Basic Algebraic Topology

Based on Kosniowski; Matveev

1. Picking up where we left off

1.1. Some Theorems

We'll breeze briskly through some more topology, then transition into knot theory topics.

Theorem 1.1. *Let Y be the quotient space of the topological space X determined by the surjective mapping $f : X \to Y$. If X is compact Hausdorff and f is closed then Y is (compact) Hausdorff.*

Corollary 1.1. *Let X be a compact Hausdorff G-space with G finite. Then X/G is a compact Hausdorff space.*

Corollary 1.2. *If X is a compact Hausdorff space and A is a closed subset of X then X/A is a compact Hausdorff space.*

2. All Together Now...

2.1. Connectedness

Definition 2.1

Let (X, τ) a topological space. Then X is said to be *connected* iff the only clopen subsets are trivial. If $S \subseteq S$, then S is said to be connected iff it is connected in the induced topology.

Equivalently, X is connected iff it cannot be expressed as the union of finitely many disjoint non-empty open subsets.

Theorem 2.1. *Let $f : X \to Y$ be continuous, and suppose X is connected. Then Y is connected as well.*

Proof. Suppose, to obtain a contradiction, that Y is disconnected. Then there exist nonempty open sets $U, V \subseteq Y$ with $U \cup V = Y$, and $U \cap V = \varnothing$. Since U, V are open and f continuous, then $f^{-1}(U)$, $f^{-1}(V)$ are open in X. Furthermore, these are disjoint nonempty subsets of X with $f^{-1}(U) \cup f^{-1}(V) = X$. Then X is disconnected, a contradiction. Hence Y is connected. ■

Theorem 2.2. *Suppose that $\{Y_j \mid j \in J\}$ is a collection of connected subsets of a space X. If $\bigcap_{j \in J} Y_j \neq \varnothing$, then $Y = \bigcup_{j \in J} Y_j$ is connected.*

Proof. Suppose U is a nonempty clopen subset of Y. Then $\exists j \in J$ s.t. $U \cap Y_j \neq \varnothing$. Hence, let $J' = \{j \in J \mid U \cap Y_j \neq \varnothing\}$. Then $\forall j' \in J'$, we have $U \cap Y_{j'}$ is clopen in the induced topology. Since $Y_{j'}$ is connected, it follows that $U \cap Y_{j'} = Y_{j'}$. Hence $U = Y$, so Y is connected. ■

Theorem 2.3. *Let X, Y be topological spaces. Then X, Y are connected iff $X \times Y$ is connected.*

Proof.

($\Rightarrow$) : Suppose X, Y are connected. $\forall x \in X,\ y \in Y,\ X \times \{y\} \cong X$ and $\{x\} \times Y \cong Y$ (and thus each is connected), and note $(X \times \{y\}) \cap (\{x\} \times Y) \neq \varnothing$, thus by theorem 1.3 their union is connected. Now, let $y \in Y$ be fixed. Observe that

$$X \times Y = \bigcup_{x \in X} (X \times \{y\}) \cap (\{x\} \times Y)$$

Hence $X \times Y$ is connected.

($\Leftarrow$) : Suppose $X \times Y$ is connected. Then since the canonical projection maps are continuous, it follows that X, Y are connected (continuous image of a connected set is connected).

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2.2. Path Connectedness

3. A Brief Discussion of Manifolds

Definition 3.1: Manifolds

Let $n \in \mathbb{Z}^{>0}$, and let (M, τ) be a topological space. Then M is called a manifold iff M is Hausdorff, and $\forall m \in M$, there exists a neighborhood N of m such that $N \cong \mathring{D}^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$.

Definition 3.2: Connected Sum

Let S_1, S_2 be compact connected 2-manifolds (surfaces), and let $D_1 \subseteq S_1$, $D_2 \subseteq S_2$ with $D_1, D_2 \cong D^2$. Let $h_1 : D_1 \to D^2$, and $h_2 : D_2 \to D^2$ be homeomorphisms. Then define $\sim$ to be an equivalence relation such that $x \sim h_2^{-1} h_1(x)$ iff $x \in \partial D_1$, and $x \sim x$ otherwise. Then $S_1 \# S_2$ is given by

$$\frac{(S_1 - \mathring{D}_1) \cup (S_2 - \mathring{D}_2)}{\sim}$$

Definition 3.3

Call a surface S^2 *orientable* if it contains no Möbius strip, and *non-orientable* otherwise. Then for $m \geq 0, n \geq 1$, we call

$$S^2 \# \left(\mathop{\#}_{i=1}^{m} T \right) = S \# mT$$

the *standard orientable surface of genus m*, and

$$S^2 \# \left(\mathop{\#}_{i=1}^{n} \mathbb{R}P^2 \right) = S \# n\mathbb{R}P^2$$

the *standard non-orientable surface of genus n*.