

A Motivating Problem
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Defining the Problem
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The Euler-Lagrange Equation

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Airlines



Figure 1: Mudd Air (adapted from [Air19])



Airline



Figure 2: Flight 1 (UPS9859), Saturday 03/23/2019

Airline, cont.



Figure 3: Flight 2 (FDX50252), Friday 03/22/2019

The difference:



Figure 4: Wind patterns at 70hPa during Flight 1



The difference:



Figure 5: Wind patterns at 70hPa during Flight 2



Question:

How do airlines calculate the best trajectory?



Key Features:

- ▶ Object of interest: a path $\mathbf{q}(t) = (x(t), y(t), z(t))$.
- ▶ Givens:
 - Start/end points $\mathbf{x}_0, \mathbf{x}_1$ (Honolulu/Anchorage)
 - A region $X \subset \mathbb{R}^3$ we can fly through (Pacific Ocean airspace)
 - Drag: $\mathbf{F}_D(t, \mathbf{q}(t), \dot{\mathbf{q}}(t))$
- ▶ Goal: find $\mathbf{q}(t)$ minimizing *total* travel time.



Defining “Cost”

- ▶ Again: drag is given by $\mathbf{F}_D(t, \mathbf{q}(t), \dot{\mathbf{q}}(t))$
- ▶ Define “instantaneous cost” function:

$$\mathcal{L}(t, \mathbf{q}(t), \dot{\mathbf{q}}(t))$$

- ▶ Then define the total “cost” of trip:

$$C[\mathbf{q}] = \int_{t_0}^{t_1} \mathcal{L}(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) \, dt$$

- ▶ What we want: a way to find “critical paths,” kind of like “critical points” from Calculus.



Suppose an optimal $\mathbf{q}(t)$ exists:

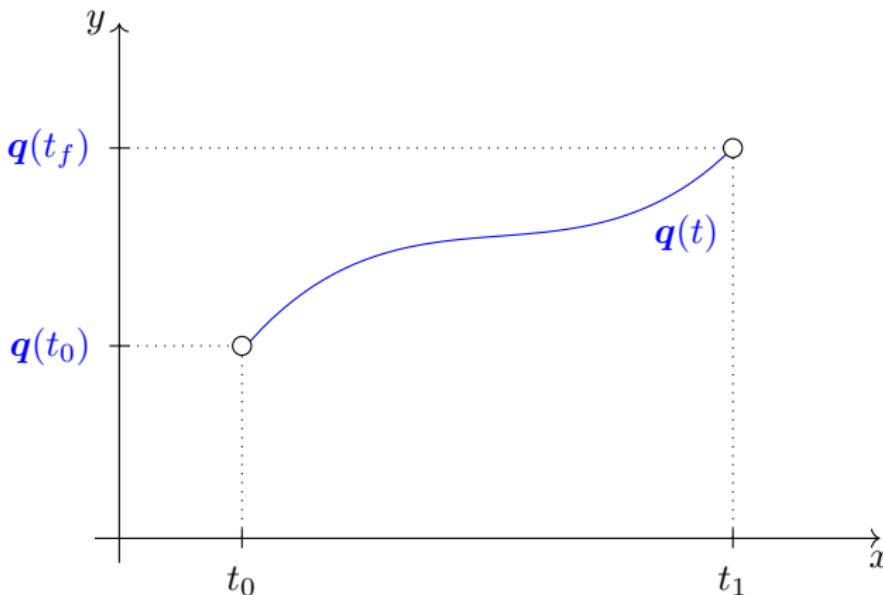


Figure 6: Optimal $\mathbf{q}(t)$



For some $\eta(t)$, add $\alpha \cdot \eta(t)$:

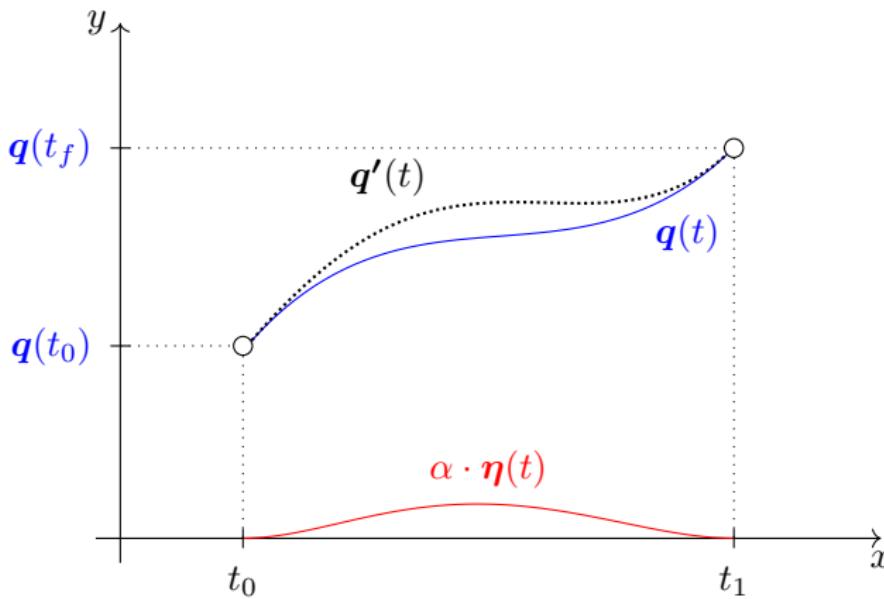


Figure 7: Add a small perturbation $\alpha \cdot \eta(t)$



For smaller α , $q'(t)$ closer to optimal:

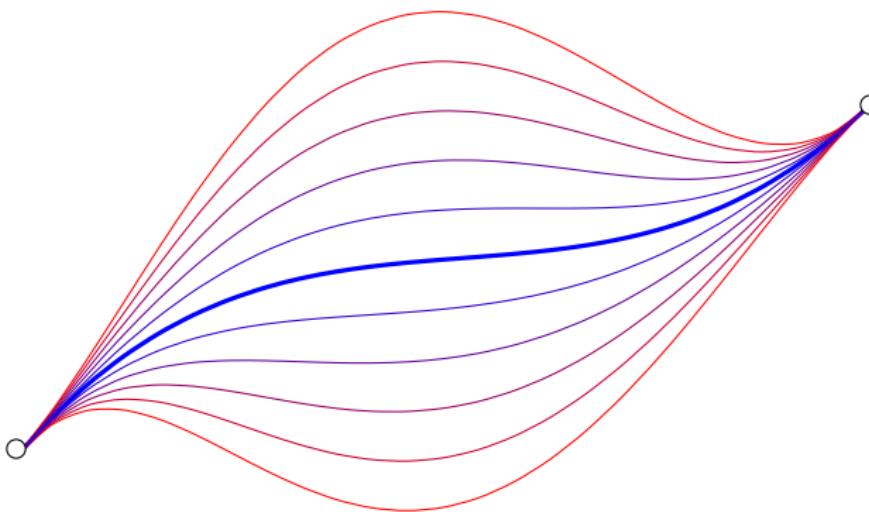


Figure 8: $q'(t)$ for various values of α



Turn C into optimizing a 1-D function:

- ▶ Consider $C[\mathbf{q}'(t)]$ as a function of just α :

$$C'(\alpha) = C[\mathbf{q}(t) + \alpha \cdot \boldsymbol{\eta}(t)].$$

This is just a map from $\mathbb{R} \rightarrow \mathbb{R}$!

- ▶ Note that $C'(0) = C[\mathbf{q}(t)]$, so

$$0 = \left. \frac{dC'(\alpha)}{d\alpha} \right|_{\alpha=0}$$

- ▶ Repeated application of the chain rule and integration by parts yields



$$\boxed{\frac{\partial \mathcal{L}}{\partial \mathbf{q}} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right) = 0}$$



Solution: Euler-Lagrange

Theorem (Euler-Lagrange)

Let X be our space of interest, and let $\mathbf{x}_0, \mathbf{x}_1 \in X$. Let $\mathbf{q}(t)$ be a path from \mathbf{x}_0 to \mathbf{x}_1 . Then if $\mathbf{q}(t)$ minimizes the “total cost function”

$$C[\mathbf{q}(t)] = \int_{t_0}^{t_1} \mathcal{L}(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) \, dt,$$

$\mathbf{q}(t)$ is a solution to the differential equation

$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right) = 0.$$



Big idea: find best path

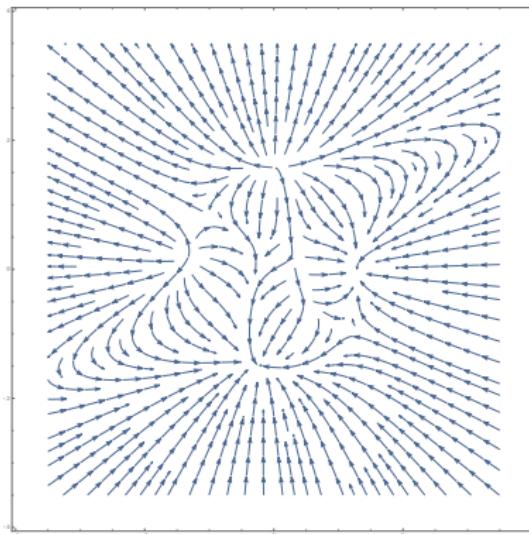
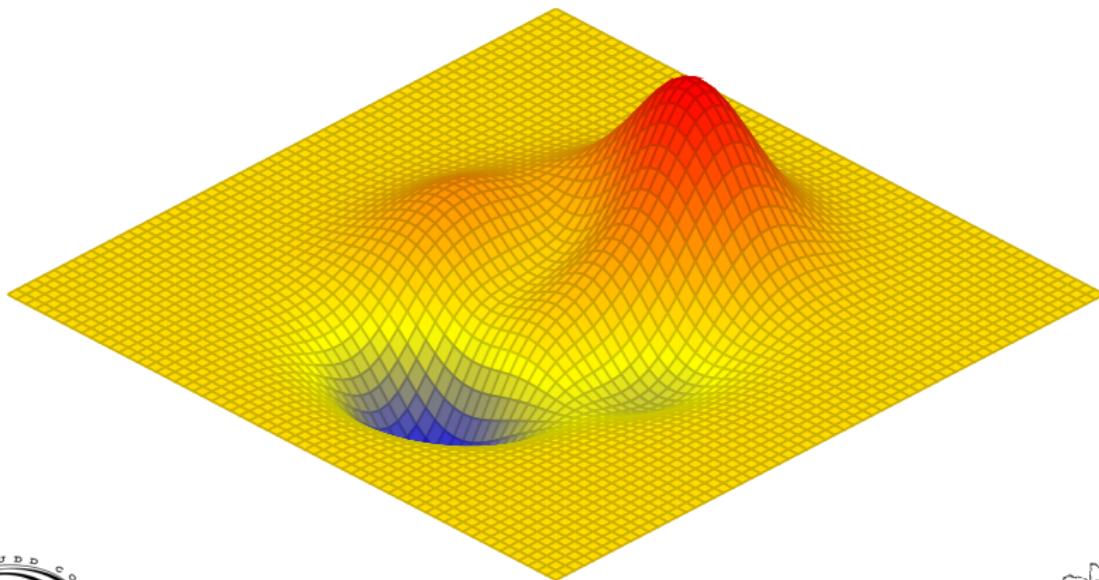


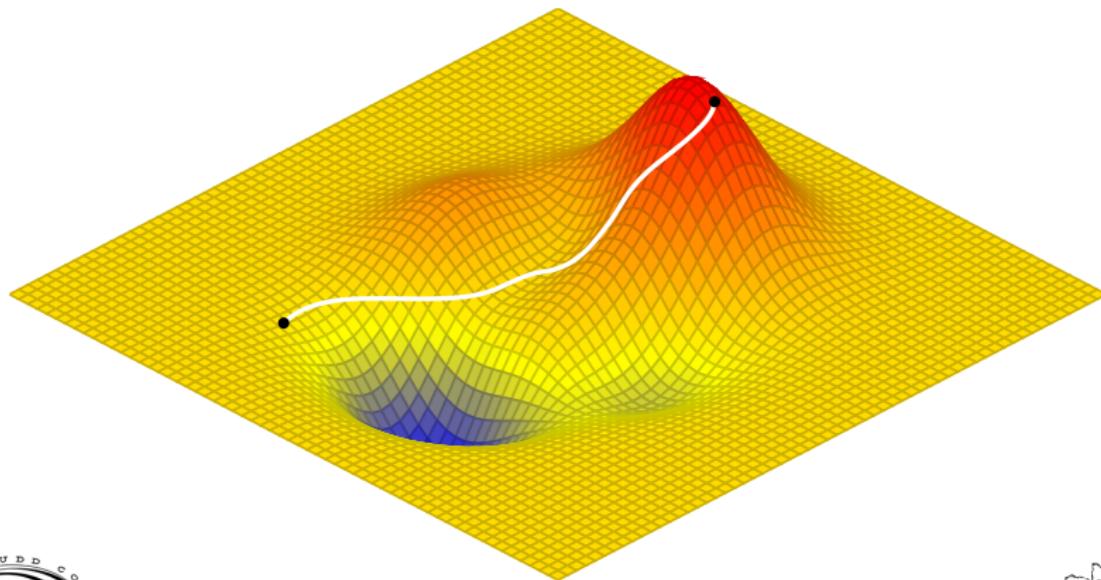
Figure 9: Drag field



Big idea, cont:



Big idea, cont:

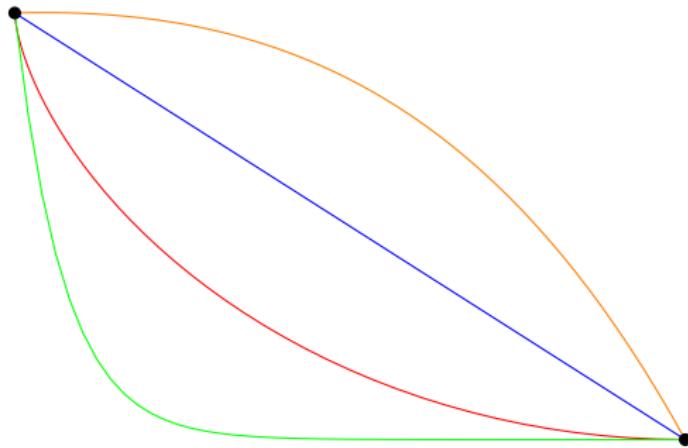


Other Examples

- ▶ Optimal control
- ▶ Geodesics & Minimal surfaces of revolution
- ▶ Lagrangian Mechanics: $\mathcal{L} = T - V$
- ▶ Brachistochrone problem



Brachistochrone



References



Airplane Movie Poster 24"x36", Mar 2019.
[Online; accessed 31. Mar. 2019].

