Terence Tao Analysis I Exercise

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D 3.1.1 Reflexive for all sets A, if $x \in A \Rightarrow x \in A$ therefore A=A is true Symmetric for sets A and B, if A=B $\Rightarrow$ B=A is true because suppose $A = B \Rightarrow (x \in A \Rightarrow x \in B) \wedge (y \in B \Rightarrow y \in A) \Rightarrow B = A$ Transitive for sets A, B, and C, $if\ A = B \wedge B = C \Rightarrow A = C$ is true because suppose $A = B \wedge B = C \Rightarrow (x \in A \Rightarrow x \in B) \wedge (y \in B \Rightarrow y \in A) \wedge (y \in B \Rightarrow y \in C) \wedge (z \in C \Rightarrow z \in B) \Rightarrow (x \in A \Rightarrow x \in B \Rightarrow x \in C) \wedge (z \in C \Rightarrow z \in B \Rightarrow z \in A) \Rightarrow A = C$

E 3.1.2 $\emptyset \neq \{\emptyset\}$ *because* $\emptyset \in \{\emptyset\}$ *but* $x \notin \emptyset$ for *all* x the same can be said about the remaining to sets when compared to the empty set and therefore the empty set is distinct to the other 3 sets. $\{\emptyset\} \neq \{\{\emptyset\}\}$ because the elements are not the same in both sets. $\{\emptyset\} \neq \{\emptyset, \{\emptyset\}\}$ because $\{\emptyset\}$ is not in both sets $\{\{\emptyset\}\} \neq \{\emptyset, \{\emptyset\}\}$ by the same logic thereby proving that all 4 sets are distinct

L 3.1.3 suppose a and b are objects, then $(x \in \{a\} \cup \{b\} \Leftrightarrow x = a \vee x = b \Leftrightarrow x \in \{a, b\}(by\ axiom\ 3.3)\ and\ y \in \{a, b\} \Leftrightarrow y = a \vee y = b \Leftrightarrow x \in \{a\} \cup \{b\}) \Rightarrow$ by definition of equality $\{a, b\} = \{a\} \cup \{b\}$ suppose A,B are sets, then $x \in A \cup B \Leftrightarrow x \in A \vee x \in B \Leftrightarrow x \in B \vee x \in A \Leftrightarrow x \in B \cup A\ therefore\ A \cup B = B \cup A\ by\ definition\ (x \in A \cup A \Leftrightarrow x \in A \vee x \in A \Leftrightarrow x \in A) \Leftrightarrow A \cup A = A\ (x \in A \cup \emptyset \Leftrightarrow x \in A \vee x \in \emptyset \Leftrightarrow x \in A$ *(By definition of an empty set, x is not in the empty set for all x therefore* $x \in \emptyset$ *is always false))* $\Leftrightarrow A \cup \emptyset = A$ $\emptyset \cup A = A \cup \emptyset = A$
it by commutitivity of union and transitivity of equality

A 3.1.11 *Suppose the axiom of replacement* $\Rightarrow P(y) \wedge y \in A \Leftrightarrow (P(y) \wedge x = y)\ for\ some\ x \in A \Leftrightarrow y \in \{z : (P(z) \wedge z = x)\ for\ some\ x \in A\} \Leftrightarrow y \in \{x \in A : P(x)\}$

A 3.2.1 *Suppose the axiom of Universal specification* $\Rightarrow$
$\{x : x\ is\ not\ an\ object\} \Rightarrow P(y)\ is\ false\ for\ every\ object\ y \Rightarrow \forall y \notin \{x : x\ is\ not\ an\ object\}$ *therefore we have an empty set as defined by Axiom 3.2 Suppose the axiom of Universal specification* $\Rightarrow ((a \wedge b\ are\ objects) \Rightarrow \exists\{x : x = a \vee x = b\}$ *which by universal specification, has the property that* $y \in \{x : x = a \vee x = b\} \Leftrightarrow y = a \vee y = b)$ *therefore implying Axiom 3.3*

E 3.2.2 $if\ A\ is\ a\ set \Rightarrow A = \emptyset \vee A \neq \emptyset$ in the case that A is equal to the empty set$\Rightarrow$ $x \notin A$ for all objects x and because A is an object$\Rightarrow$
$A \notin A$ *as required.*in the case that A is not equal to the empty set$\Rightarrow$ by the

singleton set axiom, there exists a set $\{A\}$.
since the only element in the set is A, which is a set, $A \cap \{A\} = \emptyset$by the axiom of regularity
$\Leftrightarrow (x \in A \wedge x \in \{A\} \Leftrightarrow x \in \emptyset)$ by definition of empty set, x is not in the empty set for all objects x. Therefore
$A \notin A$ has to be true because $A \in \{A\}$ *is true*
if A and B are sets$\Rightarrow$ $A \notin B \vee B \notin A$.Through proof by contradiction, Suppose $A \in B \wedge B \in A$
There exists, by the singleton set axiom, sets $\{A\} \wedge \{B\} \Rightarrow \{A\} \cap B = A \Rightarrow$
$A \notin \{A\} \cap B \Rightarrow$
$A \neq A \vee A \notin B.\ as\ A \neq A\ is\ false,\ A \notin B\ is\ true$. This results in a contradiction and therefore the negation must be true thereby proving the implication

A 3.2.3

Suppose Universal Axiom$\Rightarrow$ $\exists\{x : x\ \textit{is an object}\}$such that for every object y, $y \in \{x : x\ \textit{is an object}\} \Leftrightarrow$
y is an object. since every object y is an object, every object is in the set, thereby proving a universal set
Suppose $\exists \Omega | \forall x, x \in \Omega \Rightarrow$(By axiom of specification) $\exists\{x \in \Omega | P(x)\}$such that for every object y,
$y \in \{x \in \Omega | P(x)\} \Leftrightarrow y \in \Omega \wedge P(y) \Leftrightarrow P(y)$ (*since* $y \in \Omega$ *is always true*)
Thereby proving Universal specification and the equivalence of the two statements

E 3.3.8 $A)$*Suppose* $X \subseteq Y \subseteq Z \Rightarrow X \subseteq Z \Rightarrow \iota_{Y \to Z} \circ \iota_{X \to Y} : X \to Z\ and\ \iota_{Y \to Z} \circ \iota_{X \to Y}(x) := \iota_{Y \to Z}(\iota_{X \to Y}(x)) = \iota_{Y \to Z}(x) = x$
$\iota_{X \to Z} : X \to Z\ and\ \iota_{X \to Z}(x) := x.\ since\ \forall x \in X(\iota_{Y \to Z} \circ \iota_{X \to Y}(x) = \iota_{X \to Z}(x))$, and the domain and range are the same, by definition of equality,
$\iota_{Y \to Z} \circ \iota_{X \to Y} = \iota_{X \to Z}\ B)Suppose\ f : A \to B \Rightarrow f(x) = f(\iota_{A \to A}(x)) = \iota_{A \to A} \circ f(x)\ and\ \iota_{B \to B} \circ f(x) = \iota_{B \to B}(f(x)) = f(x)\ therefore$
$f = f \circ \iota_{A \to A} = \iota_{B \to B} \circ f\ C)Suppose\ f : A \to B\ is\ bijective \Rightarrow f \circ f^{-1}(y) = f(f^{-1}(y)) = f(x) = y\ and\ \iota_{B \to B} = y.\ therefore\ f \circ f^{-1} = \iota_{B \to B}.$
$also\ f^{-1} \circ f(x) = f^{-1}(f(x)) = f^{-1}(y) = x\ and\ \iota_{A \to A} = x\ therefore\ f^{-1} \circ f = \iota_{A \to A}\ D)Suppose\ X \cap Y = \emptyset,\ f : X \to Z,\ g : Y \to Z \Rightarrow$ (Suppose functions h and i with properties with properties shown
$\Rightarrow h \circ \iota_{X \to X \cup Y} = i \circ \iota_{X \to X \cup Y}\ and\ h \circ \iota_{Y \to X \cup Y} = i \circ \iota_{Y \to X \cup Y}) \Rightarrow h = i$ therefore h is unique. h also exists because h is a function for all of it's domain

E 3.4.1 $By\ definiton\ f^{-1}(V) = \{f^{-1}(y) : y \in V\}$,by definition of inverse image, $f^{-1}(V) = \{x \in X : f(x) \in V\}$.

But because f^{-1} is defined as the inverse of the bijective function $f \Rightarrow f^{-1}(y) = x \text{ and } f(x) = y \Rightarrow$
$f^{-1}(V) = \{f^{-1}(y) : y \in V\} = \{x \in X : y \in V\} = \{x \in X : f(x) \in V\} \text{ as required}$

E 3.4.3 $Suppose\ f(x) \in f(A \cap B) \Rightarrow x \in A \wedge x \in B \Rightarrow f(x) \in f(A)\ because\ f(A) := \{f(x) : x \in A\}\ and\ f(x) \in f(B)\ because$
$f(B) := \{f(x) : x \in B\} \Rightarrow f(x) \in f(A) \cap f(B) \Leftrightarrow f(A \cap B) \subseteq f(A) \cap f(B)$
$Suppose\ f(x) \in f(A) \cap f(B) \Rightarrow f(x) \in f(A) \wedge f(x) \in f(B) \Rightarrow x \in A \wedge x \in B \Rightarrow x \in A \cap B \Rightarrow f(x) \in f(A \cap B)\ because$
$f(A \cap B) := \{f(x) : x \in A \cap B\} \Leftrightarrow f(A) \cap f(B) \subseteq f(A \cap B)$.Since the two sets are subsets of each other, they are therefore equal.

E 3.4.7 $By\ power\ set\ axiom\ \exists\{X'|X' \subseteq X\}, \exists\{Y'|Y' \subseteq Y\}, \forall X', Y',\ \exists Y'^{X'}$
$\Rightarrow \exists\{Y'^{X'}|X' \in \{X'|X' \subseteq X\} \wedge Y' \in \{Y'|Y' \subseteq Y\}\} := A \Rightarrow \exists \bigcup A. \Rightarrow x \in \bigcup A \Leftrightarrow x \in Y'^{X'}\ for\ some\ Y'^{X'} \in A$

E 3.4.11 $X - \bigcup_{\alpha \in I} A_\alpha = \{x \in X | x \notin \bigcup_{\alpha \in I} A_\alpha\}$ (By definition of difference set)$= \{x \in X | \forall \alpha \in I, x \notin A_\alpha\}$ (By definition of Union set)
$= \{x \in X - A_\beta | \forall \alpha \in I, x \notin A_\alpha\} = \{x \in X - A_\beta | \forall \alpha \in I, x \in X - A_\alpha\} = \bigcap_{\alpha \in I}(X - A_\alpha)\ as\ required$

E 3.5.1 $Suppose\ (x, y)\ and\ (x', y')\ are\ ordered\ pairs \Rightarrow ((x, y) = (x', y') \Leftrightarrow \{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}) \Rightarrow$
$x = x' \wedge y = y'$.because suppose for contradiction, $x \neq x' \vee y \neq y' \Rightarrow when\ x \neq x', \{x\} \notin \{\{x'\}, \{x', y'\}\}$
$But\ by\ definition\ of\ equality\ \{x\} \in \{\{x'\}, \{x', y'\}\}$ therefore a contradiction.
In the case $y \neq y', \{x, y\} \notin \{\{x'\}, \{x', y'\}\}$
But by definition of equality $\{x, y\} \in \{\{x'\}, \{x', y'\}\}$
therefore a contradiction. In the final case, the same can be said and therefore also leads to a contradiction, therefore $x = x' \wedge y = y'$.
$Suppose\ x = x' \wedge y = y' \Rightarrow (\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\} \Leftrightarrow (x, y) = (x', y'))$
therefore ordered pairs defined by sets have the same properties of ordered pairs defined through definition
The cartesian product of two sets $X \times Y = \{(x, y) : x \in X, y \in Y\} = \{\{\{x\}, \{x, y\}\} : x \in X, y \in Y\}$
This is a set because for all x in X there exists a set, $\{x\}$ by the singleton set axiom. By the pair set axiom it can also be said,
$\forall x, y | x \in X \wedge y \in Y, \exists\{x, y\}$, because sets are also objects, by the pair set axiom again, $\exists\{\{x\}, \{x, y\}\}$ which is a set and therefore an object
Then by the axiom of replacement, $\exists\{z : z = \{\{\{x\}, \{x, y\}\}\} for\ some\ y \in$

$Y\ and\ for\ some\ x \in X\}$

3.5.2 $Suppose\ (x_i)_{1\leq i\leq n} = (y_i)_{1\leq i\leq n}$ $\Rightarrow$By definition of equality of functions, $x_i = y_i,\ \forall i \in \{i \in N : 1 \leq i \leq n\} \Leftrightarrow x_i = y_i,\ for\ all\ 1 \leq i \leq n$
$Suppose\ x_i = y_i\ for\ all\ 1 \leq i \leq n \Rightarrow x_i = y_i,\ \forall i \in \{i \in N : 1 \leq i \leq n\} \Rightarrow$ By definition of equality of functions, $(x_i)_{1\leq i\leq n} = (y_i)_{1\leq i\leq n}$
Therefore the iff relationship holds.
$\{(x_i)_{1\leq i\leq n} : x_i \in X_i\ for\ all\ 1 \leq i \leq n\}$is indeed a set. By the power set axiom, there exists a set of all partial functions call A that maps from
$\{i \in N : 1 \leq i \leq n\} \to \bigcup_{1\leq i\leq n} X_i$. By the axiom of specification there then exists a set B where the domain of the partial functions are ,
$\{i \in N : 1 \leq i \leq n\}$,and is surjective. By definition the functions in this set are n ordered tuples. Then by axiom of specification again, we can create a set,
$\{x \in B : x(i) \in X_i\ for\ all\ 1 \leq i \leq n\} \Leftrightarrow \{(x_i)_{1\leq i\leq n} : x_i \in X_i\ for\ all\ 1 \leq i \leq n\}\ as\ required$

3.6.1 X has equal cardinality to X because there exists the bijection $f : X \to X\ defined\ as\ x \mapsto x$.This is a bijection because
$if\ f(x) = f(x') \Rightarrow x = x'\ (injective)\ and\ \forall y \in X, \exists x \in X | f(x) = y \Leftrightarrow \forall y \in X, \exists x \in X | x = y$.This is true because $X = X\ (surjective)$
Since the function defined is surjective and injective it is therefore bijective as required.
if there is a bijection f, between X and Y$\Rightarrow$Y has equal cardinality with X because and is commutative
given X and Y have equal cardinality and Y and Z have equal cardinality$\Rightarrow$ $\exists f : X \to Y\ and\ \exists g : Y \to Z\ where\ f\ and\ g\ are\ bijections$
$\Rightarrow g \circ f : X \to Z\ \ \Rightarrow \forall z \in Z, \exists y \in Y | g(y) = z\ but\ \forall y \in Y, \exists x \in X | f(x) = y \Rightarrow \forall z \in Z, \exists x \in X | g(f(x)) = z$. Therefore a bijection between X and Z meaning X and Z have equal cardinality

3.6.2 $Suppose\ for\ contradiction\ X \neq \emptyset \Rightarrow \exists f : \emptyset \to X,\ \ Where\ f\ is\ a\ bijection \Rightarrow$ $\forall y \in X, \exists x \in \emptyset | f(x) = y$, but because there doesn't exist any x in the empty set, we get a contradiction.
therefore $X = \emptyset$

3.6.3 Base Case: When $n = 0,\ f : \emptyset \to N \Rightarrow M = 0$will make the statement, for all $1 \leq i \leq 0, f(i) \leq 0\ vacuously\ true$
Now suppose for some n, there exists a natural number M such that $f(i) \leq M\ for\ all\ 1 \leq i \leq n \Rightarrow for\ n{+}{+},\ suppose$
$g : \{i \in N : 1 \leq i \leq n{+}{+}\} \to N, for\ all\ 1 \leq i \leq n,\ g(i) := f(i)\ and\ g(n+$

$+) := M++ \Rightarrow for\ all\ 1 \leq i \leq n,\ g(i) \leq M$
$and\ g(n++) \leq M++\ because\ g(n++)+0 = (M++)+0 = M++ \Rightarrow g(i) \leq M++\ for\ all\ 1 \leq i \leq n++$
Therefore M++ satisfies the condition for n++ meaning that for all n there exists a natural number M that satisfies the given condition

4 Integers and Rationals

E 4.1.1

$Let\ (a--b)\ and\ (c--d)\ be\ integers. Then,\ a+b = a+b \Rightarrow (a--b) = (a--b)$. Therefore equality is symmetric
$Suppose\ (a--b) = (c--d) \Rightarrow a+d = b+c \Rightarrow c+b = a+d$ Because equality on natural numbers is symmetric$\Rightarrow$ $(c--d) = (a--d)$ as required showing the equality is reflexive

E 4.1.2 $suppose\ (a--b) = (a'--b') \Rightarrow a+b' = a'+b \Rightarrow b+a' = b'+a \Rightarrow (b--a) = (b'--a') \Rightarrow -(a--b) = -(a'--b')$

E 4.1.3 $Suppose\ a = (c--d) \Rightarrow -1 \times a = (0--1) \times (c--d) = (d--c) = -(c--d) = -a$

E 4.1.4 $x = (a--b),\ y = (c--d),\ z = (e--f)$
$A)x+y = (a--b)+(c--d) = (a+c)--(b+d)$
$y+x = (c--d)+(a--b) = (c+a)--(d+b)\ but,\ c+a = a+c,\ and\ b+d = d+b,\ so\ x+y = y+x$
$B)(x+y)+z = ((a+c)--(b+d))+(e--f) = (a+c+e)--(b+d+f)$
$x+(y+z) = a--b+((c+e)--(d+f)) = (a+c+e)--(b+d+f),\ therefore\ (x+y)+z = x+(y+z)$
$C)x+0 = (a--b)+(0--0) = a--b = x \Rightarrow x+0 = 0+x,\ as\ shown\ by\ A\ and\ by\ transitivity\ 0+x = x$
$D)x+(-x) = (a--b)+(b--a) = (a+b)--(a+b) = 0--0 = 0,\ and\ by\ the\ same\ logic\ as\ C,\ x+(-x) = (-x)+x = 0$
$E)xy = (a--b)\times(c--d) = (ac+bd)--(ad+bc)$
$yx = (c--d)\times(a--b) = (ca+db)--(da+cb),\ therefore\ xy = yx$
$G)x1 = 1x, by\ E$
$H)x(y+z) = (a--b)\times((c+e)--(d+f)) = (a--b)\times((c+e)--(d+f)) = (a(c+e)+b(d+f))--(a(d+f)+b(c+e))$
$xy+xz = ((ac+bd)--(ad+bc))+((ae+bf)--(af+be)) = ((ac+bd+ae+bf)--(ad+bc+af+be))\ So\ x(y+z) = xy+xz$
$I)By\ E\ x(y+z) = (y+z)x = xy+xz = yx+zx$

E 4.1.5 $by\ trichotomy\ of\ integers.\ if\ ab = 0 \Rightarrow a \vee b = 0.\ Suppose\ for\ contradiction, a = b \neq 0 \Rightarrow (a \wedge b\ is\ positive) \vee$
$(a \wedge b\ is\ negative) \vee (b\ is\ positive\ and\ a\ is\ negative) \vee (a\ is\ positive\ and\ b\ is\ negative) \Rightarrow$
$in\ the\ case\ (a \wedge b\ is\ positive)\ then\ ab\ is\ positive.$
$in\ the\ case\ (a \wedge b\ is\ negative)\ then\ ab = (-c)(-d)\ for\ positive\ c\ and\ d.\ then\ (-c)(-d) = cd, which\ is\ positive.$
$in\ the\ case\ b\ is\ positive\ and\ a\ is\ negative\ then\ ab = (-c)b = 0 - -cb = -(cb - -0) = -cb\ which\ is\ negative.$
$Same\ in\ the\ case\ a\ is\ positive\ and\ b\ is\ negative.\ So\ ab \neq 0,\ a\ contradiction.\ Therefore\ a \vee b = 0$

E 4.1.6 $ac = bc \Rightarrow ac + (-(bc)) = bc + (-(bc)) = 0 \Rightarrow ac + (-(bc)) = 0 \Rightarrow (a + -b)c = 0 \Rightarrow\ by, P4.1.8\ a - b = 0\ because$
$c \neq 0 \Rightarrow a - b + b = 0 + b \Rightarrow a = b\ as\ required$

E 4.1.8 all integers n are natural numbers. Base case n=0 is a natural number by definition
$Suppose\ as\ inductive\ hypothesis\ n\ is\ a\ natural\ number \Rightarrow n++\ is\ a\ natural\ number\ by\ definition.$But integer -1 is not a natural number. So all integers aren't natural numbers

E 4.3.1 $A)\ by\ trichotomy,\ x = 0,\ x\ is\ positive\ or,\ x\ is\ negative \Rightarrow when\ x = 0,\ |x| = 0\ so\ |x| \geq 0.\ when\ x\ is\ positive,\ |x| = x$
$x - 0 = x = |x| \Rightarrow |x| \geq 0,\ when\ x\ is\ negative,\ |x| = -x,\ since\ x\ is\ negative\ x = -y\ for\ a\ positive\ y$
$So\ -x = y\ therefore\ |x|\ is\ positive\ resulting\ in\ |x| \geq 0.$ Since for all cases of x, $|x| \geq 0$we can conclude that for a rational x, $|x| \geq 0$
$suppose\ |x| = 0 \Rightarrow$by trichotomy x is either negative positive or 0, in the negative and positive case, $|x| \neq 0\ textsoxhastoequal0$
$B) by\ C, -|x| \leq x \leq |x|\ and\ -|y| \leq y \leq |y| \Rightarrow -|x| - |y| \leq x+y \leq |x|+|y| \Rightarrow$
by trichotomy x+y is either positive negative or 0.
$when\ x+y\ is\ positive,\ |x+y| = x+y\ as\ required.\ when\ x+y\ is\ negative,\ |x+y| = -(x+y),\ so\ |x| + |y| \geq -(x+y) = |x+y| \geq -|x| - |y|\ as\ required$
$when\ x+y = 0, |x+y| = 0 \leq |x|+|y|\ as\ required.\ This\ proves\ the\ inequality.$
$C) Suppose\ -y \leq x \leq y \Rightarrow\ by\ trichotomy\ when\ y = 0,\ x = 0,\ so\ |x| \leq y.$
When y is positive, by trichotomy, x is either 0, positive or negative.
$when\ x = 0,\ |x| = 0,\ so\ |x| \leq y.\ when\ x\ is\ positive,\ |x| = x\ so\ x \leq y.\ when\ x\ is\ negative,\ |x| = -x,\ y \geq -x,\ so\ |x| \leq y.$
In the case y is negative, the statement is vacuously true. Therefore the inequality holds for all cases
$Suppose\ y \geq |x| \Rightarrow y \geq 0 \geq -y \Rightarrow$by trichotomy x is either 0, positive, or

negative. when x is positive, $|x| = x, so$
$y \geq x \geq 0 \geq -y,\ therefore\ y \geq x \geq -y.\ when\ x = 0,\ y \geq x \geq -y.\ when\ x\ is\ negative,\ |x| = -x,\ so\ -y \leq x \leq 0 \leq y$
$so\ -y \leq x \leq y$. Therefore in all cases the inequality holds
$-|x| \leq 0 \leq |x|.\ when\ x = 0\ the\ inequality\ holds.\ when\ x\ is\ positive,\ |x| = x\ therefore\ x \leq |x|\ and\ because\ x\ is\ positive,$
$-|x| \leq x \leq |x|.\ when\ x\ is\ negative,\ -x = |x|.\ so\ -|x| \leq -x \leq |x|\ which\ means\ |x| \geq x \geq -|x|.$
$Therefore\ -|x| \leq x \leq |x|\ for\ all\ rationals\ x$
$D)by\ trichotomy\ x\ and\ y\ is\ either\ positive\ negative\ or\ 0.\ when\ xy\ is\ positive,\ |xy| = xy\ and\ |x||y| = xy\ so \| xy| = |x||y|.$
The rest can be proven similarly.
$-x = -1x \Rightarrow |-x| = |-1||x| = |x|\ as\ required$
$E)x - y\ is\ a\ rational\ number,\ so\ by\ A\ |x-y| \geq 0\ therefore\ d(x,y) \geq 0$
$suppose\ d(x,y) = 0 \Rightarrow |x-y| = 0,\ but\ by\ A,\ x - y = 0 \Rightarrow x = y\ as\ required$
$suppose\ x = y \Rightarrow x - y = 0 \Rightarrow |x-y| = 0 \Rightarrow d(x,y) = 0\ as\ required$
$F)x-y = -(y-x) \Rightarrow\ by\ D\ |x-y| = |-(y-x)| = |y-x|,\ Therefore\ d(x,y) = d(y,x)$
$G)$ by B, $|(x-y)+(y-z)| = |x-z| \leq |x-y| + |y-z| \Rightarrow d(x,z) \leq d(x,y) + d(y,z)$

E 4.3.2 $A)suppose\ x = y \Rightarrow d(y,x) = 0 < \varepsilon$. Therefore x,y are $\varepsilon - close$ for every $\varepsilon > 0$
$suppose\ x,y\ are\ \varepsilon - close\ for\ every\ \varepsilon > 0 \Rightarrow d(y,x) \geq 0,\ Suppose\ d(y,x)\ is\ positive\ for\ contradiction. Then\ d(y,x) \leq d(y,x),\ so\ d(y,x)\ is\ \varepsilon = d(y,x) - close,\ but\ \varepsilon/2 > 0\ therefore\ d(y,x)\ has\ to\ be\ \varepsilon/2 - close$
$meaning\ d(y,x) \leq \varepsilon/2\ a\ contradiction.$
$now\ suppose\ d(y,x)\ is\ 0.\ Then\ d(y,x) < \varepsilon\ as\ required\ therefore\ x = y$
$B)Suppose\ x\ is\ \varepsilon{-}close\ to\ y \Rightarrow d(x,y) \leq \varepsilon,\ but\ d(x,y) = d(y,x),\ by\ P4.3.3 \Rightarrow d(x,y) = d(y,x) \leq \varepsilon \Rightarrow y\ is\ \varepsilon - close\ tox$
$C)suppose\ x\ is\ \varepsilon{-}close\ to\ y\ and\ y\ is\ \delta{-}close\ to\ z \Rightarrow d(x,y) \leq \varepsilon\ and\ d(y,z) \leq \delta \Rightarrow d(x,y) + d(y,z) \leq \varepsilon + \delta,\ but\ d(x,z) \leq d(x,y) + d(y,z)\ by\ 4.3.3$
$\Rightarrow d(x,z) \leq \varepsilon + \delta \Rightarrow x\ is\ (\varepsilon + \delta) - close\ to\ z$
$D)suppose\ x\ is\ \varepsilon - close\ to\ y\ and\ z\ is\ \delta - close\ to\ w \Rightarrow d(x,y) \leq \varepsilon\ and\ d(z,w) \leq \delta \Rightarrow d(x,y) + d(z,w) \leq \varepsilon + \delta \Rightarrow$
$by\ P4.3.3|x-y+z-w| \leq |x-y| + |z-w| = d(x,y) + d(z,w) \leq \varepsilon + \delta \Rightarrow$
$|x-y+z-w| = |x+z-(y+w)| = d(x+z, y+w)$
$\Rightarrow d(x+z, y+w) \leq \varepsilon + \delta\ as\ required.\ Since\ d(z,w) = d(w,z) \Rightarrow$
$by\ P4.3.3|x-y+w-z| \leq |x-y| + |w-z| = d(x,y) + d(w,z) \leq \varepsilon + \delta \Rightarrow$

$|x-y+w-z| = |x-z-(y-w)| = d(x-z, y-w)$
$\Rightarrow d(x-z, y-w) \leq \varepsilon + \delta$ *as required.*
$E)$*Suppose* x *and* y *are* $\varepsilon - close \Rightarrow d(x,y) \leq \varepsilon$.Suppose for contradiction that there exists an $\varepsilon' > \varepsilon$ *such that* x, y *are not* $\varepsilon - close$.
Then $d(x,y) > \varepsilon'$ *but* $\varepsilon \geq d(x,y)$, *so* $\varepsilon > \varepsilon'$, *a contradiction*. Therefore x and y are $, \varepsilon' - close,\ for\ all\ \varepsilon' > \varepsilon$
$F)$*Suppose* $d(y,x) \leq \varepsilon$ *and* $d(z,x) \leq \varepsilon$ *and* $(y \leq w \leq z$ *or* $z \leq w \leq y)$. *Suppose* $y \leq w \leq z \Rightarrow y-x \leq w-x \leq z-x$
$\Rightarrow -\varepsilon \leq y-x \leq w-x \leq z-x \leq \varepsilon$ *by* $P4.3.3 \Rightarrow -\varepsilon \leq w-x \leq \varepsilon \Rightarrow$ *by* $P4.3.3$ $|w-x| = d(w,x) \leq \varepsilon$ *as required*
Suppose $z \leq w \leq y \Rightarrow z-x \leq w-x \leq y-x \Rightarrow$ *by* $P4.3.3$ $-\varepsilon \leq z-x \leq w-x \leq y-x \leq \varepsilon \Rightarrow -\varepsilon \leq w-x \leq \varepsilon \Rightarrow$
by $P4.3.3$ $|w-x| = d(w,x) \leq \varepsilon$ *as required*
$G)$*Suppose* $d(x,y) \leq \varepsilon$ *and* z *is* $non-zero \Rightarrow |xz-yz| = |z(x-y)| = |z||x-y| = d(x,y)|z| \leq \varepsilon|z| \Rightarrow$
$|xz-yz| = d(xz, yz) \leq \varepsilon|z|$ *as required*

E 4.3.3 $A)x^n x^m = x^{n+m}$. *Base case* $n = 0 \Rightarrow x^0 x^m = x^m = x^{0+m}$ *as required*
Suppose as inductive hypothesis $x^n x^m = x^{n+m}$ *for some* $n \Rightarrow x^{n++}x^m = (x^n x)x^m = (x^n x^m)x = x^{n+m}x = x^{n+m+1}$ *as required*
Therefore the equality is true for all n.
$(x^n)^m = x^{nm}$. *Base case* $m = 0 \Rightarrow (x^n)^0 = 1 = x^0 = x^{n*0}$ *as required.*
Suppose as inductive hypothesis $(x^n)^m = x^{nm}$ *for some* $m \Rightarrow (x^n)^{m+1} = (x^n)^m * x^n = x^{nm} * x^n = x^{nm+n} = x^{n(m+1)}$ *as required*
$(xy)^n = x^n y^n$. *Base case* $n = 0 \Rightarrow (xy)^0 = 1 = 1 * 1 = x^0 * y^0$ *as required*
Suppose as inductive hypothesis $(xy)^n = x^n y^n$ *for some* $n \Rightarrow (xy)^{n+1} = (xy)^n * xy = x^n y^n * x * y$
$= x^{n+1}y^{n+1}$ *as required.*
$B)$*Suppose* $x^n = 0 \Rightarrow$ *by trichotomy* x *is either positive, negative, or* 0.
when x *is positive,* $x^1 = x > 0$. *Suppose*
as inductive hypothesis $x^n > 0$ *for some* $n \Rightarrow x^n * x > 0 * x$(positive multiplication preserves order)$\Rightarrow x^{n+1} > 0$.
Therefore for all n, *if* x *is positive* x^n *is positive, a contradiction*
therefore x *isn't positive.when* x *is negative,*a natural number is either even or odd so $m = 2n$ *or* $m = 2n+1$ *for* x^m. *in the case* $m = 2n$,
Base case $n = 0$, *so* $x^{2*0} = 1 > 0$. *Suppose as inductive hypothesis* $x^{2n} > 0$ *for some* $n \Rightarrow x^{2n} * x < 0 * x \Rightarrow x^{2n} * x * x > 0 * x * x$
$\Rightarrow x^{2n}x^2 > 0 \Rightarrow x^{2n+2} = x^{2(n+1)} > 0$, *Therefore* x^m *is positive for all even* m.
In the case $m = 2n+1$, *Base case* $n = 0$, *so* $x^{2*0+1} = x < 0$. *Suppose as*

$inductive\ hypothesis, x^{2n+1} < 0\ for\ some\ n$
$\Rightarrow x^{2n+1} * x > 0 * x$ (negative multiplication reverses order)$\Rightarrow x^{2n+1} * x * x <$
$0 * x * x \Rightarrow$
$x^{2n+1}x^2 = x^{2n+2+1} = x^{2(n+1)+1} < 0,\ Therefore\ x^m\ is\ negative\ for\ all\ odd\ m$
$so\ for\ all\ m\ if\ x\ is\ negative\ x^m\ is\ either\ positive\ or\ negative,\ a\ contradiction.$
$Therefore\ we\ can\ conclude\ x = 0$
$Suppose\ x = 0, Base\ case,\ n = 0\ is\ vacuously\ true.\ Suppose\ as\ inductive\ hypothesis\ x^n = 0\ for\ some\ n$
$\Rightarrow x^n * x = 0 * x = x^{n+1} = 0\ as\ required.$
$C)Lemma,\ if\ x\ is\ positive \Rightarrow x^n\ is\ positive\ for\ all\ natural\ numbers\ n.$
$Proof\ through\ induction.\ Base\ case$
$n = 0 \Rightarrow x^0 = 1\ as\ required.\ Suppose\ x^n\ is\ positive\ for\ some\ n \Rightarrow x^n > 0 \Rightarrow x^n * x > 0 * x \Rightarrow x^{n+1} > 0\ thereby\ closing\ the\ induction$
$C)if\ x \geq y \geq 0 \Rightarrow x^n \geq y^n \geq 0.\ for\ contradiction\ Suppose\ x \geq y \geq 0\ and\ x^n < y^n < 0\ \Rightarrow x \geq 0. \Rightarrow when\ x\ is\ positive,$
$by\ the\ above\ Lemma\ x^n > 0, a\ contradiction.\ when\ x = 0,\ x^0 = 1 > 0, and\ for\ n > 0,\ x^n = 0\ by\ B.$
In all cases we have a contradiction and therefore the original statement must be true.
$Suppose\ x > y \geq 0\ and\ n > 0 \Rightarrow x^n > y^n \geq 0$
$Proof\ by\ induction.\ Base\ case\ n = 0, so\ the\ statement\ is\ vacuously\ true.\ Suppose\ x^n > y^n \geq 0\ for\ some\ n \Rightarrow$
$x^n * x > y^n * x \geq 0\ and\ x^n * y > y^n * y \geq 0,\ but\ because\ x > y,\ x^n * x > x^n * y,\ so\ x^n * x > x^n * y > y^n * y \geq 0 \Rightarrow$
$x^{n+1} > y^{n+1} \geq 0\ as\ required$
$D)|x^n| = |x|^n.\ Base\ case\ n = 0 \Rightarrow |x^0| = |1| = 1 = |x|^0\ as\ required.$
$Suppose\ as\ inductive\ hypothesis\ |x^n| = |x|^n\ for\ some\ n \Rightarrow |x^{n+1}| = |x^n * x| = |x^n||x| = |x|^n|x| = |x|^{n+1}\ as\ required$

4.3.4 $A)x^nx^m = x^{n+m}\ for\ natural\ numbers\ n\ and\ m.\ In\ the\ case\ n\ and\ m\ is\ negative,\ x^nx^m = x^{-p}x^{-q}\ for\ positive\ p\ and\ q$
$\Rightarrow x^nx^m = (1/x^p) * (1/x^q) = 1/(x^px^q) = 1/(x^{p+q})$Because p and q are positive integers.$\Rightarrow 1/x^{p+q} = x^{-(p+q)} = x^{-p-q} = x^{n+m}\ as\ required$
$In\ the\ case\ n\ is\ negative\ and\ m\ is\ positive \Rightarrow x^{-p}x^m\ and\ x^{m-p}\ for\ positive\ p\ and\ m \Rightarrow$
$when\ m - p = 0,\ m = p$
$So\ x^{-p}x^m = x^{-m}x^m = (1/x^m) * x^m = 1 = x^{m-p}$
$when\ m - p\ is\ positive,\ m = p + c\ for\ positive\ c.\ So\ x^{-p}x^m = x^{-p}x^{p+c} = x^{-p}x^px^c = (x^p/x^p)x^c = x^c = x^{m-p} = x^{-p+m}$
$when\ m - p\ is\ negative\ p = m + c\ for\ positive\ c.\ So\ x^{-p}x^m = x^{-(m+c)}x^m =$

$x^m/x^{m+c} = x^m/(x^m x^c) = 1/x^c = x^{-c} =$
$x^{-(p-m)} = x^{-p+m}$
in the case m *is negative and* n *is positive,* $x^n x^m = x^m x^n$*, which is the same as above so equivalent proof.*
Therefore in all cases the equality holds proving it true
$(x^n)^m = x^{nm}$*, for natural numbers* n *and* m*. In the case* $-n$ *and* $-m$*, using induction*
on m *we will prove* $(x^{-n})^{-m} = x^{nm}$*. Base case* $m = 0$*,* $(x^{-n})^0 = 1 = x^0 = x^{n0}$ *as required. Suppose as inductive hypothesis*
$(x^{-n})^{-m} = x^{nm} \Rightarrow (x^{-n})^{-(m+1)} = 1/(x^{-n})^{m+1} = 1/((x^{-n})^m x^{-n}) = 1/(x^{-n})^m * x^n = (x^{-n})^{-m} * x^n = x^{nm} * x^n = (x^n)^m * x^n$
$= (x^n)^{m+1} = x^{n(m+1)}$ *as required. But* $nm = -n*-m$*, So* $(x^{-n})^{-m} = x^{nm} = x^{-n*-m}$*.*
In the case $-m$ *and* $n \Rightarrow (x^n)^{-m} = 1/(x^n)^m = 1/x^{nm} = x^{-nm} = x^{n*-m}$ *as required.*
In the case $-n$ *and* m *we have* $(x^{-n})^m = (1/x^n)^m = ((1/x)^n)^m = (1/x)^{nm} = 1/x^{nm} = x^{-nm}$ *as required*
Therefore the equality holds in all cases
$(xy)^n = x^n y^n$ *for natural number* $n \Rightarrow (xy)^{-n} = 1/(xy)^n = 1/(x^n y^n) = x^{-n}y^{-n}$ *as required*
B)Suppose $x \geq y > 0$ *and* $n > 0 \Rightarrow$ *Proof by induction. Base case* $n = 1$*,* $x^1 = x \geq y^1 = y > 0$ *as required*
Suppose as inductive hypothesis $x^n \geq y^n > 0 \Rightarrow x^n * x \geq x^n * y > 0$ *and* $x^n * y \geq y^n * y > 0 \Rightarrow x^n * x \geq x^n * y \geq y^n * y > 0 \Rightarrow$
$x^n * x \geq y^n * y > 0 \Rightarrow x^{n+1} \geq y^{n+1} > 0$ *as required*
Suppose $x \geq y > 0$ *and* $n = -p$ *for positive* $p \Rightarrow$ *Proof by induction. Base case* $p = 1$*,* $0 < x^{-1} \leq y^{-1}$ *is true because*
$x * x^{-1} \geq y * x^{-1} > 0 \Rightarrow 1 \geq y * x^{-1} > 0$ *and* $y^{-1} \geq y * y^{-1} * x^{-1} = x^{-1} > 0$ *as required*
Suppose as inductive hypothesis $0 < x^{-p} \leq y^{-p} \Rightarrow 0 < x^{-p} * y^{-1} \leq y^{-p} * y^{-1}$ *and* $0 < x^{-1} * x^{-p} \leq y^{-1} * x^{-p} \Rightarrow$
$0 < x^{-1} * x^{-p} \leq x^{-p} * y^{-1} \leq y^{-p} * y^{-1} \Rightarrow 0 < x^{-(p+1)} \leq y^{-(p+1)}$ *as required*
C)Suppose $x, y > 0, n \neq 0,$ *and* $x^n = y^n \Rightarrow x = y$*. Suppose* $x \neq y$ *for contradiction* $\Rightarrow y > x$ *or* $y < x$*.*
In the case $y > x$*,* $y > x > 0 \Rightarrow$ *if* $n > 0$*,* $y^n > x^n \geq 0$*, So* $y^n > x^n$ *a contradiction. if* $n < 0$*,* $0 < x^{-1} < y^{-1}$*,*
so $0 \leq (x^{-1})^{-n} < (y^{-1})^{-n}$*. Therefore* $y^n > x^n$ *a contradiction.*
In the case $y < x$*,* $x > y > 0$ *which is of the same form as above*

and therefore also results in a contradiction.
Since all other cases besides x=y results in a contradiction, we can say x=y
$D)|x^n| = |x|^n$ *for natural number n.* $|x^{-n}| = |(1/x)^n| = |1/x|^n$.
x is either, positive, or negative. In the case
x is negative $|1/x| = -1/x$, *so* $|x^{-n}| = |1/x|^n = (-1/x)^n = ((-x)^{-1})^n = (-x)^{-n} = |x|^{-n}$
In the case x is positive, $|1/x| = 1/x$, *so* $|x^{-n}| = |1/x|^n = (1/x)^n = x^{-n} = |x|^{-n}$.
In all cases the equality holds and therefore is true.

E 4.3.5 $N > 0 \Rightarrow$ *Proof by induction. Base case,* $N = 1 \Rightarrow 2^1 = 2 \geq 1$. *Suppose* $2^N \geq N$ *for some* $N \Rightarrow$
$2^N * 2 = 2^{N+1} \geq 2 * N = N + N$, *but* $N \geq 1$ *so* $N + N \geq N + 1 \Rightarrow 2^{N+1} \geq N + 1$ *as required.*

E 4.4.1 *Suppose* $x \geq 0 \Rightarrow x = a/b$ *for natural numbers a and b* $\Rightarrow a = mb + r$ *where* $0 \leq r < b$ *and m is a natural number*
$a/b = (mb+r)/b = m+r/b$, *but* $0 \leq r < b$ *so* $0 \leq\ r/b < 1 \Rightarrow m \leq m+r/b = a/b = x < m + 1$ *as required.*
Suppose $x < 0 \Rightarrow x = -c$ *for* $c > 0 \Rightarrow m \leq c < m + 1 \Rightarrow m < m + 1 \leq c + 1 \Rightarrow -m > -c - 1 \Rightarrow -m + 1 > -c$ *and*
$-m \geq -c$ *so* $-m+1 > -c = x \geq -m$ *as required. The inequality therefore holds for all rational x.*
Proof of uniqueness. Suppose two integers $n \neq n'$ *such that both satisfy* $m \leq x < m + 1 \Rightarrow n < n'$ *or* $n > n'$. *if* $n < n'$
$\Rightarrow n < n' \leq x < n + 1 < n' + 1$ *but* $n' = n + p$ *for positive integer p so* $p \geq 1 \Rightarrow p = 1 + c$ *where c is a natural number*
$\Rightarrow n' = n + p = n + 1 + c \Rightarrow n' \geq n + 1$ *which contradicts* $n + 1 > n'$.
if $n > n' \Rightarrow n' < n \leq x < n'+1 < n+1$ *the rest follows as above resulting in* $n \geq n' + 1$ *contradicting* $n' + 1 > n$.
Since all cases led to contradiction, $n = n'$.
Suppose $x < 0 \Rightarrow$ *the natural number* $0 > x$ *by definition*
Suppose $x \geq 0 \Rightarrow$ *exists an integer n such that* $n \leq x < n + 1 \Rightarrow 0 \leq x < n + 1 \Rightarrow 0 \leq n + 1$
it can be said that there exists a natural number N such that $N > x$ *as required.*

E 4.4.2 $A)$*Suppose for contradiction that there is a sequence of natural numbers in infinite descent* $\Rightarrow$ *Base case* $a_n \geq 0$ *for all*
Suppose as inductive hypothesis $a_n \geq k$ *for some k and all n* $\Rightarrow a_n > a_{n+1}$ *and* $a_{n+1} \geq k \Rightarrow a_n \geq a_{n+1}$ *and*
$a_{n+1} \geq k + 1 \Rightarrow a_n \geq k + 1$ *as required*

$Therefore\ a_n \geq k\ for\ all\ k \in N\ and\ all\ n \in N.\ But\ a_0 > a_1\ and\ since\ a_0 \in N,\ a_1 \geq a_0\ by\ the\ induction.\ Therefore\ a\ contradiction$
$B) Suppose\ a_n = -n\ where\ n\ is\ a\ natural\ number \Rightarrow -1 < 0 \Rightarrow -n - 1 < -n \Rightarrow -(n+1) = a_{n+1} < -n = a_n$
$for\ all\ n\ as\ required.\ Therefore\ the\ principle\ of\ infinite\ descent\ does\ not\ hold\ for\ integers.\ Suppose\ a_n = n^{-1}\ for\ natural\ number\ n \Rightarrow 0 < 1 \Rightarrow n < n+1 \Rightarrow nn^{-1}(n+1)^{-1} < (n+1)n^{-1}(n+1)^{-1} \Rightarrow$
$(n+1)^{-1} < n^{-1}\ but\ (n+1)^{-1} = a_{n+1} < n^{-1} = a_n\ as\ required.\ Therefore\ the\ principle\ of\ infinite\ descent\ does\ not\ hold\ for\ positive\ rationals$

E 4.4.3 $if\ n\ is\ a\ natural\ number \Rightarrow n = 2k+1\ or\ n = 2k\ for\ some\ natural\ number\ k.\ Proof\ by\ inductionBase\ case\ n = 0 = 2*0\ as\ required.$
$Suppose\ as\ inductive\ hypothesis\ n = 2k\ or\ n = 2k+1\ for\ some\ n \Rightarrow$
$in\ the\ case\ n = 2k,\ n+1 = 2k+1\ which\ is\ odd.\ In\ the\ case\ n = 2k+1,\ n+1 = 2k+2 = 2(k+1)\ which\ is\ even.$
$Therefore\ n+1\ is\ either\ even\ or\ odd\ closing\ the\ induction.\ Suppose\ p = 2k+1 \Rightarrow p^2 = p*p = (2k+1)(2k+1)$
$= 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1\ which\ is\ odd\ as\ required.$
$we\ shall\ prove\ if\ p, q > 0\ and\ p^n > q^n\ for\ n > 0 \Rightarrow p > q.\ Suppose\ p \leq q\ for\ contradiction \Rightarrow\ in\ the\ case\ p = q,$
$p^n = q^n.\ in\ the\ case\ p < q,\ 0 < p < q\ so\ by\ proposition\ 4.3.10\ q^n \geq p^n.\ In\ both\ cases\ we\ have\ a\ contradiction\ so\ p > q$
$Suppose\ p^2 = 2q^2 \Rightarrow p^2 = q^2 + q^2 \Rightarrow p^2 > 0\ so\ p^2 + q^2 + q^2 > q^2 + q^2 \Rightarrow 2p^2 > 2q^2 \Rightarrow p^2 > q^2 \Rightarrow p > q\ as\ required.$

5 The Real Numbers

E 5.1.1 $Suppose\ (a)_{n=1}^{\infty}\ is\ a\ cauchy\ sequence \Rightarrow there\ exists\ an\ N\ such\ that\ (a_n)_{n=N}^{\infty}\ is\ 1-steady \Rightarrow (a_n)_{n=1}^{N}\ is\ bounded$
$by\ some\ M\ according\ to\ L5.1.14 \Rightarrow\ because\ (a_n)_{n=N}^{\infty}\ is\ 1-steady,\ |a_N - a_n| \leq 1\ for\ all\ n \geq N \Rightarrow$
$we\ know\ |a_N| \leq M,\ so\ |-a_N| \leq M \Rightarrow |-a_N| + |a_N - a_n| \leq M + |a_N - a_n|\ but\ by\ the\ triangle\ inequality$
$|-a_N + a_N - a_n| \leq |-a_N| + |a_N - a_n| \Rightarrow |-a_N + a_N - a_n| = |-a_n| = |a_n| \leq M + |a_N - a_n|,\ but$
$M + |a_N - a_n| \leq M+1\ so\ |a_n| \leq M+1\ because\ |a_n| \leq M \leq M+1\ for\ all\ n \leq N\ and\ |a_n| \leq M+1\ for\ all\ n \geq N, it\ can\ be\ concluded\ that\ |a_n| \leq M +$

1 $for\ all\ n\ and\ therefore\ cauchy\ sequences\ are\ bounded.$

E 5.2.1 $Suppose\ (a_n)_{n=1}^{\infty}\ and\ (b_n)_{n=1}^{\infty}\ are\ equivalent\ sequences\ of\ rationals.$ $Suppose\ (a_n)_{n=1}^{\infty}\ is\ a\ cauchy\ sequence \Rightarrow |a_n - b_n| \leq \varepsilon\ for\ all\ n \geq N_1\ and\ |a_j - b_j| \leq \varepsilon\ for\ all\ j \geq N_1\ and$
$|a_j - a_n| \leq \varepsilon\ for\ some\ N_2\ such\ that\ n, j \geq N_2 \Rightarrow let\ N = max(N_1, N_2) \Rightarrow$ $|a_j - a_n + b_n - b_j| \leq |b_n - a_n| + |a_j - b_j| \Rightarrow |b_n - b_j| \leq |a_j - a_n + b_n - b_j| + |a_n - a_j| \leq$ $3\varepsilon\ but\ because\ all\ rational\ can\ be\ expressed\ by\ 3\varepsilon\ which\ \varepsilon\ is\ another$ $rational, it\ can\ be\ said\ that\ |b_n - b_j|\ is\ \delta - close\ for\ all\ \delta > 0$
$Therefore\ (b_n)\ is\ a\ cauchy\ sequence\ the\ converse\ is\ also\ true\ since\ it$ $follows\ the\ same\ form.$

E 5.2.2 $Suppose\ (a_n)_{n=1}^{\infty}\ and\ (b_n)_{n=1}^{\infty}\ are\ eventually\ \varepsilon - close$
$Suppose\ (a_n)_{n=1}^{\infty}\ is\ bounded \Rightarrow |a_n - b_n| \leq \varepsilon\ for\ all\ n \geq N$
$and\ |a_n| \leq M\ for\ all\ n \geq 1 \Rightarrow (b_m)_{m=1}^{N}\ is\ bounded\ so\ b_m \leq K \Rightarrow |-b_n| \leq$ $|a_n - b_n| + |-a_n| \leq \varepsilon + |-a_n| \Rightarrow$
$|b_n| \leq \varepsilon + |a_n| \leq \varepsilon + M \Rightarrow |b_n| \leq \varepsilon + M\ for\ all\ n \geq N\ and\ |b_m| \leq$ $K\ for\ all\ m \leq N\ so\ |b_m| \leq K + \varepsilon + M\ and$
$|b_n| \leq K + \varepsilon + M \Rightarrow for\ all\ k \geq 1, |b_k| \leq K + \varepsilon + M\ and\ therefore\ (b_n)_{n=1}^{\infty}$ $bounded.\ the\ converse\ can$
$be\ proved\ in\ the\ same\ way$

L5.3.14 (Kinda Important) $0 < |b_{n_0} - b_n| \leq \varepsilon/2\ and\ |b_{n_0}| > \varepsilon \Rightarrow |b_{n_0}| \leq$ $|b_{n_0} - b_n| + |b_n| \leq \varepsilon/2 + |b_n| \Rightarrow \varepsilon/2 + |b_n| \geq |b_{n_0}| > \varepsilon \Rightarrow$
$|b_n| > \varepsilon/2$

E 5.3.1 $x = x\ because\ (a_n)_{n=1}^{\infty}\ and\ (a_n)_{n=1}^{\infty}\ are\ \varepsilon - close\ for\ all\ \varepsilon >$ $0. This\ is\ because\ |a_n - a_n| = 0 < \varepsilon.$
$Therefore\ (a_n)_{n=1}^{\infty}\ and\ (a_n)_{n=1}^{\infty}\ are\ equivalent\ as\ required.$
$Suppose\ x = y \Rightarrow (a_n)_{n=1}^{\infty}\ is\ equivalent\ to\ (b_n)_{n=1}^{\infty} \Rightarrow |a_n - b_n| \leq \varepsilon\ for\ all\ n \geq$ $N, but\ |a_n - b_n| = |b_n - a_n| \Rightarrow$
$|b_n - a_n| \leq \varepsilon\ for\ all\ n \geq N \Rightarrow (b_n)_{n=1}^{\infty}\ is\ equivalent\ to\ (a_n)_{n=1}^{\infty} \Rightarrow y =$ $x\ as\ required$
$Suppose\ x = y\ and\ y = z \Rightarrow |a_n - b_n| \leq \varepsilon/2\ for\ all\ n \geq N_1\ and\ |b_n - c_n| \leq$ $\varepsilon/2\ for\ all\ n \geq N_2. Let\ N = max(N_1, N_2)$
$\Rightarrow |a_n - b_n|, |b_n - c_n| \leq \varepsilon/2\ for\ all\ n \geq N \Rightarrow |a_n - c_n| \leq |a_n - b_n| + |b_n - c_n| \leq$ $\varepsilon. Therefore\ it\ can\ be$
$concluded\ that\ x = z$

5.3.2 $|a_n| \leq M_1\ and\ |b_n| \leq M_2\ for\ all\ n \geq 1\ and\ |a_n - a_m| \leq \varepsilon'\ for\ n, m \geq$ $N_1\ and\ |b_n - b_m| \leq \delta\ for\ n, m \geq N_2 \Rightarrow$
$let\ N = max(N_1, N_2),\ so\ |a_n - a_m| \leq \varepsilon'\ and\ |b_n - b_m| \leq \delta\ for\ all\ n, m \geq$ $N.\ let\ M = max(M_1, M_2),\ |a_n|, |b_n| \leq M$

$\Rightarrow |a_nb_n - a_mb_m| \leq \varepsilon'|b_n| + \delta|a_n| + \varepsilon'\delta \Rightarrow |a_nb_n - a_mb_m| \leq \varepsilon'M + \delta M + \varepsilon'\delta$
$defining\ variables.\ \varepsilon'M, \delta M,\ and\ \varepsilon'\delta \leq \varepsilon/3 \Rightarrow \varepsilon' \leq \varepsilon/(3M), \delta \leq \varepsilon/(3M) \Rightarrow$
$let\ \varepsilon' = min(\varepsilon/3, \varepsilon/(3M))\ and\ let$
$\delta = min(1, \varepsilon/(3M)).\ This\ would\ satisfy\ the\ above\ conditions\ allowing$
$|a_nb_n - a_mb_m| \leq \varepsilon'|b_n| + \delta|a_n| + \varepsilon'\delta \Rightarrow |a_nb_n - a_mb_m| \leq \varepsilon'M + \delta M + \varepsilon'\delta \leq \varepsilon.\ Therefore\ (a_nb_n)_{n=1}^{\infty}\ is\ a\ cauchy\ sequence$
$and\ LIM_{n\to\infty}a_nb_n = xy\ is\ a\ real\ number.$
$Suppose\ x = x' \Rightarrow xy = (a_nb_n)_{n=1}^{\infty}\ and\ x'y = (a'_nb_n)_{n=1}^{\infty}\ and\ |a_n - a'_n| \leq \varepsilon/M\ for\ all\ n \geq N\ and\ |b_n| \leq M \Rightarrow$
$|a_n - a'_n||b_n| \leq \varepsilon \Rightarrow |a_nb_n - a'_nb_n| \leq \varepsilon \Rightarrow (a_nb_n)_{n=1}^{\infty}\ and\ (a'_nb_n)_{n=1}^{\infty}\ are\ equivalent$
$as\ required$

E 5.3.3 $Suppose\ a = b \Rightarrow (a)_{n=1}^{\infty}\ and\ (b)_{n=1}^{\infty}\ is\ cauchy\ since\ |a-a| = |b-b| = 0 \Rightarrow |a-b| = 0 \leq \varepsilon\ for\ all\ n \geq 1 \Rightarrow$
$Since\ 0 \leq \varepsilon\ for\ all\ \varepsilon > 0,\ it\ can\ be\ concluded\ that\ the\ two\ are\ equivalent$
$therefore\ LIM_{n\to\infty}a = LIM_{n\to\infty}b$
$Suppose\ LIM_{n\to\infty}a = LIM_{n\to\infty}b \Rightarrow |a-b| \leq \varepsilon\ for\ all\ \varepsilon > 0\ so\ a = b\ as\ required$

E 5.3.4 $Since\ (a_n)_{n=1}^{\infty}\ and\ (b_n)_{n=1}^{\infty}\ are\ equivalent,\ they\ are\ eventually\ \varepsilon-close\ for\ all\ \varepsilon > 0 \Rightarrow$
$by\ E5.2.2\ we\ have\ (b_n)_{n=1}^{\infty}\ is\ bounded\ since\ (a_n)_{n=1}^{\infty}\ is\ bounded$

E 5.3.5 $Lemma: if\ n \geq N \Rightarrow n * 1/(nN) \geq N * 1/(nN) \Rightarrow 1/N \geq 1/n$
$(1/n)_{n=1}^{\infty} \Rightarrow\ let\ \varepsilon > 0\ and\ N \geq 1\ we\ see\ that\ |1/n - 0| = |1/n| = 1/n\ for\ all\ n \geq N\ are\ less\ than\ 1/N. \Rightarrow |1/n - 0| \leq 1/N\ for\ all\ n \geq N.\ we\ have\ to\ choose\ an\ N\ such\ that\ 1/N \leq \varepsilon\ for\ any\ choice\ of\ \varepsilon > 0.\ Since\ 1 \leq \varepsilon N\ we\ have\ N \geq 1/\varepsilon.\ and\ by\ P4.4.1\ there\ always\ exists\ such\ an\ N$
$So\ we\ have\ |1/n - 0| \leq 1/N \leq \varepsilon\ for\ all\ n \geq N, showing\ that\ LIM_{n\to\infty}1/n = 0$

E 5.4.1 $if\ x \neq 0\ and\ the\ formal\ limit\ of\ some\ sequence\ (a_n)_{n=1}^{\infty} \Rightarrow$
$the\ sequence\ is\ either\ eventually\ positively\ or\ negatively\ bounded\ from\ 0$
$Suppose\ hypothesis \Rightarrow |a_n| \geq \varepsilon\ for\ all\ n \geq N_1\ and\ |a_n - a_m| \leq \varepsilon/2\ for\ all\ n, m \geq N_2 \Rightarrow let\ N = max(N_1, N_2)\ So, |a_n| \geq \varepsilon > 0\ for\ all\ n \geq N\ and\ |a_n - a_m| \leq \varepsilon/2\ for\ all\ n, m \geq N \Rightarrow a_N > 0\ or\ a_N < 0.\ In\ the\ case\ a_N > 0,\ |a_N| = a_N \geq \varepsilon \Rightarrow a_m\ is\ either\ positive\ or\ negative.\ Suppose\ a_m\ is\ negative \Rightarrow$
$|a_m| = -a_m \geq \varepsilon \Rightarrow a_N - a_m \geq 2\varepsilon > \varepsilon/2 > 0 \Rightarrow |a_N - a_m| > \varepsilon/2\ a\ contradiction.$
$Therefore\ a_m\ is\ positive\ and\ a_n \geq \varepsilon > 0\ for\ all\ n \geq N.$
$In\ the\ case\ a_N < 0,\ |a_N| = -a_N \geq \varepsilon \Rightarrow a_m\ is\ either\ negative\ or\ positive.$
$Suppose\ a_m\ is\ positive \Rightarrow |a_m| = a_m \geq \varepsilon \Rightarrow a_N \leq -\varepsilon\ and\ -a_m \leq -\varepsilon \Rightarrow a_N - a_m \leq -2\varepsilon < 0 \Rightarrow |a_N - a_m| = -(a_N - a_m) \geq 2\varepsilon >$

$\varepsilon/2$ a contradiction. Therefore a_m is negative and $a_n \leq \varepsilon < 0$ for all $n \leq N$. it can be concluded that either $a_n > 0$ for all $n \geq N$ or $a_n < 0$ for all $n \geq N$. The above can be further extended. Suppose $x = LIM_{n\to\infty} a_n$ is eventually positively bounded away from $0 \Rightarrow a_n \geq \varepsilon > 0$ for all $n \geq N$ for some $N \geq 1 \Rightarrow$ define a sequence $b_n = a_n$ for all $n \geq N$ and $b_n = \varepsilon$ for all $n < N \Rightarrow (b_n)_{n=1}^{\infty}$ is positively bound and equal to x. Therefore x is positive.Suppose $x = LIM_{n\to\infty} a_n$ is eventually negatively bounded away from $0 \Rightarrow a_n \leq -\varepsilon < 0$ for all $n \geq N$ for some $N \geq 1 \Rightarrow$ define a sequence $b_n = a_n$ for all $n \geq N$ and $b_n = -\varepsilon$ for all $n < N \Rightarrow (b_n)_{n=1}^{\infty}$ is negatively bound and equal to x. Therefore x is negative.

Finally we can conclude that if $x \neq 0$ and the formal limit of some sequence $(a_n)_{n=1}^{\infty} \Rightarrow x$ is either positive or negative $x = LIM_{n\to\infty}(a_n)$

First proving that at most one is true.

Suppose x is positive and x is $0 \Rightarrow |a_n - b_n| \leq \varepsilon/2$ and $b_n \geq \varepsilon > 0$ and $\varepsilon > 0,\ |a_n| \leq \varepsilon/4$

for all $n \geq N \Rightarrow |a_n - b_n| + |-a_n| \leq \varepsilon/2 + |a_n| \Rightarrow \varepsilon \leq |b_n| \leq \varepsilon/2 + |a_n| \Rightarrow \varepsilon/2 \leq |a_n| \leq \varepsilon/4$

$\Rightarrow \varepsilon/2 \leq \varepsilon/4,$ So $0 < \varepsilon \leq \varepsilon/2$ a contradiction

Suppose x is negative and x is $0 \Rightarrow |a_n - b_n| \leq \varepsilon/2$ and $b_n \leq -\varepsilon < 0$ and $\varepsilon > 0,\ |a_n| \leq \varepsilon/4$

for all $n \geq N \Rightarrow |a_n - b_n| + |-a_n| \leq \varepsilon/2 + |a_n|,$ and $|b_n| = -b_n \geq \varepsilon \Rightarrow \varepsilon \leq |b_n| = -b_n \leq \varepsilon/2 + |a_n|$

$\Rightarrow \varepsilon/2 \leq |a_n| \leq \varepsilon/4 \Rightarrow \varepsilon/2 \leq \varepsilon/4,$ So $0 < \varepsilon \leq \varepsilon/2$ a contradiction

Suppose x is negative and x is positive $\Rightarrow |a_n - b_n| \leq \varepsilon'/2$ for all $n \geq N_1$ and $b_n \geq \varepsilon' > 0$

and $|a_n - c_n| \leq \varepsilon'/2$ for all $n \geq N_2$ and $c_n \leq -\delta < 0 \Rightarrow$ let $\varepsilon = min(\varepsilon', \delta),$ so $|a_n - b_n|, |a_n - c_n| \leq \varepsilon/2,$

$b_n \geq \varepsilon > 0$ and $c_n \leq -\varepsilon < 0 \Rightarrow |b_n - c_n| \leq |b_n - a_n| + |a_n - c_n| \leq \varepsilon$ and $-c_n \geq \varepsilon,$ so $|b_n - c_n| = b_n - c_n \geq 2\varepsilon$

$\Rightarrow \varepsilon \geq |b_n - c_n| \geq 2\varepsilon \Rightarrow 1 \geq 2$ a contradiction

Proving at least one is true.

Suppose x isn't positive or $0 \Rightarrow x$ is either positive or negative and x is not positive $\Rightarrow x$ is negative

Suppose x isn't negative or $0 \Rightarrow x$ is either positve or negative and is not negative $\Rightarrow x$ is positive.

Suppose x isn't positive and isn't negative $\Rightarrow x$ is 0 by contrapositive

Trichotomy done

Suppose x is negative $\Rightarrow x = LIM_{n\to\infty} a_n$ for some negatively bound

$sequence\ (a_n)_{n=1}^{\infty}\ So\ -x = -LIM_{n\to\infty}a_n = LIM_{n\to\infty} - a_n\ but\ we\ have\ a_n \leq -c < 0\ for\ all\ a_n \Rightarrow -a_n \geq c > 0\ meaning\ LIM_{n\to\infty} - a_n\ is\ positively\ bound\ and\ because\ -x = LIM_{n\to\infty} - a_n,\ -x\ is\ positive\ as\ required.\ Suppose\ -x\ is\ positive \Rightarrow -x = LIM_{n\to\infty}a_n\ for\ some\ positively\ bound\ sequence\ (a_n)_{n=1}^{\infty}\ So\ x = -LIM_{n\to\infty}a_n = LIM_{n\to\infty} - a_n\ but\ we\ have\ a_n \geq c > 0\ for\ all\ a_n \Rightarrow -a_n \leq -c < 0\ meaning\ LIM_{n\to\infty} - a_n\ is\ negatively\ bound\ and\ because\ x = LIM_{n\to\infty} - a_n,\ x\ is\ negative\ as\ required.\ Suppose\ x\ and\ y\ are\ positive \Rightarrow x = LIM_{n\to\infty}a_n\ and\ y = LIM_{n\to\infty}b_n\ such\ that\ a_n \geq c > 0\ and\ b_n \geq d > 0 \Rightarrow x + y = LIM_{n\to\infty}a_n + b_n\ but\ a_n + b_n \geq c + d > 0\ so\ LIM_{n\to\infty}a_n + b_n\ is\ positively\ bound\ and\ therefore\ x+y\ is\ positive.\ xy = LIM_{n\to\infty}a_nb_n\ but\ a_nb_n \geq cd > 0\ so\ LIM_{n\to\infty}a_nb_n\ is\ positively\ bound\ and\ therefore\ xy\ is\ positive.$

5.4.2 $A)Proving\ at\ least\ one\ true\ Suppose\ x \neq y\ and\ x\ is\ not\ less\ than\ y \Rightarrow x - y \neq 0\ and\ x - y\ is\ not\ negative\ so\ by\ trichotomy\ x - y\ is\ positive\ by\ definition\ this\ means\ x > y\ Suppose\ x \neq y\ and\ x - y\ is\ not\ positive \Rightarrow by\ trichotomy\ x - y\ is\ negative \Rightarrow by\ definition\ x < y\ Suppose\ x - y\ is\ not\ positive\ and\ x - y\ is\ not\ negative \Rightarrow\ by\ trichotomy\ x - y = 0 \Rightarrow x = y\ Proving\ at\ most\ one\ is\ true\ Suppose\ x = y\ and\ x < y \Rightarrow x - y = 0\ and\ x - y\ is\ negative.\ But\ by\ trichotomy\ only\ one\ of\ these\ can\ be\ true\ a\ contradiction.\ Suppose\ x = y\ and\ x > y \Rightarrow x - y\ is\ positive\ and\ x - y = 0.\ But\ by\ trichotomy\ only\ one\ of\ these\ can\ be\ true\ a\ contradiction.\ Suppose\ x < y\ and\ x > y \Rightarrow x - y\ is\ positive\ and\ x - y\ is\ negative.\ But\ by\ trichotomy\ only\ one\ of\ these\ can\ be\ true\ a\ contradiction.$
$B)Suppose\ x < y \Rightarrow x - y\ is\ negative \Rightarrow -(x - y)\ is\ positive \Rightarrow y - x\ is\ positive \Rightarrow y > x\ as\ required\ Suppose\ y > x \Rightarrow y - x\ is\ positive \Rightarrow y - x = -(x - y)\ is\ positive \Rightarrow x - y\ is\ negative\ so\ x < y\ as\ required$
$C)x < y\ and\ y < z \Rightarrow\ x - y\ is\ negative\ and\ y - z\ is\ negative \Rightarrow -(x - y)\ is\ positive\ as\ well\ as\ -(y - z) \Rightarrow -x + y - y + z = -x + z\ which\ is\ positive\ so\ z - x\ is\ positive\ therefore\ z > x\ and\ x < z\ as\ required.$
$D)Suppose\ x < y \Rightarrow y - x\ is\ positive \Rightarrow y - z = y - x + z - z\ which\ is\ also\ positive \Rightarrow y - x + z - z = (y + z) - (x + z)\ so\ y + z > x + z\ and\ therefore\ x + z < y + z\ as\ required.$

E 5.4.3 $Given\ x\ is\ a\ real\ number \Rightarrow x = LIM_{n\to\infty}a_n. \Rightarrow |a_n - a_N| \leq 1/2\ for\ all\ n \geq N \Rightarrow |a_n - a_N| \leq 1/2$
$So\ -1/2 \leq a_n - a_N \leq 1/2 \Rightarrow a_N - 1/2 \leq a_n \leq a_N + 1/2 \Rightarrow\ define\ a\ sequence\ by\ b_n = a_N\ for$
$n < N\ and\ b_n = a_n\ for\ n \geq N \Rightarrow (b_n)_{n=1}^{\infty} = x \Rightarrow a_N - 1/2 \leq x \leq a_N + 1/2 \Rightarrow M \leq a_N < M + 1,$
$So\ M - 1 \leq x \leq M + 2.\ By\ trichotomy,\ x \geq M, x < M\ or$

$In\ the\ case\ x < M\ we\ have\ M - 1 \leq x < M.\ Let\ M - 1 = P,\ so$
$P \leq x < P + 1\ as\ required.\ In\ the\ case\ x \geq M\ we\ have\ x \geq M + 1\ or\ x <$
$M + 1$
$In\ the\ case\ of\ x \geq M + 1\ we\ have\ M + 1 \leq x < M + 2.\ Let\ M + 1 =$
$P,\ so\ P \leq x < P + 1\ as\ required$
$In\ the\ case\ of\ x < M + 1,\ we\ have\ M \leq x < M + 1\ as\ required.$

E 5.4.4 $Suppose\ x > 0 \Rightarrow N > 1/x\ by\ P5.4.13 \Rightarrow Nx > 1 \Rightarrow x >$
$1/N\ as\ required$

E 5.4.5 $Suppose\ x < y \Rightarrow y - x\ is\ positive\ so$
$y - x > 1/N = 2/2N > 0 \Rightarrow 2Ny - 2Nx > 2 \Rightarrow 2Ny > 2Nx + 2\ and\ M + 1 >$
$2Ny \geq M$
$\Rightarrow 2Ny - 2 > 2Nx\ and\ M - 1 > 2Ny - 2,\ So\ M - 1 > 2Nx \Rightarrow 2Ny \geq M >$
$M - 1 > 2Nx \Rightarrow$
$2Ny > M - 1 > 2Nx \Rightarrow y > (M - 1)/2N > x\ as\ required.$

E 5.4.6 $Suppose\ |x - y| < \varepsilon \Rightarrow -\varepsilon \leq x - y \leq \varepsilon.\ Suppose\ for\ contradiction\ that\ x -$
$y = \varepsilon \Rightarrow |x - y| = \varepsilon$
$a\ contradiction.\ So\ -\varepsilon \leq\ x - y < \varepsilon.\ Suppose\ for\ contradiction\ x - y =$
$-\varepsilon \Rightarrow -(x - y) = \varepsilon\ and$
$|x - y| = -(y - x) = \varepsilon\ a\ contradiction.\ Therefore\ -\varepsilon < x - y < \varepsilon \Rightarrow$
$y - \varepsilon < x < y + \varepsilon\ as\ required$
$Suppose\ y - \varepsilon < x < y + \varepsilon \Rightarrow -\varepsilon < x - y < \varepsilon \Rightarrow |x - y| \leq \varepsilon.\ Suppose\ |x - y| =$
$\varepsilon\ for\ contradiction \Rightarrow$
$\Rightarrow x - y\ is\ either\ positive\ or\ negative.\ In\ the\ positive\ case\ x - y = |x - y| =$
$\varepsilon\ a\ contradiction.$
$In\ the\ negative\ case\ |x - y| = -(x - y) = \varepsilon\ so\ x - y = -\varepsilon\ a\ contradiction.\ Therefore |x -$
$y| < \varepsilon\ as\ required.$
$Suppose\ |x - y| \leq \varepsilon \Rightarrow -\varepsilon \leq x - y \leq \varepsilon \Rightarrow y - \varepsilon \leq x \leq y + \varepsilon\ as\ required.$
$Suppose\ y - \varepsilon < x < y + \varepsilon \Rightarrow -\varepsilon < x - y < \varepsilon \Rightarrow |x - y| \leq \varepsilon\ as\ required.$

E 5.4.7 $if\ x \leq y + \varepsilon\ for\ all\ \varepsilon > 0 \Rightarrow x \leq y.$
$Suppose\ for\ contradiction\ that\ x > y \Rightarrow x - y\ is\ positive,\ and\ x - y \leq \varepsilon \Rightarrow$
$x - y > 1/N > 0\ for$
$some\ N\ and\ x - y \leq 1/N \Rightarrow 1/N \geq x - y > 1/N \Rightarrow 1/N > 1/N\ a\ contradiction.\ Therefore\ x \leq$
y
$Suppose\ x \leq y \Rightarrow x + \varepsilon \leq y + \varepsilon\ and\ \varepsilon > 0 \Rightarrow x + \varepsilon > x \Rightarrow x < y + \varepsilon \Rightarrow x \leq$
$y + \varepsilon\ as\ required.$
$if\ |x - y| \leq \varepsilon\ for\ all\ real\ numbers\ \varepsilon > 0 \Rightarrow x = y$
$Suppose\ for\ contradiction\ x \neq y \Rightarrow either\ x > y\ or\ x < y \Rightarrow\ In\ the\ case\ x >$
$y,\ x - y\ is\ positive$

$So\ |x-y| = x-y = c\ a\ positive\ real.\ But\ |x-y| = x-y \leq 1/N < c\ a\ contradiction.$
$In\ the\ case\ x < y,\ y-x\ is\ positive\ so\ |x-y| = |y-x| = y-x = c\ a\ positive\ real.$
$But\ |x-y| = y-x \leq 1/N < c\ a\ contradiction. Therefore\ x = y\ as\ required$

E 5.4.8 $Suppose\ a_n \leq x\ for\ all\ n \geq 1 \Rightarrow Suppose\ for\ contradiction\ that\ LIM_{n\to\infty}a_n > x \Rightarrow$
$LIM_{n\to\infty}a_n > q > x\ where\ q\ is\ a\ rational\ by$ 5.4.14 $\Rightarrow a_n \leq x < q\ so\ by$ 5.4.10 $it\ can\ be\ concluded$
$LIM_{n\to\infty}a_n < q\ but\ LIM_{n\to\infty}a_n > q \Rightarrow q < LIM_{n\to\infty}a_n < q \Rightarrow q < q\ a\ contradiction.\ Therefore$
$LIM_{n\to\infty}a_n \leq x.$
$if\ a_n \geq x\ for\ all\ n \geq 1 \Rightarrow\ LIM_{n\to\infty}a_n \geq x$
$Suppose\ for\ contradiction\ that\ LIM_{n\to\infty}a_n < x \Rightarrow LIM_{n\to\infty}a_n < q < x\ for\ some\ rational\ q \Rightarrow$
$a_n \geq x > q \Rightarrow LIM_{n\to\infty}a_n > q,\ but\ LIM_{n\to\infty}a_n < q \Rightarrow LIM_{n\to\infty}a_n > LIM_{n\to\infty}a_n\ a\ contradiction$
$Therefore\ LIM_{n\to\infty}a_n \geq x.$

E 5.5.1 $M \geq x\ and\ M \leq M'\ for\ all\ M' \geq x\ for\ all\ x \Rightarrow -M \leq -x.\ Suppose\ L \leq -x \Rightarrow -L = M' \geq x \Rightarrow$
$-L \geq M,\ So\ L \leq -M.\ Therefore\ -M\ is\ the\ inf(-E)$

E 5.5.2 $Suppose\ for\ contradiction\ if\ L < m \leq K \Rightarrow m/n\ is\ not\ an\ upper\ bound\ or\ (m-1)/n\ is\ an\ upper\ bound.\ \ we\ shall\ prove\ if\ L < p+L = m \leq K \Rightarrow m/n\ is\ not\ an\ upper\ bound,\ using\ induction\ on\ p\ Base\ case\ p = 1.\ L < L+1 = m \leq K \Rightarrow m-1 = L,\ So\ (m-1)/n\ is\ not\ an\ upper\ bound\ therefore\ m/n\ is\ not\ an\ upper\ bound.\ \ Suppose\ as\ inductive\ hypothesis\ that\ (if\ L < p+L = m \leq K \Rightarrow m/n\ is\ not\ an\ upper\ bound\ for\ some\ p) \Rightarrow Suppose\ L < p+1+L = m+1 \leq K \Rightarrow m = L\ or\ L < m \leq K.\ in\ the\ m = L\ we\ have\ m/n\ is\ not\ an\ upper\ bound\ so\ (m+1)/n\ is\ not\ an\ upper\ bound\ by\ hypothesis.\ In\ the\ second\ case\ we\ have\ m/n\ is\ not\ an\ upper\ bound\ by\ inductive\ hypothesis\ and\ by\ hypothesis\ we\ have\ (m+1)/n\ is\ not\ an\ upper\ bound.\ This\ closes\ the\ induction\ and\ it\ can\ be\ concluded\ that\ if\ L < m \leq K \Rightarrow m/n\ is\ not\ an\ upper\ bound.\ So\ L < K \leq K\ \Rightarrow K/n\ is\ not\ an\ upper\ bound\ which\ is\ a\ contradiction.\ Therefore\ we\ can\ conclude\ the\ original\ true.$

E 5.5.3 $Suppose\ m \neq m'\ for\ contradiction \Rightarrow m' > m\ or\ m' < m.\ In\ the\ case\ m' > m\ we\ have\ m'-1 \geq m \Rightarrow$
$(m'-1)/n \geq m/n,\ But\ m/n \geq x\ for\ all\ x\ in\ E\ so\ (m'-1)/n \geq x\ and\ is\ an\ upper\ bound\ therefore\ a\ contradiction.$
$In\ the\ case\ m' < m\ we\ have\ m' \leq m-1 \Rightarrow m'/n \leq (m-1)/n,\ But\ m'/n \geq$

x *for all* x *in* E *so* $(m-1)/n \geq x$
and is an upper bound therefore a contradiction. It can then be concluded that $m = m'$.

E 5.5.4 *for all* $\varepsilon > 0$ *there exists a* $1/N < \varepsilon$ *for some* $N \geq 1 \Rightarrow |q_n - q_{n'}| \leq 1/N < \varepsilon$ *for all* $n, n' \geq N$ *and is therefore cauchy.*
Suppose $LIM_{n\to\infty} q_n = S \Rightarrow |q_M - q_n| \leq 1/M$ *for all* $n \geq M$. *define a sequence such that* $b_n = q_n$ *for* $n \geq M$
and $b_n = q_M$ *for* $n < M \Rightarrow \; |q_M - b_n| = |b_n - q_M| \leq 1/M$ *for all* $n \geq 1 \Rightarrow$
$-1/M \leq b_n - q_M \leq 1/M \Rightarrow$
$q_M - 1/M \leq b_n \leq q_M + 1/M \Rightarrow q_M - 1/M \leq LIM_{n\to\infty} b_n \leq q_M + 1/M$ *but* $LIM_{n\to\infty} b_n = S \Rightarrow q_M - 1/M \leq S \leq q_M + 1/M$
$\Rightarrow |q_M - S| = |S - q_M| \leq 1/M$ *as required*

E 5.5.5 *if* z *is irrational and* q *is rational* $\Rightarrow z * q$ *is irrational*
Suppose for contradiction that $z*q$ *is rational* $\Rightarrow z*q = n/m$ *for integers* n *and* $m \Rightarrow$
$z = n/m/q \Rightarrow$
but n/m *is rational and* $1/q$ *is rational so* $n/m * 1/q$ *is also rational.*
Therefore z *is rational which is acontradiction.*
if z *is irrational and* q *is rational* $\Rightarrow q + z$ *is irrational.*
Suppose for contradicton $q+z$ *is rational* $\Rightarrow q+z = p$ *for some rational* $p \Rightarrow$
$p - q = z$, *But* $p - q$ *is rational*
So z *is rational a contradiction*
Suppose $x < y$ *and* $z^2 = 2 \Rightarrow z$ *is irrational* $\Rightarrow y - x > 0 \Rightarrow y - x > 2/2N > 0 \Rightarrow 2Ny - 2Nx > 2 > 0 \Rightarrow$
$3Ny - 3Nx > 3 > 2 > z > 0 \Rightarrow 3Ny > 3Nx + 3 > 3Nx + z \Rightarrow M + 3 + z > M + 3 > 3Ny \geq M + 2 > M + z$ *and*
$3Ny - 3 > 3Nx \Rightarrow M + z > 3Nx \Rightarrow 3Ny > M + z > 3Nx \Rightarrow y > (M + z)/3N > x$. *Since* z *is irrational so is* $(M + z)/3N$
Therefore we can conclude that there exists an irrational p *such that* $x < p < y$.

E 5.6.1 *A)Suppose* $y = x^{1/n} \Rightarrow y = sup\{z \in R : z \geq 0$ *and* $z^n \leq x\}$. *We shall prove* $y^n = x$ *by contradiction. Suppose* $y^n \neq x$
$\Rightarrow y^n > x$ *or* $y^n < x$. *In the case* $y^n < x$ *we shall first prove* $(y + \varepsilon)^n \leq y^n + M\varepsilon$ *where* M *and* ε *is positive and real. Suppose*
for contradiction that $(y + \varepsilon)^n > y^n + M\varepsilon$ *for all* $M \Rightarrow (y + \varepsilon)^n > y^n + (y + \varepsilon)^n \varepsilon/\varepsilon = (y + \varepsilon)^n + y^n$,
but $(y + \varepsilon)^n > 0$ *so* $(y + \varepsilon)^n + y^n > (y + \varepsilon)^n \Rightarrow (y + \varepsilon)^n + y^n > (y + \varepsilon)^n + y^n$ *a contradiction. We can then conclude*
that $(y + \varepsilon)^n \leq y^n + M\varepsilon$ *where* M *is positive and real.*

we have $y^n < x \Rightarrow x - y^n > 0 \Rightarrow$
$x - y^n > 1/N > 0$ *for some* $N \geq 1 \Rightarrow\ x - y^n > 1/N > 1/(N+1) > 0 \Rightarrow$
$x > y^n + 1/(N+1)$
when $M > 1$ *allow* $\varepsilon = 1/(M(N+1))$. $0 < \varepsilon < 1$ *because* $1 > 1/M$ *and* $N+1 > 1$ *so* $1 > 1/(N+1)$
therefore $0 < 1/(M(N+1)) < 1$. *we then have* $(y+\varepsilon)^n \leq y^n + 1/(N+1)$. *But* $y^n + 1/(N+1) < x \Rightarrow$
$(y+\varepsilon)^n < x$ *and therefore* $y+\varepsilon \leq y$, *a contradiction. when* $M \leq 1$ *allow* $\varepsilon = 1/(N+1) \Rightarrow M/(N+1) \leq 1/(N+1)$
$\Rightarrow y^n + M/(N+1) \leq y^n + 1/(N+1) < x \Rightarrow (y+\varepsilon)^n < x$ *so* $y + \varepsilon \leq y$ *a contradiction.*
Therefore it can be concluded that $y^n < x$ *is false.*
In the case $y^n > x$, *let* $y > \varepsilon > 0$, *we will first prove* $(y-\varepsilon)^n \geq y^n - M\varepsilon$ *for some* $M > 0$. *Suppose for contradiction*
$(y-\varepsilon)^n < y^n - M\varepsilon$ *for all* $M > 0 \Rightarrow (y-\varepsilon)^n < y^n - y^n \Rightarrow (y-\varepsilon)^n < 0$ *a contradiction. So* $(y-\varepsilon)^n \geq y^n - M\varepsilon \Rightarrow$
$y \geq y^n - x$ *or* $y < y^n - x$. *In the case* $y \geq y^n - x,\ y\ \geq y^n - x > 1/Q > 0$ *where* $Q \geq 1$.
In the case $y < y^n - x$ *we have* $y^n - x > y > 1/P > 0$ *where* $P \geq 1$. *So* $y^n - x,\ y > 1/N > 0$. *for some*
$N \geq 1 \Rightarrow y^n - 1/N > x \Rightarrow\ M \geq 1$ *or* $M < 1$. *In the case* $M \geq 1$, *let* $0 < \varepsilon = 1/(MN) < y \Rightarrow (y-\varepsilon)^n \geq y^n - 1/N$
But $y^n - 1/N > x \Rightarrow (y-\varepsilon)^n > x \Rightarrow$ *Using contradiction I will prove* $y - \varepsilon \geq z$ *for all* z.
Suppose $y - \varepsilon < z \Rightarrow (y-\varepsilon)^n < z^n < x$ *a contradiction. So* $y - \varepsilon \geq z$ *for all* $z \Rightarrow y - \varepsilon$ *is an upper bound but*
$y - \varepsilon < y$ *where* y *is the supremum. Therefore a contradiction.*
In the case $M < 1$ *let* $\varepsilon = 1/N \Rightarrow M/N < 1/N \Rightarrow (y-\varepsilon)^n \geq y^n - M/N > y^n - 1/N > x \Rightarrow (y-\varepsilon)^n > x$.
So $y-\varepsilon$ *is an upper bound and* $y-\varepsilon < y$ *where* y *is the supremum. Therefore a contradiction.*
it can then be concluded that $y^n > x$ *is also false.*
This means we are left with $y^n = x$ *as required.*
B)Suppose $y^n = x \Rightarrow x^{1/n} = (y^n)^{1/n} = z \Rightarrow z^n = y^n \Rightarrow z = y \Rightarrow y = x^{1/n}$ *as required*
C)$x^{1/n} = sup\{y \in \mathbf{R} : y \geq 0$ *and* $y^n \leq x\} \Rightarrow y = 0 \in \mathbf{R}$ *and* $y = 0 \geq 0$ *and* $y^n = 0 \leq x$. *So* $x^{1/n} \geq y = 0$ *and is therefore nonnegative. if* $x > 0 \Rightarrow x^{1/n} > 0$ *Suppose for contradiction that* $x^{1/n} \leq 0. \Rightarrow$ *In the case* $x^{1/n} < 0$ *we have* 0 *is in the set but* $x^{1/n} < 0$ *a contradiction.*

In the case $x^{1/n} = 0$ we can find $0 < 1/N < x \Rightarrow 0 < (1/N)^n < x$ so $1/N$ is in the set. But $0 < 1/N$ despite 0 being the supremum and therefore a contradiction.
It can be concluded that $x^{1/n} > 0$
if $x^{1/n} > 0 \Rightarrow x > 0$
Suppose for contradiction $x \leq 0 \Rightarrow$ In the case $x < 0,$ the set is empty and therefore does not have a supremum
a contradiction.In the case $x = 0,$ 0 is the only element in the set and therefore $0 \leq 0$ so 0 is an upper bound.
but $0 < x^{1/n}$ a contradiction. Therefore $x > 0.$
$D)$if $x > y \Rightarrow x^{1/n} > y^{1/n}$
Suppose for contradiction that $x^{1/n} < y^{1/n} \Rightarrow 0 \leq x^{1/n} < y^{1/n} \Rightarrow z = x^{1/n}$ and $w = y^{1/n} \Rightarrow z^n = x$ and $w^n = y \Rightarrow$
because $0 \leq z < w,\ z^n < w^n \Rightarrow x < y$ a contradiction. Therefore $x^{1/n} > y^{1/n}.$
Suppose $x^{1/n} > y^{1/n} \Rightarrow (x^{1/n})^n > (y^{1/n})^n \Rightarrow x > y$ as required.
$E)$if $x > 1 \Rightarrow x^{1/k}$ is a decreasing function of $k.$
first we will prove if $x > 1$ and $k \geq 1 \Rightarrow x^n > x^k$ for all $n > k.$
Proof via induction on $n.$ Base case $n = k+1,\ x > 1$ and $x^k > 0$ so $x^{k+1} > x^k$ as required.
Suppose as inductive hypothesis that $x^n > x^k$ for some $n > k \Rightarrow x > 1 \Rightarrow x^{n+1} > x^n \Rightarrow x^{n+1} > x^k$ as required.
This proves that $x^n > x^k$ for all $n > k.$
Now Suppose $x > 1 \Rightarrow x^{1/m} > 1^{1/m}$ but $1 = 1^m$ so $1^{1/m} = 1^1 \Rightarrow x^{1/m} > 1 \Rightarrow$
Suppose for contradiction $x^{1/n} \geq x^{1/k}$ for some $n > k \Rightarrow$
In the case $x^{1/n} = x^{1/k} \Rightarrow x = (x^{1/n})^n = (x^{1/k})^n$ and $x = (x^{1/k})^k = (x^{1/n})^k.$ So $(x^{1/n})^k = (x^{1/k})^n.$
But from the induction above $(x^{1/k})^n > (x^{1/n})^k$ therefore a contradiction.
In the case $x^{1/n} > x^{1/k} \Rightarrow (x^{1/n})^n = x > (x^{1/k})^n$ and $(x^{1/k})^k = x \Rightarrow (x^{1/k})^k > (x^{1/k})^n$
But from the induction above $(x^{1/k})^n > (x^{1/k})^k$ for all $n > k$ and therefore a contradiction.
it can therefore be concluded that $x^{1/n} < x^{1/k}$for all $n > k$ and so $x^{1/k}$ is indeed a decreasing function of k
Suppose $x < 1 \Rightarrow x < 1 = 1^n \Rightarrow x^{1/n} < 1 \Rightarrow$ Suppose for contradiction $x^{1/n} \leq x^{1/k}$ for some $n > k \Rightarrow (x^{1/n})^n \leq (x^{1/k})^n \Rightarrow$
$x \leq (x^{1/k})^n$ but $(x^{1/k})^k = x \Rightarrow (x^{1/k})^k \leq (x^{1/k})^n.$ But $(x^{1/k}) < 1$ and $n > k$ so $n = k + c$ where c is positive $\Rightarrow (x^{1/k})^c < 1$
$\Rightarrow (x^{1/k})^c(x^{1/k})^k < (x^{1/k})^k \Rightarrow (x^{1/k})^{c+k} = (x^{1/k})^n < (x^{1/k})^k$ a contradiction.

So it can be concluded that $x^{1/n} > x^{1/k}$ *for all* $n > k$ *and is therefore an increasing function of* k.
$F)((xy)^{1/n})^n = xy$ *and* $(x^{1/n}y^{1/n})^n = (x^{1/n})^n(y^{1/n})^n = xy \Rightarrow ((xy)^{1/n})^n = (x^{1/n}y^{1/n})^n \Rightarrow$ *by* $P4.3.12$,
$(xy)^{1/n} = x^{1/n}y^{1/n}$ *as required*
$G)((x^{1/n})^{1/m})^{mn} = ((x^{1/n})^{1/m})^m))^n = (x^{1/n})^n = x$ *and* $(x^{1/nm})^{mn} = x$ *so* $(x^{1/nm})^{mn} = ((x^{1/n})^{1/m})^{mn} \Rightarrow$
$x^{1/nm} = (x^{1/n})^{1/m}$

E 5.6.2 $A)q = a/b$ *so* $x^q = (x^{1/b})^a$, *since* $x^{1/b} > 0$ *we know* $(x^{1/b})^a > 0$ *for any integer* a. *Therefore* $x^q > 0$ *as required.*
$B)q = a/b,\ r = c/d \Rightarrow x^{q+r} = x^{a/b+c/d} = x^{(ad+cb)/bd} = (x^{1/bd})^{(ad+cb)} = (x^{1/bd})^{ad}(x^{1/bd})^{cb} = x^{ad/bd}x^{cb/bd} = x^q x^r$ *as required.*
$C)q = a/b,\ x^{-q} = x^{-a/b} = (x^{1/b})^{-a} = 1/(x^{1/b})^a = 1/(x^{a/b}) = 1/x^q$
$D)$*let* $q = a/b > 0$, *Suppose* $x > y \Rightarrow x^{1/b} > y^{1/b} \Rightarrow (x^{1/b})^a > (y^{1/b})^a \Rightarrow x^{a/b} > y^{a/b}$ *so* $x^q > y^q$.
Suppose $x^q > y^q \Rightarrow x^{a/b} > y^{a/b} \Rightarrow (x^{1/b})^a > (y^{1/b})^a$ *but* $x^{1/b}, y^{1/b} > 0$ *and so* $x^{1/b} > y^{1/b} \Rightarrow x > y$ *as required.*
$E)$*let* $x > 1, q = a/b = ad/bd$, *and* $r = c/d = cb/db$, *Suppose* $x^q > x^r \Rightarrow (x^{1/bd})^{ad} > (x^{1/bd})^{cb}$. *Suppose for contradiction*
$ad \leq cb \Rightarrow cb = ad + f$ *for some natural number* $f \Rightarrow\ (x^{1/bd})^f > 1 \Rightarrow (x^{1/bd})^f(x^{1/bd})^{ad} > (x^{1/bd})^{ad} \Rightarrow (x^{1/bd})^{cb} > (x^{1/bd})^{ad}$
a contradiction. Therefore $ad > cb \Rightarrow ad/bd > cb/bd \Rightarrow q > r$ *as required.*
Suppose $q > r \Rightarrow ad/bd > cb/db \Rightarrow ad > cb \Rightarrow x^q = (x^{1/bd})^{ad}$ *and* $x^r = (x^{1/bd})^{cb}$
In $E5.6.1$ *it has been proven that* $x^{1/bd} > 1$ *and since* $ad > cb$ $(x^{1/bd})^{ad} > (x^{1/bd})^{cb}$ *so* $x^q > x^r$ *as required.*
Let $x < 1$. *Suppose* $x^q > x^r \Rightarrow (x^{1/bd})^{ad} > (x^{1/bd})^{cb}$ *Suppose for contradiction* $ad \geq cb \Rightarrow$
$ad = cb + f$ *for some natural number* $f \Rightarrow x^{1/bd} < 1$, *so* $(x^{1/bd})^f \leq 1 \Rightarrow (x^{1/bd})^f(x^{1/bd})^{cd} \leq (x^{1/bd})^{cd} \Rightarrow$
$(x^{1/bd})^{ad} \leq (x^{1/bd})^{cd}$ *a contradiction and therefore* $ad < cb \Rightarrow ad/bd < cb/bd \Rightarrow q < r$ *as required.*
Suppose $q < r \Rightarrow ad < cb \Rightarrow x^q = (x^{1/bd})^{ad}$ *and* $x^r = (x^{1/bd})^{cb}$. *In* $E5.6.1$ *it has been proven that* $x^{1/bd} < 1 \Rightarrow$
$cb = ad + f$ *for positive* $\Rightarrow$ *Since* $x^{1/bd} < 1$, *we have* $(x^{1/bd})^f < 1 \Rightarrow (x^{1/bd})^f(x^{1/bd})^{ad} < (x^{1/bd})^{ad} \Rightarrow (x^{1/bd})^{cb} < (x^{1/bd})^{ad}$
So $x^r < x^q$ *as required*

E 5.6.3 *Suppose* x *is a real number* $\Rightarrow x = 0, x > 0, x < 0$. *In the case* $x =$

$0, |x| = 0$ *and* $(x^2)^{1/2} = (0^2)^{1/2} = (0)^{1/2} = 0$
So $|x| = (x^2)^{1/2}$ *as required. In the case* $x > 0$ *we have* $|x| = x$ *and* $(x^2)^{1/2} = (xx)^{1/2} = x^{1/2}x^{1/2} = (x^{1/2})^2$ *and by definition* $(x^{1/2})^2 = x^{2/2} = x^1 = x$, *So* $(x^2)^{1/2} = |x|$ *as required.*
In the case $x < 0$ *we have* $x = -c$ *for some positive real* $c \Rightarrow |x| = |-c| = |c| = c$ *and* $(x^2)^{1/2} = ((-c)^2)^{1/2} = (-1^2c^2)^{1/2} = (c^2)^{1/2}$ *and as shown by the case above* $(c^2)^{1/2} = c$ *so we have* $(x^2)^{1/2} = |x|$ *as required.*
In all cases $|x| = (x^2)^{1/2}$ *and therefore we can conclude* $|x| = (x^2)^{1/2}$.

6 Limits of Sequence

6.1.8

c) $\lim_{n\to\infty}(ca_n) = (\lim_{n\to\infty} c)(\lim_{n\to\infty} a_n) = c\lim_{n\to\infty} a_n$ as required

d) $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty}(a_n + (-b_n)) = \lim_{n\to\infty}(a_n) + \lim_{n\to\infty} -b_n = \lim_{n\to\infty}(a_n) - \lim_{n\to\infty} b_n = x - y$ as required

e) $b_n \neq 0$ *and* $|b_n - y| \leq |y|/2$ *for some* $n \geq N \Rightarrow$ *Suppose* $|b_n| < |y|/2$ *for contradiction. We then have* $|y| \leq |b_n - y| + |-b_n| < |y|$. This is a contradiction and therefore we can conclude $|b_n| \geq |y|/2$ and is therefore bounded away from 0. $|b_n - y| \leq \varepsilon|b_n y|$ *for all* $n \geq N \Rightarrow |b_n - y|/|b_n y| = |1/y - 1/b_n| = |b_n^{-1} - y^{-1}| \leq \varepsilon$. So therefore it can be concluded that the sequence b_n^{-1} *converges to* y^{-1}

f)$\lim_{n\to\infty} a_n/b_n = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} 1/b_n) = \lim_{n\to\infty} a_n / \lim_{n\to\infty} b_n$

g)$x = y,\ x > y,\ or\ x < y$. In the case $x = y$ we have $max(x, y) = x = y \Rightarrow |a_n - max(x, y)| \leq \varepsilon$ *and* $|b_n - max(x, y)| \leq \varepsilon \Rightarrow |max(a_n, b_n) - max(x, y)| \leq \varepsilon$ because in either case the inequality will hold true. Therefore $\lim_{n\to\infty} max(a_n, b_n) = max(x, y)$. In the case $x > y$ we have $max(x, y) = x$ *and* $x - y > 1/M \Rightarrow |b_n - y| \leq 1/2M$ *and* $|a_n - x| \leq \varepsilon \leq 1/2M \Rightarrow |(a_n - b_n) - (x - y)| \leq |y - b_n| + |a_n - x| \leq 1/M \Rightarrow 0 < x - y - 1/M \leq a_n - b_n \Rightarrow a_n > b_n \Rightarrow max(b_n, a_n) = a_n \Rightarrow |max(b_n, a_n) - max(x, y)| = |a_n - x| \leq \varepsilon$. Therefore $\lim_{n\to\infty} max(a_n, b_n) = max(x, y)$. In the case $x < y$ a proof similar to the case of $y > x$ can be shown. So in all cases we have $\lim_{n\to\infty} max(a_n, b_n) = max(x, y)$ as required

h)$x = y,\ x < y,\ or\ x > y$ In the case $x = y$ we have $min(x, y) = x = y \Rightarrow |a_n - min(x, y)| \leq \varepsilon$ *and* $|b_n - min(x, y)| \leq \varepsilon \Rightarrow |min(a_n, b_n) - min(x, y)| \leq \varepsilon$ because in all cases the inequality will hold true. Therefore $\lim_{n\to\infty} min(a_n, b_n) = min(x, y)$. In the case $x > y$ we have $min(x, y) =$

y *and* $x - y > 1/M \Rightarrow |b_n - y| \leq \varepsilon \leq 1/2M$ *and* $|a_n - x| \leq 1/2M \Rightarrow |(a_n - b_n) - (x - y)| \leq |y - b_n| + |a_n - x| \leq 1/M \Rightarrow 0 < x - y - 1/M \leq a_n - b_n \Rightarrow a_n > b_n \Rightarrow min(a_n, b_n) = b_n \Rightarrow |min(a_n, b_n) - min(x, y)| = |b_n - y| \leq \varepsilon$. Therefore it can be concluded that $\lim_{n\to\infty} min(a_n, b_n) = min(x, y)$. In the case of x¡y a proof similar to the case of y¿x can be shown. So in all cases we have $\lim_{n\to\infty} min(a_n, b_n) = min(x, y)$ as required

6.1.9

when the limit of the denominator is 0 we have $(\lim_{n\to\infty} a_n)/0$ and by definition of division, division by 0 is undefined.

6.1.10

Suppose the sequence a_n *and* b_n are eventually ε-close for every rational $\varepsilon > 0 \Rightarrow$ *let* $\varepsilon' > 0$ *be a real, we have a rational* q *such that* $\varepsilon' > q > 0. \Rightarrow |a_n - b_n| \leq \varepsilon = q < \varepsilon'$. Since ε' is a arbitrary real we can therefore conclude that the two sequences are eventually ε'-close for all $\varepsilon' > 0$ as required.

Suppose the sequence a_n *and* b_n are eventually ε-close for every real $\varepsilon > 0 \Rightarrow$ *let* $\varepsilon' > 0$ *be a rational, we have a real* r *such that* $\varepsilon' > r > 0. \Rightarrow |a_n - b_n| \leq \varepsilon = r < \varepsilon'$. Since ε' is an arbitrary rational we can therefore conclude that the two sequences are eventually ε'-close for all $\varepsilon' > 0$ as required.

6.2.1

a) we have x is a real number or $x = +\infty$ *or* $x = -\infty$. In the case x is real we have $x = x \Rightarrow x \leq x$. In the case $x = +\infty$ we have $x \leq x$ by definition. The same can be said for when $x = -\infty$

b)

6.3.1

$1 \geq 1/n$ *for any* $n \geq 1$. Suppose for contradiction that there exists an upper bound $1 > M \Rightarrow M \geq 1 \Rightarrow 1 > 1$ a contradiction and therefore it can be concluded that $1 = sup(a_n)$.

$0 < 1/N$ *for all* $n \geq 1$. Suppose for contradiction that there exists a lower bound $M > 0$. But for all $M > 0$ *there exists an* N *such that* $M > 1/N > 0$ a contradiction and therefore it can be concluded that $inf(a_n) = 0$

6.3.2

Since a_n is a sequence of real numbers, the set of all a_n is therefore a subset of $\mathbf{R}$ therefore by T6.2.12 we have $a_n \leq sup(a_n)_{n=m}^{\infty} = x$ *for all* $n \geq m$ and when M is an upper bound for a_n, $x = sup(a_n)_{n=m}^{\infty} \leq M$ finally we know $a_n \leq x$ is true so suppose $y \geq a_n$ for contradiction but then we have y is an upper bound so $y \geq x$ a contradiction. Therefore $y < a_n$ and the original statement can therefore be concluded.

6.3.3

$\varepsilon > 0,\ sup(a_n)_{n=m}^{\infty} - \varepsilon < a_N \leq sup(a_n)_{n=m}^{\infty} < sup(a_n)_{n=m}^{\infty} + \varepsilon\ for\ some\ N \geq m \Rightarrow |a_N - sup(a_n)_{n=m}^{\infty}| \leq \varepsilon$. we know that for all $n \geq N,\ a_n \geq a_N$ and because $a_n \leq sup(a_n)_{n=m}^{\infty}$ we have $a_N \leq a_n \leq sup(a_n)_{n=m}^{\infty} \Rightarrow a_N - sup(a_n)_{n=m}^{\infty} \leq a_n - sup(a_n)_{n=m}^{\infty} \leq 0\ so\ |a_N - sup(a_n)_{n=m}^{\infty}| = -a_N + sup(a_n)_{n=m}^{\infty} \geq -a_n + sup(a_n) = |a_n - sup(a_n)|$. Therefore $|a_n - sup(a_n)| \leq \varepsilon\ for\ all\ n \geq N$ and since ε is arbitrary it can be concluded that the sequence $(a_n)_{n=m}^{\infty}\ is$
$eventually\ \varepsilon - close\ to\ sup(a_n)_{n=m}^{\infty}\ for\ all\ \varepsilon > 0$ and therefore converges to $sup(a_n)_{n=m}^{\infty}$

6.3.4

Suppose $x > 1 \Rightarrow x^n x > x^n \Rightarrow x^{n+1} > x^n$ and therefore is increasing. Suppose for contradiction that the sequence has a finite upper bound M, then $\lim_{n\to\infty} x^n = sup(x^n)_{n=1}^{\infty}$ by the limit laws we then have $\lim_{n\to\infty} x^n/x^n = 1 = sup(x^n)_{n=1}^{\infty} \lim_{n\to\infty} 1/x^n = sup(x^n)_{n=1}^{\infty} \lim_{n\to\infty}(1/x)^n$ but $x > 1,\ so\ 0 < 1/x < 1$ and by P6.3.10 $\lim_{n\to\infty}(1/x)^n = 0$. We then have $1 = sup(x^n)_{n=1}^{\infty} * 0 = 0$, a contradiction. So therefore the sequence is divergent and its limit is undefined

6.4.1

we have $|a_n - c| \leq \varepsilon\ for\ all\ n \geq N \Rightarrow for\ all\ M \geq m$ we have $|a_n - c| \leq \varepsilon\ for\ some\ n \geq max(N, M)$. Because ε is chosen arbitrarily it can therefore be concluded that c is continuously adherent for all $\varepsilon > 0$ and therefore is indeed a limit point. Now suppose for contradiction that there is another limit point $c' \neq c \Rightarrow\ because\ a_n\ converges\ to\ c\ we\ have\ |a_n - c| \leq \varepsilon/2\ for\ all\ n \geq N.\ But\ c'\ is\ a\ limit\ point\ so\ |a_{n_0} - c'| \leq \varepsilon/2\ for\ some\ n_0 \geq N.$ Since $n_0 \geq N$ we have $|a_{n_0} - c| \leq \varepsilon/2 \Rightarrow |c - c'| \leq |-a_{n_0} + c| + |a_{n_0} - c'| \leq \varepsilon \Rightarrow |c - c'| \leq \varepsilon\ but\ because\ \varepsilon$ was an arbitrary real greater than 0, it can be concluded that $c = c'$ a contradiction and therefore $c = c'$ and so c is the only limit point as required.

6.4.3

c)$L^- = sup(a_N^-)_{N=m}^{\infty} \geq a_N^-$ for all $N \geq m$. But $a_N^- = inf(a_n)_{n=N}^{\infty}$ so $a_m^- = inf(a_n)_{n=m}^{\infty}\ and\ therefore\ L^- \geq\ inf(a_n)_{n=m}^{\infty}$. We also have $L^+ = inf(a_N^+)_{N=m}^{\infty} \leq a_N^+$ for all $N \geq m$. But $a_N^+ = sup(a_n)_{n=N}^{\infty}$ so $a_m^+ = sup(a_n)_{n=m}^{\infty}$ $and\ therefore\ L^+ \leq sup(a_n)_{n=m}^{\infty}$. Suppose for contradiction that $L^+ < L^- \Rightarrow$ there exists an a_N^- such that $L^+ < a_N^-$ So there exists an N' such that $a_n < a_N^- = inf(a_n)_{n=N}^{\infty}$ for all $n \geq N'$. In the case $N' \geq N$ we have an $n \geq N' \geq N$ such that $a'_N < inf(a_n)_{n=N}^{\infty}$, a contradiction. In the case $N > N'$ we have $a_N < inf(a_n)_{n=N}$ also a contradiction. Therefore it can be concluded that $L^- \leq L^+$ and so $inf(a_n)_{n=m}^\infty \leq L^- \leq L^+ \leq sup(a_n)_{n=m}^\infty$

d) Let c be a limit point of $(a_n)_{n=m}^\infty \Rightarrow |a_n - c| \leq \varepsilon$. Suppose for contradiction that $L^+ < c \; or \; L^- > c$ In the case $L^+ < c$ we have $a_n < c$ for all $n \geq N$ for some $N \geq m$ so there exists a real number q such that $c - a_n > q > 0$. but by definition of limit point, there exists an $n \geq N$ such that $|a_n - c| = |c - a_n| = c - a_n \leq q$ for some $n \geq N$, a contradiction. In the case $L^- > c$ we have $a_n > c$ for all $n \geq N$ for some $N \geq m$. so there exists a real number q such that $a_n - c > q > 0$. But by definition of limit point we have $|a_n - c| = a_n - c \leq q$ for some $n \geq N$, a contradiction. Therefore it can be concluded that $L^- \leq c \leq L^+$

e)Suppose L^+ is finite $\Rightarrow L^+ < L^+ + \varepsilon \Rightarrow a_n < L^+ + \varepsilon$ for all $n \geq N_0$ for some $N_0 \geq m$ and because $L^+ - \varepsilon < L^+$ we also have $a_n > L^+ - \varepsilon$ for all $n \geq N_1$ for some $N_1 \geq m$. Let $N = max(N_0, N_1)$ we then have $L^+ - \varepsilon < a_n < L^+ + \varepsilon$ for all $n \geq N \Rightarrow |a_n - L^+| \leq \varepsilon$ for all $n \geq N$. Now let $M \geq m \Rightarrow for \; an \; n \geq max(M, N)$ we have $|a_n - L^+| \leq \varepsilon$. Because $M \geq m$ and $\varepsilon > 0$ were chosen arbitrarily, it can be concluded that for all $\varepsilon > 0 \; and \; M \geq m$, there exists an n such that $n \geq M$ and $|a_n - L^+| \leq \varepsilon$ and so by definition we have L^+ is a limit point of the sequence $(a_n)_{n=m}^\infty$

Suppose L^- is finite $\Rightarrow L^- < L^- + \varepsilon \Rightarrow a_n < L^- + \varepsilon$ for all $n \geq N_0$ for some $N_0 \geq m$ and because $L^- - \varepsilon < L^+$ we also have $a_n > L^- - \varepsilon$ for all $n \geq N_1$ for some $N_1 \geq m$. Let $N = max(N_0, N_1)$ we then have $L^- - \varepsilon < a_n < L^- + \varepsilon$ for all $n \geq N \Rightarrow |a_n - L^-| \leq \varepsilon$ for all $n \geq N$. Now let $M \geq m \Rightarrow for \; an \; n \geq max(M, N)$ we have $|a_n - L^-| \leq \varepsilon$. Because $M \geq m$ and $\varepsilon > 0$ were chosen arbitrarily, it can be concluded that for all $\varepsilon > 0 \; and \; M \geq m$, there exists an n such that $n \geq M$ and $|a_n - L^-| \leq \varepsilon$ and so by definition we have L^- is a limit point of the sequence $(a_n)_{n=m}^\infty$

f)Let $(a_n)_{n=m}^\infty$ converge to c $\Rightarrow L^- \leq c \leq L^+$. Since a_n is a sequence of real numbers and is also convergent $|a_n| \leq M$ for some finite M $\Rightarrow -M \geq a_n \leq M \Rightarrow inf(a_n)_{n=m}^\infty \geq -M \; and \; sup(a_n)_{n=m}^\infty \leq M \Rightarrow -M \leq L^- \leq L^+ \leq M$. and is therefore finite. So $L^+ \; and \; L^-$ are limit points of $(a_n)_{n=m}^\infty$. but because the sequence converges to c, the sequence can have only one limit point, c. Since $L^+ \; and \; L^-$ are also limit points, that means $L^+ = L^- = c$ as required.

Let $L^+ = L^- = c \Rightarrow a_n < c + \varepsilon$ for all $n \geq N_0$ for some $N_0 \geq m \; and \; c - \varepsilon < a_n$ for all $n \geq N_1$ for some $N_1 \geq m$. Let $N = max(N_0, N_1) \Rightarrow c - \varepsilon \leq a_n \leq c + \varepsilon$ for all $n \geq N \Rightarrow |a_n - c| \leq \varepsilon$ for all $n \geq N$ thereby showing that $(a_n)_{n=m}^\infty$ converges to c

6.4.4

a)$sup(b_n)_{n=m}^\infty \geq b_n$ for all n but we have $b_n \geq a_n$ for all n so we can say

$sup(b_n)_{n=m}^{\infty} \geq b_n \geq a_n$ therefore $sup(b_n)_{n=m}^{\infty} \geq a_n$ and is an upper bound of a_n this means that it has to be greater than $sup(a_n)_{n=m}^{\infty}$. Therefore we have $sup(a_n)_{n=m}^{\infty} \leq sup(b_n)_{n=m}^{\infty}$

b)$inf(a_n)_{n=m}^{\infty} \leq a_n$ for all n but we have $b_n \geq a_n$ for all n so we can say $inf(a_n)_{n=m}^{\infty} \leq a_n \leq b_n$ so $inf(a_n)_{n=m} \leq b_n$ and is therefore a lower bound of b_n so this means that $inf(a_n)_{n=m} \leq inf(b_n)_{n=m}^{\infty}$

c)Since the sequences $(a_n)_{n=N}^{\infty}$ *and* $(b_n)_{n=N}^{\infty}$ for $N \geq m$ have the property $a_n \leq b_n$ we can use a to establish that $sup(a_n)_{n=N}^{\infty} \leq sup(b_n)_{n=N}^{\infty}$. This then implies that the sequence $(a_N^+)_{N=m}^{\infty}$ *and* $(b_N^+)_{N=m}^{\infty}$ have the property $a_N^+ \leq b_N^+$. So b can be used to show $inf(a_N^+)_{N=m}^{\infty} \leq inf(b_N^+)_{N=m}^{\infty}$ as required.

d)Since the sequences $(a_n)_{n=N}^{\infty}$ *and* $(b_n)_{n=N}^{\infty}$ for $N \geq m$ have the property $a_n \leq b_n$ we can use b to establish that $inf(a_n)_{n=N}^{\infty} \leq inf(b_n)_{n=N}^{\infty}$. This then implies that the sequence $(a_N^-)_{N=m}^{\infty}$ *and* $(b_N^-)_{N=m}^{\infty}$ have the property $a_N^- \leq b_N^-$. So a can be used to show $sup(a_N^-)_{N=m}^{\infty} \leq sup(b_N^-)_{N=m}^{\infty}$ as required.

6.4.5

$\lim sup_{n\to\infty} a_n \leq \lim sup_{n\to\infty} b_n \leq \lim sup_{n\to\infty} c_n$ but $\lim sup_{n\to\infty} c_n = \lim sup_{n\to\infty} a_n$ but because a_n *and* c_n both converge to L we know that $\lim sup_{n\to\infty} a_n = \lim sup_{n\to\infty} c_n = L$, therefore we have $L \leq \lim sup_{n\to\infty} b_n \leq L$ and so $\lim sup_{n\to\infty} b_n = L$. Next $\lim inf_{n\to\infty} a_n \leq \lim inf_{n\to\infty} b_n \leq \lim inf_{n\to\infty} c_n$ but because a_n *and* c_n both converge to L we know that $\lim inf_{n\to\infty} a_n = \lim inf_{n\to\infty} c_n = L$, therefore we have $L \leq \lim inf_{n\to\infty} b_n \leq L$ and so $\lim inf_{n\to\infty} b_n = L$. Since $\lim inf_{n\to\infty} b_n = L = \lim sup_{n\to\infty} b_n$ we can conclude that $(b_n)_{n=m}^{\infty}$ *converges to* L as required.

6.4.6

$a_n = -1/n$ *and* $b_n = 0$, these two sequences have the property $a_n < b_n$ for all n ≥ 1. Suppose for contradiction that $sup(a_n)_{n=1}^{\infty} \neq 0$. In the case $sup(a_n)_{n=1}^{\infty} > 0$ we have $0 > -1/n$ for all n so 0 is an upper bound, a contradiction. In the case $sup(a_n)_{n=1}^{\infty} < 0$ we have $-sup(a_n)_{n=1}^{\infty} > 0 \Rightarrow -sup(a_n)_{n=1}^{\infty} > 1/N > 0 \Rightarrow sup(a_n)_{n=1}^{\infty} < -1/N$ but $-1/N$ is in the sequence a_n and so we have a contradiction. Therefore $sup(a_n)_{n=1}^{\infty} = 0$ and so is $sup(b_n)_{n=1}^{\infty} = 0$. These sequences does not contradict with L6.4.13 because $sup(a_n)_{n=1}^{\infty} = sup(b_n)_{n=1}^{\infty}$, $inf(a_n)_{n=1}^{\infty} = -1 < 0 = inf(b_n)_{n=1}^{\infty}$, $\lim sup(a_n)_{n=1}^{\infty} = 0 = \lim sup(b_n)_{n=1}^{\infty}$, *and* $\lim inf(a_n)_{n=1}^{\infty} = -1 < 0 = \lim inf(b_n)_{n=1}^{\infty}$

6.5.1

Let $q = r/k > 0$ where $k > 0$ *and* $r > 0 \Rightarrow \lim_{n\to\infty} 1/n^q = \lim_{n\to\infty} 1/(n^{1/k})^r = \lim_{n\to\infty}(1/(n^{1/k}))^r = \lim_{n\to\infty}((1/(n^{1/k}))^{r-1}/(n^{1/k})) = \lim_{n\to\infty}(1/(n^{1/k}))^{r-1} \lim_{n\to\infty} 1/(n^{1/k}) = \lim_{n\to\infty}(1/(n^{1/k}))^{r-1} * 0 = 0$ as required

6.7.1

a)we know that $x^{q_n} > 0$ by rational exponentiation so $\lim x^{q_n} \geq 0$. Suppose for contradiction that $\lim x^{q_n} = 0 \Rightarrow |x^{q_n}| = x^{q_n} \leq \varepsilon$ for all $n \geq N$. In the case $x \geq 1$ we have $0 < 1/P < x^{-M} \leq x^{q_n}$ for some $-M \leq q_n$. But since x^{q_n} converges on 0 we would have $|x^{q_n}| = x^{q_n} \leq 1/P$ for all $n \geq N$ a contradiction. Therefore $\lim x^{q_n} > 0$. In the case $x < 1$ we have $0 < 1/P < x^M \leq x^{p^n}$ for some $M \geq p^n$. But since x^{q_n} converges on 0 we would have $|x^{q_n}| = x^{q_n} \leq 1/P$ for all $n \geq N$ a contradiction. Therefore $\lim x^{q_n} > 0$. Since both cases where 0 is the limit ended in contradiction, it can be concluded that real exponentiation on real numbers results in positive reals.

b)$\lim q_n r_n = \lim q_n \lim r_n = qr \Rightarrow \lim x^{q_n r_n} = x^{qr}$ *but* $x^{q_n r_n} = (x^{q_n})^{r_n} \Rightarrow x^q r = \lim (x^{q_n})^{r_n} = (x^q)^r$ as required.

c)$x^q = \lim x^{q_n}$ but if q_n converges to q we have $|q_n - q| \leq \varepsilon \Rightarrow |-q_n - (-q)| \leq \varepsilon$. So $-q_n$ converges to $-q$. This means we have $x^{-q} = \lim x^{-q_n} = \lim 1/x^{q_n} = 1/\lim x^{q_n} = 1/x^q$ as required.

d)Suppose $q > 0 \Rightarrow$ *Suppose* $x > y \Rightarrow x^q = \lim x^{q_n}$ *and* $y^q = \lim y^{q_n}$, but $x^{q_n} > y^{q_n}$ so $x^q = \lim x^{q_n} \geq \lim y^{q_n} = y^q$ Suppose for contradiction that $x^q = y^q \Rightarrow \lim x^{q_n}/\lim y^{q_n} = 1 \Rightarrow \lim x^{q_n}/y^{q_n} = \lim (x/y)^{q_n} = 1$, but $x/y > 1$ *and* $-M \leq q_n \Rightarrow 1 < (x/y)^{-M} \leq (x/y)^{q_n} \Rightarrow 1 < (x/y)^{-M} \leq \lim (x/y)^{q_n}$, a contradiction and therefore $x^q > y^q$

Suppose $x^q > y^q \Rightarrow$ Suppose for contradiction that $x \leq y$. By the converse above we would have $x^q < y^q$ in the case of $x < y$ and since real exponentiation is well defined, would have $x^q = y^q$ in the case of $x = y$ both are contradictions therefore we must have $x > y$

7 Series

7.1.1

a) This will be proved through induction on n. First we will prove $a_m + \sum_{i=m+1}^{p} a_i = \sum_{i=m}^{p} a_i$ for all $p \geq m$. we have $p = m + n$ where n is a natural number. We will induct on n. The base case of $n = 0$ means $a_m + \sum_{i=m+1}^{m} a_i = a_m + 0 = a_m$ but we also have $\sum_{i=m}^{m} a_i = a_m$. So the base case is true as required. Now suppose as inductive hypothesis that $a_m + \sum_{i=m+1}^{m+n} a_i = \sum_{i=m}^{m+n} a_i$ for some n $\Rightarrow a_m + \sum_{i=m+1}^{m+n+1} a_i = a_m + \sum_{i=m+1}^{m+n} a_i + a_{m+n+1} = \sum_{i=m}^{m+n} a_i + a_{m+n+1} = \sum_{i=m}^{m+n+1} a_i$ thereby closing the induction.

Starting main proof, Let $P(n)$ be the property that $if\ 0 \leq n < p -$

$m \Rightarrow \sum_{i=m}^{m} a_i + \sum_{i=m+1}^{p} a_i = \sum_{i=m}^{p} a_i$. We have the base case of $n = 0 \Rightarrow \sum_{i=m}^{m} a_i + \sum_{i=m+1}^{p} a_i = \sum_{i=m}^{m-1} a_i + a_m + \sum_{i=m+1}^{p} a_i = a_m + \sum_{i=m+1}^{p} a_i$ but we have $a_m + \sum_{i=m+1}^{p} a_i = \sum_{i=m}^{p} a_i$ thereby proving the base case. Suppose as inductive hypothesis that $\sum_{i=m}^{m+n} a_i + \sum_{i=m+n+1}^{p} a_i = \sum_{i=m}^{p} a_i$ is true for some n$\Rightarrow \sum_{i=m}^{m+n+1} a_i + \sum_{i=m+n+2}^{p} a_i = \sum_{i=m}^{m+n} a_i + a_{m+n+1} + \sum_{i=m+n+1}^{p} a_i - a_{m+n+1} = \sum_{i=m}^{m+n} a_i + \sum_{i=m+n+1}^{p} a_i = \sum_{i=m}^{p} a_i$ as required closing the induction.

b)n=m+p for some natural p. Proof through induction on p. Base case $p = 0,\ \sum_{i=m}^{m} a_i = a_m\ but\ \sum_{j=m+k}^{m+k} a_{j-k} = a_m$ thereby proving the base case. Now suppose as inductive hypothesis that $\sum_{i=m}^{m+p} a_i = \sum_{j=m+k}^{m+p+k} a_{j-k}$ for some p $\Rightarrow \sum_{i=m}^{m+p+1} a_i = \sum_{i=m}^{m+p} a_i + a_{m+p+1}\ and\ \sum_{j=m+k}^{m+p+k+1} a_{j-k} = \sum_{j=m+k}^{m+p+k} a_{j-k} + a_{m+p+1}$. Since $\sum_{j=m+k}^{m+p+k} = \sum_{i=m}^{m+p} a_i$ by inductive hypothesis, we have $\sum_{j=m+k}^{m+p+k+1} a_{j-k} = \sum_{j=m+k}^{m+p+k} a_{j-k} + a_{m+p+1} = \sum_{i=m}^{m+p} a_i + a_{m+p+1} = \sum_{i=m}^{m+p+1} a_i$ thereby closing the induction.

c)n=m+p for some natural p. Proof through induction on p. Base case $p = 0,\ \sum_{i=m}^{m} a_i + b_i = a_m + b_m\ but\ \sum_{i=m}^{m} a_i + \sum_{i=m}^{m} b_i = a_m + b_m$ thereby proving the base case. Suppose as inductive hypothesis that $\sum_{i=m}^{m+p} a_i + \sum_{i=m}^{m+p} b_i = \sum_{i=m}^{m+p} a_i + b_i \Rightarrow \sum_{i=m}^{m+p+1} a_i + \sum_{i=m}^{m+p+1} b_i = \sum_{i=m}^{m+p} a_i + \sum_{i=m}^{m+p} b_i + a_{m+p+1} + b_{m+p+1}\ and\ \sum_{i=m}^{m+p+1} a_i + b_i = \sum_{i=m}^{m+p} a_i + b_i + a_{m+p+1} + b_{m+p+1}$, but by inductive hypothesis we have $\sum_{i=m}^{m+p+1} a_i + \sum_{i=m}^{m+p+1} b_i = \sum_{i=m}^{m+p} a_i + \sum_{i=m}^{m+p} b_i + a_{m+p+1} + b_{m+p+1} = \sum_{i=m}^{m+p} a_i + b_i + a_{m+p+1} + b_{m+p+1} = \sum_{i=m}^{m+p+1} a_i + b_i$ thereby closing the induction

7.1.2

a)Since X is empty, it has 0 elements and so we will have $\sum_{x\in X} f(x) = \sum_{i=1}^{0} f(g(i))$ but by definition of finite series $\sum_{i=1}^{0} f(g(i)) = 0$ so $\sum_{x\in X} f(x) = 0$ as required.

b)$Let\ g(i) = x_0$. $\sum_{x\in X} f(x) = \sum_{i=1}^{1} f(g(i)) = f(g(1)) = f(x_0)$ as required.

c)$\sum_{y\in Y} f(g(y)) = \sum_{i=1}^{n} f(g(k(i)))$ but $g \circ k$ is a bijection from $\{i \in \mathbf{N} : 1 \leq i \leq n\}$ so by definition $\sum_{y\in Y} f(g(y)) = \sum_{i=1}^{n} f(g(k(i))) = \sum_{x\in X} f(x)$ as required.

d)define the function $g : \{j \in \mathbf{N} : 1 \leq j \leq m - n + 1\} \to X$ as $g(j) = n + j - 1$ This is obviously a bijection so it can be said that $\sum_{i\in X} a_i = \sum_{j=1}^{m-n+1} a_{n+j-1} = \sum_{i=n}^{m} a_i$ as required.

e)define the function $h : \{i \in \mathbf{N} : 1 \leq i \leq m\} \to X$ as a bijection. define the function $k : \{i \in \mathbf{N} : 1 \leq i \leq n\} \to Y$ as a bijection. define the function $g : \{i \in \mathbf{N} : 1 \leq i \leq m + n\} \to X \cup Y$ as $g(i) = h(i)\ for\ i \leq m$ and $g(i) = k(i-m)\ for\ i > m$ which is a bijection. We then have $\sum_{z\in X\cup Y} f(z) = \sum_{i=1}^{m+n} f(g(i)) = \sum_{i=1}^{m} f(g(i)) + \sum_{i=m+1}^{m+n} f(g(i)) = \sum_{i=1}^{m} f(h(i)) + \sum_{i=m+1}^{m+n} f(k(i-m)) = \sum_{i=1}^{m} f(h(i)) + \sum_{i=1}^{n} f(k(i)) = \sum_{x\in X} f(x) + \sum_{y\in Y} f(y)$ as required

f)we define a bijection $h\{i \in \mathbf{N} : 1 \leq i \leq n\} \to X$. We then have $\sum_{x\in X} f(x) + g(x) = \sum_{i=1}^{n} f(h(i)) + g(h(i)) = \sum_{i=1}^{n} f(h(i)) + \sum_{i=1}^{n} g(h(i)) = \sum_{x\in X} f(x) + \sum_{x\in X} g(x)$ as required.

7.2.1 The series is divergent because the sequence $(-1)^n$ diverges and according to the zero test means that the series $\sum_{n=1}^{\infty}(-1)^n$ is also divergent. In the examples where the tricks don't work in 1.2.2, the series are divergent and therefore undefined, so arithmetic operations can't be used on them.

7.2.2 Suppose $\sum_{n=m}^{\infty} a_n$ *converges* $\Rightarrow (S_N)_{N=m}^{\infty}$ converges to L $\Rightarrow (S_N)_{N=m}^{\infty}$ is cauchy $\Rightarrow |S_q - S_p| \leq \varepsilon$ for all $p, q \geq M$ for some $M \geq m$, but $S_q - S_p = \sum_{n=m}^{q} a_n - \sum_{n=m}^{p} a_n$. In the case $p \leq q$ we have $\sum_{n=m}^{q} a_n = \sum_{n=m}^{p} a_n + \sum_{n=p+1}^{q} a_n$ *and so* $S_q - S_p = \sum_{n=m}^{q} a_n - \sum_{n=m}^{p} a_n = \sum_{n=p+1}^{q} a_n$. Therefore $|\sum_{n=p+1}^{q} a_n| \leq \varepsilon$. In the case $p > q,\ p+1 > q$ so $|\sum_{n=p+1}^{q} a_n| = 0 \leq \varepsilon$. Therefore we have $|\sum_{n=p+1}^{q} a_n| \leq \varepsilon$ for all $p, q \geq M$. But that means $|\sum_{n=p+1}^{q} a_n| \leq \varepsilon$ for all $q \geq M+1 > M$ and $p \geq M$ since $r = p+1 \geq M+1 > M$ we can therefore say that $|\sum_{n=r}^{q} a_n| \leq \varepsilon$ for all $q, r \geq M+1$ as required.

7.2.3

By 7.2.5 we have $|\sum_{n=r}^{q} a_n| \leq \varepsilon$ for all $q, r \geq N$ for some $N \geq m$. Then for all $q = r \geq N$ we have $|\sum_{n=r}^{q} a_n| = |\sum_{n=q}^{q} a_n| = |a_q| \leq \varepsilon$ as required showing that the sequence a_n does converge to 0.

7.2.4

Suppose $\sum_{n=m}^{\infty} |a_n|$ converges $\Rightarrow |\sum_{n=p}^{q} |a_n|| \leq \varepsilon$ for all $p, q \geq N$ but we know $0 \leq |\sum_{n=p}^{q} a_n| \leq \sum_{n=p}^{q} |a_n| \Rightarrow |\sum_{n=p}^{q} a_n| \leq \sum_{n=p}^{q} |a_n| = |\sum_{n=p}^{q} |a_n|| \leq \varepsilon$. Then by 7.2.5 we have $\sum_{n=m}^{\infty} a_n$ converges.

7.2.5

a)$\lim_{N\to\infty}(S_N) = \sum_{n=m}^{\infty} a_n$ *and* $\lim_{N\to\infty}(T_N) = \sum_{n=m}^{\infty} b_n \Rightarrow \sum_{n=m}^{\infty} a_n + \sum_{n=m}^{\infty} b_n = \lim_{N\to\infty}(S_N) + \lim_{N\to\infty}(T_N) = \lim_{N\to\infty}(T_N + S_N) = \lim_{N\to\infty}(\sum_{n=m}^{N} b_n + \sum_{n=m}^{N} a_n) =$
$\lim_{N\to\infty}(\sum_{n=m}^{N} a_n + b_n) = \sum_{n=m}^{\infty}(a_n + b_n)$ as required.

b)$\lim_{N\to\infty}(S_N) = \sum_{n=m}^{\infty} a_n \Rightarrow c\sum_{n=m}^{\infty} a_n = c\lim_{N\to\infty}(S_N) = \lim_{N\to\infty}(cS_N) = \lim_{N\to\infty}(c\sum_{n=m}^{N} a_n) = \lim_{N\to\infty}(\sum_{n=m}^{N} ca_n) = \sum_{n=m}^{\infty} ca_n$ as required.

7.3.1

Suppose $\sum_{n=m}^{\infty} b_n$ converges $\Rightarrow \sum_{n=m}^{N} b_n \leq M$, but because $|a_n| \leq b_n$ we then have $\sum_{n=m}^{N} |a_n| \leq \sum_{n=m}^{N} b_n \leq M$ so $\sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n \leq M$ and because $\sum_{n=m}^{N} |a_n|$ is bounded by M for all N, by P7.3.1 it can be said that $\sum_{n=m}^{\infty} |a_n|$ is convergent. Then by definition of absolute converges we can also say $\sum_{n=m}^{\infty} a_n$ is absolutely convergent.

7.3.2

Suppose $|x| \geq 1 \Rightarrow |x^n| = |x|^n \geq 1^n = 1$ for all n and therefore does not converge to 0. It can then be concluded by the zero test that $\sum_{n=0}^{\infty} x^n$ diverges for $|x| \geq 1$.

Through induction on N we will prove $\sum_{n=0}^{N} x^n = (1-x^{N+1})/(1-x)$. For the base case of N=0 we have $\sum_{n=0}^{0} x^n = x^0 = 1$ *and* $(1-x^{N+1})/(1-x) = (1-x^1)/(1-x) = 1$, thereby proving the base case. Suppose as inductive hypothesis $\sum_{n=0}^{N} x^n = (1-x^{N+1})/(1-x)$ for some N $\Rightarrow \sum_{n=0}^{N+1} x^n = \sum_{n=0}^{N} x^n + x^{N+1} = (1-x^{N+1})/(1-x) + x^{N+1} = (1-x^{N+1})/(1-x) + (1-x)(x^{N+1})/(1-x) = ((1-x^{N+1})+(1-x)(x^{N+1}))/(1-x) = (1-x^{N+1}+x^{N+1}-x^{N+2})/(1-x) = (1-x^{N+2})/(1-x)$ as required thereby closing the induction.

Suppose $|x| < 1$, we have $\sum_{n=0}^{\infty} |x^n| = \sum_{n=0}^{\infty} |x|^n = \lim(S_N)_{N=m}^{\infty} = \lim((1-|x|^{N+1})/(1-|x|))_{N=m}^{\infty} = \lim(1-|x|^{N+1})_{N=m}^{\infty}/(1-|x|) = 1/(1-|x|)$ therefore $\sum_{n=0}^{\infty} x^n$ is absolutely convergent and $\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^n = \lim(S_N)_{N=m}^{\infty} = \lim((1-x^{N+1})/(1-x))_{N=m}^{\infty} = \lim(1-x^{N+1})_{N=m}^{\infty}/(1-x) = 1/(1-x)$ as required.

7.3.3

Suppose $\sum_{n=0}^{\infty} a_n$ is an absolutely convergent series such that $\sum_{n=0}^{\infty} |a_n| = 0$. Suppose for contradiction that $|a_p| > 0$ for some $p \Rightarrow \sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{p} |a_n| + \sum_{n=p+1}^{\infty} |a_n| = \sum_{n=0}^{p-1} |a_n| + |a_p| + \sum_{n=p+1}^{\infty} |a_n|$ but $0 \leq \sum_{n=0}^{p-1} |a_n|$, $\sum_{n=p+1}^{\infty} |a_n|$ *and* $|a_p| > 0$ so $\sum_{n=0}^{p-1} |a_n| + |a_p| + \sum_{n=p+1}^{\infty} |a_n| = \sum_{n=0}^{\infty} |a_n| > 0$, a contradiction and therefore $a_n = 0$ for all n

7.4.1

Let $Y = \{n \in \mathbf{N} : n \leq N\}$. We will prove that $f : Y \to f(Y)$ is a bijection where $f : \mathbf{N} \to \mathbf{N}$ and $f(n+1) > f(n)$. First we will prove if $n > n' \Rightarrow f(n) > f(n')$. Base case p=1, $n'+p = n'+1 > n'$, $f(n'+1) > f(n')$ by definition. Suppose as inductive hypothesis, $f(n'+p) > f(n')$ for some p $\Rightarrow f(n'+p+1) > f(n'+p)$ by definition and so by inductive hypothesis $f(n'+p+1) > f(n'+p) > f(n')$ thereby closing the induction. Suppose $f(n) = f(n') \in f(Y) \Rightarrow$ Suppose for contradiction that $n \neq n' \Rightarrow$ In the case $n > n'$, we have $f(n) > f(n')$ a contradiction. In the case $n' > n$, we have $f(n') > f(n)$ a contradiction, thereby proving injectivity. Suppose $z \in f(Y) \Rightarrow z = f(x)$ *and* $x \in Y$ and therefore proves surjectivity. Since the function is injective and surjective we can conclude that it is bijective as required.

Suppose $\sum_{n=0}^{\infty} a_n$ is an absolutely convergent series. We have $S_N =$

$\sum_{n=0}^{N} |a_n|$ and $T_M = \sum_{n=0}^{M} |a_{f(n)}| = \sum_{n\in Y} a_{f(n)}$ where Y is the set $\{y \in \mathbf{N} : y \leq M\}$, but f *is a bijection from* Y *to* $f(Y)$. So we then have $\sum_{n\in Y} |a_{f(n)}| = \sum_{i\in f(Y)} |a_i|$ we then have $f(n) \leq N$ for some N because the sequence $f(n)$ is finite and therefore bounded. It is then a subset of $\{n \in \mathbf{N} : n \leq N\}$ and so $\sum_{i\in f(Y)} |a_i| \leq \sum_{n\in\{n\in\mathbf{N}:n\leq N\}} |a_n| = \sum_{n=0}^{N} |a_n| = S_N$ but $S_N \leq L$ so $T_M \leq L$ for all M. Therefore $\sum_{n=0}^{\infty} |a_{f(n)}|$ is bounded and therefore converges. So by definition of absolute convergence we have $\sum_{n=0}^{\infty} a_{f(n)}$ is absolutely convergent.

7.5.1

Suppose for contradiction that $\lim inf_{n\to\infty} c_{n+1}/c_n = L > \lim inf_{n\to\infty} c_n^{1/n} = M \Rightarrow$ we have $L > M \geq 0$. Let $L > \varepsilon > 0 \Rightarrow 0 < L - \varepsilon < L$ so we have $c_{n+1}/c_n > L - \varepsilon$ for all $n \geq N$ for some $N \geq m \Rightarrow c_{n+1} > c_n(L-\varepsilon)$. We will prove that $c_n \geq c_N(L-\varepsilon)^{n-N}$ for all $n \geq N$ through induction. Suppose as base case p=0, so that $c^{N+p} = c^N$ *and* $c_N(L-\varepsilon)^{N+p-N} = c_N(L-\varepsilon)^0 = c_N$ this implies $c^{N+0} \geq c_N(L-\varepsilon)^{N+0-N}$ as required. Now suppose as inductive hypothesis that $c_{N+p} \geq c_N(L-\varepsilon)^{N+p-N} = c_N(L-\varepsilon)^p \Rightarrow c_{N+p+1} > c_{N+p}(L-\varepsilon)$ and $c_{N+p}(L-\varepsilon) \geq c_N(L-\varepsilon)^{p+1}$ so we have $c_{N+p+1} \geq c_N(L-\varepsilon)^{p+1}$ thereby closing the induction. Let $A = c_N(L-\varepsilon)^{-N}$ We then have $c_n^{1/n} \geq A^{1/n}(L-\varepsilon)$. But $\lim_{n\to\infty} A^{1/n}(L-\varepsilon) = (L-\varepsilon)$ so we have $\lim inf_{n\to\infty} c_n^{1/n} \geq (L-\varepsilon)$. Suppose for contradiction that $\lim inf_{n\to\infty} c_n^{1/n} < L$. Then we have $0 < q < L - \lim inf_{n\to\infty} c_n^{1/n} \leq L$ so $\lim inf_{n\to\infty} c_n^{1/n} < L - q$, but we have $\lim inf_{n\to\infty} c_n^{1/n} \geq (L-p)$ a contradiction and so $\lim inf_{n\to\infty} c_n^{1/n} \geq L$ as required.

7.5.2

we have $|(n+1)^q x^{n+1}|/|n^q x^n| = |n+1|^q|x|/|n|^q = |1+1/n|^q|x| \Rightarrow \lim_{n\to\infty} |1+1/n|^q|x| = |x| \lim_{n\to\infty} |1+1/n|^q$, but $\lim_{n\to\infty} 1/n = 0$ so $\lim_{n\to\infty} |1+1/n|^q = |1|^q = 1$ and so $\lim_{n\to\infty} |1+1/n|^q|x| = |x|$. Suppose $|x| < 1$, then by the ratio test we can conclude $\sum_{n=1}^{\infty} n^q x^n$ is absolutely convergent and by the zero text also implies $\lim_{n\to\infty} n^q x^n = 0$.

8 Infinite Sets

8.1.1

Suppose X is infinite $\Rightarrow$ Suppose for contradiction that for all proper subsets $Y \subset X$, there is no bijections between Y and X

Suppose there exists a proper subset $Y \subset X$ that has the same cardinality as X. Suppose for contradiction that X is finite. Since $Y \subset X$ we have

$\#(Y) < \#(X)$ for all Y, a contradiction. Therefore X is infinite.

8.1.2

Since the natural numbers are bounded below by 0, we have X also bounded below by 0 because all the elements of X are in **N**. Since the set X is a subset of **X** which is a subset of **R**, we can use the greatest lower bound principle to say there exists an $inf(X) \leq m$ for all $m \in X$. Suppose for contradiction that $inf(X) \notin X \Rightarrow inf(X) < m$ for all $m \in X$ *and* $n \leq inf(X) < n+1$ for some n. Suppose for contradiction that $n+1 > m$ for some $m \in X \Rightarrow n \geq m \Rightarrow m \leq inf(X)$ but we have $inf(X) < m$ a contradiction and so $n+1 \leq m$ for all $m \in X$. So $n+1$ is a lower bound of X, but $inf(X) < n+1$ a contradiction. So $inf(X) \in X$ as required.

8.1.3

Suppose for contradiction that $Z = \{x \in X : x \neq a_m \ for \ all \ m < n\}$ is finite. Let $Y = \{x \in X : x = a_m \ for \ all \ m < n\}$ Y is obviously finite so we have $Z \cup Y$ is finite, but $Z \cup Y = X$ and therefore a contradiction since X is infinite.

We have $a_n = min\{x \in X : x \neq a_m \ for \ all \ m < n\}$ and $a_{n+1} = min\{x \in X : x \neq a_m \ for \ all \ m < n+1\}$. But we have $\{x \in X : x \neq a_m \ for \ all \ m < n+1\} \subset \{x \in X : x \neq a_m \ for \ all \ m < n+1\}$ so $a_n \leq y \in \{x \in X : x \neq a_m \ for \ all \ m < n+1\}$. But $n < n+1$ so $a_n \neq y$ and we are left with $a_n < y \in \{x \in X : x \neq a_m \ for \ all \ m < n+1\}$. Since $a_{n+1} \in \{x \in X : x \neq a_m \ for \ all \ m < n+1\}$ we then have $a_n < a_{n+1}$ as required.

if $n > m \Rightarrow a_n > a_m$. We will prove this through induction. For the base case of $p = 1$ we have $a_{m+p} = a_{m+1} > a_m$ as required. Suppose as inductive hypothesis $a_{m+p} > a_m$ for some p $\Rightarrow a_{m+p+1} > a_{m+p} > a_m$ thereby closing the induction.

Suppose $n \neq m$. In the case $n > m$ we have $a_n > a_m$ and in the case $m > n$ we have $a_m > a_n$ in both cases we have $a_n \neq a_m$ as required.

a_n is in $\{x \in X : x \neq a_m \ for \ all \ m < n\}$ so $a_n \in X$ as required.

Suppose $a_n \neq x$ for all n. Then $x \neq a_m$ for all $m < n$ for any n and since $x \in X$ we have x is in $\{x \in X : x \neq a_m \ for \ all \ m < n\}$ for all n.

we have $a_n \geq n$. Proof through induction. Base case, n=0, we have $a_0 = 0 \geq 0$ as required. Now suppose as inductive hypothesis that $a_n \geq n$ for some n $\Rightarrow a_{n+1} > a_n \geq n \Rightarrow a_{n+1} > n$ so $a_{n+1} \geq n+1$ thereby closing the induction.

8.1.4

We have the set Y is either infinite or finite. In the case it is finite, $f(\mathbf{N})$

is a subset of Y and so is finite, therefore is at most countable. In the case Y is infinite, $f(\mathbf{N})$ is still a subset of Y, but $f(\mathbf{N})$ is either infinite or finite. In the case it's finite it is at most countable. In the case it is infinite, Let $A = \{n \in \mathbf{N} : f(m) \neq f(n) \ for \ all \ m < n\}$. Restrict the domain of f to A. Suppose $f(n) = f(n')$. Suppose for contradiction that $n \neq n' \Rightarrow$ In the case $n > n'$ we have $f(n') \neq f(n)$ a contradiction. In the other case we can flip the n and n' around to get the same result. Therefore $n = n'$ proving injectivity. Now suppose for contradiction that there exists $y \in f(\mathbf{N})$ such that for all $n \in A, \ f(n) \neq y$. we have $f(m) = y$ for some $m \in \mathbf{N}$. In the case $m \in A$ we have $f(m) \neq y$ a contradiction. In the case $m \notin A$, there exists a $p < m$ such that $f(p) = f(m)$. But p can be plugged in again and would result in either a contradiction or an even smaller value. Yet by the principle of infinite descent, natural numbers can't be in infinite descent and therefore we get a contradiction in both cases. Thereby proving surjectivity. Since A has a bijection with $f(\mathbf{N})$ we can conclude that A is infinite and has the same cardinality as $f(\mathbf{N})$. Since A is also a subset of x we can also conclude that A has the same cardinality as $\mathbf{N}$ and by transitivity of cardinality we can conclude that $\mathbf{N}$ has the same cardinality as $f(\mathbf{N})$ and is therefore countable, therefore at most countable as required.

8.1.5

we have X is a countable set so there exists a bijection $g : \mathbf{N} \to X$. That means $g(\mathbf{N}) = X$ so $f(g(\mathbf{N})) = f(X)$ but $f \circ g : \mathbf{N} \to Y$ and so by P8.1.8, $f(g(\mathbf{N}))$ is at most countable and therefore so is $f(X)$ as required.

8.1.6 Suppose A is at most countable. Then A is either finite or countable. In the case it is countable, there exists a bijection f between A and $\mathbf{N}$. Then f is injective and surjective, so f is injective as required. In the case A is finite we have a bijection f between A and $\{n \in \mathbf{N} : n \leq N\}$ for some $N \in \mathbf{N}$ and $\{n \in \mathbf{N} : n \leq N\} \subseteq \mathbf{N}$. So we have an injective function between A and $\mathbf{N}$ as required.

8.2.1

Since the series $\sum_{x \in X} |f(x)|$ converges, we have $\sum_{x \in X} |f(x)| = \sum_{n=0}^{\infty} |f(g(x))| = L$ for some real number L and where $g : \mathbf{N} \to X$ is a bijection when X is countable and $g : n \in \mathbf{N} : n \leq N \to X$ where N is the cardinality of X in the case X is finite. In the case X is finite, we have 2^X **GOTTA FIX can't use power set cardinality**total subsets of X and since A is finite, $\sum_{x \in A} |f(x)|$ will be a real number so $\{\sum_{x \in A} |f(x)|, A \subseteq X, A \ is \ finite\}$ will be a finite set of real numbers and therefore is bounded by some real number. That means the supremum is less than or equal to some real

number and therefore is less than ∞. In the case X is countable, a finite subset A of X, we have a bijection from $g^{-1}(A)\ to\ A$. Since $g^{-1}(A)$ is a finite subset of $\mathbf{N}$ there exists an N such that $n \in g^{-1}(A) \leq N$. Then $g^{-1}(A) \subseteq \{n \in \mathbf{N} : n \leq N\}$ and so $A \subseteq g(\{n \in \mathbf{N} : n \leq N\})$, we then have $\sum_{x \in g(\{n \in \mathbf{N}: n \leq N\})} |f(x)| = \sum_{x \in (g(\{n \in \mathbf{N}: n \leq N\}) - A)} |f(x)| + \sum_{x \in A} |f(x)|$. But $\sum_{x \in g(\{n \in \mathbf{N}: n \leq N\})} |f(x)| =$
$\sum_{n=0}^{N} |f(g(x))| \leq L$ since the series is absolutely convergent. Since
$\sum_{x \in (g(\{n \in \mathbf{N}: n \leq N\}) - A)} |f(x)|$ is positive we then have $L \geq \sum_{x \in A} |f(x)|$ for all A. So L is an upper bound of the set $\{\sum_{x \in A} |f(x)|, A \subseteq X, A\ is\ finite\}$ and therefore $sup\{\sum_{x \in A} |f(x)|, A \subseteq X, A\ is\ finite\} \leq L < \infty$ as required.

8.3.1

Base case X is a set with N=0 cardinality. The power set of X has only the empty set and therefore has cardinality 1. But $2^N = 2^0 = 1$ so the cardinality of $2^X = 2^N = 0$ as required. Suppose as inductive hypothesis that for some N. The set 2^X has cardinality 2^N where X is a set with cardinality N $\Rightarrow$ Let 2^X be the power set of X where X is a set of cardinality N+1. The set $X - \{x\}$ where $x \in X$ has cardinality of N so $2^{X-\{x\}}$ has cardinality 2^N but $2^{X-\{x\}} = \{y \in 2^X : x \notin y\}$ we also have $2^X = \{y \in 2^X : x \notin y\} \cup \{y \in 2^X : x \in y\}$. Then let $f : \{y \in 2^X : x \notin y\} \to \{y \in 2^X : x \in y\}$ be $f(A) = A - \{x\}$. Suppose $f(A) = f(A')$ $\Rightarrow$ Suppose for contradiction that $A \neq A'$ $\Rightarrow$ there exists an $x_0 \in A$ such that $x_0 \notin A'$ or there exists an $x_0 \in A'$ such that $x_0 \notin A'$ in the first case we have $x_0 \neq x$ because $x \in A\ and\ x \in A'$. So $x_0 \in A - \{x\}\ and\ x_0 \notin A' - \{x\}$. This means that $A - \{x\} \neq A' - \{x\}$ a contradiction. The other case is symmetric to the first so it can follow the same proof. Therefore $A = A'$ and the function is injective. Now suppose $y \in \{y \in 2^X : x \notin y\}$ $\Rightarrow$ since y is a subset of X and x is in X $y \cup \{x\}$ is a subset of X and so $y \cup \{x\} \in \{y \in 2^X : x \in y\}$. We therefore have $f(y \cup \{x\}) = y \cup \{x\} - \{x\} = y$ and so the function is surjective. Since we have a bijection between $\{y \in 2^X : x \notin y\}\ and\ \{y \in 2^X : x \in y\}$ the cardinality of the two are therefore equal. Since $\{y \in 2^X : x \notin y\}$ has a cardinality of 2^N we then have $\{y \in 2^X : x \in y\}$ has a cardinality of 2^N and because the two sets are disjoint we have $2^X = \{y \in 2^X : x \notin y\} \cup \{y \in 2^X : x \in y\}$ has cardinality $2^N + 2^N = 2 * 2^N = 2^{N+1}$ thereby closing the induction.

8.3.2

First we will prove that if $A \cap B = \emptyset$, $A \subseteq X$, $B \subseteq X$, and $f : X \to Y$ is injective $\Rightarrow$ $f(A) \cap f(B) = \emptyset$ Suppose the hypothesis true. Suppose for contradiction that there exists an object x such that $x \in f(A)\ and\ x \in$

$f(B) \Rightarrow x = f(a)$ *and* $x = f(b)$ for some $a \in A$ *and* $b \in B \Rightarrow f(a) = f(b)$ and by the injectivity of f, $a = b$ but $A \cap B = \emptyset$, a contradiction. Therefore $f(A) \cap f(B) = \emptyset$

We will prove through induction on n that $D_n \cap D_m = \emptyset$ for all $m < n$. Base case $n = 1$. We have $D_1 \cap D_0 = f(B-A) \cap (B-A)$ but $f(B-A) \subseteq A$. Suppose for contradiction that there is an object x such that $x \in f(B-A) \cap (B-A) \Rightarrow x \in A$ *and* $x \in B$ *and* $x \notin A$ which is a contradiction and so $x \notin f(B-A) \cap (B-A)$ for all objects x and therefore $f(B-A) \cap (B-A) = \emptyset$ as required. Now suppose as inductive hypothesis that $D_n \cap D_m = \emptyset$ for all $m < n \Rightarrow f(D_n) = D_{n+1}$ and we also have $f(D_m) = D_{m+1}$ but because $D_n \cap D_m = \emptyset$ and f is injective, we have $f(D_n) \cap f(D_m) = D_{n+1} \cap D_{m+1} = \emptyset$. So D_{n+1} Let $p = m+1$ we then have $D_n \cap D_p = \emptyset$ for all $0 < p < n+1$ but we know that $D_0 = B - A$ and is therefore has to be disjoint from $f(D_n) = D_{n+1}$ since D_{n+1} is a subset of A. We therefore have $D_n \cap D_p = \emptyset$ for all $p < n+1$ thereby closing the induction.

Suppose $n \neq m \Rightarrow$ *either* $n > m$ *or* $n < m$ In the case $n > m$ we have $D_n \cap D_m = \emptyset$ because of the above proof. In the other case we would have the same. So it can be concluded that the $D_n \cap D_m = \emptyset$ for all $n \neq m$.

Suppose $x \in A \Rightarrow g(x) = f^{-1}(x)$ *or* $g(x) = x$ In the case $g(x) = f^{-1}(x)$ we have $x \in D_n \Rightarrow f^{-1}(x) \in f^{-1}(D_n) = D_{n-1} \subseteq A \subseteq B$ when $n \geq 2$. when $n = 1$ we have $D_{n-1} = D_0 = B - A \subseteq B$ as required. In the case $g(x) = x$ we have $g(x) \in A \subseteq B$ as required. Therefore g is indeed a map from A to B.

Suppose $g(x) = g(x') \Rightarrow (x \in \bigcup_{n=1}^{\infty} D_n \text{ or } x \notin \bigcup_{n=1}^{\infty})$ *and* $(x' \in \bigcup_{n=1}^{\infty} D_n \text{ or } x' \notin \bigcup_{n=1}^{\infty})$. Suppose for contradiction that $x \in \bigcup_{n=1}^{\infty} D_n$ *and* $x' \notin \bigcup_{n=1}^{\infty} D_n \Rightarrow g(x) = f^{-1}(x)$ *and* $g(x') = x' \Rightarrow f^{-1}(x) = x'$ but we have $x \in D_n$ for some $n \geq 1$. Since $f^{-1}(D_n) = D_{n-1}$ we have $f^{-1}(x) \in D_{n-1}$. When $n \geq 2$ we have $D_{n-1} \subset \bigcup_{n=1}^{\infty} D_n$ So $f^{-1}(x) = x \in \bigcup_{n=1}^{\infty} D_n$ a contradiction. When $n = 1$ we would have $f^{-1}(x) \in B - A$ but that would mean $x \in B - A$ a contradiction since x is in A. The opposite also leads to contradiction so we have $(x, x' \in \bigcup_{n=1}^{\infty} D_n)$ *or* $(x, x' \notin \bigcup_{n=1}^{\infty} D_n)$ In the first case Suppose for contradiction $x \in D_n, x' \in D_m$ where $n \neq m \Rightarrow g(x) \in D_{n-1}, g(x') \in D_{m-1}$ which are also disjoint and therefore a contradiction. So $x, x' \in D_n$, but we know that f restricted to D_{n-1} is a bijection to D_n so its inverse is also bijective thereby injective and so since $f^{-1}(x) = f^{-1}(x') \Rightarrow x = x'$ as required. Now suppose the second case. We have $g(x) = g(x') \Rightarrow g(x) = x$ *and* $g(x') = x' \Rightarrow x = x'$ as required. Thereby proving the function injective. Suppose $y \in B \Rightarrow y \in B - A$ *or* $y \in A$. In the case $y \in B - A$, we have $y \in D_0 = f^{-1}(D_1) = g(D_1) \Rightarrow y = g(x)$ for some $x \in D_1$. In the case $y \in A$ we have $y \in \bigcup_{n=1}^{\infty} D_n$ *or* $y \notin \bigcup_{n=1}^{\infty} D_n$. In the first case we have $y \in D_n$ for some $n \geq 1$. But $g(D_{n+1}) = f^{-1}(D_{n+1}) = D_n$. So $y \in g(D_{n+1}) \Rightarrow y = g(x)$. In the second case we have $g(A - \bigcup_{n=1}^{\infty} D_n) = A - \bigcup_{n=1}^{\infty} D_n$. But $y \in A - \bigcup_{n=1}^{\infty} D_n = g(A - \bigcup_{n=1}^{\infty} D_n)$ so we have $y = g(x)$. Thereby g is also surjective. Since it is injective and surjective we therefore have g is bijective as required.

8.3.3

Suppose there is an injective map $f : A \to B$ and $g : B \to A \Rightarrow f : A \to f(A)$ is a bijection and $f(A) \subseteq B$, we also have $g : B \to g(B)$ is a bijection and $g(B) \subseteq A \Rightarrow f : g(B) \to f(g(B))\ is\ a\ bijection\ and\ f(g(B)) \subseteq f(A) \subseteq B \Rightarrow$ There is a bijection $h : f(g(B)) \to f(A)$. So $f(g(B))$ has equal cardinality with $f(A)$ but since $f(A)$ has a bijection with A, $f(A)$ has equal cardinality with A and therefore A has equal cardinality with $f(g(B))$. But $f(g(B))$ has a bijection with $g(B)$ and therefore has equal cardinality to $g(B)$ and we have $g(B)$ has equal cardinality with B so we can finally say B has equal cardinality with A.

8.3.4

Suppose X is a set. Define $f : X \to 2^X$ as $f(x) = \{x\} \Rightarrow$ Suppose $f(x) = f(x') \Rightarrow \{x\} = \{x'\} \Rightarrow x = x'$ showing that the function is inductive. Therefore we can say that X has lesser than or equal cardinality to 2^X. But we know that $X\ and\ 2^X$ cannot have equal cardinality by Cantor's theorem and so we can conclude that X has strictly less cardinality than 2^X.

Suppose A has strictly less cardinaliy than B and B has strictly less cardinality than C $\Rightarrow$ there exists functions $f : A \to B\ and\ g : B \to C$ that are injective $\Rightarrow g \circ f : A \to C$ is also injective so we can say that A has less than or equal cardinality to C. Suppose for contradiction that there exists a bijection $h : C \to A \Rightarrow f \circ h : C \to B$ is injective and so we have C and B have equal cardinality by E8.3.3. A contradiction so therefore A and C cannot have equal cardinality and we have A has strictly lesser cardinality than C as required.

8.3.5

Let X be a set $\Rightarrow$ X is either finite or infinite. In the case X is finite 2^X is finite by E8.3.1. and therefore cannot be countably infinite. In the case X is infinite we have X is either countable or uncountable. In the case X is countable 2^X cannot have equal cardinality with X and therefore cannot be countable. In the case X is uncountable, we have the map $f : X \to 2^X$ defined as $f(x) = \{x\}$ which is injective and so $f(X)$ has the same cardinality as X and is therefore uncountable. Suppose for contradiction that 2^X is at most countable $\Rightarrow$ *since* $f(X) \subseteq 2^X,\ f(X)$ is at most countable, but $f(X)$ is uncountable, a contradiction. So we have 2^X is uncountable and therefore we have 2^X is not countable in all cases

8.4.1

Suppose hypothesis true, by the axiom of choice, there exists a function that assigns to each element $x \in X$ an element $y \in Y_x$ where Y_x is defined by $Y_x = \{y \in Y : P(x, y)\}$ and $P(x, f(x))$ is true for all $x \in X$

8.4.2

Using the axiom of choice we can find a set Y such that each element y_α in the set is defined as $y_\alpha \in X_\alpha$. So the element $y_\alpha \notin X_\beta\ when\ \alpha \neq \beta$. We then have $X_\alpha \cap Y = y_\alpha$ for all α and so $\#(Y \cap X_\alpha) = 1$

8.5.1

The empty set is indeed partially order because the $\leq_\emptyset$ obeys all 3 properties given since the relation is vacuous thereby making the implications all vacuously true.

$\emptyset$ is also totally ordered because we have $\emptyset \subseteq \emptyset$ but $\emptyset$ is partially ordered as required and the implication $y, y' \in Y \Rightarrow either\ y \leq_\emptyset y'\ or\ y' \leq_\emptyset y$ is vacuous making the implication vacuously true as required.

$\emptyset$ is also well ordered because we have $\emptyset \subseteq \emptyset$ but $\emptyset$ is partially ordered and totally ordered and the implication $X \neq \emptyset\ and\ X \subseteq \emptyset \Rightarrow there\ exists\ min(X)$ is vacuous making the implication vacuously true as required.

9 Continuous functions on R

9.1.1

Let $\bar{y} \in \overline{Y} \Rightarrow |\bar{y} - y| \leq \varepsilon/2$ for some $y \in Y$. But $Y \subseteq \overline{X}$ so we have $|y - x| \leq \varepsilon/2$ for some $x \in X \Rightarrow |\bar{y} - x| \leq |\bar{y} - y| + |y - x| \leq \varepsilon \Rightarrow |\bar{y} - x| \leq \varepsilon$ and so we have $\bar{y}$ is an adherent point of X. Therefore we have $\bar{y} \in \overline{X}$. So by definition $\overline{Y} \subseteq \overline{X}$.

Let $x \in \overline{X} \Rightarrow |\bar{x} - x| \leq \varepsilon$ for some $x \in X$. But $X \subseteq Y$ so we have $|\bar{x} - x| \leq \varepsilon$ for some $x \in Y$. So $\bar{x}$ is an adherent point of Y and therefore $\bar{x} \in \overline{Y}$ and by definition we have $\overline{X} \subseteq \overline{Y}$. It can then be concluded that $\overline{X} = \overline{Y}$ as required.

9.1.2

Let $x \in X \Rightarrow |x-x| = 0 \leq \varepsilon$ and so x is an adherent point of X. Therefore $x \in \overline{X}$. So by definition $X \subseteq \overline{X}$ as required.

Let $\bar{z} \in \overline{X \cup Y} \Rightarrow |\bar{z} - z| \leq \varepsilon$ for some $z \in X \cup Y$. Suppose for contradiction that $\bar{z}$ is not adherent to X and Y. Then we have $|\bar{z} - x| > \varepsilon'$ for all $x \in X$ for some $\varepsilon' > 0$ and $|\bar{z} - y| > \delta$ for all $y \in Y$ for some $\delta > 0$. Let $m = min(\delta, \varepsilon') \Rightarrow |\bar{z} - x| > m$ and $|\bar{z} - y| > m$ for all x and y. But we have $|\bar{z} - z| \leq m$ where z is either in Y or in X. In the case it's in X we would have a contradiction, same with if it was in Y. Therefore we can conclude that $\bar{z}$ is adherent to X or Y. So $\bar{z} \in \overline{X}$ *or* $\bar{z} \in \overline{Y}$ and we then have $\bar{z} \in \overline{X} \cup \overline{Y}$. Therefore by definition we have $\overline{X \cup Y} \subseteq \overline{X} \cup \overline{Y}$.

Let $\bar{z} \in \overline{X} \cup \overline{Y} \Rightarrow \bar{z} \in \overline{X}$ *or* $\bar{z} \in \overline{Y}$. In the case $\bar{z} \in \overline{X}$ we have $|\bar{z} - x| \leq \varepsilon$ for some $x \in X$. but that means $|\bar{z} - x| \leq \varepsilon$ for some $x \in X \cup Y$ and so $\bar{z}$ is adherent to $X \cup Y$ therefore $\bar{z} \in \overline{X \cup Y}$ The case of Y and the case it's in both can be worked out similarly and therefore it can be concluded that $\bar{z} \in \overline{X \cup Y}$ and so $\overline{X} \cup \overline{Y} \subseteq \overline{X \cup Y}$. Therefore $\overline{X} \cup \overline{Y} = \overline{X \cup Y}$ as required.

Let $\bar{z} \in \overline{X \cap Y} \Rightarrow |\bar{z} - z| \leq \varepsilon$ for some z *in* X *and* Y but that means $\bar{z}$ is adherent to X and Y. Therefore we have $\bar{z} \in \overline{X}$ *and* $\bar{z} \in \overline{Y}$ so $\bar{z} \in \overline{X} \cap \overline{Y}$. It can then be concluded that $\overline{X \cap Y} \subseteq \overline{X} \cap \overline{Y}$ as required.

Suppose $X \subseteq Y$, *Let* $\bar{x} \in \overline{X} \Rightarrow |\bar{x} - x| \leq \varepsilon$ for some $x \in X$, but $x \in Y$ so we have $\bar{x}$ is an adherent point of Y. Therefore $\bar{x} \in \overline{Y}$ and so $\overline{X} \subseteq \overline{Y}$ as required.

9.1.3

Suppose $\bar{n} \in \overline{\mathbf{N}} \Rightarrow |\bar{n} - n| \leq \varepsilon$ for some $n \in \mathbf{N}$ Suppose for contradiction that $\bar{n} \notin \mathbf{N}$. Then $|\bar{n} - n| > 0$ so we have $|\bar{n} - n| > q > 0$ for some real q, but we have $|\bar{n} - n| \leq q$, a contradiction and therefore $\bar{n} \in \mathbf{N}$. So it can be concluded that $\overline{\mathbf{N}} \subseteq \mathbf{N}$. We also know $\mathbf{N} \subseteq \overline{\mathbf{N}}$. Therefore $\mathbf{N} = \overline{\mathbf{N}}$ as required.

Suppose $\bar{z} \in \overline{\mathbf{Z}} \Rightarrow |\bar{z} - z| \leq \varepsilon$ for some $z \in \mathbf{z}$ Suppose for contradiction that $\bar{z} \notin \mathbf{Z}$. Then $|\bar{z} - z| > 0$ so we have $|\bar{z} - z| > q > 0$ for some real q, but we have $|\bar{n} - n| \leq q$, a contradiction and therefore $\bar{z} \in \mathbf{Z}$. So it can be concluded that $\overline{\mathbf{Z}} \subseteq \mathbf{Z}$. We also know $\mathbf{Z} \subseteq \overline{\mathbf{Z}}$. Therefore $\mathbf{Z} = \overline{\mathbf{Z}}$ as required.

Suppose $\bar{q} \in \overline{\mathbf{Q}} \Rightarrow \bar{q} \in \mathbf{R}$ by definition and therefore $\overline{\mathbf{Q}} \subseteq \mathbf{R}$. Suppose $r \in \mathbf{R} \Rightarrow$ Suppose for contradiction that $r \notin \overline{\mathbf{Q}} \Rightarrow |r - q| > \varepsilon$ for all $q \in \mathbf{Q}$ for some $\varepsilon > 0$. But $r > r - \varepsilon$ so we have $r > p > r - \varepsilon$ for some rational p. Since $p + \varepsilon > r$ and $p - \varepsilon < r$ we have $p + \varepsilon > r > p - \varepsilon$ So $|r - p| \leq \varepsilon$ a contradiction since p is a rational. This means that $r \in \overline{\mathbf{Q}}$ and so $\mathbf{R} \subseteq \overline{\mathbf{Q}}$. Therefore it can be concluded that $\overline{\mathbf{Q}} = \mathbf{R}$ as required.

Suppose $\bar{r} \in \overline{\mathbf{R}} \Rightarrow r \in \mathbf{R}$ by definition. and so $\overline{\mathbf{R}} \subseteq \mathbf{R}$. We also have

$\mathbf{R} \subseteq \overline{\mathbf{R}}$ and so we can say $\mathbf{R} = \overline{\mathbf{R}}$ as required.

Suppose for contradiction that $x \in \overline{\emptyset} \Rightarrow |x - y| \leq \varepsilon$ for some $y \in \emptyset$ but this is a contradiction since there does not exists any objects in the empty set and so we have $x \notin \overline{\emptyset}$ for any x. Therefore $\overline{\emptyset} = \emptyset$ as required

9.1.4

Consider the set $X = \{x \in \mathbf{R} : 0 \leq x < 1\}\ and\ Y = \{1\}$. We have $X \cap Y = \emptyset$ so $\overline{X \cap Y} = \emptyset$ and since the closure of $\emptyset$ is the $\emptyset$ we then have $\overline{X \cap Y} = \emptyset$. On the other hand we have $\overline{X} = \{x \in \mathbf{R} : 0 \leq x \leq 1\}$ and $\overline{Y} = \{1\}$ so $\overline{X} \cap \overline{Y} = \{1\}$. We therefore have $\overline{X \cap Y} \neq \overline{X} \cap \overline{Y}$ as required.

9.1.5

Suppose x is an adherent point of X $\Rightarrow |x - y| \leq \varepsilon$ for some $y \in X$. We then define the sets $X_n = \{y \in X : |x-y| \leq 1/(n+1)\}$. These are obviously non-empty sets. Now using the axiom of choice we can construct a sequence where $a_n \in X_n$ for all $n \in \mathbf{N}$. We therefore have $0 \leq |x - a_n| \leq 1/(n+1)$ and when we take the limit we have $0 \leq \lim_{n\to\infty} |x-a_n| \leq 0$ so $\lim_{n\to\infty} |x-a_n| = 0$. Suppose for contradiction that $\lim_{n\to\infty} a_n \neq x \Rightarrow |x - a_n| > \varepsilon$ for some $n \geq N$ for all $N \geq 0$, but we have $|x - a_n| \leq \varepsilon$ for all $n \geq N$ for some $N \geq 0$. This is a contradiction and so we can conclude that $\lim_{n\to\infty} a_n = x$ and therefore there is a sequence consisting only of elements of X that converges to x as required.

Suppose a sequence $(a_n)_{n=0}^{\infty}$ where $a_n \in X$, converges to x $\Rightarrow |x - a_n| \leq \varepsilon$ for all $n \geq N$ for some $N \geq 0$ we therefore have $|x - a_n| \leq \varepsilon$ for $a_n \in X$ as required, showing that x is adherent to X

9.2.1

A) $((f+g)\circ h)(x) = (f+g)(h(x)) = f(h(x)) + g(h(x)). We\ also\ have (f \circ h)(x) = f(h(x))\ and\ (g \circ h)(x) = g(h(x))$. So $(f \circ h + g \circ h)(x) = f(h(x)) + g(h(x))$ as required.

B) consider the functions $f(x) = x^2,\ g(x) = x,\ h(x) = 2$ we then have $(f \circ (g+h))(x) = f(g(x) + h(x)) = f(2+x) = x^2 + 4x + 4$ and $((f \circ g) + (f \circ h))(x) = (f \circ g)(x) + (f \circ h)(x) = f(g(x)) + f(h(x)) = f(x) + f(2) = x^2 + 4$. When the two functions are taken at 1 we have $x^2 + 4 = 5\ and\ x^2 + 4x + 4 = 9$. Therefore the two functions are not equal.

C)$((f+g)\cdot h)(x) = (f+g)(x)h(x) = (f(x)+g(x))h(x) = f(x)h(x) + g(x)h(x)$ and we have $((f\cdot h)+(g\cdot h))(x) = (f\cdot h)(x) + (g \cdot h)(x) = f(x)h(x) + g(x)h(x)$ as required.

D)$(f \cdot (g+h))(x) = f(x)(g+h)(x) = f(x)(g(x)+h(x)) = f(x)g(x) + f(x)h(x)$ and we have $((f\cdot g)+(f\cdot h))(x) = (f\cdot g)(x)+(f\cdot h)(x) = f(x)g(x) + f(x)h(x)$ as required.

9.3.1

Suppose f converges to L at x_0 in E. Let $(a_n)_{n=0}^{\infty}$ be a sequence which consists entirely of elements of E and converges to x_0. Then $|f(x) - L| \leq \varepsilon$ for all $x \in E$ such that $|x - x_0| < \delta$ for some $\delta > 0$ But we also have $|a_n - x_0| \leq \delta/2$. for all $n \geq N$ for some $N \geq 0$. Since $|a_n - x_0| \leq \delta/2 < \delta$ and is in E, we have $|f(a_n) - L| \leq \varepsilon$ for all $n \geq N$. Since ε was arbitrary we can then conclude that $(f(a_n))_{n=0}^{\infty}$ converges to L.

Suppose for every sequence $(a_n)_{n=0}^{\infty}$ which consists entirely of elements of E and converges to x_0, the sequence $(f(a_n))_{n=0}^{\infty}$ converges to L. Suppose for contradiction that there exists an $\varepsilon > 0$ such that for all $\delta > 0$, $|f(x) - L| > \varepsilon$ for some $x \in E$ such that $|x - x_0| \leq \delta$. We then have $|x - x_0| < 1/(n+1)$. Consider the set $Y_n = \{x \in E : |x - x_0| < 1/n \textit{ and } |f(x) - x_0| > \varepsilon\}$ by the axiom of choice, we can construct a sequence $(a_n)_{n=0}^{\infty}$ such that $a_n \in X_n$. This sequence will have the property that all elements are in E and that $|a_n - x_0| < 1/(n+1)$. So we have $-1/(n+1) \leq a_n - x_0 \leq 1/(n+1)$ meaning $x_0 - 1/(n+1) \leq a_n \leq x_0 + 1/(n+1)$. Taking the limits we are left with $x_0 \leq \lim_{n\to\infty} a_n \leq x_0$ and so $(a_n)_{n=0}^{\infty}$ converges to L, so that would mean $(f(a_n))_{n=0}^{\infty}$ converges to L and so $|f(a_n) - L| \leq \varepsilon$, but we have $|f(a_n) - x_0| > \varepsilon$ since $a_n \in X_n$, a contradiction. Therefore f converges to L at x_0 in E.

9.4.2

Let $x_0 \in X \Rightarrow f(x_0) = c$. Now consider a sequence $(a_n)_{n=0}^{\infty}$ that converges to x_0 where $a_n \in X$. since $a_n \in X$ we have $f(a_n) = c$ so the sequence $(f(a_n))_{n=0}^{\infty}$ converges to c because $|f(a_n) - c| = 0 \leq \varepsilon$ for all $n \geq N$ for $N = 0$ as required. Since $(a_n)_{n=0}^{\infty}$ was chosen arbitrarily we can then conclude that $\lim_{x\to x_0; x\in X} f(x) = c = f(x_0)$ proving that f is continuous at x_0 and since x_0 was chosen arbitrarily, we can also say f is continuous at all $x \in X$ and therefore f is continuous as required.

Let $x_0 \in X \Rightarrow g(x_0) = x_0$. Now consider the sequence $(a_n)_{n=0}^{\infty}$ that converges to x_0 where $a_n \in X$. since $a_n \in X$ we have $g(a_n) = a_n$. We then have $|g(a_n) - x_0| = |a_n - x_0| \leq \varepsilon$ for all $n \geq N$ for some $n \geq N$ and so $(g(a_n))_{n=0}^{\infty}$ converges to x_0. we therefore have $\lim_{x\to x_0; x\in X} g(x) = x_0 = g(x_0)$ and so f is continuous at x_0 which then implies f is continuous at all $x \in X$ and therefore f is continuous as required.

9.5.1

$\lim_{x\to x_0; x\in E} f(x) = +\infty \; iff \; for \; all \; \varepsilon > 0, \; there \; exists \; \delta > 0 \; such \; that \; f(x) > \varepsilon \; for \; all \; x \in E \; such \; that \; |x - x_0| < \delta$.

$\lim_{x\to x_0; x\in E} f(x) = -\infty \; iff \; for \; all \; \varepsilon > 0, \; there \; exists \; \delta > 0 \; such \; that \; f(x) > -\varepsilon \; for \; all \; x \in E \; such \; that \; |x - x_0| < \delta$.

Consider the function $f(x) = 1/x$. We then have $f(0-) = \lim_{x \to 0; x \in \mathbf{R} \cap (-\infty, 0)} f(x)$. given a $\varepsilon > 0$, $f(x) = 1/x < -\varepsilon$ when $1/-\varepsilon > x$. So let $\delta = 1/\varepsilon$ we then have for $x \in \mathbf{R} \cap (-\infty, 0)$, *such that* $|x - 0| < 1/\varepsilon$. Since x is always negative we can say $|x| = -x < 1/\varepsilon$ and so $1/x < -\varepsilon$. But that means $f(x) < \varepsilon$. So we have $f(x) < -\varepsilon$ for all $x \in \mathbf{R} \cap (-\infty, 0)$ such that $|x| < 1/\varepsilon$. Since ε was arbitrary and $1/\varepsilon > 0$ we can then say *for all* $\varepsilon > 0$ there exists $1/\varepsilon > 0$ such that $f(x) = 1/x < -\varepsilon$ for all $x \in \mathbf{R} \cap (-\infty, 0)$ such that $|x| < 1/\varepsilon$. So by definition we have $f(0-) = -\infty$ as required.

for $f(0+)$ a similar proof can be done to show $f(0+) = +\infty$

9.6.1

A) the function $f(x) = (x - 1.5)^2$ satisfies this condition. It doesn't contradict the maximum principle because $(1, 2)$ is an open interval rather than a closed one

B) the function $f(x) = 1/(x + 1)$ satisfies this condition. It doesn't contradict the maximum principle because $[0, 2)$ is not closed on both sides as is required.

C) the function $f(x) = 0$ when $x = -1, 1$ and $f(x) = x$ when $x \in (-1, 1)$ satisfies this condition. It doesn't contradict the maximum principle because f is not continuous on the given interval.

D) the function $f(x) = 0 \text{ when } x = 0 \text{ and } f(x) = 1/x$ otherwise has no upper or lower bound. This doesn't contradict the maximum principle because the function is not continuous which are requirements.

9.7.1

Since m and M are defined as the supremum and infinium of a continuous function, by the maximum principle we have $f(x_{max}) = M$ for some $x_{max} \in [a, b]$ and $f(x_{min}) = m$ for some $x_{min} \in [a, b]$. We have three cases, either $x_{min} = x_{max},\ x_{min} < x_{max},\ or\ x_{min} > x_{max}$. Suppose $x_{min} = x_{max}$, this would mean $f(x_{min}) = f(x_{max})$ and so $y = f(x_{min})$ so we have $x_{min} \in [a, b]$ such that $f(x_{min}) = y$. In the case $x_{min} < x_{max}$, consider the set $[x_{min}, x_{max}]$ this is obviously a subset of $[a, b]$ and so $f|_{[x_{min}, x_{max}]} \to \mathbf{R}$ is a continuous function. But from the intermediate value theorem, given a y between $f(x_{min}, f(x_{max})$, there exists a $c \in [x_{min}, x_{max}] \subseteq [a, b]$ such that $f(c) = y$ as required. The case of $x_{min} > x_{max}$ has a similar proof.

Suppose $x \in f([a, b]) \Rightarrow m \leq x \leq M$ by definition. That means $x \in [m, M]$ and so $f([a, b]) \subseteq [m, M]$. Suppose $y \in [m, M]$ by the proof above, there exists an $x \in [a, b]$ such that $f(x) = y$ and so by definition of $f([a, b])$ we have $y \in f([a, b])$ therefore $[m, M] \subseteq f([a, b])$. So $f([a, b]) = [m, M]$ as required.

9.7.2

Consider the function $f(x)-x$ this function is continuous because $f(x)$ *and* x is continuous. Since the domain is $[0,1]$ we have $0 \leq f(0)$ and $-1 \leq f(1)-1 \leq 0$ so we have $f(1)-1 \leq 0 \leq f(0)$. So by the intermediate value theorem there exists a $c \in [0,1]$ such that $f(c)-c=0$ and so $f(c)=c$ as required.

9.9.1

Suppose $(a_n)_{n=1}^{\infty}$ is equivalent to $(b_n)_{n=1}^{\infty}$. Then $|a_n - b_n| \leq \varepsilon$ for all $n \geq N$ for some $N \geq 1$ but that also means $-\varepsilon \leq a_n - b_n \leq \varepsilon$. So $-\varepsilon \leq \lim_{n\to\infty}(a_n - b_n) \leq \varepsilon$ but since ε is arbitrary and greater than 0, we have $0 \leq \lim_{n\to\infty}(a_n - b_n) \leq 0$. Therefore $\lim_{n\to\infty}(a_n - b_n) = 0$

Suppose $\lim_{n\to\infty}(a_n - b_n) = 0$. Then $|a_n - b_n| \leq \varepsilon$ for all $n \geq N$. So by definition. $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent as required.

9.9.2

Suppose f is uniformly continuous on X. Allow $\varepsilon > 0$ to be fixed. We have for some $\delta > 0$, $|f(x)-f(x_0)| \leq \varepsilon$ given that $x, x_0 \in X$ and $|x-x_0| \leq \delta$ Now consider two equivalent sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ with all elements in X. That means we have $|x_n - y_n| \leq \delta$ for all $n \geq N$ for some $N \geq 0$. Then $|f(x_n) - f(y_n)| \leq \varepsilon$ since x_n and y_n are in X. Since ε was arbitrary it can then be said for all $\varepsilon > 0$, there exists $N \geq 0$ such that $|f(x_n) - f(y_n)| \leq \varepsilon$ for all $n \geq N$. Therefore $(f(x_n))_{n=1}^{\infty}$ is equivalent to $(f(y_n))_{n=1}^{\infty}$ as required.

converse can be proven with contradiction.

10 Differentiation of Functions

10.1.1

Since f is differentiable at x_0 we have $\lim_{x\to x_0; X-\{x_0\}}(f(x)-f(x_0))/(x-x_0) = L$ for some real number L. This means that for all sequences $(a_n)_{n=1}^{\infty}$ that converge to x_0 and consists of $a_n \in X - \{x_0\}$, we have $((f(a_n) - f(x_0))/(a_n - x_0))_{n=1}^{\infty}$ converges to L. But since x_0 is a limit point of Y, we have, there exists $(a_n)_{n=1}^{\infty}$ where $a_n \in Y - \{x_0\} \subseteq X - \{x_0\}$ such that $(a_n)_{n=1}^{\infty}$ converges to x_0. Therefore we have $((f(a_n) - f(x_0))/(a_n - x_0))_{n=1}^{\infty}$ also converges to L. Since the sequence that converges to x_0 was chosen arbitrarily in Y. It can be concluded that $\lim_{x\to x_0; X-\{x_0\}}(f(x)-f(x_0))/(x-x_0) = L$ as required.

10.1.2

Suppose f is differentiable at x_0 on X with derivative L. Let $\varepsilon > 0$, there exists a $\delta > 0$ such that $|(f(x)-f(x_0))/(x-x_0)-L| \leq \varepsilon$ for all $x \in X-\{x_0\}$ such that $|x-x_0| < \delta$. We then have $|(f(x)-f(x_0))/(x-x_0)-L||x-x_0| \leq \varepsilon|x-x_0|$ But $|(f(x)-f(x_0))/(x-x_0)-L||x-x_0| = |(f(x)-f(x_0))(x-x_0)/(x-x_0)-L(x-x_0)| \leq \varepsilon|x-x_0|$. Now consider the case $x \neq x_0$. Then $|(f(x)-f(x_0))-L(x-x_0)| = |(f(x)-(f(x_0)+L(x-x_0))| \leq \varepsilon|x-x_0|$ since $x-x_0 \neq 0$. Now consider the case $x = x_0$ we then have $x-x_0 = 0$ and so $\varepsilon|x-x_0| = 0$ but we also have $|(f(x)-f(x_0))-L(x-x_0)| = |f(x_0)-f(x_0)-L*(0)| = 0$ and so $|(f(x)-(f(x_0)+L(x-x_0))| = 0 \leq \varepsilon|x-x_0| = 0$ we can therefore say that $|(f(x)-(f(x_0)+L(x-x_0))| \leq \varepsilon|x-x_0|$ for all $x \in X$ such that $|x-x_0| < \delta$. as required.

Suppose (b). In the case $x \neq x_0$ we have $|(f(x)-(f(x_0)+L(x-x_0))|/|x-x_0| = |(f(x)-f(x_0))/(x-x_0)-L| \leq \varepsilon|x-x_0|/|x-x_0| = \varepsilon$. We can therefore say, for all $x \in X-\{x_0\}$ such that $|x-x_0| \leq \delta$, $|(f(x)-f(x_0))/(x-x_0)-L| \leq \varepsilon$ and so by definition, f is differentiable at $x_0 \in X$ with derivative L.

10.1.3

Suppose f is differentiable at x_0. We then have $\lim_{x\to x_0;x\in X-\{x_0\}}(f(x)-f(x_0))/(x-x_0) = L$ for some real L. Now consider $\lim_{x\to x_0;x\in X-\{x_0\}} x-x_0$ This is obviously equal to 0 and by the limit law we have $\lim_{x\to x_0;x\in X-\{x_0\}}((f(x)-f(x_0))/(x-x_0))*(x-x_0) = \lim_{x\to x_0;x\in X-\{x_0\}}((f(x)-f(x_0)) = L*0 = 0$ and since $\lim_{x\to x_0;x\in X-\{x_0\}}((f(x)-f(x_0)) = \lim_{x\to x_0;x\in X-\{x_0\}} f(x)$ $-\lim_{x\to x_0;x\in X-\{x_0\}} f(x_0) = \lim_{x\to x_0;x\in X-\{x_0\}} f(x)-f(x_0)$, we have $\lim_{x\to x_0;x\in X-\{x_0\}} f(x)-f(x_0) = 0$ and so $\lim_{x\to x_0;x\in X-\{x_0\}} f(x) = f(x_0)$. Therefore f is continuous at x_0.

10.2.1

Suppose $x_0 \in (a,b)$ and f is differentiable at x_0 and f attains either a local maximum or local minimum at x_0. In the case x_0 is a local maximum, there exists a $\delta_0 > 0$ such that $f|_{X\cap(x_0-\delta_0,x_0+\delta_0)}$ attains a maximum at x_0. So we have $f(x) \leq f(x_0)$ for all $x \in X$ such that $|x-x_0| < \delta_0$. We also have $\lim_{x\to x_0;x\in X-\{x_0\}}(f(x)-f(x_0))/(x-x_0) = L$. Therefore the left limit and right limit is equal to L. Since $(f(x)-f(x_0))/(x-x_0) > 0$ in the left limit we have $\lim_{x\to x_0;x\in X-\{x_0\}}(f(x)-f(x_0))/(x-x_0) \geq 0$ and in the right limit it would be $\lim_{x\to x_0;x\in X-\{x_0\}}(f(x)-f(x_0))/(x-x_0) \leq 0$. Since the two limits are equal we must have L=0.

The minimum can be proven similarly

10.2.2

The function $-|x|$ is such a function. Consider the left limit, $\lim_{x\to 0;x\in(-\infty,0)}(-|x|+|0|)/(x-0)$. Since $x < 0$, $(-|x|+|0|)/(x-0) = 1$ and so the left limit is equal to 1. Now consider the right limit, $\lim_{x\to 0;x\in(0,\infty)}(-|x|+|0|)/(x-0)$. Since $x>0$, $(-|x|+|0|)/(x-0)=-1$. The left and right limits are not equivalent and therefore the limit is not continuous at x_0 and therefore can't be differentiable at 0. This does not contradict with P10.2.6 because it doesn't satisfy the function being differentiable at 0 which is required by the hypothesis.

10.2.3

consider the function $f(x)=x^3$, we have a derivative of 0 at 0, as shown by $\lim_{x\to 0;x\in X-\{x_0\}}(x^3-0)/(x-0)=\lim_{x\to 0;x\in X-\{x_0\}}x^2=0$, But there is no local maximum because when $x>0$ we have $x^3>0$. There is also no local minima because when $x<0$, $x^2>0$ and so $x^3<0$. This doesn't contradict P10.2.6 because it is an implication. So the hypothesis doesn't have to be true for it's conclusion to be true.

10.3.1

Suppose f is monotone increasing and f is differentiable at x_0. Then $f(x)-f(x_0)\le 0$ when $x-x_0<0$ so we have $f(x)/x\le f(x_0)/x_0$ and so $(f(x)-f(x_0))/(x-x_0)\ge 0$. We also have $f(x)-f(x_0)\ge 0$ when $x-x_0>0$ so we have $(f(x)-f(x_0))/(x-x_0)\ge 0$ and so $\lim_{x\to x_0;x\in X-\{x_0\}}(f(x)-f(x_0))/(x-x_0)\ge 0$ because $(f(x)-f(x_0))/(x-x_0)\ge 0$ across the given domain. Therefore $f'(x_0)\ge 0$

A similar proof can be used for monotone decreasing.

10.4.1

A)Consider the function $f(y)=y^n$ This is strictly monotone increasing because $y^n>z^n$ when $y>z$. Consider a sequence $(a_n)_{n=1}^{\infty}$ which converges to $y_0\in(0,\infty)$ we then have $(\lim_{n\to\infty}(a_n))^3=y_0^3=(\lim_{n\to\infty}(a_n)^3)=\lim_{n\to\infty}f(a_n)$ so we have $\lim_{n\to\infty}f(a_n)=f(y_0)$. This means that the function is continuous. Since $g(x)$ is the inverse of $f(x)$ we know by P9.8.3 that it is also strictly monotone increasing and continuous as required.

B)We know that $f(y)$ is differentiable along its domain due to the power rule, specifically $f'(y)=ny^{n-1}$ this function is never 0. We can then use the inverse theorem. So by the inverse theorem we have $g'(x)=1/f'(y)$ where $x=f(y)$. We then have $g'(x)=1/(ny^{n-1})$ but $x=y^n$ so $y=x^{1/n}$. Therefore $g'(x)=1/(ny^{n-1})=1/(n((x^{1/n})^{n-1})=1/nx^{1-1/n}=(1/n)x^{1-1/n}$ as required.

10.5.1

By newton's approximation we have $\delta>0$ such that $|g(x)-(g(x_0)+L(x-x_0))|\le|L/2||x-x_0|$ for all $x\in X$ such that $|x-x_0|\le\delta$ Since $g(x_0)=0$ we then have $|g(x)-L(x-x_0)|\le|L/2||x-x_0|$. Now suppose for contradiction that $g(x)=0$ for some x, we then have $|g(x)-L(x-x_0)|=$

$|-L(x-x_0)| = |L||x-x_0| \leq |L/2||x-x_0|$. So, $|L| \leq |L/2|$ which is a contradiction. Therefore $g(x) \neq 0$ as required.

Now consider a sequence $(a_n)_{n=0}^{\infty}$ that converges to x_0 and is entirely in $(X \cap (x_0-\delta, x_0+\delta)) - \{x_0\}$. We have $f'(x_0) = \lim_{n\to\infty}(f(a_n)-f(x_0))/(a_n - x_0)$ and $g'(x_0) = \lim_{n\to\infty}(g(a_n)-g(x_0))/(a_n-x_0) = \lim_{n\to\infty} g(a_n)/(a_n - x_0)$. Since $g(a_n) \neq 0$, so $g(a_n)/(a_n-x_0) \neq 0$ and $g'(x_0) \neq 0$ we can use limit law and get $f'(x_0)/g'(x_0) = \lim_{n\to\infty} f(a_n)(a_n-x_0)/(a_n-x_0)g(a_n) = \lim_{n\to\infty} f(a_n)/g(a_n)$. So we have for all sequences $(a_n)_{n=0}^{\infty}$ that converges to x_0, $(f(a_n)/g(a_n))_{n=1}^{\infty}$ converges to $f'(x_0)/g'(x_0)$ as required.

10.5.2 Example 1.2.12 does not contradict L'Hopital's rule because $f(x)$ is not differentiable at 0, since it's derivative at 0 has the term 0^-3 which is undefined.

11 The Riemann Integral

11.1.1 Since X is bounded we know that X therefore has a supremum and infinium. So we have $x \in X$ is $inf(X) \leq x \leq sup(X)$. Now we have a few cases $sup(X) \in X \; and \; inf(X) \in X$, $sup(X) \notin X \; and \; inf(X) \notin X$, $sup(X) \in X \; and \; inf(X) \notin X$, $sup(X) \notin X \; and \; inf(X) \in X$. In the case $sup(X) \in X \; and \; inf(X) \in X$, the interval $[inf(X), sup(X)]$ is subset of X. Now suppose $x \in X$. Then $inf(X) \leq x \leq sup(X)$ and therefore $x \in [inf(X), sup(X)]$. So we can conclude that $X \subseteq [inf(X), sup(X)]$. We then have $[inf(X), sup(X)] = X$ and so X is a bounded interval. In the case $sup(X) \notin X \; and \; inf(X) \notin X$, $inf(X) < x < sup(X)$. Consider the interval $(inf(X), sup(X))$, X is obviously a subset of it. Now suppose $x \in (inf(X), sup(X))$. Then $i < x < s$ for some i and s in X. This is true because the negation means x is either an upper bound or lower bound which is a contradiction. Since i and s are in X we have an interval $[i, s]$ which is a subset of X. Since $x \in [i, s]$, x is also in X. It can therefore be concluded that $(inf(X), sup(X)) \subseteq X$ and so $X = (inf(X), sup(X))$ as required. In the last two cases a similar proof to the one above can be used. So in all cases X is a bounded interval as required.

Suppose X is a bounded interval. It is obviously bounded. We also have all $a \leq x \leq b$ are in X where a and b are the bounds of the interval. Now suppose $x, y \in X \; and \; x < y$. Suppose $z \in [x, y]$. Then $x \leq z \leq y$. but we have $a \leq x \leq z \leq y \leq b$ and so $z \in X$. Therefore $[x, y] \subseteq X$ as required.

11.1.2

Suppose I and J are bounded intervals. Then I and J are both bounded and connected. Consider the intersection $I \cap J$ and $x \in I \cap J$. So x is in both I and J, therefore $|x| \leq M$ and $|x| \leq N$ where M and N are bounds of I and J respectively. Define $P = max(M, N)$. We then have $|x| \leq P$ and so $I \cap J$ is bounded. Now suppose $x, y \in I \cap J$ where $x < y$. Since I and J are connected we have $[x.y] \subseteq I$ and $[x, y] \subseteq J$ because $x, y \in I$ and $x, y \in J$. Now suppose $z \in [x, y]$. Then $z \in I \cap J$ therefore $[x, y] \subseteq I \cap J$. So we can then say $I \cap J$ is connected. Since $I \cap J$ is bounded and connected. We can then say $I \cap J$ is a bounded interval as required.

11.2.1

P' is a partition of I which is finer than **P**. Suppose $X \in$ **P'**, then $X \subseteq J$ where $J \in$ **P**. Now suppose $x \in X$, then $x \in J$ and f is constant on J so we have $f(x) = c$ for some constant c. Therefore f is constant on X and since X is an arbitrary element of the partition **P'** we can say, for all $X \in$ **P'**, f is constant on X and so by definition, *f is piecewise constant with respect to* **P'** as required.

11.2.2

Since f and g are piecewise constant functions on I, we have a partition **P** of I such that f is piecewise constant with respect to **P** and we have a partition **P'** of I such that g is piecewise constant with respect to **P'**. Consider the new partition of I, **P#P'**. This partition is finer than both **P** and $\mathbf{P}'$ so by lemma 11.2.7, f and g is piecewise constant with respect to **P#P'**. So now consider the function $(f + g)$.Let $x \in J$ for some $J \in$ **P#P'**. Since $f|_J(x) = c$ for some constant c and $g|_J(x) = d$ for some constant d, we have $(f + g)|_J(x) = d + c$ and so $(f + g)$ is constant on J. Since J is an arbitrary element of the partition **P#P'**, we can then say, for all $J \in$ **P#P'**, (f+g) is constant on J. Since **P#P'** is a partition of I we have $(f + g)$ is a piecewise constant function on I as required. All other algebraic operations on f and g can be proven similarly.

11.2.3

Consider the partition **P#P'** $= C$ of I. Suppose $P \in$ **P** and let $Q_P = \{N \in C : N \subseteq P\}$. Let $x \in P$. we then have $x \in J$ for some $J \in C$ but $J \subseteq L$ for some $L \in$ **P**. Suppose for contradiction $L \neq P$ but since $L, P \in$ **P** and $x \in L$ and $x \in P$, we have a contradiction since **P** is a partition. By definition of partitions, x can only be in one of the sets. So $L = P$. Therefore $J \subseteq P$. So $J \in Q_P$. This means that for all $x \in P$, there exists a $J \in Q_P$ such that $x \in J$ and since $J \in C$, means x lies in only one $J \in Q_P$. Therefore Q_P is a partition of P. We then have $|P| = \sum_{J \in Q_P} |J|$ by

T11.1.13. Now consider the sum $\sum_{L\in C} c_L|L|$. Where c_L is the constant value of $f|_L$. Since $C = \bigcup_{P\in\mathbf{P}} Q_P$, we have $\sum_{L\in C} c_L|L| = \sum_{P\in\mathbf{P}} \sum_{J\in Q_P} c_J|J|$. But because $f|_P$ is constant and $J \subseteq P$, $c_J = c_P$ for all $J \in Q_P$. So we have $\sum_{P\in\mathbf{P}} \sum_{J\in Q_P} c_J|J| = \sum_{P\in\mathbf{P}} \sum_{J\in Q_P} c_P|J| = \sum_{P\in\mathbf{P}} c_P \sum_{J\in Q_P} |J| = \sum_{P\in\mathbf{P}} c_P|P|$. That means $\sum_{P\in\mathbf{P}} c_P|P| = \sum_{L\in C} c_L|L|$ but $C = \mathbf{P}\#\mathbf{P'}$ and so by definition we have $p.c. \int_{[\mathbf{P}]} f = p.c. \int_{[\mathbf{P}\#\mathbf{P'}]} f$. It can also be shown similarly that $p.c. \int_{[\mathbf{P'}]} f = p.c. \int_{[\mathbf{P}\#\mathbf{P'}]} f$. So we have $p.c. \int_{[\mathbf{P'}]} f = p.c. \int_{[\mathbf{P}]} f$ as required.

11.3.1

Suppose f majorizes g and g majorizes h. Then $f(x) \geq g(x) \geq h(x)$ so $f(x) \geq h(x)$ for all $x \in I$ and therefore f majorizes h. as required

Suppose f and g majorize each other. Then $f(x) \geq g(x)$ and $f(x) \leq g(x)$ for all $x \in I$ we then have $f(x) = g(x)$ for all $x \in I$ and so $f = g$ as required.

11.3.2

Suppose f majorizes g. Then $f(x) \geq g(x)$. So $f(x) + h(x) \geq g(x) + h(x)$ but by definition $f(x) + h(x) = (f + h)(x) \geq g(x) + h(x) = (g + h)(x)$ and so we have $f + h$ majorizes $g + h$ as required. Now consider $(f \cdot h)(x)$ and $(g \cdot h)(x)$. We have the cases $h(x) < 0$, and $h(x) \geq 0$. In the case of $h(x) \geq 0$ we have $f(x) * h(x) \geq g(x) * h(x)$ but In the case of $h(x) < 0$ we would have $f(x) * h(x) \leq g(x) * h(x)$. We can then coclude that (fh) only maximizes (gh) when $h(x) \geq 0$ for all $x \in I$. The same can be said with a constant multiple c.

11.3.3

Since f is a piecewise function and $f(x) \geq f(x)$ we have f is a p.c. function that majorizes f and therefore $p.c. \int_I f \in \{p.c. \int_I g : g\ is\ a\ p.c.\ function\ that\ majorizes\ f\}$. Consider $\overline{\int}_I f$. Suppose for contradiction that $\overline{\int}_I f < p.c. \int_I f$. Then there exists a $\int_I g$ such that $\overline{\int}_I f \leq p.c. \int_I g < p.c. \int_I f$. This g majorizes f so we have $g(x) \geq f(x)$. But by integration laws we then have $p.c. \int_I g \geq p.c. \int_I f$ a contradiction and therefore $\overline{\int}_I f = p.c. \int_I f$. The same can be said for $\underline{\int}_I f$ and so we have $\underline{\int}_I f = \overline{\int}_I f = p.c. \int_I f$. Then by definition f is Riemann integrable and $\int_I f = \overline{\int}_I f = p.c. \int_I f$ as required.

11.3.4

We have $g(x) \geq f(x)$ and $p.c. \int_I g = p.c. \int_{[\mathbf{P}]} g = \sum_{J\in\mathbf{P}} c_J|J|$. Since $c_J \geq (sup_{x\in J}(f(x))$, because $c_J = g(x)$ where $x \in J$ and $g(x) \geq f(x)$ for all x and is therefore greater than all $f(x)$ for $x \in J$ as well, we have $c_J|J| > (sup_{x\in J}(f(x))|J|$ and so by series law we have $\sum_{J\in\mathbf{P}:J\neq\emptyset} c_J|J| \geq \sum_{J\in\mathbf{P}:J\neq\emptyset}(sup_{x\in J}(f(x))|J|$ as required. We are allowed to exclude $\emptyset$ from the left summation since it contributes 0 to the summation. The minorization can be proved similarly

11.4.2

Suppose for contradiction that $f(x_0) > 0$ for some $x_0 \in [a,b]$, But since f is continuous, we would have $|f(x) - f(x_0)| \leq f(x_0)/2$ for all $x \in X - \{x_0\}$ such that $|x - x_0| < \delta$ for some $\delta > 0$. But that means we have $f(x_0)/2 \leq f(x) \leq 3f(x_0)/2$. So now consider the partition $\{[a, x_0 - \delta], (x_0 - \delta, b]\}$ of $[a,b]$ The functions restricted to the two sets are Riemann integrable. Now consider the partition $\{(x_0 - \delta, x_0 + \delta)[x_0 + \delta, b]\}$ The function restricted to these two sets are also Riemann integrable. But we have $f(x_0)/2 \leq f(x)$ for all $x \in (x_0 - \delta, x_0 + \delta)$ and so $\int_{(x_0-\delta,x_0+\delta)} f|_{(x_0-\delta,x_0+\delta)} \geq \int_{(x_0-\delta,x_0+\delta)} f(x_0)/2$ and by the integration laws we have $\int_{(x_0-\delta,x_0+\delta)} f|_{(x_0-\delta,x_0+\delta)} + \int_{[x_0+\delta,b]} f|_{[x_0+\delta,b]} + \int_{[a,x_0-\delta]} f|_{[a,x_0-\delta]} = \int_{[a,b]} f$. Since $f(x) \geq 0$ for all $x \in [a,b]$ we then have $\int_{(x_0-\delta,x_0+\delta)} f|_{(x_0-\delta,x_0+\delta)} + \int_{[x_0+\delta,b]} f|_{[x_0+\delta,b]} + \int_{[a,x_0-\delta]} f|_{[a,x_0-\delta]} = \int_{[a,b]} f \geq 0 + \int_{(x_0-\delta,x_0+\delta)} f(x_0)/2 + 0$. But $\int_{(x_0-\delta,x_0+\delta)} f(x_0)/2 = (f(x_0)/2)|(x_0 - \delta, x_0 + \delta)| > 0$. Therefore we have $\int_{[a,b]} f \geq \int_{(x_0-\delta,x_0+\delta)} f(x_0)/2 > 0$ a contradiction. Therefore $f(x) = 0$ for all $x \in [a,b]$ as required

11.5.1

There exists a partition of I such that $f|_J$ is continuous on J for all $J \in \mathbf{P}$. Since I is bounded J is also bounded so we have $f|_J$ is Riemann Integrable, and because $J \subseteq I$ we have the function $F_J(x)$ which is defined as 0 if $x \notin J$ and $f(x)$ if $x \in J$ These functions are also Riemann integrable and so $\sum_{J \in \mathbf{P}} F_J(x)$ is also Riemann integrable, but we have $\sum_{J\in\mathbf{P}} F_J(x) = f(x)$ and so $f(x)$ is Riemann integrable.

11.6.1

Let $a < b$ represent the bounds of the bounded interval. We then have $0 < \varepsilon < (b-a)/2$. Since $a+\varepsilon$ and $b-\varepsilon$ are both between a and b as well as $a+\varepsilon$ and $b-\varepsilon$, the interval $[a+\varepsilon, b-\varepsilon] \in I$. The function $f|_{[a+\varepsilon,b-\varepsilon]}$ is now closed, bounded and monotone and so is riemann integrable. Now consider the function $h : I \to \mathbf{R}$ where $h(x) = f(x)$ when $x \in [a+\varepsilon, b-\varepsilon]$ and $h(x) = M$, where M is the upper bound of f, when $x \notin [a+\varepsilon, b-\varepsilon]$. This function obviously majorizes f so we have $\overline{\int}_I f \leq \int_I h = \int_{[a+\varepsilon,b-\varepsilon]} f|_{[a+\varepsilon,b-\varepsilon]} + 2\varepsilon M$. Using similar reasoning we can also get $\underline{\int}_I f \geq \int_{[a+\varepsilon,b-\varepsilon]} f|_{[a+\varepsilon,b-\varepsilon]} - 2\varepsilon M$. Therefore $\overline{\int}_I f - \underline{\int}_I f \leq 4\varepsilon M$. But ε is arbitrary so we must have $\overline{\int}_I f - \underline{\int}_I f \leq 0$. So $\overline{\int}_I f \leq \underline{\int}_I f$. But $\overline{\int}_I f \geq \underline{\int}_I f$ and so $\overline{\int}_I f = \underline{\int}_I f$ as required.

11.9.2

consider $F(x) - G(x)$ over the interval I. Since I is bounded, we have

$inf(I) \leq x \leq sup(I)$ for all $x \in I$. By the mean value theorem we then have $(F(x)-G(x)-(F(inf(I))-G(inf(I))) = (F'(x_0)-G'(x_0)))(x-inf(I))$ for some $inf(I) \leq x_0 \leq x$. But $F'(x_0) = f(x_0) = G'(x_0)$ and so $(F(x)-G(x))-(F(inf(I))-G(inf(I))) = 0(x-inf(I)) = 0$ Therefore $F(x)-F(inf(I)) = G(x)-G(inf(I))$. Therefore $F(x) = G(x)-G(inf(I))+F(inf(I))$ and since $-G(inf(I))+F(inf(I))$ is a constant, we have what is required.

11.10.1

Since $F\ and\ G$ is differentiable across $[a,b]$, it is also continuous across the same interval. Therefore $F\ and\ G$ is Riemann Integrable. We then have $FG'\ and\ F'G$ is also Riemann integrable since it is the product of two Riemann integrable functions. By the product rule, we have $(FG)' = FG'+F'G$. Then $\int_{[a,b]}(FG)' = \int_{[a,b]} FG' + \int_{[a,b]} F'G$. That means $\int_{[a,b]} FG' = \int_{[a,b]}(FG)' - \int_{[a,b]} F'G$, but by the fundamental theorem of calculus $\int_{[a,b]}(FG)' = F(b)G(b) - F(a)G(a)$ and so $\int_{[a,b]} FG' = F(b)G(b) - F(a)G(a) - \int_{[a,b]} F'G$ as required.

11.10.2

Consider $x, y \in \phi^{-1}(J)$ such that $x < y$, $x, y \in [a,b]$ as well by definition of inverse. Now consider the interval $[x,y]$ and let $z \in [x,y]$. Since ϕ is monotone increasing we have $\phi(x) \leq \phi(z) \leq \phi(y)$. But $\phi(x)\ and\ \phi(y)$ are both in J and since J is a partition, it is also an interval and thereby connected. That means $[\phi(x), \phi(y)] \subseteq J$ and because $\phi(z) \in [\phi(x), \phi(y)]$, we have $\phi(z) \in J$. Therefore $z \in \phi^{-1}(J)$ and so $[x,y] \subseteq \phi^{-1}(J)$. We can then conclude that $\phi^{-1}(J)$ is indeed connected.

We have $f(x) = c_J$ when $x \in J$. Now consider the composition $f \circ \phi$ on the interval $\phi_{-1}(J)$ and $x \in \phi^{-1}(J)$. By definition we have $\phi(x) \in J$ and so $(f \circ \phi)(x) = f(\phi(x)) = c_J$. Therefore the constant value of $f \circ \phi$ on the interval $\phi_{-1}(J)$ is indeed c_J as required.

We know $\phi_{-1}(J) \subseteq [a,b]$. Now suppose $x \in [a,b]$, then $\phi(x) \in [\phi(a), \phi(b)]$, then $\phi(x) \in J$ for some $J \in \mathbf{P}$ and so $x \in \phi^{-1}(J)$. Now suppose for contradiction that, for some $x \in \phi^{-1}(J)$, $x \in \phi^{-1}(J')$ where $J \neq J'$. By definition of partitions we have for all $y \in J$, $y \notin J'$, but $\phi(x) \in J$ and $\phi(x) \in J'$, a contradiction and therefore $J = J'$. Thus we can conclude that $\{\phi^{-1}(J) : J \in \mathbf{P}\}$ is indeed a partition of $[a,b]$ as required.

Let $Q \in \mathbf{Q}$, then $Q = \phi^{-1}(J)$ for some $J \in \mathbf{P}$. Consider $f \circ \phi|_Q$, we have $f \circ \phi|_Q(x) = c_J$. Since Q was arbitrary, we can then conclude that for all $Q \in \mathbf{Q}$, $f \circ \phi$ is constant valued on Q and is thus piecewise constant with respect to $\mathbf{Q}$ as required.

Let $J \in \mathbf{P}$. J is a bounded interval so, let c and d be the left and right endpoints of the interval respectively. We then have the closure of

$\phi^{-1}(J)$ being $[\phi^{-1}(c), \phi^{-1}(d)]$. Since $\phi^{-1}(J)$ is a bounded interval we have $\phi^{-1}(c)$ *and* $\phi^{-1}(d)$ are left and right endpoints of $\phi^{-1}(J)$. So $\phi[\phi^{-1}(J)] = \phi(\phi^{-1}(d)) - \phi(\phi^{-1}(c)) = d - c$, but $|J| = d - c$. Thus we have $\phi[\phi^{-1}(J)] = |J|$ as required.

11.10.3

Since $f(x)$ is riemann integrable, we have $\int_{[a,b]} \overline{f} \le \overline{\int}_{[a,b]} f + \varepsilon$ where $\overline{f}$ is a piecewise constant function that majorizes f. Let $\mathbf{P}$ be a partition such that $\overline{f}$ is piecewise constant with respect to $\mathbf{P}$. We then have $\int_{[a,b]} \overline{f} = \sum_{J \in \mathbf{P}} c_J |J|$. Now consider the partition of $[-b, -a]$ defined as $\mathbf{Q} = \{h^{-1}(J) : J \in \mathbf{P}\}$ where $h : [-b, -a] \to [a, b]$ is defined as $h(x) = -x$. We then have $g(x) = f(-x) \le \overline{f}(-x)$ for all $x \in [-b, -a]$. So $\overline{f} \circ h$ majorizes g. We will now prove $\overline{f} \circ h$ is piecewise constant with respect to $\mathbf{P}$. Suppose $x \in h^{-1}(J)$, then $h(x) \in J$ and so $\overline{f}(h(x)) = c_J$. Since $x \in h^{-1}(J)$ was arbitrary, we can conclude that, for all $x \in h^{-1}(J)$, $\overline{f}(h(x)) = c_J$. Therefore $\overline{f} \circ h$ is indeed piecewise constant with respect to $\mathbf{P}$ and therefore $\overline{\int}_{[-b,-a]} g \le \int_{[-b,-a]} \overline{f} \circ h = \sum_{Q \in \mathbf{Q}} c_Q |Q|$. But we have a bijection $h : \mathbf{Q} \to \mathbf{P}$. This is a bijection because, suppose $Q \in \mathbf{Q}$, then $h(Q) = h(h^{-1}(J)) = J \in \mathbf{P}$. Suppose $Q \ne Q'$, then suppose for contradiction that $h(Q) = h(Q')$, we then have $J = J'$ for some $J, J' \in \mathbf{P}$, but $\mathbf{P}$ is a partition and so $J = J'$, but that would mean $h^{-1}(J) = h^{-1}(J')$ a contradiction and therefore $h(Q) = h(Q')$. Now suppose $J \in \mathbf{P}$, by definition of $\mathbf{Q}$, $h^{-1}(J) \in \mathbf{Q}$ and $h(h^{-1}(J)) = J$ as required. Thereby concluding that the function is indeed a bijection. We therefore have $\sum_{J \in \mathbf{P}} c_J |J| = \sum_{Q \in \mathbf{Q}} c_{h(Q)} |h(Q)|$, but $c_Q = c_{h(Q)}$ because suppose $x \in Q$, then $x \in h^{-1}(J)$ for some $J \in \mathbf{P}$ and so $h(x) \in J$, therefore $\overline{f}(h(x)) = c_J$, so $c_Q = c_J$. We also have $h(Q) = h(h^{-1}(J)) = J$ and so $c_{h(Q)} = c_J$. Suppose $Q \in \mathbf{Q}$. We know Q is a bounded interval so we have c and d as left and right endpoints respectively. $|Q| = d - c$, but since $h(x) = -x$ we have $|J| = |h(Q)| = -c - -d = d - c$ so $|J| = |Q|$. Therefore $\sum_{J \in \mathbf{P}} c_J |J| = \sum_{Q \in \mathbf{Q}} c_Q |Q|$. Then $\int_{[-b,-a]} \overline{f} \circ h = \int_{[a,b]} \overline{f}$. So $\overline{\int}_{[-b,-a]} g \le \overline{\int}_{[a,b]} f + \varepsilon$. Doing a similar proof we can also show $\underline{\int}_{[-b,-a]} g \ge \underline{\int}_{[a,b]} f - \varepsilon$ and so $\underline{\int}_{[a,b]} f - \varepsilon \le \underline{\int}_{[-b,-a]} g \le \overline{\int}_{[-b,-a]} g \le \overline{\int}_{[a,b]} f + \varepsilon$. but since ε is arbitrary we have $\underline{\int}_{[a,b]} f \le \underline{\int}_{[-b,-a]} g \le \overline{\int}_{[-b,-a]} g \le \overline{\int}_{[a,b]} f$ and since $\underline{\int}_{[a,b]} f = \overline{\int}_{[a,b]} f$ because f is riemann integrable, we have $\underline{\int}_{[-b,-a]} g = \overline{\int}_{[-b,-a]} g = \int_{[a,b]} f$. Thus $\int_{[-b,-a]} g = \int_{[a,b]} f$ as required.