Misc. Categories and Introduction to Universals

Lecture 3

1 Introduction

Recall the definition of a natural transformation.

Definition 1.1: Natural Transformation

Let $\mathbf{B}, \mathbf{C}$ be categories, and let $\mathcal{F}, \mathcal{G} : \mathbf{C} \to \mathbf{D}$. Let η be called a *natural transformation* if

i) For all $c \in \mathbf{C}$, η assigns a morphism $\eta_c : \mathcal{F}_o(c) \to \mathcal{G}_o(c)$ (known as the *component* of η at c), such that

ii) $\forall f : c \to c'$, $\eta_{c'} \circ \mathcal{F}_a(f) = \mathcal{G}_a(f) \circ \eta_{c'}$

when we first introduced natural transformations, we hinted that natural transformations can be thought of as morphisms on a category with functors as the objects. We return to this topic now, seeking a more formal understanding.

Definition 1.2: Bullet composition

Let $\mathbf{C}, \mathbf{B}$ be categories, and let $\mathcal{R}, \mathcal{S}, \mathcal{T}, \ldots : \mathbf{C} \to \mathbf{B}$. Then let $\sigma : \mathcal{R} \stackrel{\bullet}{\to} \mathcal{S}$ and $\tau : \mathcal{S} \stackrel{\bullet}{\to} \mathcal{T}$ be natural transformations. Then define $\tau \bullet \sigma : \mathcal{R} \stackrel{\bullet}{\to} \mathcal{T}$ such that $\forall c \in \mathbf{C}$, we have

$$(\tau \bullet \sigma)_c = \tau_c \circ \sigma_c.$$

Then the composite $\tau \bullet \sigma$ is natural.

$\bullet$ is associative, and for each $\mathcal{T}$, has an identity transformation, namely $1_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$, with $c \mapsto 1_{\mathcal{T}c}$. Thus, the functors themselves carry the structure of a category.

Definition 1.3: Functor Category

Let $\mathbf{B}, \mathbf{C}$ be categories. Then we construct a *functor category* $\mathbf{B}^{\mathbf{C}} = \mathrm{Funct}(\mathbf{C}, \mathbf{B})$ with objects

$$\mathrm{ob}\big(\mathbf{B}^{\mathbf{C}}\big) = \{T : \mathbf{C} \to \mathbf{B}\}$$

and morphisms

$$\hom\big(\mathbf{B}^{\mathbf{C}}\big) = \big\{\tau \bigm| \tau : \mathcal{S} \stackrel{\bullet}{\to} \mathcal{T},\ \tau \text{ is natural}\big\}.$$

with composition defined by $\bullet$. We'll consider a few examples:

(a) Let X a finite set, and $\mathbf{B}$ a category. Then $\mathbf{B}^X$ is the set of all functions from X to $\mathbf{B}$.

One might wonder whether we can find another definition of composition that

2 Comma Categories

Comma categories will play a large role in studying Adjoint functors in the future, so we'll take some time here to discuss them in detail. Essentially, Comma categories serve as a way of connecting two functors that share the same codomain category, by constructing the category of morphisms between their images.

Definition 2.1: Comma Category

Let $\mathbf{C}, D, E$ be categories, and let $\mathcal{F}, G$ be functors with $\mathcal{F} : D \to C$, and $\mathcal{G} : E \to C$. Then the *comma category*, denoted by

$$(\mathcal{F} \downarrow \mathcal{G}) \quad \text{or} \quad (\mathcal{F}, \mathcal{G})$$

is defined as follows:

$$\mathrm{ob}((\mathcal{F} \downarrow \mathcal{G})) = \{\langle d, e, f\rangle \mid d \in \mathrm{ob}(\mathbf{D}), e \in \mathrm{ob}(\mathbf{E}), f : \mathcal{F}_o(d) \to \mathcal{G}_o(e)\},$$

which can be expressed diagramatically by

$$\begin{tikzcd}
\mathcal{F}_o(d) \arrow[d, "f"] \\
\mathcal{G}_o(e)
\end{tikzcd}$$

Figure 1: Objects of $(\mathcal{F} \downarrow \mathcal{G})$

and

$$\mathrm{hom}((\mathcal{F} \downarrow \mathcal{G})) = \{\langle g, h\rangle \mid g : d \to d', h : e \to e'\}$$

such that

$$\begin{tikzcd}
\mathcal{F}_o(d) \arrow[r, "\mathcal{F}_a(g)"] \arrow[d, "f"'] & \mathcal{F}_o(d') \arrow[d, "f'"] \\
\mathcal{G}_o(e) \arrow[r, "\mathcal{G}_a(h)"] & \mathcal{G}_o(e')
\end{tikzcd}$$

Figure 2: Morphisms of $(\mathcal{F} \downarrow \mathcal{G})$

commutes.

Essentially, we're