

## Rare events probability estimation

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MAISON DE LA SIMULATION

# Outline

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- ▣ Problem definition
- ▣ Brute-force estimation using Monte Carlo sampling
- ▣ Importance sampling
- ▣ Isoprobabilistic transformation
- ▣ Most-probable-failure-point(s)-based methods

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# Problem definition

## Given

- a *random vector* with known probability distribution:

$$\mathbf{X} \sim F_{\mathbf{X}}$$

modelling the uncertainty attached to a component of interest.

- and a *performance model* that characterize its state :

$$g: \mathbf{x} \mapsto g(\mathbf{x})$$

with the convention :

- if  $g(\mathbf{x}) \leq 0$ , then the system *fails* ;
- otherwise, it is *safe*.

## Objective

Quantify the component safety level  
in the form of a *(subjective) failure probability*.

« *subjective probability* » = A probability that is conditioned by *assumptions/choices*  
(probabilistic + performance models)

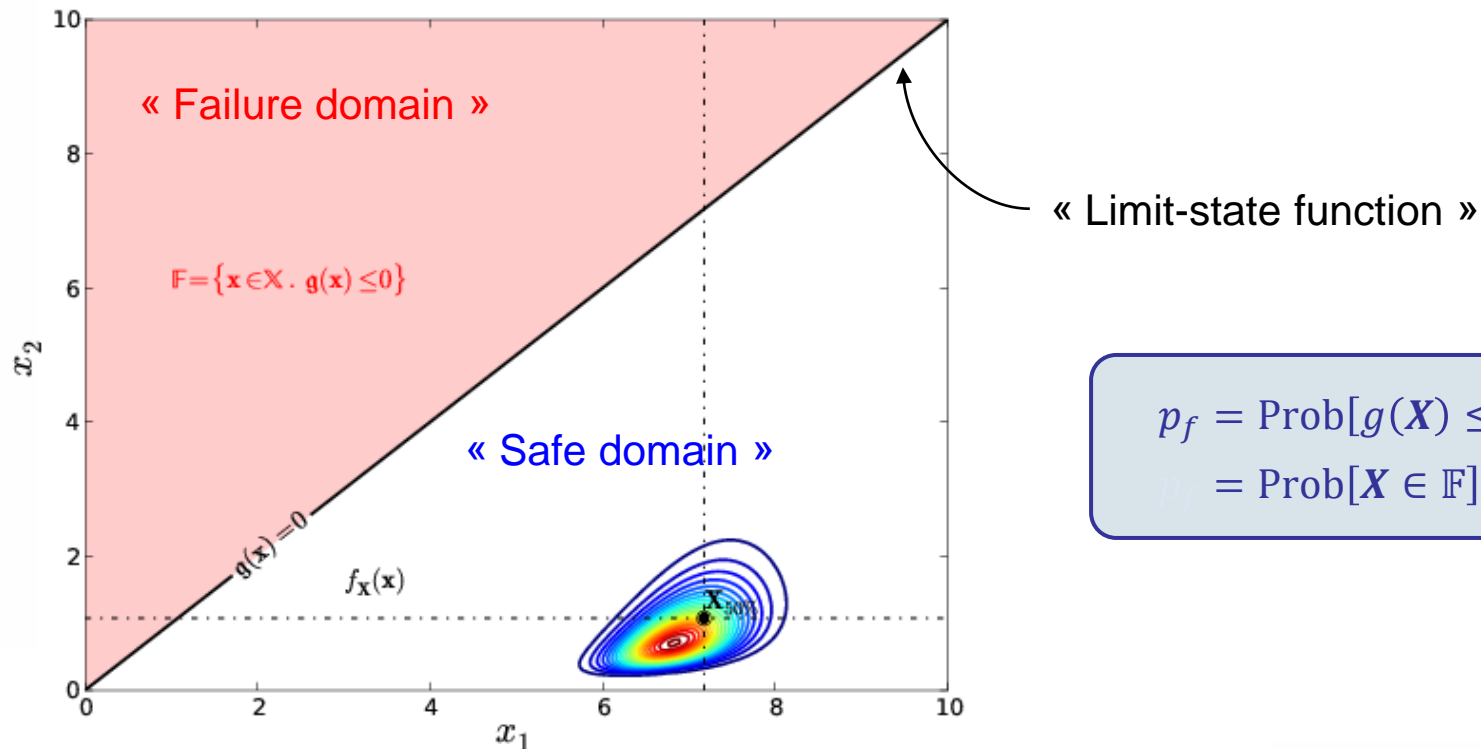
# Problem definition

## Input

- Ex : Consider a simple *capacity vs demand* example:

$$g(r, s) = r - s$$

with  $R \sim \mathcal{LN}(\lambda_R, \zeta_R)$  and  $S \sim \mathcal{LN}(\lambda_S, \zeta_S)$  and composed with a Normal copula whose shape matrix only term is  $\rho_0 = 0,525$ .



$$p_f = \text{Prob}[g(X) \leq 0] \\ = \text{Prob}[X \in \mathbb{F}]$$

# Problem definition

## ☐ Definitions for the failure probability

- The failure probability is essentially defined as the *value of the CDF of the safety margin  $G \equiv g(X)$  at point 0*:

$$p_f = F_G(0) = \int_{-\infty}^0 f_G(t) dt$$

but  $G$ 's distribution is rarely *known*.

- It also rewrites as the sum of  $X$ 's PDF *over the failure domain  $\mathbb{F}$* :

$$p_f = \int_{\mathbb{F}=\{x \in \mathbb{X}: g(x) \leq 0\}} f_X(x) dx$$

- It eventually rewrites as the expectation of the *failure indicator function  $\mathbb{I}_{\mathbb{F}}$*  over the support  $\mathbb{X}$  of the input probability distribution:

$$p_f = \int_{\mathbb{X}} \mathbb{I}_{\mathbb{F}}(x) f_X(x) dx = \mathbb{E}[\mathbb{I}_{\mathbb{F}}(X)]$$

# Problem definition

## ⌘ Premise

- $G \equiv g(\mathbf{X})$ 's distribution is *rarely known* (it is in some specific cases: linear combinations of independent random variables, univariate composite distributions)
- *Numerical integration techniques* (e.g. quadrature rules) are not suitable for integrating indicator functions (their precision is often less than the probability's order of magnitude).

## ⌘ Dedicated methods

- Brute-force estimation using (intensive) Monte Carlo sampling
- Approximation methods
- Advanced, reduced variance, Monte Carlo sampling methods (not covered in this tutorial)
- Surrogate-model-based methods (not covered in this tutorial)

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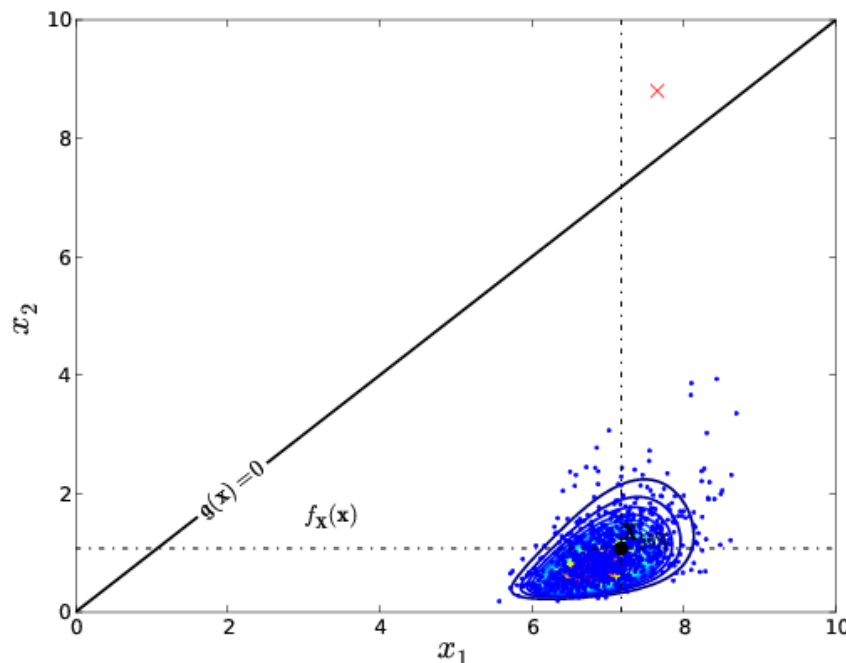


# Brute-force Monte Carlo estimation

## Principle

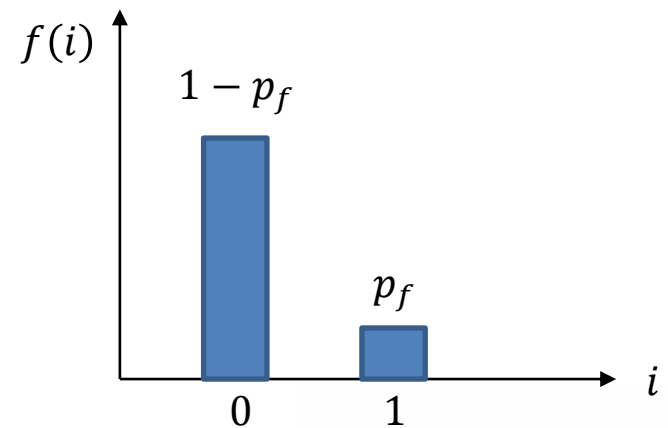
- The crude Monte Carlo estimator of the failure probability is *the empirical average of the Bernoulli failure experiment* :

$$\hat{P}_{f,\text{MCS}} = \frac{1}{N} \sum_{i=1}^N \mathbb{I}_{\text{F}}(\mathbf{X}^{(i)})$$



where:

$$\mathbb{I}_{\text{F}}(\mathbf{X}) \sim \text{Ber}(p_f)$$



# Brute-force Monte Carlo estimation

## ⌚ Convergence

- According to the *central limit theorem* (CLT), this estimator is unbiased and converges as follows :

$$\hat{P}_{f,\text{MCS}} \underset{N \rightarrow \infty}{\sim} \mathcal{N} \left( p_f, \sqrt{\frac{p_f(1-p_f)}{N}} \right)$$

- Rule of thumb:** before applying the CLT, make sure that:

$$\min\{Np_f; N(1-p_f)\} \geq 10$$

- The asymptotic distribution enables the calculation of  $(1 - \alpha)$ -confidence intervals:

$$\hat{p}_{f,\text{MCS}} + \Phi^{-1} \left( \frac{\alpha}{2} \right) \sqrt{\frac{p_f(1-p_f)}{N}} \leq p_f \leq \hat{p}_{f,\text{MCS}} + \Phi^{-1} \left( 1 - \frac{\alpha}{2} \right) \sqrt{\frac{p_f(1-p_f)}{N}}$$

- Ex : For  $1 - \alpha = 95\%$ ,  $\Phi^{-1} \left( \frac{\alpha}{2} \right) \approx -1,96$  and  $\Phi^{-1} \left( 1 - \frac{\alpha}{2} \right) \approx +1,96$ .

# Brute-force Monte Carlo estimation

## Convergence

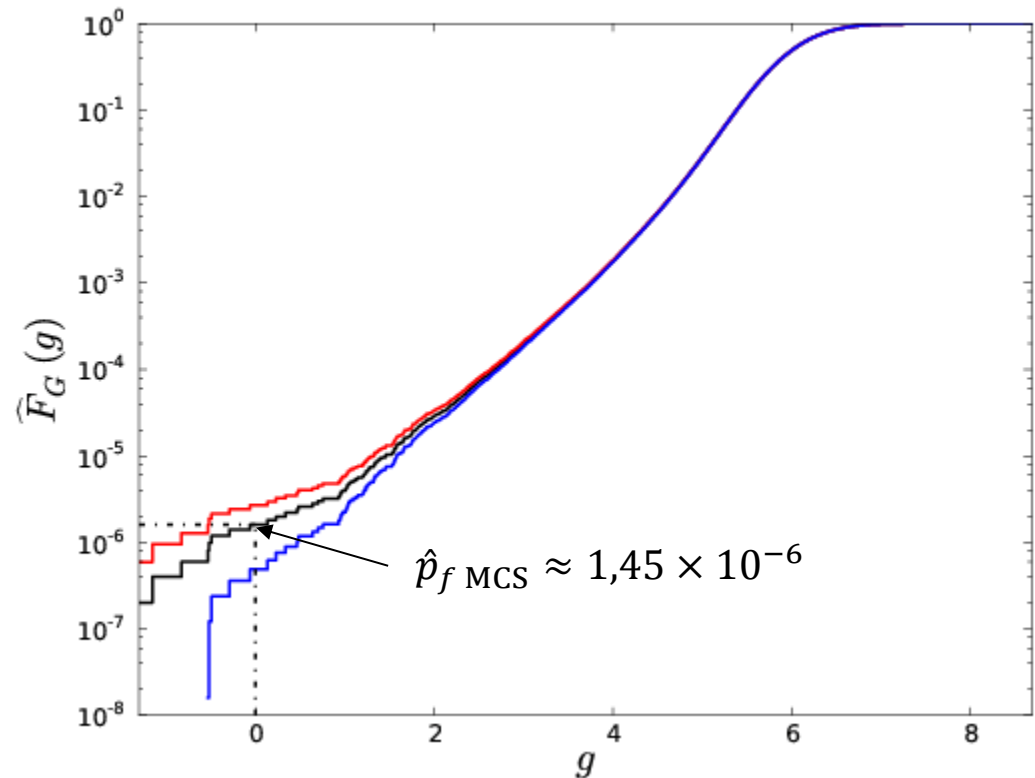
- The *required sample size* drastically increases as the probability gets low:

For a given *10% target coefficient of variation*

$$\delta = \sqrt{\frac{1 - p_f}{N p_f}} \approx \frac{1}{\sqrt{N p_f}}$$

$$p_f \approx 10^{-k} \Rightarrow N_{\min} \approx 10^{k+2}$$

$p_f$	$N_{\min}$
$10^{-2}$	10 000
$10^{-3}$	100 000
$10^{-4}$	1 000 000



# Brute-force Monte Carlo estimation

## ⌕ Pros & cons

- + Unbiased, reference, estimator
- + Easy to implement (at least for the statistics part)
- + Rich result (it is then possible to build a good approximation of  $G$ 's CDF)
- + Highly distributable over high-performance computing means (e.g. over a cluster)
- Slow convergence : requires important computing resources

## ⌕ When should it be used?

- When you don't have a choice (when no other more clever method is applicable)!
- When the performance function is fast to evaluate :
  - *Simple closed-form expressions* ;
  - *HPC resources* available.

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- ▣ Isoprobabilistic transformation
- ▣ Most-probable-failure-point(s)-based methods

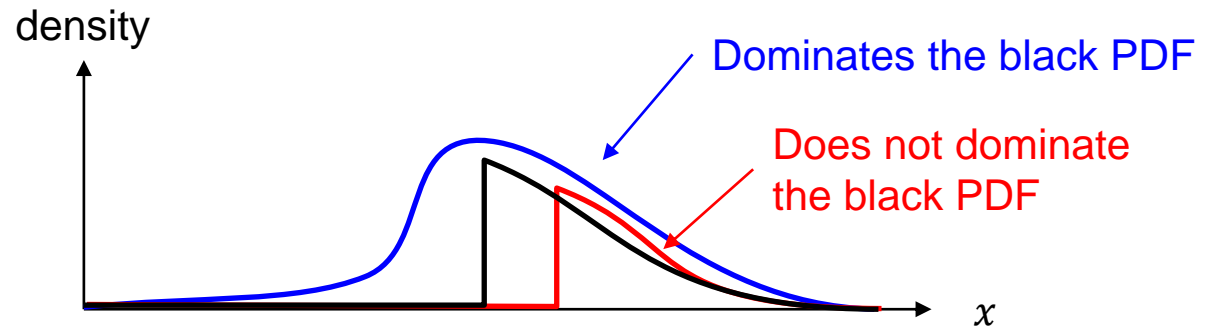
# Importance sampling

## ⌘ Principle

- Let  $H$  denote some *instrumental probability distribution* with PDF  $h$  satisfying the following *dominance condition* :

$$h(x) = 0 \Rightarrow \mathbb{I}_F(x)f_X(x) = 0$$

that would ideally make the failure event of interest *more frequent*.



- The failure probability rewrites:

$$p_f = \int_{\mathbb{X}} \mathbb{I}_F(x)f_X(x)dx = \int_{\mathbb{X}} \frac{\mathbb{I}_F(x)f_X(x)}{h(x)}h(x)dx$$

$$p_f = \mathbb{E}_Z \left[ \frac{\mathbb{I}_F(Z)f_X(Z)}{h(Z)} \right]$$

# Importance sampling

## ⌘ Use & properties

- Given an  $N$ -sample:

$$\mathbf{Z} = \{\mathbf{Z}^{(i)}, \quad i = 1, \dots, N\} \sim h$$

- The importance sampling *estimator* reads:

$$\hat{P}_{f,h} = \frac{1}{N} \sum_{i=1}^N \frac{\mathbb{I}_{\mathbb{F}}(\mathbf{Z}^{(i)}) f_X(\mathbf{Z}^{(i)})}{h(\mathbf{Z}^{(i)})}$$

- and *converges* according to the central limit theorem:

$$\hat{P}_{f,h} \underset{N \rightarrow \infty}{\sim} \mathcal{N}(p_f, \sigma_{p_f})$$

- The estimation *variance obviously depends on  $h$* :

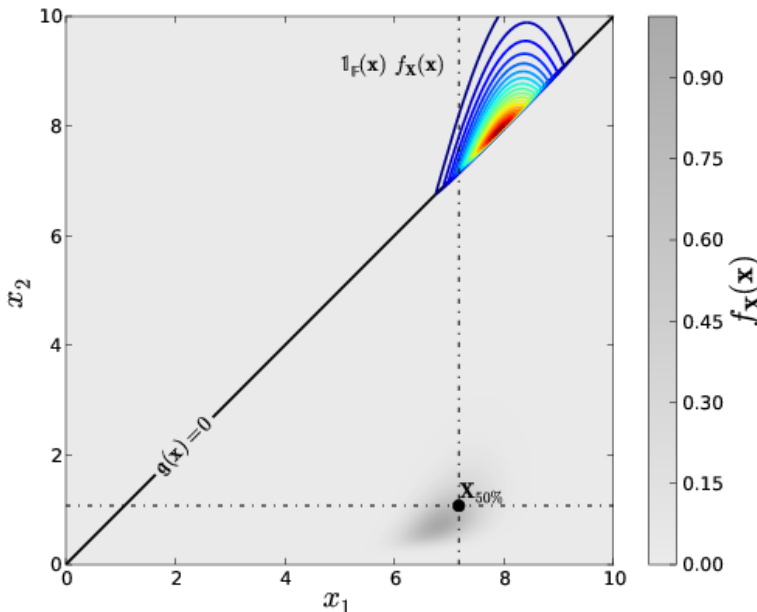
$$\sigma_{p_f}^2 = \frac{1}{N} \left( \mathbb{E}_{\mathbf{Z}} \left[ \frac{\mathbb{I}_{\mathbb{F}}(\mathbf{Z}) f_X^2(\mathbf{Z})}{h^2(\mathbf{Z})} \right] - p_f^2 \right)$$

# Importance sampling

## ⌚ Choosing $h$ ?

- Any distribution provided the *dominance condition* holds.
- The best instrumental PDF yields a *zero estimation variance* and reads:

$$h^*(x) = \frac{\mathbb{I}_F(x) f_X(x)}{p_f}$$



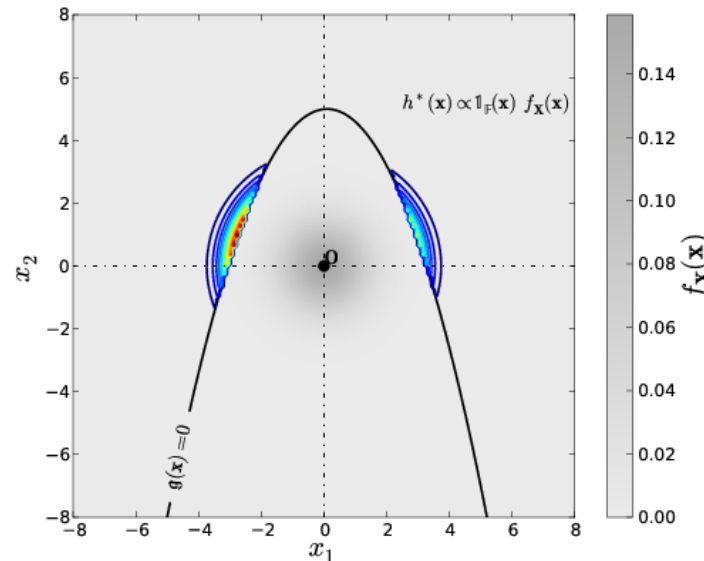
- impractical because its *normalizing constant* is the sought probability  $p_f$ !
- confirms *intuition* :
  - it is the probability distribution of the input parameters yielding failure.
  - it barely satisfies the *dominance condition*.



# Importance sampling

## ⌕ A fundamental concept in reliability analysis

- The objective is to explore *the tail of the safety margin's probability distribution* (the lower tail in our case:  $p_f \equiv \text{Prob}[G \leq 0]$ )...
- Using a *biased sampling technique for the input* in order to make failure much more frequent...
- And ideally, by sampling *only and exhaustively* failed situations (i.e. without forgetting any (significant) area of the failure domain).



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- ▣ **Isoprobabilistic transformation**
- ▣ Most-probable-failure-point(s)-based methods

# Isoprobabilistic transformation

## ⌘ Motivation

- *Spherical distributions* are invariant by rotation

$$\mathbf{U} \sim \mathbf{R}\mathbf{U}, \quad \forall \mathbf{R} \in \mathcal{SP}_n(\mathbb{R})$$

thus enabling analytical developments (to come).

## ⌘ Available transformations

- Independent copula → Componentwise transformations
- Elliptical copula → Generalized Nataf transformation
- Any other composed distribution → Rosenblatt transformation

The *choice for the most-suitable transformation*  
is *automatic* in OpenTURNS.

- Further readings: Lebrun & Dutfoy (2009a,b,c).

# Isoprobabilistic transformation

## ⌘ Standard space properties

- Given the *components order* in the input distribution and the Cholesky factor are fixed, the transformation is *unique* and *bijective* (it is *invertible*).
- The *probability measure is preserved*, hence the *failure probability* in the standard space equals the failure probability in the original (physical) space.

**ATTENTION :** This does not hold for approximations though.

- The transformed performance function is defined by composition:

$$g^\circ(\mathbf{u}) = g(\mathbf{x}) = (g \circ T^{-1})(\mathbf{u}), \quad \mathbf{u} \in \mathbb{R}^n$$

- This, in turns, enables the definition of the *failure domain* in the standard space:

$$\mathbb{F}^\circ = \{\mathbf{u} \in \mathbb{R}^n : g^\circ(\mathbf{u}) \leq 0\}$$

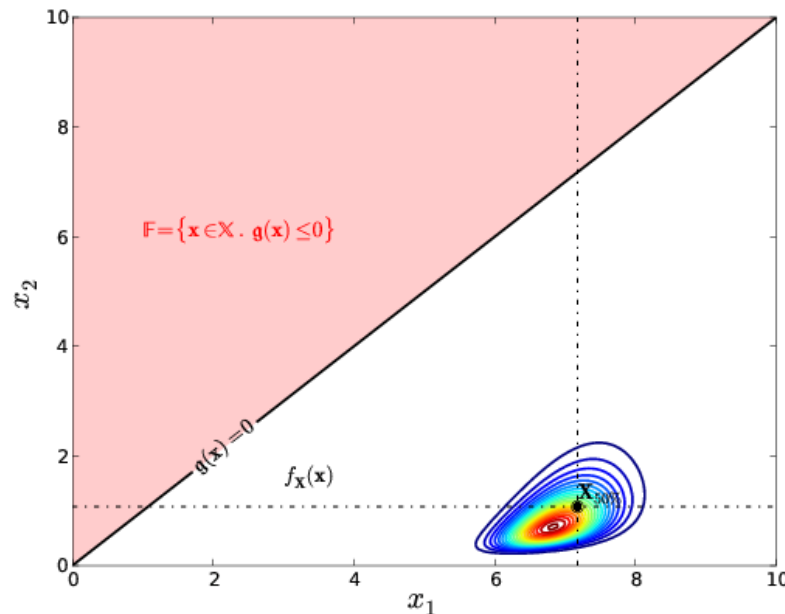
# Isoprobabilistic transformation

## Standard space properties

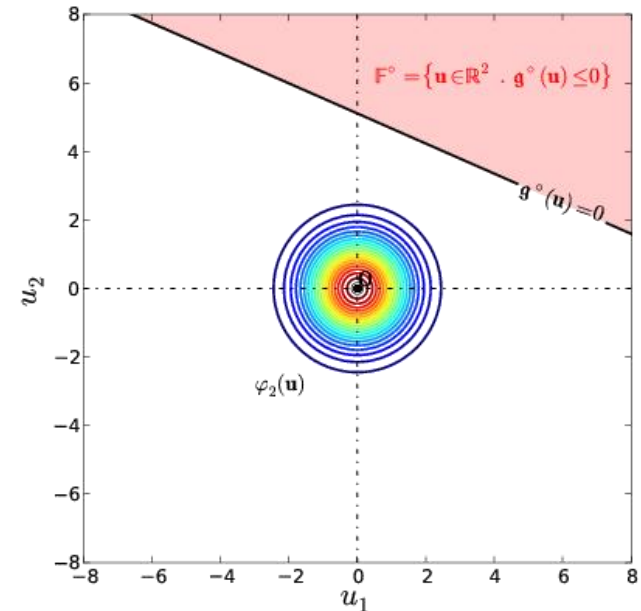
- Ex : Consider the *capacity vs demand* example:

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where the variables  $R \sim \mathcal{LN}(\lambda_R, \zeta_R)$  and  $S \sim \mathcal{LN}(\lambda_S, \zeta_S)$  are composed with a Normal copula with shape parameter  $\rho_0 = 0,525$ .



Physical space



Standard space

# Outline

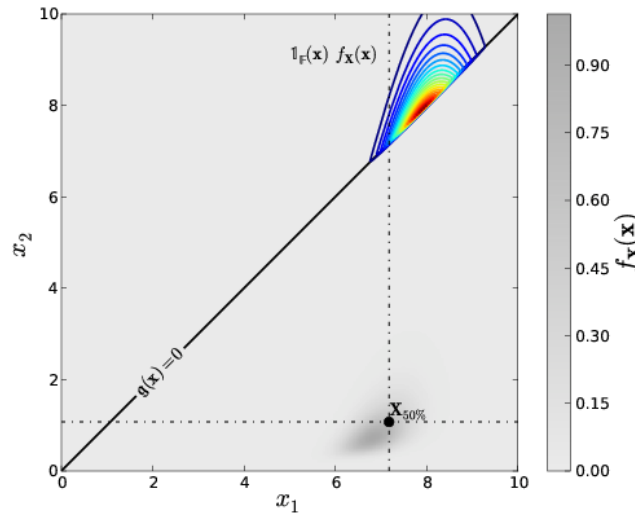
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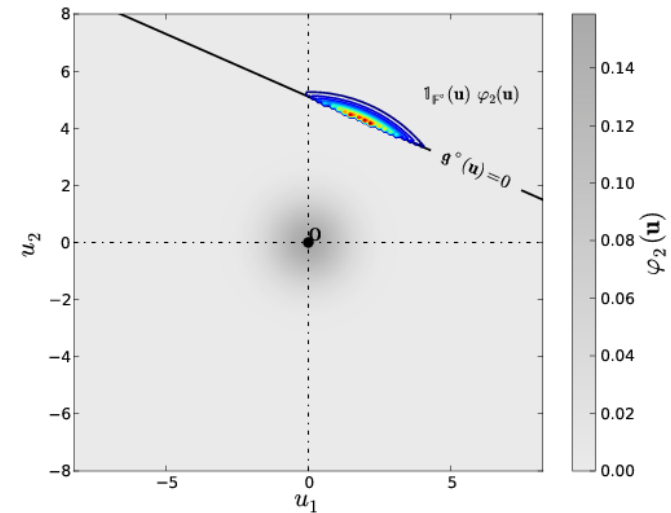
# MPFP: FORM, SORM, P\*-IS & FORM- $\Sigma$

## ⌚ Most probable failure point(s)

- Let's get back to the *optimal importance sampling concept*:



Physical space



Standard space

- We define the most probable failure point(s) as *the mode(s) of the optimal instrumental distribution*:

$$\mathbf{u}^* = \arg \max_{\mathbf{u} \in \mathbb{R}^n} \mathbb{I}_{F^*}(\mathbf{u}) \varphi_n(\mathbf{u})$$

- The solution for this optimization problem is *not necessarily unique*, although it is often the case in many applications (e.g. in structural mechanics).

# MPFP: FORM, SORM, P\*-IS & FORM-Σ

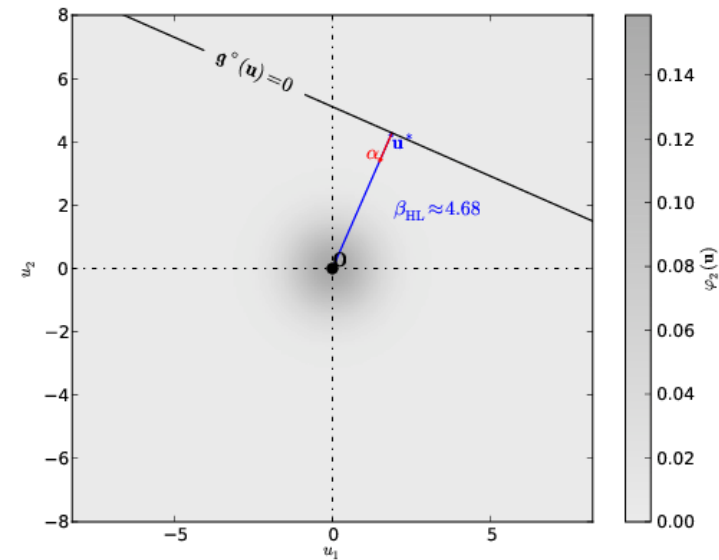
## ☐ Most probable failure point(s)

- Let's work on the definition:

$$\mathbf{u}^* = \arg \max_{\mathbf{u} \in \mathbb{R}^n} \mathbb{I}_{F^\circ}(\mathbf{u}) \varphi_n(\mathbf{u})$$

$$\mathbf{u}^* = \arg \max_{\mathbf{u} \in \mathbb{R}^n} \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2} \mathbf{u}^T \mathbf{u}\right) : g^\circ(\mathbf{u}) \leq 0$$

$$\mathbf{u}^* = \arg \min_{\mathbf{u} \in \mathbb{R}^n} \mathbf{u}^T \mathbf{u} : g^\circ(\mathbf{u}) \leq 0$$



- This is then equivalent to searching the *failure point(s) in the standard space that are the closest to the origin.*



## ☐ Search algorithms (constrained optimization)

- The *Abdo-Rackwitz algorithm* exploits the specificities of the problem at hand:
  - The objective function is quadratic.
  - The constraint is nonlinear, but it is linearized at each step based on the information brought by the gradient.
  - The optimization steps (the moves amplitude) can either be *fixed* (small) or *optimized* (variable) using merit rules such as Goldstein-Armijo's.
  - The algorithm converges when the current point satisfies both:
    - $g(\mathbf{u}^*) = 0$  (the point is on the limit-state surface)
    - $\nabla_{\mathbf{u}} g^{\circ}(\mathbf{u}^*) \parallel \mathbf{u}^*$  (the gradient of the constraint is colinear to that of the objective function)
- The *COBYLA* (Constrained Optimization BY Linear Approximations) algorithm is an interesting alternative when the partial derivatives of the performance function are hard to estimate (using finite differences schemes).

# MPFP: FORM, SORM, P\*-IS & FORM-Σ

## ☐ First-order reliability method (FORM)

- Assumption: the most probable failure point is *unique*.
- The performance function is linearized at the MPFP:

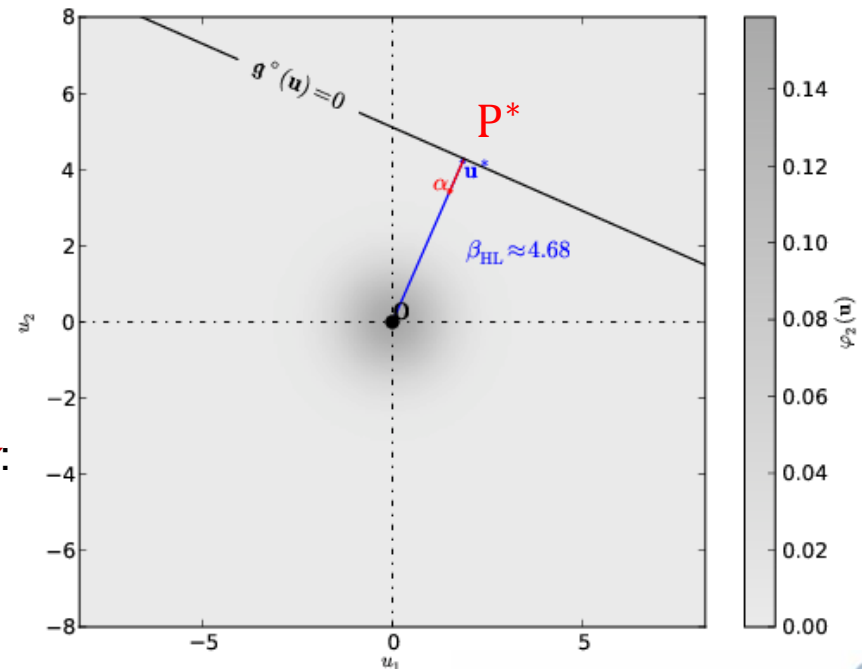
$$g_{1,u^*}^\circ(\mathbf{u}) = g^\circ(\mathbf{u}^*) + \nabla_{\mathbf{u}} g^\circ(\mathbf{u}^*)^T (\mathbf{u} - \mathbf{u}^*) = \nabla_{\mathbf{u}} g^\circ(\mathbf{u}^*)^T (\mathbf{u} - \mathbf{u}^*)$$

- We introduce the *unit orientation vector*:

$$\boldsymbol{\alpha} = \frac{\nabla_{\mathbf{u}} g^\circ(\mathbf{u}^*)}{\|\nabla_{\mathbf{u}} g^\circ(\mathbf{u}^*)\|_2}$$

- And the *Hasofer-Lind reliability index*:

$$\beta_{\text{HL}} = -\boldsymbol{\alpha}^T \mathbf{u}^* = \overline{OP^*}$$



## ☐ First-order reliability method (FORM)

- The *approximate failure domain in the standard space* rewrites:

$$\begin{aligned}\mathbb{F}_{1,u^*}^\circ &= \{\mathbf{u} \in \mathbb{R}^n : g_{1,u^*}^\circ(\mathbf{u}) \leq 0\} \\ &= \{\mathbf{u} \in \mathbb{R}^n : \nabla_{\mathbf{u}} g^\circ(\mathbf{u}^*)^T (\mathbf{u} - \mathbf{u}^*) \leq 0\} \\ &= \{\mathbf{u} \in \mathbb{R}^n : \boldsymbol{\alpha}^T (\mathbf{u} - \mathbf{u}^*) \leq 0\} \\ &= \{\mathbf{u} \in \mathbb{R}^n : \boldsymbol{\alpha}^T \mathbf{u} + \beta_{\text{HL}} \leq 0\}\end{aligned}$$

- So that we obtain the following first-order approximation of the failure probability:

$$\begin{aligned}p_{f\ 1,u^*} &= \text{Prob}[\boldsymbol{\alpha}^T \mathbf{U} + \beta_{\text{HL}} \leq 0] \\ &= \text{Prob}[Z \leq -\beta_{\text{HL}}], \text{ with } Z = \boldsymbol{\alpha}^T \mathbf{U} \sim \mathcal{N}(0, 1)\end{aligned}$$

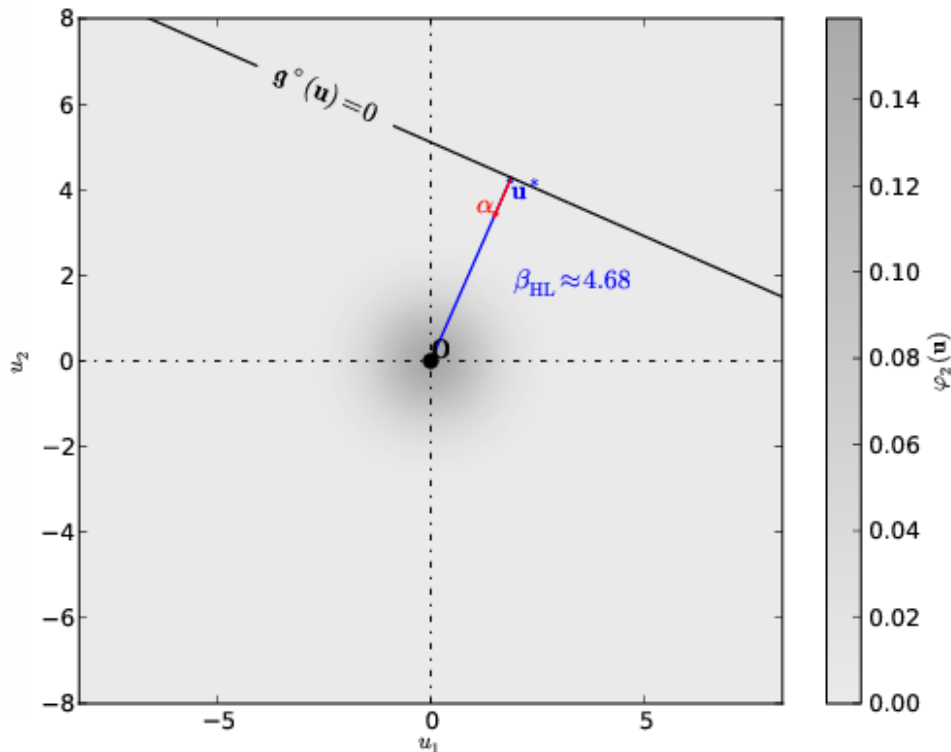
- Hence:

$$p_{f\ 1,u^*} = \Phi(-\beta_{\text{HL}})$$

# MPFP: FORM, SORM, P\*-IS & FORM- $\Sigma$

## ☐ First-order reliability method (FORM)

- Ex : (capacity vs demand example)



$$\beta_{HL} \approx 4,68$$
$$p_{f1,u^*} \approx 1,44 \times 10^{-6}$$

- The limit-state surface being linear in the standard space, in this particular case, FORM is the reference solution.
- *Generally speaking, this is only an approximation.*

## ☐ SORM: *accounting for local curvatures*

- Assuming we can compute *the second-order partial derivatives of the performance function* in the standard space.
- On peut prendre en compte l'*éventuelle courbure de l'état-limite* en poussant le développement de Taylor à l'ordre 2 au voisinage du  $P^*$  :

$$g_{2,u^*}^\circ = g^\circ(\mathbf{u}^*) + \nabla_{\mathbf{u}} g^\circ(\mathbf{u}^*)^T (\mathbf{u} - \mathbf{u}^*) + \frac{1}{2} (\mathbf{u} - \mathbf{u}^*)^T \nabla_{\mathbf{u}\mathbf{u}} g^\circ(\mathbf{u}^*) (\mathbf{u} - \mathbf{u}^*)$$

- In case the standard space is spanned by Gaussian variables, *Breitung* has shown the following *asymptotic result* :

$$p_{f 2,u^*} \xrightarrow{\beta_{HL} \rightarrow +\infty} \Phi(-\beta_{HL}) \prod_{i=1}^n \frac{1}{\sqrt{1 + \beta_{HL} \kappa_i}}$$

where  $\kappa_i$  are the *curvatures* calculated from the Hessian matrix. This result is *valid* as soon as  $1 + \beta_{HL} \kappa_i \geq 0, \quad i = 1, \dots, n$ .

- Lebrun & Dutfoy (2009a) generalized the approximation to *spherical distributions*.

# MPFP: FORM, SORM, P\*-IS & FORM-Σ

## 🔒 P\*-IS: MPFP(s)-centered importance sampling

- Another correction can be obtained by *importance sampling with an instrumental PDF centered at the identified MPFP(s)*.
- For this, the state-of-the-art consists in using a Gaussian instrumental distribution:

$$\varphi_{n,u^*}(\mathbf{u}) = \varphi_n(\mathbf{u} - \mathbf{u}^*) = \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{(\mathbf{u} - \mathbf{u}^*)^T(\mathbf{u} - \mathbf{u}^*)}{2}\right)$$

- In this case, the failure probability *estimator* simplifies:

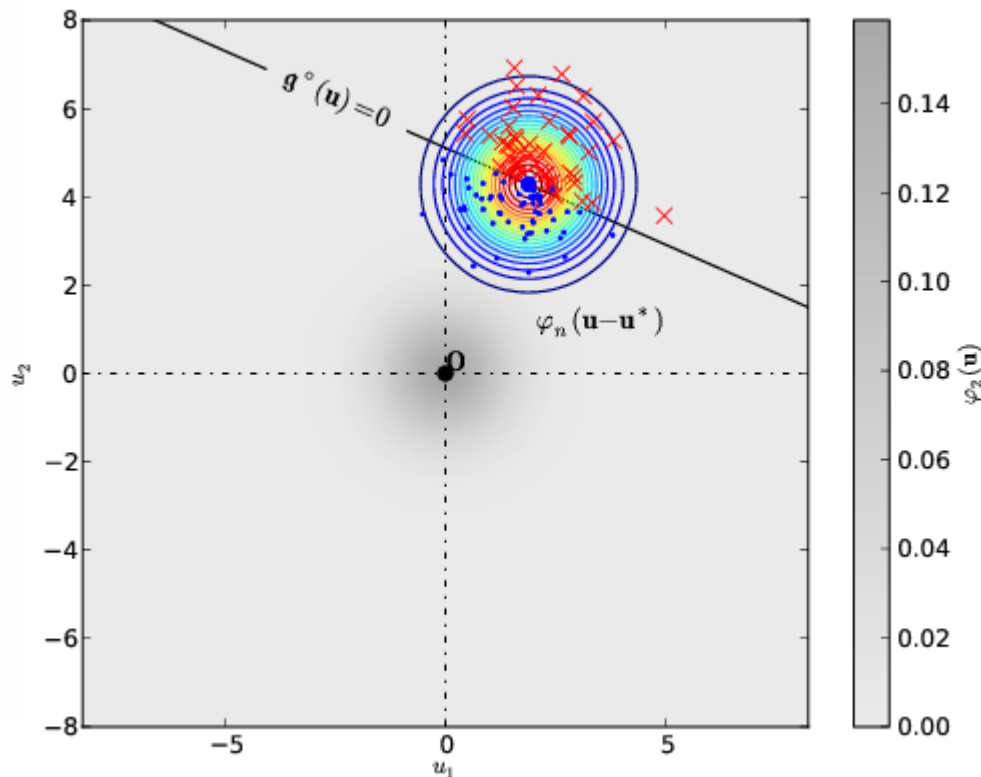
$$\hat{P}_{f,u^*IS} = \frac{\exp(-\beta_{HL}^2/2)}{N} \sum_{i=1}^N \mathbb{I}_{\mathbb{F}^\circ}(\mathbf{Z}^{(i)}) \exp(-\mathbf{Z}^{(i)T} \mathbf{u}^*)$$

- It is unbiased if the *dominance condition* of  $\varphi_{n,u^*}$  over  $\mathbb{I}_{\mathbb{F}^\circ} \times \varphi_n$  holds (*unicity of the MPFP?*).
- It « *converges* » *much faster*, because the sampled points fails with a probability that is close to 50% (say between 10 and 90%).

# MPFP: FORM, SORM, P\*-IS & FORM- $\Sigma$

## ☐ P\*-IS: MPFP(s)-centered importance sampling

- Ex : (capacity vs demand example)



$p_{f,\mathbf{u}^*IS} \approx 1,49 \times 10^{-6}$   
up to a 10%  
coefficient of variation.

- Convergence is obtained with only **600 additional runs of  $g$** ...
- Compared to  $10^8$  runs for crude Monte Carlo that gave:  
 $\hat{p}_{f,MCS} \approx 1,45 \times 10^{-6}$   
up to a **10% coefficient of variation**.

# MPFP: FORM, SORM, P\*-IS & FORM-Σ

## FORM : importance factors

- The unit direction vector indicates how the *reliability index evolves with respect to the MPFP coordinates*:

$$\beta_{\text{HL}} = -\boldsymbol{\alpha}^T \mathbf{u}^* = \sum_{i=1}^n -\alpha_i u_i^* \Rightarrow \alpha_i = -\frac{\partial \beta_{\text{HL}}}{\partial u_i^*}$$

- In case *the distribution has a non-independent copula* though, each standard variable  $u_i$  is a function of several original (physical) variable  $x_i$ , so that the  $\alpha_i$ 's *are difficult to read*.



## FORM : importance factors

- In case the copula is Normal, Lemaire (2009) defined the following *corrected importance factors*:

$$\gamma_i = \frac{1}{\|\boldsymbol{\gamma}\|_2} \sigma_{X_i} \left. \frac{\partial g}{\partial x_i} \right|_{\mathbf{x}=\mathbf{x}^*}, \quad i = 1, \dots, n$$

- In the more general case, Lebrun & Dutfoy (2009c) proposed another more general, although unsigned, definition:

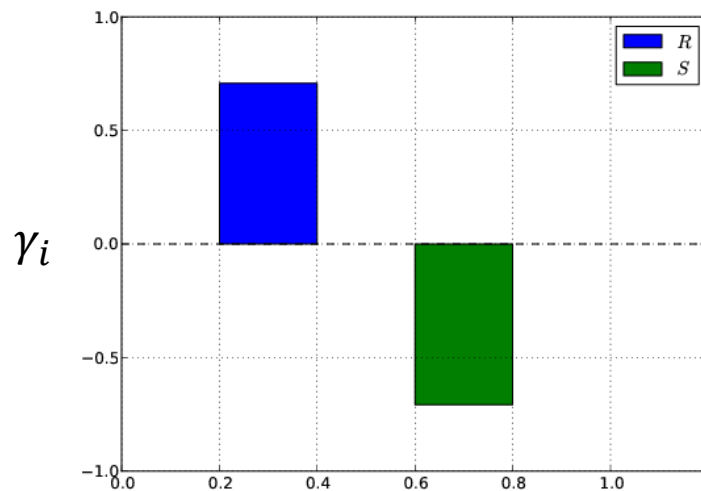
$$\gamma_i^2 = \frac{w_i^2}{\|\mathbf{w}\|_2^2}, \quad i = 1, \dots, n$$

$$\mathbf{w} = \begin{pmatrix} E^{-1}(F_{X_1}(x_1)) \\ \vdots \\ E^{-1}(F_{X_n}(x_n)) \end{pmatrix}$$

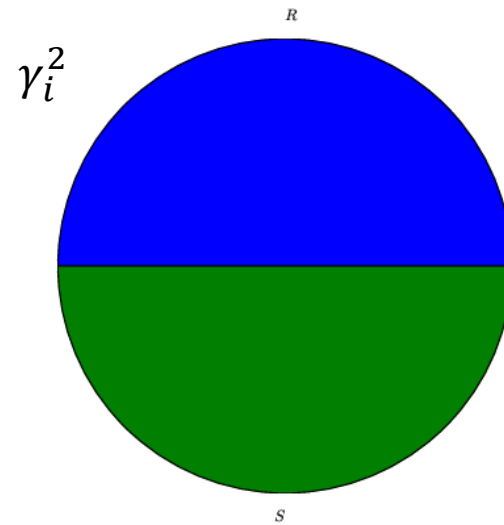
# MPFP: FORM, SORM, P\*-IS & FORM- $\Sigma$

## FORM : importance factors

- These results are often presented in either one or both of these two charts:



*Signed bar chart*



*Pie chart*

$\alpha_i$  or  $\gamma_i$  **positive**  $\Rightarrow X_i$  is a **capacity variable**  
 $\alpha_i$  or  $\gamma_i$  **negative**  $\Rightarrow X_i$  is a **demand variable**

*The quadratic sum equals 1.*  
*Qualitative comparison of the importance of variables w.r.t. failure.*

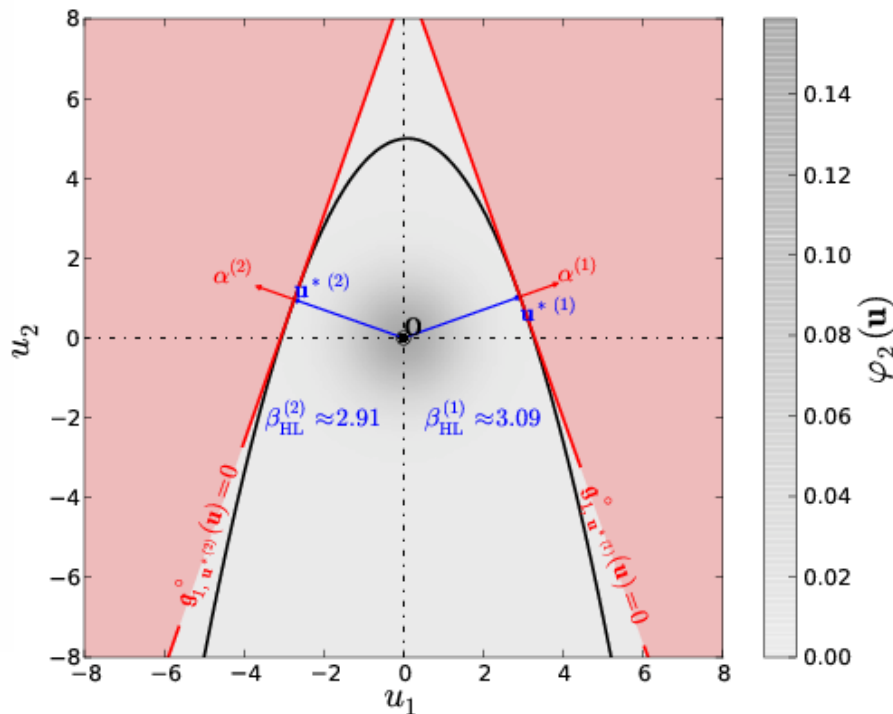
# MPFP: FORM, SORM, P\*-IS & FORM- $\Sigma$

## FORM : Multiple design points

- Ex : Consider the following limit-state function:

$$g(x_1, x_2) = b - x_2 - \kappa(u_1 - e)^2$$

where  $\mathbf{X} = \mathbf{U} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ ,  $b = 5$ ,  $\kappa = 0,5$  and  $e = 0,1$ .



- PMSOft implements *an algorithm that is devoted to the search of multiple  $P^*$*  using:
  - a smart reset strategy;
  - exclusion balls around the already found design points.
  - See Der Kiureghian & Dakessian (1998).
- However the enumeration may lack completeness.

# MPFP: FORM, SORM, P\*-IS & FORM-Σ

## ☐ FORM-Σ : Serial combination of linear limit-states

- Input: *for the  $n_{P^*}$  identified MPFPs* :
  - reliability indices:  $\beta_{HL} = (\beta_{HL}^{(i)}, i = 1, \dots, n_{P^*})$
  - importance factors in the standard space:  $\mathbf{A} = (\alpha^{(i)}, i = 1, \dots, n_{P^*})$
- Objective : combine these results into a single probability, the one associated to the *serial system* formed by the contributors.
- Solution:

$$p_{f,1\Sigma} = \text{Prob} \left[ \mathbf{U} \in \bigcup_{i=1}^{n_{P^*}} \left\{ \mathbf{u} \in \mathbb{R}^n : \alpha^{(i)T} \mathbf{u} + \beta_{HL}^{(i)} \leq 0 \right\} \right] = 1 - \Phi_{n_{P^*}}(\beta_{HL}; \mathbf{0}, \rho)$$

where:

$$\rho_{ij} = \alpha^{(i)T} \alpha^{(j)}, \quad i, j = 1, \dots, n_{P^*}$$

are the « *pairwise limit-states' correlation* » ( $-1 \leq \rho \leq 1$ ).

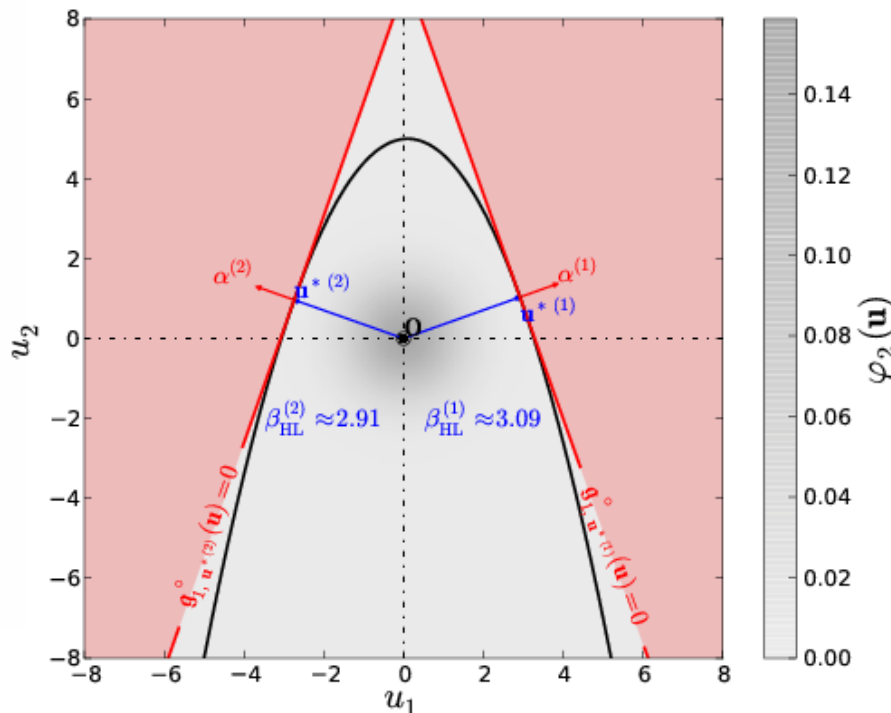
# MPFP: FORM, SORM, P\*-IS & FORM-Σ

## FORM-Σ : Serial combination of linear limit-states

- Ex : Consider the following limit-state function:

$$g(x_1, x_2) = b - x_2 - \kappa(u_1 - e)^2$$

where  $\mathbf{X} = \mathbf{U} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ ,  $b = 5$ ,  $\kappa = 0,5$  and  $e = 0,1$ .



The correlation between the two limit-states is:

$$\rho_{12} = \boldsymbol{\alpha}^{(1)T} \boldsymbol{\alpha}^{(2)} \approx -0,78$$

Hence the *first-order approximation* of the serial system failure probability is:

$$p_{f1,\Sigma} = 1 - \Phi_2 \left( \begin{pmatrix} 3,09 \\ 2,91 \end{pmatrix}; 0, \begin{bmatrix} 1 & -0,78 \\ -0,78 & 1 \end{bmatrix} \right) \\ \approx 2,82 \times 10^{-3}$$

The crude Monte Carlo estimate is:

$$\hat{p}_{f,\text{MCS}} \approx 3,12 \times 10^{-3}$$

Up to a *10% coefficient of variation*.

# MPFP: FORM, SORM, P\*-IS & FORM- $\Sigma$

## ⌕ Pros & Cons

- + The most probable failure point concept is interesting because:
  - + Coordinates  $\rightarrow$  singular configuration(s) of the system.
  - + Importance factors  $\rightarrow$  give clues for improving reliability.
- + Affordable computational cost.
- The most probable failure point concept is dangerous:
  - non-unicity risk (FORM, SORM & basic P\*-IS) ;
  - non-completeness risk (FORM- $\Sigma$ ).
- Missing (FORM, SORM, FORM- $\Sigma$ ) or subjective (P\*-IS) error metric.

## ⌕ When should it be used?

- As a first *approximation* ;
- *Confirmed by an expert judgement* about the identified failure modes.

# Conclusions

- Reliability methods aim at estimating the safety level attached to a component in the form of a **subjective failure probability** :

$$p_f = \text{Prob}[\text{failure} \mid \text{model}]$$

- The methods reviewed in this presentation are implemented in **OpenURNS** & **Uranie**.
- Crude Monte Carlo sampling** enables exploring the model:
  - without requiring any assumption,
  - at a great computational expense though (HPC may help).
- Most-probable-failure-point(s)-based techniques** enable:
  - a reduction of the computational effort (even if HPC may still help);
  - a deeper investigation of the system thanks to:
    - the most probable failure points coordinates;
    - the importance factors.

# Further readings

- Ditlevsen, O. & Madsen, H. (1996).  
*Structural reliability methods*.  
John Wiley & Sons.
- Lebrun, R. & Dutfoy, A. (2009a).  
A generalization of the Nataf transformation to distributions with elliptical copula.  
*Prob. Eng. Mech.*, 24, 172–178.
- Lebrun, R. & Dutfoy, A. (2009b).  
An innovating analysis of the Nataf transformation from the copula viewpoint.  
*Prob. Eng. Mech.*, 24, 312–320.
- Lebrun, R. & Dutfoy, A. (2009c).  
Do Rosenblatt and Nataf isoprobabilistic transformations really differ?  
*Prob. Eng. Mech.*, 2009, 24, 577–584.
- Lemaire, M. (2009).  
*Structural reliability*.  
Wiley. 480 pp.