The Euler-Lagrange Equation

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Airlines

Figure 1: Mudd Air

Airline

Figure 2: Flight 1 (UPS9859), Saturday 03/23/2019

Airline, cont.

Figure 3: Flight 2 (FDX50252), Friday 03/22/2019

Some statistics:

- Total flight distance:
  - Flight 1: 4551km
  - Flight 2: 4670km
- Total flight time:
  - Flight 1: 5h 30min 7s
  - Flight 2: 5h 14min 6s
- Flight 2 went $\sim$ 100km further, but arrived $\sim$ 15 faster!

The difference:

The difference:

Figure 4: Wind patterns at 70hPa during Flight 1

The difference:

Figure 5: Wind patterns at 70hPa during Flight 2

Big idea: find best path

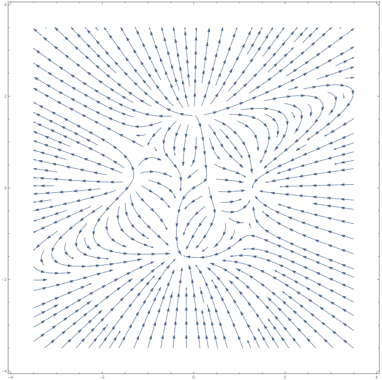
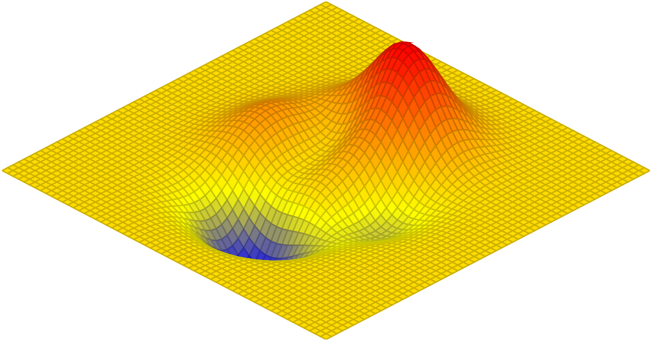
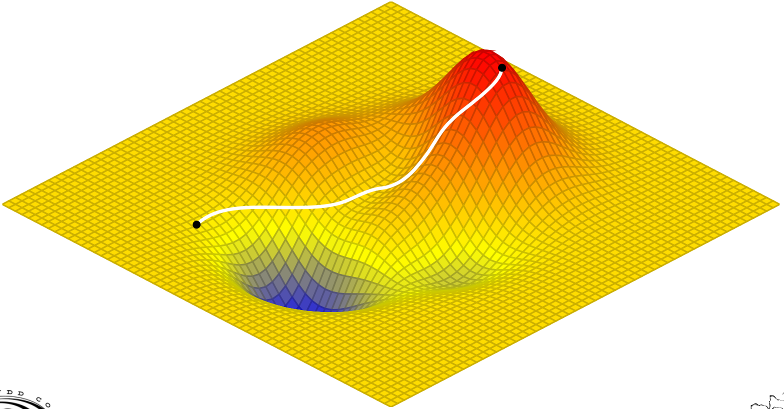


Figure 6: Drag field

Big idea, cont:



Big idea, cont:



What's essential about this problem?

- Object of interest: a path $\boldsymbol{q}(t) = (x(t), y(t), z(t))$.
- Givens:
  - Region we can fly over (Pacific Ocean)
  - Start/end points: (Honolulu/Anchorage)
  - Drag: $\mathbf{F}_D(t,\, \boldsymbol{q}(t),\, \dot{\boldsymbol{q}}(t))$
- Goal: find $\boldsymbol{q}(t)$ minimizing *total* travel time.

Defining "Cost"

- Drag: $\mathbf{F}_D(t,\, \boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))$
- Define "instantaneous cost" function:

$$\mathcal{L}(t,\, \boldsymbol{q}(t),\, \dot{\boldsymbol{q}}(t))$$

- Total "cost" of trip:

$$C[\boldsymbol{q}] = \int_{t_0}^{t_1} \mathcal{L}(t,\, \boldsymbol{q}(t),\, \dot{\boldsymbol{q}}(t))\ \mathrm{d}t$$

- What we want: an analogue of the first derivative test

Euler-Lagrange

Theorem (Euler-Lagrange)

Let $\boldsymbol{q}(t) : \mathbb{R} \to \mathbb{R}^n$ be a path. Then if $\boldsymbol{q}(t)$ is an extreme value of the functional

$$C[\boldsymbol{q}] = \int_{t_0}^{t_1} \mathcal{L}(t, \boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)) \, \mathrm{d}t$$

then $\boldsymbol{q}$ is a solution to the differential equation

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right) = 0.$$

Suppose an optimal $\boldsymbol{q}(t)$ exists:

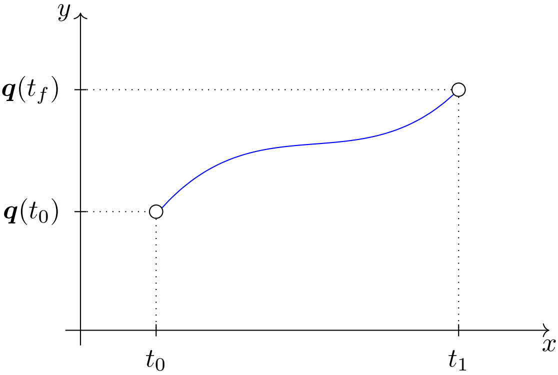


Figure 7: Optimal $\boldsymbol{q}(t)$

Suppose an optimal $\boldsymbol{q}(t)$ exists:

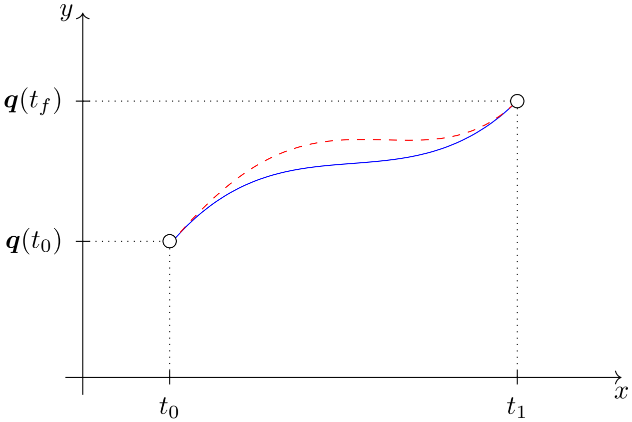


Figure 8: Add a small perturbation $\varepsilon \cdot \boldsymbol{\eta}(t)$

Shortest Time Path