Category Theory, Topology, and Knots
Notes from my Independent Study

(Or: *To All the Proofs I've Ever Loved)*

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1. Basic Category Theory

1.1 Introduction

Often in mathematics, we are interested in considering collections of objects with structure, and how that structure is modified or preserved when mapping to some other object. Category theory makes this a little more formal. First, we begin with some definitions.

1.1.1 Meta-objects

Definition 1.1.1: Metagraph

A *metagraph* consists of any collection (note: does not mean a set! See *proper classes*) of *objects* $o_1, o_2, \ldots$ (not necessarily countable), and *arrows* $a_1, a_2, \ldots$, together two operations that allow us to put the two in correspondence:

Definition 1.1.2: Domain & Codomain

Domain, which assigns to each arrow a_i an object $o_j = \mathrm{dom}(a_i)$, and *Codomain*, which assigns to each arrow a_i an object $o_k = \mathrm{cod}(a_i)$.

As is convention, we denote the above with one of the following diagrams:

$$a_i : o_j \to o_k \qquad\qquad o_j \xrightarrow{a_i} o_k$$

Figure 1.1: Equivalent representations of the above

Of course, these diagrams can become quite complicated:

$$\begin{tikzcd}
o_1 \arrow[dr, "a_3"] \arrow[drr, bend left, "a_1"] \arrow[ddr, bend right, "a_2"'] & & o_2 \arrow[d, "a_8"] \\
& o_3 \arrow[ur, "a_9"] \arrow[r, "a_4"] \arrow[d, "a_5"] & o_4 \arrow[d, "a_6"] \\
& o_5 \arrow[r, "a_7"] & o_6
\end{tikzcd}$$

Figure 1.2: Complicated diagram

Using these metagraphs, we can construct a *metacategory*:

Definition 1.1.3: Metacategory

A *metacategory* is a metagraph, with the addition of two more operations:

Identity, which maps each object to an "identity" arrow:

$$id(o) = 1_o : o \to o;$$

Composition, which maps a pair of arrows a_1, a_2 (with $\mathrm{dom(a_1)} = \mathrm{cod(a_2)}$) to a "composite" arrow, $a_1 \circ a_2$, such that composition is associative:

$$\begin{tikzcd}
\mathrm{dom}(a_2) \arrow[r, "a_2"] \arrow[dr, "a_2 \circ a_1"] & \mathrm{cod}(a_2) = \mathrm{dom}(a_1) \arrow[d, "a_1"] \\
& \mathrm{cod}(a_1)
\end{tikzcd}$$

and for all arrows $a_1 : o_1 \to o_2$, $a_2 : o_2 \to o_3$, $\exists$ an identity arrow 1_{o_2} such that

$$\begin{tikzcd}
o_1 \arrow[r, "a_1"] \arrow[dr, "a_1"'] & o_2 \arrow[d, "1_{o_2}"] \\
& o_2
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
o_2 \arrow[d, "1_{o_2}"'] \arrow[dr, "a_2"] & \\
o_2 \arrow[r, "a_2"] & o_3
\end{tikzcd}$$

Figure 1.3: Identity arrows

commute.

1.1.2 Categories, proper

In order to actually work with the objects we're used to commonly seeing in mathematics, we'll have to narrow our scope a bit. Notably, instead of considering a general *collection* of objects, we'll instead restrict ourselves to just sets. Hence, we can't consider the category of all categories, and whatnot. We skip the definition of a directed graph ("diagram scheme"), and jump straight to categories. Basically, we summarize the metacategory properties, just specifying that we're using sets now:

Definition 1.1.4: Category

A *category* C consists of

1) A set $\mathrm{ob}(C)$ of *objects*,
2) A set $\hom(C)$ of *arrows*,
3) A function $\mathrm{id} : \mathrm{ob}(C) \to \hom(C)$, by $o \mapsto 1_o$,
4) And a function $\circ : \hom(C) \times_{\mathrm{ob}(C)} \hom(C) \to \hom(C)$ (with $\times_{\mathrm{ob}(C)}$ giving composable pairs) with $(a_1, a_2) \mapsto a_1 \circ a_2$

A few terms:

(a) A category with every arrow identity is called *discrete*.

(b) A group is a category with just a single object. Here, the object really just represents the "group itself." The arrows are morphisms, which we can think of as being "left multiply by a" or "right multiply by a."

1.1.3 Functors

One key aspect of category theory that will be of particular interest to us is how we can translate structure from one space to another. This is the basis for, among other things, the field of representation theory. Here, we want to not only to put objects in correspondence with each other, but also preserve the morphism structure on them. In this sense, a functor is a morphism of categories.

Definition 1.1.5: Functor

Let C, D be categories, and let $c \in \text{ob}(C)$, $f, g \in \text{hom}(C)$. Then call $\mathcal{F} = (\mathcal{F}_o, \mathcal{F}_a)$ a functor if

$$\mathcal{F}_o : \text{ob}(C) \to \text{ob}(D) \qquad \mathcal{F}_a : \text{hom}(C) \to \text{hom}(D)$$

such that the following essential properties are preserved:

(a) $\forall (f : c \to c') \in \text{hom}(C)$, $\mathcal{F}_a(f) : \mathcal{F}_o(c) \to \mathcal{F}_o(c')$, with

$$\mathcal{F}_a(1_c) = 1_{\mathcal{F}_o(c)}$$

and

$$\mathcal{F}_a(g \circ f) = \mathcal{F}_a(g) \circ \mathcal{F}_a(f).$$

before we go on, let's take a moment to really think what's going on here in terms of our diagrams. Loosely speaking, we're finding ways of folding our diagrams in on themselves such that we don't put two pairs of antiparallel arrows together. This is best expressed with a picture:

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[loop above, "1_a"] & b \arrow[loop above, "1_b"] \\
d \arrow[u, "h"] \arrow[r, "g"'] \arrow[loop below, "1_d"] & c \arrow[u, "i"'] \arrow[loop below, "1_c"]
\end{tikzcd}$$

Figure 1.4: Example starting category C

Suppose we want to map to the following category D:

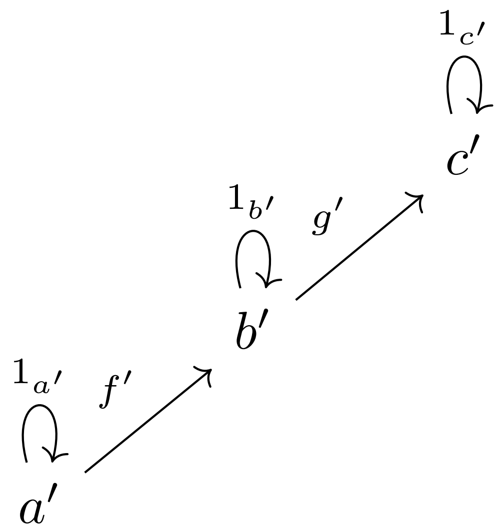

Figure 1.5: Mapped-to category D

One way we could do so would be to define the following functor:

$$\begin{aligned}\mathcal{F}_o(d) &= a'\\ \mathcal{F}_o(a) &= b'\\ \mathcal{F}_o(c) &= b'\\ \mathcal{F}_o(b) &= c'\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}_a(1_a) &= 1_{b'} & \qquad\qquad \mathcal{F}_a(f) &= g'\\ \mathcal{F}_a(1_c) &= 1_{c'} & \mathcal{F}_a(i) &= g'\\ \mathcal{F}_a(1_d) &= 1_{a'} & \mathcal{F}_a(g) &= f'\\ \mathcal{F}_a(1_b) &= 1_{c'} & \mathcal{F}_a(h) &= f'\end{aligned}$$

visually, we can interpret this as follows:

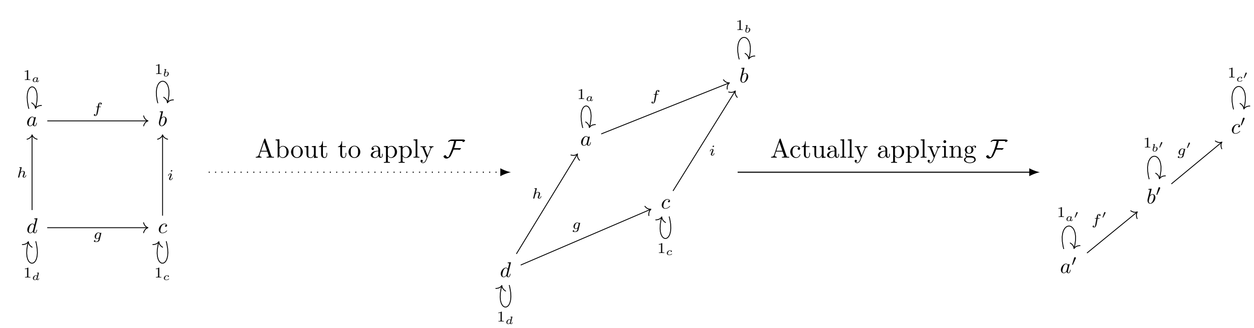

Figure 1.6: Demonstration of "folding"

Do in "class" — attempt to prove this is a valid way of thinking about stuff.

Just as we have "surjective" and "injective" functions, so too do we have "full" and "faithful" functors.

Definition 1.1.6

A functor $\mathcal{F} : C \to D$ is called *full* if $\forall c, c' \in \text{ob}(C)$, $\forall g : \mathcal{F}_o(c) \to \mathcal{F}_o(c')$, $\exists f : c \to c'$ s.t. $g = \mathcal{F}_a(f)$. Note that this is *not* the same thing as being surjective with respect to morphisms and objects. In the case of objects, it is quite clear that it need not be surjective. With respect to morphisms, this is not quite so. However it *does* have to be surjective on $\hom(\mathcal{F}_o(\text{ob}(C)))$. Basically, if we forget about all the objects with only the identity morphism defined on them (call this resulting category C'), then fullness requires that C' map to an isomorphic directed sub-graph in D under $\mathcal{F}$.

Definition 1.1.7

A functor $\mathcal{F} : C \to D$ is called *faithful* if $\forall c, c' \in C$, $\forall f_1, f_2 : c \to c'$, then $\mathcal{F}_a(f_1) = \mathcal{F}_a(f_2) : \mathcal{F}_o(c) \to \mathcal{F}_o(c')$ implies $f_1 = f_2$. Again, $\mathcal{F}$ need not be injective on $\text{ob}(D)$, $\hom(D)$. The object case is fairly apparent, but with respect to morphisms, it's again a little more subtle. Essentially, faithfullness is only requiring that if we have two morphims *with the same domain and codomain* in C, we can't "squish" them together in D. However, morphisms with different domains and codomains can be assigned to the same morphism in D.

Definition 1.1.8

Subcategories are defined as one might expect a subset of objects and arrows such that for each arrow, we have both domain and codomain, for each object, we have identity arrow, and for each composable pair, we have their composite.

1.2 Natural Transformations

As it turns out, it's turtles all the way down. We now want to discuss morphsims of functors.

Definition 1.2.1

Let C, D be categories, and let $\mathcal{F}, \mathcal{G} : C \to D$ be functors. Then call η a natural transformation if

1) $\forall c \in C$, η assigns to c a morphism $\eta_c : \mathcal{F}_o(c) \to \mathcal{G}_o(c)$, called the *component* of η at c. This η_c must satisfy
2) $\forall f : c \to c'$, we have that $\eta_{c'} \circ \mathcal{F}_a(f) = \mathcal{G}_a(f) \circ \eta_c$

$$\begin{tikzcd}
\mathcal{F}_o(c) \arrow[r, "\eta_c"] \arrow[d, "\mathcal{F}_a(f)"'] & \mathcal{G}_o(c) \arrow[d, "\mathcal{G}_a(f)"] \\
\mathcal{F}_o(c') \arrow[r, "\eta_{c'}"] & \mathcal{G}_o(c')
\end{tikzcd}$$

Figure 1.7: Natural transformation

Essentially, this "glues" diagrams of functors together such that moving around the "image" one the $\mathcal{F}$ side and then hopping over to the $\mathcal{G}$ side leaves you in the same place as first hopping over to the $\mathcal{G}$ side and then taking an analogous path there (draw on board maybe). If η is invertible (that is, if every component of η is invertible), then we call η a *natural isomorphism* or *natural equivalence*.

Example: determinants, ring homomorphisms, and matrices. Can either apply (matrix) ring homomorphism first, then take det, or take det, then apply ring (matrix entries) homomorphism. Either gets the same result.

1.3 Monics, Epics, and Zeros.

Epic indeed. One might be wondering, *how the heck does this whole "groups as categories with one objec" thing play out? How do we distinguish between two morphisms, if they have the same domain and codomain?* All fantastic questions. Let's jump right into it.

In category theory, our primary focus will largely be on *morphisms*. As MacLane states, this is part of the power of Category Theory — instead of thinking about properties in terms of element-by-element treatments, we can instead think of them in terms of morphisms that sort of "take us between states." hence, we bring a slew of definitions.

Definition 1.3.1: Invertible morphisms

Let C a category, and $f : c \to c' \in \hom(C)$. Call f *invertible* if $\exists f^{-1} : c' \to c \in \hom(C)$ s.t. $f \circ f^{-1} = 1_{c'}$, and $f^{-1} \circ f = 1_c$. Then f^{-1} is unique, and f, f^{-1} are called *isomorphisms*.

Definition 1.3.2: Isomorphism

Call $c, c' \in \text{ob}(C)$ *isomorphic* if there is an isomorphism between them.

Definition 1.3.3: Monic

Let $a, c, c' \in \text{ob}(C)$, and let $m \in \hom_C(c', a)$. Call m *monic* if $\forall f_1, f_2 \in \hom_C(c, c')$, $m \circ f_1 = m \circ f_2 \implies f_1 = f_2$. Basically, m never jumbles up parallel arrows from c to c', provided you apply it *after* the arrows. It is always *left*-cancellable, and sort of "always preserves information in the domain."

Definition 1.3.4: Epi

Epis are defined similarly, but are *right*-cancellable. In some sense, it never loses information in the co-domain.

Definition 1.3.5: Right & Left Inverses

Right and left inverses are defined in the expected manner. Let C a category, and $f \in \hom_C(c, c')$. Then $s \in \hom_C(c', c)$ is called a *right* inverse or *section* of f if $f \circ s = 1_{c'}$, and a *left* inverse or *retraction* if $s \circ f = 1_c$.

If f has a right inverse, it is epi, if it has a left inverse, it is monic. However the converse of these statements do not necessarily hold. Am still slightly confused about that; remember to ask Prof. Nelson.

Let $g \in \hom_C(c', c)$, $h \in \hom_C(c, c')$. Then if $g \circ h = 1_c$, call g a *split epi*, h a *split monic*, and $f = h \circ g$, call f idempotent.

Definition 1.3.6: Terminal

Call an object c *terminal* if $\forall c' \in \mathrm{ob}(C)$, $\exists! f : c' \to c$.

Definition 1.3.7: Initial

Call an object c *initial* if $\forall c' \in \mathrm{ob}(C)$, $\exists! f : c \to c'$.

Definition 1.3.8: Null

Call an object c *null* if it is both initial and terminal.

2. Duality in Category Theory

2.1 Introduction

First, we'll talk about about some of the hom-set stuff we didn't really get much time to touch on last time.

2.1.1 Hom-Sets

As as we will see, *hom-sets* play a big role in understanding functors. For example, calling a functor *full* is equivalent to it being *surjective* on a particular hom-set, and similar with faithful functors and injectivity.

Definition 2.1.1: Hom-set

Let $\mathbf{C}$ be a category, and $a, b \in \mathrm{ob}(\mathbf{C})$. Then define the *hom-set* of (a, b) by

$$\hom_{\mathbf{C}}(a, b) = \{f \mid f \in \hom(\mathbf{C}),\ f : a \to b\}.$$

This suggests the following (equivalent) formulation of the category theory axioms:

Category Axioms (hom-set version)

(i) A small *category* is a set of objects $a, b, c, \ldots$ together with

(ii) A function that assigns to each ordered pair $\langle a, b\rangle$ a set $\hom_{\mathbf{C}}(a, b)$, and

(iii) A function *composition* for each ordered triple $\langle a, b, c\rangle$ with

$$\circ : \hom_{\mathbf{C}}(b, c) \times \hom_{\mathbf{C}}(a, b) \to \hom_{\mathbf{C}}(a, c)$$

(iv) For each $b \in \mathrm{ob}(\mathbf{C})$, $\hom_{\mathbf{C}}(b, b)$ contains at least one element 1_b satisfying the "unit" axioms (see: right / left composition by unit)

(v) Hom-sets are pairwise disjoint. This assures $\mathrm{dom}, \mathrm{cod}$ are well-defined for all morphisms.

In this context, we can define a functor in terms of hom-sets:

Definition 2.1.2: Functor (redux)

Let $\mathcal{T} : \mathbf{C} \to \mathbf{B}$ be defined with the usual object functor $\mathcal{T}_o$, together with a collection of functions

$$\mathcal{T}^{a,b} : \hom_{\mathbf{C}}(a, b) \to \hom_{\mathbf{B}}(\mathcal{T}_o(a), \mathcal{T}_o(b))$$

then $\mathcal{T}$ is *full* when every such $\mathcal{T}^{a,b}$ is surjective, and *faithful* when injective.

On to duality.

2.2 Duality

2.2.1 Motivation

Recall that last time, we defined functors between categories with

$$\mathcal{T} : \mathbf{C} \to \mathbf{B}$$

if

$$\mathcal{T} = \begin{cases} \mathcal{T}_o : \text{ob}(\mathbf{C}) \to \text{ob}(\mathbf{B}) \\ \mathcal{T}_a : \hom(\mathbf{C}) \to \hom(\mathbf{B}) \end{cases}$$

such that for all $c \in \text{ob}(\mathbf{C})$, $\mathcal{T}(\text{id}c) = \text{id}\mathcal{T}_o(c)$, and for all $f, g \in \hom(\mathbf{C})$, $\mathcal{T}_a(g \circ f) = \mathcal{T}_a(g) \circ \mathcal{T}_a(f)$. However, as it turns out, this is a bit of a restrictive framework — we could imagine plenty of scenarios in which we might want to study something that *almost* looks like a functor, except that

$$\mathcal{T}_a(g \circ f) = \mathcal{T}_a(f) \circ \mathcal{T}_a(g).$$

such an object is called a *contravariant* functor, and we will examine them in more depth below. But first, note the fundamental similarity between the statements above — if we had objects a, b, c with morphisms f, g, h such that the following diagram commutes,

$$\begin{tikzcd} a \arrow[d, "f"'] \arrow[dr, "h = g \circ f"] & \\ b \arrow[r, "g"'] & c \end{tikzcd}$$

Figure 2.1: Example diagram

then the contravariant functor would create a diagram similar to

$$\begin{tikzcd} \mathcal{T}_o(a) & \\ \mathcal{T}_o(b) \arrow[u, "\mathcal{T}_a(f)"] & \mathcal{T}_o(c) \arrow[l, "\mathcal{T}_a(g)"] \arrow[ul, "\mathcal{T}_a(h) = \mathcal{T}_a(f) \circ \mathcal{T}_a(g)"'] \end{tikzcd}$$

Figure 2.2: Example diagram

certainly these two structures should be thought of as "similar" in some sense — if there's any justice in the world, we might even expect that some theorems we prove about functors in general will translate into guarantees about these so-called "contravariant functors." Indeed, this is the case: but to make it formal, we need to introduce the idea of *duality*, which will prove surprisingly powerful.

2.2.2 Basic definitions

As you should now expect, we'll build up from axioms:

Definition 2.2.1: Atomic Statements

Let $\mathbf{C}$ be a category. Then if $a, b \in \text{ob}(\mathbf{C})$, $f, g \in \hom(\mathbf{C})$, an *atomic statement* is a statement of the form:

(a) $a = \text{dom}(f)$ or $b = \text{cod}(f)$

(b) $\text{id}a$ is the identity map on a

(c) g can be composed with f to yield $h = g \circ f$.

That is, an atomic statement is just a statement about the axiomatic properties of categories.

From these, we can build phrases of *statements* using the formal grammar defined by propositional logic.

Definition 2.2.2: Sentences

A *sentence* is a statement (see above) in which we have no free variables; that is every variable is "bound" or "defined." For instance, the statement "for all $f \in \text{hom}(\mathcal{C})$ there exists $a, b \in \text{ob}(\mathcal{C})$ with $f : a \to b$" forms a sentence, while "$f : a \to b$" is an extreme case of one that does not (in the latter, we have no idea what any of the variables are actually referring to). In the context of category theory, the collection of sentences built out of atomic statements are known as *ETAC* ("the elementary theory of an abstract category").

Now, we introduce the concept of *duality*:

Definition 2.2.3: Duality

Let Σ be a statement of ETAC. Then the *dual* of Σ is intuitively the statement "in reverse," and is typically denoted by Σ^*. This can be formalized as simply flipping every "domain" statement into a "codomain" statement, and replacing "$h = g \circ f$" with "$h = f \circ g$." Some examples of duals are given below:

Statement Σ	Dual Statement Σ^*
$f : a \to b$	$f : b \to a$
$a = \text{dom}(f)$	$a = \text{cod}(f)$
$i = \text{id}a$	$i = \text{id}a$
$h = g \circ f$	$h = f \circ g$
f is monic	f is epic
u is a right inverse of h	u is a left inverse of h
f is invertible	f is invertible
t is a terminal object	t is an initial object

Note that $\Sigma^{**} = \Sigma$, and that if we prove some theorem about a statement Σ, the dual statement Σ^* can be proven as well.

2.2.3 Contravariance and Opposites

We might ask ourselves: what happens if we dual *every* statement in $\mathbf{C}$? What would some of the resulting objects' properties be? This is the focus of the next section.

Definition 2.2.4: Dual Category

Let $\mathbf{C}$ be a category. Then call $\mathbf{C}^*$ (also denoted $\mathbf{C}^{\text{op}}$) the *dual* or *opposite* category iff for each statement Σ about $\mathbf{C}$, Σ^* holds about $\mathbf{C}^*$.

This results in the following properties:

Properties of the Dual Category

1) $\mathbf{C}$ and $\mathbf{C}^*$ have the same objects.
2) We can put each $f \in \hom(\mathbf{C})$ into a one-to-one relationship with $f^* \in \hom(\mathbf{C}^*)$.
3) For each $f \in \hom(\mathbf{C})$, $\operatorname{dom}(f) = \operatorname{cod}(f^*)$, and $\operatorname{cod}(f) = \operatorname{dom}(f^*)$.
4) For composable g, f, $(g \circ f)^* = f^* \circ g^*$.
5) If Σ^* is true in $\mathbf{C}$, then Σ is true in $\mathbf{C}^*$.

Recall our definition of contravariant functors on page 2. The important quality that we saw with contravariant functors was that they *reverse the order* of morphism composition. One might note that this sounds an awful lot like a dual property — and indeed, there is a connection here. We examine this in the theorem below.

Theorem 2.2.1 (Contravariant functors and duality)**.** *Let* $\mathbf{C}, \mathbf{B}$ *be categories, and let* $T : \mathbf{C} \to \mathbf{B}$ *be a contravariant functor. Then* T *can be expressed as a covariant functor from* $\mathbf{C}^* \to \mathbf{B}$

Proof. Let $a, b, c \in \operatorname{ob}(\mathbf{C})$, and let $f : a \to b$, $g : b \to c$. Then if $\mathcal{T} : \mathbf{C} \to \mathbf{B}$ is a contravariant functor, let $\overline{\mathcal{T}} : \mathbf{C}^* \to \mathbf{B}$ be defined by

$$\mathcal{T} f = \overline{\mathcal{T}} f^*$$

for all $f \in \hom(\mathbf{C})$. Then note that

$$\begin{aligned} \mathcal{T}(g \circ f) &= \overline{\mathcal{T}}((g \circ f)^*) \\ \mathcal{T}(f) \circ \mathcal{T}(g) &= \overline{\mathcal{T}}(f^* \circ g^*) \\ &= \overline{\mathcal{T}}(f^*) \circ \overline{\mathcal{T}}(g^*) \end{aligned}$$

thus, $\overline{\mathcal{T}}$ is a covariant functor from $\mathbf{C}^*$ to $\mathbf{B}$. ■

similarly, by the principle of duality, any covariant functor from $\mathbf{C} \to \mathbf{B}$ can be thought of as a contravariant functor from $\mathbf{C}^* \to \mathbf{B}$.

We look at an interesting example:

Definition 2.2.5: Hom-functors

Let $\mathbf{C}$ be a category with small hom-sets. Then since each hom-set is small, for every $a \in \operatorname{ob}(\mathbf{C})$, define the *covariant hom-functor*

$$\hom_{\mathbf{C}}(a, -) : \mathbf{C} \to \mathbf{Set}$$

such that the object function gives

$$b \mapsto \hom_{\mathbf{C}}(a, b)$$

and arrow function

$$[k : b \to b'] \mapsto \Big[\hom_{\mathbf{C}}(a, k) : \hom_{\mathbf{C}}(a, b) \to \hom_{\mathbf{C}}(a, b')\Big]$$

where the RHS of the above is defined by $f \mapsto k \circ f$ for each $f : a \to b$. Since the notation above is cumbersome, MacLane suggests instead using $k_\star$ ("composition with k on the left", or "the map induced by k").

Similarly, we define the *contravariant hom-functor* by, for each $b \in \operatorname{ob}\mathbf{C}$,

$$\hom_{\mathbf{C}}(-, b) : \mathbf{C}^* \to \mathbf{Set}$$

with arrow function

$$[g : a \to a'] \mapsto \Big[\hom_{\mathbf{C}}(g, b) : \hom_{\mathbf{C}}(a', b) \to \hom_{\mathbf{C}}(a, b)\Big]$$

defined by $f \mapsto f \circ g$. Again, omitting b, this is often written as $g^\star$. In summary,

$$k_\star f = k \circ f \qquad g^\star f = f \circ g$$

and the following diagram commutes.

$$\begin{tikzcd}
\hom_{\mathbf{C}}(a', b) \arrow[r, "g_\star"] \arrow[d, "k^\star"'] & \hom_{\mathbf{C}}(a, b) \arrow[d, "k_\star"] \\
\hom_{\mathbf{C}}(a', b') \arrow[r, "g^\star"] & \hom_{\mathbf{C}}(a, b')
\end{tikzcd}$$

2.2.4 Products of Categories

We define the category analog of the cartesian product:

Definition 2.2.6: Product of Categories

Let $\mathbf{B}, \mathbf{C}$ be categories. We construct the *product* of $\mathbf{B}$ and $\mathbf{C}$ as follows:

$$\operatorname{ob}(\mathbf{B} \times \mathbf{C}) = \operatorname{ob}(\mathbf{B}) \times \operatorname{ob}(\mathbf{C})$$

and

$$\hom(\mathbf{B} \times \mathbf{C}) = \hom(\mathbf{B}) \times \hom(\mathbf{C}).$$

composition is defined in the obvious manner. For all pairs of objects $\langle b.c \rangle$, $\langle b', c' \rangle$, $\langle b'', c'' \rangle$, and pairs of arrows $\langle f : b \to b', g : c \to c' \rangle$, $\langle f' : b' \to b'', g' : c' \to c'' \rangle$, then if

$$\langle b, c \rangle \xrightarrow{\langle f, g \rangle} \langle b', c' \rangle \xrightarrow{\langle f', g' \rangle} \langle b'', c'' \rangle$$

then we write

$$\langle f', g' \rangle \circ \langle f, g \rangle = \langle f' \circ f, g' \circ g \rangle$$

We can define *projection* functors in the obvious manner as well:

Definition 2.2.7

Consider functors P, Q with

$$P : \mathbf{B} \times \mathbf{C} \to \mathbf{B} \qquad Q : \mathbf{B} \times \mathbf{C} \to \mathbf{C}$$

such that, for all $\langle f, g\rangle \in \mathrm{ob}(\mathbf{B}\times\mathbf{C}), \hom(\mathbf{B}\times\mathbf{C})$,

$$P\langle f, g\rangle = f, \qquad Q\langle f, g\rangle = g.$$

Here, we will see the first of many descriptions of a "universal" property.

Theorem 2.2.2 (Look-ahead)**.** *Let* $\mathbf{D}$ *be a category, and* $\mathcal{R}, \mathcal{T}$ *be any two functors with* $\mathcal{R} : D \to B$, $\mathcal{T} : \mathbf{D} \to \mathbf{C}$. *Then* $\exists! \mathcal{F} : \mathbf{D} \to \mathbf{B}\times\mathbf{C}$ *such that the following diagram commutes:*

$$\begin{tikzcd}
 & \mathbf{D} \arrow[dl, "\mathcal{R}"'] \arrow[d, dotted, "\mathcal{F}"] \arrow[dr, "\mathcal{T}"] & \\
\mathbf{B} & \mathbf{B}\times\mathbf{C} \arrow[l, "\mathcal{P}"'] \arrow[r, "\mathcal{Q}"] & \mathbf{C}
\end{tikzcd}$$

Figure 2.3: Uniqueness of inclusion

Proof. (Sketch) For the diagram to commute, for all $h \in \hom(\mathbf{D})$, we must have $\mathcal{F} = \langle \mathcal{R}h, \mathcal{T}h\rangle$. The universality follows pretty trivially. ∎

Similarly to products of categories, we define products of functors:

Definition 2.2.8: Functor products

Let $\mathcal{U} : \mathbf{B} \to \mathbf{B}'$, $\mathcal{V} \to \mathbf{C} \to \mathbf{C}'$. Then we say $\mathcal{U}$ and $\mathcal{V}$ have a product $\mathcal{U}\times\mathcal{V} : \mathbf{B}\times\mathbf{C} \to \mathbf{B}'\times\mathbf{C}'$ if

$$(\mathcal{U}\times\mathcal{V})_o(\langle b, c\rangle) = \langle \mathcal{U}_o a, \mathcal{V}_o b\rangle \qquad (\mathcal{U}\times\mathcal{V})_a(\langle f, g\rangle) = \langle \mathcal{U}_a f, \mathcal{V}_a g\rangle$$

equivalently described by the following commutative diagram:

$$\begin{tikzcd}
\mathbf{B} \arrow[d, "\mathcal{U}"'] & \mathbf{B}\times\mathbf{C} \arrow[l, "\mathcal{P}"'] \arrow[r, "\mathcal{Q}"] \arrow[d, dotted, "\mathcal{U}\times\mathcal{V}"] & \mathbf{C} \arrow[d, "\mathcal{V}"] \\
\mathbf{B}' & \mathbf{B}'\times\mathbf{C}' \arrow[l, "\mathcal{P}'"'] \arrow[r, "\mathcal{Q}'"] & \mathbf{C}'
\end{tikzcd}$$

Figure 2.4: Functor products

Note that since functors are morphisms on categories, then $\times$ itself is a functor on small categories:

$$\times : \mathbf{Cat}\times\mathbf{Cat} \to \mathbf{Cat}.$$

In the above section, we've concerned ourself with functors mapping from a category to a product category (e.g., $\mathcal{F} : \mathbf{D} \to \mathbf{B}\times\mathbf{C}$). We will now examine the "dual" concept, that functors from a product category to a category.

Definition 2.2.9: Bifunctor

Let $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ be categories. Let $\mathcal{S} : \mathbf{B} \times \mathbf{C} \to \mathbf{D}$. Then $\mathcal{S}$ is called a *bifunctor*.

Simply put, a bifunctor is just a functor of two arguments.

Let $\mathcal{S}$ be the bifunctor given in the definition above. Then if we fix one of its arguments, we get something that is effectively a single-argument functor, similarly to how fixing an argument of a two-variable function yields something that's "effectively" a single-variable function. This process is described by the following theorem:

Theorem 2.2.3. *Let $\mathbf{B}, \mathbf{C}$, and $\mathbf{D}$ be categories. For all objects $c \in \mathrm{ob}(\mathbf{C})$ and $b \in \mathrm{ob}(\mathbf{B})$, let*

$$\mathcal{L}_c : \mathbf{B} \to \mathbf{D}, \qquad \mathcal{M}_b : \mathbf{C} \to \mathbf{D}$$

be functors such that $\mathcal{M}_b(c) = \mathcal{L}_c(b)$ for all b and c. Then there exists a bifunctor $\mathcal{S} : \mathbf{B} \times \mathbf{C} \to \mathbf{D}$ with $S(-, c) = \mathcal{L}_c$ for all c and $S(b, -) = \mathcal{M}_b$ for all b if and only if for every pair of arrows $f : b \to b'$ and $g : c \to c'$ one has

$$\mathcal{M}_{b'}(g) \circ \mathcal{L}_c(f) = \mathcal{L}_{c'}(f) \circ \mathcal{M}_b(g) \tag{2.1}$$

These equal arrows (2.1) in $\mathbf{D}$ are then the value $\mathcal{S}(f, g)$ of the arrow function of $\mathcal{S}$ at f and g.

3. Basic Topology

(Sung to the tune of *Time in a Bottle*)
If I could save Klein in a Bottle
The first thing that I'd like to glue
Is an edge to an edge
Of a projective plane and then #
With another $P2$

3.1 Introduction

First, a motivating quote.

> "Point set topology is a disease from which later generations will regard themselves as having recovered" -Henri Poincaré

That's... not exactly a ringing endorsement. Why did Henri Poincaré have such a low opinion of point-set topology? Well, loosely speaking, because it's just not the right tool for the job, especially when compared to *algebraic topology.*

As it turns out, lots of topics in topology can be simplified by attaching algebraic objects to topological spaces, and proving that certain properties of this object correspond naturally to properties of our topological space. The vehicle by which we navigate between the two is, as one might expect, Category Theory. First, we give a brief summary of basic concepts in algebraic topology, before moving into the Homology presentation given in Matveev.

3.1.1 Basic Point-Set Topology

As in most branches of mathematics, our object of study here will be some collection of sets, together with some *structure* we can associate with them. In Elementary Algebra, this takes the form of *group* and *ring* operations, and later the respective homomorphisms preserving them. In Elementary Analysis, this (loosely speaking) took the form of a *distance metric*, and the properties it bestowed on sets. Analogous to our study of homomorphisms in Algebra, we often studied *continuous functions* in Analysis, and the properties of sets that they preserved. Note the resemblance between the two expressions:

$\varphi(g_1 \oplus g_2) = \varphi(g_1) \otimes \varphi(g_2)$	$d(x,y) < \delta \implies d'(f(x), f(y)) < \varepsilon$
A homomorphism $\varphi : G \to H$	A continuous function $f : (E, d) \to (E', d')$

while the analogy doesn't hold exactly, in both cases, we have some particular class of functions such that structure in one space is preserved in the image. In the case of homomorphisms, the group operation in the first group is "respected" by the homomorphism upon mapping into the second. In the case of continuous functions, our equation is essentially stating that the function respects things being "close" to one another (our map doesn't destroy the properties of the distance metric). We can think of this as stipulating that we didn't "tear" our starting space at all. This is best visualized by thinking about our continuous functions not as their *graphs* (as we are often used to), but rather as *maps* that deform the input domain in various manners to yield the image. As an example, one might think of the function $f(x) = x^2$ as the action of folding $\mathbb{R}$ on itself, and stretching the edges out towards infinity (this is often a strategy employed in visualizing complex-valued functions).

One might wonder what sorts of interesting discoveries we could make by generalizing our starting premises on the right-hand-side, so that we could make our questions more similar to those on the left.

That is, similarly to how we defined distance metrics so as to generalize the *key* properties of Euclidean distance, so too will we generalize the idea of *continuity of a function.* This is the central idea of basic topology. Now, all we need is a good place to start. Recall the following theorem of Analysis:

Theorem 3.1.1. *Let (E, d) and (E', d') be metric spaces. Then a function $f : E \to E'$ is said to be* continuous *iff for all open sets $U \subseteq M_2$, we have $f^{-1}(U)$ is open in M_1.*

Note, this theorem makes no guarantees about the image of an open set being open in the codomain. Really, we can make our image as "jagged" as we want (within reason), provided we fold and deform our domain in a smooth manner. But it does indicate to us open sets appear to be intimately tied to the idea of "smooth" deformations. In particular, noting that the theorem is an if and only if, we might wonder what would happen were we to discard the distance metric entirely, and take the above as a *definition* instead of as a *theorem.* There's just one catch: if we really want to discard the metric, how are we going to define the notion of an "open" set? Topologies provide the answer.

3.1.2 Some definitions

Definition 3.1.1

Let X be a set, and let $\mathcal{U}$ be a collection of subsets of X satisfying the following:

(i) $\varnothing \in \mathcal{U}$, $X \in \mathcal{U}$.

(ii) For all $U_1, U_2 \in \mathcal{U}$, we have $U_1 \cap U_2 \in \mathcal{U}$ (by induction, we obtain closure under finite intersections).

(iii) For any subset $\{U_i \mid i \in I\} \subseteq \mathcal{U}$, we have

$$\bigcup_{i \in I} U_i = \boldsymbol{U} \in \mathcal{U}$$

($\mathcal{U}$ is closed under arbitrary unions).

then $\mathcal{U}$ is called a *topology* for X, and $(X, \mathcal{U})$ is called a *topological space.* We call the elements of $\mathcal{U}$ the *open sets of* $(X, \mathcal{U})$.

note that a topology is thus a particular kind of *algebra of sets* under the binary operations $\cup, \cap$, with identity $\varnothing$ for $\cup$, and X for $\cap$. Note that $(\cup, \varnothing)$, $(\cap, X)$ are duals of each other, in the sense that for any sentence S built out of atomic propositions about our set algebra, if S is true, then the statement we obtain by

1. Replacing each $\cup$ with $\cap$ and each $\cap$ with $\cup$,
2. Interchanging each $\varnothing$ and X, and
3. Reversing inclusions

must also be true. Less relevantly (but maybe an object of interest), observe that if we replace "arbitrary unions" with "countable unions", and further require closure under complementation, then we obtain a σ-algebra.

As it turns out, this definition of a topology is more general than that given by distance metrics. Whereas every distance metric gives rise to a topology, there are topologies that are not *metrizable*, meaning they do not arise from any metric on a set. We list a few common topologies. Let $(X, \mathcal{U})$ be a topological space. Then

1. If $\mathcal{U} = \{\varnothing, X\}$, we call $\mathcal{U}$ the *concrete* or *indiscrete topology.*

2. If $\mathcal{U} = \mathcal{P}(X)$ (i.e., every subset of X is open), then we call $\mathcal{U}$ the *discrete topology*. Note the direct connection to the discrete metric.
3. Suppose $\mathcal{U} = \{\varnothing, X\} \cup \left\{U \subseteq X \;\middle|\; \left|\overline{U}\right| < \infty\right\}$. That is, $X, \varnothing$, and all subsets of X with finite compliment. Then call $\mathcal{U}$ the *finite complement topology*.
4. Let $X = \mathbb{R}$, and $\mathcal{U} = \{\varnothing, \mathbb{R}\} \cup \{(x, \infty) \mid x \in \mathbb{R}\}$

As an exercise, I'll give a proof for some of these. Note that below, I'll be using overline to denote the compliment of a set, a notation I'll discard quickly once we start talking about compliments. But for now, it's the most convenient.

1. Trivial
2. Also trivial
3. Observe $\varnothing, X \in \mathcal{U}$ by definition. We first prove that $\mathcal{U}$ is closed under finite intersections. Let $U_1, U_2 \in \mathcal{U}$. Then $\overline{U_1 \cap U_2} = \overline{U_1} \cup \overline{U_2}$ (De Morgan's Laws). The union of two finite sets is finite, hence $\overline{U_1 \cap U_2}$ is finite and so $U_1 \cap U_2 \in \mathcal{U}$. Now, let $U' = \{U_i \mid i \in I\} \subseteq \mathcal{U}$. Then
$$\overline{\bigcup_{i \in I} U_i} = \bigcap_{i \in I} \overline{U_i}$$
which is an intersection of finite sets, and is thust finite itself. Hence $\mathcal{U}$ is closed under arbitrary unions. It follows that $(X, \mathcal{U})$ is a topological space. ■
4. The non-trivial elements of $\mathcal{U}$ inhereit a total order by inclusion from the total order on $\mathbb{R}$. We add $\varnothing, \mathbb{R}$ to the total order by putting $\mathbb{R} = \sup(\mathcal{U})$, $\varnothing = \inf(\mathcal{U})$. The closure properties follow. ■

We define some noteworthy sets.

Definition 3.1.2: Interior

Let $(X, \mathcal{U})$ be a topological space. Let $Y \subseteq X$. Let $U' = \{U_i \in \mathcal{U} \mid i \in I, U_i \subseteq Y\}$ be the set of all open sets contained in Y. Then call the *interior* of Y (sometimes denoted Y°)
$$\operatorname{int}(Y) = \bigcup_{i \in I} U_i.$$

we have open sets — what's next, closed sets???? Yeah, uh, you got me there.

Definition 3.1.3: Closed sets

Let $(X, \mathcal{U})$ be a topological space. Let $C \subseteq X$, and let $X - C$ denote the compliment of C in X. Then call C *closed* iff $X - C$ is open. **VERY VERY IMPORTANT:** a set can be both open and closed. In fact, in the discrete topology, every set is both.

From the principle of duality (or, alternatively, De Morgan's Laws), the dual statements of the topology axioms hold for closed sets. That is,

i. $X, \varnothing$ are closed
ii. The set of closed sets is closed (haha) under finite unions
iii. The set of closed sets is closed under arbitrary intersections

Analogously to the interior of a set, we define its closure. Note closure is, in a sense, the dual of the interior operator.

Definition 3.1.4

Let $(X,\mathcal{U})$ be a topological space. Let $Y \subseteq X$. Then let

$$V' = \{V_i \mid i \in I, X - V_i \in \mathcal{U}, Y \subseteq V_i\}$$

that is, the set of all closed sets containing Y. Then call the *closure* of Y (denoted $\overline{Y}$)

$$\text{clos}(Y) = \bigcap_{i \in I} V_i$$

Finally, we define the boundary of a set as

Definition 3.1.5

Let $(X,\mathcal{U})$ a topological space. Let $Y \subseteq X$. Then define the boundary of Y, denoted ∂Y, by $\partial Y = \overline{Y} - Y$.

One might wonder why we've chosen to recycle the partial derivative notation to denote the boundary. The intuitiion is as follows (this lightly paraphrased from a stackexchange answer by Terry Tao):

> Let S be a smooth, bounded body. Then the surface area $|\partial S|$ is the derivative of the volume $|S_r|$ of the r-neighborhoods S_r at $r = 0$:
>
> $$|\partial S| = \frac{\mathrm{d}}{\mathrm{d}r} |S_r| \bigg|_{r=0}$$
>
> Here's the part I don't understand: he then goes on to say "More generally, one intuitively has the Newton quotient-like formula"
>
> $$\partial S = \lim_{h \to 0^+} \frac{S_h \setminus S}{h}$$
>
> "the right-hand side does not really make formal sense, but certainly one can view $S_h \setminus S$ as a $[0, h]$-bundle over ∂S for h sufficiently small (in particular, the radius of curvature of S)."

Finally, we define the neighborhood of a point x.

Definition 3.1.6: Neighborhood

Let $(X,\mathcal{U})$ be a topological space. Let $N \subseteq X$, and let $x \in N$. Then call N a neighborhood of x if $x \in N^\circ$.

Note the following properties:

1. clos and int are idempotent.
2. $X - Y^\circ = \overline{X - Y}$
3. $\partial Y = \varnothing \iff Y$ is clopen.
4. Let $U \in \mathcal{U}$. Then $Y = \overline{U} \iff Y = \overline{U^\circ}$ (proof: note $U = U^\circ$).
5. For each point $x \in X$, there is at least one neighborhood of x, namely X.
6. If M and N are neighborhoods of x, then so is $N \cap M$ (use closure under finite intersections).

3.1.3 Continuous functions and Induced Topologies

As promised, we'll now define a continuous function between two topological spaces.

Definition 3.1.7: Continuous Function

A function $f : X \to Y$ between two topological spaces is said to be *continuous* if for every open set $U \subset Y$, the inverse image $f^{-1}(U)$ is open in X (the same holds for closed sets). These are the morphisms in **Top**.

If f is bijective with a continuous inverse, then call f a *homeomorphism.*

Question: how can we make the following into a trivial statement of category theory?

> Let Y be a topological space with the property that for every topological space X, all functions $f : X \to Y$ are continuous. Prove that Y has the concrete topology.

Answer: use the forgetful functor and the universal mapping principle?

Definition 3.1.8: Open / closed maps

Let $f : X \to Y$. If the image of every open set is open, f is said to be *an open map* (define closed maps analogously). Note, open mappings are not necessarily continuous.

And lastly, a convenient theorem.

Theorem 3.1.2. *Let $(X,\mathcal{U})$, $(Y,\mathcal{V})$ be topological spaces, with $X \cong Y$ by a homemorphism $h : X \to Y$. Then for every point $x \in X$, $X - \{x\} \cong Y - \{h(x)\}$ (that is, removing a point preserves the homeomorphism).*

Intuitively, we think of homeomorphisms as encoding information about how two spaces can be stretched and deformed into one another, without tearing / gluing pieces together / apart. Note, however, that this does not necessarily correspond to our intuitive understanding of "tearing / gluing" — were I to hand you a circular segment of rope, and ask you to construct the following knot

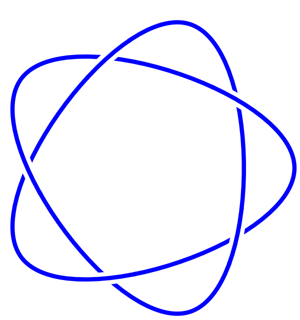

Figure 3.1: cinquefoil

you'd say it couldn't be done. However that's not exactly true. If we weren't constrained by the pesky physical details of the real world, we could simply pass the rope through itself to yield the desired knot. This is what is meant in topology by a homeomorphism — all that has to be conserved are, loosely speaking, "adjacency" relationships. On the flip side, we might also note that this knot can hardly be said to "not differ at all" from a circle. But in a topological sense, they are homeomorphic. Maybe we'll talk about ambient isotopy next week.

Now, induced topologies. Loosely speaking, this works kind of like a group action, except by a topological space on a subset of the underlying set. It's worth noting that this analogy might be *highly* tenuous.

Definition 3.1.9: Induced topologies

Let $(X,\mathcal{U})$ be a topological space, and $S \subseteq X$. Let $\mathcal{U}' = \{U \cap S \mid U \in \mathcal{U}\}$. Then we call $\mathcal{U}'$ *the topology on S induced by the topology of X* (note the desired topological axioms follow from the properties of $\mathcal{U}$ and the commutativity of union / intersection on sets). If S has the induced topology, we call S a subspace of X.

One thing to note about this definition is that an open set in the induced topology need not be open in the original topology. Really, what we've essentially done is "project" the topological structure of some space on to one of its subsets. Naturally, this can't give us information about the open-ness of a set selected from $\mathcal{U}'$ were we to imbed it back into $\mathcal{U}$. However, if we do know something about how S is related to X from the get go, *then* we can guarantee a stronger condition:

Theorem 3.1.3. *If $S \in \mathcal{U}$, then $\mathcal{U}' \subseteq \mathcal{U}$. That is, the opens sets of S are open in X. The proof follows directly from the definition of induced topology.*

3.1.4 Quotient and Product spaces

Whereas the induced topology can be thought of as arising from an injetive map $\iota : S \to X$, we will now consider the topology arising from a surjective map $q : X \to Y$.

Definition 3.1.10: Quotient topology

Let $f : (X,\mathcal{U}) \to Y$ be surjective as a function of sets. Then define the *quotient topology* on Y by

$$\mathcal{U}' = \left\{U \subseteq Y \;\middle|\; f^{-1}(U) \in \mathcal{U}\right\}.$$

The topology axioms follow from properties of the inverse image. Once Y has been endowed with the quotient topology, f becomes a continuous map.

Theorem 3.1.4. *Let $f : X \to Y$ be a mapping and suppose that Y has the quotient topology with respect to X. Let Z be a topological space. Then a mapping $g : Y \to Z$ is continuous iff gf is continuous.*

In our definition of a quotient topology, because Y is a set intially without structure, we don't really care what exactly its elements are — really, all that matters to us is that our mapping is surjective (Y only assumes structure once we define the quotient topology). Hence, there's no reason why we can't treat all such Y as if they were subsets of X (every surjective mapping from X to Y can be factored through an isomorphism of Y with a subset of X). This, together with the suspicious name "quotient topology," leads us to question whether we can view this as some form of *mod* operation. Indeed, we can. Recall the standard definition of an equivalence relation:

Definition 3.1.11

Let X be a set. Then an *equivalence relation* $\sim$ is a binary relation such that

(a) $\sim$ is reflexive,

(b) $\sim$ is symmetric, and

(c) $\sim$ is transitive.

For each $x \in X$, we define the *equivalence class of* x under $\sim$ by $[x] = \{y \in X \mid x \sim y\}$. The set of equivalence classes of X under $\sim$ is usually denoted $X/\sim$.

For $f : X \to X/\sim$ defined by $f(x) = [x]$, then $X/\sim$ together with the quotient topology is said to be "obtained from X by topological identification."

There is one other way in which we can think about quotient spaces. Consider an action on a topological space by a group G. Then define the equivalence relation $x \sim y \iff y \in \mathcal{O}_X(x)$ (i.e., they are in the same orbit under the action). Then we can give X/G the quotient topology.

At this point, we've considered various methods of constructing topological spaces from existing ones. In the induced topology, we thought of giving a set with no structure of its own additional structure by including it in a large topological space. In the quotient topology, we found ways of taking the structure of a topological space and imposing it on some other set. Now, we consider the construction in which we have *two* topological spaces, and we want to construct a new one from the two.

Definition 3.1.12

Let $(X, \mathcal{U}), (Y, \mathcal{V})$ be topological spaces. Define the topological product $X \times Y$ by $(X \times Y,\ \mathcal{U} \otimes \mathcal{V})$, where

$$\mathcal{U} \otimes \mathcal{V} = \left\{ \bigcup_{i \in I} W_i \;\middle|\; i \in I,\ \text{and } W_i \in \mathcal{U} \times \mathcal{V} \right\}$$

that is, the closure under union of all products of open sets in X and Y.

3.1.5 Compactness and Hausdorffness

In analysis, a central question was "what properties of our spaces are preserved under continuous functions?" To this, we answered Completeness, Connectedness, and (in some cases) compactness. Now, we'll perform the analogous identification for homeomorphisms. But first, we need to define the actual properties we want to prove the invariance of. First comes compactness, which I like to think of as "the next-best thing to being finite."

Definition 3.1.13: Covers

Let X a set, and let $S \subseteq X$. Then suppose we have $V = \{V_i \subseteq X \mid i \in I\}$, and $S \subseteq \bigcup_{i \in I} V_i$. Then V is called a *cover* of S. If I is finite, then we call V a *finite* cover. If X is a topological space with topology $\mathcal{U}$, and $V \subseteq \mathcal{U}$, then we call V an *open* cover. If we have $V' \subseteq V$ a cover of S, then we call V' a *subcover*.

now, the familiar definition of compactness.

Definition 3.1.14

Let $(X, \mathcal{U})$ a topological space, and let $S \subseteq X$. Then S is said to be *compact* if every open cover of S has a finite subcover.

As one might expect, a subspace $S \subseteq X$ is compact iff it is compact under the induced topology.

3.2 Picking up where we left off

3.2.1 Compactness

Last time, we finished by giving some basic definitions of comapctness, and whatnot. We'll begin with a small exercise to shake some of the cobwebs loose.

(a) Suppose that X has the finite complement topology. Show that X is compact. Show that each subset of X is compact.

(b) Prove that a topological space is compact if and only if whenever $\{C_j \mid j \in J\}$ is a collection of closed sets with $\bigcap_{j \in J} C_j = \varnothing$ then there is a finite subcollection $\{C_k \mid k \in K\}$ such that $\bigcap_{k \in K} C_k = \varnothing$.

(c) Let $\mathcal{F}$ be the topology on $\mathbb{R}$ defined by $U \in \mathcal{F}$ iff $\forall s \in U, \exists t > s$ s.t. $[s, t) \subseteq U$. Prove that the subset $[0, 1]$ of $\mathcal{F}$ is not compact.

Now

(a) Let $U = \{U_i \mid i \in I\}$ be an open cover of X. Let $U_i \in U$. Then $X - U_i$ is finite. For every $x_j \in X - U_i$, $\exists U_j \in U$ s.t. $x \in U_j$ (because U is a cover). Then the set consisting of U_i and the U_j is a finite subcover, thus X is compact. Let $Y \subseteq X$. Then let $V = \{V_k \mid k \in K\}$ be an open cover of Y. Proceed an analogous argument to the above to obtain Y compact.

(b) Let X be a topological space, and suppose X is compact. Let $C = \left\{C_j \;\middle|\; j \in J, C_j = \overline{C_j}\right\}$ (i.e., the C_j are closed), and suppose

$$\bigcap_{j \in J} C_j = \varnothing.$$

By De Morgan's laws,

$$\bigcup_{j \in J} X - C_j = X$$

since C_j are all closed, then $X - C_j$ are open, hence this is an open cover of X, and there exists a finite subcover. Apply De Morgan's laws again to yield the desired result.

(c) Let $\varepsilon > 0$ be given. Let U be given by the open cover

$$U = \left\{\left[1 - \frac{1}{2^i}, 1 - \frac{1}{2^{i+1}}\right) \;\middle|\; i = 0, 1, \ldots\right\} \cup \{[1, 1 + \varepsilon)\}$$

and note that all the sets in U are disjoint, and that they cover $[0, 1]$. From disjointness, it follows there is no finite subcover.

3.2.2 A brief review of projections

On our first pass through, we didn't treat projection maps in a lot of depth, so we'll very briefly revisit them here.

Definition 3.2.1: Projection Maps

Let X, Y be topological spaces. Then define $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$ by

$$\pi_X(x, y) = x \qquad \pi_Y(x, y) = y$$

π_X and π_Y are referred to as the *product projections*. Note that both are continuous.

3.3 Compactness, Continued

Theorem 3.3.1. *Let (X, τ) be a topological space, and let $S \subseteq X$. Then S is compact in (X, τ) iff S is compact under the induced topology.*

Proof. Forwards direction is trivial. For the backwards direction, suppose S is compact in the induced topology. Let $U = \{U_i \mid i \in I\}$ be an open cover of S in (X, τ). Then $V = \{V_i = U_i \cap S \mid i \in I\}$ is an open cover of S in the induced topology, and hence by compactness there exists a finite subcover $V' = \{V_{i_k} \mid i_k \in I, k = 1, \ldots, n\}$. Now, take $U' = \{U_{i_k} \mid i_k \in I, k = 1, \ldots, n\}$. Then U' is a finite subcover of U. Since U was taken to be arbitrary, this implies S is comapct. ■

In the metrizable topologies we encountered in Real Analysis, we proved that continuous functions preserve compactness. However, we will now show that the same result holds in a general topological space.

Theorem 3.3.2 (Continuity and Compactness)**.** *Let $f : (X, \tau) \to (Y, \upsilon)$ be a continuous map. Let $S \subseteq X$ be a compact subspace. Then $f(S)$ is compact in Y.*

Proof. Let $V = \{V_i \mid i \in I\} \subseteq \upsilon$ be an open cover of $f(S)$. Because f is continuous, $U = \left\{f^{-1}(V_i) \;\middle|\; i \in I\right\}$ is a collection of open sets covering $f^{-1}(f(S)) \supseteq S$. Since S is compact, there exists a finite subcover $U' = \left\{f^{-1}(V_{i_k}) \;\middle|\; i_k \in I, k = 1, \ldots, n\right\}$ covering $f^{-1}(f(S))$. Then $V' = \{V_{i_k} \mid i_k \in I, k = 1, \ldots, n\}$ is a finite subcover of V. Thus $f(S)$ is compact in Y. ■

By virtue of the properties of continuous functions that we proved last time, some nice results follow immediately:

Corollary 3.3.1.

(a) *Any closed interval in $\mathbb{R}$ is compact.*

(b) *If X and Y are homeomorphic, then X is compact iff Y is.*

(c) *If X is compact, and Y is any set, then Y with the quotient topology induced by $f : X \to Y$ is compact.*

For completeness, we list some closure properties of compact subspaces:

Theorem 3.3.3. *Let (X, τ) be a topological space. Let $S = \{S_i \mid i \in I\} \subseteq$ be the collection of compact subspaces of X. Then*

(a) *If $S_1, S_2 \in S$, then $S_1 \cup S_2 \in S$ (union of two compact subspaces is compact). It follows by induction that any finite union of compact subspaces is compact.*

(b) *It is not the case that in an arbitrary topological space, an arbitrary intersection of compact spaces is compact (we need Hausdorffness). But for finite intersections, things work out.*

Theorem 3.3.4. *Let (X, τ) be a compact topological space, and let $S \subseteq X$ be closed. Then S is compact.*

Proof. Let $U = \{U_i \mid i \in I\}$ be an open cover of S. Let $U_0 = X - S$. Then U_0 is open, and $U \cup \{U_0\}$ covers X. Then since X is compact, there exists a finite subcover $U' = \{U_i \mid i \in I \cup \{0\}\}$. Take $U'' = U' - \{U_0\}$ to obtain a finite subcover of U. ■

I'm proud to have written this proof without looking at the one in the book at all, only to find later that they're basically identical.

Theorem 3.3.5. *Let X, Y be topological spaces. Then X, Y are compact iff $X \times Y$ is compact.*

Proof.

($\Rightarrow$): Suppose X, Y are compact. WTS $X \times Y$ is compact as well. Let $W = \{W_i \mid i \in I\}$ be an open cover of $X \times Y$. Note that $\forall y \in Y$, $X \times \{y\}$ is homeomorphic to X.

($\Leftarrow$): Suppose $X \times Y$ is compact. Let $U = \{U_i \mid i \in I\}$ be an open cover of X, and $V = \{V_j \mid j \in J\}$ be an open cover of Y. Then W given by

$$W = \left\{ \bigcup_{k \in K} W_k \,\middle|\, W_k \in U \times V \right\}$$

is an open cover of $X \times Y$, and thus admits a finite subcover:

$$W' = \{W_\ell \mid \ell \in L;\ |L| < \infty\} \subseteq W$$

Apply a similar trick something something boom

■

3.4 Hausdorff Spaces

Hausdorffness is an important property in Topology that essentially allows us to separate things from each other (our space is not "infinitely bunched-up" somewhere).

Definition 3.4.1

Let (X, τ) be a topological space. Then call X *Hausdorff* iff for all $x, y \in X$ such that $x \neq y$, there exist open sets U_x, U_y with $x \in U_x$, $y \in U_y$, and $U_x \cap U_y = \varnothing$.

Note that by a simple $\varepsilon/2$ argument, it follows that all metrizable spaces are Hausdorff.

Definition 3.4.2: T_k spaces

For $k = 0, 1, 2, 3, 4$, call X a T_k space if it satisfies the k-th condition below (indexing starts at 0):

T_0: For all $x, y \in X$ $(x \neq y)$, there is an open set U containing one but not the other (i.e., $x \in U$ and $y \notin U$, or $y \in U$ and $x \notin U$).

T_1: For all $x, y \in X$ $(x \neq y)$, there are open sets U, V such that $x \in U$, $y \in V$, and $x \notin V$, $y \notin U$.

T_2: For all x, y in X $(x \neq y)$, there are open sets U, V such that $x \in U$, $y \in V$, and $U \cap V = \varnothing$ (there are disjoint neighborhoods about x and y).

T_3: X is T_1, and for all closed subsets F and points $x \notin F$, there exist open sets U, V such that $F \subseteq U$, $x \in V$, and $U \cap V = \varnothing$.

T_4: X is T_1, and for all pairs of disjoint closed subsetes F_1, F_2, there exist open sets U, V such that $F_1 \subseteq U$, $F_2 \subseteq V$

Naturally, if X and Y are homeomorphic topological spaces, and X is T_k, then Y is T_k as well. As an exercise, we construct spaces that are T_j (for $j = 0, \ldots, 4$) that are not $T_{i>j}$.

(X_0): Let $X_0 = (\mathbb{R}^{\geq 0}, \tau)$, where $\tau = \left\{[0,t) \;\middle|\; t \in \mathbb{R}^{\geq 0}\right\}$. Note that τ is indeed a topology on $\mathbb{R}^{\geq 0}$. Note X_0 is not T_1.

(X_1): Let $X_1 = (X, \tau)$ where

Theorem 3.4.1. *A space X is T_1 iff every point of X is closed.*

Proof. Suppose (X, τ) is T_1. Let $x \in X$ be arbitrary, and let $y \in X - \{x\}$. Then $\exists U_y \in \tau$ with $y \in U_y$, but $x \notin U_y$. Hence

$$\bigcup_{y \in X - \{x\}} U_y = X - \{x\}$$

is open, and so $\{x\}$ is closed.

Now suppose $\{x\}$ is closed. Then T_1 follows immediately. ■

An important theorem:

Theorem 3.4.2. *Let A be a compact subset of a Hausdorff space X. Then A is closed.*

Proof. We define the following (bizarre) open cover. For all $a \in A$, for all $x \in X - A$, there exist disjoint open sets U_a, V_a such that $a \in U_a$, and $x \in V_a$. Then $U = \{U_a\}$ is an open cover of A, and thus contains a finite subcover $U' = \{U_{a'} \mid a' \in \mathcal{A} \subseteq A, |\mathcal{A}| < \infty\}$, and corresponding V'. Then note that

$$V_x = \bigcap_{a' \in A} V_{a'}$$

is an open set (closure under finite intersections) such that $V_x \cap A = \varnothing$. Hence

$$X - A = \bigcup_{x \in X} V_x$$

and so $X - A$ is open. Then A is closed. ■

Now, it is time for another Very Important Theorem™.

Theorem 3.4.3. *Let (X, τ), (Y, υ) be topological spaces, with X compact and Y Hausdorff. Let $f : X \to Y$ be continuous. Then f is a homeomorphism iff f is a bijection.*

Proof.

($\Rightarrow$) : Suppose f is a bijection. Then f^{-1} exists, and $ff^{-1} = \mathrm{id}Y$, $f^{-1}f = \mathrm{id}X$. WTS f^{-1} is continuous. Let $S \subseteq X$ be closed. Then S is compact, hence $\left(f^{-1}\right)^{-1}(S) = f(S)$ is compact in Y. Thus $f(S)$ is closed as well, hence f^{-1} is continuous. Thus f is a homeomorphism.

($\Leftarrow$) : Suppose f is a homeomorphism. Then f is a bijection.

■

This part is actually not finished.

3.4.1 Some Theorems

We'll breeze briskly through some more point-set topology, then move on to the foundations of algebraic topology.

Theorem 3.4.4. *Let Y be the quotient space of the topological space X determined by the surjective mapping $f : X \to Y$. If X is compact Hausdorff and f is closed then Y is (compact) Hausdorff.*

Corollary 3.4.1. *Let X be a compact Hausdorff G-space with G finite. Then X/G is a compact Hausdorff space.*

Corollary 3.4.2. *If X is a compact Hausdorff space and A is a closed subset of X then X/A is a compact Hausdorff space.*

4. Towards Algebraic Topology

4.1 All Together Now...

4.1.1 Connectedness

Definition 4.1.1

Let (X, τ) a topological space. Then X is said to be *connected* iff the only clopen subsets are trivial. If $S \subseteq S$, then S is said to be connected iff it is connected in the induced topology.

Equivalently, X is connected iff it cannot be expressed as the union of finitely many disjoint nonempty open subsets.

Theorem 4.1.1. *Let $f : X \to Y$ be continuous, and suppose X is connected. Then Y is connected as well.*

Proof. Suppose, to obtain a contradiction, that Y is disconnected. Then there exist nonempty open sets $U, V \subseteq Y$ with $U \cup V = Y$, and $U \cap V = \varnothing$. Since U, V are open and f continuous, then $f^{-1}(U)$, $f^{-1}(V)$ are open in X. Furthermore, these are disjoint nonempty subsets of X with $f^{-1}(U) \cup f^{-1}(V) = X$. Then X is disconnected, a contradiction. Hence Y is connected. ■

Theorem 4.1.2. *Suppose that $\{Y_j \mid j \in J\}$ is a collection of connected subsets of a space X. If $\bigcap_{j\in J} Y_j \neq \varnothing$, then $Y = \bigcup_{j\in J} Y_j$ is connected.*

Proof. Suppose U is a nonempty clopen subset of Y. Then $\exists j \in J$ s.t. $U \cap Y_j \neq \varnothing$. Hence, let $J' = \{j \in J \mid U \cap Y_j \neq \varnothing\}$. Then $\forall j' \in J'$, we have $U \cap Y_{j'}$ is clopen in the induced topology. Since $Y_{j'}$ is connected, it follows that $U \cap Y_{j'} = Y_{j'}$. Hence $U = Y$, so Y is connected. ■

Theorem 4.1.3. *Let X, Y be topological spaces. Then X, Y are connected iff $X \times Y$ is connected.*

Proof:

- $(\Rightarrow)$: Suppose X, Y are connected. $\forall x \in X,\ y \in Y,\ X \times \{y\} \cong X$ and $\{x\} \times Y \cong Y$ (and thus each is connected), and note $(X \times \{y\}) \cap (\{x\} \times Y) \neq \varnothing$, thus by theorem 1.3 their union is connected. Now, let $y \in Y$ be fixed. Observe that

$$X \times Y = \bigcup_{x\in X} (X \times \{y\}) \cap (\{x\} \times Y)$$

  Hence $X \times Y$ is connected.

- $(\Leftarrow)$: Suppose $X \times Y$ is connected. Then since the canonical projection maps are continuous, it follows that X, Y are connected (continuous image of a connected set is connected). ■

4.1.2 Path Connectedness

We now introduce a new, stronger notion of *connectedness* that allows us to treat somewhat pathological examples, such as the Topologist's Sine Curve (seen below):

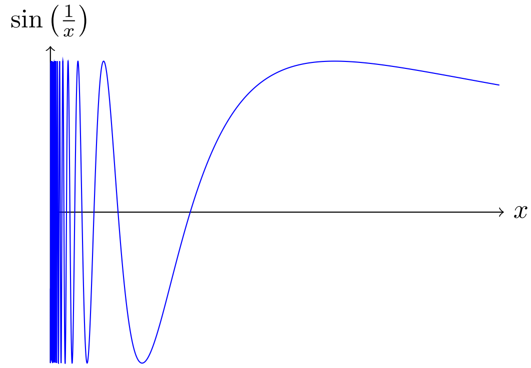

Figure 4.1: Topolgist's Sine Curve

Here, if we take the topological space $X = \left\{(x, y) \mid y = \sin\left(\frac{1}{x}\right)\right\} \cup \{(0,0)\}$ under the induced topology with respect to $\mathbb{R}^2$, then we see that X is connected, but clearly, there's some…funny business going on near the origin. Topologically, there's no problem — it's impossible to separate the origin out from the rest of the points of the curve; no matter how close we get, we'll see infinitely many points of the curve wobbling around. But clearly, there's no way to actually "link" the origin into the graph. This second point is the idea we want to capture with path connectedness. But first, we need to define the idea of a path.

Definition 4.1.2: Path

Let X be a topological space, and let $[0,1] \subseteq \mathbb{R}$. Then if $f : [0,1] \to X$ is a continuous function, we call f a *path*. **Important Note:** $f([0,1])$ is *not* the path; rather the mapping f itself is the path. $f([0,1])$ is called a *curve*.

In order to apply paths in full, we will first need some handy lemmas.

Lemma 4.1.1. *Let f be a path in X, and let $\overline{f}$ be defined by $\overline{f}(t) = f(1-t)$. Then $\overline{f}$ is also a path in X.*

Proof. Let $g : [0,1] \to [0,1]$ be defined by $g(t) = 1 - t$. Then g is continuous. Hence $\overline{f} = f \circ g$ is continuous. ∎

For the next lemma, we actually need another lemma first:

Lemma 4.1.2 (Gluing Lemma)**.** *Let W, X be topological spaces and suppose that $W = A \cup B$ with A, B both closed subsets of W. If $f : A \to X$ and $g : B \to X$ are continuous functions such that for all $w \in A \cap B$ $f(w) = g(w)$, then $h : W \to X$ defined by*

$$h(w) = \begin{cases} f(w) & \text{if } w \in A, \\ g(w) & \text{if } w \in B \end{cases}$$

is a continuous function.

Proof. Note that h is well-defined, and let U a closed set in X. Then

$$\begin{aligned} h^{-1}(U) &= \left(h^{-1}(U) \cap A\right) \cup \left(h^{-1}(U) \cap B\right) \\ &= f^{-1}(U) \cup g^{-1}(U) \end{aligned}$$

but $f^{-1}(U)$ and $g^{-1}(U)$ are closed, so $h^{-1}(U)$ is closed, thus h is a continuous function. ■

It is worth remarking that an equivalent claim could be proven for A, B open.

Lemma 4.1.3. *Let X be a topological space, and let f, g be paths in X such that $f(1) = g(0)$. Then define the* concatenation *of f and g (denoted $f * g$) to be the path in X such that*

$$f * g(t) = \begin{cases} f(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

is a path in X.

Proof. Take $A = [0, 1/2]$, $B = [1/2, 1]$, and apply the Gluing Lemma. ■

Definition 4.1.3: Path connectedness

A space X is said to be *path connected* if given any two points $x_0, x_1 \in X$, there is a path in X from x_0 to x_1.

Some nice straightforward results follow from the fact that path connectedness is defined in terms of continuous functions:

Theorem 4.1.4. *Let X be a path-connected topological space, and let f be a continuous mapping to a topological space Y. Then $f(X)$ is path-connected.*

Proof. Let $u, v \in Y$. Then $\exists a, b \in X$ s.t. $f(a) = u,\ f(b) = v$. Since X is path connected, there exists a path g in X from a to b. Then since f is continuous, $f \circ g$ is a path from u to v in Y. ■

Theorem 4.1.5. *Supposet that $\{Y_i \mid i \in I\}$ is a family of path connected sets. Then if*

$$\bigcap_{i \in I} Y_i \neq \varnothing,$$

then $Y = \bigcup_{i \in I} Y_i$ is path-connected.

Proof. Let $a, b \in Y$, and let $c \in \bigcap_{i \in I} Y_i$. Then $\exists i, j \in I$ s.t. $a \in Y_i, b \in Y_j$. Note that $c \in Y_i, Y_j$ as well. Then take a path f in Y_i from a to c, and a path g in Y_j from b to c. Then the path $h = f * g$ is a path from a to c in Y. ■

Theorem 4.1.6. *Let X, Y be topological spaces. Then X and Y are path connected iff $X \times Y$ is path connected.*

Proof.

($\Rightarrow$) : Suppose X, Y are path connected. $\forall x \in X,\ y \in Y$, $X \times \{y\} \cong X$ and $\{x\} \times Y \cong Y$ (and thus each is path connected), and note $(X \times \{y\}) \cap (\{x\} \times Y) \neq \varnothing$, thus by theorem 2.5 their union is path connected. Now, let $y \in Y$ be fixed. Observe that

$$X \times Y = \bigcup_{x \in X} (X \times \{y\}) \cap (\{x\} \times Y)$$

Hence $X \times Y$ is path connected.

($\Leftarrow$) : Suppose $X \times Y$ is path connected. Then since the canonical projection maps are continuous, it follows that X, Y are path connected (continuous image of a path connected set is path connected). ■

Theorem 4.1.7. *Every path connected space connected. Not every connected space is path connected. Oh, also, any non-empty open connected subset of $\mathbb{R}$ is path connected.*

4.2 A Brief Discussion of Manifolds

Definition 4.2.1: Manifolds

Let $n \in \mathbb{Z}^{>0}$, and let (M, τ) be a topological space. Then M is called a manifold iff M is Hausdorff, and $\forall m \in M$, there exists a neighborhood N of m such that $N \cong \mathring{D}^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$.

Definition 4.2.2: Connected Sum

Let S_1, S_2 be compact connected 2-manifolds (surfaces), and let $D_1 \subseteq S_1$, $D_2 \subseteq S_2$ with $D_1, D_2 \cong D^2$. Let $h_1 : D_1 \to D^2$, and $h_2 : D_2 \to D^2$ be homeomorphisms. Then define $\sim$ to be an equivalence relation such that $x \sim h_2^{-1} h_1(x)$ iff $x \in \partial D_1$, and $x \sim x$ otherwise. Then $S_1 \# S_2$ is given by

$$\frac{(S_1 - \mathring{D}_1) \cup (S_2 - \mathring{D}_2)}{\sim}$$

Definition 4.2.3

Call a surface S^2 *orientable* if it contains no Möbius strip, and *non-orientable* otherwise. Then for $m \geq 0, n \geq 1$, we call

$$S^2 \# \left(\mathop{\#}_{i=1}^{m} T \right) = S \# mT$$

the *standard orientable surface of genus m*, and

$$S^2 \# \left(\mathop{\#}_{i=1}^{n} \mathbb{R}P^2 \right) = S \# n\mathbb{R}P^2$$

the *standard non-orientable surface of genus n*.

4.3 Homotopy

We are now ready to introduce a concept that will be of fundamental importance later on. In a nutshell, this is the idea of *equivalent maps*, as motivated by the following uncharacteristically-unprofessional diagram:

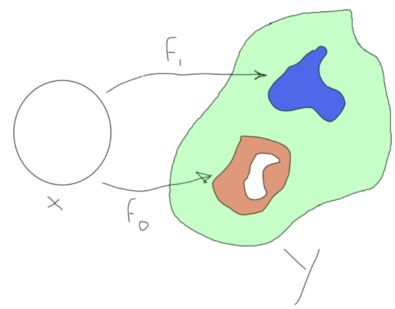

Figure 4.2: I'm not as good with inkscape as Prof. Nelson is

f_0 and f_1 are, in some sense, fundamentally not the same as each other, as one contains a hole while the other does not. To make this precise, we define the idea of homotopy equivalence:

Definition 4.3.1

Let X, Y be topological spaces, and let $f_0, f_1 : X \to Y$. Then say f_0, f_1 are *homotopic* iff there is a continuous map $F : X \times I \to Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. In this case we write $f_0 \simeq f_1$, and call the map F a *homotopy* between f_0 and f_1. For each $t \in [0, 1]$, we denote $F(x, t)$ by $f_t(x)$.

Some part of me really just wants to draw a commutative diagram here, but it doesn't quite feel appropriate. Anyways, note that a homotopy essentially amounts to a path through the space of continuous maps between topological spaces. Also, observe that our definition might not yet be perfectly desirable, as it allows us to make paths homotopic to points. Hence, we define the notion of a *relative homotopy*.

Definition 4.3.2

Let X be a topological space, and let $A \subseteq X$. Suppose that $f_0, f_1 : X \to Y$ are continuous. Then we say f_0 and f_1 are *homotopic relative to* A iff there exists a homotopy $F : X \times I \to Y$ between f_0 and f_1 such that $F(a, t)$ does not depend on t for any $a \in A$. That is, $\forall a \in A, t \in I$, $F(a, t) = f_0(a)$. In this case, we say F is a *homotopy relative to* A, and we write $f_0 \simeq f_1(\mathrm{rel}A)$, or $f_0 \simeq_{\mathrm{rel}A} f_1$.

Less rigorously, this essentially amounts to the homotopy *leaving* A *alone*. Note that if $A = \{0, 1\}$, homotopy relative to A means that we can't always get some map "over an obstacle."

Lemma 4.3.1. $\simeq_{\mathrm{rel}\,A}$ *is an equivalence relation on* $\hom(X, Y)$.

Proof. First, observe that $\simeq_{\mathrm{rel}\,A}$ is reflexive: take $F(x, t) = f_0(x)$. Symmetry follows from the fact that if $F(x, t)$ is a homotopy, then $F(x, 1 - t)$ is a homotopy. Finally, note that if we have $f \simeq_{\mathrm{rel}\,A} g$ and $g \simeq_{\mathrm{rel}\,A} h$, then we can simply concatenate the homotopys from f to g and g to h to yield one from f to h. ■

We can now apply this new shiny tool to some classifications of topological spaces.

Definition 4.3.3

Let X, Y be topological spaces. We say X and Y are *of the same homotopy type* if there exist continuous maps $f : X \to Y$, $g : Y \to X$ such that $gf \simeq 1 : X \to X$, $fg \simeq 1 : Y \to Y$.

Ok, there *really* feels like there's some category nonsense going on here. Anyways, note that we did *not* use relative homotopy here. That was 100% intentional — in a sense, we want homotopy equivalence to encode information about retraction and stretching that can be lossy in ways that homeomorphism cannot. Homeomorphism is rigid about exact equivalence, homotopy equivalence lets us fudge things a little bit. For instance, a möbius band is certainly not homeomorphic to a cylinder, whereas the two are both homotopy equivalent to a circle.

Definition 4.3.4

A space X is said to be *contractible* if it is homotopy equivalent to a point.

Definition 4.3.5

A subset A of a topological space X is called a *retract* of X iff there is a continuous map $r : X \to A$ such that $r\imath = 1 : A \to A$, where $\imath$ is the inclusion map. Equivalently, $r|A = 1$. Under these conditions, r is called an *inclusion* map.

Definition 4.3.6

A subset $A \subseteq X$ is called a *deformation retract* of X if there is a retraction $r : X \to A$ such that $\imath r \simeq 1 : X \to X$.

If A is a deformation retract of X, it follows that A and X are homotopy equivalent.

Definition 4.3.7

Let X be a topological space, and $A \subseteq X$. Then call A a *strong deformation retract* iff there is a retraction $r : X \to A$ such that $\imath r \simeq_{\text{rel } A} 1 : X \to X$.

Essentially, a strong deformation retract is a way of deforming X within itself to A, while keeping A fixed.

4.4 Towards the fundamental group

4.4.1 Group Structure of Paths

Recall that we defined the concatenation of two paths f and g to be $f*g$, provided $f(1) = g(0)$. In preparation for a discussion of the fundamental group, we wish to investigate this further. In particular, we'll be interested in looking at the extent to which equivalence classes of paths (under homotopy relative to a particular choice of A) exhibit the structure of a group. This, ultimately, will be what allows us to escape the sadness of point-set topology, and transition to some truly beautiful mathematics.

Definition 4.4.1

Let X be a topological space, and let f, g be paths in X. We say f and g are *equivalent* iff f and

g are homotopic relative to $\{0, 1\}$, in which case we write $f \sim g$. The equivalence classes of f are denoted by $[f]$.

Note that if f and g are equivalent, then the homotopy is some continuous function $F : I \times I \to X$ such that

$$\begin{aligned} F(t,0) &= f_0(t) \quad \text{and} \quad F(t,1) = f_1(t) \quad t \in I \\ F(0,s) &= f_0(0) \quad \text{and} \quad F(1,s) = f_0(1) \quad s \in I \end{aligned}$$

Hence, in a bit of an abuse of notation, we write $F : f_0 \sim f_1$.

Theorem 4.4.1. *Let X be a topological space, and let f, g, h be paths in X. Then*

1) $[f][g] = [f * g]$
2) $[f]([g][h]) = ([f][g])[h]$ *(whenever the product is defined — note though that in general, $(f * g) * h \neq f * (g * h)$. But, we have the next-best thing: if $f(1) = g(0)$ and $g(1) = h(0)$, then $(f * g) * h \sim f * (g * h)$).*
3) *If $x \in X$, then define $\epsilon_x : I \to X$ by $\epsilon_x(t) = x$. Then $[\epsilon_x][f] = [f] = [f][\epsilon_y]$ if f begins at x and ends at y.*
4) *Let $x = f(0)$, and $y = f(1)$. Then $[f][\overline{f}] = [\epsilon_x]$, and $[\overline{f}][f] = [\epsilon_y]$.*

Hence, we see that the equivalence classes of paths on X *almost* behave like a group. If only multiplication were always defined, and we had $x = y$...

4.4.2 The Fundamental Group

The trick is to make it happen.

Definition 4.4.2

Let X be a topological space, and let f be a path in X. Then f is said to be *closed* if $f(0) = f(1)$. If $f(0) = f(1) = x$, then we say f is *based* at x.

Oh look. We made it happen. How, you might ask? Well, observe:

Definition 4.4.3: Fundamental Group

Let X be a topological space, and let $x \in X$. Let f be some closed path based at x. Then define the *fundamental group* of X with base point x (denoted $\pi(X, X)$) to be $[f]$.

That this is a group follows immediately from Theorem 5.1. Buckle your seatbelts everyone, things are about to get *really* snazzy.

Theorem 4.4.2. *Let $x, y \in X$. If there is a path in X from x to y, then $\pi(X, x)$ and $\pi(X, y)$ are isomorphic.*

Proof. Let f be a path in X from x to y. Then for all $g \in \pi(X, x)$, $[\overline{f}] * [g] * [f] = [\overline{f} * g * f]$ is an equivalence class of paths in $\pi(X, y)$. Hence, define $\varphi : \pi(X, x) \to \pi(X, y)$ by

$$\varphi_f([g]) = [\overline{f} * g * f].$$

Clearly, this is bijective (we can define $\varphi_f^{-1}(h) = [f * h * \overline{f}]$, and note that $\varphi_f^{-1}\varphi = 1_{\pi(X,x)}$, $\varphi_f\varphi_f^{-1} = 1_Y$), and inherits the group structure of $\pi(X,x)$ in its image (since it is a conjugation). Thus it is a homomorphism. ■

Corollary 4.4.1. *Let X be a path connected topological space. Then for all $x, y \in X$, $\pi(X,x) \cong \pi(X,y)$.*

Note that the requirement that X be *path* connected is essential.

4.4.3 Continuous Functions & Fundamental Groups

We want to understand how continuous functions affect the fundamental group. First, we have the following three facts:

Lemma 4.4.1. *Let X, Y be topological spaces, and let $\varphi : X \to Y$ be continuous. Then*

1) If f, g are paths in X, then φf, φg are paths in Y.

2) If $f \sim g$, then $\varphi f \sim \varphi g$

3) If f is a closed path in X based at $x \in X$, then φf is a closed path in Y based at $\varphi(x)$

From these facts, we might deduce that continuous maps are, on some level, really just performing a sort of group homomorphism.

Definition 4.4.4: Induced Homomorphism

Let X, Y be topological spaces, and let $\varphi : X \to Y$ be continuous. Then define the *induced homomorphism* as follows: for all $x \in X$, define $\varphi_* : \pi(X,x) = \pi(Y, \varphi(x))$ such that if f is a path based on x,

$$\varphi_*([f]) = [\varphi f].$$

We referred to this as a homomorphism, so it better actually be one. Let's prove it!

Theorem 4.4.3. *Let X, Y be topological spaces, and let $\varphi : X \to Y$ be continuous. Then φ_* is a homomorphism of groups.*

Proof. Let $[f], [g] \in \pi(X,x)$. Then

$$\begin{aligned}
\varphi_*([f] * [g]) &= \varphi_*([f * g]) \\
&= [\varphi(f * g)] \\
&= [\varphi(f) * \varphi(g)] \qquad (*) \\
&= [\varphi(f)] * [\varphi(g)]
\end{aligned}$$

note that $(*)$ follows by simply cutting any representative path halfway through, and doing the work. Hence, φ_* is a group homomorphism. ■

A number of nice properties follow.

Theorem 4.4.4. *Let X, Y, Z be topological spaces, and let $\varphi : X \to Y$, $\psi : Y \to Z$ be continuous maps. Then*

1) $(\psi\varphi)_* = \psi_*\varphi_*$

2) If $1 : X \to X$ is the identity map then 1_* is the identity homomorphism.

3) If φ is a homeomorphism, then $\varphi_* : \pi(X, x) \to \pi(Y, \varphi(x))$ is an isomorphism of groups.

Instead of proving these, we make a remark about functors. Recall the following definition:

Definition 4.4.5: Functor

Let $\mathbf{C}, \mathbf{D}$ be categories, and let $c \in \mathrm{ob}(C)$, $f, g \in \hom(C)$. Then call $\mathcal{F} = (\mathcal{F}_o, \mathcal{F}_a)$ a functor if

$$\mathcal{F}_o : \mathrm{ob}(\mathbf{C}) \to \mathrm{ob}(\mathbf{D}) \qquad \mathcal{F}_a : \hom(\mathbf{C}) \to \hom(\mathbf{D})$$

such that the following essential properties are preserved:

(a) $\forall (f : c \to c') \in \hom(\mathbf{C})$, $\mathcal{F}_a(f) : \mathcal{F}_o(c) \to \mathcal{F}_o(c')$, with

$$\mathcal{F}_a(1_c) = 1_{\mathcal{F}_o(c)}$$

and

(b)

$$\mathcal{F}_a(g \circ f) = \mathcal{F}_a(g) \circ \mathcal{F}_a(f).$$

Or alternatively, with homsets:

Definition 4.4.6: Functor (version 2)

Let $\mathcal{T} : \mathbf{C} \to \mathbf{B}$ be defined with the usual object functor $\mathcal{T}_o$, together with a collection of functions

$$\mathcal{T}^{a,b} : \hom_{\mathbf{C}}(a, b) \to \hom_{\mathbf{B}}(\mathcal{T}_o(a), \mathcal{T}_o(b))$$

then $\mathcal{T}$ is *full* when every such $\mathcal{T}^{a,b}$ is surjective, and *faithful* when injective.

We can now frame the results obtained above in the language of category theory.

Theorem 4.4.5. *Let $\mathbf{Top}_*$ denote the category with objects (X, x) where X is a topological space, and x is a selected base point. Define the morphisms on $\mathbf{Top}_*$ to be continuous maps of the form $\varphi : (X, x) \to (Y, y)$, where we require $\varphi(x) = y$. Then the map $\mathcal{F} : \mathbf{Top}_* \to \mathbf{Grp}$ taking each (X, x) to the corresponding fundamental group is a functor.*

Proof. Take

$$\mathcal{F}_o((X, x)) = \pi(X, x) \qquad \mathcal{F}_a(\varphi) = \varphi_*.$$

And note that indeed, as we defined above, if $\varphi : (X, x) \to (Y, y)$, then $\varphi_* : \pi(X, x) \to \pi(Y, y)$, and $\mathcal{F}_a(1_{(X,x)}) = 1_*$, and indeed $\mathcal{F}_a(\psi \circ \varphi) = \mathcal{F}_a(\psi) \circ \mathcal{F}_a(\varphi)$. Hence $\mathcal{F}$ is a functor. ∎

5. More Fundamental Group (Unfinished)

5.1 Solutions to Some Exercises from Kosniowski

5.1.1 Exercises 15.6

(a) Show that two paths f, g from x to y give rise to the same isomorphism from $\pi(X, x)$ to $\pi(X, y)$ (i.e., $u_f = u_g$) iff $[g * \overline{f}] \in \mathbf{Z}(\pi(X, x))$, where $\mathbf{Z}(G)$ denotes the center.

(b) Let $u_f : \pi(X, x) \to \pi(X, y)$ be the isomorphism determined by a path from x to y. Prove that u_f is independent of f iff $\pi(X, x)$ is abelian.

Solutions

(a) We have

$$u_f([\gamma]) = [\overline{f} * \gamma * f] \qquad\qquad u_g([\gamma]) = [\overline{g} * \gamma * g].$$

Thus, $u_f = u_g$ iff $\forall [\gamma] \in \pi(X, x)$

$$\begin{aligned}
u_f([\gamma]) &= u_g([\gamma]) \\
[\overline{f} * \gamma * f] &= [\overline{g} * \gamma * g] \\
[g][\overline{f} * \gamma * f][\overline{f}] &= [g][\overline{g} * \gamma * g][\overline{f}] \\
[g * \overline{f} * \gamma * f * \overline{f}] &= [g * \overline{g} * \gamma * g * \overline{f}] \\
[g * \overline{f} * \gamma] &= [\gamma * g * \overline{f}]
\end{aligned}$$

iff $[g * \overline{f}] \in \mathbf{Z}(\pi(X, x))$. ■

(b) u_f is independent of f iff $\forall [f], [g] \in \pi(X, x)$, $u_f = u_g$, which (by the exercise above) occurs iff $[g * \overline{f}] \in \mathbf{Z}(\pi(X, x))$, after which point it is a straightforward exercise in Algebra to show that $\pi(X, x)$ is necessarily abelian (follows readily because $[f], [g]$ were taken to be arbitrary).

5.1.2 Exercisess 15.11

(a) Give an example of an injective continuous map $\varphi : X \to Y$ for which φ_* is not injective.

(b) Give an example of a surjective continuous map $\varphi : X \to Y$ for which φ_* is not surjective.

(c) Prove that if $\varphi : X \to Y$ is continuous and f is a path from x to y then $\varphi_* u_f = u_{\varphi(f)}\varphi$, where both sides of the equation above are maps of the form $f : \pi(X, x) \to \pi(Y, \varphi(y))$, and u_f, $u_{\varphi(f)}$ are the isomorphisms of fundamental groups determined by f and $\varphi(f)$.

(d) Prove that two continuous mappings $\varphi, \psi : X \to Y$, with $\varphi(x_0) = \psi(x_0)$ for some point $x_0 \in X$, induce the same homomorphism from $\pi(X, x_0)$ to $\pi(Y, \varphi(x_0))$ if φ and ψ are homotopic relative to x_0.

(e) Suppose that A is a retract of X with retraction $r : X \to A$. Prove that $i_* : \pi(A, a) \to \pi(X, a)$ is a momomorphism (where $i : A \to X$ denotes inclusion) and that $r_* : \pi(X, a) \to \pi(A, a)$ is an epimorphism for any point $a \in A$.

(f) With the notation of (e), above, suppose that $i_*\pi(A, a) \trianglelefteq \pi(X, a)$. Prove that $\pi(X, a)$ is the direct product of the subgroups image (i_*) and kernel (r_*).

(g) Prove that if A is a strong deformation retract of X then the inclusion map $i : A \to X$ induces an isomorphism $i_* : \pi(A, a) \to \pi(X, a)$ for any point $x \in A$.

(h) Show that if $\varphi : X \to X$ is a continuous map with $\varphi \simeq 1$ then $\varphi_* : \pi(X, x_0) \to \pi(X, \varphi(x_0))$ is an isomorphism for each point $x_0 \in X$.

Solutions:

(a) Consider $\varphi : S^1 \hookrightarrow D^2$. Note $\pi(S^1, x) \cong \mathbb{Z}$, while $\pi(D^2, x) = 0$. Now $\ker(\varphi_*) = \mathbb{Z}$ so φ_* is not injective.

(b) Consider $\varphi : D^2 \twoheadrightarrow S^1$. Then φ_* is the inclusion map of the trivial group into $\mathbb{Z}$, which is clearly not onto.

(c) Ok, I'm pretty sure the claim is supposed to actually be proving that $\varphi_* u_f = u_{\varphi(f)} \varphi'_*$, where $\varphi_* : \pi(X, y) \to \pi(Y, \varphi(y))$, and $\varphi'_* : \pi(X, x) \to \pi(X, \varphi(x))$. Proving the claim is equivalent to showing that the following diagram commutes:

$$\begin{tikzcd}
\pi(X,x) \arrow[r, "u_f"] \arrow[d, "\varphi'_*"'] & \pi(X,y) \arrow[d, "\varphi_*"] \\
\pi(Y,\varphi(x)) \arrow[r, "u_{\varphi(f)}"'] & \pi(Y,\varphi(y))
\end{tikzcd}$$

Figure 5.1: Commutative diagram for $\varphi_* u_f = u_{\varphi(f)} \psi_*$

First, note that $\varphi(f)$ is a path from $\varphi(x)$ to $\varphi(y)$. Let $[g] \in \pi(X, x)$. Then

$$\begin{aligned}
\varphi_*(u_f([g])) &= \varphi_*([\overline{f} * g * f]) \\
&= [\varphi(\overline{f} * g * f)] \\
&= [\overline{\varphi(f)} * \varphi(g) * \varphi(f)] \\
&= u_{\varphi(f)}([\varphi(g)]) \\
&= u_{\varphi(f)}(\varphi'_*([g]))
\end{aligned}$$

as desired.

(d) We have $\varphi, \psi : X \to Y$, and $\exists x_0 \in X$ s.t. $\varphi(x_0) = \psi(x_0)$, and $\varphi \simeq_{\text{rel } \{x_0\}} \psi$. Thus, $\exists F : X \times [0,1] \to Y$ such that $\forall x \in X$, $F(x, 0) = \varphi(x)$, $F(x, 1) = \psi(x)$, and $F(x_0, t) = \varphi(x_0) = \psi(x_0)$ for all $t \in [0, 1]$.

(e)

(f)

Some more theorems

Theorem 5.1.1. *Let $\varphi, \psi : X \to Y$ be continuous mappings between topological spaces and let $F : \varphi \simeq \psi$ be a homotopy. If $f : I \to Y$ is the path from $\varphi(x_0)$ to $\psi(x_0)$ given by $f(t) = F(x_0, y)$ then the homomorphisms*

$$\varphi_* : \pi(X, x_0) \to \pi(Y, \varphi(x_0)) \qquad \textit{and} \qquad \psi_* : \pi(X, x_0) \to \pi(Y, \psi(x_0))$$

are related by $\psi_* = u_f \varphi_*$, *where* u_f *is the isomorphism from* $\pi(Y, \varphi(x_0))$ *to* $\pi(Y, \psi(x_0))$ *determined by the path* f.

Proof. Let $[g] \in \pi(X, x_0)$. WTS $[\psi g] = [\overline{f} * \phi g * f]$. Observe that ■

6. Homology and Cohomology

(This chapter based off Munkres). In the following chapter, we'll make some connections between algebraic topology (particularly, homology theory) and differential topology. The link will be differential forms, a formalism we construct as a natural way of treating the problem of differentiation and integration of general functions on k-manifolds in $\mathbb{R}^n$.

6.1 Differential Forms

In our treatment, we will take a coordinate-free approach to differential forms. This confers the benefits of allowing us to work in a coordinate-free manner — as we will see, just as we used *operators* to define a coordinate-free approach to linear maps, so too will we use *tensors* here to work with differential forms. However, this comes at the cost of requiring that we first treat Multilinear algebra, and stuff. Thankfully, Munkres' text is cohesive and clear in this regard.

6.1.1 Multilinear Algebra

Just as the concept of a linear map lies at the heart of Linear Algebra, the idea of a *multilinear map* (commonly referred to as a "tensor") is the fundamental principle of multilinear algebra. The definitions to follow should make it clear that these maps are a natural and straightforward extension of the ideas we encounter in linear algebra. First, recall the definition of a linear functional.

Definition 6.1.1: Linear Functional

Let $\mathcal{V}$ be a vector space over a filed $\mathbb{F}$, and let $f : \mathcal{V} \to \mathbb{F}$. Then f is said to be *linear* if $\forall v_1, v_2 \in \mathcal{V}$, $\forall c \in \mathbb{F}$, we have

1. $f(cv_1) = cf(v_1)$
2. $f(v_1 + v_2) = f(v_1) + f(v_2)$.

This idea generalizes naturally to a multilinear map:

Definition 6.1.2: Tensor

Let $\mathcal{V}$ be a vector space over a field $\mathbb{F}$, and let $\mathcal{V}^k = \bigoplus_{i=1}^k V$. Then a function $f : V^k \to \mathbb{F}$ is said to be **linear in the i^{th} variable** if, given fixed vectors v_j for $j \neq i$, the function

$$T(v) = f(v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_k)$$

is a linear functional. f is said to be **multilinear** if it is linear in the i^{th} variable for each i. Such an f is also called a **k-tensor** on $\mathcal{V}$, or a **tensor of order** k on $\mathcal{V}$. We denote the set of all k-tensors on $\mathcal{V}$ by $\mathcal{L}^k(\mathcal{V})$. Note that in $\mathcal{L}^1(\mathcal{V})$, we recover the dual space of $\mathcal{V}$.

We now show that, analogously to with linear transformations, tensors are determined uniquely by their values on basis elements.

Theorem 6.1.1. *Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be a basis for $\mathcal{V}$. Then if $f, g \in \mathcal{L}^k(\mathcal{V})$, and if $\forall I = (i_1, \ldots, i_k)$ where*

$\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$, we have

$$f(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}) = g(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k})$$

then $f = g$.

It's worth noting that we place no restrictions on the ordering of I, and similarly do not mind if it contains repeated elements.

Proof. Let $(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ be an arbitrary k-tuple of vectors in $\mathcal{V}$. Then we can express each $\mathbf{v}_i$ by

$$\mathbf{v}_i = \sum_{j=1}^{n} c_{ij}\mathbf{e}_i.$$

Hence

$$\begin{aligned}
f(\mathbf{v}_1, \ldots, \mathbf{v}_k) &= f\left(\sum_{j_1=1}^{n} c_{1j_1}\mathbf{e}_{j_1}, \mathbf{v}_2, \ldots, \mathbf{v}_k\right) \\
&= \sum_{j_1=1}^{n} c_{1j_1} f(\mathbf{e}_{j_1}, \mathbf{v}_2, \ldots, \mathbf{v}_k) \\
&= \sum_{j_1=1}^{n}\sum_{j_2=1}^{n} c_{1j_1}c_{2j_2} f(\mathbf{e}_{j_1}, \mathbf{e}_{j_2}, \ldots, \mathbf{v}_k) \\
&= \sum_{j_1=1}^{n} c_{1j_1} \sum_{j_2=1}^{n} c_{2j_2} \quad \cdots \quad \sum_{j_k=1}^{n} c_{kj_k} f(\mathbf{e}_{j_1}, \mathbf{e}_{j_2}, \ldots, \mathbf{e}_{j_k}) \\
&= \sum_{1 \le j_1, \ldots, j_k \le n} c_{1j_1}c_{2j_2}\ldots c_{kj_k} f(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_k}) \\
&= \sum_{1 \le j_1, \ldots, j_k \le n} c_{1j_1}c_{2j_2}\ldots c_{kj_k} g(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_k}) \\
&= g(\mathbf{v}_1, \ldots, \mathbf{v}_k)
\end{aligned}$$

as desired. ■

Perhaps the main takeaway here is that indices can be a bit of a mess.[1] Hence, I'll elect to formally define k-tuples of indices below, so that we don't have to keep definining them whenever we want to use them.

Definition 6.1.3: Indexing tuples

Let $\mathcal{I}(n) = \{1, \ldots, n\}$. Then if $I = (i_1, \ldots, i_k)$, where $\{i_1, \ldots, i_k\} \subseteq \mathcal{I}(n)$ (with some elements possibly repeated), we call I a **k-index**. We'll denote the set of all k-indices chosen from n elements by $\mathcal{I}_k(n)$.

Let I be a k-index. Then whenever the meaning is understood, we will denote a tuple of vectors like $(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ by $\mathbf{v}_I$, and similarly for other objects as is appropriate.

One might recall from linear algebra that for a vector space $\mathcal{V}$, $\mathcal{L}^1(\mathcal{V})$ is itself a vector space, called the *dual space* of $\mathcal{V}$. In fact, it turns out that $\mathcal{L}^k(\mathcal{V})$ is a vector space for all k. As such, we would like to endow it with a basis. We define a good candidate in the following definition (which is also sort of a theorem):

[1] I tried to think of a way to index cleverly so as to express this more cleanly, but the best I got was turning the $c_{1j_1}c_{2j_2}\cdots c_{kj_k}$ into a $\prod_{i=1}^{k} c_{ij_i}$, which is definitely not better.

Definition 6.1.4: Elementary k-tensor

Let $\mathcal{V}$ be a vector space with basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$, and let $I \in \mathcal{I}_k(n)$. Then there exists a unique $\phi_I \in \mathcal{L}^k(\mathcal{V})$ such that $\forall J \in \mathcal{I}_k(n)$,

$$\phi_I(\mathbf{e}_J) = \begin{cases} 0 & \text{if } I \neq J \\ 1 & \text{if } I = J \end{cases}.$$

We call such a ϕ_I an **elementary k-tensor on $\mathcal{V}$**. Note the sense in which these can be loosely thought of as a generalization of the kronecker delta.

N.B. — Recall that by $\mathbf{e}_J$, we mean $(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_k})$.

In claiming existence and uniqueness, we really ought to provide a proof. And so we shall.

Proof. If we can show existence, uniqueness will follow from the theorem stated at the beginning of the chapter. Hence, we show existence. First, note that for $k = 1$, ϕ_i is just the kronecker delta with one entry fixed:

$$\phi_i(\mathbf{e}_j) = \delta_{ij}$$

thus, for the general case, we simply take

$$\begin{aligned} \phi_I(\mathbf{v}_1, \ldots, \mathbf{v}_k) &= \prod_{\alpha=1}^{k} \phi_{i_\alpha}(\mathbf{v}_i) \\ &= [\phi_{i_1}(\mathbf{v}_1)] \cdot [\phi_{i_2}(\mathbf{v}_2)] \cdots [\phi_{i_k}(\mathbf{v}_k)] \end{aligned}$$

Now, note that

$$\phi_I(\mathbf{e}_J) = \prod_{\alpha=1}^{k} [\phi_{i_\alpha}(\mathbf{e}_{j_\alpha})]$$

which is 1 iff $i_\alpha = j_\alpha$ for all α, and 0 otherwise. Hence, we have the desired property for ϕ_I. ■