Basic Algebraic Topology

Based on Kosniowski; Matveev

1. Picking up where we left off

1.1. Compactness

Last time, we finished by giving some basic definitions of comapctness, and whatnot. We'll begin with a small exercise to shake some of the cobwebs loose.

(a) Suppose that X has the finite complement topology. Show that X is compact. Show that each subset of X is compact.

(b) Prove that a topological space is compact if and only if whenever $\{C_j \mid j \in J\}$ is a collection of closed sets with $\bigcap_{j\in J} C_j = \varnothing$ then there is a finite subcollection $\{C_k \mid k \in K\}$ such that $\bigcap_{k\in K} C_k = \varnothing$.

(c) Let $\mathcal{F}$ be the topology on $\mathbb{R}$ defined by $U \in \mathcal{F}$ iff $\forall s \in U, \exists t > s$ s.t. $[s,t) \subseteq U$. Prove that the subset $[0,1]$ of $\mathcal{F}$ is not compact.

Now

(a) Let $U = \{U_i \mid i \in I\}$ be an open cover of X. Let $U_i \in U$. Then $X - U_i$ is finite. For every $x_j \in X - U_i$, $\exists U_j \in U$ s.t. $x \in U_j$ (because U is a cover). Then the set consisting of U_i and the U_j is a finite subcover, thus X is compact. Let $Y \subseteq X$. Then let $V = \{V_k \mid k \in K\}$ be an open cover of Y. Proceed an analogous argument to the above to obtain Y compact.

(b) Let X be a topological space, and suppose X is compact. Let $C = \left\{C_j \;\middle|\; j \in J, C_j = \overline{C_j}\right\}$ (i.e., the C_j are closed), and suppose

$$\bigcap_{j\in J} C_j = \varnothing.$$

By De Morgan's laws,

$$\bigcup_{j\in J} X - C_j = X$$

since C_j are all closed, then $X - C_j$ are open, hence this is an open cover of X, and there exists a finite subcover. Apply De Morgan's laws again to yield the desired result.

(c) Let $\varepsilon > 0$ be given. Let U be given by the open cover

$$U = \left\{ \left[1 - \frac{1}{2^i}, 1 - \frac{1}{2^{i+1}}\right) \;\middle|\; i = 0, 1, \dots \right\} \cup \{[1, 1+\varepsilon)\}$$

and note that all the sets in U are disjoint, and that they cover $[0,1]$. From disjointness, it follows there is no finite subcover.

1.2. A brief review of projections

On our first pass through, we didn't treat projection maps in a lot of depth, so we'll very briefly revisit them here.

Definition 1.1: Projection Maps

Let X, Y be topological spaces. Then define $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$ by

$$\pi_X(x, y) = x \qquad \pi_Y(x, y) = y$$

π_X and π_Y are referred to as the *product projections*. Note that both are continuous.

2. Compactness, Continued

Theorem 2.1. *Let (X, τ) be a topological space, and let $S \subseteq X$. Then S is compact in (X, τ) iff S is compact under the induced topology.*

Proof. Forwards direction is trivial. For the backwards direction, suppose S is compact in the induced topology. Let $U = \{U_i \mid i \in I\}$ be an open cover of S in (X, τ). Then $V = \{V_i = U_i \cap S \mid i \in I\}$ is an open cover of S in the induced topology, and hence by compactness there exists a finite subcover $V' = \{V_{i_k} \mid i_k \in I, k = 1, \ldots, n\}$. Now, take $U' = \{U_{i_k} \mid i_k \in I, k = 1, \ldots, n\}$. Then U' is a finite subcover of U. Since U was taken to be arbitrary, this implies S is comapct. ■

In the metrizable topologies we encountered in Real Analysis, we proved that continuous functions preserve compactness. However, we will now show that the same result holds in a general topological space.

Theorem 2.2 (Continuity and Compactness)**.** *Let $f : (X, \tau) \to (Y, \upsilon)$ be a continuous map. Let $S \subseteq X$ be a compact subspace. Then $f(S)$ is compact in Y.*

Proof. Let $V = \{V_i \mid i \in I\} \subseteq \upsilon$ be an open cover of $f(S)$. Because f is continuous, $U = \left\{ f^{-1}(V_i) \,\middle|\, i \in I \right\}$ is a collection of open sets covering $f^{-1}(f(S)) \supseteq S$. Since S is compact, there exists a finite subcover $U' = \left\{ f^{-1}(V_{i_k}) \,\middle|\, i_k \in I, k = 1, \ldots, n \right\}$ covering $f^{-1}(f(S))$. Then $V' = \{V_{i_k} \mid i_k \in I, k = 1, \ldots, n\}$ is a finite subcover of V. Thus $f(S)$ is compact in Y. ■

By virtue of the properties of continuous functions that we proved last time, some nice results follow immediately:

Corollary 2.1.

(*a*) *Any closed interval in $\mathbb{R}$ is compact.*

(*b*) *If X and Y are homeomorphic, then X is compact iff Y is.*

(*c*) *If X is compact, and Y is any set, then Y with the quotient topology induced by $f : X \to Y$ is compact.*

For completeness, we list some closure properties of compact subspaces:

Theorem 2.3. *Let (X, τ) be a topological space. Let $S = \{S_i \mid i \in I\} \subseteq$ be the collection of compact subspaces of X. Then*

(*a*) *If $S_1, S_2 \in S$, then $S_1 \cup S_2 \in S$ (union of two compact subspaces is compact). It follows by induction that any finite union of compact subspaces is compact.*

(*b*) *It is not the case that in an arbitrary topological space, an arbitrary intersection of compact spaces is compact (we need Hausdorffness). But for finite intersections, things work out.*

Theorem 2.4. *Let (X, τ) be a compact topological space, and let $S \subseteq X$ be closed. Then S is compact.*

Proof. Let $U = \{U_i \mid i \in I\}$ be an open cover of S. Let $U_0 = X - S$. Then U_0 is open, and $U \cup \{U_0\}$ covers X. Then since X is compact, there exists a finite subcover $U' = \{U_i \mid i \in I \cup \{0\}\}$. Take $U'' = U' - \{U_0\}$ to obtain a finite subcover of U. ■

I'm proud to have written this proof without looking at the one in the book at all, only to find later that they're basically identical.

Theorem 2.5. *Let X, Y be topological spaces. Then X, Y are compact iff $X \times Y$ is compact.*

Proof.

($\Rightarrow$): Suppose X, Y are compact. WTS $X \times Y$ is compact as well. Let $W = \{W_i \mid i \in I\}$ be an open cover of $X \times Y$. Note that $\forall y \in Y$, $X \times \{y\}$ is homeomorphic to X.

($\Leftarrow$): Suppose $X \times Y$ is compact. Let $U = \{U_i \mid i \in I\}$ be an open cover of X, and $V = \{V_j \mid j \in J\}$ be an open cover of Y. Then W given by

$$W = \left\{ \bigcup_{k \in K} W_k \;\middle|\; W_k \in U \times V \right\}$$

is an open cover of $X \times Y$, and thus admits a finite subcover:

$$W' = \{W_\ell \mid \ell \in L;\ |L| < \infty\} \subseteq W$$

Apply a similar trick something something boom

■

3. Hausdorff Spaces

Hausdorffness is an important property in Topology that essentially allows us to separate things from each other (our space is not "infinitely bunched-up" somewhere).

Definition 3.1

Let (X, τ) be a topological space. Then call X *Hausdorff* iff for all $x, y \in X$ such that $x \neq y$, there exist open sets U_x, U_y with $x \in U_x$, $y \in U_y$, and $U_x \cap U_y = \varnothing$.

Note that by a simple $\varepsilon/2$ argument, it follows that all metrizable spaces are Hausdorff.

Definition 3.2: T_k spaces

For $k = 0, 1, 2, 3, 4$, call X a T_k space if it satisfies the k-th condition below (indexing starts at 0):

T_0: For all $x, y \in X$ $(x \neq y)$, there is an open set U containing one but not the other (i.e., $x \in U$ and $y \notin U$, or $y \in U$ and $x \notin U$).

T_1: For all $x, y \in X$ $(x \neq y)$, there are open sets U, V such that $x \in U$, $y \in V$, and $x \notin V$, $y \notin U$.

T_2: For all x, y in X $(x \neq y)$, there are open sets U, V such that $x \in U$, $y \in V$, and

$U \cap V = \varnothing$ (there are disjoint neighborhoods about x and y).

T_3: X is T_1, and for all closed subsets F and points $x \notin F$, there exist open sets U, V such that $F \subseteq U$, $x \in V$, and $U \cap V = \varnothing$.

T_4: X is T_1, and for all pairs of disjoint closed subsetes F_1, F_2, there exist open sets U, V such that $F_1 \subseteq U$, $F_2 \subseteq V$

Naturally, if X and Y are homeomorphic topological spaces, and X is T_k, then Y is T_k as well. As an exercise, we construct spaces that are T_j (for $j = 0, \ldots, 4$) that are not $T_{i>j}$.

(X_0): Let $X_0 = (\mathbb{R}^{\geq 0}, \tau)$, where $\tau = \left\{[0, t) \;\middle|\; t \in \mathbb{R}^{\geq 0}\right\}$. Note that τ is indeed a topology on $\mathbb{R}^{\geq 0}$. Note X_0 is not T_1.

(X_1): Let $X_1 = (X, \tau)$ where

Theorem 3.1. *A space X is T_1 iff every point of X is closed.*

Proof. Suppose (X, τ) is T_1. Let $x \in X$ be arbitrary, and let $y \in X - \{x\}$. Then $\exists U_y \in \tau$ with $y \in U_y$, but $x \notin U_y$. Hence

$$\bigcup_{y \in X - \{x\}} U_y = X - \{x\}$$

is open, and so $\{x\}$ is closed.

Now suppose $\{x\}$ is closed. Then T_1 follows immediately. ∎

An important theorem:

Theorem 3.2. *Let A be a compact subset of a Hausdorff space X. Then A is closed.*

Proof. We define the following (bizarre) open cover. For all $a \in A$, for all $x \in X - A$, there exist disjoint open sets U_a, V_a such that $a \in U_a$, and $x \in V_a$. Then $U = \{U_a\}$ is an open cover of A, and thus contains a finite subcover $U' = \{U_{a'} \mid a' \in \mathcal{A} \subseteq A, |\mathcal{A}| < \infty\}$, and corresponding V'. Then note that

$$V_x = \bigcap_{a' \in A} V_{a'}$$

is an open set (closure under finite intersections) such that $V_x \cap A = \varnothing$. Hence

$$X - A = \bigcup_{x \in X} V_x$$

and so $X - A$ is open. Then A is closed. ∎

Now, it is time for another Very Important Theorem™.

Theorem 3.3. *Let (X, τ), (Y, υ) be topological spaces, with X compact and Y Hausdorff. Let $f : X \to Y$ be continuous. Then f is a homeomorphism iff f is a bijection.*

Proof.

($\Rightarrow$) : Suppose f is a bijection. Then f^{-1} exists, and $ff^{-1} = \mathrm{id}_Y$, $f^{-1}f = \mathrm{id}_X$. WTS f^{-1} is continuous. Let $S \subseteq X$ be closed. Then S is compact, hence $\left(f^{-1}\right)^{-1}(S) = f(S)$ is compact in Y. Thus $f(S)$ is closed as well, hence f^{-1} is continuous. Thus f is a homeomorphism.

($\Leftarrow$) : Suppose f is a homeomorphism. Then f is a bijection.

∎