The Euler-Lagrange Equation

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Airlines

- Time is fuel; fuel is money.

Airline

Figure: Flight 1 (UPS9859), Saturday 03/23/2019

Airline, cont.

Figure: Flight 2 (FDX50252), Friday 03/22/2019

Some statistics:

- Total flight distance:
  - Flight 1: 4551km
  - Flight 2: 4670km
- Total flight time:
  - Flight 1: 5h 30min 7s
  - Flight 2: 5h 14min 6s
- Flight 2 went $\sim$ 100km further, but arrived $\sim$ 15 faster!

The difference:

The difference:

Figure: Wind patterns at 70hPa during Flight 1

The difference:

Figure: Wind patterns at 70hPa during Flight 2

What's essential about this problem?

- Object of interest: a path $\mathbf{x}(t) = (x(t), y(t), z(t))$.
- Givens:
  - Region we can fly over (Pacific Ocean)
  - Start/end points (Honolulu/Anchorage)
  - Drag function: $\mathbf{F}_D(t,\, \mathbf{x}(t),\, \dot{\mathbf{x}}(t))$
- Goal: find $\mathbf{x}(t)$ minimizing *total* travel time.

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- Goal: find $\mathbf{x}(t)$ minimizing *total* travel time.
  - Important: total travel time is an "accumulated cost"

Translating to the General Case

Flight Optimization:

- Object of interest:
  - Path $\mathbf{x}(t)$
- Givens:
  - Domain: region of globe
  - End points: HNL, ANC
  - Drag: $\mathbf{F}(t, \mathbf{x}(t),\ \mathbf{v}(t))$
- Goal: find $\mathbf{x}(t)$ minimizing travel time

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General:

- Object of interest
  - A path $\boldsymbol{q} : [t_0, t_f] \to X$
- Givens:
  - Domain: some space X
  - End points: $\boldsymbol{q}(t_0)$, $\boldsymbol{q}(t_f)$
  - Cost func: $\mathcal{L}(t, \boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))$
-

Overview

- We want to find some optimal path $\boldsymbol{q}(t)$ satisfying

Wind

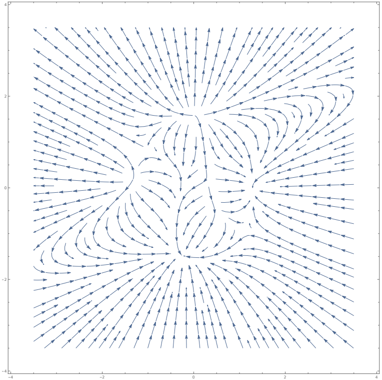
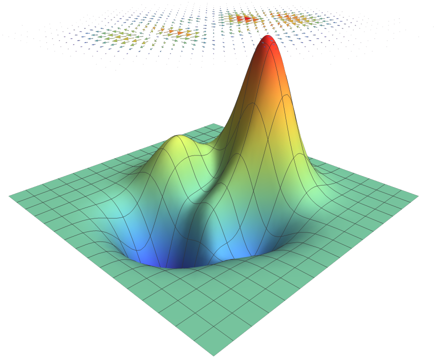
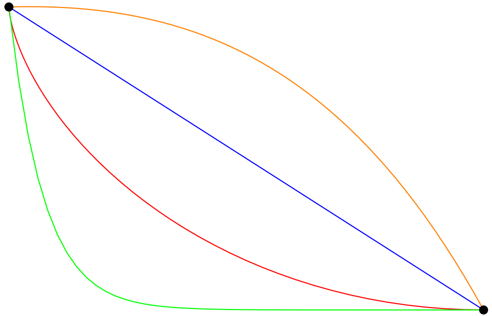


Figure: Wind Vector Field

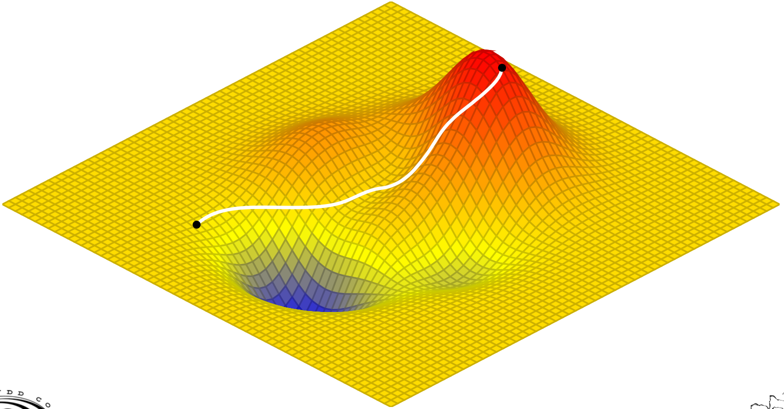
"Cost" function



Shortest Time Path



Shortest Path



The statement

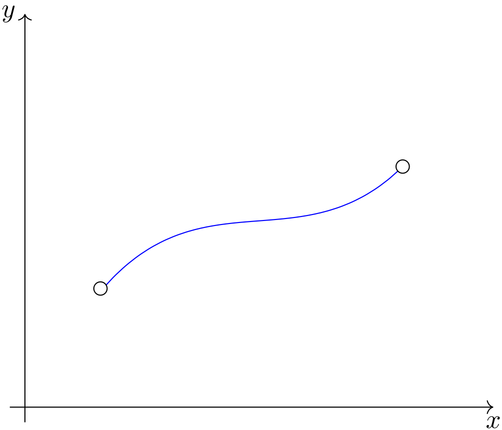
Theorem (Euler-Lagrange)

Let $\boldsymbol{q}(t) : \mathbb{R} \to \mathbb{R}^n$ *be a path. Then if* $\boldsymbol{q}(t)$ *is an extreme value of the functional*

$$S(\boldsymbol{q}) = \int_a^b \mathcal{L}(t, \boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)) \ \mathrm{d}t$$

then $\boldsymbol{q}$ *is a solution to the differential equation*

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right) = 0$$


y
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