.

b) Suppose hypothesis. We then have

$$f * g(x) = \int_{-\infty}^{\infty} f(y)g(x - y)dy = \int_{[-1,1]} g(y)f(x - y)dy =$$

$$\int_{[-1,-\delta]} g(y)f(x - y) + \int_{[-\delta,\delta]} g(y)f(x - y)dy + \int_{[\delta,1]} g(y)f(x - y)dy$$

From the hypothesis we have $|f(x) - f(x - y)| < \varepsilon \rightarrow f(x) - \varepsilon < f(x - y) < f(x) + \varepsilon$ for $y \in (-\delta, \delta)$. Also $|f(x)| \leq M$. Due to f and g being continuous on $\mathbf{R}$, the Riemann integral allows

$$\int_{[-\delta,\delta]} g(y)f(x - y)dy = \int_{(-\delta,\delta)} g(y)f(x - y)dy$$

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In the case $f(x) \geq 0$ we have

$$\int_{(-\delta,\delta)} g(y)f(x-y)dy - f(x) \leq \int_{(-\delta,\delta)} g(y)(f(x)+\varepsilon)dy - f(x) \leq$$

$$f(x) + \varepsilon - f(x) = \varepsilon \leq \varepsilon(1+2M)$$

where our second to last inequality comes from part a and our last inequality is due to $f(x)$ being bounded. Also

$$\int_{(-\delta,\delta)} g(y)f(x-y)dy - f(x) \geq \int_{(-\delta,\delta)} g(y)(f(x)-\varepsilon)dy - f(x) \geq$$

$$(1-2\varepsilon)(f(x)-\varepsilon) - f(x) = f(x) - 2f(x)\varepsilon - \varepsilon + 2\varepsilon^2 - f(x) =$$

$$\varepsilon(2\varepsilon - 1 - 2f(x)) \geq -\varepsilon(1+2f(x)) \geq -\varepsilon(1+2M)$$

where our last inequality comes from $f(x)$ being bounded Thus we conclude that for $f(x) \geq 0$ $|\int_{[-\delta,\delta]} g(y)f(x-y)dy - f(x)| \leq (1+2M)\varepsilon$. Now in the case $f(x) < 0$, we have

$$\int_{(-\delta,\delta)} g(y)f(x-y)dy - f(x) \leq \int_{(-\delta,\delta)} g(y)(f(x)+\varepsilon)dy - f(x) \leq$$

$$f(x)(1-2\varepsilon) + \varepsilon - f(x) = (1-2f(x))\varepsilon \leq \varepsilon(1+2M)$$

where once again the final inequality comes from f being bounded and the second the last is from part a. Also

$$\int_{(-\delta,\delta)} g(y)f(x-y)dy - f(x) \geq \int_{(-\delta,\delta)} g(y)(f(x)-\varepsilon)dy - f(x) \geq$$

$$f(x) - \varepsilon - f(x) \geq -\varepsilon \geq -\varepsilon - 2M\varepsilon = -\varepsilon(1+2M)$$

once again meaning that $|\int_{[-\delta,\delta]} g(y)f(x-y)dy - f(x)| \leq (1+2M)\varepsilon$. Therefore we can conclude that

$$\left| \int_{[-\delta,\delta]} g(y)f(x-y)dy - f(x) \right| \leq (1+2M)\varepsilon$$

for all x. Now we then have from part a

$$\left| \int_{[-1,-\delta]} g(y)f(x-y)dy \right| + \left| \int_{[\delta,1]} g(y)f(x-y)dy \right| \leq$$

$$\int_{[-1,-\delta]} |g(y)||f(x-y)|dy + \int_{[\delta,1]} |g(y)||f(x-y)|dy \le$$

$$M\left(\int_{[-1,-\delta]} g(y)dy + \int_{[\delta,1]} g(y)dy\right) \le 2\varepsilon(1-\delta)M \le 2M\varepsilon$$

thus we have

$$|f*g(x) - f(x)| \le \left|\int_{[-\delta,\delta]} g(y)f(x-y)dy - f(x)\right| + \left|\int_{[-1,-\delta]} g(y)f(x-y)dy\right|$$

$$+ \left|\int_{[\delta,1]} g(y)f(x-y)dy\right| \le (1+2M)\varepsilon + 2M\varepsilon = (1+4M)\varepsilon$$

as required.

3.8.7

Suppose hypothesis. Let $\varepsilon > 0$. By exercise 3.8.3 we have $f(x)$ is bounded by some $M > 0$ and also it is uniformly continuous so $|f(x) - f(y)| < \varepsilon/(1+4M)$ for all $|x-y| < \delta$ for some $0 < \delta < 1$. By Lemma 3.8.8 we can $(\varepsilon/(1+4M), \delta)$-approximate the identity with a polynomial $g(x)$. Also by theorem 3.8.14 we have

$$|f*g(x) - f(x)| \le (1+4M)\frac{\varepsilon}{1+4M} = \varepsilon$$

for all $x \in [0,1]$ since $f*g(x)$ is a polynomial on $[0,1]$ by Lemma 3.8.13 we are now done.

3.8.8

Suppose hypothesis. Let $P(x)$ be a polynomial. Then $P(x) = \sum_{n=0}^{N} a_n x^n$. We then have

$$\int_{[0,1]} f(x)P(x)dx = \int_{[0,1]} f(x)\sum_{n=0}^{N} a_n x^n dx = \sum_{n=0}^{N}\int_{[0,1]} f(x)a_n x^n dx =$$

$$\sum_{n=0}^{N} a_n \int_{[0,1]} f(x)x^n dx = 0$$

where the last equality comes from $\int_{[0,1]} f(x)x^n dx = 0$ for all n. By corollary 3.8.15 there is a polynomial such that $|P(x) - f(x)| \le \varepsilon$ on $[0,1]$. Suppose there doesn't exist a polynomial such that $P(x) = f(x)$. Then for all $P(x)$, we have $|P(x) - f(x)| > 0$. So we have $|P(x) - f(x)| > \lambda > 0$. But we have for some $P(x)$, $|P(x) - f(x)| \le \lambda/2$ by 3.8.15, a contradiction. Therefore we

have a polynomial such that $P(x) = f(x)$. Putting this polynomial into the integral we have

$$\int_{[0,1]} f(x)^2 dx = 0$$

but since $f(x)^2 \geq 0$ and $f(x)$ is continuous, we must have $f(x) = 0$ on $[0, 1]$.

3.8.9

Let ε and $x \in \mathbf{R}$. We shall prove for when either $x = 0$ or $x = 1$ because the other cases can be easily proved using a δ which is smaller than $\min(|x - a|, |x - b|)$ In the case $x = 0$, we have $F(0) = 0$ and since $f(x)$ is continuous we know that $|F(0) - F(y)| = |F(0) - f(y)| < \varepsilon$ for all $y \in [0, 1]$ such that $|0 - y| < \delta$ for some δ. Now for $y < 0$, we consider $\delta < y < 0$ and see that $F(y) = 0$ by definition of F. Thus $|F(0) - F(y)| = 0 < \varepsilon$ and so for all $|0 - y| < \delta$ we have $|F(0) - F(y)| < \varepsilon$ proving continuity at $x = 0$. In the case $x = 1$ we follow the same logic. Thus we see that at all x the function $F(x)$ is continuous as required.

4 Power Series

4.1.1

Suppose hypothesis

a) By the root test and positivity of R we have

$$\limsup_{n \to \infty} |c_n(x - a)^n|^{1/n} = \limsup_{n \to \infty} |c_n|^{1/n}|(x - a)|^{n/n} =$$

$$\limsup_{n \to \infty} |c_n|^{1/n}|(x - a)| > \limsup_{n \to \infty} |c_n|^{1/n} R = \frac{\limsup_{n \to \infty} |c_n|^{1/n}}{\limsup_{n \to \infty} |c_n|^{1/n}} = 1$$

and thus we can conclude that the series does indeed diverge for $|x - a| > R$.

b) Similar to above we have

$$\limsup_{n \to \infty} |c_n(x - a)^n|^{1/n} = \limsup_{n \to \infty} |c_n|^{1/n}|(x - a)|^{n/n} =$$

$$\limsup_{n \to \infty} |c_n|^{1/n}|(x - a)| < \limsup_{n \to \infty} |c_n|^{1/n} R = \frac{\limsup_{n \to \infty} |c_n|^{1/n}}{\limsup_{n \to \infty} |c_n|^{1/n}} = 1$$

and thus we can conclude that the series does indeed absolutely converge for $|x - a| < R$.

c) Since $f^{(n)}(x) = c_n(x-a)^n$ is continuous, it is thus also bounded on the compact interval $[a-r, a+r]$. We then have $||f^{(n)}||_\infty = |c_n r^n|$. Since $r < R$, we have by b that

$$\sum_{n=0}^{\infty} |c_n r^n|$$

does indeed converge. That means by the M-test we can conclude that $\sum_{n=0}^{\infty} c_n(x-a)^n$ does indeed converge uniformly to f and is also continuous as required.

d) Consider the functions

$$f'_{(N)} = \sum_{n=0}^{N} c_n(x-a)^n$$

Clearly the function is continuous and differentiable on the given interval with derivative

$$f'_{(N)} = \sum_{n=1}^{N} c_n n(x-a)^{n-1}$$

Now consider

$$f'(x) = \sum_{n=1}^{\infty} c_n n(x-a)^{n-1}$$

We have $||c_n n(x-a)^{n-1}||_\infty = n|c_n|r^{n-1}$. This results in

$$\sum_{n=1}^{\infty} n|c_n|r^{n-1} = \frac{1}{r}\sum_{n=1}^{\infty} n|c_n|r^n$$

By the root test we have

$$\limsup_{n\to\infty} |nc_n r^n|^{1/n} = \limsup_{n\to\infty} |nc_n|^{1/n} r <$$

$$\limsup_{n\to\infty} |nc_n|^{1/n} R = \frac{\limsup_{n\to\infty} |nc_n|^{1/n}}{\limsup_{n\to\infty} |c_n|^{1/n}} =$$

$$\frac{\limsup_{n\to\infty} |n|^{1/n} \limsup |c_n|^{1/n}}{\limsup_{n\to\infty} |c_n|^{1/n}} = 1$$

where the second to last equality is true due to both the $\limsup$ being convergent and also having a positive sequence. We thus see that

$$\sum_{n=1}^{\infty} n|c_n|r^{n-1}$$

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converges which means by the M-test, we can conclude that

$$f'(x) = \sum_{n=1}^{\infty} c_n n(x-a)^{n-1}$$

uniformly converges to $f'(x)$. We also have that $f(x)$ where $|x-a| < r$ exist due to part b. Thus by Theorem 3.7.1 we can conclude that because the sum converges uniformly to $f(x)$, that means $f(x)$ is differentiable with derivative $f'(x)$ on $[a-r, a+r]$. Since r was chosen arbitrarily as $0 < r < R$. This also means that $f(x)$ is differentiable for all $x \in (a-R, a+R)$.

e)

4.1.2