Basic Algebraic Topology

Based on Kosniowski; Matveev

1. Picking up where we left off

1.1. Some Theorems

We'll breeze briskly through some more point-set topology, then move on to the foundations of algebraic topology.

Theorem 1.1. *Let Y be the quotient space of the topological space X determined by the surjective mapping $f : X \to Y$. If X is compact Hausdorff and f is closed then Y is (compact) Hausdorff.*

Corollary 1.1. *Let X be a compact Hausdorff G-space with G finite. Then X/G is a compact Hausdorff space.*

Corollary 1.2. *If X is a compact Hausdorff space and A is a closed subset of X then X/A is a compact Hausdorff space.*

2. All Together Now...

2.1. Connectedness

Definition 2.1

Let (X, τ) a topological space. Then X is said to be *connected* iff the only clopen subsets are trivial. If $S \subseteq S$, then S is said to be connected iff it is connected in the induced topology.

Equivalently, X is connected iff it cannot be expressed as the union of finitely many disjoint non-empty open subsets.

Theorem 2.1. *Let $f : X \to Y$ be continuous, and suppose X is connected. Then Y is connected as well.*

Proof. Suppose, to obtain a contradiction, that Y is disconnected. Then there exist nonempty open sets $U, V \subseteq Y$ with $U \cup V = Y$, and $U \cap V = \varnothing$. Since U, V are open and f continuous, then $f^{-1}(U)$, $f^{-1}(V)$ are open in X. Furthermore, these are disjoint nonempty subsets of X with $f^{-1}(U) \cup f^{-1}(V) = X$. Then X is disconnected, a contradiction. Hence Y is connected. ■

Theorem 2.2. *Suppose that $\{Y_j \mid j \in J\}$ is a collection of connected subsets of a space X. If $\bigcap_{j \in J} Y_j \neq \varnothing$, then $Y = \bigcup_{j \in J} Y_j$ is connected.*

Proof. Suppose U is a nonempty clopen subset of Y. Then $\exists j \in J$ s.t. $U \cap Y_j \neq \varnothing$. Hence, let $J' = \{j \in J \mid U \cap Y_j \neq \varnothing\}$. Then $\forall j' \in J'$, we have $U \cap Y_{j'}$ is clopen in the induced topology. Since $Y_{j'}$ is connected, it follows that $U \cap Y_{j'} = Y_{j'}$. Hence $U = Y$, so Y is connected. ■

Theorem 2.3. *Let X, Y be topological spaces. Then X, Y are connected iff $X \times Y$ is connected.*

Proof:

($\Rightarrow$) : Suppose X, Y are connected. $\forall x \in X,\ y \in Y$, $X \times \{y\} \cong X$ and $\{x\} \times Y \cong Y$ (and thus each is connected), and note $(X \times \{y\}) \cap (\{x\} \times Y) \neq \varnothing$, thus by theorem 1.3 their union is connected. Now, let $y \in Y$ be fixed. Observe that

$$X \times Y = \bigcup_{x \in X} (X \times \{y\}) \cap (\{x\} \times Y)$$

Hence $X \times Y$ is connected.

($\Leftarrow$) : Suppose $X \times Y$ is connected. Then since the canonical projection maps are continuous, it follows that X, Y are connected (continuous image of a connected set is connected). ■

2.2. Path Connectedness

We now introduce a new, stronger notion of *connectedness* that allows us to treat somewhat pathological examples, such as the Topologist's Sine Curve (seen below):

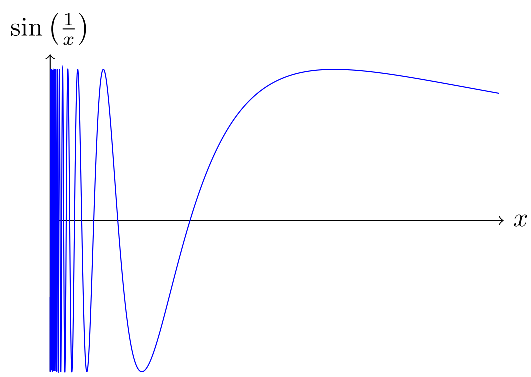

Figure 1: Topolgist's Sine Curve

Here, if we take the topological space $X = \left\{(x, y) \mid y = \sin\left(\frac{1}{x}\right)\right\} \cup \{(0,0)\}$ under the induced topology with respect to $\mathbb{R}^2$, then we see that X is connected, but clearly, there's some. . . funny business going on near the origin. Topologically, there's no problem — it's impossible to separate the origin out from the rest of the points of the curve; no matter how close we get, we'll see infinitely many points of the curve wobbling around. But clearly, there's no way to actually "link" the origin into the graph. This second point is the idea we want to capture with path connectedness. But first, we need to define the idea of a path.

Definition 2.2: Path

Let X be a topological space, and let $[0, 1] \subseteq \mathbb{R}$. Then if $f : [0, 1] \to X$ is a continuous function, we call f a *path*. **Important Note:** $f([0, 1])$ is *not* the path; rather the mapping f itself is the path. $f([0, 1])$ is called a *curve*.

In order to apply paths in full, we will first need some handy lemmas.

Lemma 2.1. *Let f be a path in X, and let $\overline{f}$ be defined by $\overline{f}(t) = f(1-t)$. Then $\overline{f}$ is also a path in X.*

Proof. Let $g : [0,1] \to [0,1]$ be defined by $g(t) = 1 - t$. Then g is continuous. Hence $\overline{f} = f \circ g$ is continuous. ∎

For the next lemma, we actually need another lemma first:

Lemma 2.2 (Gluing Lemma)**.** *Let W, X be topological spaces and suppose that $W = A \cup B$ with A, B both closed subsets of W. If $f : A \to X$ and $g : B \to X$ are continuous functions such that for all $w \in A \cap B$ $f(w) = g(w)$, then $h : W \to X$ defined by*

$$h(w) = \begin{cases} f(w) & \text{if } w \in A, \\ g(w) & \text{if } w \in B \end{cases}$$

is a continuous function.

Proof. Note that h is well-defined, and let U a closed set in X. Then

$$\begin{aligned} h^{-1}(U) &= \big(h^{-1}(U) \cap A\big) \cup \big(h^{-1}(U) \cap B\big) \\ &= f^{-1}(U) \cup g^{-1}(U) \end{aligned}$$

but $f^{-1}(U)$ and $g^{-1}(U)$ are closed, so $h^{-1}(U)$ is closed, thus h is a continuous function. ∎

It is worth remarking that an equivalent claim could be proven for A, B open.

Lemma 2.3. *Let X be a topological space, and let f, g be paths in X such that $f(1) = g(0)$. Then define the* concatenation *of f and g (denoted $f * g$) to be the path in X such that*

$$f * g(t) = \begin{cases} f(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

is a path in X.

Proof. Take $A = [0, 1/2]$, $B = [1/2, 1]$, and apply the Gluing Lemma. ∎

Definition 2.3: Path connectedness

A space X is said to be *path connected* if given any two points $x_0, x_1 \in X$, there is a path in X from x_0 to x_1.

Some nice straightforward results follow from the fact that path connectedness is defined in terms of continuous functions:

Theorem 2.4. *Let X be a path-connected topological space, and let f be a continuous mapping to a topological space Y. Then $f(X)$ is path-connected.*

Proof. Let $u, v \in Y$. Then $\exists a, b \in X$ s.t. $f(a) = u,\ f(b) = v$. Since X is path connected, there exists a path g in X from a to b. Then since f is continuous, $f \circ g$ is a path from u to v in Y. ∎

Theorem 2.5. *Supposet that $\{Y_i \mid i \in I\}$ is a family of path connected sets. Then if*

$$\bigcap_{i \in I} Y_i \neq \varnothing,$$

then $Y = \bigcup_{i \in I} Y_i$ is path-connected.

Proof. Let $a, b \in Y$, and let $c \in \bigcap_{i \in I} Y_i$. Then $\exists i, j \in I$ s.t. $a \in Y_i, b \in Y_j$. Note that $c \in Y_i, Y_j$ as well. Then take a path f in Y_i from a to c, and a path g in Y_j from b to c. Then the path $h = f * g$ is a path from a to c in Y. ∎

Theorem 2.6. *Let X, Y be topological spaces. Then X and Y are path connected iff $X \times Y$ is path connected.*

Proof.

($\Rightarrow$) : Suppose X, Y are path connected. $\forall x \in X,\ y \in Y$, $X \times \{y\} \cong X$ and $\{x\} \times Y \cong Y$ (and thus each is path connected), and note $(X \times \{y\}) \cap (\{x\} \times Y) \neq \varnothing$, thus by theorem 2.5 their union is path connected. Now, let $y \in Y$ be fixed. Observe that

$$X \times Y = \bigcup_{x \in X} (X \times \{y\}) \cap (\{x\} \times Y)$$

Hence $X \times Y$ is path connected.

($\Leftarrow$) : Suppose $X \times Y$ is path connected. Then since the canonical projection maps are continuous, it follows that X, Y are path connected (continuous image of a path connected set is path connected). ∎

Theorem 2.7. *Every path connected space connected. Not every connected space is path connected. Oh, also, any non-empty open connected subset of $\mathbb{R}$ is path connected.*

3. A Brief Discussion of Manifolds

Definition 3.1: Manifolds

Let $n \in \mathbb{Z}^{>0}$, and let (M, τ) be a topological space. Then M is called a manifold iff M is Hausdorff, and $\forall m \in M$, there exists a neighborhood N of m such that $N \cong \mathring{D}^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$.

Definition 3.2: Connected Sum

Let S_1, S_2 be compact connected 2-manifolds (surfaces), and let $D_1 \subseteq S_1$, $D_2 \subseteq S_2$ with $D_1, D_2 \cong D^2$. Let $h_1 : D_1 \to D^2$, and $h_2 : D_2 \to D^2$ be homeomorphisms. Then define $\sim$ to be an equivalence relation such that $x \sim h_2^{-1} h_1(x)$ iff $x \in \partial D_1$, and $x \sim x$ otherwise. Then $S_1 \# S_2$ is given by

$$\frac{(S_1 - \mathring{D}_1) \cup (S_2 - \mathring{D}_2)}{\sim}$$

Definition 3.3

Call a surface S^2 *orientable* if it contains no Möbius strip, and *non-orientable* otherwise. Then for $m \geq 0, n \geq 1$, we call

$$S^2 \# \left(\mathop{\#}_{i=1}^{m} T \right) = S \# mT$$

the *standard orientable surface of genus* m, and

$$S^2 \# \left(\mathop{\#}_{i=1}^{n} \mathbb{R}P^2 \right) = S \# n\mathbb{R}P^2$$

the *standard non-orientable surface of genus* n.

4. Homotopy

We are now ready to introduce a concept that will be of fundamental importance later on. In a nutshell, this is the idea of *equivalent maps*, as motivated by the following uncharacteristically-unprofessional diagram:

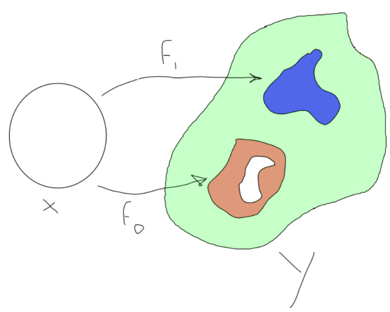

Figure 2: I'm not as good with inkscape as Prof. Nelson is

f_0 and f_1 are, in some sense, fundamentally not the same as each other, as one contains a hole while the other does not. To make this precise, we define the idea of homotopy equivalence:

Definition 4.1

Let X, Y be topological spaces, and let $f_0, f_1 : X \to Y$. Then say f_0, f_1 are *homotopic* iff there is a continuous map $F : X \times I \to Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. In this case we write $f_0 \simeq f_1$, and call the map F a *homotopy* between f_0 and f_1. For each $t \in [0, 1]$, we denote $F(x, t)$ by $f_t(x)$.

Some part of me really just wants to draw a commutative diagram here, but it doesn't quite feel appropriate. Anyways, note that a homotopy essentially amounts to a path through the space of continuous maps between topological spaces. Also, observe that our definition might not yet be perfectly desirable, as it allows us to make paths homotopic to points. Hence, we define the notion of a *relative homotopy*.

Definition 4.2

Let X be a topological space, and let $A \subseteq X$. Suppose that $f_0, f_1 : X \to Y$ are continuous. Then we say f_0 and f_1 are *homotopic relative to* A iff there exists a homotopy $F : X \times I \to Y$ between f_0 and f_1 such that $F(a,t)$ does not depend on t for any $a \in A$. That is, $\forall a \in A, t \in I$, $F(a,t) = f_0(a)$. In this case, we say F is a *homotopy relative to* A, and we write $f_0 \simeq f_1(\mathrm{rel}A)$, or $f_0 \simeq_{\mathrm{rel}A} f_1$.

Less rigorously, this essentially amounts to the homotopy *leaving* A *alone.* Note that if $A = \{0,1\}$, homotopy relative to A means that we can't always get some map "over an obstacle."

Lemma 4.1. $\simeq_{\mathrm{rel}\,A}$ *is an equivalence relation on* $\hom(X,Y)$.

Proof. First, observe that $\simeq_{\mathrm{rel}\,A}$ is reflexive: take $F(x,t) = f_0(x)$. Symmetry follows from the fact that if $F(x,t)$ is a homotopy, then $F(x, 1-t)$ is a homotopy. Finally, note that if we have $f \simeq_{\mathrm{rel}\,A} g$ and $g \simeq_{\mathrm{rel}\,A} h$, then we can simply concatenate the homotopys from f to g and g to h to yield one from f to h. ∎

We can now apply this new shiny tool to some classifications of topological spaces.

Definition 4.3

Let X, Y be topological spaces. We say X and Y are *of the same homotopy type* if there exist continuous maps $f : X \to Y$, $g : Y \to X$ such that $gf \simeq 1 : X \to X$, $fg \simeq 1 : Y \to Y$.

Ok, there *really* feels like there's some category nonsense going on here. Anyways, note that we did *not* use relative homotopy here. That was 100% intentional — in a sense, we want homotopy equivalence to encode information about retraction and stretching that can be lossy in ways that homeomorphism cannot. Homeomorphism is rigid about exact equivalence, homotopy equivalence lets us fudge things a little bit. For instance, a möbius band is certainly not homeomorphic to a cylinder, whereas the two are both homotopy equivalent to a circle.

Definition 4.4

A space X is said to be *contractible* if it is homotopy equivalent to a point.

Definition 4.5

A subset A of a topological space X is called a *retract* of X iff there is a continuous map $r : X \to A$ such that $r\imath = 1 : A \to A$, where $\imath$ is the inclusion map. Equivalently, $r|A = 1$. Under these conditions, r is called an *inclusion* map.

Definition 4.6

A subset $A \subseteq X$ is called a *deformation retract* of X if there is a retraction $r : X \to A$ such that $\imath r \simeq 1 : X \to X$.

If A is a deformation retract of X, it follows that A and X are homotopy equivalent.

Definition 4.7

Let X be a topological space, and $A \subseteq X$. Then call A a *strong deformation retract* iff there is a retraction $r : X \to A$ such that $\imath r \simeq_{\text{rel } A} 1 : X \to X$.

Essentially, a strong deformation retract is a way of deforming X within itself to A, while keeping A fixed.

5. Towards the fundamental group

5.1. Group Structure of Paths

Recall that we defined the concatenation of two paths f and g to be $f * g$, provided $f(1) = g(0)$. In preparation for a discussion of the fundamental group, we wish to investigate this further. In particular, we'll be interested in looking at the extent to which equivalence classes of paths (under homotopy relative to a particular choice of A) exhibit the structure of a group. This, ultimately, will be what allows us to escape the sadness of point-set topology, and transition to some truly beautiful mathematics.

Definition 5.1

Let X be a topological space, and let f, g be paths in X. We say f and g are *equivalent* iff f and g are homotopic relative to $\{0, 1\}$, in which case we write $f \sim g$. The equivalence classes of f are denoted by $[f]$.

Note that if f and g are equivalent, then the homotopy is some continuous function $F : I \times I \to X$ such that

$$\begin{aligned} F(t,0) &= f_0(t) \quad \text{and} \quad F(t,1) = f_1(t) \qquad t \in I \\ F(0,s) &= f_0(0) \quad \text{and} \quad F(1,s) = f_0(1) \qquad s \in I \end{aligned}$$

Hence, in a bit of an abuse of notation, we write $F : f_0 \sim f_1$.

Theorem 5.1. *Let X be a topological space, and let f, g, h be paths in X. Then*

1) $[f][g] = [f * g]$

2) $[f]([g][h]) = ([f][g])[h]$ *(whenever the product is defined — note though that in general, $(f * g) * h \neq f * (g * h)$. But, we have the next-best thing: if $f(1) = g(0)$ and $g(1) = h(0)$, then $(f * g) * h \sim f * (g * h)$).*

3) If $x \in X$, then define $\epsilon_x : I \to X$ by $\epsilon_x(t) = x$. Then $[\epsilon_x][f] = [f] = [f][\epsilon_y]$ if f begins at x and ends at y.

4) Let $x = f(0)$, and $y = f(1)$. Then $[f][\overline{f}] = [\epsilon_x]$, and $[\overline{f}][f] = [\epsilon_y]$.

Hence, we see that the equivalence classes of paths on X *almost* behave like a group. If only multiplication were always defined, and we had $x = y$...

5.2. The Fundamental Group

The trick is to make it happen.

Definition 5.2

Let X be a topological space, and let f be a path in X. Then f is said to be *closed* if $f(0) = f(1)$. If $f(0) = f(1) = x$, then we say f is *based* at x.

Oh look. We made it happen. How, you might ask? Well, observe:

Definition 5.3: Fundamental Group

Let X be a topological space, and let $x \in X$. Let f be some closed path based at x. Then define the *fundamental group* of X with base point x (denoted $\pi(X, X)$) to be $[f]$.

That this is a group follows immediately from Theorem 5.1. Buckle your seatbelts everyone, things are about to get *really* snazzy.

Theorem 5.2. *Let $x, y \in X$. If there is a path in X from x to y, then $\pi(X, x)$ and $\pi(X, y)$ are isomorphic.*

Proof. Let f be a path in X from x to y. Then for all $g \in \pi(X, x)$, $[\overline{f}] * [g] * [f] = [\overline{f} * g * f]$ is an equivalence class of paths in $\pi(X, y)$. Hence, define $\varphi : \pi(X, x) \to \pi(X, y)$ by

$$\varphi_f([g]) = [\overline{f} * g * f].$$

Clearly, this is bijective (we can define $\varphi_f^{-1}(h) = [f * h * \overline{f}]$, and note that $\varphi_f^{-1}\varphi = 1_{\pi(X,x)}$, $\varphi_f \varphi_f^{-1} = 1_Y$), and inherits the group structure of $\pi(X, x)$ in its image (since it is a conjugation). Thus it is a homomorphism. ■

Corollary 5.1. *Let X be a path connected topological space. Then for all $x, y \in X$, $\pi(X, x) \cong \pi(X, y)$.*

Note that the requirement that X be *path* connected is essential.

5.3. Continuous Functions & Fundamental Groups

We want to understand how continuous functions affect the fundamental group. First, we have the following three facts:

Lemma 5.1. *Let X, Y be topological spaces, and let $\varphi : X \to Y$ be continuous. Then*

1) *If f, g are paths in X, then φf, φg are paths in Y.*
2) *If $f \sim g$, then $\varphi f \sim \varphi g$*
3) *If f is a closed path in X based at $x \in X$, then φf is a closed path in Y based at $\varphi(x)$*

From these facts, we might deduce that continuous maps are, on some level, really just performing a sort of group homomorphism.

Definition 5.4: Induced Homomorphism

Let X, Y be topological spaces, and let $\varphi : X \to Y$ be continuous. Then define the *induced homomorphism* as follows: for all $x \in X$, define $\varphi_* : \pi(X, x) = \pi(Y, \varphi(x))$ such that if f

is a path based on x,

$$\varphi_*([f]) = [\varphi f].$$

We referred to this as a homomorphism, so it better actually be one. Let's prove it!

Theorem 5.3. *Let X, Y be topological spaces, and let $\varphi : X \to Y$ be continuous. Then φ_* is a homomorphism of groups.*

Proof. Let $[f], [g] \in \pi(X, x)$. Then

$$\begin{aligned} \varphi_*([f] * [g]) &= \varphi_*([f * g]) \\ &= [\varphi(f * g)] \\ &= [\varphi(f) * \varphi(g)] \qquad (*) \\ &= [\varphi(f)] * [\varphi(g)] \end{aligned}$$

note that $(*)$ follows by simply cutting any representative path halfway through, and doing the work. Hence, φ_* is a group homomorphism. ∎

A number of nice properties follow.

Theorem 5.4. *Let X, Y, Z be topological spaces, and let $\varphi : X \to Y$, $\psi : Y \to Z$ be continuous maps. Then*

1) $(\psi\varphi)_* = \psi_*\varphi_*$

2) If $1 : X \to X$ is the identity map then 1_* is the identity homomorphism.

3) If φ is a homeomorphism, then $\varphi_* : \pi(X, x) \to \pi(Y, \varphi(x))$ is an isomorphism of groups.

Instead of proving these, we make a remark about functors. Recall the following definition:

Definition 5.5: Functor

Let $\mathbf{C}, \mathbf{D}$ be categories, and let $c \in \mathrm{ob}(C)$, $f, g \in \hom(C)$. Then call $\mathcal{F} = (\mathcal{F}_o, \mathcal{F}_a)$ a functor if

$$\mathcal{F}_o : \mathrm{ob}(\mathbf{C}) \to \mathrm{ob}(\mathbf{D}) \qquad \mathcal{F}_a : \hom(\mathbf{C}) \to \hom(\mathbf{D})$$

such that the following essential properties are preserved:

(a) $\forall (f : c \to c') \in \hom(\mathbf{C})$, $\mathcal{F}_a(f) : \mathcal{F}_o(c) \to \mathcal{F}_o(c')$, with

$$\mathcal{F}_a(1_c) = 1_{\mathcal{F}_o(c)}$$

and

(b)

$$\mathcal{F}_a(g \circ f) = \mathcal{F}_a(g) \circ \mathcal{F}_a(f).$$

Or alternatively, with homsets:

Definition 5.6: Functor (version 2)

Let $\mathcal{T} : \mathbf{C} \to \mathbf{B}$ be defined with the usual object functor $\mathcal{T}_o$, together with a collection of functions

$$\mathcal{T}^{a,b} : \hom_{\mathbf{C}}(a,b) \to \hom_{\mathbf{B}}(\mathcal{T}_o(a), \mathcal{T}_o(b))$$

then $\mathcal{T}$ is *full* when every such $\mathcal{T}^{a,b}$ is surjective, and *faithful* when injective.

We can now frame the results obtained above in the language of category theory.

Theorem 5.5. *Let* $\mathbf{Top}_*$ *denote the category with objects* (X,x) *where* X *is a topological space, and* x *is a selected base point. Define the morphisms on* $\mathbf{Top}_*$ *to be continuous maps of the form* $\varphi : (X,x) \to (Y,y)$*, where we require* $\varphi(x) = y$*. Then the map* $\mathcal{F} : \mathbf{Top}_* \to \mathbf{Grp}$ *taking each* (X,x) *to the corresponding fundamental group is a functor.*

Proof. Take

$$\mathcal{F}_o((X,x)) = \pi(X,x) \qquad\qquad \mathcal{F}_a(\varphi) = \varphi_*.$$

And note that indeed, as we defined above, if $\varphi : (X,x) \to (Y,y)$, then $\varphi_* : \pi(X,x) \to \pi(Y,y)$, and $\mathcal{F}_a(1_{(X,x)}) = 1_*$, and indeed $\mathcal{F}_a(\psi \circ \varphi) = \mathcal{F}_a(\psi) \circ \mathcal{F}_a(\varphi)$. Hence $\mathcal{F}$ is a functor. ■