Basic Algebraic Topology

Based on Notes by Sergey Matveev

1. Introduction

First, a motivating quote.

> "Point set topology is a disease from which later generations will regard themselves as having recovered" -Henri Poincaré

As it turns out, lots of topics in topology can be simplified by attaching algebraic constructs to different topological spaces, and proving that certain properties of our group (or what have you) correspond naturally to properties of our topological space. The vehicle by which we navigate between the two is, as one might expect, Category Theory. First, we give a brief summary of basic concepts in point-set topology, before moving into the Homology Theory presentation given in Matveev.

1.1. Basic Point-Set Topology

As in most branches of mathematics, our object of study here will be some collection of sets, together with some *structure* we can associate with them. In Elementary Algebra, this takes the form of *group* and *ring* operations, and later the respective homomorphisms preserving them. In Elementary Analysis, it (loosely speaking) took the form of a *distance metric*, and the properties it bestowed on sets. Analogous to our study of homomorphisms in Algebra, we often studied *continuous functions* in Analysis, and the properties of sets that they preserved. Note the resemblance between the two expressions:

$$\varphi(g_1 \oplus g_2) = \varphi(g_1) \otimes \varphi(g_2) \qquad\qquad d(x,y) < \delta \implies d'(f(x), f(y)) < \varepsilon$$

A homomorphism $\varphi : G \to H$ A continuous function $f : (E, d) \to (E', d')$

while the analogy doesn't hold exactly, in both cases, we have some particular class of functions such that structure in one space is preserved in the image. In the case of homomorphisms, the group operation in the first group is "respected" by the homomorphism upon mapping into the second. In the case of continuous functions, our equation is essentially stating that we don't "tear" our starting space at all. This is best visualized by thinking about our continuous functions not as their *graphs* (as we are often used to), but rather as *maps* that deform the input domain in various manners to yield the image. As an example, one might think of the function $f(x) = x^2$ as the action of folding $\mathbb{R}$ on itself, and stretching the edges out towards infinity (this is often a strategy employed in visualizing complex-valued functions).

One might wonder what sorts of interesting discoveries we could make by generalizing our starting premises on the right-hand-side, so that we could make our questions more similar to those on the left. That is, similarly to how we defined distance metrics so as to generalize the *key* properties of Euclidean distance, so too will we generalize the idea of *continuity of a function.* This is the central idea of basic topology. Now, all we need is a good place to start. Recall the following theorem of Analysis:

Definition 1.1

Let (E, d) and (E', d') be metric spaces. Then a function $f : E \to E'$ is said to be *continuous* iff for all open sets $U \subseteq M_2$, we have $f^{-1}(U)$ is open in M_1

Of course, this makes no guarantees about the image of an open set being open in the codomain. Really, we can make our image as "jagged" as we want (within reason), provided we fold and deform our domain in a smooth way. But it does indicate to us open sets appear to be intimately tied to the idea of "smooth" deformations — and that it might be fruitful to pursue an understanding of our space that does not depend on the details of a particular metric, but rather just on relationships between open sets. Hence, we define a topology as follows:

Definition 1.2

Let X be a set, and let $\mathcal{U}$ be a collection of subsets of X satisfying the following:

(i) $\varnothing \in \mathcal{U}$, $X \in \mathcal{U}$.

(ii) For all $U_1, U_2 \in U$, $U_1 \cap U_2 \in \mathcal{U}$ (closure under finite intersections).

(iii) For any subset $\{U_i \mid i \in I\} \subseteq \mathcal{U}$, we have

$$\bigcup_{i \in I} U_i = \boldsymbol{U} \in \mathcal{U} \qquad \text{(closure under arbitrary unions)}$$

then $\mathcal{U}$ is called a *topology* for X, and $(X,\mathcal{U})$ is called a *topological space*. We call the elements of $\mathcal{U}$ the *open sets of* $(X,\mathcal{U})$.

note that a topology is thus a particular kind of *algebra of sets* under the binary operations $\cup, \cap$, with identity $\varnothing$ for $\cup$, and X for $\cap$. Note that $(\cup, \varnothing)$, $(\cap, X)$ are duals of each other, in the sense that for any sentence S built out of atomic propositions of our set algebra, if S is true, then the statement we obtain by

1. Replacing each $\cup$ with $\cap$ and each $\cap$ with $\cup$,
2. Interchanging each $\varnothing$ and X, and
3. Reversing inclusions

must also be true. Note that if we replace "arbitrary unions" with "countable unions", and further require closure under complementation, then we obtain a σ-algebra.

As it turns out, this definition of a topology is more general than that given by distance metrics. Whereas every distance metric gives rise to a topology, there are topologies that are not *metrizable*, meaning they do not arise from any metric on a set. We list a few common topologies. Let $(X,\mathcal{U})$ be a topological space. Then

1. If $\mathcal{U} = \{\varnothing, X\}$, we call $\mathcal{U}$ the *concrete* or *indiscrete topology.*
2. If $\mathcal{U} = \mathcal{P}(X)$ (i.e., every subset of X is open), then we call $\mathcal{U}$ the *discrete topology.* Note the analog to the discrete metric.
3. Suppose $\mathcal{U} = \{\varnothing, X\} \cup \left\{U \subseteq X \;\middle|\; \left|\overline{U}\right| < \infty\right\}$. That is, $X, \varnothing$, and all subsets of X with finite compliment. Then call $\mathcal{U}$ the *finite complement topology.*
4.

our definition of continuity follows identically to in a metric space:

Definition 1.3

A function $f : X \to Y$ between two topological spaces is said to be *continuous* if for every oepn set $U \subset Y$, the inverse image $f^{-1}(U)$ is open in X. The same holds for closed subsets. Continuous functions are composable to yield another continuous function. If f is bijective with a continuous inverse, then call f a *homeomorphism*.