# Tutorial (week 12)

All the answers can be given with pseudocode or with Python. Remember that most of the time, many solutions are possible.

1. Characterize each of the following recurrence equations using the master theorem (assuming that $T(n) = c$ for $n < d$, for constants $c > 0$ and $d \geq 1$).

   a) $T(n) = 2T(n/2) + \log n$
   b) $T(n) = 8T(n/2) + n^2$
   c) $T(n) = 16T(n/2) + (n \log n)^4$
   d) $T(n) = 7T(n/3) + n$
   e) $T(n) = 9T(n/3) + n^3 \log n$

   **Solution:**

   a) (case 1) we have $a = 2$, $b = 2$ and $f(n) = \log n$.
   Thus, $T(n)$ is $O(n)$ since $\log_b(a) = 1$ and $f(n)$ is $O(n^{1-\epsilon})$.
   b) (case 1) we have $a = 8$, $b = 2$ and $f(n) = n^2$.
   Thus, $T(n)$ is $O(n^3)$ since $\log_b(a) = 3$ and $f(n)$ is $O(n^{3-\epsilon})$.
   c) (case 2) we have $a = 16$, $b = 2$ and $f(n) = (n \log n)^4$.
   Thus, $T(n)$ is $O(n^4 \log^5 n)$ since $\log_b(a) = 4$ and $f(n)$ is $O(n^4 \log^k n)$ for $k = 4$.
   d) (case 1) we have $a = 7$, $b = 3$ and $f(n) = n$.
   Thus, $T(n)$ is $O(n^{\log^3 7})$ since $\log_b(a) \simeq 1.77$ and $f(n)$ is $O(n^{1.77-\epsilon})$.
   e) (case 3) we have $a = 9$, $b = 3$ and $f(n) = n^3 \log n$.
   Thus, $T(n)$ is $O(n^3 \log n)$ since $\log_b(a) = 2$ and $f(n)$ is $\Omega(n^{2+\epsilon})$.

2. Use the divide-and-conquer algorithm, to compute $J \cdot I$, where $J = \texttt{10110011}$ and $I = \texttt{10111010}$ in binary. Show your work.

   **Solution:**
   First, note that $J = \texttt{10110011}$ is $1 + 2 + 16 + 32 + 128 = 179$, and that $I = \texttt{10111010}$ is $2 + 8 + 16 + 32 + 128 = 186$, thus we should obtain $J \cdot I = 33294 = \texttt{1000001000001110}$.

   In order to compute the multiplication of two integers, we divide them into two parts (lower bits and higher bits): $I_h = \texttt{1011}$, $I_l = \texttt{1010}$, $J_h = \texttt{1011}$, $J_l = \texttt{0011}$.

   We then use the following formula (with $n = 8$):

   $$I \cdot J = I_h J_h 2^n + [(I_h - I_l) \cdot (J_l - J_h) + I_h J_h + I_l J_l] 2^{n/2} + I_l J_l.$$

   so that we have only three $n/2$-bit multiplications to perform: $I_h \cdot J_h$, $I_l \cdot J_l$ and $(I_h - I_l) \cdot (J_l - J_h)$.

   $$I_h \cdot J_h = \texttt{1011} \cdot \texttt{1011} = \texttt{1111001} = 121$$
   $$I_l \cdot J_l = \texttt{1010} \cdot \texttt{0011} = \texttt{11110} = 30$$
   $$(I_h - I_l) \cdot (J_l - J_h) = \texttt{0001} \cdot -\texttt{1000} = 1 \cdot -8 = -8$$

and one can indeed verify that

$$121 \cdot 2^8 + (-8 + 121 + 30) \cdot 2^4 + 30 = 33294$$

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Computation of $\texttt{1011} \cdot \texttt{1011}$: we have $I_h = \texttt{10}$, $I_l = \texttt{11}$, $J_h = \texttt{10}$, $J_l = \texttt{11}$, with:

$$\begin{aligned} I_h \cdot J_h &= \texttt{10} \cdot \texttt{10} = \texttt{100} = 4 \\ I_l \cdot J_l &= \texttt{11} \cdot \texttt{11} = \texttt{1001} = 9 \\ (I_h - I_l) \cdot (J_l - J_h) &= -\texttt{1} \cdot \texttt{1} = -1 \cdot 1 = 1 \end{aligned}$$

which leads to $\texttt{1011} \cdot \texttt{1011} = 4 \cdot 2^4 + (-1 + 4 + 9) \cdot 2^2 + 9 = 121$.

Computation of $\texttt{1010} \cdot \texttt{0011}$: we have $I_h = \texttt{10}$, $I_l = \texttt{10}$, $J_h = \texttt{00}$, $J_l = \texttt{11}$, with:

$$\begin{aligned} I_h \cdot J_h &= \texttt{10} \cdot \texttt{00} = \texttt{0} = 0 \\ I_l \cdot J_l &= \texttt{10} \cdot \texttt{11} = \texttt{110} = 6 \\ (I_h - I_l) \cdot (J_l - J_h) &= \texttt{0} \cdot \texttt{11} = 0 \cdot 3 = 0 \end{aligned}$$

which leads to $\texttt{1010} \cdot \texttt{0011} = 0 \cdot 2^4 + (0 + 0 + 6) \cdot 2^2 + 6 = 30$.

Computation of $\texttt{0001} \cdot \texttt{1000}$: we have $I_h = \texttt{00}$, $I_l = \texttt{01}$, $J_h = \texttt{10}$, $J_l = \texttt{00}$, with:

$$\begin{aligned} I_h \cdot J_h &= \texttt{00} \cdot \texttt{10} = \texttt{0} = 0 \\ I_l \cdot J_l &= \texttt{01} \cdot \texttt{00} = \texttt{0} = 0 \\ (I_h - I_l) \cdot (J_l - J_h) &= -\texttt{1} \cdot -\texttt{10} = -1 \cdot -2 = 2 \end{aligned}$$

which leads to $\texttt{0001} \cdot \texttt{1000} = 0 \cdot 2^4 + (2 + 0 + 0) \cdot 2^2 + 0 = 8s$.

3. Use Strassen's matrix multiplication algorithm to multiply the matrices

$$X = \begin{pmatrix} 3 & 2 \\ 4 & 8 \end{pmatrix} \qquad Y = \begin{pmatrix} 1 & 5 \\ 9 & 6 \end{pmatrix}$$

**Solution:**
Using Strassen's matrix multiplication, we have

$$\begin{aligned} A = 3, \quad B = 2, \quad C = 4, \quad D = 8 \\ E = 1, \quad F = 5, \quad G = 9, \quad H = 6 \end{aligned}$$

We then compute the terms $S_i$:

$$\begin{aligned} S_1 &= A(F - H) = -3, \\ S_2 &= (A + B)H = 30, \\ S_3 &= (C + D)E = 12, \\ S_4 &= D(G - E) = 64, \end{aligned} \qquad \begin{aligned} S_5 &= (A + D)(E + H) = 77 \\ S_6 &= (B - D)(G + H) = -90 \\ S_7 &= (A - C)(E + F) = -6 \end{aligned}$$

Given these 7 matrices $S_i$, we can compute:

$$\begin{aligned} I &= S_5 + S_6 + S_4 - S_2 = 77 - 90 + 64 - 30 = 21 \\ J &= S_1 + S_2 = -3 + 30 = 27 \\ K &= S_3 + S_4 = 12 + 64 = 76 \\ L &= S_1 - S_7 - S_3 + S_5 = -3 + 6 - 12 + 77 = 68, \end{aligned}$$

4. A complex number $a + bi$, where $i = \sqrt{-1}$, can be represented by the pair $(a, b)$. Describe a method performing only three real-number multiplications to compute the pair $(e, f)$ representing the product of $a + bi$ and $c + di$.

**Solution:**
With a naive decomposition, we have that $(a + bi) \cdot (c + di) = ac - bd + (ad + bc)i$, which requires the computation of 4 multiplications: $ac$, $bd$, $ad$ and $bc$.

The goal is thus to find a different decomposition that would allow us to save at least one multiplication. First, note that

$$(a - b) \cdot (c - d) = ac - ad - bc + bd = ac + bd - (ad + bc)$$

which gives us:

$$(ad + bc) = ac + bd - (a - b) \cdot (c - d)$$

Therefore, we can write:

$$(a + bi) \cdot (c + di) = ac - bd + (ad + bc)i = ac - bd + [ac + bd - (a - b) \cdot (c - d)]i$$

which requires only 3 multiplications: $ac$, $bd$, and $(a - b) \cdot (c - d)$.

5. What is the maximal set from the following set of points:

$$\{(7, 2), (3, 1), (9, 3), (4, 5), (1, 4), (6, 9), (2, 6), (5, 7), (8, 6)\}$$

**Solution:**
Applying the divide-and-conquer approach, we first divide in two sets (the median being the point $(5, 7)$):

$$\begin{aligned} L &= \{(3, 1), (4, 5), (1, 4), (2, 6)\} \\ G &= \{(7, 2), (9, 3), (6, 9), (5, 7), (8, 6)\} \end{aligned}$$

and we find recursively (again using divide-and-conquer approach) the maximal set from these two sets:

$$\begin{aligned} \text{for } L{:}&\{(2, 6), (4, 5)\} \\ \text{for } G{:}&\{(6, 9), (8, 6), (9, 3)\} \end{aligned}$$

Then, we take the leftmost point of the maximal set of $G$, which is $X = (6, 9)$, and we remove from the maximal set of $L$ all the points that are dominated by $X$. We can see that both $(2, 6)$ and $(4, 5)$ are dominated by $X$. Thus, only $\{(6, 9), (8, 6), (9, 3)\}$ remains as maximal set.

6. Give an example of a set of $n$ points in the plane such that every point is a maximum point, that is, no point in this set is dominated by any other point in this set.

   **Solution:**
   By using the following set of points $\{(1, n), (2, n-1), \ldots, (i, n-i+1), \ldots, (n, 1)\}$, we can see that no point is dominated by any other point in the set.

   Indeed, if we take the list of points as an ordered list, we can see that for a point $I = (i, n-i+1)$, all the points preceding $I$ in the list will have a smaller $x$ coordinate than $I$ and a greater $y$ coordinate than $I$. Conversely, all the points after $I$ in the list will have a greater $x$ coordinate than $I$ and a smaller $y$ coordinate than $I$.

7. Consider the recurrence equation, $T(n) = 2T(n-1) + 1$, for $n > 1$, where $T(n) = 1$ for $n = 1$. Prove that $T(n)$ is $O(2^n)$.

   **Solution:**
   TODO

$$\begin{aligned} T(n) &= 2T(n-1) + 1 \\ &= 2(2T(n-2) + 1) + 1 = 2^2 T(n-2) + 1 + 2 \\ &= 2^2(2T(n-3) + 1) + 1 + 2 = 2^3 T(n-3) + 1 + 2 + 2^2 \\ &= \cdots \\ &= 2^i T(n-i) + \sum_{j=0}^{i-1} 2^j \end{aligned}$$

   We remark that $T(n-i)$ is the base case when $i = n-1$. Thus, by choosing $i = n-1$, we have:

$$T(n) = 2^{n-1} + \sum_{j=0}^{n-2} 2^j = 2^{n-1} + 2^{n-1} - 1 = 2^n - 1.$$

   Now, we assume that $T(n) = 2^n - 1$. This is true for $n = 1$ as $T(1) = 1$. Then

$$T(n+1) = 2T(n) + 1 = 2(2^n - 1) = 2^{n+1} - 1$$

   which proves that $T(n) = 2^n - 1$ and thus $T(n)$ is $O(2^n)$.

8. A very common problem in computer graphics is to approximate a complex shape with a bounding box. For a set, $S$, of $n$ points in 2−dimensional space, the idea is to find the smallest rectangle, $R$, with sides parallel to the coordinate axes that contains all the points in $S$. Once $S$ is approximated by such a bounding box, we can often speed up lots of computations that involve $S$. For example, if $R$ is completely obscured some object in the foreground, then we don't need to render any of $S$. Likewise, if we shoot a virtual ray and it completely misses $R$, then it is guarantee to completely miss $S$. So doing comparisons with $R$ instead of $S$ can often save time. But this savings is wasted if we spend a lot of time constructing $R$; hence, it would be ideal to have a fast way of computing a bounding box, $R$, for a set, $S$, of $n$ points in the plane. Note that the construction of $R$ can be reduced to two instances of the problem of simultaneously finding the minimum and the maximum in a set of $n$ numbers; namely, we need only do this for the x-coordinates in $S$ and then for the y-coordinates in

$S$. Therefore, design a divide-and-conquer algorithm for finding both the minimum and the maximum element of $n$ numbers using no more than $3n/2$ comparisons.

**Solution:**
Divide the set of numbers into $n/2$ groups of two elements each. With $n/2$ comparisons we will determine a minimum and a maximum in each group (by simply comparing the two elements in each group). This allows use to separate the maximums and minimums for each group. Then, we find the minimum of the minimums and the maximum of the maximums using a standard algorithm that scans through all value to find the minimum/maximum. These additional scans use $n/2$ comparisons each ($n/2$ for maximums and $n/2$ for minimums).

## Hints

- **Question 5: Recall when a point is dominated by another.**
- **Question 6: Review the concept of when one point dominates another.**
- **Question 7: Use a good induction hypothesis, such as $T(n) = 2^n - 1$.**
- **Question 8: Think about what happens in the base case.**