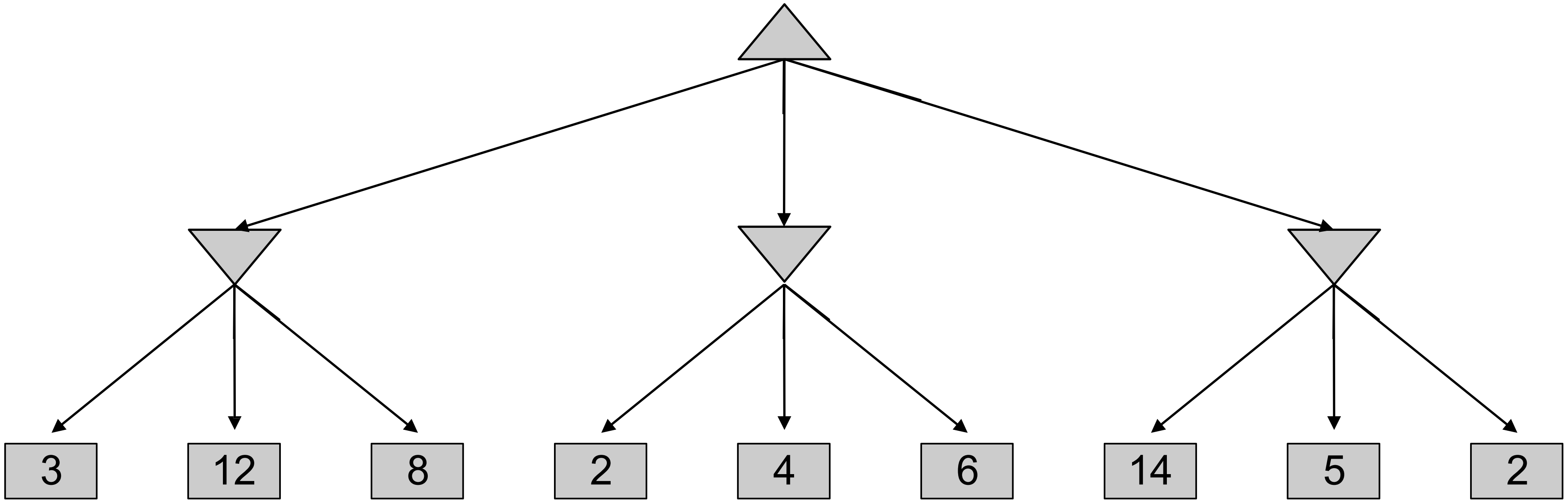
# Fundamentals of Artificial Intelligence

## Lecture-5
## Uncertainty and Utilities

**Debela Desalegn**

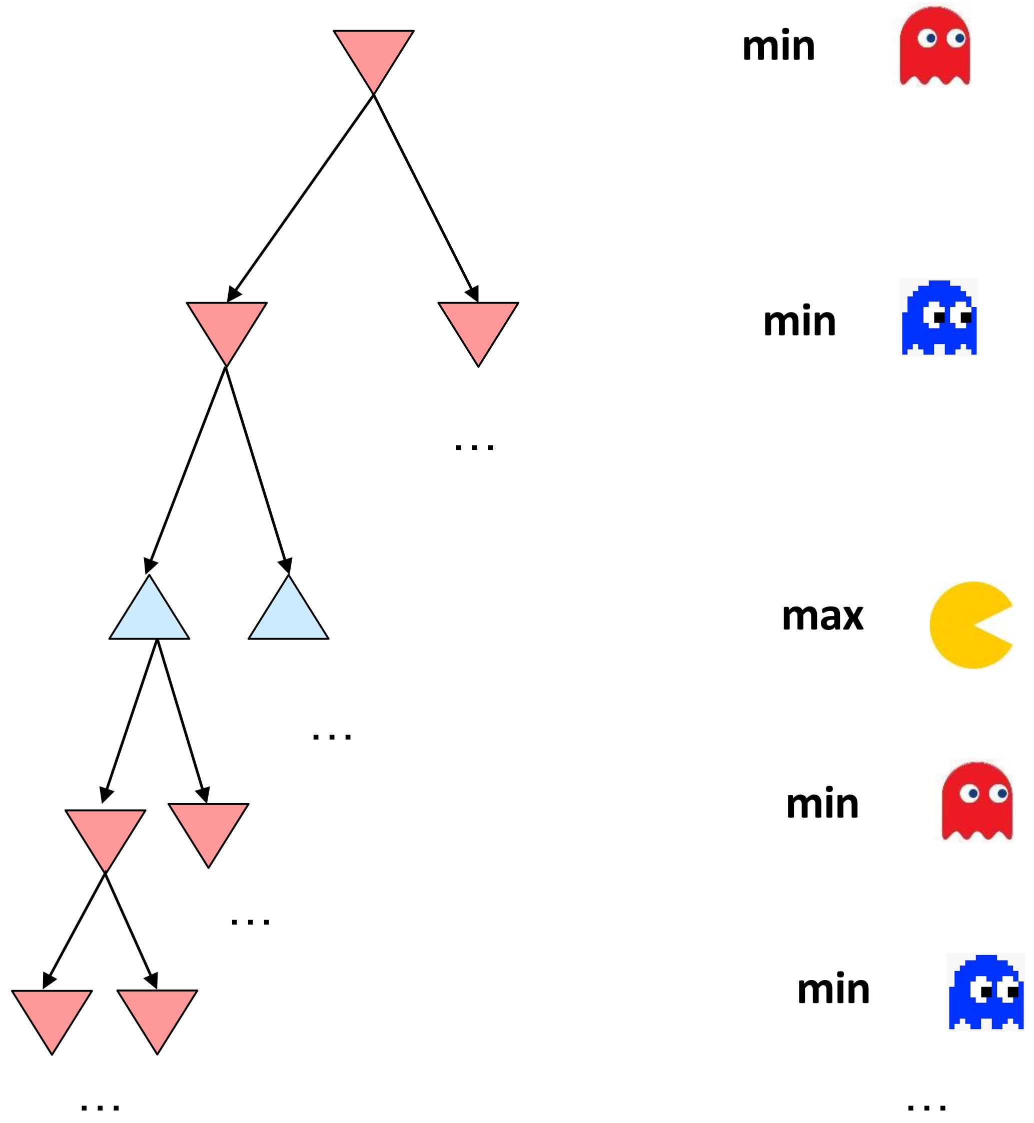
**April 06, 2025**

# Recall: Minimax

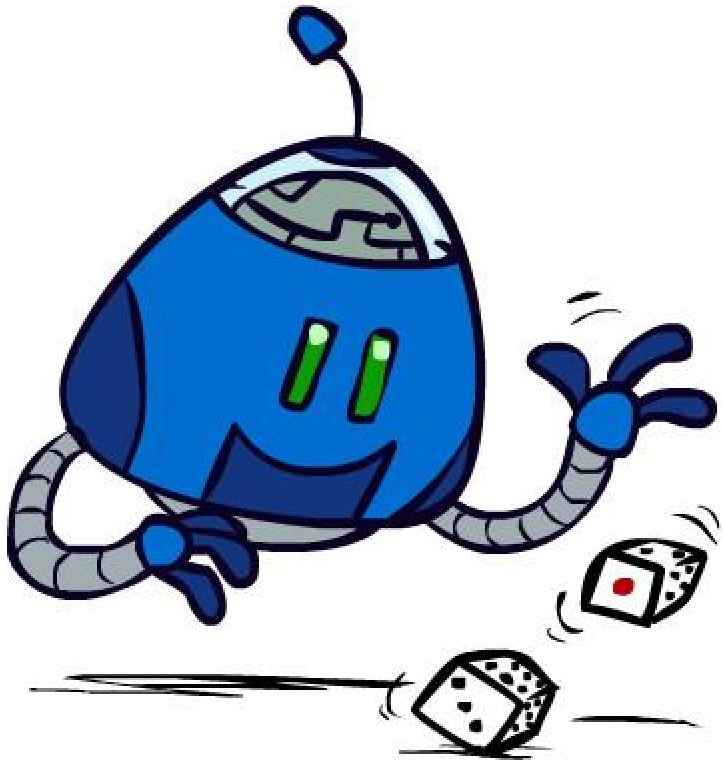


# MiniMiniMax and Emerging Coordination

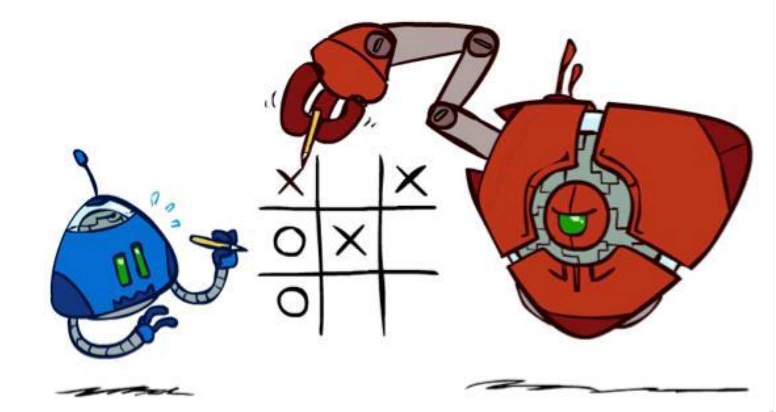
- Minimax can be extended to more than 2 players
  - e.g. 2 ghosts and 1 pacman
- Result: even though the 2 ghosts independently run their own MiniMiniMax search, they will naturally coordinate because:
  - They optimize the same objective
  - They know they optimize the same objective (i.e. they know the other ghost is also a minimizer)

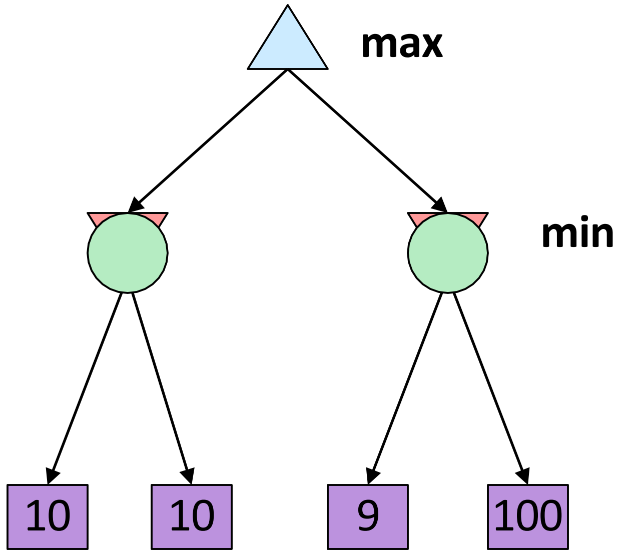


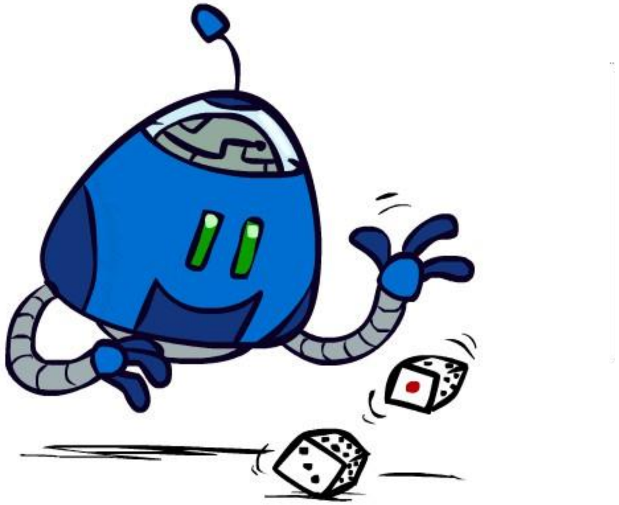
# Uncertain Outcomes



# Worst-Case vs. Average Case





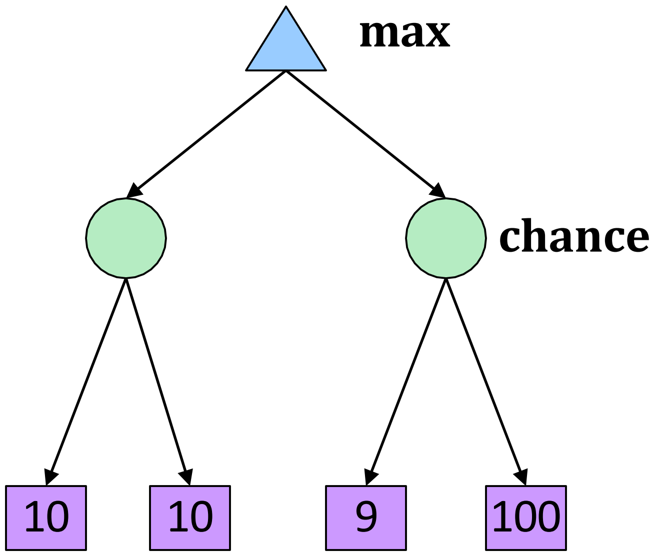
Idea: Uncertain outcomes controlled by chance, not an adversary!

# Expectimax Search

- Why wouldn't we know what the result of an action will be?
  - Explicit randomness: rolling dice
  - Unpredictable opponents: the ghosts respond randomly
  - Actions can fail: when moving a robot, wheels might slip

- Values should now reflect average-case (expectimax) outcomes, not worst-case (minimax) outcomes
- Expectimax search: compute the average score under optimal play
  - Max nodes as in minimax search
  - Chance nodes are like min nodes but the outcome is uncertain
  - Calculate their expected utilities
  - I.e. take weighted average (expectation) of children

- Later, we'll learn how to formalize the underlying uncertain-result problems as Markov Decision Processes



# Expectimax Pseudocode

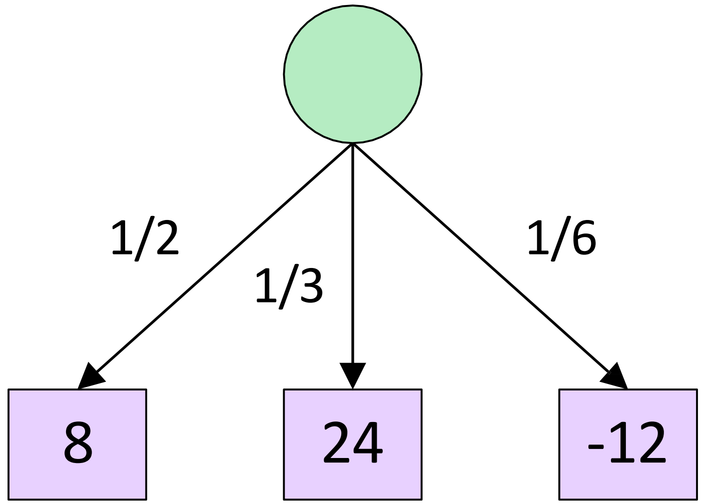
```
def value(state):
    if the state is a terminal state: return the state's utility
    if the next agent is MAX: return max-value(state)
    if the next agent is EXP: return exp-value(state)
```

```
def max-value(state):
    initialize v = -∞
    for each successor of state:
        v = max(v, value(successor))
    return v
```

```
def exp-value(state):
    initialize v = 0
    for each successor of state:
        p = probability(successor)
        v += p * value(successor)
    return v
```

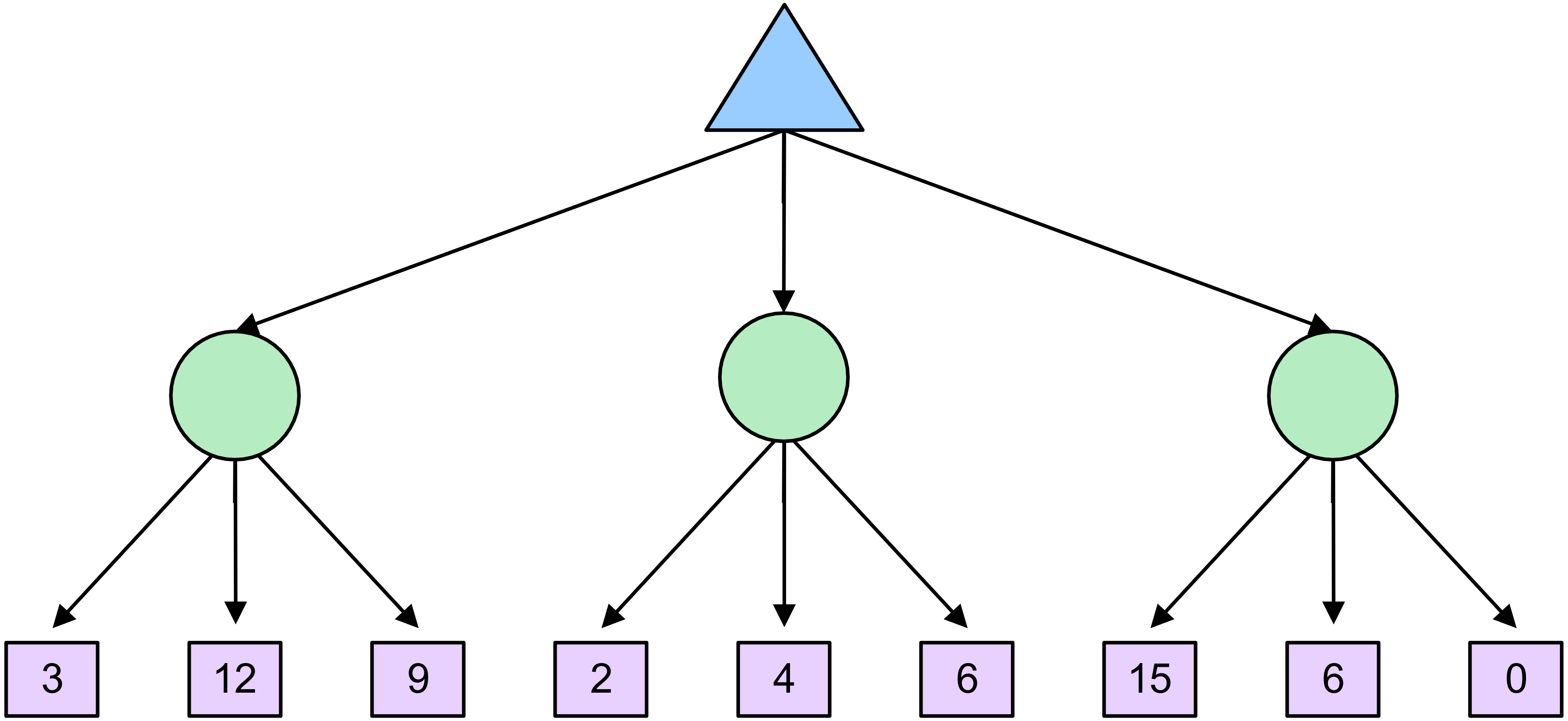
# Expectimax Pseudocode

```
def exp-value(state):
    initialize v = 0
    for each successor of state:
        p = probability(successor)
        v += p * value(successor)
    return v
```

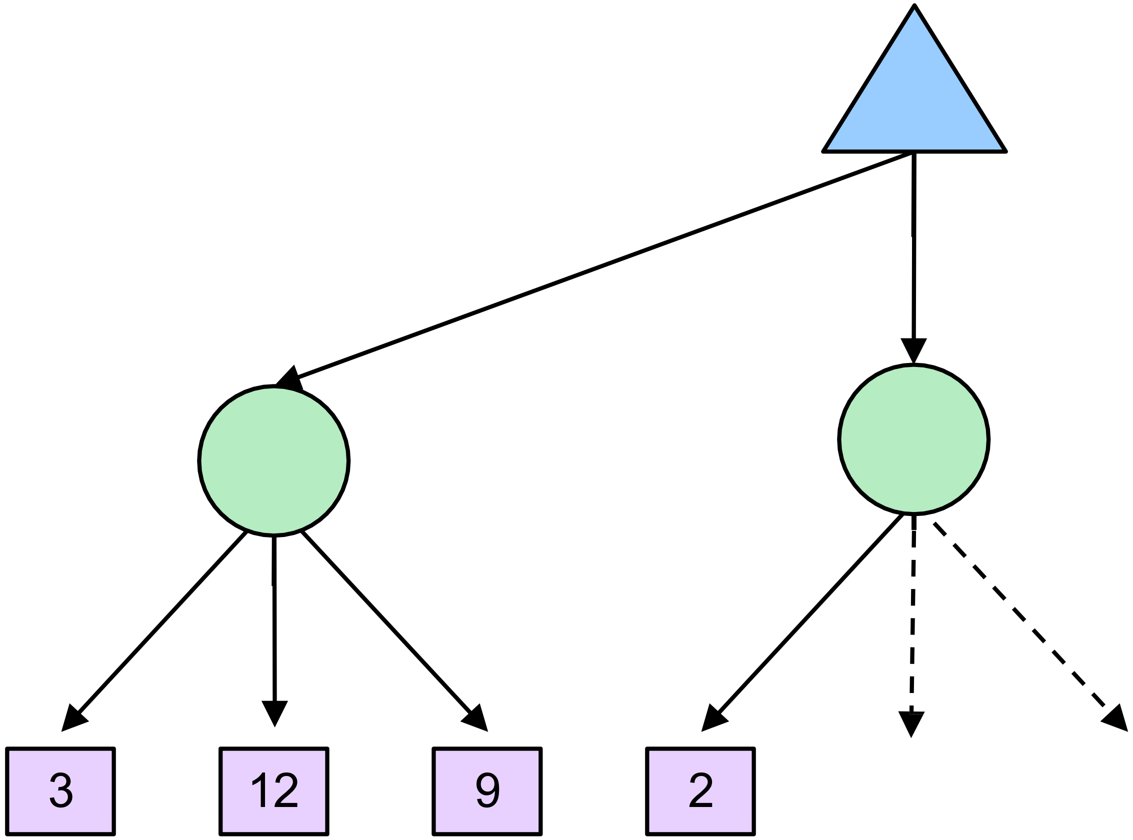


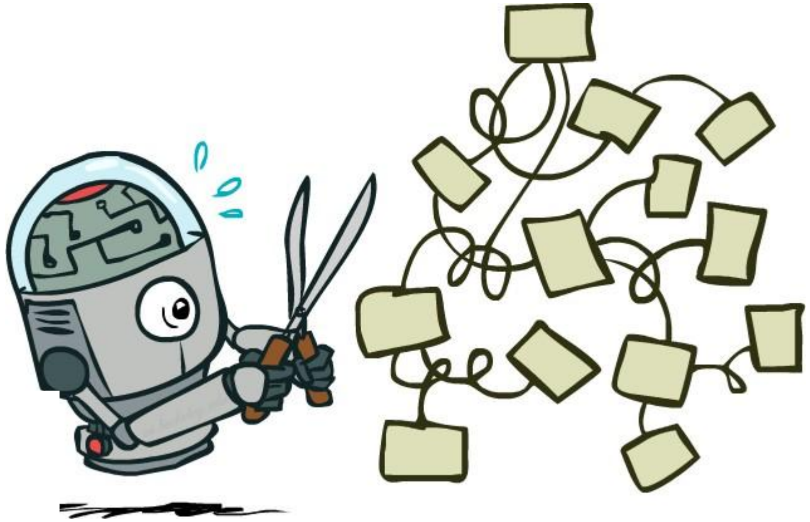

v = (1/2) (8) + (1/3) (24) + (1/6) (-12) = 10

# Expectimax Example

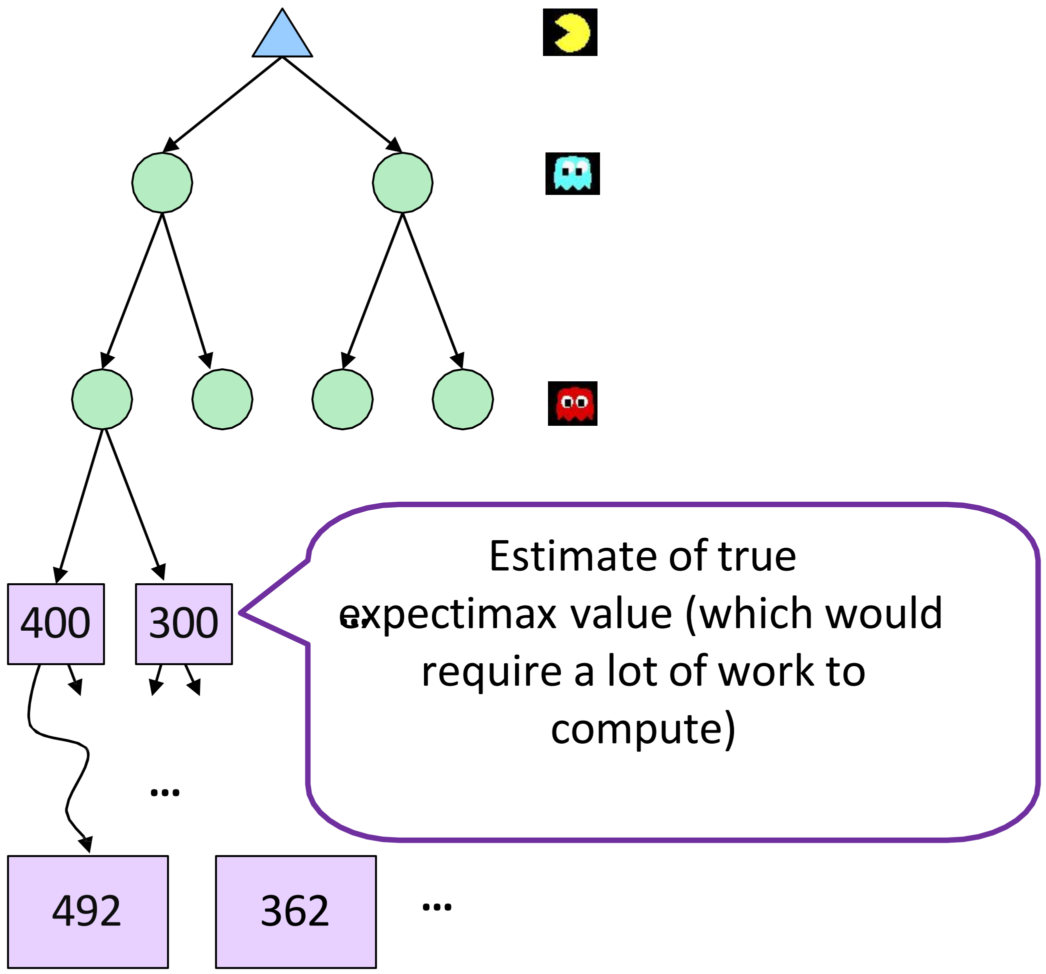


# Expectimax Pruning?

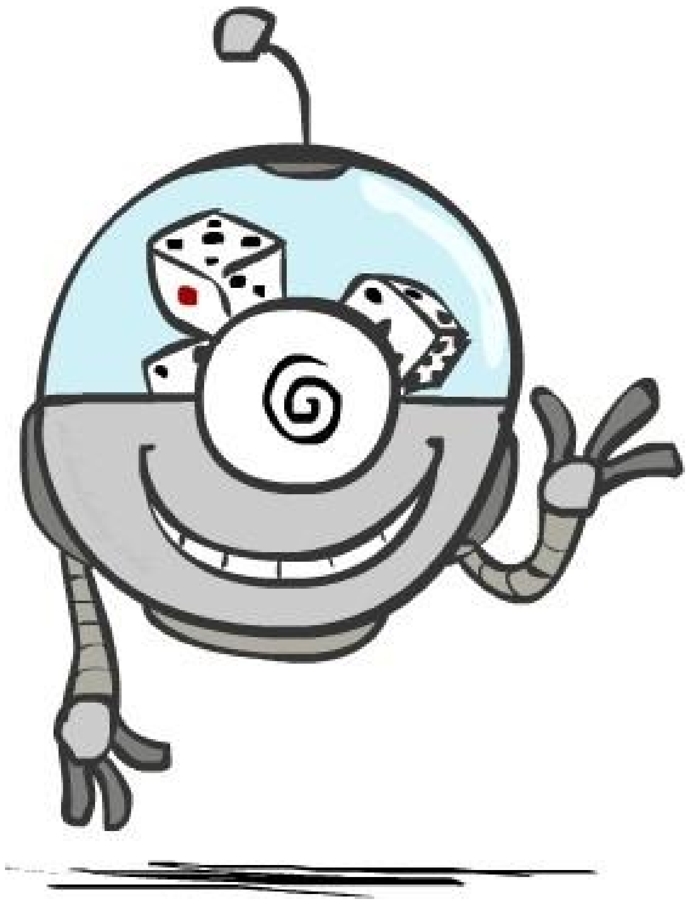


# Depth-Limited Expectimax



# Probabilities



# Reminder: Probabilities

- A random variable represents an event whose outcome is unknown
- A probability distribution is an assignment of weights to outcomes

- Example: Traffic on freeway
  - Random variable: T = whether there's traffic
  - Outcomes: T in {none, light, heavy}
  - Distribution: P(T=none) = 0.25, P(T=light) = 0.50, P(T=heavy) = 0.25
- Some laws of probability (more later):
  - Probabilities are always non-negative
  - Probabilities over all possible outcomes sum to one

- As we get more evidence, probabilities may change:
  - P(T=heavy) = 0.25, P(T=heavy | Hour=8am) = 0.60
  - We'll talk about methods for reasoning and updating probabilities later

0.25

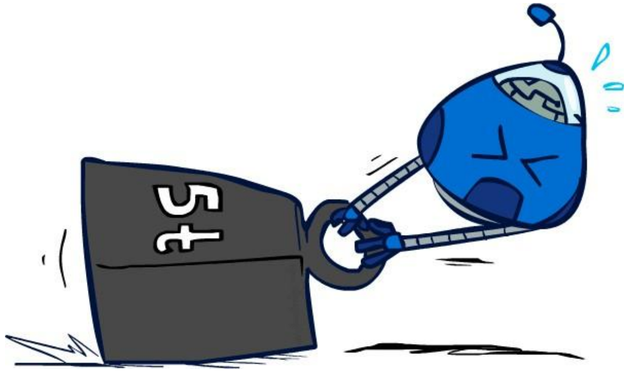
0.50

0.25

# Reminder: Expectations

- The expected value of a function of a random variable is the average, weighted by the probability distribution over outcomes

- Example: How long to get to the airport?



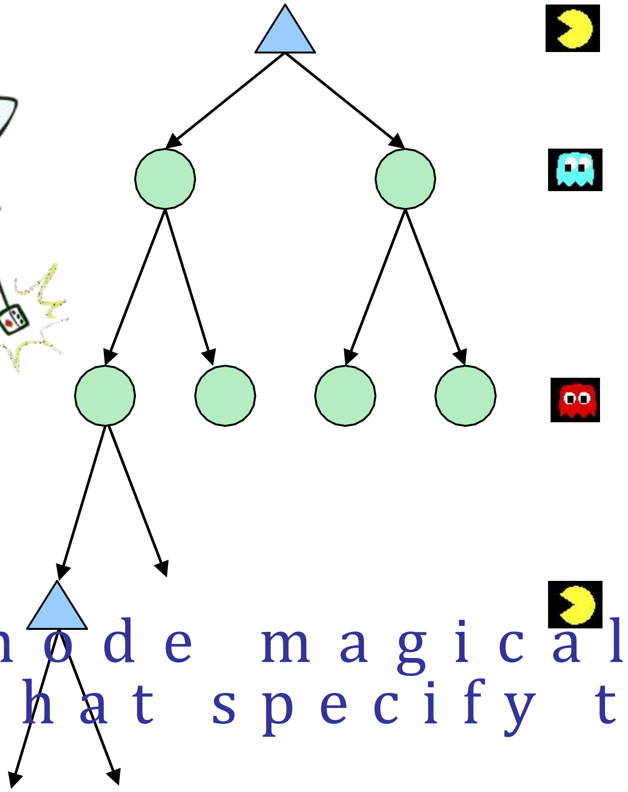

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Time: | 20 min | + | 30 min | + | 60 min | | 35 min |
| | x | | x | | x | | |
| Probability: | 0.25 | | 0.50 | | 0.25 | | |



# What Probabilities to Use?

- In expectimax search, we have a probabilistic model of how the opponent (or environment) behave in any state
  - Model could be a simple uniform distribution (roll a die)
  - Model could be sophisticated and require a great deal of computation
  - We have a chance node for any outcome out of our control: opponent or environment
  - The model might say that adversarial actions are likely!

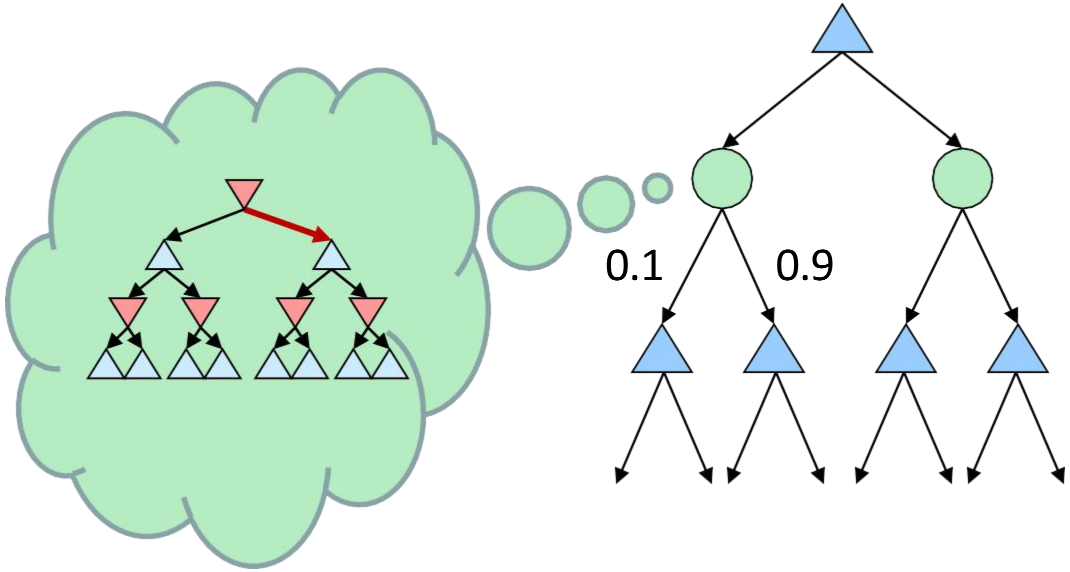
- For now, assume each chance node magically comes along with probabilities that specify the distribution over its outcomes



*Having a probabilistic belief about another agent's action does not mean that the agent is flipping any coins!*

# Quiz: Informed Probabilities

- Let's say you know that your opponent is actually running a depth 2 minimax, using the result 80% of the time, and moving randomly otherwise
- Question: What tree search should you use?




- Answer: Expectimax!

- To figure out EACH chance node's probabilities, you have to run a simulation of your opponent
- This kind of thing gets very slow very quickly
- Even worse if you have to simulate your opponent simulating you…
- … except for minimax, which has the nice property that it all collapses into one game tree

# Modeling Assumptions

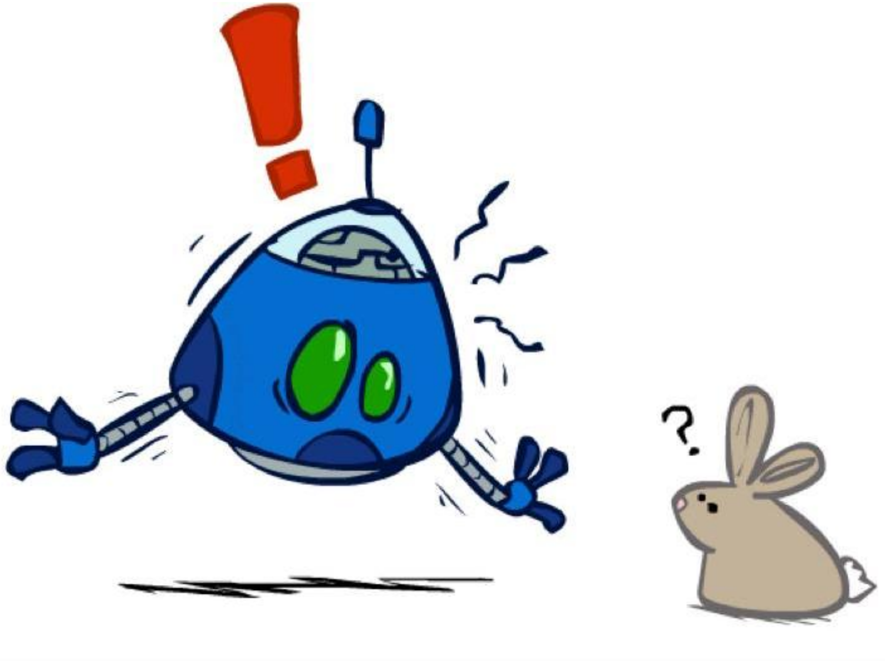
# The Dangers of Optimism and Pessimism

## Dangerous Optimism

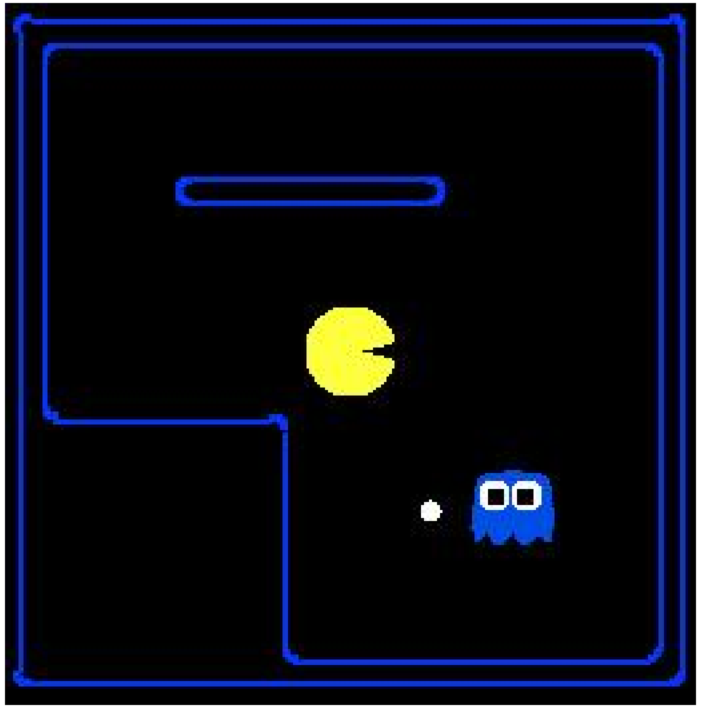
Assuming chance when the world is adversarial


## Dangerous Pessimism

Assuming the worst case when it's not likely



# Assumptions vs. Reality

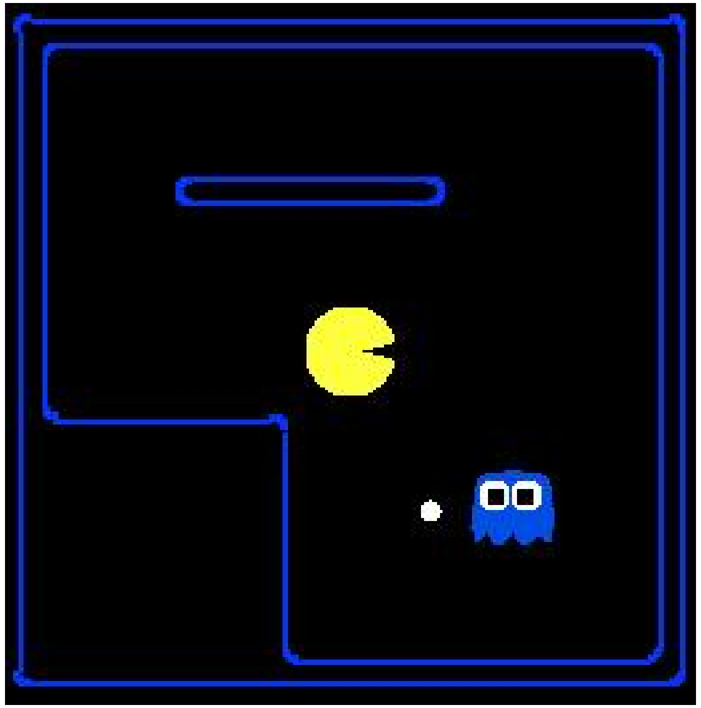


| | Adversarial Ghost | Random Ghost |
|---|---|---|
| Minimax Pacman | | |
| Expectimax Pacman | Won 1/5<br>Avg. Score: -303 | Won 5/5<br>Avg. Score: 503 |

Results from playing 5 games

Pacman used depth 4 search with an eval function that avoids trouble
Ghost used depth 2 search with an eval function that seeks Pacman

# Assumptions vs. Reality



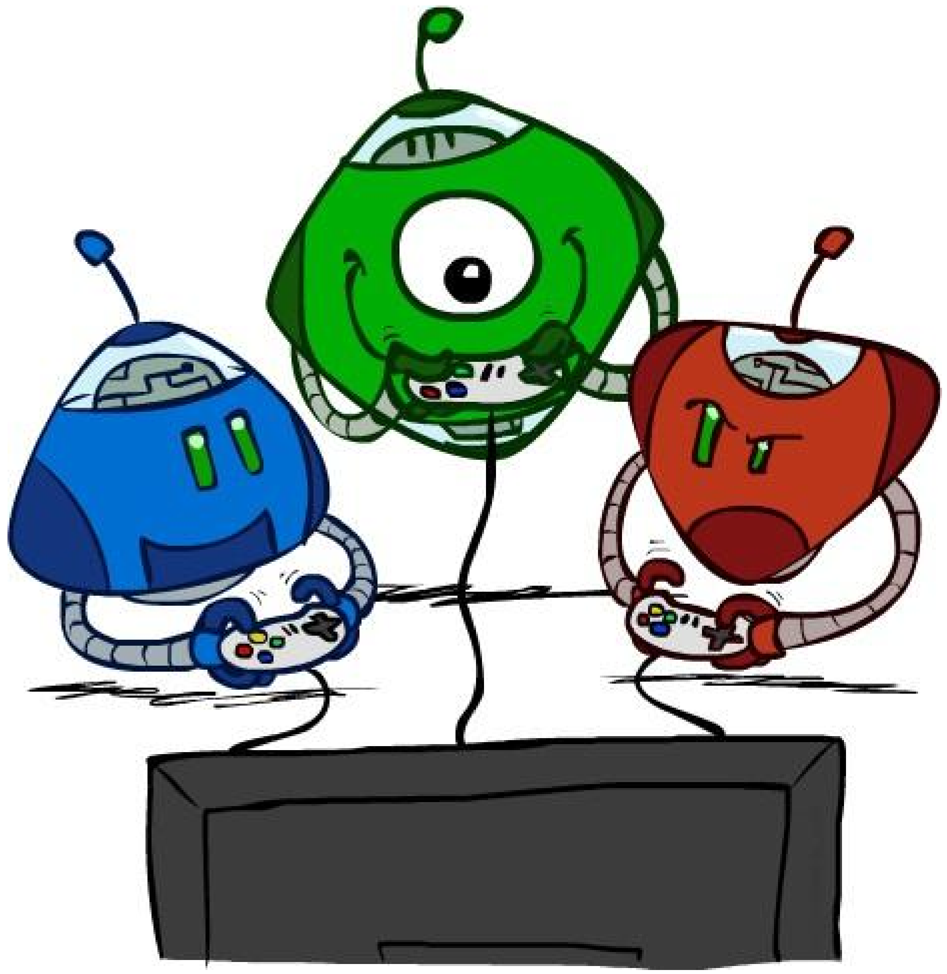
| | Adversarial Ghost | Random Ghost |
|---|---|---|
| Minimax Pacman | Won 5/5<br>Avg. Score: 483 | Won 5/5<br>Avg. Score: 493 |
| Expectimax Pacman | Won 1/5<br>Avg. Score: -303 | Won 5/5<br>Avg. Score: 503 |

Results from playing 5 games

Pacman used depth 4 search with an eval function that avoids trouble
Ghost used depth 2 search with an eval function that seeks Pacman

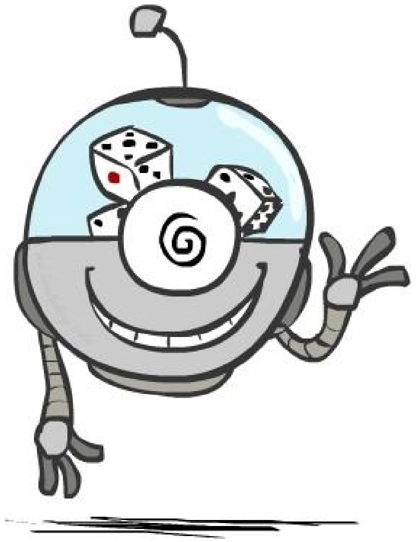
[Demos: world assumptions (L7D3,4,5,6)]

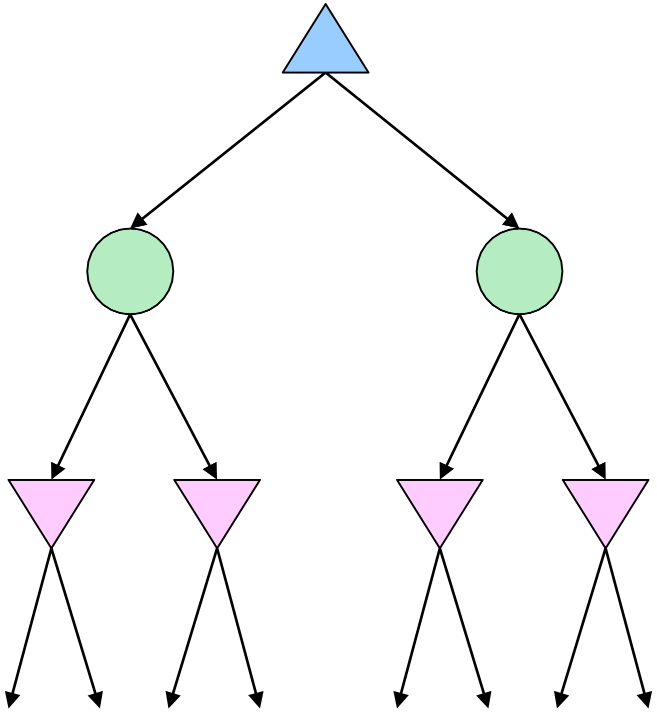
# Other Game Types



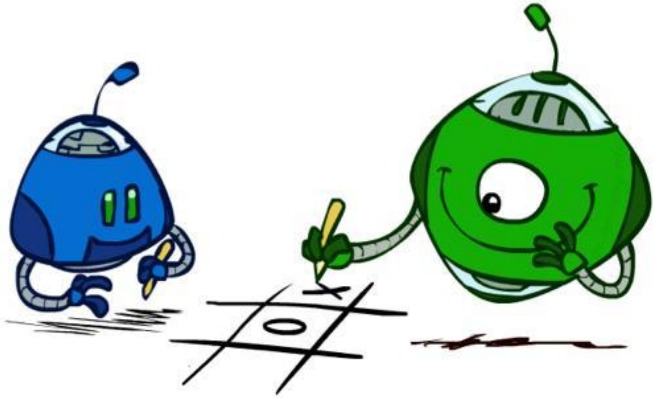
# Mixed Layer Types

- E.g. Backgammon
- Expectiminimax
  - Environment is an extra "random agent" player that moves after each min/max agent
  - Each node computes the appropriate combination of its children



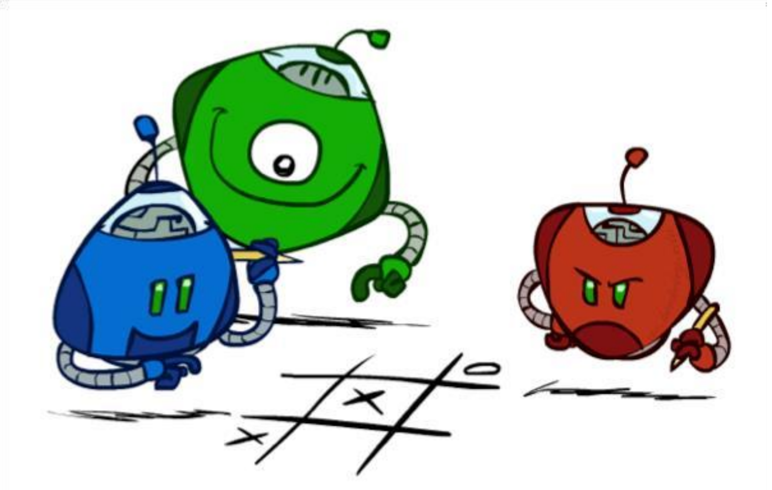


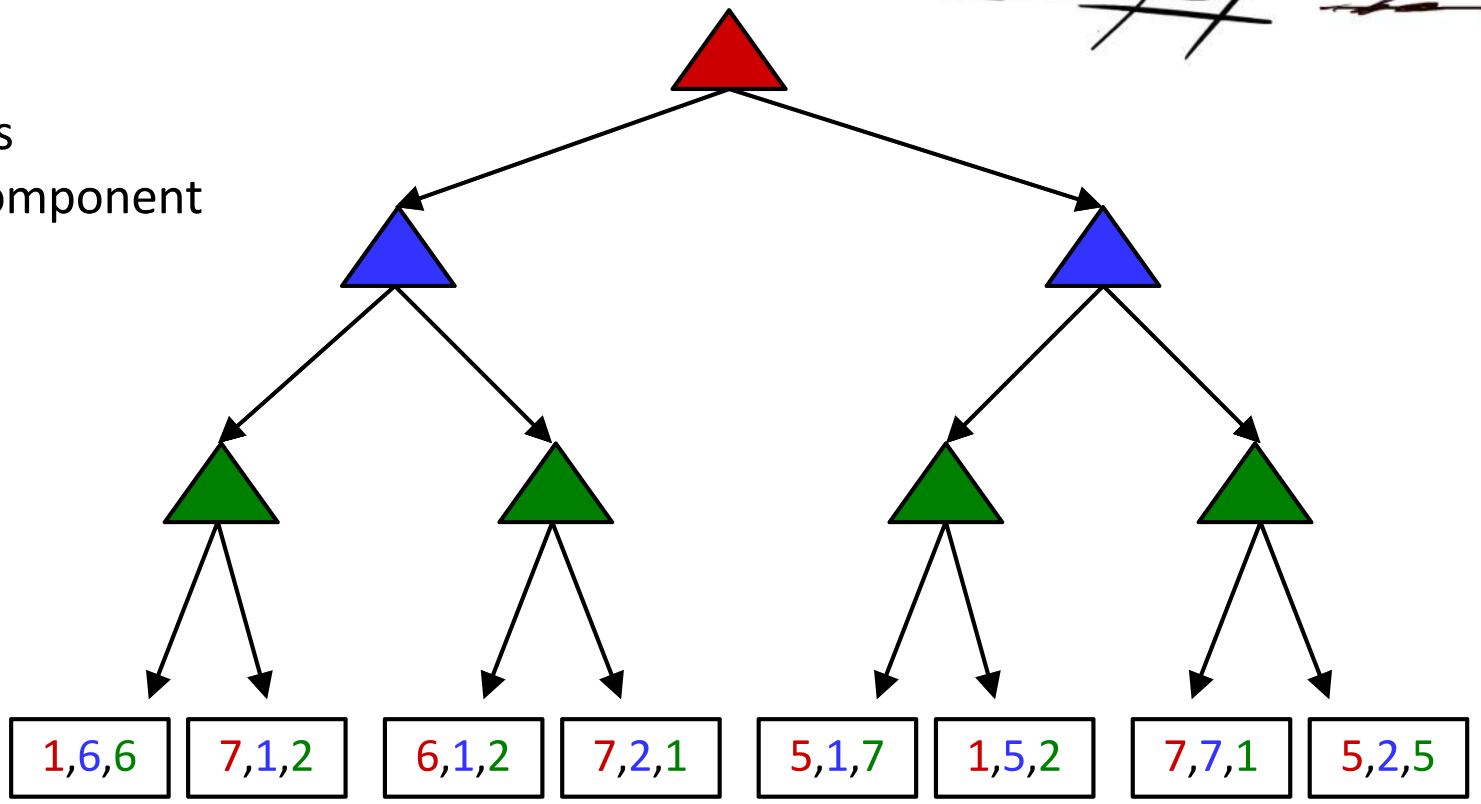
# Multi-Agent Utilities



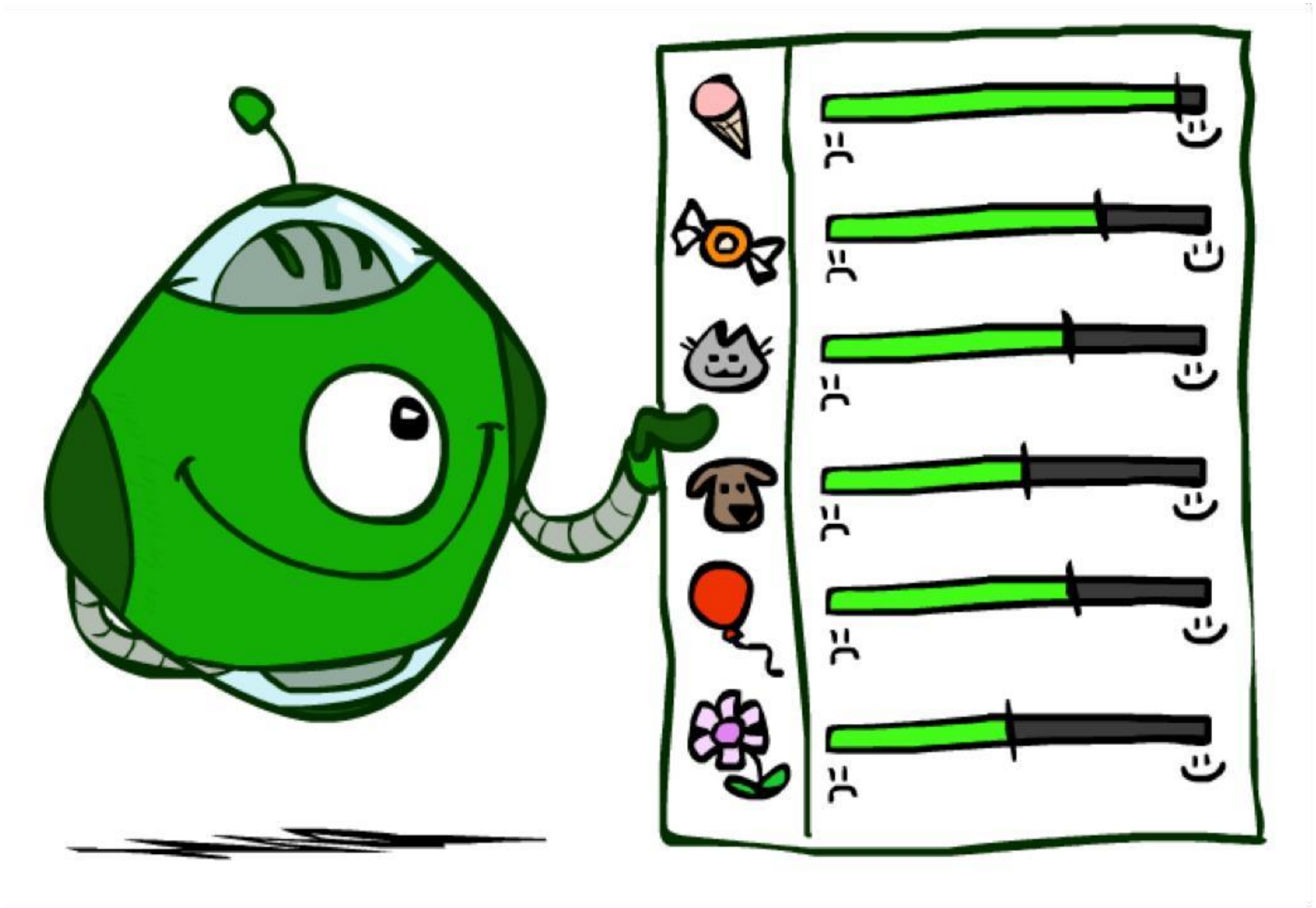
- What if the game is not zero-sum, or has multiple players?
- Generalization of minimax:

  Terminals have utility tuples
  - Node values are also utility tuples
  - Each player maximizes its own component
  - Can give rise to cooperation and competition dynamically…





# Utilities



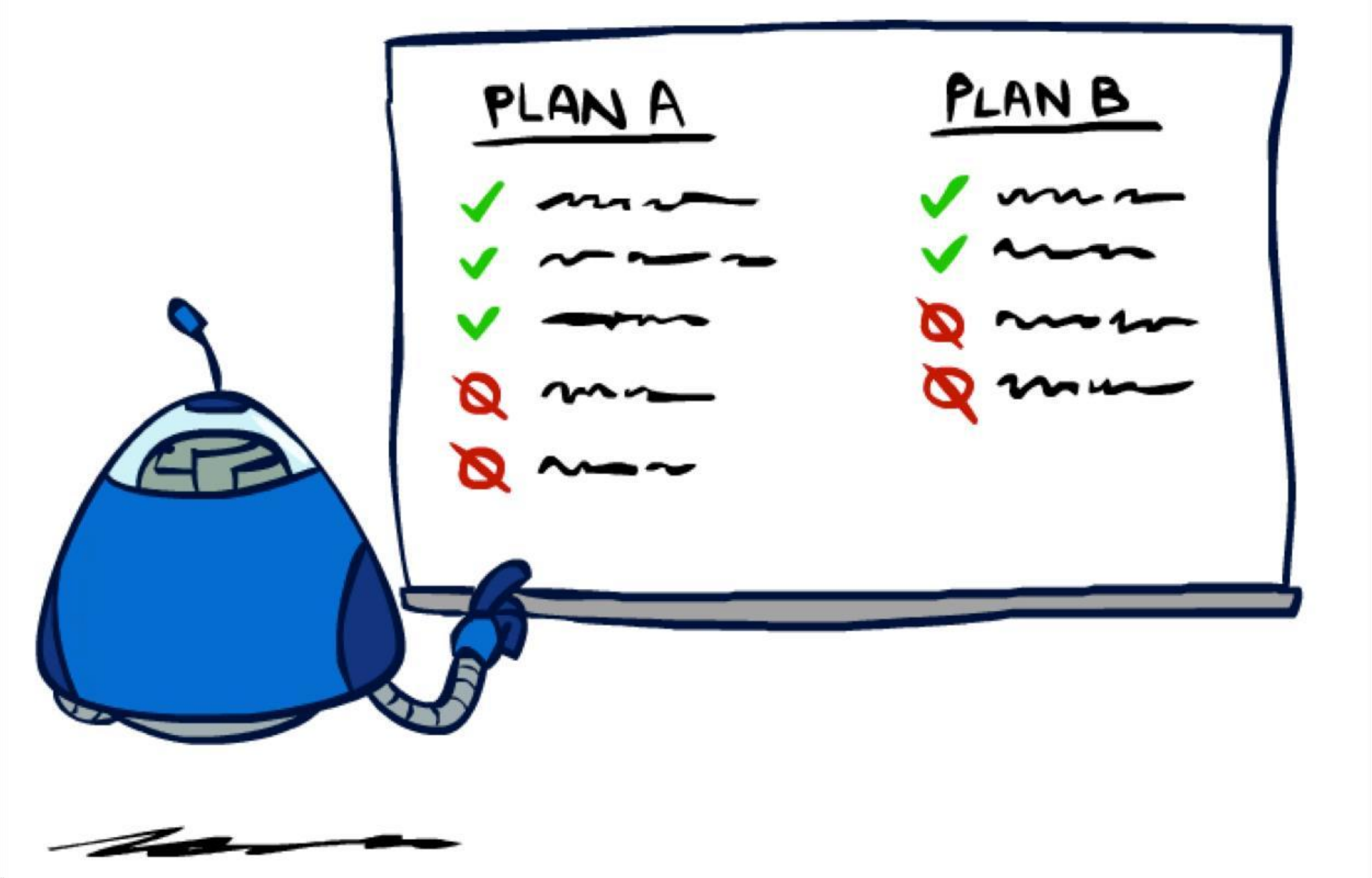
# Maximum Expected Utility

- Why should we average utilities?   Why not minimax?

- Principle of maximum expected utility:
  - A rational agent should chose the action that maximizes its expected utility, given its knowledge

- Questions:
  - Where do utilities come from?
  - How do we know such utilities even exist?
  - How do we know that averaging even makes sense?
  - What if our behavior (preferences) can't be described by utilities?

# Rationality



# Rational Preferences

## The Axioms of Rationality

Orderability

$$(A \succ B) \vee (B \succ A) \vee (A \sim B)$$

Transitivity

$$(A \succ B) \wedge (B \succ C) \Rightarrow (A \succ C)$$

Continuity

$$A \succ B \succ C \Rightarrow \exists p\ [p, A;\ 1-p, C] \sim B$$

Substitutability

$$A \sim B \Rightarrow [p, A;\ 1-p, C] \sim [p, B; 1-p, C]$$

Monotonicity

$$A \succ B \Rightarrow$$
$$(p \geq q \Leftrightarrow [p, A;\ 1-p, B] \succeq [q, A;\ 1-q, B])$$

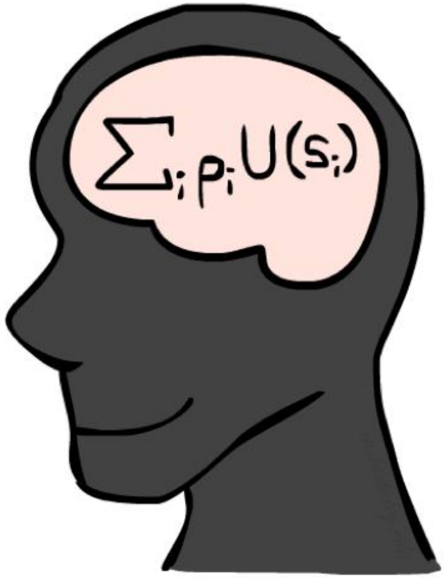

Theorem: Rational preferences imply behavior describable as maximization of expected utility

# MEU Principle

- Theorem [Ramsey, 1931; von Neumann & Morgenstern, 1944]
  - Given any preferences satisfying these constraints, there exists a real-valued function U such that:

$$U(A) \geq U(B) \;\Leftrightarrow\; A \succeq B$$

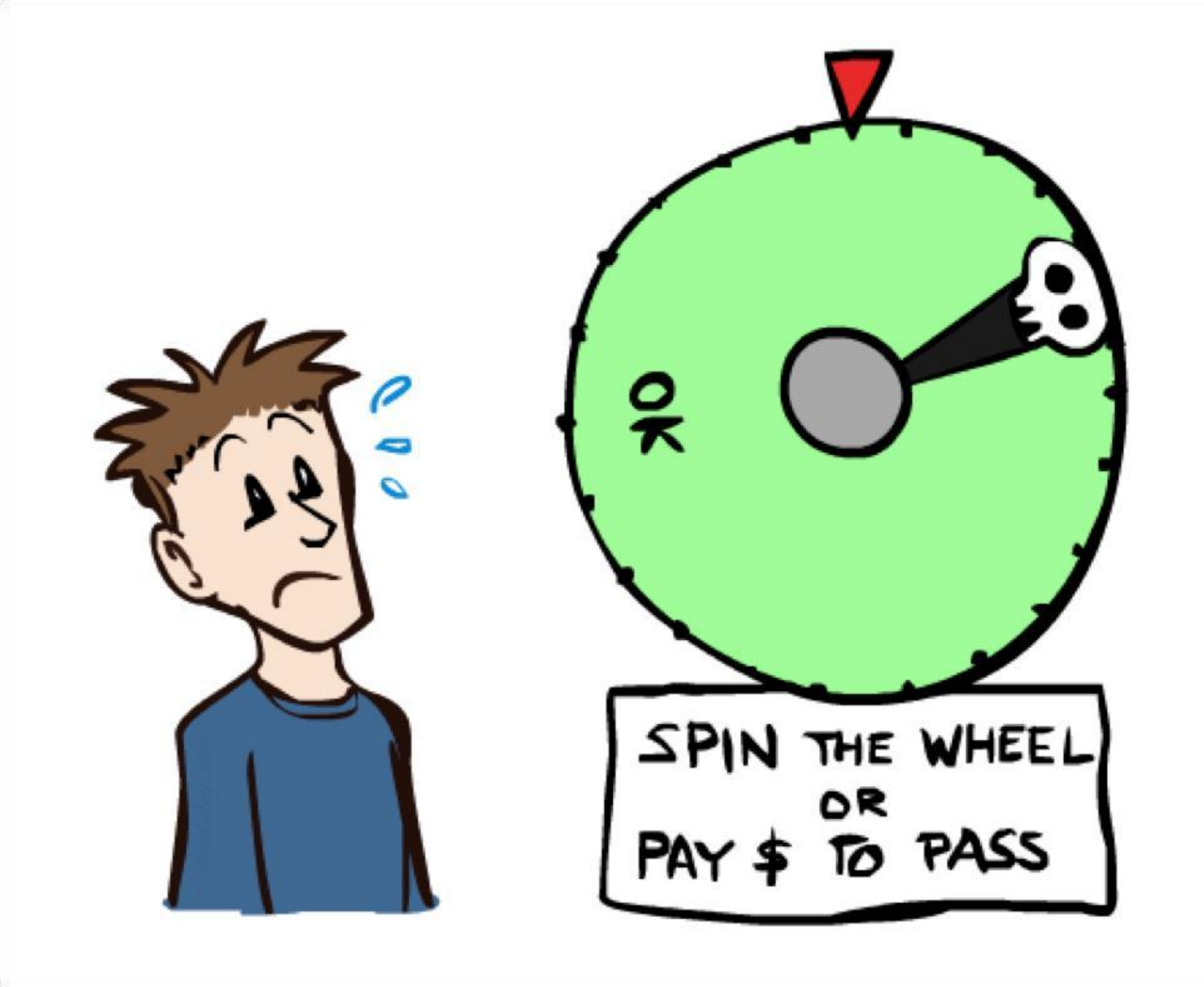
$$U([p_1, S_1; \;\ldots\; ; \; p_n, S_n]) = \sum_i p_i U(S_i)$$




  - I.e. values assigned by U preserve preferences of both prizes and lotteries!

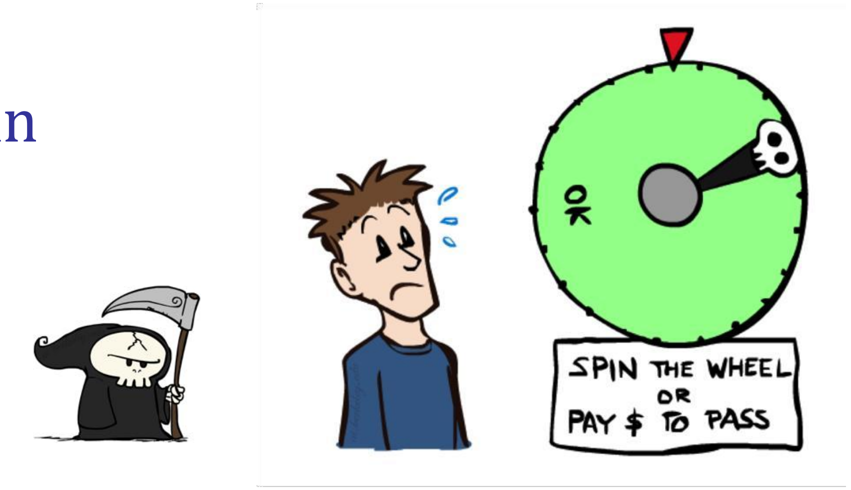
- Maximum expected utility (MEU) principle:
  - Choose the action that maximizes expected utility
  - Note: an agent can be entirely rational (consistent with MEU) without ever representing or manipulating utilities and probabilities
  - E.g., a lookup table for perfect tic-tac-toe, a reflex vacuum cleaner

# Human Utilities



# Human Utilities

- Utilities map states to real numbers. Which numbers?
- Standard approach to assessment (elicitation) of human utilities:
  - Compare a prize A to a standard lottery $L_p$ between
    - "best possible prize" $u_+$ with probability p
    - "worst possible catastrophe" $u_-$ with probability 1-p
  - Adjust lottery probability p until indifference: $A \sim L_p$
  - Resulting p is a utility in [0,1]



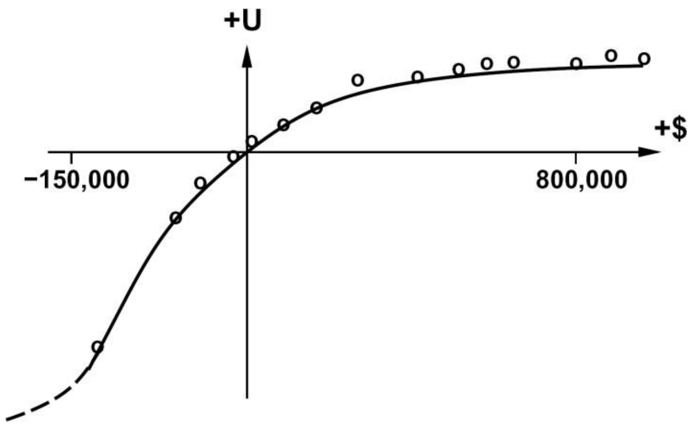

*Pay $30* $\sim$

0.999999 → *No change*

0.000001 → *Instant death*

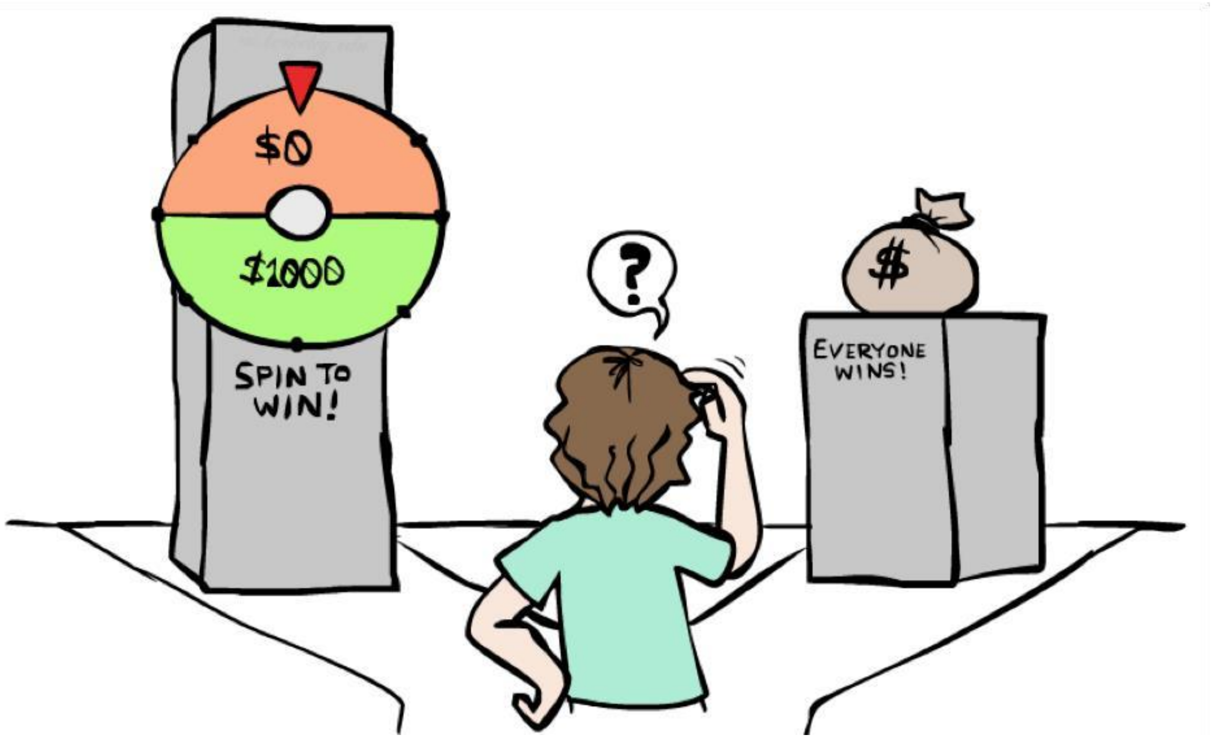
# Money

- Money <u>does not</u> behave as a utility function, but we can talk about the utility of having money (or being in debt)
- Given a lottery L = [p, \$X; (1-p), \$Y]
  - The expected monetary value EMV(L) is p*X + (1-p)*Y
  - U(L) = p*U(\$X) + (1-p)*U(\$Y)
  - Typically, U(L) < U( EMV(L) )
  - In this sense, people are risk-averse
  - When deep in debt, people are risk-prone



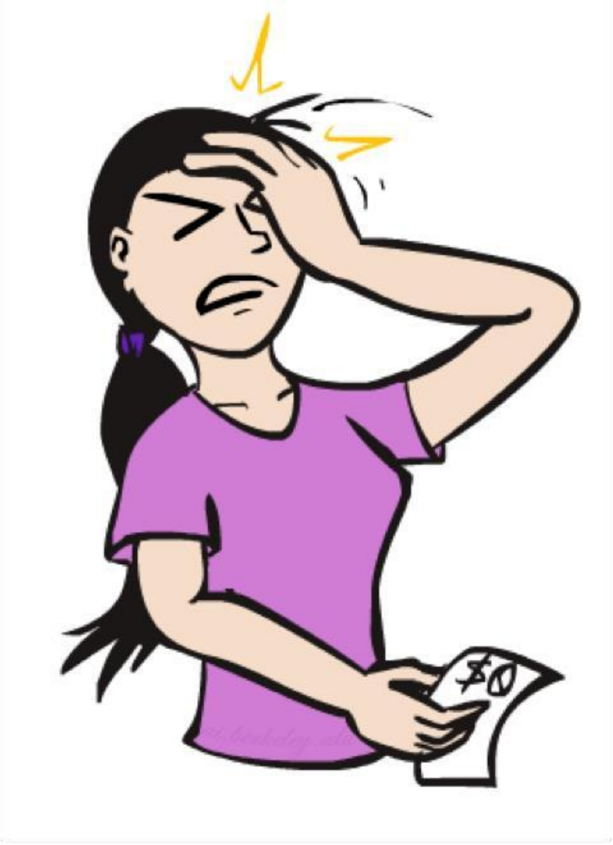
# Example: Insurance

- Consider the lottery [0.5, $1000; 0.5, $0]
- What is its expected monetary value?  ($500)
- What is its certainty equivalent?
  - Monetary value acceptable in lieu of lottery
  - $400 for most people
- Difference of $100 is the insurance premium
  - There's an insurance industry because people will pay to reduce their risk
  - If everyone were risk-neutral, no insurance needed!
- It's win-win: you'd rather have the $400 and the insurance company would rather have the lottery (their utility curve is flat and they have many lotteries)



# Example: Human Rationality?

- Famous example of Allais (1953)
  - A: [0.8, $4k;    0.2, $0]
  - B: [1.0, $3k;    0.0, $0]
  - C: [0.2, $4k;    0.8, $0]
  - D: [0.25, $3k;    0.75, $0]
- Most people prefer B > A, C > D
- But if U($0) = 0, then
  - B > A      U($3k) > 0.8 U($4k)
  - C > D      0.8 U($4k) > U($3k)



# Next Time: CSP