Project 1 - FYS3150

Josef Ayman M.
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List a link to your github repository here!

PROBLEM 1

We check that $u(x) = 1 - (1 - e^{-10})x - e^{-10x}$ is an exact solution to our Poisson equation.
We simplify it:

$$u(x) = 1 - x + e^{-10}x - e^{-10x}$$

Then we try to find the second derivative:

$$u'(x) = -1 + e^{-10} + 10e^{-10x}u''(x) = -100e^{-10x}$$

Which when fits the source term $-(-100e^{-10x}) = f(x) = 100e^{-10x}$
And the boundary conditions:

$$u(0) = 1 - 0 + 0 - 1 = 0u(1) = 1 - 1 + e^{-10} - e^{-10} = 0$$

PROBLEM 2

In problem 2 we will write a program that:

- Defines a vector of x values.
- Evaluates the exact solution at each point.
- Writes the results to a file.
- It's recommended to use the *armadillo* library.

We can include figures using the `figure` environment. Whenever we include a figure or table, we *must* make sure to actually refer to it in the main text, e.g. something like this: "In figure 1 we show . . . ".

PROBLEM 3

Discretization of the Domain

- The domain x ranges from 0 to 1.
- We divide the domain into N equally spaced intervals, where h is the spacing between grid points, $h = \frac{1}{N}$.
- Define grid points x_i as:

$$x_i = ih \quad \text{for } i = 0, 1, 2, \ldots, N$$

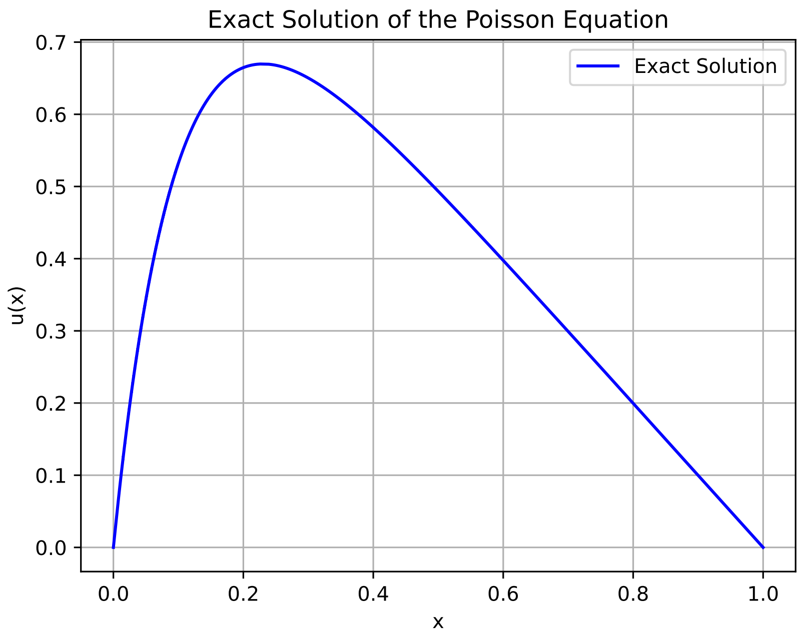

FIG. 1. Write a descriptive caption here that explains the content of the figure. Note the font size for the axis labels and ticks — the size should approximately match the document font size.

Let v_i be the approximation to $u(x_i)$.

To approximate the second derivative $\frac{d^2u}{dx^2}$ at a point x_i, we use the **finite difference method**. The central difference approximation for the second derivative is:

$$\frac{d^2u}{dx^2} \approx \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2}$$

Substitute into the Poisson Equation

$$-\frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} = f(x_i)$$

Simplify
Rearrange the equation to solve for the discrete approximation:

$$v_{i+1} - 2v_i + v_{i-1} = -h^2 f_i$$

or

$$-\frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} = f_i \tag{1}$$

Boundary Conditions
In the problem, the boundary conditions are $u(0) = 0$ and $u(1) = 0$. Therefore:

$$v_0 = 0 v_N = 0$$

Putting it all together, the discretized version of the Poisson equation is:

$$v_{i+1} - 2v_i + v_{i-1} = -h^2 \cdot 100e^{-10x_i}$$

for $i = 1, 2, \ldots, N-1$, with boundary conditions:

$$v_0 = 0 \quad \text{and} \quad v_N = 0$$

PROBLEM 4

To rewrite the discretized Poisson equation as a matrix equation, we start from the discretized second derivative:

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f_i$$

This can be written in matrix form as $\mathbf{A}\vec{v} = \vec{g}$, where $\mathbf{A}$ looks like:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

g_i is our right hand of the equation.

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix}$$

$$\begin{bmatrix} 2v_1 & -v_2 & 0 & 0 \\ -v_1 & 2v_2 & -v_3 & 0 \\ 0 & -v_2 & 2v_3 & -v_4 \\ 0 & 0 & -v_3 & 2v_4 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix}$$

We now showed that we can rewrite the equation in the matrix form $\mathbf{A}\vec{v} = \vec{g}$