Statistical Testing of Least-Squares Quantities

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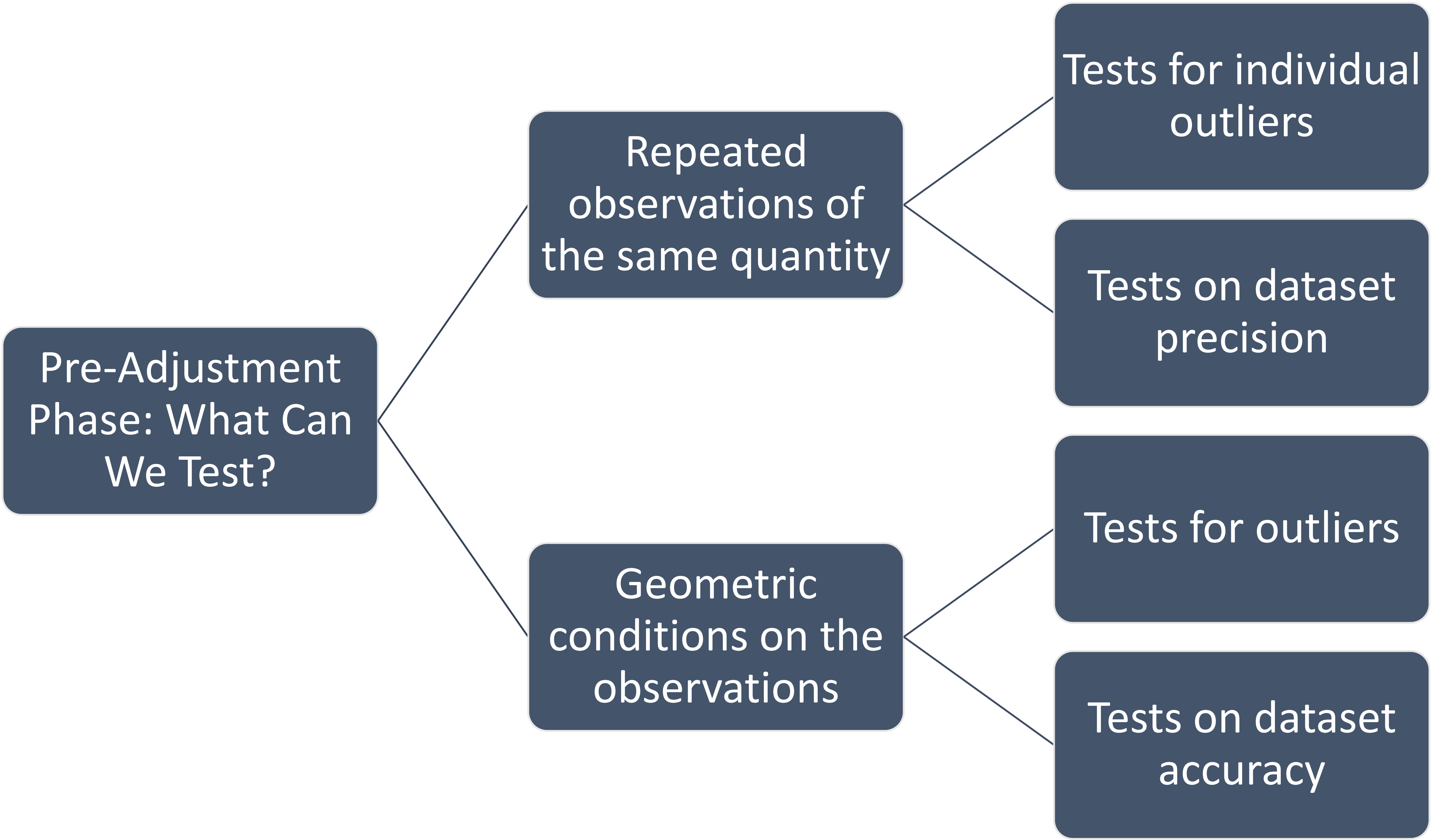
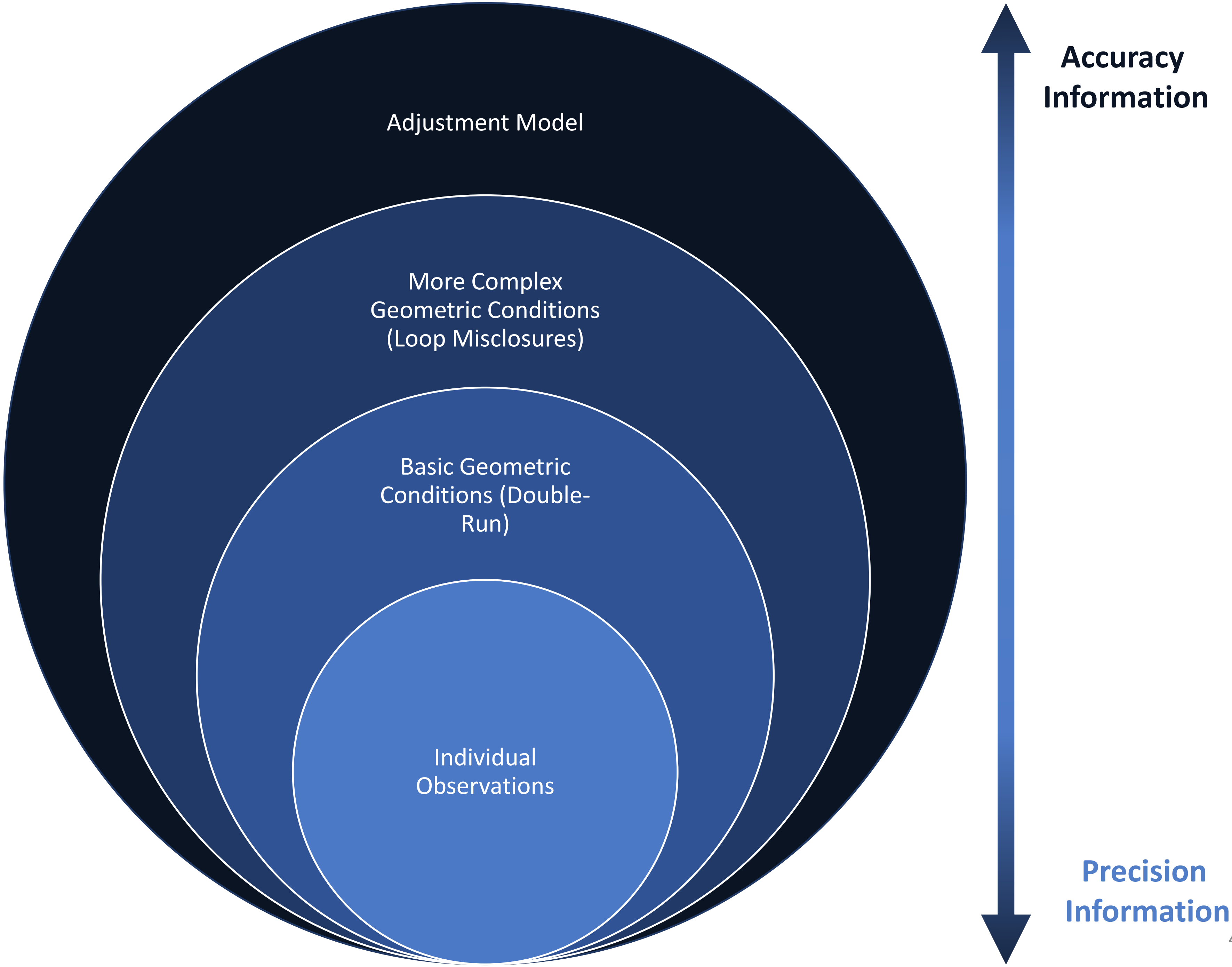
Planning Phase: What Can We Test?

$\boldsymbol{D_{\widehat{x}\widehat{x}}}$	$\boldsymbol{D_{\widehat{v}\widehat{v}}}$	$\boldsymbol{D_{\widehat{\ell}\widehat{\ell}}}$


Pre-Adjustment Phase: What Can We Test?
Repeated observations of the same quantity
Geometric conditions on the observations
Tests for individual outliers
Tests on dataset precision
Tests for outliers
Tests on dataset accuracy

Revisiting Precision and Accuracy

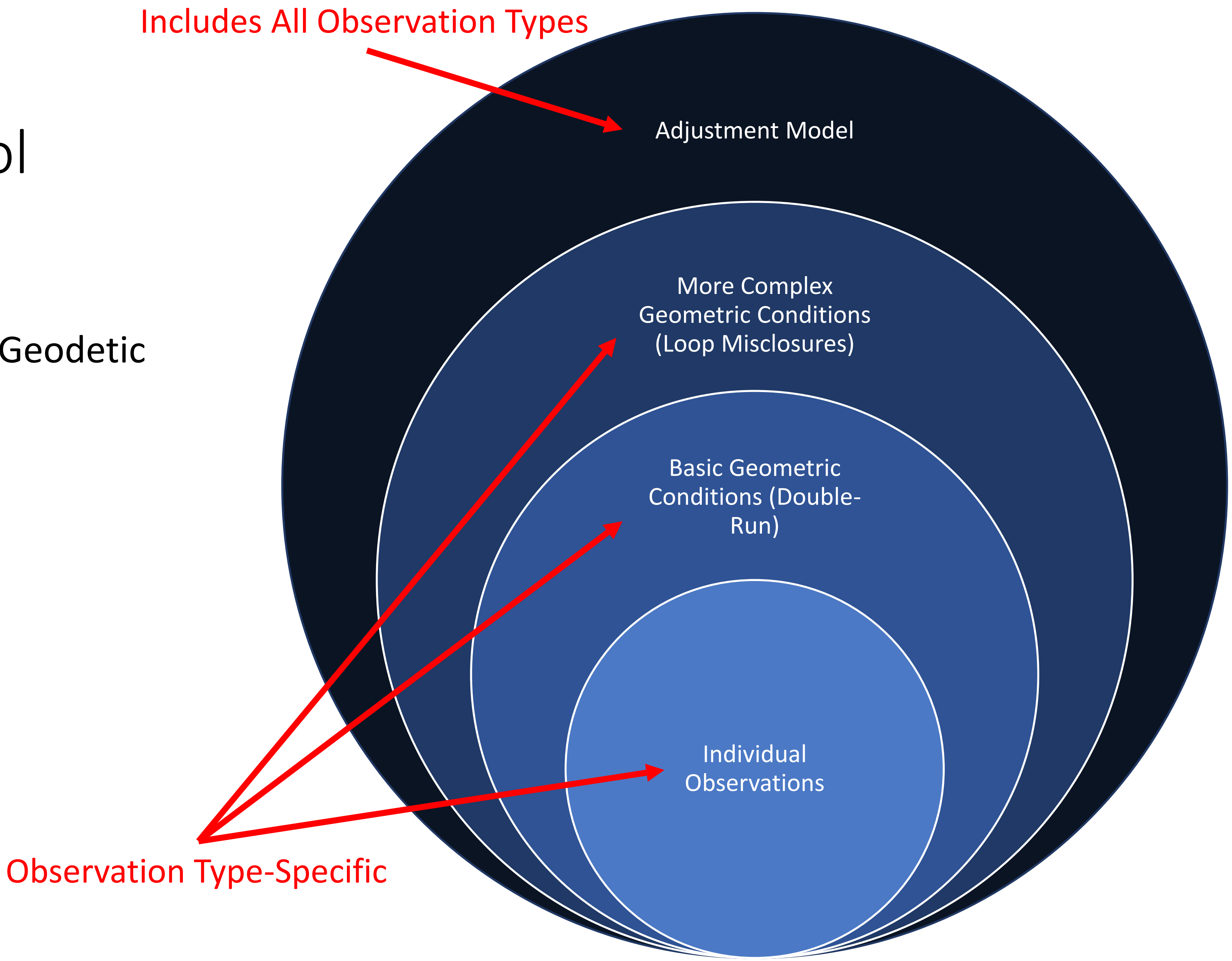
- Precision describes how ***repeatable*** an observation is
- Accuracy describes how close an observation is *to the* ***truth***


Adjustment Model
More Complex Geometric Conditions (Loop Misclosures)
Basic Geometric Conditions (Double-Run)
Individual Observations
Accuracy Information
Precision Information

The Challenge: Geodetic Control Networks

Types of Observations in Geodetic Control Networks:

1. Horizontal Directions
2. Vertical Angles
3. Slope Distances
4. Height Differences
5. GPS Baselines



Post-Adjustment Phase: What Can We Test?

$\widehat{\boldsymbol{v}}$	$\widehat{\boldsymbol{x}}$	$\hat{\sigma}_0^2$

Testing the *A Posteriori* Variance Factor

$$dof \cdot \hat{\sigma}_0^2 \sim \chi^2(dof)$$

$$H_0 : \hat{\sigma}_0^2 = \sigma_0^2$$
$$H_a : \hat{\sigma}_0^2 \neq \sigma_0^2$$

Example:

After an adjustment with 40 redundant observations, the *a posteriori* variance factor is determined to be 2.5. *What does this say about the accuracy of the observations?*

Modification: The *a posteriori* variance factor is 0.8.

Testing Least-Squares Residuals

$$\hat{\boldsymbol{v}} \sim \boldsymbol{N}(\boldsymbol{0}, \boldsymbol{D}_{\hat{\boldsymbol{v}}\hat{\boldsymbol{v}}})$$

$$\frac{\hat{v}_i}{\sigma_{\hat{v}_i}} \sim N(0, 1)$$

$$\frac{\hat{v}_i}{\hat{\sigma}_{\hat{v}_i}} \sim T(dof)$$

Example:

After an adjustment with 2 redundant observations, the least-squares residual vector was calculated to be $\hat{\boldsymbol{v}} = [0.02 \quad -0.08 \quad 0.06]^T$ with a covariance matrix of

$$\boldsymbol{D}_{\hat{\boldsymbol{v}}\,\hat{\boldsymbol{v}}} = \begin{bmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{bmatrix} cm^2$$

Are any of these residuals outliers?

Modification: The *a priori* variance factor is 1.2.

Testing Least-Squares Unknowns

$$\hat{\boldsymbol{x}} \sim \boldsymbol{N}(\widetilde{\boldsymbol{x}}, \boldsymbol{D}_{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}})$$

You can test hypotheses on individual unknowns, but this doesn’t account for correlation between your estimated unknowns.

Linear Hypothesis Testing

$$H_0: \boldsymbol{Hx} = \boldsymbol{w_h}$$
$$H_a: \boldsymbol{Hx} \neq \boldsymbol{w_h}$$

Example: We have a triangulation network with two unknown points (A and B). Point A is an old benchmark with the coordinates (1000, 2000), and we want to test if it has shifted.

$$H_0: \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_A \\ Y_A \\ X_B \\ Y_B \end{bmatrix} = \begin{bmatrix} 1000 \\ 2000 \end{bmatrix}$$

Linear Hypothesis Testing

$$\boldsymbol{H\hat{x}} \sim \boldsymbol{N}\left(\boldsymbol{H\tilde{x}}, \boldsymbol{HD_{\hat{x}\hat{x}}H^T}\right)$$

$$\boldsymbol{H\hat{x}} - \boldsymbol{w_h} \sim \boldsymbol{N}\left(\boldsymbol{0}, \boldsymbol{HD_{\hat{x}\hat{x}}H^T}\right)$$

$$(\boldsymbol{H\hat{x}} - \boldsymbol{w_h})^{\boldsymbol{T}}\left(\boldsymbol{HD_{\hat{x}\hat{x}}H^T}\right)^{-1}(\boldsymbol{H\hat{x}} - \boldsymbol{w_h}) \sim \chi^2(h)$$

Linear Hypothesis Testing Example

We have a triangulation network with two unknown points (A and B). Point A is an old benchmark with the coordinates (1000, 2000), and we want to test if it has shifted.

Our least-squares estimates of the coordinates of A and B is $\hat{\boldsymbol{x}} = [1000.025 \quad 2000.015 \quad 300 \quad 400]^T$ with a covariance matrix of

$$\boldsymbol{D}_{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}} = \begin{bmatrix} 9 & 4 & 0 & 0 \\ 4 & 81 & 0 & 0 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & -1 & 9 \end{bmatrix} mm^2$$

A Posteriori Linear Hypothesis Testing

$$(\boldsymbol{H}\hat{\boldsymbol{x}} - \boldsymbol{w_h})^T \left(\boldsymbol{H}\boldsymbol{D}_{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}}\boldsymbol{H}^T\right)^{-\mathbf{1}} (\boldsymbol{H}\hat{\boldsymbol{x}} - \boldsymbol{w_h}) \sim \chi^2(h)$$

$$\hat{\boldsymbol{v}}^T \boldsymbol{P} \hat{\boldsymbol{v}} \sim \chi^2(dof)$$

$$\frac{(\boldsymbol{H}\hat{\boldsymbol{x}} - \boldsymbol{w_h})^T \left(\boldsymbol{H}\boldsymbol{D}_{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}}\boldsymbol{H}^T\right)^{-\mathbf{1}} (\boldsymbol{H}\hat{\boldsymbol{x}} - \boldsymbol{w_h})/h}{\hat{\sigma}_0^2} \sim F(h, dof)$$

Linear Hypothesis Testing Example

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The system had an *a posteriori* variance factor of 2, with 10 redundant observations in the network.