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Zero-Dimensional Dynamical Systems, Formal Languages, and Universality*

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Abstract. We measure the complexity of dynamical systems on zero-dimensional compact metric spaces by the complexity of formal languages, which these systems generate on clopen partitions of the state space. We show that in the classes of recursive, context-sensitive, context-free, regular, etc., languages there exist universal dynamical systems which yield, by factor maps, all dynamical systems of the class. Universal systems are not unique, but in every class there exists a smallest universal system.

1. Introduction

The behavior of many a dynamical system is elucidated when a symbolic model for it is found. In this case the dynamical system in question is shown to be a factor of a symbolic system, usually a one-sided or a two-sided subshift. This method works in general. According to a generalized Alexandrov theorem (see, e.g., [2]), every dynamical system on a compact metric space is a factor of some dynamical system on a discontinuum, i.e., on a zero-dimensional perfect compact metric space. For this reason, dynamical systems on a discontinuum are quite diversified. Besides subshifts they include, for example, cellular automata and adding machines. In general, they can be represented as automata networks, i.e., countable systems of interconnected finite automata (see, e.g., [8]).

In this paper we introduce a classification of dynamical systems on zero-dimensional spaces using the concepts of formal language theory. Suppose that we observe the development of a dynamical system in time, but we are unable to distinguish points which are too close to each other. We have an observable, whose range is a finite alphabet. It is natural to assume that this observable is continuous, i.e., that the sets of the

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corresponding partition are clopen (closed and open). Every trajectory of the system then generates a sequence of symbols, and the set of all these sequences is a language.

Given any family $\mathfrak{L}$ of languages (e.g., regular or recursive), we say that a dynamical system has languages of class $\mathfrak{L}$ (or that it is of class $\mathfrak{L}$) if every clopen partition of its state space generates a language in $\mathfrak{L}$. The languages obtained are in one-to-one correspondence with one-sided subshifts, and the corresponding subshifts are factors of the given dynamical system. Thus the classification is based on the complexity of one-sided subshifts, which can be obtained as factors of the dynamical system in question.

In topological dynamics, a dynamical system is called universal if every dynamical system is its factor. This means that it is the most complex dynamical system imaginable, since every other system might be constructed from it by collapsing points. There exists a universal minimal dynamical system. This is a minimal dynamical system on a compact Hausdorff space, such that every minimal dynamical system on a compact Hausdorff space is its factor (see [1]).

We say that a dynamical system is universal for a family of languages $\mathfrak{L}$ if it is of class $\mathfrak{L}$, and every dynamical system of class $\mathfrak{L}$ is its factor. We give conditions on a family of languages $\mathfrak{L}$ which guarantee the existence of a universal dynamical system of class $\mathfrak{L}$. Universal systems are not unique, but under these conditions there exists a unique smallest universal system. In particular, our results apply to the classes of recursive, context-sensitive, context-free, and regular languages. They apply also to two subclasses of regular languages, namely, the periodic languages, whose subshifts are countable, and the bounded periodic languages, whose subshifts are finite. There exists also a smallest universal system for the family of transitive periodic languages, which is just the adding machine with infinite multiplicities.

2. Zero-Dimensional Spaces and Systems

A compact metric space is zero-dimensional if every two different points have disjoint clopen neighborhoods. A space is perfect if it has no isolated points. Zero-dimensional spaces are usually constructed as product or inverse limit spaces. Denote by $\mathbb{N} = \{0, 1, \ldots\}$ the set of nonnegative integers, let $(A_i)_{i\in\mathbb{N}}$ be a sequence of finite alphabets, and suppose that every A_i has at least two elements. Their product $X = \prod_{i\in\mathbb{N}} A_i$ is the set of all sequences $x = (x_i)_{i\in\mathbb{N}}$, where $x_i \in A_i$. If every A_i is equipped with the discrete topology, their product is a metrizable topological space. A metric which yields the product topology is for example $d(x, y) = 2^{-n}$, where $n = \min\{i \in \mathbb{N}; x_i \neq y_i\}$. The product space is a discontinuum, i.e., a zero-dimensional, perfect, compact metric space. Every two discontinua are homeomorphic. When all $A_i = A$ are equal, we get a special case of the power space $X = A^{\mathbb{N}}$.

If $(A_i)_{i\in\mathbb{N}}$ is a sequence of finite alphabets, and $(h_i\colon\ A_{i+1} \to A_i)_{i\in\mathbb{N}}$ is a sequence of mappings, their inverse limit space is

$$X = \lim_{i}(A_i, h_i) = \left\{ x \in \prod_{i\in\mathbb{N}} A_i;\ \ h_i(x_{i+1}) = x_i \right\}.$$

With the metric d inherited from the product space, we have $d(x, y) < 2^{-n}$ iff $x_n = y_n$. The inverse limit space is a closed subspace of the product space $\prod_i A_i$, hence it is compact and zero-dimensional. However, it need not be perfect. Every zero-dimensional compact metric space is homeomorphic to some inverse limit space (see [10]).

A clopen partition of a zero-dimensional space X is a surjective continuous function $\alpha\colon X \to A$, where A is a finite set with the discrete topology. Thus $\{\alpha^{-1}(a);\, a \in A\}$ is a finite collection of nonempty disjoint clopen sets, whose union is the full space. The diameter of a partition $\alpha\colon X \to A$ is $\mathrm{diam}(\alpha) = \max\{\mathrm{diam}(\alpha^{-1}(a));\, a \in A\}$. A clopen partition $\alpha\colon X \to A$ is finer than a clopen partition $\beta\colon X \to B$ (we write $\alpha \geq \beta$) if there exists a map $h\colon A \to B$ with $h\alpha = \beta$. The join $\alpha \vee \beta\colon X \to C$ of clopen partitions $\alpha\colon X \to A$ and $\beta\colon X \to B$ is defined by $(\alpha \vee \beta)(x) = (\alpha(x), \beta(x))$. Its range C is a subset of $A \times B$.

Definition 1. We say that $(\alpha_i\colon X \to A_i)_{i\in\mathbb{N}}$ is a separating sequence of clopen partitions if $\lim_{i\to\infty} \mathrm{diam}(\alpha_i) = 0$, and every α_{i+1} is finer than α_i.

If $X = \lim_i(A_i, h_i)$ is an inverse limit space, denote by $\alpha_i\colon X \to A_i$ the projection $\alpha_i(x) = x_i$. Since $h_i\alpha_{i+1} = \alpha_i$, $(\alpha_i)_{i\in\mathbb{N}}$ is a separating sequence of clopen partitions. Conversely, let X be a zero-dimensional, compact metric space, and let $(\alpha_i\colon X \to A_i)_{i\in\mathbb{N}}$ be a separating sequence of clopen partitions with $h_i\colon A_{i+1} \to A_i$ satisfying $h_i\alpha_{i+1} = \alpha_i$. Define $\alpha\colon X \to \lim_i(A_i, h_i)$ by $\alpha(x)_i = \alpha_i(x)$. Then α is a homeomorphism.

Definition 2. A (zero-dimensional) dynamical system (X, f) is a continuous map $f\colon X \to X$ of a nonempty zero-dimensional compact metric space X to itself. A homomorphism $\varphi\colon (X, f) \to (Y, g)$ of dynamical systems is a continuous map $\varphi\colon X \to Y$ such that $g\varphi = \varphi f$. A factor map is a homomorphism which is surjective. A conjugacy is a homomorphism which is bijective.

Dynamical systems on product spaces can be viewed as automata networks, i.e., countable networks of finite automata operating synchronously. The automaton at site $i \in \mathbb{N}$ is given by a finite set A_i of its inner states, a finite set of inputs $K_i \subseteq \mathbb{N}$, and a transition function $f_i\colon \prod_{j\in K_i} A_j \to A_i$. The state of an automata network is given by the inner states of all automata, so the state space of the network is $X = \prod_i A_i$. The global transition function $f\colon X \to X$ is given by $f(x)_i = f_i(x|K_i)$. Thus X is a product space, and f is a continuous function. By compactness, every continuous self-map of a product space can be obtained in this way.

New dynamical systems may be constructed as infinite products or inverse limits of dynamical systems. Let $(X_i, d_i)_{i\in\mathbb{N}}$ be a sequence of metric spaces with bounded diameter $\mathrm{diam}(X_i) \leq c$. Their product is the metric space (X, d), where $X = \prod_{i\in\mathbb{N}} X_i$, and $d(u, v) = \sum_{i=0}^{\infty} d_i(u_i, v_i)2^{-i}$. If all X_i are zero-dimensional compact metric spaces, then so is their product. If $(X_i)_{i\in\mathbb{N}}$ is a sequence of metric spaces with bounded diameter, and $h_i\colon X_{i+1} \to X_i$ are continuous mappings, then their inverse limit is

$$X = \lim_i(X_i, h_i) = \left\{x \in \prod_{i\in\mathbb{N}} X_i;\ h_i(x_{i+1}) = x_i\right\}.$$

Again, if all X_i are zero-dimensional and compact, then so is their inverse limit. If $(X_i, f_i)_{i\in\mathbb{N}}$ is a sequence of dynamical systems, their product is the system (X, f), where

$X = \prod_i X_i$, and $f(x)_i = f_i(x_i)$. If $(h_i\colon (X_{i+1}, f_{i+1}) \to (X_i, f_i))_{i\in\mathbb{N}}$ is a sequence of homomorphisms, their inverse limit is the dynamical system $(X, f) = \lim_i(X_i, f_i, h_i)$, where $X = \lim_i(X_i, h_i)$, and $f(x)_i = f_i(x_i)$.

3. Language Classification

If A is a finite alphabet and $n \in \mathbb{N}$, denote by A^n the set of words over A of length n, $A^* = \bigcup_{n\in\mathbb{N}} A^n$ is the set of finite words, and $\overline{A^*} = A^* \cup A^{\mathbb{N}}$. For $u = u_0u_1u_2\cdots \in \overline{A^*}$, denote by $|u|$ its length ($0 \leq |u| \leq \infty$), and denote by λ the word of zero length. Write $u \sqsubseteq v$ if u is a subword of v, i.e., if there exist $0 \leq i \leq j < |v|$ with $u = v_i \cdots v_j$. Denote by $u_{|i} = u_0 \cdots u_{i-1}$ the initial subword of u of length i. For $u \in A^*$ denote by $u^n \in A^*$ the repetition of u n times, and by $u^\infty \in A^{\mathbb{N}}$ its infinite repetition.

A language over A is any subset $L \subseteq A^*$. A (one-sided) full shift is a dynamical system $(A^{\mathbb{N}}, \sigma)$ on the power space $A^{\mathbb{N}}$ given by $\sigma(u)_i = u_{i+1}$. A subshift (over A) is any subsystem of the full shift, i.e., a dynamical system (Σ, σ), where $\Sigma \subseteq A^{\mathbb{N}}$ is a nonempty closed subset of $A^{\mathbb{N}}$ satisfying $\sigma(\Sigma) \subseteq \Sigma$ (the inclusion might be strict).

A nonempty language $L \subseteq A^*$ is right central if it is closed under subwords, i.e., $(\forall u \in L)(\forall v \sqsubseteq u)(v \in L)$, and extendable to the right, i.e., $(\forall u \in L)(\exists a \in A)(ua \in L)$. The adherence of a right central language $L \subseteq A^*$ is the subshift $\mathcal{A}(L) = \{u \in A^{\mathbb{N}}; (\forall v \in A^*)(v \sqsubseteq u \Rightarrow v \in L)\}$. The language of a subshift $\Sigma \subseteq A^{\mathbb{N}}$ is the right central language $\mathcal{L}(\Sigma) = \{u \in A^*; (\exists v \in \Sigma)(u \sqsubseteq v)\}$. We have $\mathcal{AL}(\Sigma) = \Sigma$ and $\mathcal{LA}(L) = L$, so right central languages and subshifts are in one-to-one correspondence (see [3]).

If (X, f) is a dynamical system, and if $\alpha\colon X \to A$ is a clopen partition, denote by $\alpha^{\mathbb{N}}\colon (X, f) \to (A^{\mathbb{N}}, \sigma)$ the homomorphism defined by $\alpha^{\mathbb{N}}(x)_i = \alpha f^i(x)$. Denote by $\Sigma_\alpha(X, f) = \alpha^{\mathbb{N}}(X) \subseteq A^{\mathbb{N}}$ the subshift generated by α on X, so that $\alpha^{\mathbb{N}}\colon (X, f) \to (\Sigma_\alpha(X, f), \sigma)$ is a factor map. Put $\mathcal{L}_\alpha(X, f) = \mathcal{L}(\Sigma_\alpha(X, f))$ and $\mathcal{L}^n_\alpha(X, f) = \mathcal{L}_\alpha(X, f) \cap A^n$ for $n > 0$. In particular, if $\Sigma \subseteq A^{\mathbb{N}}$ is a subshift and $\alpha_0\colon A^{\mathbb{N}} \to A$ is the projection $\alpha_0(x) = x_0$, then $\mathcal{L}_{\alpha_0}(\Sigma, \sigma) = \mathcal{L}(\Sigma)$. If $\alpha\colon (X, f) \to (\Sigma, \sigma)$ is a factor map, then $\Sigma = \Sigma_{\alpha_0}(X, f)$, where $\alpha_0(x) = \alpha(x)_0$.

Definition 3. Let $\mathfrak{L}$ be a family of languages. We say that a dynamical system (X, f) is of class $\mathfrak{L}$ (or that it has languages of class $\mathfrak{L}$) if, for every clopen partition $\alpha\colon X \to A$, the language $\mathcal{L}_\alpha(X, f)$ belongs to $\mathfrak{L}$.

Subshifts and factor maps form a category, which admits the pullback construction (see [14]). If $\Sigma_1 \subseteq A_1^{\mathbb{N}}$, $\Sigma_2 \subseteq A_2^{\mathbb{N}}$ are subshifts, and $h_1\colon (\Sigma_1, \sigma) \to (\Sigma_3, \sigma)$, $h_2\colon (\Sigma_2, \sigma) \to (\Sigma_3, \sigma)$ are factor maps, denote the projections by $\Sigma_0 = \{(u_i, v_i) \in (A_1 \times A_2)^{\mathbb{N}};\ u \in \Sigma_1,\ v \in \Sigma_2,\ h_1(u) = h_2(v)\}$ and $\pi_1\colon \Sigma_0 \to \Sigma_1, \pi_2\colon \Sigma_0 \to \Sigma_2$. Then $\Sigma_0 \subseteq (A_1 \times A_2)^{\mathbb{N}}$ is a subshift, which is called the pullback of h_1 and h_2.

Definition 4. Let $\mathfrak{L}$ be a family of languages. We say that $\mathfrak{L}$ is closed under factors if, for every factor map $h\colon (\Sigma_1, \sigma) \to (\Sigma_2, \sigma)$, $\mathcal{L}(\Sigma_2)$ belongs to $\mathfrak{L}$, whenever $\mathcal{L}(\Sigma_1)$ belongs to $\mathfrak{L}$. We say that $\mathfrak{L}$ is closed under pullbacks if the pullback of every pair of factor maps between subshifts of class $\mathfrak{L}$ is of class $\mathfrak{L}$.

Proposition 1. *Let $\mathfrak{L}$ be a family of languages closed under factors. Then the following conditions are equivalent*:

1. (X, f) *is of class* $\mathfrak{L}$.
2. *There exists a separating sequence* $(\alpha_i\colon X \to A_i)_{i\in\mathbb{N}}$ *of clopen partitions, such that, for every* i, $\mathcal{L}_{\alpha_i}(X, f)$ *belongs to* $\mathfrak{L}$.
3. (X, f) *is conjugate to the inverse limit of subshifts of class* $\mathfrak{L}$.

Proof. $1 \Rightarrow 2$ and $2 \Leftrightarrow 3$ are clear. To prove $2 \Rightarrow 1$, let $\beta\colon X \to B$ be a finite clopen partition. For every $x \in X$ there exists an integer n_x such that $W_x = \alpha_{n_x}^{-1}\alpha_{n_x}(x) \subseteq \beta^{-1}\beta(x)$. Let $\{W_x; x \in A\}$ be a finite subcover of $\{W_x; x \in X\}$, and $n = \max\{n_x; x \in A\}$. Then $\alpha_n \geq \beta$, so we have a map $h\colon A_n \to B$ with $h\alpha_n = \beta$, and $h^{\mathbb{N}}\colon (\Sigma_{\alpha_n}(X, f), \sigma) \to (\Sigma_\beta(X, f), \sigma)$ is a factor map, so $\mathcal{L}_\beta(X, f)$ belongs to $\mathfrak{L}$. $\square$

4. Universal Systems

Definition 5. Let $\mathfrak{L}$ be a family of languages. We say that a dynamical system (X, f) is a universal system of class $\mathfrak{L}$ if it is of class $\mathfrak{L}$, and every dynamical system of class $\mathfrak{L}$ is its factor. We say that a universal system (X, f) of class $\mathfrak{L}$ has the extension property if, for all subshifts $\Sigma_A \subseteq A^{\mathbb{N}}$, $\Sigma_B \subseteq B^{\mathbb{N}}$ of class $\mathfrak{L}$, and all factor maps $h\colon (\Sigma_A, \sigma) \to (\Sigma_B, \sigma)$ and $\beta\colon (X, f) \to (\Sigma_B, \sigma)$, there exists a factor map $\alpha\colon (X, f) \to (\Sigma_A, \sigma)$ with $h\alpha = \beta$.

We now prove that, for many classes of languages, a universal system with the extension property exists and that it can be embedded into any other universal system of the same class. We fix for every $n > 0$ an alphabet A_n with n elements and consider only subshifts over these alphabets. This ensures that we have only a continuum of all subshifts. As in the theory of categories we consider diagrams of subshifts and factor maps, defined by graphs. A graph is a system $G = (V, E, s, t)$ where V is a set of vertices, E is a set of edges, and $s, t\colon E \to V$ are the source and target maps. A path from a vertex j to a vertex k is a sequence of edges $e_0, \ldots, e_m$ such that $s(e_0) = j$, $t(e_m) = k$, and $t(e_i) = s(e_{i+1})$ for $0 \leq i < m$. We consider only graphs without cycles, i.e., we assume that there is no path from i to i for any vertex i. A diagram over G consists of subshifts $(\Sigma_i)_{i\in V}$, and factor maps $(h_i\colon \Sigma_{s(i)} \to \Sigma_{t(i)})_{i\in E}$ such that whenever $e_0, \ldots, e_m$ and $d_0, \ldots, d_n$ are two paths with common source $s(e_0) = s(d_0)$ and target $t(e_m) = t(d_n)$, the compositions $h_{e_m} \cdots h_{e_0} = h_{d_n} \cdots h_{d_0}$ are equal (the commutativity condition). In particular we consider (finite) forks which are diagrams over graphs $V = \{0, \ldots, m\}$, $E = \{1, \ldots, m\}$, $s(i) = 0$, $t(i) = i$ for $0 < i \leq m$, where $m \geq 0$. If $m = 0$, then the fork consists only of the subshift Σ_0. If $m = 1$, then the fork consists only of a factor map $h_1\colon \Sigma_0 \to \Sigma_1$. A chain is a diagram over the graph with $V = E = \mathbb{N}$, $s(i) = i + 1$, $t(i) = i$, so it consists of factor maps $h_i\colon \Sigma_{i+1} \to \Sigma_i$, $i \geq 0$.

Theorem 1. *Let $\mathfrak{L}$ be a family of languages closed under factors and pullbacks, such that there exists only countably many subshifts of class $\mathfrak{L}$. Then there exists a universal system of class $\mathfrak{L}$ with the extension property.*

Proof. Since for any two subshifts there are only countably many factor maps between them, there exist only countably many finite forks with subshifts of class $\mathfrak{L}$. Consider now a sequence of forks which contains every fork infinitely many times. Thus we have integers $r_i, i \geq 0$, subshifts $\Sigma_{ij}, 0 \leq j \leq r_i$, and factor maps $h_{ij}\colon \Sigma_{i0} \to \Sigma_{ij}, 0 < j \leq r_i$. We construct by induction an infinite diagram containing all chains and all forks. We start at step 0 with the empty diagram. Suppose that at step n we have constructed a diagram with factor maps $f_e\colon X_{s(e)} \to X_{t(e)}$, edges $e \in E_n = \{0, \ldots, m-1\}$, and vertices $V_n = \{0, \ldots, p-1\}$. In step $n+1$ we extend the diagram by adding the nth fork to all possible places where its targets occur. This means that when $k_1, \ldots, k_{r_n} \in V_n$ are vertices such that $X_{k_i} = \Sigma_{ni}$ for $1 \leq i \leq r_n$, then we add a new vertex p and new edges $m, \ldots, m + r_n - 1$, and put $X_p = \Sigma_{n0}$, $f_{m+i-1} = h_{ni}$ for $0 \leq i \leq r_n$. We perform this extension, however, only if we obtain a diagram, i.e., if the commutativity condition is satisfied. There are only finitely many sequences $k_1, \ldots, k_{r_n}$ in V_n satisfying this condition and we perform the extension for every one of them. In particular, if for the nth fork we have $r_n = 0$, then this procedure amounts to adding the single subshift $X_p = \Sigma_{n0}$ and no factor maps. We obtain thus a diagram with vertices $V_{n+1} = \{0, \ldots, p' - 1\}$ and edges $E_{n+1} = \{0, \ldots, m' - 1\}$. Since for every edge $e \in E_{n+1}$ we have $s(e) > t(e)$, the diagram contains no cycles. Note that for some n this procedure need not be applicable, so that the graph would not be extended. In particular, the empty graph can be extended only with a fork of size $r_n = 0$ consisting of a single subshift. We now have the union of all these graphs with vertices $V = \bigcup_n V_n = \mathbb{N}$ and vertices $E = \bigcup_n E_n = \mathbb{N}$. For $n > 0$ put now

$$X_{|n} = \left\{ x \in \prod_{i=0}^{n-1} X_i;\ (\forall e \in E)(s(e) < n \Rightarrow f_e(x_{s(e)}) = x_{t(e)}) \right\}$$

and denote by $\pi_n\colon X_{|(n+1)} \to X_{|n}$ the projections $\pi_n(x_0, \ldots, x_{n+1}) = (x_0, \ldots, x_n)$. We claim that $(X, f) = \lim_n (X_{|n}, \sigma, \pi_n)$ is a universal system of class $\mathfrak{L}$ with the extension property. Since $\mathfrak{L}$ is closed under pullbacks, $(X_{|n}, \sigma)$ is a subshift of class $\mathfrak{L}$. If $\alpha\colon X \to A$ is a clopen partition, then α is uniformly continuous and there exist n and $\alpha_{|n}\colon X_{|n} \to A$, such that $\alpha(x) = \alpha_{|n}(x_0, \ldots, x_{n-1})$, so $\Sigma_\alpha(X, f) = \Sigma_{\alpha_{|n}}(X_{|n}, \sigma)$ is of class $\mathfrak{L}$. Thus (X, f) has languages of class $\mathfrak{L}$. To show that (X, f) is universal, consider a dynamical system (Y, g) of class $\mathfrak{L}$. By Proposition 1, there exist subshifts (Y_i, σ) of class $\mathfrak{L}$, and factor maps $g_i\colon (Y_{i+1}, \sigma) \to (Y_i, \sigma)$ such that (Y, g) is conjugate to $\lim_i (Y_i, \sigma, g_i)$. Since our starting sequence of forks contains every fork infinitely many times, it contains a subsequence of forks Y_0, $g_0\colon Y_1 \to Y_0$, $g_1\colon Y_2 \to Y_1, \ldots$. It follows that the diagram constructed contains the chain of g_i, i.e., there exists an increasing sequence $e_i \in E$ such that $f_{e_i} = g_i$. We thus have the projection $\pi\colon (X, f) \to \lim_i (X_{t(e_i)}, \sigma, f_{e_i})$, and $\lim_i (X_{t(e_i)}, \sigma, f_{e_i})$ is conjugate to (Y, g). Thus (Y, g) is a factor of (X, f). To show that (X, f) has the extension property, assume that $\Sigma_A \subseteq A^{\mathbb{N}}$, $\Sigma_B \subseteq B^{\mathbb{N}}$ are subshifts of class $\mathfrak{L}$, and that $\beta\colon (X, f) \to (\Sigma_B, \sigma)$ and $h\colon (\Sigma_A, \sigma) \to (\Sigma_B, \sigma)$ are factor maps. Since β is uniformly continuous, there exist n and $\beta_n\colon X_{|n} \to B$ such that $\beta(x) = \beta_n(x_0 \cdots x_{n-1}) = \beta_n \pi_{|n}(x)$ where $\pi_{|n}\colon X \to X_{|n}$ is a projection. Denote by Σ_C the pullback of β_n and h, and by $\nu\colon \Sigma_C \to X_{|n}$, $\pi_A\colon \Sigma_C \to \Sigma_A$ the projections. For

$i < n$ denote the projections by $\pi_{ni}\colon X_{|n} \to X_i$. Consider the fork $\pi_{ni}\nu\colon \Sigma_C \to X_i$, $0 \le i < n$. As this fork occurs infinitely many times in our starting sequence, there exists $m > n$ such that $X_m = \Sigma_C$ and $\pi_{ni}\nu\colon X_m \to X_i$, $0 \le i < n$, are the edges of the diagram satisfying the commutativity condition. Thus $\nu(x_m) = (x_0 \cdots x_{n-1})$, so $\nu\mu_m = \pi_{|n}$ where $\mu_m\colon X \to X_m$ is the projection. Define $\alpha\colon X \to \Sigma_A$ by $\alpha(x) = \pi_A(x_m) = \pi_A\mu_m(x)$. Then $h\alpha = h\pi_A\mu_m = \beta_n\nu\mu_m = \beta_n\pi_{|n} = \beta$.

$$\begin{tikzcd}
X \arrow[r, "\mu_m"] \arrow[d, "\pi_i"] \arrow[dr, "\pi_{|n}"] & X_m \arrow[r, "\pi_A"] \arrow[d, "\nu"] & \Sigma_A \arrow[d, "h"] \\
X_i & X_{|n} \arrow[l, "\pi_{ni}"'] \arrow[r, "\beta_n"] & \Sigma_B
\end{tikzcd}$$

□

Theorem 2. *Let $\mathfrak{L}$ be a family of languages closed under factors, let (X, f), (Y, g) be universal systems of class $\mathfrak{L}$, and suppose that (X, f) has the extension property. Then every factor map $\psi\colon (Y, g) \to (X, f)$ has a cross section, i.e., there exists an injective homomorphism $\varphi\colon (X, f) \to (Y, g)$, with $\psi\varphi = Id_X$.*

Proof. Let $(\alpha_i\colon X \to A_i)_{i\in\mathbb{N}}$ be a separating sequence of clopen partitions with $h_i\colon A_{i+1} \to A_i$ satisfying $h_i\alpha_{i+1} = \alpha_i$. Let $\beta_0\colon Y \to B_0$ be any partition finer than $\alpha_0\psi$ with $\operatorname{diam}(\beta_0) < 1$, so we have a map $\nu_0\colon B_0 \to A_0$ with $\nu_0\beta_0 = \alpha_0\psi$. We continue the construction by induction. Suppose we have a partition $\beta_{i-1}\colon Y \to B_{i-1}$ and a map $\nu_{i-1}\colon B_{i-1} \to A_{i-1}$ with $\nu_{i-1}\beta_{i-1} = \alpha_{i-1}\psi$ and $\operatorname{diam}(\beta_{i-1}) < 2^{-i+1}$. Define $B_i' = \{(a, b) \in A_i \times B_{i-1}; h_{i-1}(a) = \nu_{i-1}(b)\}$, and $\beta_i'\colon Y \to B_i'$ by $\beta_i'(y) = (\alpha_i\psi(y), \beta_{i-1}(y))$. Let $\beta_i\colon Y \to B_i$ be any clopen partition finer than β_i' with $\operatorname{diam}(\beta_i) < 2^{-i}$. Then we have maps $\nu_i\colon B_i \to A_i$ and $k_{i-1}\colon B_i \to B_{i-1}$ with $\nu_{i-1}k_{i-1} = h_{i-1}\nu_i$, $\nu_i\beta_i = \alpha_i\psi$, and $k_{i-1}\beta_i = \beta_{i-1}$. Thus $(\beta_i\colon Y \to B_i)_{i\in\mathbb{N}}$ is a separating sequence of clopen partitions, so (Y, g) is conjugate to $\lim_i(\Sigma_{\beta_i}(Y, g), \sigma, k_i^{\mathbb{N}})$. Since (X, f) has the extension property, there exist clopen partitions $\gamma_i\colon X \to B_i$, such that $\gamma_i^{\mathbb{N}}\colon (X, f) \to (\Sigma_{\beta_i}(Y, g), \sigma)$ is a factor map, and $\nu_i\gamma_i = \alpha_i$. It follows that there exists a homomorphism $\varphi\colon (X, f) \to (Y, g)$ with $\beta_i\varphi = \gamma_i$. Then for every i we have $\alpha_i = \nu_i\gamma_i = \nu_i\beta_i\varphi = \alpha_i\psi\varphi$, so $\psi\varphi = Id_X$.

$$\begin{tikzcd}
Y \arrow[r, "\beta_i"] \arrow[d, shift left, "\psi"] & B_i \arrow[r, "k_{i-1}"] \arrow[d, "\nu_i"] & B_{i-1} \arrow[d, "\nu_{i-1}"] \\
X \arrow[u, shift left, "\varphi"] \arrow[ur, "\gamma_i"] \arrow[r, "\alpha_i"] & A_i \arrow[r, "h_{i-1}"] & A_{i-1}
\end{tikzcd}$$

□

Theorem 3. *Let $\mathfrak{L}$ be a family of languages closed under factors. Then every two universal systems of class $\mathfrak{L}$ with the extension property are conjugate.*

Proof. Suppose that (X, f), (Y, g) are universal systems of class $\mathfrak{L}$ with the extension property. Let $\alpha_0\colon X \to A_0$ be any clopen partition. Since (Y, g) is universal, there exists a factor map $k\colon (Y, g) \to \Sigma_{\alpha_0}(X, f)$ and a clopen partition $\beta_0\colon Y \to A_0$ defined by $\beta_0(y) = k(y)_0$, so $\Sigma_{\beta_0}(Y, g) = \Sigma_{\alpha_0}(X, f)$. We continue the construction by induction. Let $\alpha_n\colon X \to A_n$, $\beta_n\colon Y \to A_n$ be clopen partitions with $\Sigma_{\beta_n}(Y, g) = \Sigma_{\alpha_n}(X, f)$. If n is even, pick a clopen partition $\beta_{n+1}\colon Y \to A_{n+1}$ finer than β_n with $\operatorname{diam}(\beta_{n+1}) < 2^{-n-1}$, so we have a map $h_n\colon A_{n+1} \to A_n$ with $h_n\beta_{n+1} = \beta_n$. Since (X, f) has the extension property, there exists a clopen partition $\alpha_{n+1}\colon X \to A_{n+1}$ with $h_n\alpha_{n+1} = \alpha_n$, and $\Sigma_{\beta_{n+1}}(Y, g) = \Sigma_{\alpha_{n+1}}(X, f)$. If n is odd, we construct first a partition $\alpha_{n+1}\colon X \to A_{n+1}$ with $\operatorname{diam}(\alpha_{n+1}) < 2^{-n-1}$, and then $\beta_{n+1}\colon Y \to A_{n+1}$. (The distinction between even and odd n is not essential; it suffices that both these cases occur infinitely many times.) It follows that both $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ are separating sequences of clopen partitions, and both (X, f) and (Y, g) are conjugate to $\lim_n(\Sigma_{\alpha_n}(X, f), \sigma, h_n)$. □

Thus Theorem 3 says that the universal system with the extension property is unique up to a conjugacy, and Theorem 2 says that it is a subsystem of any universal system (with or without the extension property).

5. Hierarchy of Language Classes

In complexity theory, various classes of languages are defined by different kind of computational devices (see, e.g., [17] or [16]). A language is regular (**REG**) if it can be recognized by a (deterministic or nondeterministic) finite automaton. It is context-free (**CF**) if it can be recognized by a nondeterministic push-down automaton. It is nondeterministic real time (**NREAL**) if it can be recognized by a nondeterministic (multitape) Turing machine in real time, i.e., in a number of steps equal to the length of the given word. It is context-sensitive (**CS**) if it can be recognized by a nondeterministic Turing machine in linear space. It is recursive (**REC**) if it can be recognized by a (deterministic or nondeterministic) Turing machine.

For dynamical purposes, two subclasses of the class of regular languages seem to be useful. We say that a (right central) regular language is periodic (**PER**) if its corresponding subshift is countable, and that it is bounded periodic (**BPER**) if its corresponding subshift is finite. We have

$$\mathbf{BPER} \subset \mathbf{PER} \subset \mathbf{REG} \subset \mathbf{CF} \subset \mathbf{NREAL} \subseteq \mathbf{CS} \subset \mathbf{REC}$$

(it is not known whether **NREAL** is a proper subclass of **CS**). All these classes are closed under factors and pullbacks so Theorems 1–3 apply to them.

Two-sided subshifts with regular languages are just sofic systems introduced by Weiss [18]. Nonregular and even nonrecursive subshifts occur naturally in the study of unimodal mappings on the real interval (see [7], [11], [19]). Many examples of language classification are provided by cellular automata. The trivial cellular automaton $f(x)_i = 0$ on $X = \{0, 1\}^{\mathbb{Z}}$ has bounded periodic languages. Indeed, its one-column language of $\alpha(x) = x_0$ is $\mathcal{L}_\alpha(X, f) = \{10^\infty, 0^\infty\}$. The cellular automaton $f(x)_i = x_{i-1}x_ix_{i+1}$ has periodic languages, in particular, $\mathcal{L}_\alpha(X, f) = \{1^\infty\} \cup \{1^n0^\infty; n \geq 0\}$. Blanchard and Maass [4] show that the intricate class of Coven aperiodic cellular automata (e.g.,

$f(x)_i = x_i + x_{i+1}(1 - x_{i+2}) \bmod 2$) have regular languages. Gilman [9] shows that every cellular automaton has context-sensitive languages and gives an example of a non-context-free cellular automaton $f(x)_i = x_{i+1}x_{i+2}$. Its one-column language consists of all $u \in \{0,1\}^{\mathbb{N}}$ such that if $u_i = 1$ and $u_{i+1} = 0$, then $u_j = 0$ for all $i < j < 2i$. For other examples see [12]. Beyond cellular automata we have a system with periodic languages $f(0^\infty) = 0^\infty$, $f(0^i 1x) = 0^i x$, where $x \in \{0,1\}^{\mathbb{N}}$, and a regular system with infinite topological entropy $f(x)_i = x_{2i}$.

Example 1 (Morse Sequence). The subshift $\Sigma = cl\{\sigma^i(W); i \geq 0\} \subseteq \{0,1\}^{\mathbb{N}}$, where *cl* is the topological closure and $W = 01101001\ldots$ is the Morse sequence, is of class **NREAL**.

Proof. Define $W^{(0)} = 0$, $W^{(n+1)} = W^{(n)}\widehat{W^{(n)}}$, so that $W = \lim_{i\to\infty} W^{(i)}$ (here $\hat{u}$ is obtained from u by reverting all its bits). The subshift Σ consists of all sequences, whose every subword occurs in W. Its language $\mathcal{L}(\Sigma)$ consists of all words which occur in W. It is easy to see that if $u \in \{0,1\}^*$, and $|W^{(n-1)}| < |u| \leq |W^{(n)}|$, then $u \in \mathcal{L}(\Sigma)$ iff $u \sqsubseteq W^{(n+3)}$, and $|W^{(n+3)}| = 16|W^{(n-1)}| \leq 16|u|$. Thus to recognize whether $u \in \mathcal{L}(\Sigma)$, it suffices to generate $W^{(n+3)}$, guess nondeterministically an integer $i \leq 16|u|$, and verify whether $u_j = W^{(n+3)}_{i+j}$ for $j < |u|$. This takes time linear in the length of u. By a result of Book and Greibach [5], the languages recognizable nondeterministically in linear time are recognizable nondeterministically in real time. □

Example 2 (Adding Machines). Let $k = (k_i)_{i\in\mathbb{N}}$ be a sequence of integers such that $k_i > 1$ for every i. Put $X = \prod_i X_i$, where $X_i = \{0, 1, \ldots, k_i - 1\}$, and define $f\colon X \to X$ by $f(x)_i = x_i$, if there exists $j < i$ with $x_j \neq k_j - 1$, and $f(x)_i = x_i + 1 \bmod k_i$ otherwise. Then (X, f) is a bounded periodic system with no periodic point (see [6]).

Example 3 (A Turing Machine). Let $X = \{0,1\}^{\mathbb{Z}} \times \{0,1\}^{\mathbb{Z}}$, and let $f(u, v) = (\sigma(u), \sigma(v))$ if $v_0 = 0$, and $f(u, v) = (\sigma^{-1}(u), \sigma(v))$ if $v_0 = 1$. Then (X, f) is of class **NREAL** (see [15] and [13]).

6. Transitive Systems

A language property of a different kind is transitivity. We say that a language $L \subseteq A^*$ is transitive if for every $u, v \in L$ there exists $w \in L$ such that $uwv \in L$. Clearly, dynamical systems with transitive languages are just topologically transitive systems. The class of transitive languages is closed under factors, but it is not closed under pullbacks. Moreover, the set of transitive subshifts is not countable, so Theorem 1 does not apply. Nevertheless, the adding machine (Example 2) with infinite multiplicities is a universal system with the extension property for the family of transitive periodic languages. (Note that a transitive language is periodic iff it is bounded periodic and a subshift with transitive periodic languages consists of a single periodic orbit.)

Given a sequence of integers $k = (k_i)_{i\in\mathbb{N}}$ with $k_i > 1$, define the multiplicity $m(p)$ of a prime p as the sum of the powers to which p occurs as a factor in all the k_i, allowing

$m(p)$ to be infinite, if p occurs as a factor in infinitely many k_i. Buescu and Stewart [6] prove that two adding machines are conjugate iff they have the same multiplicity functions.

Theorem 4. *The adding machine whose multiplicities are all infinite is the universal system with the extension property for the class of transitive periodic languages.*

Proof. Let (X, f) be the adding machine with infinite multiplicities, and let (Y, g) be a system with transitive periodic languages. Then for every clopen partition $\beta\colon Y \to B$, $\Sigma_\beta(Y, g)$ consists of a single periodic orbit, so it is a factor of $X_{|n}$ for high enough n. □

Note that transitive periodic languages are not closed under pullbacks, so this condition in the assumption of Theorem 1 is not necessary. We do not know whether a universal system exists also for the class of regular transitive systems.

Theorem 5. *In the classes of systems with bounded periodic, periodic, and regular languages, universal systems are not unique up to conjugacy.*

Proof. In the appropriate class $\mathfrak{L}$ of languages, let (X, f) be the universal system with the extension property, let (Z, h) be the universal transitive periodic system from Theorem 4, and let $(Y, g) = (X \cup Z, f \cup h)$ be their disjoint union. If $\alpha_i\colon X \to A_i$ is a separating sequence of clopen partitions, with $h_i\colon A_{i+1} \to A_i$ satisfying $h_i \alpha_{i+1} = \alpha_i$, then $\Sigma_{\alpha_i}(X, f)$ has a periodic point, and there exists a homomorphism $\gamma_i\colon (Z, h) \to \Sigma_{\alpha_i}(X, f)$. Moreover, we can choose γ_i so that $h_i^{\mathbb{N}} \gamma_{i+1} = \gamma_i$. Thus $\gamma\colon (Z, h) \to (X, f)$ defined by $\alpha_i^{\mathbb{N}} \gamma = \gamma_i$ is a homomorphism, so $\langle Id_X, \gamma\rangle\colon (Y, g) \to (X, f)$ is a factor map, so (Y, g) is a universal system of class $\mathfrak{L}$ too. Define now a factor map $\psi\colon (Y, g) \to (\Sigma_B, \sigma) = (\{\overline{0}, \overline{1}\}, \sigma)$ by $\psi(X) = \overline{0}$, $\psi(Z) = \overline{1}$. If $h\colon (\Sigma_A, \sigma) \to (\Sigma_B, \sigma)$ is a factor map such that $h^{-1}(\overline{1})$ is nontransitive, then there is no factor map $\varphi\colon (Y, g) \to (\Sigma_A, \sigma)$ with $h\varphi = \psi$. Thus (Y, g) does not have the extension property, so it is not conjugate to (X, f). □

The universal systems contructed here work for dynamical systems which are continuous mappings, whose acting semigroup is $\mathbb{N}$. This is why one-sided subshifts have been used. The results could be modified for homeomorphisms, whose acting group is $\mathbb{Z}$, just by using the two-sided subshifts instead.

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