Neglect-zero effects in Dynamic Semantics

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Abstract. The article presents a bilateral update semantics for epistemic modals which captures their discourse dynamics [54] as well as their potential to give rise to FC inferences [58]. The latter are derived as neglect-zero effects as in [3]. Neglect-zero is a tendency in human cognition to disregard structures that verify sentences by virtue of an empty witness set. The upshot of modelling the neglect-zero tendency in a dynamic setting is a notion of dynamic logical consequence which makes interesting predictions concerning possible divergences between everyday and logico-mathematical reasoning.

Keywords: Dynamic Semantics · Epistemic MIGHT · Free choice disjunction · Bilateral negation · Post-supposition.

1 Introduction

In Free Choice (FC) inferences, conjunctive meanings are derived from disjunctive modal sentences contrary to the prescriptions of classical logic:

(1) Deontic FC [35]
  a. You may go to the beach or to the cinema.
  b. $\rightsquigarrow$ You may go to the beach and you may go to the cinema.

(2) Epistemic FC [58]
  a. Mr. X might be in Victoria or in Brixton.
  b. $\rightsquigarrow$ Mr. X might be in Victoria and he might be in Brixton.

[3] presented a formal account of FC inferences in a bilateral state-based modal logic (BSML). The novel hypothesis at the core of the proposal was that FC and related inferences are a straightforward consequence of a tendency in human cognition to neglect models that verify sentences by virtue of some empty configurations (*zero-models*). Using tools from team semantics [51, 57], [3] showed that the tendency to neglect zero-models (*neglect-zero* tendency) derives FC inferences (when interpreting disjunctions speakers associate each disjunct with a non-empty possibility) and their cancellation under negation. The latter result relied on the adopted bilateralism, where each connective comes with an assertion and a rejection condition, and negation is defined in terms of the latter notion [49, 45, 31].

In this article I will present a Bilateral Update Semantics (BiUS), building on [54]'s update semantics for epistemic MIGHT, where the neglect-zero tendency is explicitly formalised in a dynamic setting. The resulting system will derive ignorance and epistemic FC inferences as neglect-zero effects, as in [3], as well as capture the dynamics of epistemic modals in discourse, as in [54].

One crucial difference with respect to classical dynamic semantics [28, 54, 29] concerns the treatment of negation. Like BSML, BiUS adopts a bilateral notion of negation validating double negation elimination. When extended to the first-order case, BiUS has therefore the potential to provide an account of Barbara Partee's bathroom example (as explained at the end of Section 3.3, example (38)). The dynamic notion of logical consequence defined by BiUS further makes interesting predictions concerning the impact of neglect-zero on everyday reasoning and its deviation from classical logic.

The next section presents static BSML and its main motivation; Section 3 introduces BiUS, its core results and applications; Section 4 discusses some potential applications of the dynamic implementation of neglect-zero to the psychology of everyday reasoning; and Section 5 concludes.

2 Bilateral state-based modal logic (BSML)

In team semantics formulas are interpreted with respect to a set of points of evaluation (a team) rather than single points [51, 57]. In a team based modal logic [52, 39], teams are sets of possible worlds. BSML is a bilateral version of a team-based modal logic [3, 8] where teams are interpreted as information states. Bilateralism in this context means that we model assertion and rejection conditions rather than truth. In BSML, inferences relate speech acts rather than propositions and therefore might diverge from classical semantic entailments.

- Classical modal logic: $M, w \models \phi,$ where $w \in W$
- Team-based modal logic: $M, t \models \phi,$ where $t \subseteq W$
- Bilateral state-based modal logic (BSML):

$$M, s \models \phi, \text{ “}\phi \text{ is assertable in information state } s\text{”}, \quad \text{with } s \subseteq W$$
$$M, s \mathrel{=\!\!|} \phi, \text{ “}\phi \text{ is rejectable in information state } s\text{”}, \quad \text{with } s \subseteq W$$

2.1 Motivation

BSML was developed as part of a larger project with the goal to arrive at a formal account of a class of natural language inferences which diverge from classical entailments but also from canonical conversational implicatures. These include ignorance inference in modified numerals [24, 16, 47, 4] and epistemic indefinites [32, 5, 7] and phenomena of free choice in indefinites and disjunction [35, 18, 58].

Let us focus on the case of FC inferences triggered by disjunction as in (3).

(3) You may (A or B) $\rightsquigarrow$ You may A

The logical counterpart of (3) is not valid in standard deontic logic [55]:

(4) $\Diamond(\alpha \vee \beta) \rightarrow \Diamond\alpha$ [Free Choice (FC) principle]

Plainly making the FC principle valid, for example by adding it as an axiom, would not do because we would be able to derive $\Diamond b$ from any other $\Diamond a$ as shown in (5) [35].

(5) 1. $\Diamond a$ [assumption]
2. $\Diamond(a \vee b)$ [from 1, by classical reasoning]
3. $\Diamond b$ [from 2, by FC principle]

The step leading to 2 in (5) uses the following classically valid principle:

(6) $\Diamond\alpha \rightarrow \Diamond(\alpha \vee \beta)$

The natural language counterpart of (6), however, seems invalid [44]:

(7) You may post this letter $\not\rightsquigarrow$ You may post this letter or burn it.

Thus our intuitions in natural language are in direct opposition to the principles of classical logic.

Many solutions have been proposed to solve the paradox of free choice (see [41] for a recent overview). Pragmatic **neo-Gricean** solutions derive FC inferences as conversational implicatures, i.e., pragmatic inferences derived as the product of rational interactions between cooperative language users [27, 38, 46, 21]. On this view the step leading to 3 is unjustified. In **grammatical** solutions, instead, FC inferences result from the (optional) application of covert grammatical operators [20, 15, 9, 10]. Again the step leading to 3 is unjustified and the paradox is solved without the need to modify the logical system. **Semantic** solutions by contrast typically change the logic. FC inferences are treated as semantic entailments [48, 2, 11, 25]. The step leading to 3 is justified, but then it is the step leading to 2 which is no longer valid [2] or transitivity fails [25].

The main hypothesis behind the BSML solution is that FC inferences are neither the result of conversational reasoning (as proposed in neo-gricean approaches) nor the effect of optional applications of grammatical operators (as in the grammatical view). Rather they are a straightforward consequence of something else speakers do in conversation. Namely, when interpreting a sentence they create structures representing reality, pictures of the world [33] and in doing so they systematically neglect structures which (vacuously) verify the sentence by virtue of some empty configuration. This tendency, which [3] calls *neglect-zero*, follows from the expected difficulty of the cognitive operation of evaluating truths with respect to empty witness sets [12].

Models which verify a sentence by virtue of some empty set are called *zero-models*. As an illustration [3] discusses the following examples:

(8) Every square is black.

a. Verifier: [■, ■, ■]
b. Falsifier: [■, □, ■]
c. Zero-models: [ ]; [△, △, △]; [◇, ▲, ♠]

(9) Less than three squares are black.

a. Verifier: [■, □, ■]
b. Falsifier: [■, ■, ■]
c. Zero-models: [ ]; [△, △, △]; [◇, ▲, ♠]

The interpretation of (8) and (9) leads to the creation of structures representing reality, some verifying the sentence (the models depicted in (a)), some falsifying it (the models in (b)). The neglect-zero hypothesis states that zero-models, the ones represented in (c), are usually kept out of consideration. Zero-models are neglected because they are cognitively taxing, as confirmed by findings from number cognition [42, 12]. This difficulty can be argued to further explain the special status of the zero among the natural numbers [42]; the existential import effects operative in the logic of Aristotle (the inference from *every A is B* to *some A is B*); and why downward-monotonic quantifiers (e.g., *less than n squares*) are more difficult to process than upward-monotonic ones (e.g., *more than n squares*) [12]. Since empty witnesses in zero-models encode the absence of objects, they are more detached from sensory experience and therefore harder to conceive. The inference from the perception of absence to the truth of a sentence brings in additional costs, which results in a systematic dispreference for zero-models, a neglect-zero tendency. The idea at the core of [3] is that FC and related inferences, just like the Aristotelian existential import effects, are a consequence of such neglect-zero tendency assumed to be operative among language users in ordinary conversations.

Like neo-Gricean solutions, the neglect-zero approach views FC inferences as pragmatic inferences, albeit not of the conversational implicature kind. Like semantic solutions, [3] modifies classical logic, but not to derive FC inferences as semantic entailments, but rather to formally describe the pragmatic factors responsible for these inferences and isolate their impact in a rigorous way. As we will see, BSML formally defines a pragmatic enrichment function $[\ ]^+$ and generates FC inferences only for enriched $[\Diamond(a \vee b)]^+$. When enriched, this formula is no longer derivable from $\Diamond a$. Like in grammatical approaches, the paradox in (5) is then solved as a case of equivocation:

(10)
1. $\Diamond a$
2. $\Diamond(a \vee b) \neq [\Diamond(a \vee b)]^+$
3. $\qquad\qquad\qquad \Diamond b$

[3] gives two pieces of evidence in favor of the neglect-zero solution to the paradox of FC.

Argument from empirical coverage Although pragmatic, grammatical and semantic accounts can derive the basic FC inference (labeled as Narrow Scope FC below), FC sentences give rise to complex inference patterns when embedded under logical operators: FC inferences systematically disappear in negative contexts (Dual Prohibition), but are embeddable under universal quantifiers (Universal FC); furthermore they arise under double negation (Double Negation FC) and also when disjunction takes wide scope with respect to the modal operator (Wide scope FC):

(11) Narrow Scope FC [35]

a. You may go to the beach or to the cinema.
$\rightsquigarrow$ You may go to the beach and you may go to the cinema.

b. $\Diamond(\alpha \vee \beta) \rightsquigarrow \Diamond\alpha \wedge \Diamond\beta$

(12) Dual Prohibition [6]

a. You are not allowed to eat the cake or the ice-cream.
$\rightsquigarrow$ You are not allowed to eat either one.

b. $\neg\Diamond(\alpha \vee \beta) \rightsquigarrow \neg\Diamond\alpha \wedge \neg\Diamond\beta$

(13) Universal FC [13]

a. All of the boys may go to the beach or to the cinema.
$\rightsquigarrow$ All of the boys may go to the beach and all of the boys may go to the cinema.

b. $\forall x\Diamond(\alpha \vee \beta) \rightsquigarrow \forall x(\Diamond\alpha \wedge \Diamond\beta)$

(14) (Embedded) Double Negation FC [26]

a. Exactly one girl cannot take Spanish or Calculus.
$\rightsquigarrow$ One girl can take neither of the two and each of the others can choose between them.

b. $\exists x(\neg\Diamond(\alpha(x) \vee \beta(x)) \wedge \forall y(y \neq x \rightarrow \neg\neg\Diamond(\alpha(y) \vee \beta(y)))) \rightsquigarrow$
$\exists x(\neg\Diamond\alpha(x) \wedge \neg\Diamond\beta(x) \wedge \forall y(y \neq x \rightarrow (\Diamond\alpha(y) \wedge \Diamond\beta(y))))$

(15) Wide Scope FC [58]

a. Detectives may go by bus or they may go by boat.
$\rightsquigarrow$ Detectives may go by bus and may go by boat.

b. Mr. X might be in Victoria or he might be in Brixton.
$\rightsquigarrow$ Mr. X might be in Victoria and might be in Brixton.

c. $\Diamond\alpha \vee \Diamond\beta \rightsquigarrow \Diamond\alpha \wedge \Diamond\beta$

As shown in [3] these patterns are captured by the neglect-zero approach implemented in BSML. Most other approaches instead need additional assumptions as summarised in Table 1.[1] [2]

[1] Among the exceptions to this claim is [25]. See [3] for a comparison.

[2] Consider approaches in the grammatical tradition. Dual Prohibition cases are not derived directly but are explained by appealing to variations of the Strongest Meaning Hypothesis [17]. To account for wide scope FC inferences, which again cannot be generated by (recursive) applications of grammatical exhaustification, different strategies must be employed (see [9, 10]). As for the case of double negation FC, as discussed in detail in [26], pages 147-149, by recursive exhaustification only we cannot capture the so-called ALL-OTHERS-FREE-CHOICE inference displayed in (14). Inclusion-based grammatical accounts [9, 10], given some additional assumptions

	NS FC	**Dual Prohib**	**Universal** FC	**Double Neg** FC	**WS** FC
Neo-Gricean	yes	yes	no	?	no
Grammatical	yes	y/n	yes	y/n	y/n
Semantic	yes	no	yes	no	no
Neglect-zero	yes	yes	yes	yes	yes

Table 1. Comparison with competing accounts of FC inference

Argument from cognitive plausibility Disjunction in natural language can give rise to different pragmatic effects:

(16) You may have coffee or tea.

a. Ignorance: $\neg K\Diamond\alpha \wedge \neg K\Diamond\beta$ (speaker doesn't know which)

b. FC inference: $\Diamond\alpha \wedge \Diamond\beta$ (you may choose which)

c. Scalar implicature: $\neg\Diamond(\alpha \wedge \beta)$ (you may not have both)

On neo-Gricean and grammatical accounts, FC inferences and scalar implicatures are viewed as originating from a common source: Gricean reasoning or the application of covert grammatical operators, see Table 2. The experimental literature however has shown remarkable differences between FC and scalar inferences. The former are more robust and easier to process than the latter [14, 53] and are acquired earlier [50].

	processing cost	**acquisition**
FC inference	low	early
scalar implicature	high	late

The neglect-zero hypothesis has the potential to arrive at a principled explanation of these differences. On this view, FC inferences are not akin to scalar implicatures. FC follows from the assumption that when interpreting sentences language users neglect zero-models. Zero-models are neglected because cognitively taxing. Thus FC inferences result from a tendency to avoid a cognitive difficulty. Their low processing cost and early acquisition are therefore expected on this view. However, the question of how to model scalar implicatures in BSML remains. In particular, if the modelling leads to correct predictions in terms of processing and acquisition. This is one of the issues left open in [3].

about alternatives, can derive the inference for 'exactly one' sentences but need further modifications to account for similar readings in the case of sentences using 'exactly two' or higher. In a logic-based account like BSML, the ALL-OTHERS-FREE-CHOICE reading in all these variants can be captured simply by validating dual prohibition ($\neg\Diamond(\alpha\vee\beta) \rightsquigarrow \neg\Diamond\alpha\wedge\neg\Diamond\beta$) and double negation FC ($\neg\neg\Diamond(\alpha\vee\beta) \rightsquigarrow \Diamond\alpha\wedge\Diamond\beta$). The former allows us to derive the blue part in the inference below and the latter the red part:

(i) $\exists x(\neg\Diamond(\alpha(x) \vee \beta(x)) \wedge \forall y(y \neq x \rightarrow \neg\neg\Diamond(\alpha(y) \vee \beta(y)))) \rightsquigarrow$
$\exists x(\neg\Diamond\alpha(x) \wedge \neg\Diamond\beta(x) \wedge \forall y(y \neq x \rightarrow (\Diamond\alpha(y) \wedge \Diamond\beta(y))))$

	Ignorance inference	**FC inference**	**Scalar implicature**
Neo-Gricean	reasoning	reasoning	reasoning
Grammatical	debated	grammatical	grammatical
Neglect-zero	neglect-zero	neglect-zero	—

Table 2. Comparison with neo–Gricean and grammatical view

2.2 BSML: formal definitions

The target language is the language of propositional modal logic enriched with the non-emptiness atom, NE, from team logic [57], which [3] uses to define the pragmatic enrichment function $[\ ]^+$.

Definition 1 (Language). *The language L_{BSML} is defined recursively as*

$$\phi := p \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \Diamond\phi \mid \text{NE}$$

where $p \in PROP$, a countable set of propositional variables.

A Kripke model for L_{BSML} is a triple, $M = \langle W, R, V \rangle$, where W is a set of worlds, R is an accessibility relation on W and V is a world-dependent valuation function for the elements of $PROP$.

Formulas in the language are interpreted in models M with respect to a state $s \subseteq W$. Both support, $\models$, and anti-support , $=\!\mid$, conditions are specified. On the intended interpretation $M, s \models \phi$ stands for 'formula ϕ is assertable in s' and $M, s =\!\mid \phi$ stands for 'formula ϕ is rejectable in s', where s stands for the information state of the relevant speaker. $R[w]$ refers to the set $\{v \in W \mid wRv\}$.

Definition 2 (Semantic clauses).

$$\begin{aligned}
M, s &\models p \quad \textit{iff} \quad \forall w \in s : V(w,p) = 1 \\
M, s &=\!\mid p \quad \textit{iff} \quad \forall w \in s : V(w,p) = 0 \\
M, s &\models \neg\phi \quad \textit{iff} \quad M, s =\!\mid \phi \\
M, s &=\!\mid \neg\phi \quad \textit{iff} \quad M, s \models \phi \\
M, s &\models \phi \vee \psi \quad \textit{iff} \quad \exists t, t' : t \cup t' = s \ \&\ M, t \models \phi \ \&\ M, t' \models \psi \\
M, s &=\!\mid \phi \vee \psi \quad \textit{iff} \quad M, s =\!\mid \phi \ \&\ M, s =\!\mid \psi \\
M, s &\models \phi \wedge \psi \quad \textit{iff} \quad M, s \models \phi \ \&\ M, s \models \psi \\
M, s &=\!\mid \phi \wedge \psi \quad \textit{iff} \quad \exists t, t' : t \cup t' = s \ \&\ M, t =\!\mid \phi \ \&\ M, t' =\!\mid \psi \\
M, s &\models \Diamond\phi \quad \textit{iff} \quad \forall w \in s : \exists t \subseteq R[w] : t \neq \emptyset \ \&\ M, t \models \phi \\
M, s &=\!\mid \Diamond\phi \quad \textit{iff} \quad \forall w \in s : M, R[w] =\!\mid \phi \\
M, s &\models \text{NE} \quad \textit{iff} \quad s \neq \emptyset \\
M, s &=\!\mid \text{NE} \quad \textit{iff} \quad s = \emptyset
\end{aligned}$$

[3] adopts the standard abbreviation: $\Box\phi := \neg\Diamond\neg\phi$, and therefore derives the following interpretation for the necessity modal:

$$M, s \models \Box\phi \text{ iff for all } w \in s : R[w] \models \phi$$
$$M, s \dashv \Box\phi \text{ iff for all } w \in s : \text{there is a } t \subseteq R[w] : t \neq \emptyset \;\&\; t \dashv \phi$$

Logical consequence is defined as preservation of support.[3]

Definition 3 (Logical consequence). *$\phi \models \psi$ iff for all M, s : $M, s \models \phi \Rightarrow M, s \models \psi$*

In this framework we can further define team-sensitive restrictions on the accessibility relation.

Definition 4 (Team-sensitive constraints on R).

- *R is indisputable in (M, s) iff for all $w, v \in s : R[w] = R[v]$*
- *R is state-based in (M, s) iff for all $w \in s : R[w] = s$*

An accessibility relation R is state-based in a model-state pair (M, s) if all and only worlds in s are R-accessible within s. An accessibility relation R is indisputable in a model-state pair (M, s) if any two worlds in s access exactly the same set of worlds according to R. Clearly if R is state-based, R is also indisputable.

[3] proposes to use these constraints to capture the difference between epistemic and deontic modal verbs. In BSML, if we adopt a state-based accessibility relation, we can capture the infelicity of so-called *epistemic contradictions* [56], while preserving the non-factivity of $\Diamond$:

1. Epistemic contradiction: $\Diamond\alpha \wedge \neg\alpha \models \bot$ (if R is state-based)
2. Non-factivity: $\Diamond\alpha \not\models \alpha$

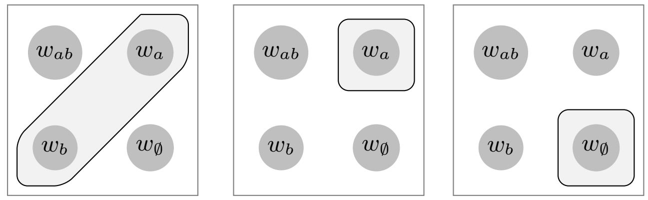
This motivates the assumption of a state-based R for epistemic modal verbs - the assertion of the epistemic possibility of a proposition conjoined with its negation as in (17) is indeed infelicitous [54, 56, 30, 40], but not for deontic ones, which don't give rise to similar infelicities, see (18):

(17) #It might be raining but it is not raining.

(18) You may smoke, but you don't smoke.

The accessibility relation in the case of deontic modals can at most be indisputable. Assuming s represents the information state of the relevant speaker, a state-based R leads to an interpretation of the modal as a quantifier over the epistemic possibilities the speaker entertains. An indisputable R instead only means that the speaker is fully informed about R, so, if R represents a deontic accessibility relation, indisputability means that the speaker is fully informed about (or has full authority on) what propositions are obligatory or allowed, as for example is arguably the case in performative uses.

[3] For a proof-theory of BSML and related systems see [8].



(a) Verifier (b) Zero-model (c) Falsifier

Fig. 1. Models for $(a \vee b)$.

2.3 Neglect-zero effects in BSML

In BSML, a state s supports a disjunction iff s is the union of two substates, each supporting one of the disjuncts. As an illustration consider the states represented in Figure 1. In these pictures w_a stands for a world where only a is true, w_b only b, etc. The disjunction $(a \vee b)$ is supported by the first two states, but not by 3(c) because the latter consists of $w_\emptyset$, a world where both a and b are false. The state in 1(b) supports $(a \vee b)$, because we can find suitable substates supporting each disjunct: the state itself, supporting a, and the empty state, vacuously supporting b. State 1(b) is then an example of a *zero-model* for $(a \vee b)$, a model which verifies the formula by virtue of an empty witness. Using NE, [3] defines a notion of *pragmatic enrichment*, whose core effect is to disallow such zero-models. A state s supports a pragmatically enriched disjunction $[\alpha \vee \beta]^+$ iff s is the union of two *non-empty* substates, each supporting one of the disjuncts. Such enriched disjunctions thus require both their disjuncts to be live possibilities [58, 23]. The pragmatic enrichment function is recursively defined for formulas in the NE-free fragment of the language as follows:

Definition 5 (Pragmatic enrichment function).

$$\begin{aligned}
[p]^+ &= p \wedge \text{NE} \\
[\neg\alpha]^+ &= \neg[\alpha]^+ \wedge \text{NE} \\
[\alpha \vee \beta]^+ &= ([\alpha]^+ \vee [\beta]^+) \wedge \text{NE} \\
[\alpha \wedge \beta]^+ &= ([\alpha]^+ \wedge [\beta]^+) \wedge \text{NE} \\
[\Diamond\alpha]^+ &= \Diamond[\alpha]^+ \wedge \text{NE}
\end{aligned}$$

The main result of this research is that in BSML $[\,]^+$-enrichments have non-trivial effects only when applied to positive disjunctions. This, in combination with the adopted notion of modality, derives FC inferences for pragmatically enriched formulas while no undesirable side effects obtain with other configurations, notably under single negation:

- Narrow scope FC: $[\Diamond(\alpha \vee \beta)]^+ \models \Diamond\alpha \wedge \Diamond\beta$
- Dual Prohibition: $[\neg\Diamond(\alpha \vee \beta)]^+ \models \neg\Diamond\alpha \wedge \neg\Diamond\beta$
- Double Negation: $[\neg\neg\Diamond(\alpha \vee \beta)]^+ \models \Diamond\alpha \wedge \Diamond\beta$

– Wide scope FC: $[\Diamond\alpha \vee \Diamond\beta]^+ \models \Diamond\alpha \wedge \Diamond\beta$ [if R is indisputable]

Notice that an indisputable R is required for deriving wide scope FC inferences. This means that wide scope FC is always predicted for epistemic modals, which, leading to epistemic contradiction, require an accessibility relation which is state-based and therefore indisputable (see Definition 4). Deontic modals instead only lead to wide scope FC inference in certain contexts, namely when the assumption of indisputability is justified.[4] These are contexts where the speaker is assumed to be fully informed about what is obligatory or allowed, for example in some performative uses of the verb. A further prediction of BSML is that cases of overt FC cancellations like (19)-(20) have to be treated as cases of wide scope FC where the assumption of indisputability is not warranted. In both cases, the prediction is arguably borne out [22, 36].

(19) You may eat the cake or the ice-cream, I don't know which.

(20) You may either eat the cake or the ice-cream, it depends on what John has taken.

Finally notice that in this framework neglect-zero effects can be isolated and literal meanings, ruled by classical logic, can be recovered. We can indeed model the global suspension of neglect-zero effects using BSML$^{\emptyset}$, the NE-free fragment of BSML, which behaves like classical modal logic (CML).

$$\alpha \models_{BSML} \beta \text{ iff } \alpha \models_{CML} \beta \qquad \text{[if } \alpha, \beta \text{ are NE-free]}$$

In BSML$^{\emptyset}$, which captures logico-mathematical reasoning, zero-models are always allowed and play an essential role. Paraphrasing Whitehead, we can conjecture that the use of zero-models 'is only forced on us by the needs of cultivated modes of thought'.[5]

> 'The point about zero is that we do not need to use it in the operations of daily life. No one goes out to buy zero fish. It is in a way the most civilized of all the cardinals, and its use is only forced on us by the needs of cultivated modes of thought.' (A.N. Whitehead quoted by [42]).

[4] By assuming a non-indisputable accessibility relation we can also account for the lack of FC inference in the following arguably wide scope disjunction cases discussed in [41]:

(i) a. It is OK for John to have ice-cream or it is OK for him to have cake.
b. It's conceivable that she will call or it's conceivable that she will write.

[5] This conjecture needs to be qualified. We do engage with zero-models in our daily life, for example when interpreting sentences with downward entailing quantifiers which can only be verified by zero-models, e.g., *I have zero ideas of how to prove this* or *I went to the store to buy fish, but they didn't have any, so we'll have no fish for dinner tonight*. Downward entailing quantifiers (*no/zero*) however are more costly to process than their upward entailing counterparts (*some*), a fact which can be taken to confirm the cognitive difficulty of engaging with zero-models.

We will return to the role of zero-models in everyday and logico-mathematical reasoning in section 4.

3 An Update Semantics for epistemic FC

In this section we introduce, BiUS, a bilateral update semantics for epistemic FC. Our point of departure is Veltman's update semantics for epistemic MIGHT [54, 19]. One of the goals of [54] was to account for dynamic effects of epistemic modals in discourse. Veltman observed a difference between sequences like (21-a) and (21-b): 'it is quite normal for one's expectations to be overruled by the facts - that is what is going on in the first sequence. But once you know something, it is a bit silly to pretend that you still expect something else, which is what is going on in the second' (see [54], page 223).

(21) Veltman's sequences
   a. Maybe this is Frank Veltman's example. It isn't his example!
   b. ?This is not Frank Veltman's example! Maybe it's his example.

As we saw in the previous section, BSML can capture the infelicity of epistemic contradictions (example (17)) by modelling epistemic modals via a state-based accessibility relation. But the distinction illustrated in (21) could not be explained. BiUS will remedy to this deficiency: it will capture the discourse dynamics of epistemic modals but also their FC potential, which was not addressed in [54]:

(22) Narrow scope epistemic FC
   a. Mr. X might be in Victoria *or* in Brixton. $\rightsquigarrow$ Mr. X might be in Victoria *and* he might be in Brixton.
   b. $\Diamond(\alpha \vee \beta) \rightsquigarrow \Diamond\alpha \wedge \Diamond\beta$

(23) Wide scope epistemic FC
   a. Mr. X might be in Victoria *or* he might be in Brixton. $\rightsquigarrow$ Mr. X might be in Victoria *and* might be in Brixton.
   b. $\Diamond\alpha \vee \Diamond\beta \rightsquigarrow \Diamond\alpha \wedge \Diamond\beta$

The FC inferences in (22)-(23) will be derived as neglect-zero effects as in BSML. The upshot of modelling neglect-zero in a dynamic setting is a notion of dynamic consequence which will make interesting predictions concerning possible divergences between everyday and logico-mathematical reasoning, as will be discussed in section 4.

3.1 Veltman's Update Semantics for epistemic modals

Veltman presented an Update Semantics (US) for a propositional language with an additional unary operator (here $\Diamond$) expressing epistemic MIGHT [54, 19]. The language is defined as follows precluding Boolean operations on the MIGHT-formulas.

Definition 6 (Syntax). *The languages L_{PL} and L_{US} are defined recursively as:*

$$\begin{aligned} L_{PL}: \ & \alpha := p \mid \neg\alpha \mid \alpha \vee \alpha \mid \alpha \wedge \alpha \\ L_{US}: \ & \phi := \alpha \mid \Diamond\alpha \end{aligned}$$

where $p \in PROP$, a countable set of propositional variables.

Models are pairs $M = \langle W, V\rangle$, where W is a set of possible worlds and V is a world-dependent valuation function. Information states, $s \subseteq W$, are sets of possible worlds. Formulas in L_{US} denote functions from states to states.

Definition 7 (Updates).

$$\begin{aligned} s[p] &= s \cap \{w \in W \mid V(p,w) = 1\} \\ s[\phi \wedge \psi] &= s[\phi] \cap s[\psi] \\ s[\phi \vee \psi] &= s[\phi] \cup s[\psi] \\ s[\neg\phi] &= s - s[\phi] \\ s[\Diamond\phi] &= s, \ \textit{if } s[\phi] \neq \emptyset; \\ &= \emptyset, \ \textit{otherwise} \end{aligned}$$

Support and logical consequence are defined in terms of the update function, as standard in dynamic semantics. A state s supports a formula ϕ, $s \models \phi$, if updating s with the formula does not lead to any change.

Definition 8 (Support).

$$s \models \phi \textit{ iff } s[\phi] = s$$

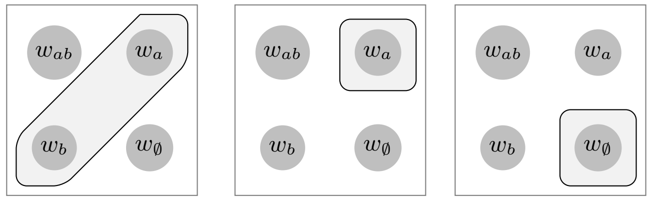
A formula is a consequence of a sequence of premisses, $\phi_1, \ldots, \phi_n \models \psi$ iff any state resulting from an update with the premisses, supports the conclusion.[6]

Definition 9 (Logical consequence).

$$\phi_1, \ldots, \phi_n \models \psi \textit{ iff for all } s : s[\phi_1]\ldots[\phi_n] \models \psi$$

In Figure 2 we give some illustrations for $PROP = \{a, b\}$. As in BSML, also in US the information state depicted in 2(b) is predicted to be a zero-model for $(a \vee b)$. It supports the formula by virtue of an empty witness set. Our goal here is to extend US with a notion of neglect-zero enrichments whose core effect is again to rule out such zero-models. As we saw in the previous section [3] defined pragmatic enrichment in terms of a conjunction with NE. This strategy however will not work in the context of an update semantics. A natural interpretation for an update with NE would be $s[\text{NE}] = s$, if $s \neq \emptyset$; $\emptyset$ (or undefined) otherwise. But

[6] It is worth mentioning that this is only one of the notions of logical consequence discussed in [54]. In fact, [54] eventually adopts a version which does not quantify over states.



(a) $\not\models a$; $\models a \vee b$ (b) $\models a$; $\models a \vee b$ (c) $\models \neg a$; $\models \neg(a \vee b)$

Fig. 2. Illustrations of support in Veltman's update semantics

the conjunction $\phi \wedge \text{NE}$, given Veltman's semantics for $\wedge$, would only rule out the empty state as a possible input for the formula. To model dynamic pragmatic enrichments we will instead introduce a complex expression ϕ^{NE}, where NE is interpreted as a **post-supposition**, i.e., a constraint that needs to be satisfied *after* the update with the relevant sentence:

(24) $s[\phi^{\text{NE}}] = s[\phi]$, if $s[\phi] \neq \emptyset$; undefined ($\#$) otherwise

Compare the notion of a post-supposed NE with the more familiar notion of a **presupposition** $\phi_{[\psi]}$ which must be satisfied *before* the update (in the local context), and with **Veltman's** MIGHT, $\Diamond\phi$, which expresses the same non-emptiness requirement as ϕ^{NE} but differs in the produced output state: s (rather than $s[\phi]$) if the requirement is satisfied; $\emptyset$ (rather than $\#$) otherwise:

- Post-supposed NE: $s[\phi^{\text{NE}}] = s[\phi]$, if $s[\phi] \neq \emptyset$; undefined ($\#$) otherwise
- Presupposition: $s[\phi_{[\psi]}] = s[\phi]$, if $s \models \psi$; undefined ($\#$) otherwise
- Veltman's MIGHT: $s[\Diamond\phi] = s$, if $s[\phi] \neq \emptyset$; $\emptyset$ otherwise

It is easy to see that modal disjunction (ignorance) and epistemic FC inferences are straightforwardly derived for enriched disjunctions defined in terms of post-supposed NE:

(25) $\alpha^{\text{NE}} \vee \beta^{\text{NE}} \models \Diamond\alpha \wedge \Diamond\beta$

(26) $\Diamond(\alpha^{\text{NE}} \vee \beta^{\text{NE}}) \models \Diamond\alpha \wedge \Diamond\beta$

But what about negation? Under negation (enriched) disjunction should behave classically:

(27) Mr X is not in A or B $\rightsquigarrow$ Mr X is not in A and he is not in B.

(28) Mr X cannot be in A or B $\rightsquigarrow$ Mr X cannot be in A and he cannot be in B.

Standard dynamic negation ($s[\neg\phi] = s - s[\phi]$) gives wrong results here. The formula in (29) would never be supported by any state. For example, it would be undefined in $\{w_\emptyset\}$, a state which would support $\neg(a \vee b)$:

(29) $\neg(a^{\text{NE}} \vee b^{\text{NE}})$

To fix this problem the update semantics we will introduce below adopts a bilateral notion of negation, as in BSML, defined in terms of a rejection update function $[\;]^r$ as in (30):

(30) $s[\neg\phi] = s[\phi]^r \;\&\; s[\neg\phi]^r = s[\phi]$

Let us have a closer look.

3.2 Bilateral Update Semantics (BiUS)

We work with the language of propositional modal logic extended with ϕ^{NE}, expressing a post-supposed requirement of non-emptiness.[7]

Definition 10 (Syntax). *The language L_{BiUS} is recursively defined as:*

$$\phi := p \mid \neg\phi \mid (\phi \vee \phi) \mid (\phi \wedge \phi) \mid \Diamond\phi \mid \phi^{\text{NE}}$$

with $p \in PROP$, a countable set of propositional variables.

Models, $M = \langle W, V\rangle$, and states, $s \subseteq W$, are defined as above. Formulas again denote functions from states to states. In Definition 11, only the clauses for post-supposed NE (clause 5) and negation (clause 6) are new. As explained above, an update with ϕ^{NE} returns the input state s updated with ϕ, if $s[\phi]$ is defined and different from $\emptyset$; undefined otherwise. Negation is defined in terms of a recursively defined rejection update function $[\;]^r$. Notice that in the rejection update for ϕ^{NE} the contribution of NE is trivialized (clause $5'$).

Definition 11 (Updates).

1. $s[p] = s \cap \{w \in W \mid V(p,w) = 1\}$
2. $s[\phi \wedge \psi] = s[\phi] \cap s[\psi]$
3. $s[\phi \vee \psi] = s[\phi] \cup s[\psi]$
4. $s[\Diamond\phi] = s$, *if* $s[\phi] \neq \emptyset$; $\emptyset$, *if* $s[\phi] = \emptyset$; *undefined (#) otherwise*
5. $s[\phi^{\text{NE}}] = s[\phi]$, *if* $s[\phi] \neq \emptyset$; *undefined (#) otherwise*
6. $s[\neg\phi] = s[\phi]^r$

where $[\phi]^r$ is recursively defined as follows:

1′ $s[p]^r = s \cap \{w \in W \mid V(p,w) = 0\}$

[7] The language of BiUS allows Boolean operations on $\Diamond$-formulas in contrast to Veltman's L_{US}, which precluded iteration and embedding of the $\Diamond$-operator. Because of this restriction, US validated idempotence ($s[\phi] = s[\phi][\phi]$) and monotonicity ($s \subseteq t$ implies $s[\phi] \subseteq t[\phi]$), which instead are not generally valid in BiUS. The adoption of a more liberal language is motivated by our linguistic goals. For example we want to explain wide scope free choice and the interpretation of *might* under negation. Some of our results however will depend on idempotence and monotonicity and, therefore, will only be valid for a fragment of the language.

2′ $s[\phi \wedge \psi]^r = s[\phi]^r \cup s[\psi]^r$
3′ $s[\phi \vee \psi]^r = s[\phi]^r \cap s[\psi]^r$
4′ $s[\Diamond\phi]^r = s$, *if* $s[\phi]^r = s$; $\emptyset$, *if* $s[\phi]^r \neq s$; *undefined (#) otherwise*
5′ $s[\phi^{\mathrm{NE}}]^r = s[\phi]^r$
6′ $s[\neg\phi]^r = s[\phi]$

and $s \neq x$ *means* s *is a state different from* x *(i.e., it excludes #) and* $x \cup y$ *and* $x \cap y$ *are defined only if both* x *and* y *are defined.*

Support is defined as above.

Definition 12 (Support).

$$s \models \phi \text{ iff } s[\phi] = s$$

A formula is a consequence of a sequence of premisses, $\phi_1, \ldots, \phi_n \models \psi$ iff any state resulting from an update with the premisses, *if defined*, supports the conclusion.

Definition 13 (Logical consequence).

$$\phi_1, \ldots, \phi_n \models \psi \text{ iff for all } s : s[\phi_1] \ldots [\phi_n] \text{ defined } \Rightarrow s[\phi_1] \ldots [\phi_n] \models \psi$$

At last we define neglect-zero enrichments in terms of ϕ^{NE}. Pragmatically enriching an NE-free formula α, $|\alpha|^+$, consists in adding the post-supposition of NE to any subformula of α.

Definition 14 (Dynamic pragmatic enrichment). *For* NE*-free* α, $|\alpha|^+$ *defined as follows:*

$$\begin{aligned} |p|^+ &= p^{\mathrm{NE}} \\ |\neg\alpha|^+ &= (\neg|\alpha|^+)^{\mathrm{NE}} \\ |\alpha \vee \beta|^+ &= (|\alpha|^+ \vee |\beta|^+)^{\mathrm{NE}} \\ |\alpha \wedge \beta|^+ &= (|\alpha|^+ \wedge |\beta|^+)^{\mathrm{NE}} \\ |\Diamond\alpha|^+ &= (\Diamond|\alpha|^+)^{\mathrm{NE}} \end{aligned}$$

For example, $|p \vee \neg q|^+ = (p^{\mathrm{NE}} \vee (\neg q^{\mathrm{NE}})^{\mathrm{NE}})^{\mathrm{NE}}$.

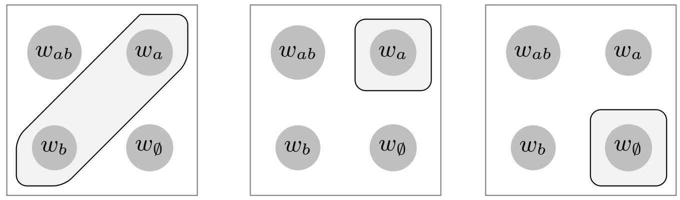
3.3 Results

It is easy to see that BiUS matches the predictions of [3] with respect to epistemic FC, while at the same time captures the discourse dynamic of epistemic MIGHT as in [54].

As in [3], we derive ignorance and FC inferences for pragmatically enriched formulas, while no undesirable side effects obtain under negation:[8][9]

[8] Proofs are in appendix. See also Figure 3 for illustrations.

[9] Notice that Modal disjunction and Negation 1 only hold for α and β of the restricted language L_{US}. Counterexamples in the unrestricted L_{BiUS} involve formulas which violate idempotence, such as epistemic contradictions. E.g., $|(p \wedge \Diamond\neg p) \vee p|^+ \not\models \Diamond(p \wedge \Diamond\neg p)$ (counterexample to Modal Disjunction), and $|\neg(\neg(p \wedge \Diamond\neg p) \vee \neg p)|^+ \not\models \neg\neg(p \wedge \Diamond\neg p)$ (counterexample to Negation 1).



(a) $\not\models a$; $\models |a \vee b|^+$ (b) $\models a$; $\not\models |a \vee b|^+$ (c) $\models \neg|a \vee b|^+$

Fig. 3. Illustrations of supporting states in BiUS

1. Modal disjunction (ignorance): $|\alpha \vee \beta|^+ \models \Diamond\alpha \wedge \Diamond\beta$ (if $\alpha, \beta \in L_{US}$)

(31) Mr. X is in Victoria or in Brixton.
$\rightsquigarrow$ Mr. X might be in Victoria and he might be in Brixton.

2. Narrow scope epistemic FC: $|\Diamond(\alpha \vee \beta)|^+ \models \Diamond\alpha \wedge \Diamond\beta$

(32) Mr. X might be in Victoria or in Brixton.
$\rightsquigarrow$ Mr. X might be in Victoria and he might be in Brixton.

3. Wide scope epistemic FC: $|\Diamond\alpha \vee \Diamond\beta|^+ \models \Diamond\alpha \wedge \Diamond\beta$

(33) Either Mr. X might be in Victoria or he might be in Brixton.
$\rightsquigarrow$ Mr. X might be in Victoria and he might be in Brixton.

4. Negation 1: $|\neg(\alpha \vee \beta)|^+ \models \neg\alpha \wedge \neg\beta$ (if $\alpha, \beta \in L_{US}$)

(34) Mr X is not in Victoria or in Brixton
$\rightsquigarrow$ Mr X is not in Victoria and he is not in Brixton.

5. Negation 2: $|\neg\Diamond(\alpha \vee \beta)|^+ \models \neg\Diamond\alpha \wedge \neg\Diamond\beta$

(35) Mr X cannot be in Victoria or in Brixton
$\rightsquigarrow$ Mr X cannot be in Victoria and he cannot be in Brixton.

We also match the predictions of [54] with respect to Veltman's sequences, deriving a difference between $\Diamond\alpha, \neg\alpha \not\models \bot$ and $\neg\alpha; \Diamond\alpha$ with the latter leading to contradiction and the former, instead, being consistent (update does not necessarily lead to the state of absurdity), although incoherent (no non-empty supporting state), exactly as in [54]:

(36) Veltman's sequences

a. Maybe this is Frank Veltman's example. It isn't his example!
$\Diamond\alpha, \neg\alpha \not\models \bot$

b. ?This is not Frank Veltman's example! Maybe it's his example.
$\neg\alpha, \Diamond\alpha \models \bot$

Note instead that the ordering of the conjuncts does not matter in the following versions of epistemic contradictions, which were not expressible in Veltman's original system and which are here both predicted to be incoherent:

(37) Epistemic contradictions
  a. ?It might be raining but it isn't raining.
     $\Diamond\alpha \wedge \neg\alpha \not\models \bot$, but incoherent
  b. ?It isn't raining but it might be raining.
     $\neg\alpha \wedge \Diamond\alpha \not\models \bot$, but incoherent

In fact the ϕ^{NE}-free fragment of BiUS is equivalent to US if we preclude iterations and embedding of the *might*-operator.

$$\alpha_1, \dots, \alpha_n \models_{BiUS} \beta \text{ iff } \alpha_1, \dots, \alpha_n \models_{US} \beta \qquad [\text{if } \alpha_1, ..., \alpha_n, \beta \in L_{US}]$$

And the non-modal and ϕ^{NE}-free fragment of BiUS is equivalent to classical logic:

$$\alpha_1, \dots, \alpha_n \models_{BiUS} \beta \text{ iff } \alpha_1, \dots, \alpha_n \models_{CL} \beta \qquad [\text{if } \alpha_1, \dots, \alpha_n, \beta \in L_{PL}]$$

The latter fact means that in BiUS we can isolate neglect-zero effects just like in BSML.

In contrast to other standard dynamic systems, however, BiUS validates double negation elimination (also for non-eliminative ϕ):

- Double Negation Elimination: $\neg\neg\phi \equiv \phi$[10]

Thus BiUS also validates double negation FC (see example (14) and footnote 2 for motivation):

- Double Negation FC: $|\neg\neg\Diamond(\alpha \vee \beta)|^+ \models \Diamond\alpha \wedge \Diamond\beta$

Moreover, when applied to dynamic systems for anaphora (e.g., [29]) bilateral negation can give us a treatment of Partee's bathroom example [37]:

(38) a. Either there is no bathroom in this house or it's in a funny place.
  b. $\neg\exists x Px \vee Qx$

In [29], example (38) is problematic because the last occurrence of x in (38-b) is not bound by $\exists x$, since dynamic negation neutralises the dynamic potential of existential quantifiers in its scope.

Let us assume an account of the existential quantifier, conjunction and disjunction as in [29] in combination with a bilateral notion of negation as in BiUS:

- $s[\exists x\phi] = \bigcup_{d\in D}(s[x/d][\phi])$
- $s[\phi \wedge \psi] = s[\phi][\psi]$
- $s[\phi \vee \psi] = \{i \in s \mid i \text{ survives in } (s[\phi] \cup s[\neg\phi][\psi])\}$

[10] *Proof:* $s[\neg\neg\phi] = s[\neg\phi]^r = s[\phi]$.

– $s[\neg\phi] = s[\phi]^r$

Then no matter what rejection clause one assumes for $\exists x$, the last occurrence of x in (38-b) would be bound by $\exists x$, as illustrated by the coloured parts in (39):

(39) $s[\neg\exists xPx] \cup s[\neg\neg\exists xPx][Qx] = s[\neg\exists xPx] \cup s[\neg\exists xPx]^r[Qx] = s[\neg\exists xPx] \cup s[\exists xPx][Qx]$

The full development of a quantified version of BiUS which can capture besides FC and related inferences also cross-sentential and donkey anaphora and their interactions with modality [29, 1] must be left to future work.

4 Everyday vs logico-mathematical reasoning

People often reason contrary to the prescriptions of classical logic. One hypothesis arising from this research is that at least in part the divergence between everyday and logico-mathematical reasoning might be due to a neglect-zero tendency. While zero-models tend to be neglected in conversation, they play a crucial role in logico-mathematical reasoning.

According to our hypothesis there are three kinds of reasonings [let $\alpha_1, ...\alpha_n, \beta$ range over NE-free formulas]:

1. *Zero-free reasonings:* classically valid reasonings which do not involve zero-models:
$$\alpha_1, ..., \alpha_n \models \beta \; \& \; |\alpha_1|^+, ..., |\alpha_n|^+ \models |\beta|^+$$
2. *Neglect-zero fallacies:* classically invalid reasonings which are valid if we neglect zero-models, e.g., ignorance and FC inferences:
$$\alpha_1, ..., \alpha_n \not\models \beta \; \& \; |\alpha_1|^+, ..., |\alpha_n|^+ \models |\beta|^+$$
3. *Zero-reasonings:* classically valid reasonings which rely on zero-models:
$$\alpha_1, ..., \alpha_n \models \beta \; \& \; |\alpha_1|^+, ..., |\alpha_n|^+ \not\models |\beta|^+$$

The hypothesis that zero-models are cognitively taxing leads to various predictions. For example, zero-reasonings should be harder for non-logically trained reasoners than zero-free reasonings. In what follows we discuss two examples illustrating these predictions.

Consider first **Disjunction Introduction**:

(40) A. THEREFORE, A OR B.

A rule-based theory which assumes that human reasoners apply the rules of Natural Deduction would predict that if asked to formulate conclusions from premise A reasoners should mention A OR B. Past experiments however showed that people who are not trained in logic do not spontaneously produce the disjunction [34]. Classical model-based theories of reasoning which link the difficulty of a reasoning solely to the amount of models involved in the reasoning process also fail to account for this fact [34, 43]. In these theories, the premise leads to the construction of a model validating A. But, classically, any verifier of A is also a verifier of A OR B and so by employing a single model the conclusion A OR B should in principle be available to the everyday reasoner. Our neglect-zero hypothesis, instead, has a ready explanation of why this is not the case. A minimal verifier of A is also a verifier for A OR B but only if we allow the possibility of an empty witness for the second disjunct. Since a zero-model is involved we correctly predict that the inference is not spontaneously drawn. Disjunction introduction is indeed an example of a *zero-reasoning*, classically valid but relying on zero-models:

– $\alpha \models \alpha \vee \beta$, but $|\alpha|^+ \not\models |\alpha \vee \beta|^+$

Consider now the following two versions of **Disjunctive Syllogism** in which the ordering of the premises is reversed:

(41) A OR B; NOT A. THEREFORE, B.

(42) NOT A; A OR B. THEREFORE, B.

Both reasonings are classically valid:

– $\alpha \vee \beta, \neg\alpha \models \beta$
– $\neg\alpha, \alpha \vee \beta \models \beta$

But only (42) involves a zero-model. Any state resulting from an update with $\neg\alpha$, is a zero-model for the disjunction $\alpha \vee \beta$. This means that the sequence of updates $s[|\neg\alpha|^+][|\alpha \vee \beta|^+]$ is never defined, no matter what s is, leading to explosion:

– $|\neg\alpha|^+, |\alpha \vee \beta|^+ \models \bot$

If we reverse the ordering of the premises, with the disjunction first as in (41), the update is instead unproblematic and the output state supports the conclusion:

– $|\alpha \vee \beta|^+, |\neg\alpha|^+ \models |\beta|^+$ (and $\not\models \bot$)

(42) is then predicted to be harder than (41). We leave to future work the experimental testing of this prediction. Let me stress that this last prediction relies on the *dynamic* notion of logical consequence defined in Definition 13 and, therefore, if experimentally confirmed, would constitute independent motivation for the implementation of neglect-zero effects in a dynamic setting.

5 Conclusion

We presented an update semantics for epistemic modals capturing both their discourse dynamics and their potential to give rise to FC inferences. The latter were derived as neglect-zero effects as in [3]. In future work we intend to extend the system to the first order case and to further study and experimentally test its predictions on the impact of neglect-zero on reasoning and interpretation.

Acknowledgements I would like to thank two anonymous reviewers for their insightful comments which led to substantial improvements. I am also grateful to Marco Degano for discussion and to Bo Flachs for his help with some of the proofs.

Appendix

BiUS, US and PL

Theorem 1. *$\alpha_1, \dots, \alpha_n \models_{BiUS} \beta$ iff $\alpha_1, \dots, \alpha_n \models_{US} \beta$ [if $\alpha_1, ..., \alpha_n, \beta \in L_{US}$]*

Proof. We only need to check the case of negation, i.e. show that $s[\gamma]^r = s - s[\gamma]$ for all s and $\gamma \in L_{PL}$ (recall that $\Diamond$ cannot appear in the scope of negation in L_{US}). We prove this by induction on the complexity of γ.

(i) $s[p]^r = s \cap \{w \in W \mid V(p,w) = 0\} = s - \{w \in W \mid V(p,w) = 1\} = s - W[p] = s - s[p]$
(ii) $s[\alpha \wedge \beta]^r = s[\alpha]^r \cup s[\beta]^r =_{IH} (s - s[\alpha) \cup (s - s[\beta]) = s - (s[\alpha] \cap s[\beta]) = s - s[\alpha \wedge \beta]$
(iii) $s[\alpha \vee \beta]^r = s[\alpha]^r \cap s[\beta]^r =_{IH} (s - s[\alpha]) \cap (s - s[\beta]) = s - (s[\alpha \cup s[\beta) = s - s[\alpha \vee \beta]$
(iv) $s[\neg\alpha]^r = s[\alpha]$. Since $s[\alpha] \subseteq s$ by eliminativity, $s[\alpha] = s - (s - s[\alpha]) =_{IH} s - s[\alpha]^r = s - s[\neg\alpha]$.

Theorem 2. *$\alpha_1, \dots, \alpha_n \models_{BiUS} \beta$ iff $\alpha_1, \dots, \alpha_n \models_{PL} \beta$ [if $\alpha_1, \dots, \alpha_n, \beta \in L_{PL}$]*

Proof. This follows from the fact that in $BiUS$ (just like in US, see [54], page 231), all $\alpha \in L_{PL}$ are such that for any s, $s[\alpha] = s \cap W[\alpha]$.

Ignorance and free choice

The proofs of the facts below use the following lemmas.

Lemma 1. *For $\alpha \in L_{BiUS}$ and NE-free, and any state s.*

(i) If $s[|\alpha|^+]$ is defined, then $s[|\alpha|^+] = s[\alpha]$
(ii) If $s[|\alpha|^+]^r$ is defined, then $s[|\alpha|^+]^r = s[\alpha]^r$

Proof. By an easy double induction on the complexity of α.

Lemma 2. *For $\alpha \in L_{US}$ and any state s.*

(i) Idempotence: $s[\alpha] = s[\alpha][\alpha]$ and $s[\neg\alpha] = s[\neg\alpha][\neg\alpha]$
(ii) Monotonicity: $s \subseteq t$ implies $s[\alpha] \subseteq t[\alpha]$
(iii) Downward closure of $\neg\alpha$: $s \subseteq t$ implies $t[\neg\alpha] = t \Rightarrow s[\neg\alpha] = s$.

Proof. These properties are consequences of the following two facts: (a) in L_{US} all *might*-formulas have the form $\Diamond\alpha$, where α is $\Diamond$-free; (b) all $\Diamond$-free α (i.e., $\alpha \in L_{PL}$) are such that for all s, $s[\alpha] = s \cap W[\alpha]$.

Lemma 3 (Eliminativity). *For $\phi \in L_{BiUS}$ and any state s.*

If $s[\phi]^{(r)}$ is defined, then $s[\phi]^{(r)} \subseteq s$

Fact 1 (Modal Disjunction) $|\alpha \vee \beta|^+ \models \Diamond\alpha \wedge \Diamond\beta$ *(if $\alpha, \beta \in L_{US}$)*

Proof. Suppose $s[|\alpha \vee \beta|^+]$ is defined. Then $s[|\alpha \vee \beta|^+] = s[|\alpha|^+] \cup s[|\beta|^+]$ with both $s[|\alpha|^+]$ and $s[|\beta|^+]$ defined and $\neq \emptyset$. By Lemma 1 we have $s[|\alpha|^+] = s[\alpha] \neq \emptyset$. From $s[|\alpha|^+] \subseteq s[|\alpha \vee \beta|^+]$ it follows $s[\alpha] \subseteq s[|\alpha \vee \beta|^+]$. By monotonicity of α (Lemma 2) we conclude $s[\alpha][\alpha] \subseteq s[|\alpha \vee \beta|^+][\alpha]$. Since $s[\alpha][\alpha] = s[\alpha] \neq \emptyset$ (by idempotence of α), we conclude $s[|\alpha \vee \beta|^+][\alpha] \neq \emptyset$. But then $s[|\alpha \vee \beta|^+] \models \Diamond\alpha$. Similarly for $\Diamond\beta$.

For a counterexample to Modal Disjunction with $\alpha \notin L_{US}$, let α be $(p \wedge \Diamond\neg p)$. Then $\{w_p, w_\emptyset\}[|(p \wedge \Diamond\neg p) \vee p|^+] = \{w_p\}$ is defined but does not support $\Diamond(p \wedge \Diamond\neg p)$. Thus $|(p \wedge \Diamond\neg p) \vee p|^+ \not\models \Diamond(p \wedge \Diamond\neg p)$.

Fact 2 (Narrow Scope FC) $|\Diamond(\alpha \vee \beta)|^+ \models \Diamond\alpha \wedge \Diamond\beta$

Proof. Suppose $s[|\Diamond(\alpha \vee \beta)|^+]$ is defined. Then $s[|\Diamond(\alpha \vee \beta)|^+] = s[\Diamond|(\alpha \vee \beta)|^+] = s \neq \emptyset$ and $s[|\alpha \vee \beta|^+] = s[|\alpha|^+] \cup s[|\beta|^+] \neq \emptyset$. It follows that $s[|\alpha|^+] \neq \emptyset \neq s[|\beta|^+]$. By Lemma 1 we conclude $s[\alpha] \neq \emptyset$. Hence $s[|\Diamond(\alpha \vee \beta)|^+][\Diamond\alpha] = s$ and thus $s[|\Diamond(\alpha \vee \beta)|^+] \models \Diamond\alpha$. Similarly for $\Diamond\beta$.

Fact 3 (Wide Scope FC) $|\Diamond\alpha \vee \Diamond\beta|^+ \models \Diamond\alpha \wedge \Diamond\beta$

Proof. Suppose $s[|\Diamond\alpha \vee \Diamond\beta|^+]$ is defined. Then $s[|\Diamond\alpha \vee \Diamond\beta|^+] = s[|\Diamond\alpha|^+ \vee |\Diamond\beta|^+] = s[|\Diamond\alpha|^+] \cup s[|\Diamond\beta|^+] = s \neq \emptyset$. Hence both $s[|\Diamond\alpha|^+]$ and $s[|\Diamond\beta|^+]$ are defined which means $s[\Diamond|\alpha|^+] = s[\Diamond|\beta|^+] = s \neq \emptyset$. It follows that $s[|\alpha|^+] \neq \emptyset \neq s[|\beta|^+]$. By Lemma 1 we conclude $s[\alpha] \neq \emptyset$. Hence $s[|\Diamond\alpha \vee \Diamond\beta|^+][\Diamond\alpha] = s$ and thus $s[|\Diamond\alpha \vee \Diamond\beta|^+] \models \Diamond\alpha$. Similarly for $\Diamond\beta$.

Fact 4 (Negation 1) $|\neg(\alpha \vee \beta)|^+ \models \neg\alpha \wedge \neg\beta$ *(if $\alpha, \beta \in L_{US}$)*

Proof. Suppose $s[|\neg(\alpha \vee \beta)|^+]$ is defined. Then $s[|\neg(\alpha \vee \beta)|^+] = s[|\alpha|^+ \vee |\beta|^+]^r = s[|\alpha|^+]^r \cap s[|\beta|^+]^r$. By Lemma 1 we have $s[|\alpha|^+]^r = s[\alpha]^r = s[\neg\alpha] \neq \emptyset$. From $s[|\neg(\alpha \vee \beta)|^+] \subseteq s[|\alpha|^+]^r$ we have then $s[|\neg(\alpha \vee \beta)|^+] \subseteq s[\neg\alpha]$. By idempotence, $s[\neg\alpha] = s[\neg\alpha][\neg\alpha]$, and by downword closure, $s[|\neg(\alpha \vee \beta)|^+] = s[|\neg(\alpha \vee \beta)|^+][\neg\alpha]$. Hence $s[|\neg(\alpha \vee \beta)|^+] \models \neg\alpha$. Similarly for $\neg\beta$.

Fact 5 (Negation 2) $|\neg\Diamond(\alpha \vee \beta)|^+ \models \neg\Diamond\alpha \wedge \neg\Diamond\beta$

Proof. Suppose $s[|\neg\Diamond(\alpha \vee \beta)|^+]$ is defined. Then $s[|\neg\Diamond(\alpha \vee \beta)|^+] = s[\Diamond|(\alpha \vee \beta)|^+]^r \neq \emptyset$. This means that $s[\Diamond|(\alpha \vee \beta)|^+]^r = s$ and so also $s[|(\alpha \vee \beta)|^+]^r = s[|\alpha|^+ \vee |\beta|^+]^r = s[|\alpha|^+]^r \cap s[|\beta|^+]^r = s$. By Lemma 1, $s[|\alpha|^+]^r = s[\alpha]^r$ and so $s \subseteq s[\alpha]^r$. By eliminativity, $s[\alpha]^r = s$ and so $s[\Diamond\alpha]^r = s$ and $s[\neg\Diamond\alpha] = s$. Hence $s[|\neg\Diamond(\alpha \vee \beta)|^+] \models \neg\Diamond\alpha$. Similarly for $\neg\Diamond\beta$.

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