Duality in Category Theory

Lecture 2

1 Introduction

First, we'll talk about about some of the hom-set stuff we didn't really get much time to touch on last time.

1.1 Hom-Sets

Hi here have a hom-set

2 Duality

We start with some definitions:

Definition 1: Atomic Statements

Let $\mathcal{C}$ be a category. Then if $a, b \in \text{ob}(\mathcal{C})$, $f, g \in \text{hom}(\mathcal{C})$, an *atomic statement* is a statement of the form:

(a) $a = \text{dom}(f)$ or $b = \text{cod}(f)$

(b) id_a is the identity map on a

(c) g can be composed with f to yield $h = g \circ f$.

That is, an atomic statement is just a statement about the axiomatic properties of categories.

From these, we can build phrases of *statements* in Σ, using the formal grammar defined by propositional logic.

Definition 2: Sentences

A *sentence* is a statement (see above) in which we have no free variables; that is every variable is "bound" or "defined." For instance, the statement "for all $f \in \text{hom}(\mathcal{C})$ there exists $a, b \in \text{ob}(\mathcal{C})$ with $f : a \to b$" forms a sentence, while "For all b, $f : a \to b$" does not. In the latter, we can't be sure what a, f are referring to.

Test of theorem

Theorem 2.1. *Let $\mathcal{B}, \mathcal{C}$, and $\mathcal{D}$ be categories. For all objects $c \in \text{ob}(\mathcal{C})$ and $b \in \text{ob}(\mathcal{B})$, let*

$$\mathcal{L}_c : \mathcal{B} \to \mathcal{D}, \qquad \mathcal{M}_b : \mathcal{C} \to \mathcal{D}$$

be functors such that $\mathcal{M}_b(c) = \mathcal{L}_c(b)$ for all b and c. Then there exists a bifunctor $\mathcal{S} : \mathcal{B} \times \mathcal{C} \to \mathcal{D}$ with $S(-, c) = \mathcal{L}_c$ for all c and $S(b, -) = \mathcal{M}_b$ for all b if and only if for every pair of arrows $f : b \to b'$ and $g : c \to c'$ one has

$$\mathcal{M}_{b'}(g) \circ \mathcal{L}_c(f) = \mathcal{L}_{c'}(f) \circ \mathcal{M}_b(g) \tag{1}$$

These equal arrows (1) in $\mathcal{D}$ are then the value $\mathcal{S}(f, g)$ of the arrow function of $\mathcal{S}$ at f and g.