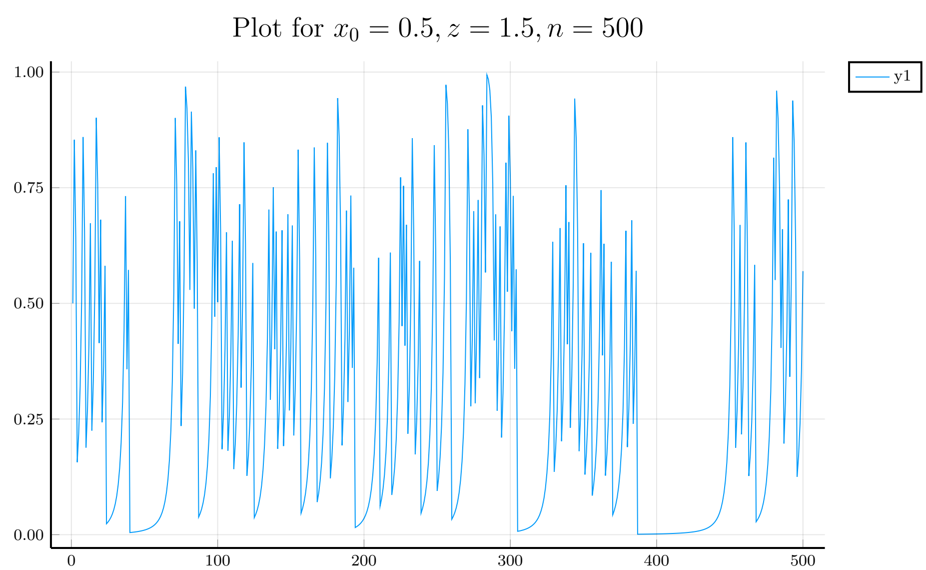
References

[BBG+04] Vieri Benci, C. Bonanno, Stefano Galatolo, G. Menconi, and M. Virgilio. Dynamical systems and computable information. *Discrete & Continuous Dynamical Systems - B*, 4(4):935, 2004.

In this paper, the authors examine the connections between information theory and dynamical systems. Particular attention is payed to the concept of *entropy*, paticularly in the contexts of *strings in a measure space* and the *relative entropy of a measure* (defined by the Kullback-Leibler divergence of an ergodic measure with respect to the standard Lebesgue measure). The authors then prove that certain conditions on the entropy of the measure can be used to obtain bounds on the amount of information required to describe orbits of the dynamical system. To this end, the authors present both theoretical results and "experimental" results (computations) for certain dynamical systems, e.g. the so-called *Manneville map* $T : [0,1] \to [0,1]$ defined by

$$T(x) = (x + x^z) \bmod 1$$

where $z > 1$ (example plotted below)



Overall: This paper provides an interesting second perspective on how classical ideas in computing intersect with dynamical systems. In some ways the analysis appears to be more tractable than that appearing in the papers we have read that examine the dynamics of turing machines. However, it is likely that our paper will not emphasize the ideas in this paper as much — if we employ it, it will probably be primarily for the introduction to information theory in ergodic systems it gives, and less for the actual results proved.

[DKB04] Jean-Charles Delvenne, Petr Kurka, and Vincent Blondel. Decidability and Universality in Symbolic Dynamical Systems. *arXiv*, Apr 2004.

In this paper, the authors examine generalizations of abstract Turing machines to the case of dynamical systems on symbolic spaces (compact metric spaces whose clopen sets form a countable basis). Since every compact metric space can be realized as the continuous image of the Cantor set, we can think of any symbolic space (up to homeomorphism) as being represented by a closed subset of the Cantor set. As a first example, the authors consider consider a simple dynamical-systems-flavored statement

of the halting problem: namely, given an *effective* system $f : X \to X$ and clopen subsets $U, V \subseteq X$, does there exist $x \in U$, $n \in \mathbb{N}$ such that $f^{(n)}(x) \in V$? Similar formulations are given for other variants of the halting problem, with some discussion of corresponding automata, as well as the concept of decidability.

After these introductory concepts, the authors give the definition of "computational universality" that they employ in their analysis. In addition to examples from Turing machines, the authors demonstrate how the definition applies to cases such as *cellular automata*, *tag systems*, and *Collatz functions*. They then prove some results linking the computational power of a system to its dynamical properties (e.g., Proposition 6: given an effective system such that for any transition function, the "observation system" (defined earlier in the paper) has clopen basins, then the system is decidable). Some results relating to shadowing properties and equicontinuity are also proven. Finally, the authors conclude with an example of a chaotic system that is computationally universal (previously thought to be impossible), and discuss how the shadowing property might provide a better heuristic for computational universality.

[KCG94] Pascal Koiran, Michel Cosnard, and Max Garzon. Computability with low-dimensional dynamical systems. *Theoretical Computer Science*, 132(1):113–128, Sep 1994.

Inspired by first-order recurrent neural networks, the authors examine the extent to which the computational universality of piecewise-linear maps is dependent on the dimension of the space. They prove that there exists functions in PL_2 (piecewise-linear maps from $\mathbb{R}^2$ to $\mathbb{R}^2$) that are universal (in the sense that they can simulate an arbitrary Turing machine), but are not capable of simulating arbitrary cellular automata unless the initial configuration is finite. By contrast, they show that PL_1 maps are not Turing-complete by employing a result that states 1D PL maps cannot have infinitely-many period-k cycles for infinitely-many k. However, they show that if the hypotheses are weakened to allow *countably* PL maps or continuous piecewise-monotone functions, then we do obtain Turing completeness.

[Kůr97] Petr Kůrka. On topological dynamics of Turing machines. *Theoretical Computer Science*, 174(1):203–216, Mar 1997.

This paper defines two dynamical systems based on the motion of the Turing Machine head and analyzes their behavior. It is interesting because it analyzes Turing Machines as dynamical systems, just like the Siegelmann paper analyzes dynamical systems as models of computation.

[Moo90] Cristopher Moore. Unpredictability and undecidability in dynamical systems. *Phys. Rev. Lett.*, 64(20):2354–2357, May 1990.

This paper analyzes the decidability of questions about three-dimensional dynamical systems. It is interesting because it examines the complexity of questions we might want to ask about dynamical systems.

[SF98] Hava T. Siegelmann and Shmuel Fishman. Analog computation with dynamical systems. *Physica D*, 120(1):214–235, Sep 1998.

This paper presents a way to understand the behavior of continuous dynamical systems as a form of computation. It is interesting because it talks about dynamical systems as models of computation, and analyzes them using tools from computational complexity.

[TZ05] J. V. Tucker and J. I. Zucker. Computable total functions on metric algebras, universal algebraic specifications and dynamical systems. *Journal of Logic and Algebraic Programming*, 62(1):71–108, Jan 2005.