LE/ESSE 2640: Adjustment calculus

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Lecture 9:Non- linear Parametric Adjustment

Non-linear parametric adjustment

- Functional models
  - The math function: $L = F(X)$
  - Observation equations: $\boldsymbol{l} + \boldsymbol{v} = F(\widehat{\boldsymbol{x}})$
  - Residual equations: $\boldsymbol{v} = \hat{\boldsymbol{l}} - \boldsymbol{l} = F(\widehat{\boldsymbol{x}}) - \boldsymbol{l}$

Non-linear parametric adjustment

- Stochastic models
  - $\boldsymbol{L} \sim N(\bar{\boldsymbol{l}}, \Sigma_{\boldsymbol{LL}})$
  - $\boldsymbol{L}$: Observation vector (Variable)
  - $\bar{\boldsymbol{l}}$: the expectation of $\boldsymbol{l}$
  - $\Sigma_{\boldsymbol{LL}}$: variance and covariance of $\boldsymbol{L}$
  - $Q = \frac{1}{\sigma_0^2} \Sigma_{\boldsymbol{LL}}$: the cofactor matrix
  - $P = \sigma_0^2 \Sigma_{\boldsymbol{LL}}{}^{-1}$: the weight matrix

Non-linear parametric adjustment

- Least squares solutions
  - Find such $\hat{\boldsymbol{x}}$ that the cost function: $\phi = \boldsymbol{\nu}^T P \boldsymbol{\nu}$ min

Non-linear parametric adjustment

- $\phi = \boldsymbol{v}^T P \boldsymbol{v} \quad \boldsymbol{v} = \hat{\boldsymbol{l}} - \boldsymbol{l} = F(\widehat{\boldsymbol{x}}) - \boldsymbol{l}$
- Linearization at the initial values
- $\hat{\boldsymbol{l}} = F(\widehat{\boldsymbol{x}}^0) + \frac{\partial F}{\partial \widehat{\boldsymbol{x}}}|_{\widehat{\boldsymbol{x}} = \widehat{\boldsymbol{x}}^0} \delta \widehat{\boldsymbol{x}}$
- Recall the linear case: $\hat{\boldsymbol{l}} = A\widehat{\boldsymbol{x}} + \mathrm{C}$, $\widehat{\boldsymbol{x}} = N^{-1} A^T P(\boldsymbol{l} - \mathrm{C})$
- $A = \frac{\partial F}{\partial \widehat{\boldsymbol{x}}}|_{\widehat{\boldsymbol{x}} = \widehat{\boldsymbol{x}}^0} \quad \mathrm{C} = F(\widehat{\boldsymbol{x}}^0)$
- $\delta \widehat{\boldsymbol{x}} = N^{-1} A^T P[\boldsymbol{l} - F(\widehat{\boldsymbol{x}}^0)]$

Non-linear parametric adjustment

- $\widehat{\boldsymbol{x}} = \widehat{\boldsymbol{x}}^0 + \delta\widehat{\boldsymbol{x}}$
- $\boldsymbol{\nu} = F(\widehat{\boldsymbol{x}}^0) + \frac{\partial F}{\partial \widehat{\boldsymbol{x}}}|_{\widehat{\boldsymbol{x}}=\widehat{\boldsymbol{x}}^0}\delta\widehat{\boldsymbol{x}} - \boldsymbol{l}$
- $\boldsymbol{\nu} = J\delta\widehat{\boldsymbol{x}} - [\boldsymbol{l} - F(\widehat{\boldsymbol{x}}^0)]$
- $\boldsymbol{\nu} = A\delta\widehat{\boldsymbol{x}} - [\boldsymbol{l} - F(\widehat{\boldsymbol{x}}^0)]$
- $\hat{\sigma}_0^2 = \frac{\boldsymbol{\nu}^T P \boldsymbol{\nu}}{n-m}$

Variance-covariance matrices

- Quantities: $\boldsymbol{l}$ (observations), $\widehat{\boldsymbol{x}}$ (parameters), $\boldsymbol{v}$ (residuals), $\hat{\boldsymbol{l}}$ (adjusted observations)

	$\boldsymbol{l}$	$\widehat{\boldsymbol{x}}$	$\boldsymbol{v}$	$\hat{\boldsymbol{l}}$
$\boldsymbol{l}$	Σ_{ll}			
$\widehat{\boldsymbol{x}}$				
$\boldsymbol{v}$				
$\hat{\boldsymbol{l}}$				

Variance-covariance matrices

- Derivation of variance-covariance matrices
  - Express the quantities as functions of measurement vector
  - Derivation of variance-covariance matrices based on error propagation

Variance-covariance matrices

- Express the quantities as functions of measurement vector
- $\boldsymbol{l} = \boldsymbol{l}$
- $\delta\widehat{\boldsymbol{x}} = N^{-1}A^T P[\boldsymbol{l} - F(\widehat{\boldsymbol{x}}^0)]$
- $\widehat{\boldsymbol{x}} = \widehat{\boldsymbol{x}}^0 + \delta\widehat{\boldsymbol{x}} = \widehat{\boldsymbol{x}}^0 + N^{-1}A^T P[\boldsymbol{l} - F(\widehat{\boldsymbol{x}}^0)]$
- $\boldsymbol{\nu} = A\delta\widehat{\boldsymbol{x}} - [\boldsymbol{l} - F(\widehat{\boldsymbol{x}}^0)] = (AN^{-1}A^T P - \mathrm{I}\,)[\boldsymbol{l} - F(\widehat{\boldsymbol{x}}^0)]$
- $\hat{\boldsymbol{l}} = \boldsymbol{l} + \boldsymbol{\nu} = \boldsymbol{l} + (AN^{-1}A^T P - \mathrm{I}\,)[\boldsymbol{l} - F(\widehat{\boldsymbol{x}}^0)]$
- $\hat{\boldsymbol{l}} = \boldsymbol{l} + \boldsymbol{\nu} = AN^{-1}A^T P\boldsymbol{l} - (AN^{-1}A^T P - \mathrm{I}\,)F(\widehat{\boldsymbol{x}}^0)$

Variance-covariance matrices

- Derivation of variance-covariance matrices based on error propagation
- $\widehat{\boldsymbol{x}} = \widehat{\boldsymbol{x}}^0 + \delta\widehat{\boldsymbol{x}} = \widehat{\boldsymbol{x}}^0 + N^{-1}A^T P[\boldsymbol{l} - F(\widehat{\boldsymbol{x}}^0)]$
- Derive $\Sigma_{\widehat{\boldsymbol{x}}\widehat{\boldsymbol{x}}}$, given $\Sigma_{\boldsymbol{ll}}$
- $\Sigma_{\widehat{\boldsymbol{x}}\widehat{\boldsymbol{x}}} = (N^{-1}A^T P)\Sigma_{\boldsymbol{ll}}(N^{-1}A^T P)^T$

$$= N^{-1}A^T P\sigma_0^2 P^{-1}PAN^{-1}$$

$$= \sigma_0^2 N^{-1}A^T PAN^{-1} = \sigma_0^2 N^{-1}$$

Variance-covariance matrices

- $\boldsymbol{\nu} = A\delta\widehat{\boldsymbol{x}} - [\boldsymbol{l} - F(\widehat{\boldsymbol{x}}^0)] = (AN^{-1}A^TP - \mathrm{I}\,)[\boldsymbol{l} - F(\widehat{\boldsymbol{x}}^0)]$
- $\Sigma_{\boldsymbol{\nu\nu}} = (AN^{-1}A^TP - \mathrm{I})\Sigma_{\boldsymbol{ll}}(AN^{-1}A^TP - \mathrm{I})^T$

$$= (AN^{-1}A^TP - \mathrm{I})\sigma_0^2P^{-1}(AN^{-1}A^TP - \mathrm{I})^T$$
$$= \sigma_0^2(AN^{-1}A^T - P^{-1})(PAN^{-1}A^T - \mathrm{I})$$
$$= \sigma_0^2(AN^{-1}A^TPAN^{-1}A^T - P^{-1}PAN^{-1}A^T - AN^{-1}A^T + P^{-1})$$
$$= \sigma_0^2(AN^{-1}A^T - AN^{-1}A^T - AN^{-1}A^T + P^{-1})$$

$$\Sigma_{\boldsymbol{\nu\nu}} = \sigma_0^2P^{-1} - \sigma_0^2AN^{-1}A^T$$
$$\Sigma_{\boldsymbol{\nu\nu}} = \Sigma_{\boldsymbol{ll}} - \sigma_0^2AN^{-1}A^T$$

Variance-covariance matrices

- $\hat{\boldsymbol{l}} = AN^{-1}A^TP\boldsymbol{l} - (AN^{-1}A^TP - \mathrm{I})F(\hat{\boldsymbol{x}}^0)$
- $\Sigma_{\hat{\boldsymbol{l}}\hat{\boldsymbol{l}}} = (AN^{-1}A^TP)\Sigma_{\boldsymbol{ll}}(AN^{-1}A^TP)^T$
  $= (AN^{-1}A^TP)\sigma_0^2P^{-1}(AN^{-1}A^TP)^T$
  $= \sigma_0^2AN^{-1}A^TPAN^{-1}A^T$
  $= \sigma_0^2AN^{-1}A^T$

Variance-covariance matrices

- $\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}^0 + \delta\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}^0 + N^{-1}A^T P[\boldsymbol{l} - F(\hat{\boldsymbol{x}}^0)]$
- Derive $\Sigma_{\boldsymbol{l}\hat{\boldsymbol{x}}}$, $\Sigma_{\hat{\boldsymbol{x}}\boldsymbol{l}}$ given $\Sigma_{\boldsymbol{ll}}$
- $\Sigma_{\hat{\boldsymbol{x}}\boldsymbol{l}} = (N^{-1}A^T P)\Sigma_{\boldsymbol{ll}} = N^{-1}A^T P \sigma_0^2 P^{-1}$
- $\Sigma_{\hat{\boldsymbol{x}}\boldsymbol{l}} = \sigma_0^2 N^{-1}A^T$
- $\Sigma_{\boldsymbol{l}\hat{\boldsymbol{x}}} = {\Sigma_{\hat{\boldsymbol{x}}\boldsymbol{l}}}^T = \sigma_0^2 A N^{-1}$

Variance-covariance matrices

- $\boldsymbol{\nu} = (AN^{-1}A^TP - \mathrm{I}\,)[\boldsymbol{l} - F(\widehat{\boldsymbol{x}}^0)]$
- Derive $\Sigma_{\boldsymbol{l\nu}}$, $\Sigma_{\boldsymbol{\nu l}}$ given $\Sigma_{\boldsymbol{ll}}$
- $\Sigma_{\boldsymbol{\nu l}} = (AN^{-1}A^TP - \mathrm{I}\,)\Sigma_{\boldsymbol{ll}}$

$$= (AN^{-1}A^TP - \mathrm{I})\sigma_0^2P^{-1}$$

$$= \sigma_0^2AN^{-1}A^T - \sigma_0^2P^{-1}$$

$$= \sigma_0^2AN^{-1}A^T - \Sigma_{\boldsymbol{ll}}$$

$$\Sigma_{\boldsymbol{l\nu}} = {\Sigma_{\boldsymbol{\nu l}}}^T = \sigma_0^2AN^{-1}A^T - \Sigma_{\boldsymbol{ll}}$$

Variance-covariance matrices

- $\hat{\boldsymbol{l}} = AN^{-1}A^TP\boldsymbol{l} - (AN^{-1}A^TP - \mathrm{I}\,)F(\widehat{\boldsymbol{x}}^0)$
- Derive $\Sigma_{\boldsymbol{l}\hat{\boldsymbol{l}}}$, $\Sigma_{\hat{\boldsymbol{l}}\boldsymbol{l}}$ given $\Sigma_{\boldsymbol{l}\boldsymbol{l}}$
- $\Sigma_{\hat{\boldsymbol{l}}\boldsymbol{l}} = (AN^{-1}A^TP)\Sigma_{\boldsymbol{l}\boldsymbol{l}}$

$$= (AN^{-1}A^TP)\sigma_0^2P^{-1}$$

$$= \sigma_0^2AN^{-1}A^T$$

$$\Sigma_{\boldsymbol{l}\hat{\boldsymbol{l}}} = {\Sigma_{\hat{\boldsymbol{l}}\boldsymbol{l}}}^T = \sigma_0^2AN^{-1}A^T$$

Variance-covariance matrices

- $\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}^0 + \delta\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}^0 + N^{-1}A^TP[\boldsymbol{l} - F(\hat{\boldsymbol{x}}^0)]$
- $\boldsymbol{\nu} = (AN^{-1}A^TP - \mathrm{I}\,)[\boldsymbol{l} - F(\hat{\boldsymbol{x}}^0)]$
- Derive $\Sigma_{\boldsymbol{\nu}\hat{\boldsymbol{x}}}$, $\Sigma_{\hat{\boldsymbol{x}}\boldsymbol{\nu}}$ given $\Sigma_{\boldsymbol{ll}}$
- $$\begin{aligned}\Sigma_{\hat{\boldsymbol{x}}\boldsymbol{\nu}} &= (N^{-1}A^TP)\Sigma_{\boldsymbol{ll}}\big((AN^{-1}A^TP - \mathrm{I}\,)\big)^T \\ &= (N^{-1}A^TP)\sigma_0^2P^{-1}(PAN^{-1}A^T - \mathrm{I}\,) \\ &= \sigma_0^2N^{-1}A^TPAN^{-1}A^T - \sigma_0^2N^{-1}A^T \\ &= \mathbf{0}\end{aligned}$$
- $\Sigma_{\boldsymbol{\nu}\hat{\boldsymbol{x}}} = \mathbf{0}$

Variance-covariance matrices

- $\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}^0 + \delta\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}^0 + N^{-1}A^TP[\boldsymbol{l} - F(\hat{\boldsymbol{x}}^0)]$
- $\hat{\boldsymbol{l}} = AN^{-1}A^TP\boldsymbol{l} - (AN^{-1}A^TP - \mathrm{I}\,)F(\hat{\boldsymbol{x}}^0)$
- Derive $\Sigma_{\hat{\boldsymbol{l}}\hat{\boldsymbol{x}}}$, $\Sigma_{\hat{\boldsymbol{x}}\hat{\boldsymbol{l}}}$ given $\Sigma_{\boldsymbol{ll}}$
- $$\begin{aligned}\Sigma_{\hat{\boldsymbol{x}}\hat{\boldsymbol{l}}} &= (N^{-1}A^TP)\Sigma_{\boldsymbol{ll}}(AN^{-1}A^TP)^T \\ &= (N^{-1}A^TP)\sigma_0^2P^{-1}PAN^{-1}A^T \\ &= \sigma_0^2N^{-1}A^TPAN^{-1}A^T \\ &= \sigma_0^2N^{-1}A^T\end{aligned}$$
- $\Sigma_{\hat{\boldsymbol{l}}\hat{\boldsymbol{x}}} = \sigma_0^2AN^{-1}$

Variance-covariance matrices

- $\boldsymbol{\nu} = (AN^{-1}A^TP - \mathrm{I}\,)[\boldsymbol{l} - F(\widehat{\boldsymbol{x}}^0)]$
- $\hat{\boldsymbol{l}} = AN^{-1}A^TP\boldsymbol{l} - (AN^{-1}A^TP - \mathrm{I}\,)F(\widehat{\boldsymbol{x}}^0)$
- Derive $\Sigma_{\boldsymbol{\nu}\hat{\boldsymbol{l}}}$, $\Sigma_{\hat{\boldsymbol{l}}\boldsymbol{\nu}}$ given $\Sigma_{\boldsymbol{ll}}$
- $$\begin{aligned}\Sigma_{\hat{\boldsymbol{l}}\boldsymbol{\nu}} &= (AN^{-1}A^TP)\Sigma_{\boldsymbol{ll}}\left((AN^{-1}A^TP - \mathrm{I}\,)\right)^T \\ &= (AN^{-1}A^TP)\sigma_0^2P^{-1}(PAN^{-1}A^T - \mathrm{I}\,) \\ &= \sigma_0^2AN^{-1}A^TPAN^{-1}A^T - \sigma_0^2AN^{-1}A^T \\ &= \mathbf{0}\end{aligned}$$
- $\Sigma_{\boldsymbol{\nu}\hat{\boldsymbol{l}}} = \mathbf{0}$

Variance-covariance matrices

- Quantities: $\boldsymbol{l}$ (observations), $\widehat{\boldsymbol{x}}$ (parameters), $\boldsymbol{v}$ (residuals), $\hat{\boldsymbol{l}}$ (adjusted observations)

	$\boldsymbol{l}$	$\widehat{\boldsymbol{x}}$	$\boldsymbol{v}$	$\hat{\boldsymbol{l}}$
$\boldsymbol{l}$	Σ_{ll}	$\sigma_0^2 AN^{-1}$	$\sigma_0^2 AN^{-1}A^T - \Sigma_{ll}$	$\sigma_0^2 AN^{-1}A^T$
$\widehat{\boldsymbol{x}}$	$\sigma_0^2 N^{-1}A^T$	$\sigma_0^2 N^{-1}$	$\mathbf{0}$	$\sigma_0^2 N^{-1}A^T$
$\boldsymbol{v}$	$\sigma_0^2 AN^{-1}A^T - \Sigma_{ll}$	$\mathbf{0}$	$\Sigma_{ll} - \sigma_0^2 AN^{-1}A^T$	$\mathbf{0}$
$\hat{\boldsymbol{l}}$	$\sigma_0^2 AN^{-1}A^T$	$\sigma_0^2 AN^{-1}$	$\mathbf{0}$	$\sigma_0^2 AN^{-1}A^T$