Linear Algebra

Recap

- We have learned a about several objects that come from linear algebra. Vectors, and matrices are examples of these objects.

Linearity

- The word “linear” is used to describe many different things in math that have some common threads.
- A function $f(x)$ is linear if it follows these two criteria:
  - For some constant a, $f(ax) = a\, f(x)$
  - For any x_1 and x_2, $f(x_1 + x_2) = f(x_1) + f(x_2)$
- This may be written more succinctly as $f(x)$ is linear if and only if for all x_1, x_2 and constants a and b,

$$f(ax_1 + bx_2) = af(x_1) + bf(x_2)$$

Notation

- We will use something called Dirac notation named after the physicist Paul Dirac.
- If I call v a vector, then I write it as $|v\rangle$
- We will learn some more parts of Dirac notation as we learn some more about linear algebra.

Vectors

- Vectors have many different examples.
  - We have used them as columns of numbers like $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
  - Those familiar with physics may recognize vectors as things with magnitude and direction.
    - Example, velocity: travelling 20 miles per hour (magnitude), north-east (direction).
  - A more complete definition is that a vector is part of something called a vector space

Vector Spaces

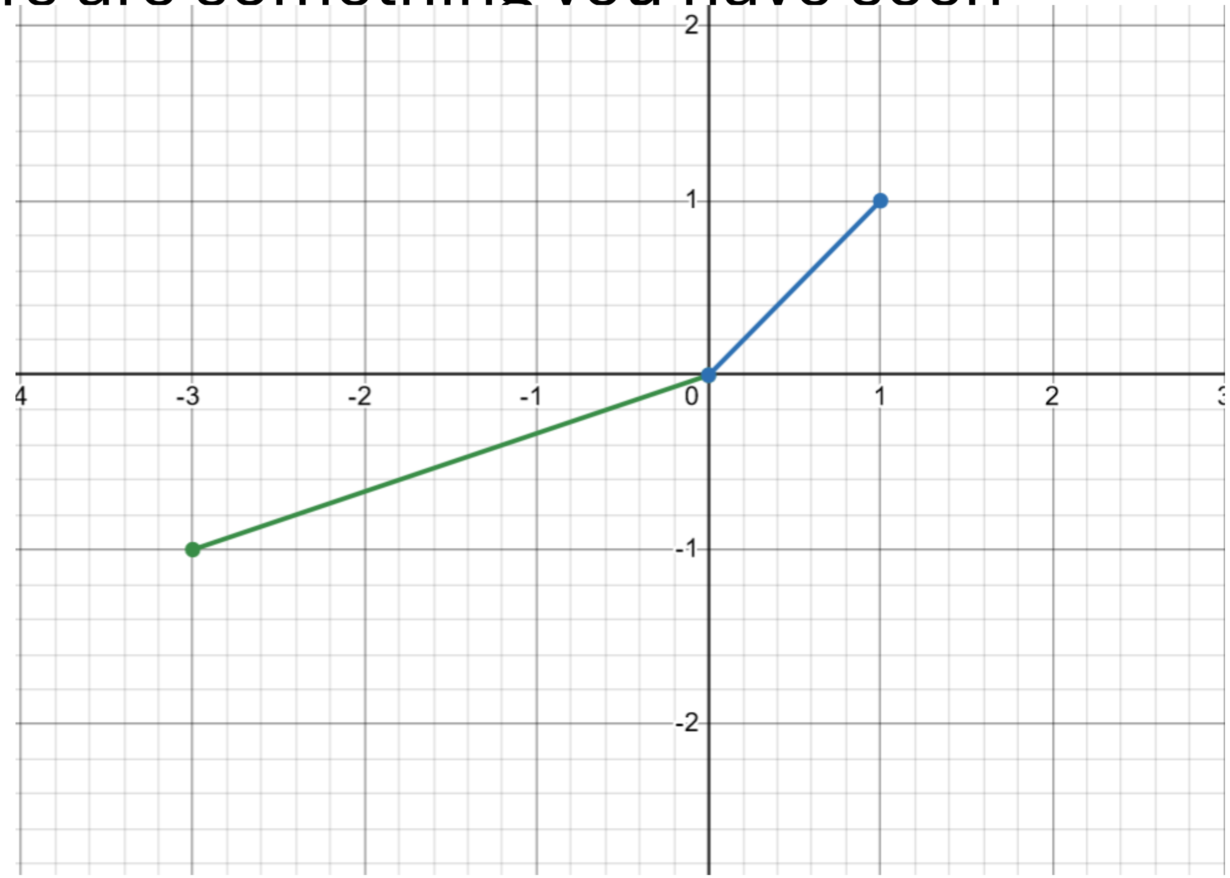
- A vector space is a set of vectors with the following rule.
  - If $|v\rangle$ and $|w\rangle$ are vectors and in a vector space V then $a|v\rangle + b|w\rangle$is another vector in the vector space (a and b are scalars)

Vector Space Example

- Claim: All ordered pairs of real numbers form a vector space.
  - Consider 2 vectors $|v\rangle = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $|w\rangle = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ and 2 real numbers a and b.
  - $a \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + b \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} a\, v_1 + bw_1 \\ a\, v_2 + bw_2 \end{bmatrix}$
  - $\begin{bmatrix} a\, v_1 + bw_1 \\ a\, v_2 + bw_2 \end{bmatrix}$ is an ordered pair of real numbers since multiplying and adding real numbers produces a real number.

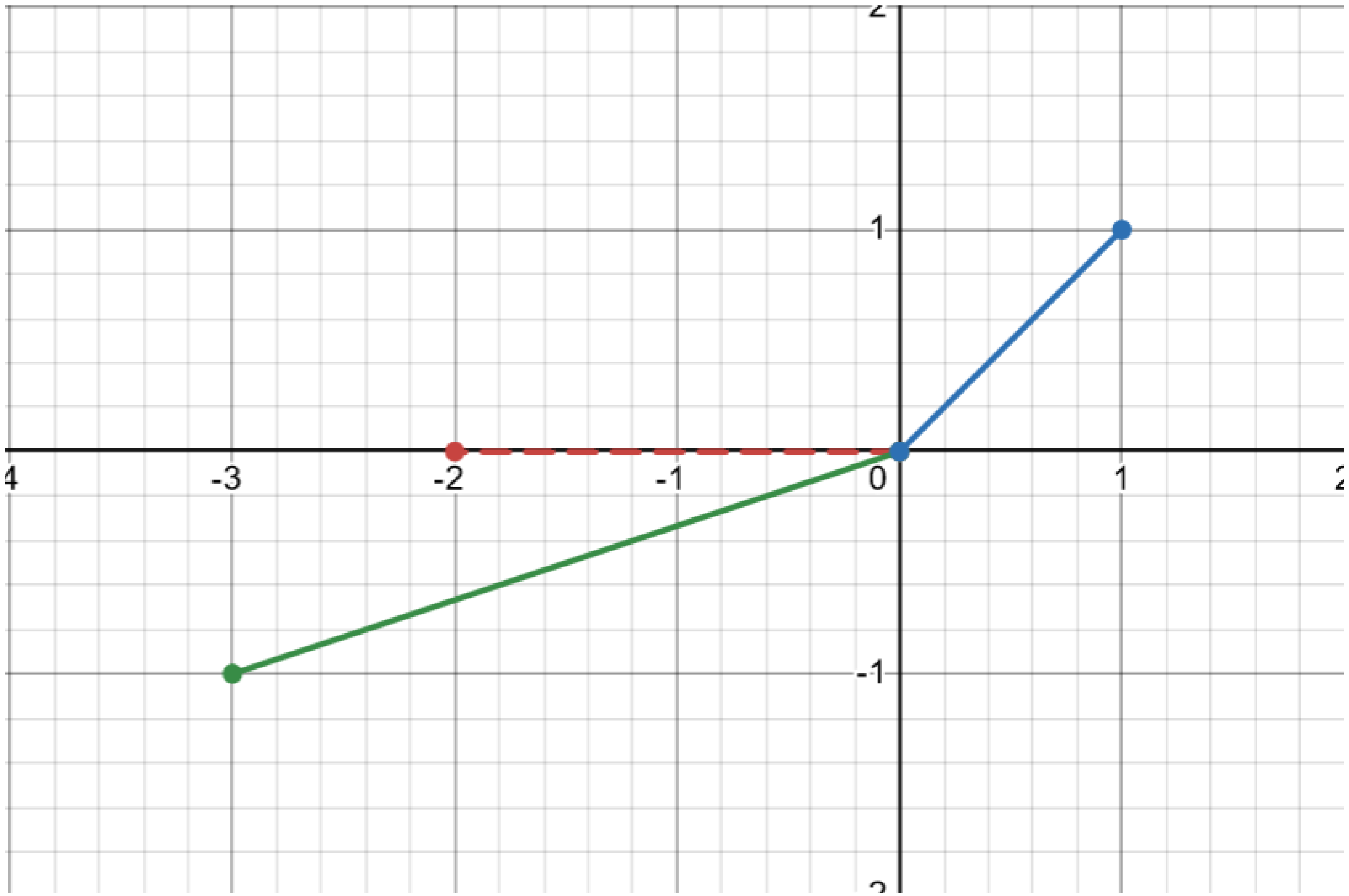
Vector Space Examples

- Ordered pairs of real numbers are something you have seen before, a cartesian plane.
- Here is a plot of 2 vectors the green represents the vector $(1, 1)$. The blue represents the vector (-3, -1)



Vector Space Example

- The sum of these two vectors is in red and given by:
$$\begin{bmatrix} -3 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$



Other vector spaces.

- Triplets of numbers, as might describe a point in 3D space, form a vector space.
- In fact, an ordered collection of n real numbers is a vector in the vector space of n ordered real numbers.

Abstract Vector Space

- Vectors may also be something that are less obvious. An example of this is a polynomial.
- Consider all of the polynomials of degree at most 3:
  - Examples: $5, \quad x^2, \quad 4x, \quad 5x^3 + 2x, \quad etc$
- These polynomials form a vector space: let's see:
  - $|v_1\rangle = a_1\, x^3 + b_1\, x^2 + c_1 x + d_1$
  - $|v_2\rangle = a_2 x^3 + b_2 x^2 + c_2\, x + d_2$
  - $|v_1\rangle + |v_2\rangle = (a_1\, x^3 + b_1\, x^2 + c_1 x + d_1) + (a_2 x^3 + b_2 x^2 + c_2\, x + d_2) = (a_1 + a_2)x^3 + (b_1 + b_2)x^2 + (c_1 + c_2)\, x + (d_1 + d_2)$
  - Still a polynomial of degree at most 3.
- Polynomials of degree at most n form a vector space.

Linear Independence/linear dependance

- Two vectors $|v\rangle$ and $|w\rangle$are linearly dependent if there exists some scalar a for which $a|v\rangle = |w\rangle$.
  - They are linearly independent if they are not linearly dependent.
- Example: $|v\rangle = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, |w\rangle = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ These are linearly dependent because $|v\rangle = -1 \cdot |w\rangle$
- Example 2: $|v\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |w\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ These are not linearly dependent because there is no scalar a for which $a|v\rangle = |w\rangle$.

Linear combinations

- A linear combination of 2 vectors $|v\rangle$and $|w\rangle$ is the sum $a|v\rangle + b|w\rangle$ where a and b are scalars

Basis vectors

- Basis vectors are a set of vectors for which any vector in the vector space they belong to may be expressed as a linear combination of the basis vectors.
- Example:
  - $|v\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |w\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
  - $|v\rangle$and $|w\rangle$ form a basis for all ordered pairs of real numbers.
  - For some new vector $|u\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$, that vector is just $a|v\rangle + b|w\rangle$

Basis Vectors

- The dimension of a vector space is the minimum number of basis vectors required to be able to produce any vector in that space.
- Any set of n linearly independent vectors form a basis for a dimension n vector space.

Inner Product

- An inner product is a special type of operation between two vectors but can take on different forms.
  - For two vectors $|v\rangle, |w\rangle$ it is sometime written in Dirac notation as $\langle v|w\rangle$
- An inner product takes two vectors, and does the following:
  - Produces one scalar value
  - $\langle v|w\rangle = \langle w|v\rangle$ (symmetry)
  - Linearity, (u is another vector), $\langle av + bu|\, w\rangle = a\langle v|w\rangle + b\langle u|w\rangle$
  - Positive semi-definite $\langle v|v\rangle \geq 0$

Dot Product

- A common inner product for ordered lists of numbers is called the dot product and is calculated as follows.
  - $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = v_1 w_1 + v_2 w_2$
  - $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = v_1 w_1 + v_2 w_2 + v_3 w_3$
  - This pattern continues to longer vectors.

The Transpose

- The transpose of a vector $|v\rangle = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is given by $|\mathrm{v}\rangle^T = \langle v| = [v_1 \quad v_2]$
- This is handy because the dot product is given by $|v\rangle^T|v\rangle = \langle v|v\rangle$ = $[v_1 \quad v_2]\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ which gives us the dot product through matrix multiplication.