Basic Algebraic Topology

Based on Kosniowski; Matveev

(Sung to the tune of *Time in a Bottle*)
If I could save Klein in a Bottle
The first thing that I'd like to glue
Is an edge to an edge
Of a projective plane and then #
With another $P2$

1. Introduction

First, a motivating quote.

> "Point set topology is a disease from which later generations will regard themselves as having recovered" -Henri Poincaré

That's... not exactly a ringing endorsement. Why did Henri Poincaré have such a low opinion of point-set topology? Well, loosely speaking, because it's just not the right tool for the job, especially when compared to *algebraic topology.*

As it turns out, lots of topics in topology can be simplified by attaching algebraic objects to topological spaces, and proving that certain properties of this object correspond naturally to properties of our topological space. The vehicle by which we navigate between the two is, as one might expect, Category Theory. First, we give a brief summary of basic concepts in algebraic topology, before moving into the Homology presentation given in Matveev.

1.1. Basic Point-Set Topology

As in most branches of mathematics, our object of study here will be some collection of sets, together with some *structure* we can associate with them. In Elementary Algebra, this takes the form of *group* and *ring* operations, and later the respective homomorphisms preserving them. In Elementary Analysis, this (loosely speaking) took the form of a *distance metric*, and the properties it bestowed on sets. Analogous to our study of homomorphisms in Algebra, we often studied *continuous functions* in Analysis, and the properties of sets that they preserved. Note the resemblance between the two expressions:

$$\varphi(g_1 \oplus g_2) = \varphi(g_1) \otimes \varphi(g_2) \qquad\qquad d(x,y) < \delta \implies d'(f(x), f(y)) < \varepsilon$$

A homomorphism $\varphi : G \to H$ A continuous function $f : (E, d) \to (E', d')$

while the analogy doesn't hold exactly, in both cases, we have some particular class of functions such that structure in one space is preserved in the image. In the case of homomorphisms, the group operation in the first group is "respected" by the homomorphism upon mapping into the second. In the case of continuous functions, our equation is essentially stating that the function respects things being "close" to one another (our map doesn't destroy the properties of the distance metric). We can think of this as stipulating that we didn't "tear" our starting space at all. This is best visualized by thinking about our continuous functions not as their *graphs* (as we are often used to), but rather as *maps* that deform the input domain in various manners to yield the image. As an example, one might think of the function $f(x) = x^2$ as the action of folding $\mathbb{R}$ on itself, and stretching the edges out towards infinity (this is often a strategy employed in visualizing complex-valued functions).

One might wonder what sorts of interesting discoveries we could make by generalizing our starting premises on the right-hand-side, so that we could make our questions more similar to those on the left. That is, similarly to how we defined distance metrics so as to generalize the *key* properties of Euclidean distance, so too will we generalize the idea of *continuity of a function*. This is the central idea of basic topology. Now, all we need is a good place to start. Recall the following theorem of Analysis:

Theorem 1.1. *Let (E, d) and (E', d') be metric spaces. Then a function $f : E \to E'$ is said to be* continuous *iff for all open sets $U \subseteq M_2$, we have $f^{-1}(U)$ is open in M_1.*

Note, this theorem makes no guarantees about the image of an open set being open in the codomain. Really, we can make our image as “jagged” as we want (within reason), provided we fold and deform our domain in a smooth manner. But it does indicate to us open sets appear to be intimately tied to the idea of “smooth” deformations. In particular, noting that the theorem is an if and only if, we might wonder what would happen were we to discard the distance metric entirely, and take the above as a *definition* instead of as a *theorem*. There’s just one catch: if we really want to discard the metric, how are we going to define the notion of an “open” set? Topologies provide the answer.

1.2. Onwards to Algebraic Topology

Definition 1.1

Let X be a set, and let $\mathcal{U}$ be a collection of subsets of X satisfying the following:

(i) $\varnothing \in \mathcal{U}$, $X \in \mathcal{U}$.

(ii) For all $U_1, U_2 \in \mathcal{U}$, we have $U_1 \cap U_2 \in \mathcal{U}$ (by induction, we obtain closure under finite intersections).

(iii) For any subset $\{U_i \mid i \in I\} \subseteq \mathcal{U}$, we have

$$\bigcup_{i \in I} U_i = \boldsymbol{U} \in \mathcal{U}$$

($\mathcal{U}$ is closed under arbitrary unions).

then $\mathcal{U}$ is called a *topology* for X, and $(X, \mathcal{U})$ is called a *topological space*. We call the elements of $\mathcal{U}$ the *open sets of* $(X, \mathcal{U})$.

note that a topology is thus a particular kind of *algebra of sets* under the binary operations $\cup, \cap$, with identity $\varnothing$ for $\cup$, and X for $\cap$. Note that $(\cup, \varnothing)$, $(\cap, X)$ are duals of each other, in the sense that for any sentence S built out of atomic propositions about our set algebra, if S is true, then the statement we obtain by

1. Replacing each $\cup$ with $\cap$ and each $\cap$ with $\cup$,
2. Interchanging each $\varnothing$ and X, and
3. Reversing inclusions

must also be true. Less relevantly (but maybe an object of interest), observe that if we replace “arbitrary unions” with “countable unions”, and further require closure under complementation, then we obtain a σ-algebra.

As it turns out, this definition of a topology is more general than that given by distance metrics. Whereas every distance metric gives rise to a topology, there are topologies that are not *metrizable*, meaning they do not arise from any metric on a set. We list a few common topologies. Let $(X, \mathcal{U})$ be a topological space. Then

1. If $\mathcal{U} = \{\varnothing, X\}$, we call $\mathcal{U}$ the *concrete* or *indiscrete topology*.
2. If $\mathcal{U} = \mathcal{P}(X)$ (i.e., every subset of X is open), then we call $\mathcal{U}$ the *discrete topology*. Note the direct connection to the discrete metric.
3. Suppose $\mathcal{U} = \{\varnothing, X\} \cup \left\{U \subseteq X \,\middle|\, \left|\overline{U}\right| < \infty\right\}$. That is, $X, \varnothing$, and all subsets of X with finite compliment. Then call $\mathcal{U}$ the *finite complement topology*.
4. Let $X = \mathbb{R}$, and $\mathcal{U} = \{\varnothing, \mathbb{R}\} \cup \{(x, \infty) \mid x \in \mathbb{R}\}$

As an exercise, I'll give a proof for some of these. Note that below, I'll be using overline to denote the compliment of a set, a notation I'll discard quickly once we start talking about compliments. But for now, it's the most convenient.

1. Trivial
2. Also trivial
3. Observe $\varnothing, X \in \mathcal{U}$ by definition. We first prove that $\mathcal{U}$ is closed under finite intersections. Let $U_1, U_2 \in \mathcal{U}$. Then $\overline{U_1 \cap U_2} = \overline{U_1} \cup \overline{U_2}$ (De Morgan's Laws). The union of two finite sets is finite, hence $\overline{U_1 \cap U_2}$ is finite and so $U_1 \cap U_2 \in \mathcal{U}$. Now, let $U' = \{U_i \mid i \in I\} \subseteq \mathcal{U}$. Then

$$\overline{\bigcup_{i \in I} U_i} = \bigcap_{i \in I} \overline{U_i}$$

which is an intersection of finite sets, and is thust finite itself. Hence $\mathcal{U}$ is closed under arbitrary unions. It follows that $(X, \mathcal{U})$ is a topological space. ■

4. The non-trivial elements of $\mathcal{U}$ inhereit a total order by inclusion from the total order on $\mathbb{R}$. We add $\varnothing, \mathbb{R}$ to the total order by putting $\mathbb{R} = \sup(\mathcal{U})$, $\varnothing = \inf(\mathcal{U})$. The closure properties follow. ■

We define some noteworthy sets.

Definition 1.2: Interior

Let $(X, \mathcal{U})$ be a topological space. Let $Y \subseteq X$. Let $U' = \{U_i \in \mathcal{U} \mid i \in I, U_i \subseteq Y\}$ be the set of all open sets contained in Y. Then call the *interior* of Y (sometimes denoted Y°)

$$\operatorname{int}(Y) = \bigcup_{i \in I} U_i.$$

we have open sets — what's next, closed sets???? Yeah, uh, you got me there.

Definition 1.3: Closed sets

Let $(X, \mathcal{U})$ be a topological space. Let $C \subseteq X$, and let $X - C$ denote the compliment of C in X. Then call C *closed* iff $X - C$ is open. **VERY VERY IMPORTANT:** a set can be both open and closed. In fact, in the discrete topology, every set is both.

From the principle of duality (or, alternatively, De Morgan's Laws), the dual statements of the topology axioms hold for closed sets. That is,

i. $X, \varnothing$ are closed

ii. The set of closed sets is closed (haha) under finite unions

iii. The set of closed sets is closed under arbitrary intersections

Analogously to the interior of a set, we define its closure. Note closure is, in a sense, the dual of the interior operator.

Definition 1.4

Let $(X, \mathcal{U})$ be a topological space. Let $Y \subseteq X$. Then let

$$V' = \{V_i \mid i \in I, X - V_i \in \mathcal{U}, Y \subseteq V_i\}$$

that is, the set of all closed sets containing Y. Then call the *closure* of Y (denoted $\overline{Y}$)

$$\operatorname{clos}(Y) = \bigcap_{i \in I} V_i$$

Finally, we define the boundary of a set as

Definition 1.5

Let $(X, \mathcal{U})$ a topological space. Let $Y \subseteq X$. Then define the boundary of Y, denoted ∂Y, by $\partial Y = \overline{Y} - Y$.

One might wonder why we've chosen to recycle the partial derivative notation to denote the boundary. The intuitiion is as follows (this lightly paraphrased from a stackexchange answer by Terry Tao):

> Let S be a smooth, bounded body. Then the surface area $|\partial S|$ is the derivative of the volume $|S_r|$ of the r-neighborhoods S_r at $r = 0$:
>
> $$|\partial S| = \frac{\mathrm{d}}{\mathrm{d}r} |S_r| \bigg|_{r=0}$$
>
> Here's the part I don't understand: he then goes on to say "More generally, one intuitively has the Newton quotient-like formula"
>
> $$\partial S = \lim_{h \to 0^+} \frac{S_h \setminus S}{h}$$
>
> "the right-hand side does not really make formal sense, but certainly one can view $S_h \setminus S$ as a $[0, h]$-bundle over ∂S for h sufficiently small (in particular, the radius of curvature of S)."

Finally, we define the neighborhood of a point x.

Definition 1.6: Neighborhood

Let $(X, \mathcal{U})$ be a topological space. Let $N \subseteq X$, and let $x \in N$. Then call N a neighborhood of x if $x \in N^\circ$.

Note the following properties:

1. clos and int are idempotent.
2. $X - Y^\circ = \overline{X - Y}$
3. $\partial Y = \varnothing \iff Y$ is clopen.
4. Let $U \in \mathcal{U}$. Then $Y = \overline{U} \iff Y = \overline{U^\circ}$ (proof: note $U = U^\circ$).

5. For each point $x \in X$, there is at least one neighborhood of x, namely X.
6. If M and N are neighborhoods of x, then so is $N \cap M$ (use closure under finite intersections).

1.3. Continuous functions and Induced Topologies

As promised, we'll now define a continuous function between two topological spaces.

Definition 1.7: Continuous Function

A function $f : X \to Y$ between two topological spaces is said to be *continuous* if for every open set $U \subset Y$, the inverse image $f^{-1}(U)$ is open in X (the same holds for closed sets). These are the morphisms in **Top**.

If f is bijective with a continuous inverse, then call f a *homeomorphism*.

Question: how can we make the following into a trivial statement of category theory?

> Let Y be a topological space with the property that for every topological space X, all functions $f : X \to Y$ are continuous. Prove that Y has the concrete topology.

Answer: use the forgetful functor and the universal mapping principle?

Definition 1.8: Open / closed maps

Let $f : X \to Y$. If the image of every open set is open, f is said to be *an open map* (define closed maps analogously). Note, open mappings are not necessarily continuous.

And lastly, a convenient theorem.

Theorem 1.2. *Let $(X,\mathcal{U})$, $(Y,\mathcal{V})$ be topological spaces, with $X \cong Y$ by a homemorphism $h : X \to Y$. Then for every point $x \in X$, $X - \{x\} \cong Y - \{h(x)\}$ (that is, removing a point preserves the homeomorphism).*

Intuitively, we think of homeomorphisms as encoding information about how two spaces can be stretched and deformed into one another, without tearing / gluing pieces together / apart. Note, however, that this does not necessarily correspond to our intuitive understanding of "tearing / gluing" — were I to hand you a circular segment of rope, and ask you to construct the following knot

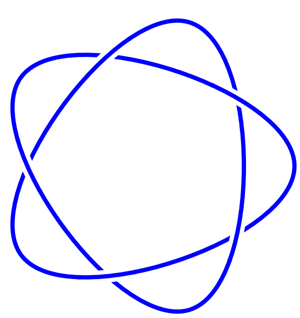

Figure 1: cinquefoil

you'd say it couldn't be done. However that's not exactly true. If we weren't constrained by the pesky physical details of the real world, we could simply pass the rope through itself to yield the desired knot.

This is what is meant in topology by a homeomorphism — all that has to be conserved are, loosely speaking, "adjacency" relationships. On the flip side, we might also note that this knot can hardly be said to "not differ at all" from a circle. But in a topological sense, they are homeomorphic. Maybe we'll talk about ambient isotopy next week.

Now, induced topologies. Loosely speaking, this works kind of like a group action, except by a topological space on a subset of the underlying set. It's worth noting that this analogy might be *highly* tenuous.

Definition 1.9: Induced topologies

Let $(X,\mathcal{U})$ be a topological space, and $S \subseteq X$. Let $\mathcal{U}' = \{U \cap S \mid U \in \mathcal{U}\}$. Then we call $\mathcal{U}'$ *the topology on S induced by the topology of X* (note the desired topological axioms follow from the properties of $\mathcal{U}$ and the commutativity of union / intersection on sets). If S has the induced topology, we call S a subspace of X.

One thing to note about this definition is that an open set in the induced topology need not be open in the original topology. Really, what we've essentially done is "project" the topological structure of some space on to one of its subsets. Naturally, this can't give us information about the open-ness of a set selected from $\mathcal{U}'$ were we to imbed it back into $\mathcal{U}$. However, if we do know something about how S is related to X from the get go, *then* we can guarantee a stronger condition:

Theorem 1.3. *If $S \in \mathcal{U}$, then $\mathcal{U}' \subseteq \mathcal{U}$. That is, the opens sets of S are open in X. The proof follows directly from the definition of induced topology.*

1.4. Quotient and Product spaces

Whereas the induced topology can be thought of as arising from an injetive map $\iota : S \to X$, we will now consider the topology arising from a surjective map $q : X \to Y$.

Definition 1.10: Quotient topology

Let $f : (X,\mathcal{U}) \to Y$ be surjective as a function of sets. Then define the *quotient topology* on Y by

$$\mathcal{U}' = \left\{U \subseteq Y \;\middle|\; f^{-1}(U) \in \mathcal{U}\right\}.$$

The topology axioms follow from properties of the inverse image. Once Y has been endowed with the quotient topology, f becomes a continuous map.

Theorem 1.4. *Let $f : X \to Y$ be a mapping and suppose that Y has the quotient topology with respect to X. Let Z be a topological space. Then a mapping $g : Y \to Z$ is continuous iff gf is continuous.*

In our definition of a quotient topology, because Y is a set intially without structure, we don't really care what exactly its elements are — really, all that matters to us is that our mapping is surjective (Y only assumes structure once we define the quotient topology). Hence, there's no reason why we can't treat all such Y as if they were subsets of X (every surjective mapping from X to Y can be factored through an isomorphism of Y with a subset of X). This, together with the suspicious name "quotient topology," leads us to question whether we can view this as some form of *mod* operation. Indeed, we can. Recall the standard definition of an equivalence relation:

Definition 1.11

Let X be a set. Then an *equivalence relation* $\sim$ is a binary relation such that

(a) $\sim$ is reflexive,

(b) $\sim$ is symmetric, and

(c) $\sim$ is transitive.

For each $x \in X$, we define the *equivalence class of* x under $\sim$ by $[x] = \{y \in X \mid x \sim y\}$. The set of equivalence classes of X under $\sim$ is usually denoted $X/\sim$.

For $f : X \to X/\sim$ defined by $f(x) = [x]$, then $X/\sim$ together with the quotient topology is said to be "obtained from X by topological identification."

There is one other way in which we can think about quotient spaces. Consider an action on a topological space by a group G. Then define the equivalence relation $x \sim y \iff y \in \mathcal{O}_X(x)$ (i.e., they are in the same orbit under the action). Then we can give X/G the quotient topology.

At this point, we've considered various methods of constructing topological spaces from existing ones. In the induced topology, we thought of giving a set with no structure of its own additional structure by including it in a large topological space. In the quotient topology, we found ways of taking the structure of a topological space and imposing it on some other set. Now, we consider the construction in which we have *two* topological spaces, and we want to construct a new one from the two.

Definition 1.12

Let $(X, \mathcal{U}), (Y, \mathcal{V})$ be topological spaces. Define the topological product $X \times Y$ by $(X \times Y,\ \mathcal{U} \otimes \mathcal{V})$, where

$$\mathcal{U} \otimes \mathcal{V} = \left\{ \bigcup_{i \in I} W_i \;\middle|\; i \in I, \text{ and } W_i \in \mathcal{U} \times \mathcal{V} \right\}$$

that is, the closure under union of all products of open sets in X and Y.

1.5. Compactness and Hausdorffness

In analysis, a central question was "what properties of our spaces are preserved under continuous functions?" To this, we answered Completeness, Connectedness, and (in some cases) compactness. Now, we'll perform the analogous identification for homeomorphisms. But first, we need to define the actual properties we want to prove the invariance of. First comes compactness, which I like to think of as "the next-best thing to being finite."

Definition 1.13: Covers

Let X a set, and let $S \subseteq X$. Then suppose we have $V = \{V_i \subseteq X \mid i \in I\}$, and $S \subseteq \bigcup_{i \in I} V_i$. Then V is called a *cover* of S. If I is finite, then we call V a *finite* cover. If X is a topological space with topology $\mathcal{U}$, and $V \subseteq \mathcal{U}$, then we call V an *open* cover. If we have $V' \subseteq V$ a cover of S, then we call V' a *subcover*.

now, the familiar definition of compactness.

Definition 1.14

Let $(X, \mathcal{U})$ a topological space, and let $S \subseteq X$. Then S is said to be *compact* if every open cover of S has a finite subcover.

As one might expect, a subspace $S \subseteq X$ is compact iff it is compact under the induced topology.