The Euler-Lagrange Equation

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Airlines

Figure 1: Mudd Air; Image adapted from [Air19]

Airline

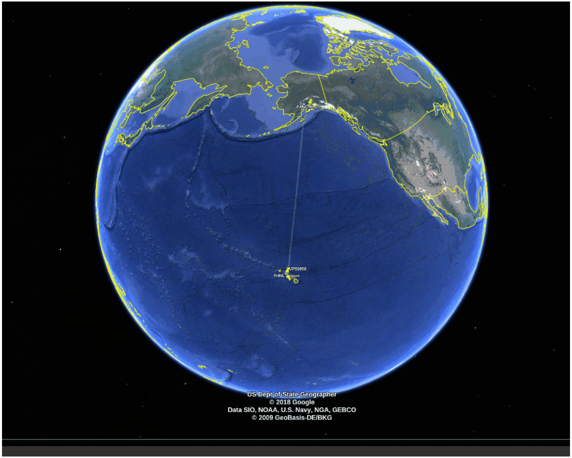


Figure 2: Flight 1 (UPS9859), Saturday 03/23/2019

Airline, cont.

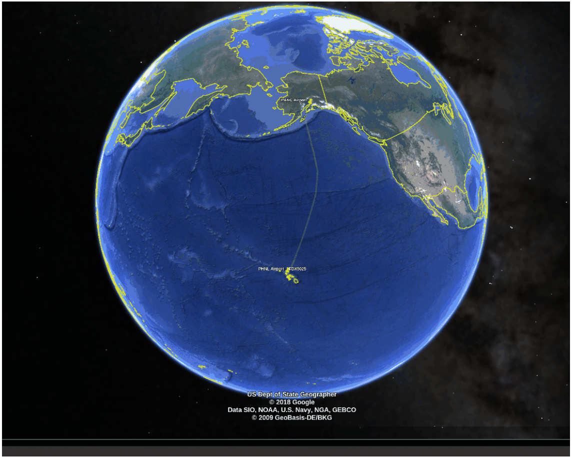


Figure 3: Flight 2 (FDX50252), Friday 03/22/2019

The difference:

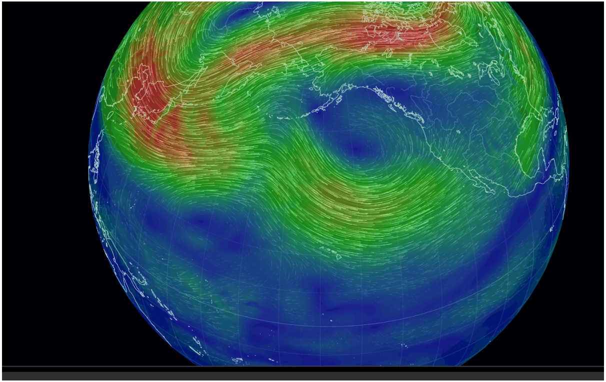


Figure 4: Wind patterns at 70hPa during Flight 1

The difference:

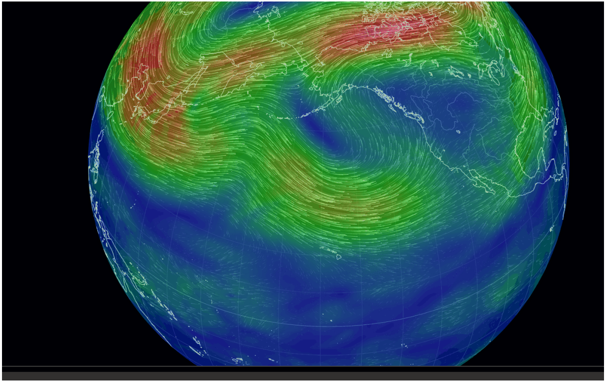


Figure 5: Wind patterns at 70hPa during Flight 2

Question:

How do airlines calculate the best trajectory?

Big idea: find best path

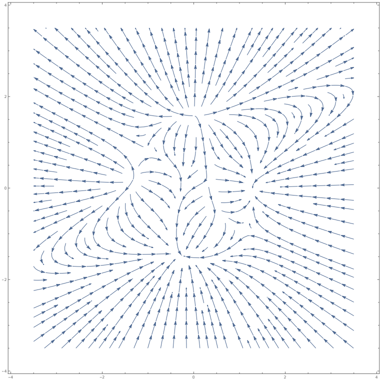
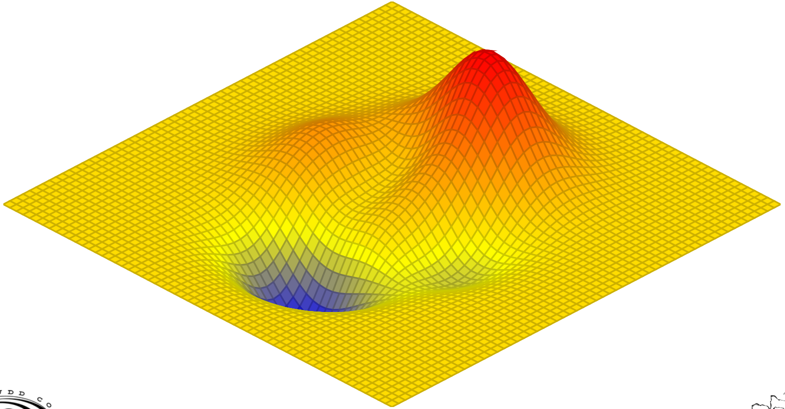
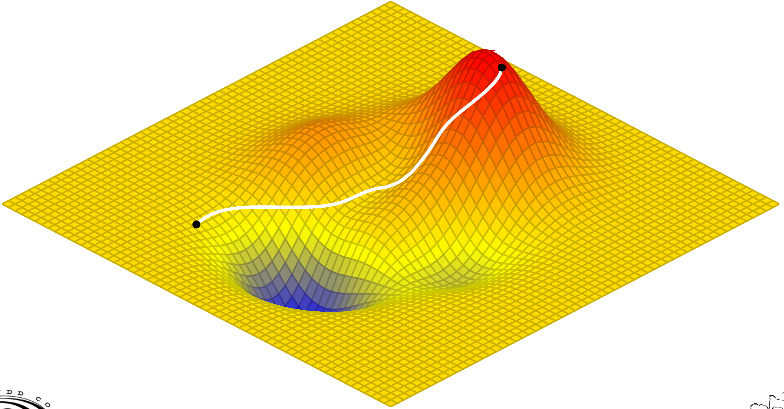


Figure 6: Drag field

Big idea, cont:



Big idea, cont:



What's essential about this problem?

- Object of interest: a path $\boldsymbol{q}(t) = (x(t), y(t), z(t))$.
- Givens:
  - Region we can fly over (Pacific Ocean)
  - Start/end points: (Honolulu/Anchorage)
  - Drag: $\mathbf{F}_D(t,\, \boldsymbol{q}(t),\, \dot{\boldsymbol{q}}(t))$
- Goal: find $\boldsymbol{q}(t)$ minimizing *total* travel time.

Defining "Cost"

- Drag: $\mathbf{F}_D(t,\, \boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))$
- Define "instantaneous cost" function:

$$\mathcal{L}(t,\, \boldsymbol{q}(t),\, \dot{\boldsymbol{q}}(t))$$

- Total "cost" of trip:

$$C[\boldsymbol{q}] = \int_{t_0}^{t_1} \mathcal{L}(t,\, \boldsymbol{q}(t),\, \dot{\boldsymbol{q}}(t))\ \mathrm{d}t$$

- What we want: an analogue of the first derivative test

Solution: Euler-Lagrange

Theorem (Euler-Lagrange)

Let X be our space of interest, and let $\mathbf{x}_0$, $\mathbf{x}_1 \in X$. Let $\boldsymbol{q}(t)$ be a path from $\mathbf{x}_0$ to $\mathbf{x}_1$. Then if $\boldsymbol{q}(t)$ minimizes the "total cost function"

$$C[\boldsymbol{q}(t)] = \int_{t_0}^{t_1} \mathcal{L}(t, \boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)) \ \mathrm{d}t,$$

$\boldsymbol{q}(t)$ is a solution to the differential equation

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right) = 0.$$

Overview (don't sweat the details):

1. Suppose an optimal $\boldsymbol{q}(t)$ exists.
2. Take a non-zero path $\boldsymbol{\eta}(t)$, with $\eta(t_0) = 0 = \eta(t_f)$.
3. For all $\alpha \neq 0$, the path $\boldsymbol{q'}(t) = \boldsymbol{q}(t) + \alpha \cdot \boldsymbol{\eta}(t)$ is not optimal.
4. Thus, $C[\boldsymbol{q'}(t)]$ is minimized when $\alpha = 0$.
5. Applying the first derivative test in α, we can show

$$\left.\frac{\mathrm{d}C}{\mathrm{d}\alpha}\right|_{\alpha=0} = \int_{t_0}^{t_1} \left[\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right)\right] \eta(t) \ \mathrm{d}t = 0$$

and the result follows.

Suppose an optimal $\boldsymbol{q}(t)$ exists:

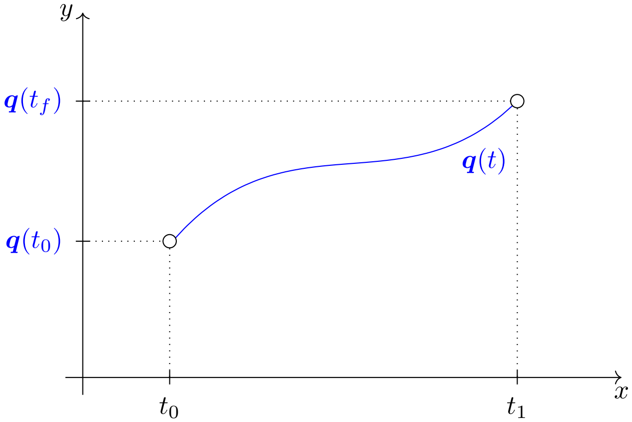


Figure 7: Optimal $\boldsymbol{q}(t)$

For some $\boldsymbol{\eta}(t)$, add $\alpha \cdot \boldsymbol{\eta}(t)$:

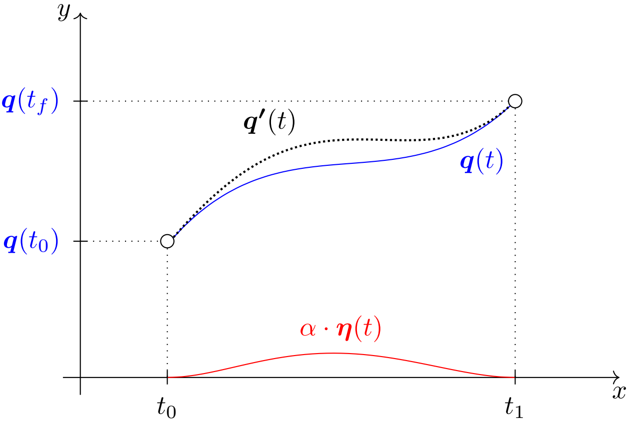


Figure 8: Add a small perturbation $\alpha \cdot \boldsymbol{\eta}(t)$

$\boldsymbol{q}'(t)$ for various values of α:

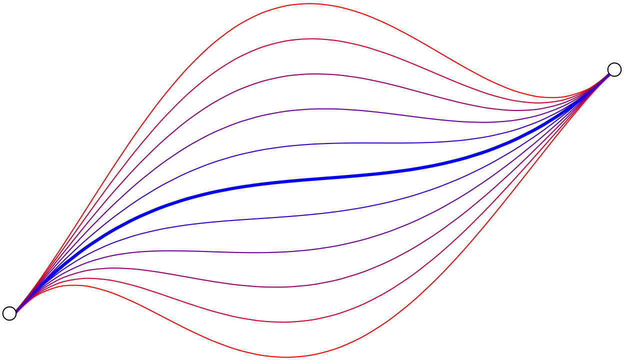


Figure 9: $\boldsymbol{q}'(t)$ for various values of α

Turn C into optimizing a 1-D function:

- Consider $C[\boldsymbol{q'}(t)]$ as a function of just α:

  $$C'(\alpha) = C[\boldsymbol{q}(t) + \alpha \cdot \boldsymbol{\eta}(t)].$$

  This is just a map from $\mathbb{R} \to \mathbb{R}$!

- Note that $C'(0) = C[\boldsymbol{q}(t)]$, so

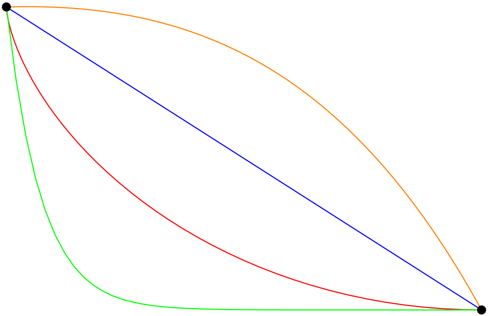
  $$0 = \left.\frac{\mathrm{d}C'(\alpha)}{\mathrm{d}\alpha}\right|_{\alpha=0}$$

- Repeated application of the chain rule and integration by parts yields the desired result!

Examples

- Lagrangian Mechanics
- Brachistochrone problem
- Optimal control
- Geodesics
- Minimal surfaces of revolution

Brachistochrone



References

Airplane Movie Poster 24”x36”, Mar 2019.
[Online; accessed 31. Mar. 2019].