Basic Category Theory

(Pseudo)-lecture 1

Introduction

Often in mathematics, we are interested in considering collections of objects with structure, and how that structure is modified or preserved when mapping to some other object. Category theory makes this a little more formal. First, we begin with some definitions.

Meta-objects

Definition 0.1 (Metagraph)**.** A *metagraph* consists of any collection (note: does not mean a set! See *proper classes*) of *objects* $o_1, o_2, \ldots$ (not necessarily countable), and *arrows* $a_1, a_2, \ldots$, together two operations that allow us to put the two in correspondence:

Definition 0.2 (Domain & Codomain)**.** *Domain*, which assigns to each arrow a_i an object $o_j = \text{dom}(a_i)$, and *Codomain*, which assigns to each arrow a_i an object $o_k = \text{cod}(a_i)$.

As is convention, we denote the above with one of the following diagrams:

$$a_i : o_j \to o_k \qquad\qquad o_j \xrightarrow{a_i} o_k$$

Figure 1: Equivalent representations of the above

Of course, these diagrams can become quite complicated:

$$\begin{tikzcd}
o_1 \arrow[dr, "a_3"] \arrow[rrd, bend left, "a_1"] \arrow[ddr, bend right, "a_2"'] & & o_2 \arrow[d, "a_8"] \\
& o_3 \arrow[ur, "a_9"] \arrow[r, "a_4"] \arrow[d, "a_5"] & o_4 \arrow[d, "a_6"] \\
& o_5 \arrow[r, "a_7"] & o_6
\end{tikzcd}$$

Figure 2: Complicated diagram

Using these metagraphs, we can construct a *metacategory*:

Definition 0.3 (Metacategory)**.** A *metacategory* is a metagraph, with the addition of two more operations:

Identity, which maps each object to an "identity" arrow:

$$id(o) = 1_o : o \to o;$$

Composition, which maps a pair of arrows a_1, a_2 (with $\text{dom}(\mathrm{a}_1) = \text{cod}(\mathrm{a}_2)$) to a "composite" arrow, $a_1 \circ a_2$, such that composition is associative:

$$\begin{tikzcd}
\text{dom}(a_2) \arrow[r, "a_2"] \arrow[dr, "a_2 \circ a_1"] & \text{cod}(a_2) = \text{dom}(a_1) \arrow[d, "a_1"] \\
& \text{cod}(a_1)
\end{tikzcd}$$

and for all arrows $a_1 : o_1 \to o_2$, $a_2 : o_2 \to o_3$, $\exists$ an identity arrow 1_{o_2} such that

$$\begin{tikzcd} o_1 \arrow[r, "a_1"] \arrow[dr, "a_1"'] & o_2 \arrow[d, "1_{o_2}"] \\ & o_2 \end{tikzcd} \qquad\qquad \begin{tikzcd} o_2 \arrow[d, "1_{o_2}"] \arrow[dr, "a_2"] & \\ o_2 \arrow[r, "a_2"] & o_3 \end{tikzcd}$$

Figure 3: Identity arrows

commute.

Categories, proper

In order to actually work with the objects we're used to commonly seeing in mathematics, we'll have to narrow our scope a bit. Notably, instead of considering a general *collection* of objects, we'll instead restrict ourselves to just sets. Hence, we can't consider the category of all categories, and whatnot. We skip the definition of a directed graph ("diagram scheme"), and jump straight to categories. Basically, we summarize the metacategory properties, just specifying that we're using sets now:

Definition 0.4 (Category)**.** A *category* C consists of

1) A set $\mathrm{ob}(C)$ of *objects*,
2) A set $\hom(C)$ of *arrows*,
3) A function $\mathrm{id} : \mathrm{ob}(C) \to \hom(C)$, by $o \mapsto 1_o$,
4) And a function $\circ : \hom(C) \times_{\mathrm{ob}(C)} \hom(C) \to \hom(C)$ (with $\times_{\mathrm{ob}(C)}$ giving composable pairs) with $(a_1, a_2) \mapsto a_1 \circ a_2$

A few terms:

(a) A category with every arrow identity is called *discrete*.

(b) A group is a category with just a single object. Here, the object really just represents the "group itself." The arrows are morphisms, which we can think of as being "left multiply by a" or "right multiply by a."

Functors

One key aspect of category theory that will be of particular interest to us is how we can translate structure from one space to another. This is the basis for, among other things, the field of representation theory. Here, we want to not only to put objects in correspondence with each other, but also preserve the morphism structure on them. In this sense, a functor is a morphism of categories.

Definition 0.5 (Functor)**.** Let C, D be categories, and let $c \in \mathrm{ob}(C)$, $f, g \in \hom(C)$. Then call $\mathcal{F} = (\mathcal{F}_o, \mathcal{F}_a)$ a functor if

$$\mathcal{F}_o : \mathrm{ob}(C) \to \mathrm{ob}(D) \qquad \mathcal{F}_a : \hom(C) \to \hom(D)$$

such that the following essential properties are preserved:

(a) $\forall (f : c \to c') \in \hom(C)$, $\mathcal{F}_a(f) : \mathcal{F}_o(c) \to \mathcal{F}_o(c')$, with

$$\mathcal{F}_a(1_c) = 1_{\mathcal{F}_o(c)}$$

and

$$\mathcal{F}_a(g \circ f) = \mathcal{F}_a(g) \circ \mathcal{F}_a(f).$$

before we go on, let's take a moment to really think what's going on here in terms of our diagrams. Loosely speaking, we're finding ways of folding our diagrams in on themselves such that we don't put two pairs of antiparallel arrows together. This is best expressed with a picture:

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[loop above, "1_a"] & b \arrow[loop above, "1_b"] \\
d \arrow[u, "h"] \arrow[r, "g"'] \arrow[loop below, "1_d"] & c \arrow[u, "i"'] \arrow[loop below, "1_c"]
\end{tikzcd}$$

Figure 4: Example starting category C

Suppose we want to map to the following category D:

$$\begin{tikzcd}
& & c' \arrow[loop above, "1_{c'}"] \\
& b' \arrow[ur, "g'"] \arrow[loop above, "1_{b'}"] & \\
a' \arrow[ur, "f'"] \arrow[loop above, "1_{a'}"] & &
\end{tikzcd}$$

Figure 5: Mapped-to category D

One way we could do so would be to define the following functor:

$$\mathcal{F}_o(d) = a'$$
$$\mathcal{F}_o(a) = b'$$
$$\mathcal{F}_o(c) = b'$$
$$\mathcal{F}_o(b) = c'$$

and

$$\mathcal{F}_a(1_a) = 1_{b'} \qquad \mathcal{F}_a(f) = g'$$
$$\mathcal{F}_a(1_c) = 1_{c'} \qquad \mathcal{F}_a(i) = g'$$
$$\mathcal{F}_a(1_d) = 1_{a'} \qquad \mathcal{F}_a(g) = f'$$
$$\mathcal{F}_a(1_b) = 1_{c'} \qquad \mathcal{F}_a(h) = f'$$

visually, we can interpret this as follows:

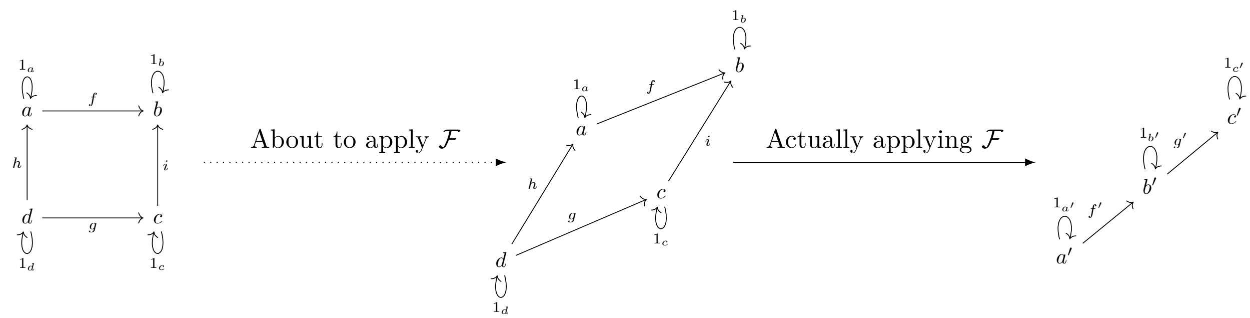

Figure 6: Demonstration of "folding"

Do in "class" — attempt to prove this is a valid way of thinking about stuff.

Just as we have "surjective" and "injective" functions, so too do we have "full" and "faithful" functors.

Definition 0.6. A functor $\mathcal{F} : C \to D$ is called *full* if $\forall c, c' \in \text{ob}(C)$, $\forall g : \mathcal{F}_o(c) \to \mathcal{F}_o(c')$, $\exists f : c \to c'$ st $g = \mathcal{F}_a(f)$. Note that this is *not* the same thing as being surjective with respect to morphisms and objects. In the case of objects, it is quite clear that it need not be surjective. With respect to morphisms, this is not quite so. However it *does* have to be surjective on $\hom(\mathcal{F}_o(\text{ob}(C)))$. Basically, if we forget about all the objects with only the identity morphism defined on them (call this resulting category C'), then fullness requires that C' map to an isomorphic directed sub-graph in D under $\mathcal{F}$.

Definition 0.7. A functor $\mathcal{F} : C \to D$ is called *faithful* if $\forall c, c' \in C$, $\forall f_1, f_2 : c \to c'$, then $\mathcal{F}_a(f_1) = \mathcal{F}_a(f_2) : \mathcal{F}_o(c) \to \mathcal{F}_o(c')$ implies $f_1 = f_2$. Again, $\mathcal{F}$ need not be injective on $\text{ob}(D)$, $\hom(D)$. The object case is fairly apparent, but with respect to morphisms, it's again a little more subtle. Essentially, faithfullness is only requiring that if we have two morphims *with the same domain and codomain* in C, we can't "squish" them together in D. However, morphisms with different domains and codomains can be assigned to the same morphism in D.

Definition 0.8. Subcategories are defined as one might expect a subset of objects and arrows such that for each arrow, we have both domain and codomain, for each object, we have identity arrow, and for each composable pair, we have their composite.

Natural Transformations

As it turns out, it's turtles all the way down. We now want to discuss morphsims of functors.

Definition 0.9. Let C, D be categories, and let $\mathcal{F}, \mathcal{G} : C \to D$ be functors. Then call η a natural transformation if

1) $\forall c \in C$, η assigns to c a morphism $\eta_c : \mathcal{F}_o(c) \to \mathcal{G}_o(c)$, called the *component* of η at c. This η_c must satisfy

2) $\forall f : c \to c'$, we have that $\eta_{c'} \circ \mathcal{F}_a(f) = \mathcal{G}_a(f) \circ \eta_c$

$$\begin{tikzcd}
\mathcal{F}_o(c) \arrow[r, "\eta_c"] \arrow[d, "\mathcal{F}_a(f)"'] & \mathcal{G}_o(c) \arrow[d, "\mathcal{G}_a(f)"] \\
\mathcal{F}_o(c') \arrow[r, "\eta_{c'}"] & \mathcal{G}_o(c')
\end{tikzcd}$$

Figure 7: Natural transformation

Essentially, this "glues" diagrams of functors together such that moving around the "image" one the $\mathcal{F}$ side and then hopping over to the $\mathcal{G}$ side leaves you in the same place as first hopping over to the $\mathcal{G}$ side and then taking an analogous path there (draw on board maybe). If η is invertible (that is, if every component of η is invertible), then we call η a *natural isomorphism* or *natural equivalence.*

Example: determinants, ring homomorphisms, and matrices. Can either apply (matrix) ring homomorphism first, then take det, or take det, then apply ring (matrix entries) homomorphism. Either gets the same result.

Monics, Epics, and Zeros.

Epic indeed. One might be wondering, *how the heck does this whole "groups as categories with one objec" thing play out? How do we distinguish between two morphisms, if they have the same domain and codomain?* All fantastic questions. Let's jump right into it.

In category theory, our primary focus will largely be on *morphisms.* As MacLane states, this is part of the power of Category Theory — instead of thinking about properties in terms of element-by-element treatments, we can instead think of them in terms of morphisms that sort of "take us between states." hence, we bring a slew of definitions.

Definition 0.10 (Invertible morphisms)**.** Let C a category, and $f : c \to c' \in \hom(C)$. Call f *invertible* if $\exists f^{-1} : c' \to c \in \hom(C)$ st $f \circ f^{-1} = 1_{c'}$, and $f^{-1} \circ f = 1_c$. Then f^{-1} is unique, and f, f^{-1} are called *isomorphisms.*

Definition 0.11 (Isomorphism)**.** Call $c, c' \in \mathrm{ob}(C)$ *isomorphic* if there is an isomorphism between them.

Definition 0.12 (Monic)**.** Let $a, c, c' \in \mathrm{ob}(C)$, and let $m \in \hom_C(c', a)$. Call m *monic* if $\forall f_1, f_2 \in \hom_C(c, c')$, $m \circ f_1 = m \circ f_2 \implies f_1 = f_2$. Basically, m never jumbles up parallel arrows from c to c', provided you apply it *after* the arrows. It is always *left*-cancellable, and sort of "always preserves information in the domain."

Definition 0.13 (Epi)**.** Epis are defined similarly, but are *right*-cancellable. In some sense, it never loses information in the co-domain.

Definition 0.14 (Right & Left Inverses)**.** Right and left inverses are defined in the expected manner. Let C a category, and $f \in \hom_C(c, c')$. Then $s \in \hom_C(c', c)$ is called a *right* inverse or *section* of f if $f \circ s = 1_{c'}$, and a *left* inverse or *retraction* if $s \circ f = 1_c$.

If f has a right inverse, it is epi, if it has a left inverse, it is monic. However the converse of these statements do not necessarily hold. Am still slightly confused about that; remember to ask Prof. Nelson.

Let $g \in \hom_C(c', c)$, $h \in \hom_C(c, c')$. Then if $g \circ h = 1_c$, call g a *split epi*, h a *split monic*, and $f = h \circ g$, call f idempotent.

Definition 0.15 (Terminal)**.** Call an object c *terminal* if $\forall c' \in \mathrm{ob}(C)$, $\exists! f : c' \to c$.

Definition 0.16 (Initial)**.** Call an object c *initial* if $\forall c' \in \text{ob}(C)$, $\exists! f : c \to c'$.

Definition 0.17 (Null)**.** Call an object c *null* if it is both initial and terminal.