Duality in Category Theory

Lecture 2

1 Introduction

First, we'll talk about about some of the hom-set stuff we didn't really get much time to touch on last time.

1.1 Hom-Sets

As as we will see, *hom-sets* play a big role in understanding functors. For example, calling a functor *full* is equivalent to it being *surjective* on a particular hom-set, and similar with faithful functors and injectivity.

Definition 1.1: Hom-set

Let $\mathbf{C}$ be a category, and $a, b \in \text{ob}(\mathbf{C})$. Then define the *hom-set* of (a, b) by

$$\hom_{\mathbf{C}}(a, b) = \{f \mid f \in \hom(\mathbf{C}),\ f : a \to b\}.$$

This suggests the following (equivalent) formulation of the category theory axioms:

Category Axioms (hom-set version)

(i) A small *category* is a set of objects $a, b, c, \ldots$ together with

(ii) A function that assigns to each ordered pair $\langle a, b\rangle$ a set $\hom_{\mathbf{C}}(a, b)$, and

(iii) A function *composition* for each ordered triple $\langle a, b, c\rangle$ with

$$\circ : \hom_{\mathbf{C}}(b, c) \times \hom_{\mathbf{C}}(a, b) \to \hom_{\mathbf{C}}(a, c)$$

(iv) For each $b \in \text{ob}(\mathbf{C})$, $\hom_{\mathbf{C}}(b, b)$ contains at least one element 1_b satisfying the "unit" axioms (see: right / left composition by unit)

(v) Hom-sets are pairwise disjoint. This assures dom, cod are well-defined for all morphisms.

In this context, we can define a functor in terms of hom-sets:

Definition 1.2: Functor (redux)

Let $\mathcal{T} : \mathbf{C} \to \mathbf{B}$ be defined with the usual object functor $\mathcal{T}_o$, together with a collection of functions

$$\mathcal{T}^{a,b} : \hom_{\mathbf{C}}(a, b) \to \hom_{\mathbf{B}}(\mathcal{T}_o(a), \mathcal{T}_o(b))$$

then $\mathcal{T}$ is *full* when every such $\mathcal{T}^{a,b}$ is surjective, and *faithful* when injective.

On to duality.

2 Duality

2.1 Motivation

Recall that last time, we defined functors between categories with

$$\mathcal{T} : \mathbf{C} \to \mathbf{B}$$

if

$$\mathcal{T} = \begin{cases} \mathcal{T}_o : \mathrm{ob}(\mathbf{C}) \to \mathrm{ob}(\mathbf{B}) \\ \mathcal{T}_a : \hom(\mathbf{C}) \to \hom(\mathbf{B}) \end{cases}$$

such that for all $c \in \mathrm{ob}(\mathbf{C})$, $\mathcal{T}(\mathrm{id}_c) = \mathrm{id}_{\mathcal{T}_o(c)}$, and for all $f, g \in \hom(\mathbf{C})$, $\mathcal{T}_a(g \circ f) = \mathcal{T}_a(g) \circ \mathcal{T}_a(f)$. However, as it turns out, this is a bit of a restrictive framework — we could imagine plenty of scenarios in which we might want to study something that *almost* looks like a functor, except that

$$\mathcal{T}_a(g \circ f) = \mathcal{T}_a(f) \circ \mathcal{T}_a(g).$$

such an object is called a *contravariant* functor, and we will examine them in more depth below. But first, note the fundamental similarity between the statements above — if we had objects a, b, c with morphisms f, g, h such that the following diagram commutes,

$$\begin{tikzcd} a \arrow[d, "f"'] \arrow[dr, "h = g \circ f"] & \\ b \arrow[r, "g"'] & c \end{tikzcd}$$

Figure 1: Example diagram

then the contravariant functor would create a diagram similar to

$$\begin{tikzcd} \mathcal{T}_o(a) & \\ \mathcal{T}_o(b) \arrow[u, "\mathcal{T}_a(f)"] & \mathcal{T}_o(c) \arrow[l, "\mathcal{T}_a(g)"] \arrow[ul, "\mathcal{T}_a(h) = \mathcal{T}_a(f) \circ \mathcal{T}_a(g)"'] \end{tikzcd}$$

Figure 2: Example diagram

certainly these two structures should be thought of as “similar” in some sense — if there's any justice in the world, we might even expect that some theorems we prove about functors in general will translate into guarantees about these so-called “contravariant functors.” Indeed, this is the case: but to make it formal, we need to introduce the idea of *duality*, which will prove surprisingly powerful.

2.2 Basic definitions

As you should now expect, we'll build up from axioms:

Definition 2.1: Atomic Statements

Let $\mathbf{C}$ be a category. Then if $a, b \in \mathrm{ob}(\mathbf{C})$, $f, g \in \hom(\mathbf{C})$, an *atomic statement* is a statement of the form:

(a) $a = \mathrm{dom}(f)$ or $b = \mathrm{cod}(f)$

(b) id_a is the identity map on a

(c) g can be composed with f to yield $h = g \circ f$.

That is, an atomic statement is just a statement about the axiomatic properties of categories.

From these, we can build phrases of *statements* using the formal grammar defined by propositional logic.

Definition 2.2: Sentences

A *sentence* is a statement (see above) in which we have no free variables; that is every variable is "bound" or "defined." For instance, the statement "for all $f \in \hom(\mathcal{C})$ there exists $a, b \in \mathrm{ob}(\mathcal{C})$ with $f : a \to b$" forms a sentence, while "$f : a \to b$" is an extreme case of one that does not (in the latter, we have no idea what any of the variables are actually referring to). In the context of category theory, the collection of sentences built out of atomic statements are known as *ETAC* ("the elementary theory of an abstract category").

Now, we introduce the concept of *duality*:

Definition 2.3: Duality

Let Σ be a statement of ETAC. Then the *dual* of Σ is intuitively the statement "in reverse," and is typically denoted by Σ^*. This can be formalized as simply flipping every "domain" statement into a "codomain" statement, and replacing "$h = g \circ f$" with "$h = f \circ g$." Some examples of duals are given below:

Statement Σ	Dual Statement Σ^*
$f : a \to b$	$f : b \to a$
$a = \mathrm{dom}(f)$	$a = \mathrm{cod}(f)$
$i = \mathrm{id}_a$	$i = \mathrm{id}_a$
$h = g \circ f$	$h = f \circ g$
f is monic	f is epic
u is a right inverse of h	u is a left inverse of h
f is invertible	f is invertible
t is a terminal object	t is an initial object

Note that $\Sigma^{**} = \Sigma$, and that if we prove some theorem about a statement Σ, the dual statement Σ^* can be proven as well.

2.3 Contravariance and Opposites

We might ask ourselves: what happens if we dual *every* statement in $\mathbf{C}$? What would some of the resulting objects' properties be? This is the focus of the next section.

Definition 2.4: Dual Category

Let $\mathbf{C}$ be a category. Then call $\mathbf{C}^*$ (also denoted $\mathbf{C}^{\mathrm{op}}$) the *dual* or *opposite* category iff for each statement Σ about $\mathbf{C}$, Σ^* holds about $\mathbf{C}^*$.

This results in the following properties:

Properties of the Dual Category

1) $\mathbf{C}$ and $\mathbf{C}^*$ have the same objects.

2) We can put each $f \in \hom(\mathbf{C})$ into a one-to-one relationship with $f^* \in \hom(\mathbf{C}^*)$.

3) For each $f \in \hom(\mathbf{C})$, $\operatorname{dom}(f) = \operatorname{cod}(f^*)$, and $\operatorname{cod}(f) = \operatorname{dom}(f^*)$.

4) For composable g, f, $(g \circ f)^* = f^* \circ g^*$.

5) If Σ^* is true in $\mathbf{C}$, then Σ is true in $\mathbf{C}^*$.

Recall our definition of contravariant functors on page 2. The important quality that we saw with contravariant functors was that they *reverse the order* of morphism composition. One might note that this sounds an awful lot like a dual property — and indeed, there is a connection here. We examine this in the theorem below.

Theorem 2.1 (Contravariant functors and duality)**.** *Let $\mathbf{C}, \mathbf{B}$ be categories, and let $T : \mathbf{C} \to \mathbf{B}$ be a contravariant functor. Then T can be expressed as a covariant functor from $\mathbf{C}^* \to \mathbf{B}$*

Proof. Let $a, b, c \in \operatorname{ob}(\mathbf{C})$, and let $f : a \to b$, $g : b \to c$. Then if $\mathcal{T} : \mathbf{C} \to \mathbf{B}$ is a contravariant functor, let $\overline{\mathcal{T}} : \mathbf{C}^* \to \mathbf{B}$ be defined by

$$\mathcal{T} f = \overline{\mathcal{T}} f^*$$

for all $f \in \hom(\mathbf{C})$. Then note that

$$\begin{aligned} \mathcal{T}(g \circ f) &= \overline{\mathcal{T}}((g \circ f)^*) \\ \mathcal{T}(f) \circ \mathcal{T}(g) &= \overline{\mathcal{T}}(f^* \circ g^*) \\ &= \overline{T}(f^*) \circ \overline{T}(g^*) \end{aligned}$$

thus, $\overline{\mathcal{T}}$ is a covariant functor from $\mathbf{C}^*$ to $\mathbf{B}$. ■

similarly, by the principle of duality, any covariant functor from $\mathbf{C} \to \mathbf{B}$ can be thought of as a contravariant functor from $\mathbf{C}^* \to \mathbf{B}$.

We look at an interesting example:

Definition 2.5: Hom-functors

Let $\mathbf{C}$ be a category with small hom-sets. Then since each hom-set is small, for every $a \in \operatorname{ob}(\mathbf{C})$, define the *covariant hom-functor*

$$\hom_{\mathbf{C}}(a, -) : \mathbf{C} \to \mathbf{Set}$$

such that the object function gives

$$b \mapsto \hom_{\mathbf{C}}(a, b)$$

and arrow function

$$[k : b \to b'] \mapsto \Big[\hom_{\mathbf{C}}(a, k) : \hom_{\mathbf{C}}(a, b) \to \hom_{\mathbf{C}}(a, b')\Big]$$

where the RHS of the above is defined by $f \mapsto k \circ f$ for each $f : a \to b$. Since the notation above is cumbersome, MacLane suggests instead using $k_\star$ ("composition with k on the left", or "the map induced by k").

Similarly, we define the *contravariant hom-functor* by, for each $b \in \operatorname{ob}\mathbf{C}$,

$$\hom_{\mathbf{C}}(-, b) : \mathbf{C}^* \to \mathbf{Set}$$

with arrow function

$$[g : a \to a'] \mapsto \Big[\hom_{\mathbf{C}}(g, b) : \hom_{\mathbf{C}}(a', b) \to \hom_{\mathbf{C}}(a, b)\Big]$$

defined by $f \mapsto f \circ g$. Again, omitting b, this is often written as $g^\star$. In summary,

$$k_\star f = k \circ f \qquad g^\star f = f \circ g$$

and the following diagram commutes.

$$\begin{tikzcd}
\hom_{\mathbf{C}}(a', b) \arrow[r, "g_\star"] \arrow[d, "k^\star"'] & \hom_{\mathbf{C}}(a, b) \arrow[d, "k_\star"] \\
\hom_{\mathbf{C}}(a', b') \arrow[r, "g^\star"] & \hom_{\mathbf{C}}(a, b')
\end{tikzcd}$$

2.4 Products of Categories

We define the category analog of the cartesian product:

Definition 2.6: Product of Categories

Let $\mathbf{B}, \mathbf{C}$ be categories. We construct the *product* of $\mathbf{B}$ and $\mathbf{C}$ as follows:

$$\mathrm{ob}(\mathbf{B} \times \mathbf{C}) = \mathrm{ob}(\mathbf{B}) \times \mathrm{ob}(\mathbf{C})$$

and

$$\hom(\mathbf{B} \times \mathbf{C}) = \hom(\mathbf{B}) \times \hom(\mathbf{C}).$$

composition is defined in the obvious manner. For all pairs of objects $\langle b.c \rangle$, $\langle b', c' \rangle$, $\langle b'', c'' \rangle$, and pairs of arrows $\langle f : b \to b', g : c \to c' \rangle$, $\langle f' : b' \to b'', g' : c' \to c'' \rangle$, then if

$$\langle b, c \rangle \xrightarrow{\langle f, g \rangle} \langle b', c' \rangle \xrightarrow{\langle f', g' \rangle} \langle b'', c'' \rangle$$

then we write

$$\langle f', g' \rangle \circ \langle f, g \rangle = \langle f' \circ f, g' \circ g \rangle$$

We can define *projection* functors in the obvious manner as well:

Definition 2.7

Consider functors P, Q with

$$P : \mathbf{B} \times \mathbf{C} \to \mathbf{B} \qquad Q : \mathbf{B} \times \mathbf{C} \to \mathbf{C}$$

such that, for all $\langle f, g \rangle \in \mathrm{ob}(\mathbf{B} \times \mathbf{C}), \hom(\mathbf{B} \times \mathbf{C})$,

$$P\langle f, g \rangle = f, \qquad Q\langle f, g \rangle = g.$$

Here, we will see the first of many descriptions of a "universal" property.

Theorem 2.2 (Look-ahead)**.** *Let* $\mathbf{D}$ *be a category, and* $\mathcal{R}, \mathcal{T}$ *be any two functors with* $\mathcal{R} : D \to B$, $\mathcal{T} : \mathbf{D} \to \mathbf{C}$. *Then* $\exists! \mathcal{F} : \mathbf{D} \to \mathbf{B} \times \mathbf{C}$ *such that the following diagram commutes:*

$$\begin{tikzcd} & \mathbf{D} \arrow[dl, "\mathcal{R}"'] \arrow[d, dotted, "\mathcal{F}"] \arrow[dr, "\mathcal{T}"] & \\ \mathbf{B} & \mathbf{B} \times \mathbf{C} \arrow[l, "\mathcal{P}"'] \arrow[r, "\mathcal{Q}"] & \mathbf{C} \end{tikzcd}$$

Figure 3: Uniqueness of inclusion

Proof. (Sketch) For the diagram to commute, for all $h \in \hom(\mathbf{D})$, we must have $\mathcal{F} = \langle \mathcal{R}h, \mathcal{T}h \rangle$. The universality follows pretty trivially. ■

Similarly to products of categories, we define products of functors:

Definition 2.8: Functor products

Let $\mathcal{U} : \mathbf{B} \to \mathbf{B}'$, $\mathcal{V} \to \mathbf{C} \to \mathbf{C}'$. Then we say $\mathcal{U}$ and $\mathcal{V}$ have a product $\mathcal{U} \times \mathcal{V} : \mathbf{B} \times \mathbf{C} \to \mathbf{B}' \times \mathbf{C}'$ if

$$(\mathcal{U} \times \mathcal{V})_o(\langle b, c \rangle) = \langle \mathcal{U}_o a, \mathcal{V}_o b \rangle \qquad (\mathcal{U} \times \mathcal{V})_a(\langle f, g \rangle) = \langle \mathcal{U}_a f, \mathcal{V}_a g \rangle$$

equivalently described by the following commutative diagram:

$$\begin{tikzcd} \mathbf{B} \arrow[d, "\mathcal{U}"'] & \mathbf{B} \times \mathbf{C} \arrow[l, "\mathcal{P}"'] \arrow[r, "\mathcal{Q}"] \arrow[d, dotted, "\mathcal{U} \times \mathcal{V}"] & \mathbf{C} \arrow[d, "\mathcal{V}"] \\ \mathbf{B}' & \mathbf{B}' \times \mathbf{C}' \arrow[l, "\mathcal{P}'"'] \arrow[r, "\mathcal{Q}'"] & \mathbf{C}' \end{tikzcd}$$

Figure 4: Functor products

Note that since functors are morphisms on categories, then $\times$ itself is a functor on small categories:

$$\times : \mathbf{Cat} \times \mathbf{Cat} \to \mathbf{Cat}.$$

In the above section, we've concerned ourself with functors mapping from a category to a product category (e.g., $\mathcal{F} : \mathbf{D} \to \mathbf{B} \times \mathbf{C}$). We will now examine the "dual" concept, that functors from a product category to a category.

Definition 2.9: Bifunctor

Let $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ be categories. Let $\mathcal{S} : \mathbf{B} \times \mathbf{C} \to \mathbf{D}$. Then $\mathcal{S}$ is called a *bifunctor*.

Simply put, a bifunctor is just a functor of two arguments.

Let $\mathcal{S}$ be the bifunctor given in the definition above. Then if we fix one of its arguments, we get something that is effectively a single-argument functor, similarly to how fixing an argument of a two-variable function yields something that's "effectively" a single-variable function. This process is described by the following theorem:

Theorem 2.3. *Let* $\mathbf{B}, \mathbf{C}$*, and* $\mathbf{D}$ *be categories. For all objects* $c \in \mathrm{ob}(\mathbf{C})$ *and* $b \in \mathrm{ob}(\mathbf{B})$*, let*

$$\mathcal{L}_c : \mathbf{B} \to \mathbf{D}, \qquad \mathcal{M}_b : \mathbf{C} \to \mathbf{D}$$

be functors such that $\mathcal{M}_b(c) = \mathcal{L}_c(b)$ *for all* b *and* c*. Then there exists a bifunctor* $\mathcal{S} : \mathbf{B} \times \mathbf{C} \to \mathbf{D}$ *with* $S(-, c) = \mathcal{L}_c$ *for all* c *and* $S(b, -) = \mathcal{M}_b$ *for all* b *if and only if for every pair of arrows* $f : b \to b'$ *and* $g : c \to c'$ *one has*

$$\mathcal{M}_{b'}(g) \circ \mathcal{L}_c(f) = \mathcal{L}_{c'}(f) \circ \mathcal{M}_b(g) \tag{1}$$

These equal arrows (1) *in* $\mathbf{D}$ *are then the value* $\mathcal{S}(f, g)$ *of the arrow function of* $\mathcal{S}$ *at* f *and* g*.*