Where the Wild Knots Are

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Abstract

The new work in this document can be broken down into two main parts.

In the first, we introduce a formalism for viewing the *signed Gauss code* for virtual knots in terms of an action of the symmetric group on a countable set. This is achieved by creating a "standard unknot" whose diagram contains countably-many crossings, and then representing tame knots in terms of the action of permutations with finite support; wild knots with topologically discrete crossing sets can be encoded by permutations for which the "finite support" condition is dropped. We present some preliminary computational results regarding the group operation given by this encoding, but do not explore it in detail. We then discuss some of the main challenges to working with this representation, and finish with a discussion of directions for future work.

To make the encoding above formal, we require the aforementioned "unknot with a countable sequence of crossings;" building up the machinery to work with these kinds of objects is the focus of the second part of the project. Note that initially, the presence of infinitely-many crossing might appear to be a contradiction to the finiteness constraint in Reidemeister's theorem; we show that this is not the case, and introduce the notion of *feral points* to represent areas of our diagrams in which it is not immediately obvious whether the knot is *wild* or *tame*. Our countable-crossing unknot possesses such a point. We employ uniform convergence to create sufficient conditions for guaranteeing the preservation of ambient isotopy under limits, and resolve a seeming contradiction given by the wild arc of Fox-Artin. Finally, we show that any knot (wild or tame) whose crossings are topologically discrete in a 2D diagram is ambient isotopic to a countable union of polygonal segments, and discuss implications for extending Reidemeister's theorem in this context.

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Acknowledgments

I am grateful to Edward P. Moore, my English teacher throughout much of high school. He never stopped believing in me or encouraging me to do the same. Thank you Mr. Moore — I'm sorry about the page count. It seems that despite your best efforts, I still have a tendency to write a lot.

Thank you to Professor Francis Su and Professor Sam Nelson for all their guidance throughout this project. And thank you to Professor Ken Fandell for encouraging me to view math (like art) as an opportunity for creativity and self-expression. Finally, thank you to all of the incredible professors I've had at the Claremont Colleges. Your dedication to your students and your work will always inspire me.

Preface

A copy of this document and of all of its source code can be found at https://github.com/redpanda1234/thesis-public.

Given the scope of this project, despite our best efforts, there are probably a few typos. If any are spotted, we highly encourage the reader to submit an issue report[1] or contact the author directly at fkobayashi@g.hmc.edu. Similarly if one discovers any factual inaccuracies.

Notation

We record some notational conventions used throughout this document. First, a note on some non-standard glyphs:

> **Note:** we will sometimes make use of glyphs from the two Japanese phonetic alphabet systems, ひらがな (IPA:[a] [çiraga↓na]) and カタカナ (IPA: [kataka↓na]). The motivation is that (a) more commonly-employed glyph systems (e.g., Greek / Roman) are heavily overloaded in mathematics, and (b) the ひらがな characters つ, の, ゆ, and め look qualitatively similar to
>
> - Planar isotopy (つ),
> - Reidemeister I (の),
> - Reidemeister II (ゆ), and
> - Reidemeister III (め)
>
> respectively, so they seemed a convenient alternative in the context of knot theory.
>
> It's worth mentioning that it's possible to typeset these symbols without having to use XeLaTeX or LuaLaTeX by using the `newunicodechar`

[1] URL: https://github.com/redpanda1234/thesis-public/issues

package and manually specifying the code point in `udmj30`.[b] The author has made a small package for LaTeX that does this; it can be found on Github.[c]

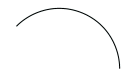
[a] See https://en.wikipedia.org/wiki/International_Phonetic_Alphabet and https://en.wikipedia.org/wiki/Help:IPA/Japanese

[b] See https://tex.stackexchange.com/a/171614

[c] https://github.com/redpanda1234/kana.sty

Table of Symbols

Notation	Meaning	Page (if applc.)
つ	Reidemeister "0" move (planar isotopy)	
の	Reidemeister I move	
ゆ	Reidemeister II move	
め	Reidemeister III move	
ラ	Generic Reidemeister move	
ら	Alt. to ラ	
K	Generally reserved for knots	
K	Generally reserved for *polygonal* knots	
S^n	The standard n-sphere	
$\mathbb{D}^n$	The standard n-ball ($\partial\mathbb{D}^n = S^{n-1}$)	
$\triangle$	Used for simplices	
$\langle A\rangle$	Sometimes used for convex hull(A)	
$(a, b)_{\circlearrowleft}$	Open interval in S^1	151
$[a, b]_{\circlearrowleft}$	Closed interval in S^1	151
$a \prec^{\circlearrowleft} b \prec^{\circlearrowleft} c$	$b \in (a, c)_{\circlearrowleft}$	151
$\mathcal{T}_{\text{std}}$	The standard topology on $\mathbb{R}^n$	
$[a, b]$	$\{x \in \mathbb{R} \mid a \leq x \leq b\}$	
(a, b)	$\{x \in \mathbb{R} \mid a < x < b\}$	
$B_r(x)$	Ball of radius r centered at x	
$(x_\alpha)_{\alpha\in\square}$	Seq. of x indexed by $\alpha \in \square$	
$(x_\alpha)_{\alpha\in\square} \subseteq X$	Shorthand for a seq. of points in X.	
$x_\alpha \nearrow x$	x_α increase to x	
$x_\alpha \to x$	x_α converges to x	
$x_\alpha \xrightarrow{u} x$	x_α converges to x uniformly	
$x_\alpha \searrow x$	x_α decrease to x	
$\overline{A}$	Closure of A	
A^c	Set complement of A	
$X \setminus A$	$\{x \in X \mid x \notin A\}$	
$X - A$	Alt. to $X \setminus A$	
$A \subseteq X$	A is a subset of X	
$A \subsetneq X$	A is a proper subset of X	



$f : A \hookrightarrow B$	f is injective from A to B
$f : A \twoheadrightarrow B$	f is surjective from A to B
$f : A \hookrightarrow\!\!\!\!\twoheadrightarrow B$	f is a bijection between A and B
$f\|_S$	f restricted to S
$\overleftarrow{f}(A)$	Inverse image of A under f
$\overrightarrow{f}(A)$	Image of A under f
$\mathbb{Z}^{>0}$	Positive integers
$\mathbb{Z}^{\geq 0}$	Non-negative integers
$\mathbb{R}^{>0}$	Positive reals
$\mathbb{R}^{\geq 0}$	Non-negative reals
$\mathbb{Z}_+$	Alt. for $\mathbb{Z}^{>0}$
$\mathbb{R}_+$	Alt. for $\mathbb{R}^{>0}$
s.t.	Such that
囧	Contradiction[2]
$\blacksquare$	QED
$\square$	QED for small proofs (e.g. claims, sketches)
$\Diamond$	Used to denote the end of definitions, etc.
[IPA]	Occasionally used for IPA pronuncation
〔words〕	Used to help parse large noun phrases

Table 1 Some notational conventions

Flipbook

A flipbook showing the construction of a $(7, 2)$ knot has been provided in the right margin for the reader's entertainment.

Document Formatting

Throughout this document we will occasionally use a *leftbar* environment to visually distinguish some parts of the document from others. E.g., in providing a recap of a series of proofs, we might do something like

> **Recap:** The above results are rather technical, but they are important because Lemma 1 gives [. . .], which will allow us to show [. . .] later.

If pursuing an iff proof, it will likely be formatted as follows:

($\Rightarrow$) : We want to show $A \Longrightarrow B$. [. . .]

($\Leftarrow$) : We want to show $B \Longrightarrow A$. [. . .]

[2] https://en.wiktionary.org/wiki/%E5%9B%A7#Chinese

We mentioned this in the notation table, but we'll do so here again. Sometimes, if there is a particularly nasty-to-parse noun phrase, we'll wrap it in 〔tortoise shell brackets〕[3] to make the sentence easier to read (we hope).

Finally, wherever possible, we have sought to insert hyperlinks for cross-referenced material (e.g., theorems, citations, equations, etc.) so that readers using a PDF copy can navigate it more easily. The coloring scheme is the default for the `hyperref` package, which is as follows.

- Red for linkcolor,
- Black for anchorcolor,
- Green for citecolor,
- Cyan for filecolor,
- Red again for menucolor,
- Cyan again for runcolor, and
- Magenta for urlcolor.

[3] きっこ う([kik̚ko̞ː]).

Chapter 1

Introduction

The night Max wore his wolf suit and made mischief of one kind
and another
his mother called him "WILD THING!"
and Max said "I'LL EAT YOU UP!"
so he was sent to bed without eating anything.

—Maurice Sendak, *Where the Wild Things Are*

Before we begin: if the reader has not yet read the preface, we would like to take this opportunity to mention its existence. Among other things, it contains a table of notation (Table 1) and a brief note on the formatting of the document.

1.1 Two Overviews of the Project

We will give two birds-eye views of some of the questions we've examined over the course of this project. The first (Section 1.2) is presented in *chronological* order, so as to capture more of the overarching motivation to the topics we pursued. We hope this will help to ground the reader in a sense of how each component of the project grew out of the previous ones, and that it can also help orient new researchers who are investigating similar questions to the ones we pursued. Note, by virtue of this choice in presentation, it might be harder to get a sense for "where things are going" while reading this section.

To address this, the second (Section 1.3, Page 8) contains more technical detail, and stays much closer to the final layout of topics within the document. Essentially, it is meant to function as an annotated table of contents, listing big takeaways and/or theorems from each section of the report. It is also much terser than the first, owing to its different objective.

Naturally there is some redundancy between these two summaries, but we hope that their different focuses will afford clarity to the reader in navigating the content. On a final note, we should mention that careful definitions of all the concepts presented below can be found later; the ones here are occasionally given in loose terms to help keep the pacing brisk.

1.2 A Chronological Overview

At the outset, our goals were as follows.

(1) Give clear/engaging exposition on the foundations of knot theory, paying close attention to rigor.

(2) Explore the extent to which unknotting moves give us way to define new algebraic structure on the category of knots. In particular, we were interested in the loosely "multiplicative" structure of the connected sum, and wanted to see if we could find an underlying "addition" operation corresponding to it.

We leave it to the reader to judge our handiwork regarding (1). Regarding (2), after a semester of largely-fruitless exploratory work, it became clear that this was too ambitious for a 1-year project.[1]

However, there was at least one intriguing result that came out of our attacks. In attempting to define all modifications of the Gauss code in terms of group actions, we needed a way to treat all knots as if they contained the same number of crossings. Essentially, this was because we needed a way to interpret what "flip crossing 17" would mean if we only had a 3-crossing knot. This led us to the following example of a knot that *appears* to be ambient-isotopic to the unknot, despite having infinitely-many crossings (shown in Fig. 1.1).

[1] That's not to say we did nothing — we have some interesting exploratory results, but most of them are computational, as working theoretically turned out to hinge on the following example (which led us down the bottomless rabbit hole of *wild knot theory*).

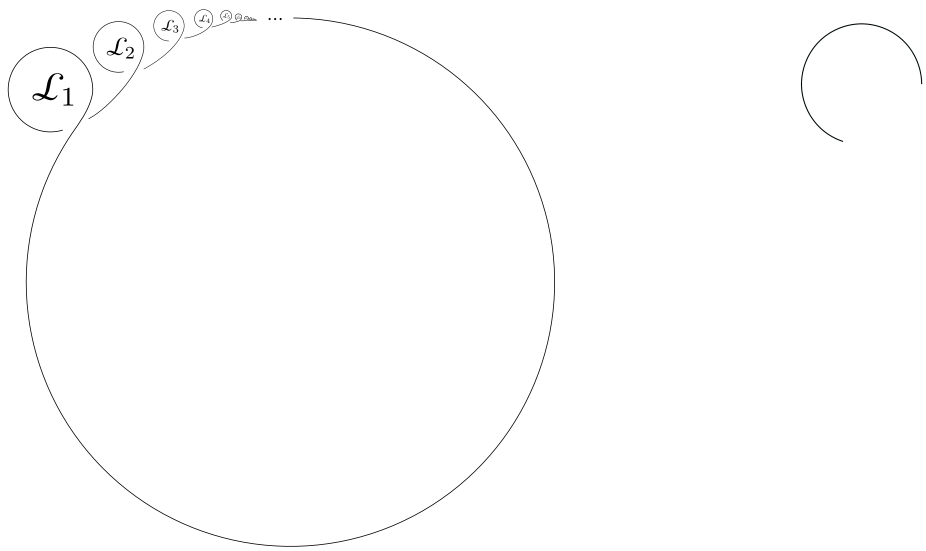


Figure 1.1 A strange unknot...

We'll show this is a valid embedding of S^1 when we talk about it in depth in Section 5.1.[2] Note, the proof we will give in that section differs from the sketch we're about to describe; the latter came first chronologically (and ends up being a bit more powerful), but is messier to work with.

Anyways: The loose idea is that we can create a uniformly convergent sequence of ambient isotopies yielding Fig. 1.1 in the limit. We can show that the associated homeomorphisms are uniformly convergent, and that their inverses are as well. Then, after arguing bijectivity, we can apply the fact that a uniform limit of continuous functions is continuous to get a homeomorphism in the limit. Finally, we apply a uniform convergence argument to the overall ambient isotopy to show it is continuous.

This example was particularly puzzling, as it seemed to create inconsistencies in the common definitions of *tame knots*, a handful of which are given below:

Common Definition 1. We say a knot $K : S^1 \hookrightarrow \mathbb{R}^3$ is *tame* iff it is ambient isotopic to a polygonal knot. $\Diamond$

Common Definition 2. We say a knot $K : S^1 \hookrightarrow \mathbb{R}^3$ is *tame* iff for every point x on K, there exists a neighborhood U_x such that the pair $(U_x, K \cap U_x) \cong$ the

[2] In particular, see Theorem 5.3

standard (ball, diameter) pair. ◊

Common Definition 3. We say a knot $K : S^1 \hookrightarrow \mathbb{R}^3$ is *tame* iff it can be thickened to an embedding of a solid torus. ◊

Common Definition 4. We say a knot $K : S^1 \hookrightarrow \mathbb{R}^3$ is *tame* iff it has finitely many crossings. ◊

In particular: we were claiming that our K satisfied Common Definition 1 (which would also imply satisfying Common Definition 2), but it was in no way obvious that it satisfied Common Definition 3. Further, Common Definition 4 seemed to be outright violated. Hence the project shifted focus to examining the foundations of Knot Theory in an attempt to verify

1. Whether or not any inconsistency was actually present, and
2. If so, where? If not, why didn't we think so?

For (a), the first plan of attack was to try and determine whether there was a flaw in our theorems. One was not forthcoming to us.[3]

Having failed to find an error in our proofs (but being convinced at the time that a flaw must exist), we set about trying to disprove our claim by finding another context where the same technique yielded a contradiction. This led us to the following example of Fox and Artin (1948), here drawn as an arc[4] made of countably-many line segments (Fig. 1.2)

[3] Of course, this does not mean an error does not exist (the reader is encouraged to get in touch if the find a problem), but we are fairly confident in our techniques, especially given that we provided multiple approaches (E.g., Theorem 5.3, Section 5.1.1).

[4] An *arc* is an embedding of $[0, 1] \hookrightarrow \mathbb{R}^3$ (as opposed to a knot, which is an embedding of $S^1 \hookrightarrow \mathbb{R}^3$).

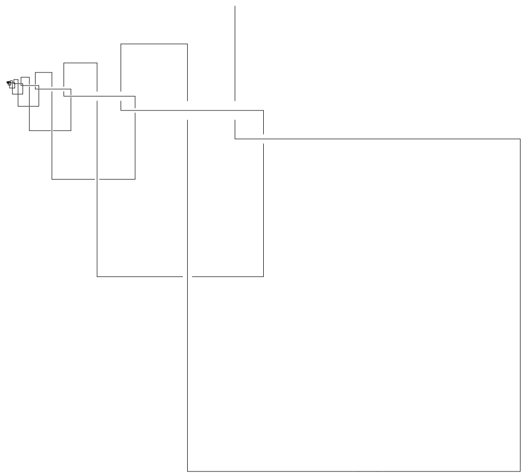


Figure 1.2 The Wild Arc of Fox-Artin

Interestingly, although it's possible to use Reidemeister II moves to undo any finite number of the "stitches" in the above, this knot is provably *inequivalent* to any tame knot. To show this, Fox and Artin developed an invariant for "tameness" based on the fundamental group of a nested sequence of closed neighborhoods of the wild point.[5] The idea is that by showing that the induced homomorphisms at each step are necessarily nontrivial, one can show no ambient isotopy to a tame arc exists.

At this point, we took a deep dive into investigating the question of when two *arbitrary* knots are ambient-isotopic. In particular, we wanted to know when we could use countable sequences of Reidemeister moves. This became the focus of most the remainder of the project.

As it turns out, this was far harder than we suspected. In fact, across multiple books, papers, and conversations with knot theorists, we heard sentiments like the following:

- "It would be fair to say that we know next to nothing about wild knots. Hence, we will not discuss them. We leave the problem to future generations." -An unnamed book

- "I feel like I really know nothing about wild knots. . . and I think most knot theorists would say the same (about themselves as well as me)." -An unnamed knot theorist

[5] We reproduce their argument in Example 7.2.

And so on. Further, a book we just recently discovered that treats the general topic of embeddings of manifolds (Daverman and Venema (2009)) has this to say about the requirements for approaching the topic:

> "What background is needed for reading this text? Chiefly, a knowledge of piecewise linear topology. [Although] to be honest, [we must also] presume extensive understanding of both general and algebraic topology [...] as well. In an attempt to limit our presumptions, we [...] shall take as granted the results from two fairly standard texts on general and algebraic topology [...] each of which can be treated quite effectively in a year-long graduate course. Unfortunately, even [these] turn out to be insufficient for all our needs."

Naturally, all of this would be sure to strike fear into the heart of any undergraduate attempting to tackle the problem, particularly one who has yet to actually complete a single formal course in Topology. Thus, it is a good thing we did not discover these problems until near the completion of the project.

Being blithely unaware of the dangers ahead, we forged dutifully onward. Due to some confusion with the definition of a locally-finite simplicial complex, we ended up focusing a large amount of our effort on studying the question of which knots can be converted (through ambient isotopy) to a countable union of polygonal segments. The source of the confusion was as follows: If one ignores the weak topology[6] and views simplicial complexes solely in terms of partitioning $\mathbb{R}^n$, then it becomes possible to find "locally-finite" "simplicial complexes" realizing many wild knots as chains of 1-simplices (with the wild points being included as separate 0-simplices). The prototypical example is to do something like the following:

> For all $n \in \mathbb{N}$, let $a_n = \left\{\frac{1}{n+1}\right\}$, $b_n = \left\{\frac{1}{n}\right\}$, and $I_n = \left[\frac{1}{n+1}, \frac{1}{n}\right]$. Also let $I_\infty = \{0\}$. Then
>
> $$K = \{I_\infty\} \cup \bigcup_{n \in \mathbb{N}} \{a_n, b_n, I_n\}$$
>
> satisfies the following properties:
>
> 1. For all $\sigma \in K$, every face τ of σ is also an element of K
> 2. For all $\sigma_0, \sigma_1 \in K$, if $\sigma_0 \cap \sigma_1 \neq \varnothing$, then $\tau = \sigma_0 \cap \sigma_1$ is a sub-face

[6] Or, in our case, (a) simply does not know about its existence, followed in short time by (b) even upon learning about its existence, does not understand its significance

of both σ_0 and σ_1, and

3. For all 0-simplices $\sigma \in K$, σ is a vertex of at most finitely-many simplices in K.

At first glance, one might protest that I_∞ should really be a face of *infinitely-many* of the 1-simplices. Morally, maybe yes. But definitionally, no, this is actually not the case. I_∞ is only incident in one simplex, namely itself. Indeed, one can see that for all $n \in \mathbb{N}$, $I_n \cap I_\infty = a_n \cap I_\infty = b_n \cap I_\infty = \varnothing$. We can create analogues in higher dimensions; E.g., an $\mathbb{R}^2$ example is given by the following.

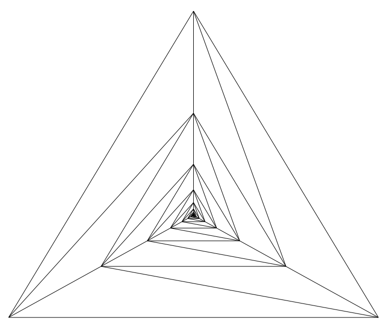


Figure 1.3 A "locally-finite" simplicial complex.

With this as motivation, we built up some formal machinery for working directly with diagrams of wild knots provided that the crossing points remain transverse and discrete.[7] This allowed us to show in Theorem 9.7 that *all* such knots are ambient isotopic to representatives comprised of a countable union of polygonal segments. This was very exciting, since it seemed it could open the door to attempting to generalize Reidemeister's theorem to countable sequences of moves. We already had an *if* direction from Theorem 7.4; we were interested in an *only if*. Unfortunately, we did not have time to see it through. This would be an interesting direction for future work.

Having taken a deep foray into the topic of wild embeddings & built up more machinery, at long last we returned back to our countable Reidemeister I example, now much more confident in its correctness. Using it as the cornerstone, we built up some basic definitions for our desired algebraic framework (Section 5.2), and created some basic tools for performing computational

[7] For those familiar with the subject matter, this relaxes our axioms for "regular diagrams" to allow possibly *countably* many crossings.

search.[8]

We should emphasize that the primary strengths of this new formalism do not stem from it delivering us a new understanding of the equivalence problem (although its possible that connections will arise in the future), but rather from the fact that it seems to offer a natural language for studying unknotting moves. We would be very interested in seeing future work in this direction.

This more-or-less wraps up the timeline for the project. In composing this report, we were given an opportunity to reflect on what the "heart" of the project was. From the above, it might seem like an answer is hard to glean, given the wide range of tangents we embarked on. We would agree. In fact, we think that in it's own way, the [“slipperiness” of identifying the correct road forward in absence of established machinery] was the defining feature of our work. We found that often, as soon as we stepped outside the scope of Reidemeister's theorem, the standard intuition we used to approach knot theory was fundamentally challenged at every turn. There were many theorems we took for granted — e.g., "ambient isotopy and ambient orientation-preserving homeomorphism are equivalent" — that turned out to contain hidden PL hypotheses that we did not quite understand the significance of.

Thus, in choosing how to present our findings, we decided to strive for being as encyclopedic as possible, to help orient those interested in future work. To that end we have placed particular attention on (a) pointing out ways in which this perspective might be useful in studying tame embeddings, (b) drawing attention to important counterexamples that challenge our intuition, and (c) striving to be maximally rigorous in working with our proofs.

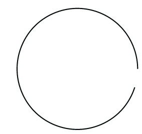
We hope they find the reader well.

1.3 The More Technical Overview

The document is structured as follows.

(I) Chapter 1: In this chapter (which you are reading right now), we give two high-level overviews of the project, including one that has a more personal/chronological feel to it (Section 1.2), as well as a more detailed "table of contents," which includes, among other things, a recursive reference to its own contents (Section 1.3).

[8] The code is available on Github: https://github.com/redpanda1234/permutation-knots



(II) Part I: Fundamentals of Knot Theory. Here, we give an overview of the big picture ideas in knot theory, and list some basic definitions. We encourage those more familiar with the topics to read Chapter 2 but skip the rest.

1) In Chapter 2, we discuss the knot equivalence problem, with particular focus on an analogy with determining equivalence of arithmetic expressions. We discuss the differences between these two contexts (e.g., the absence of a good simplification algorithm for knots), as well as what kinds of additional structure on the knot category might help get around some of these difficulties in the future. This serves to motivate the approach taken in Chapter 5

2) In Chapter 3, we provide the standard definitions for knots, ambient isotopy (highlighting the problems with choosing something like *isotopy* instead as our definition of equivalence), tameness, regular diagrams, orientation, and so on. This is targeted mainly at those who are new to the topic; experts will probably find nothing surprising. One thing of note is that we place emphasis on highlighting the fact that going from “working with knots” to “working with knot diagrams” should not be taken for granted.

(III) Part II: Combinatorial Representations. Here, we discuss what we call *combinatorial representations*, i.e. ways of abstracting information in knot diagrams to strings that can be manipulated purely algebraically / combinatorially.

4) In Chapter 4, we introduce the signed Gauss code for a knot diagram. We discuss the Gauss code encoding of Reidemeister moves, especially the planarity constraint of Reidemeister II. A significant portion of the exposition is dedicated to discussing what we have called the *diagram graph* (Definition 4.3), which is a graph constructed from the Gauss code that is particularly natural for computational manipulation. In particular, we prove that it has a unique planar embedding, and thus can be used to verify proposed Reidemeister II moves (we give a sketch for a greedy algorithm performing this check).

Finally, we finish by introducing virtual knots as a way to avoid the planarity concerns of Reidemeister II, and work instead in a more purely combinatorial context. Discussion of the forbidden moves leads us to *briefly* mentioning unknotting moves, and the intriguing sense in which they encode "recipes" for how to build knots from the unknot.

5) In Chapter 5, motivated by the discussion of unknotting moves, we build up basic definitions for our formalism that connects Gauss codes to actions of the symmetric group on a countable set. The desire here is to flesh out the idea of unknotting moves as "building" knots and "converting them into each other." Unfortunately, it seems like on a purely theoretical level, it's not possible to reconcile knot equivalence with the group structure in a sensible way. Nonetheless, we did see some unexpected patterns in performing computer searches with `sage`.

   Underpinning all of our results in this chapter is the example of an unknot with a countable number of crossings. To show this is valid, we use a slightly simpler version of the uniform convergence proof we give later (Theorem 7.4). To be extra sure that the result is valid (even if our proof turns out to secretly contain a flaw), we construct a C^1 embedding for our example of interest (Section 5.1.1), which suffices to guarantee it is tame (see Appendix A).

(IV) In Part III, we shift our focus to general topological embeddings, withe the goal of getting a better understanding our examples of countable Reidemeister I moves.

6) In Chapter 6, we clarify common definitions for tameness and wildness that we have found in the literature, reconciling those that we can with the terms used when studying more general embeddings of m-manifolds into n-manifolds. We try to be particularly cognizant of which category we are working in at all times.

7) In Chapter 7, we begin building up machinery for our later work in studying ambient isotopy for general topological embeddings. We develop two tools (strand separation and uniform converence) which prove useful for working with wild knots whose wild points are topologically discrete. We do not build machinery for working with everywhere-wild knots.

8) In Chapter 8, we apply these tools in the case of $\mathbb{R}^2$, and show that all curves $K : S^1 \hookrightarrow \mathbb{R}^2$ are ambient isotopic (this will be important when we move to $\mathbb{R}^3$ in Chapter 9). In Section 8.2, we

discuss the pathologies that can arise in diagrams in $\mathbb{R}^2$, which we refer as *feral* behavior.

9) In Chapter 9, we use the techniques of Chapter 7 and Chapter 8 to study ambient isotopies in $\mathbb{R}^3$. The loose idea is that as long as the crossing points in our diagrams are toplogically discrete, we have a bunch of strands that essentially act like they're curves embedded in $\mathbb{R}^2$ (because, after all, they don't cross). This reduces the behavior to results covered by Chapter 8. By then showing we can also constrain the behavior of our embeddings *near* crossing points, we can then show that if a wild knot has a diagram with topologically discrete crossings, then it is ambient isotopic to a representative comprised of a countable union of polygonal segments. We conclude with some directions for future work, and offer a very brief sketch of how one might build an analogue to Reidemeister's theorem in this context.

10) Lastly, in Chapter 10, we summarize the results of the project, and discuss possible ways of turning our "crossing-discrete wild knots" into a category more directly analogous to the PL case. This concludes the main body of the document.

(V) Part IV is the appendix. Here, we include some miscellany that didn't fit particularly well into any parts of the main document.

A) Appendix A contains two extra feral knots that we have parameterized by a C^1 embedding.

B) Appendix B contains a basic crash-course in PL Topology, with an emphasis on the "crash" part.

C) Appendix C includes misc. data from the project. This includes tables for the cycle representations computed in Chapter 5.

Part I

Fundamentals of Knot Theory

Chapter 2

Motivation

That very night in Max's room a forest grew
and grew–
and grew until his ceiling hung with vines
and the walls became the world all around

—Maurice Sendak, *Where the Wild Things Are*

One of the most fascinating things about knot theory is the disconnect between the relative ease of posing a question and the great difficulty of providing a rigorous answer to it. Granted, many mathematical fields are like this — but knot theory is somewhat curious in the *extremity* of the mismatch. Many of the most fundamental problems in the field can be boiled down to ideas that are accessible to any lay-person, and yet are quite challenging to approach mathematically.

Understanding *why* is the goal of this chapter of the document. As a motivating example, we discuss the problem of *knot equality* through analogy with equality of arithmetic strings. This analogy ends up also serving as the motivation for our proposal that studying *unknotting moves* could offer new insights into the structure of the knot category.

In Chapter 3, we give background definitions, with a focus on emphasizing the fact that relationships between *knots* and *knot diagrams* are subtle, and should not be taken for granted. As we will see later on, loosening what we call "an admissible diagram" can give us some valuable tools for understanding Knots.

Lastly, in Section 3.3 we offer a brief discussion of polygonal knots. Mainly, this is a gallery of some pictures; rigorous treatment is left to the section in the appendix about PL Topology and our examination of tameness in Part III.

The remainder of the discussion in this chapter is presented at a high level, with just a few (optional) formal definitions.[1] We hope the ideas remain both 〔accessible to a non-technical audience〕 and 〔interesting for experts〕.

2.1 The Big Picture

PICTURE

Figure 2.1 A Bad Joke

In mathematics, a *knot* is an embedding of a circle into another space.[2] Intuitively, think of taking a rope and twisting it around in space in all sorts of ways, finally fusing the ends together so that we get a closed loop:

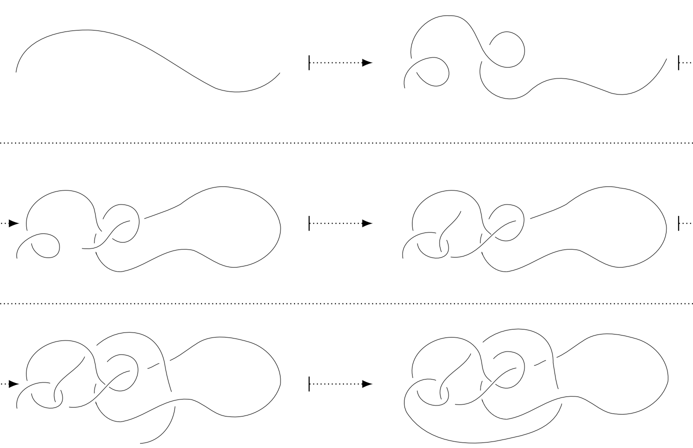
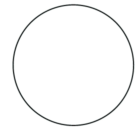


Figure 2.2 Constructing a knot

[1] We hope the more rigor-oriented readers will be patient in tolerating some imprecision for the time being, but if not, formal definitions can be found in Chapter 3, Page 31.

[2] In loose terms, "embedding" means that (a) if we zoom in closely enough to our knot, it looks like a line, and (b) if we walk all the way around in one direction, we get back to where we started. This is made formal in Section 3.1.



Note, our loop does not need to have any twists to be considered a knot — a regular old circle is a perfectly valid knot! We call this the *unknot*, and we'll see that it has some interesting properties later (e.g., it acts like the number 0 for a knot "addition" operation).

We say two knots K_1, K_2 are *equivalent* (denoted $K_1 \cong K_2$) if we can deform K_1 into K_2 without cutting the rope and gluing it back together. For example, the left two knots in the diagram below are equivalent, and both are distinct from the knot on the right.[3]

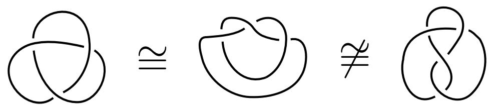
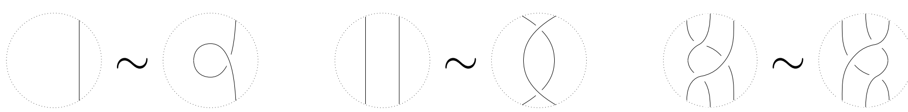


Figure 2.3 Two equivalent knots and one inequivalent one

One of the central questions in knot theory is "given diagrams D_1 and D_2 how do we determine whether they represent the same knot?" If starting from first principles, this question is **HARD** to approach mathematically; in fact it requires a bit of topological knowledge to even formulate the question properly. Thankfully, in practice we don't usually need to think about any of that because of a theorem proven by Reidemeister (1927) and, independently, Alexander and Briggs (1926). In essence, they were able to show that two "well-behaved" diagrams[4] D_1, D_2 represent the same knot iff D_1 can be turned into D_2 by a sequence of the following so-called *Reidemeister moves:*



which are creatively referred to (in left-to-right order) as "Reidemeister I," "Reidemeister II," and "Reidemeister III," respectively.[5] Note, while Reidemeister III might look complicated, it's really just saying that we can move one strand between the crossing formed by two other strands.

[3] At this point, it is worth noting that there's a technical distinction between a *diagram* for a knot and the actual knot itself. We'll return to this later when we define things rigorously, but gist is that knots live in $\mathbb{R}^3$ and diagrams are projections onto $\mathbb{R}^2$.

[4] Again, we'll discuss what "well-behaved" means in great detail during Chapter 3, but it boils down to "the string only crosses itself in an X shape."

[5] We also include another move, which allows us to bend the string arbitrarily as long as we don't introduce a crossing.

On a theoretical level, this is a very elegant characterization of knot equivalence. However, in practice, determining equivalence is still quite challenging. Even if D_1, D_2 are relatively simple and both represent the same knot, the sequence of moves relating the two can be quite long. It's even worse when $K_1 \ncong K_2$, because then we have to prove a negative result: namely, that there *does not exist* a sequence of Reidemeister moves takes D_1 to D_2! Again, this is usually quite hard. Though Algorithms deciding the problem do exist, they are currently far too inefficient to be practical. For those who are familiar with Complexity Theory: It has been proven (Hass et al. (1998)) that a special case of knot equality is at least NP. Reidemeister-based algorithms for the general problem have runtimes like $O(k \uparrow\uparrow n)$ (Lackenby (2016)), which makes even an NP solution seem out of reach for now.

But why? This seems like it should be easy! Our objects are very tangible, the space we're working in ($\mathbb{R}^3$) is well-behaved, and we aren't asking for anything too fancy — just a simple way to determine equality. How can we understand the source of this difficulty? Here, an analogy with something more familiar will be helpful.

2.2 An Analogy

Let's say I hand you two integers n, m. Would you be able to tell me if $n = m$? Most likely the answer is yes. For instance, if I said

$$n = 8 \qquad\qquad m = 31$$

you'd probably be able to distinguish n and m. In particular, they're uh... not... the same number.

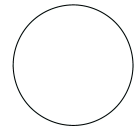
But now let's frame the same problem in a slightly different way. Suppose I had given you something like

$$n = ((1+3)+3)+1 \qquad\qquad m = (((3 \cdot 3) \cdot 3) + (2 \cdot 2)) \cdot 1$$

instead. Could you still determine if $n = m$? Maybe mental arithmetic isn't your forte, but in theory the answer is "yes" — these equations give us the same solutions for n, m as before, we just have to do a little bit of extra work to see it.

$$\begin{aligned} n &= (4+3)+1 & m &= ((9 \cdot 3) + (2 \cdot 2)) \cdot 1 \\ &= 7+1 & m &= (27 + (2 \cdot 2)) \cdot 1 \\ &= 8 & m &= (27+4) \cdot 1 \end{aligned}$$

For all intents and purposes these are cosmetic differences; they don't make the problem too much harder to solve than it was before. We can just simplify the expressions, see that the left gives 8 and the right gives 31, and then we know $8 \neq 31$ so we can go on our way.



This continues to hold even when the expressions get comically large, e.g.

$$n = 2 \cdot (2 \cdot (8 \cdot (21 - 38) - 7 \cdot 6) + (4 \cdot (21 + (-10 \cdot (2 - 4))) + 7 \cdot 28))$$
$$m = (2 \cdot (-3 \cdot (1500)) - 4) - (4 \cdot (18 - (8 \cdot (10 - (3 - 12))))) \cdot 15 + 5 \cdot (25 \cdot (3 \cdot 4) - 101)$$

While simplifying these by hand might be tedious and unpleasant, it's certainly something we could get a computer to do. In fact, programs for this task are fairly efficient, being able to handle expressions far larger than the above with near-instantaneous results.[6]

The point of these examples is the following insight: When we say things like "$n = 8, m = 31$," we are often really thinking about *equivalence classes* of arithmetic expressions that can be converted into each other using the standard rules of algebra.[7] Hence, when we say things like

$$8 = ((1 + 3) + 3) + 1$$

we really mean

$$8 \in [8] \qquad \text{and} \qquad ((1 + 3) + 3) + 1 \in [8].$$

In a deep sense, this is the same kind of problem we're tackling with knots. We're given two "expressions" (knot diagrams) representing our objects of

[6] To be precise: We claim we can verify equality of two n-bit integer arithmetic expressions in $O\big(n \log^2(n)\big)$ time — not too much worse than the $O(n)$ time for a direct equality check. We would like to acknowledge Jonathan Hayase for helping to produce the argument below.

Sketch: First, note that given any two k-bit numbers, $+, -$ are $O(k)$ and $\times$ is $O(k \log k)$. Hence, without loss of generality the worst case for our problem is an input of the form $I = x_1 \times x_2 \times x_3 \times \cdots \times x_m$ where $m < n$ and each of the x_i are of the same length (requires some finagling to argue).

Observe that multiplying two k-bit numbers yields (at most) a $2k$-bit integer. Hence, employing a divide-and-conquer approach, we can perform the total computation for I recursively by computing $(x_1 \times x_2)$, $(x_3 \times x_4)$, $\ldots$, $(x_{m-1} \times x_m)$, and then using this to compute $((x_1 \times x_2) \times (x_3 \times x_4))$, and so on. At the ℓ^{th} pass, we multiply $2^{\log(n)-\ell}$ pairs of numbers, each of length 2^ℓ bits, and there are $\log(n)$ passes total. This gives us a final runtime of $\sum_{\ell=1}^{\log(n)-1} 2^{\log(n)-\ell} \cdot 2^\ell \log(2^\ell) = n \cdot \sum_{\ell=1}^{\log(n)-1} \ell = n \cdot \frac{(\log(n)-1)\cdot\log(n)}{2}$, which is $O(n \log^2(n))$.

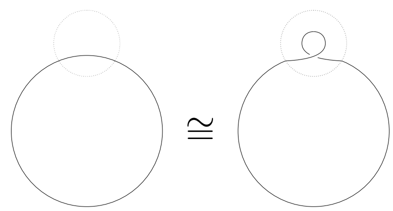
We run this algorithm twice (once for each side of the equation), and then finally check the resulting products for equality, which runs in $O(n)$. This gets us a total runtime of $O(n \log^2(n))$, which is quite fast.

[7] Recall, we denote the equivalence class of x by $[x]$.

interest, and we want to know whether they can be converted into each other by a sequence of rules (Reidemeister moves) that preserve equivalence. In this sense, Reidemeister moves can be thought of as analogous to things like "adding 0 to both sides" or "rearranging terms." That is,

$$0 = 0 + (1 - 1)$$

is philosophically similar to



Of course, the analogy isn't perfect. If it were, then we could just import all of our techniques for simplifying expressions in $\mathbb{Z}$ to the context of Knots and the problem would be solved! In the next section, we'll scrutinize some of the differences that cause the analogy to break down. Understanding exactly where we lose the structure we rely on in $\mathbb{Z}$ will be essential in motivating the questions we'll examine throughout the rest of this document. Hence, we now turn our attention to these concerns.

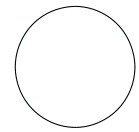
2.3 Where the Analogy Breaks Down

First we define some helpful vocabulary to make concepts like "simplest form" a bit more rigorous. Note, the terms below are sometimes used with different meanings in different fields.

Definition 2.1 (Normal Form)**.** Let S be a set of strings with an equivalence relation $\sim$ and a length function ℓ.[8] Let $s \in S$. Then we say s is in *normal form* (also sometimes called *simplest form*) if there exists no $s' \in S$ such that $s' \sim s$ and $\ell(s') < \ell(s)$. ◊

Basically, a normal form for some s is the shortest possible expression equivalent to s. In the integers, we'd say the string $\lceil 1 + 1 \rceil$ is not in normal form, as it requires 3 characters to write out (two numerals and a + sign),

[8] ℓ just counts the number of symbols that appear in our written representation of elements of S. E.g., if the string $\texttt{123} \in S$, we'd have $\ell(\texttt{123}) = 3$.



while [2] only requires 1 character to write. For knots, one of the most natural choices for ℓ is the number of crossings in the knot.

Note, we do not require normal forms to be unique; e.g., if we're working with two-variable polynomial strings, then $x+y$ and $y+x$ would both be considered normal forms. They are equivalent under $\sim$, but not identical as strings. We employ another definition when we care about uniqueness:

Definition 2.2 (Canonical Form)**.** Let S be a set of strings with an equivalence relation $\sim$. Index the equivalence classes of $S/\sim$ by some set I. For each $i \in I$, select a representative element s_i from $[s]_i$. Then we say s_i is the *canonical form* for all $s \in [s]_i$. $\diamondsuit$

If one wants to think about canonical form more tangibly in terms of a simplification process, the definition can be restated in terms of functions:

Definition 2.3 (Canonicalization)**.** Let S be a set of strings with an equivalence relation $\sim$. Then a *canonicalization* is a map $c : S \to S$ such that

1. For all $s \in S$, $c(c(s)) = c(s)$, and
2. For all $s_1, s_2 \in S$, we have $s_1 \sim s_2 \Longleftrightarrow c(s_1) = c(s_2)$. $\diamondsuit$

The correspondence to Definition 2.2 comes with the observation that for all s, $c(s)$ satisfies the requirements given in Definition 2.2 for the canonical element of $[s]$.

Ok, now we turn to examining the differences between $\mathbb{Z}$ and knots. There are two big ones we'll focus for now:

(1) We have no efficient algorithm for reducing knot diagrams to normal forms using Reidemeister moves.[9]

(2) What's more, even if there *were* such an algorithm, it would not give us a canonical form. This is because normal forms for knots are not unique, in contrast to the situation in $\mathbb{Z}$ and $\mathbb{Q}$.

Let's expand on these points. Regarding (1): Note that when we simplified our integer arithmetic expressions, each line had fewer terms than the ones above. E.g.,

$$\begin{aligned} m &= (((3 \cdot 3) \cdot 3) + (2 \cdot 2)) \cdot 1 \\ &= ((9 \cdot 3) + (2 \cdot 2)) \cdot 1 \end{aligned}$$

[9]Here, "simplifying" means finding an equivalent diagram with fewer crossings.

$$
\begin{aligned}
&= (27 + (2 \cdot 2)) \cdot 1 \\
&= (27 + 4) \cdot 1 \\
&= 31 \cdot 1 \\
&= 31.
\end{aligned}
$$

That is, each step of the simplification process took us strictly closer to a normal form. This is not an inherent property of simplification algorithms in general. If, for instance, we were working in arithmetic expressions over $\mathbb{Q}$, then we could have situations like the following:

$$\ell = \frac{1}{3} + \frac{1}{2}.$$

In order to simplify this expression symbolically, we actually have to make it more complicated first. In particular, making the terms share a common denominator requires adding extra symbols.

$$
\begin{aligned}
&= \frac{1 \cdot 2}{3 \cdot 2} + \frac{1}{2} \\
&= \frac{1 \cdot 2}{3 \cdot 2} + \frac{1 \cdot 3}{2 \cdot 3} \\
&= \frac{(1 \cdot 2) + (1 \cdot 3)}{3 \cdot 2} \\
&= \frac{2 + (1 \cdot 3)}{3 \cdot 2} \\
&= \frac{2 + 3}{3 \cdot 2} \\
&= \frac{2 + 3}{6} \\
&= \frac{5}{6}.
\end{aligned}
$$

We encounter a similar situation with Knots. Consider the following example from Kauffman and Lambropoulou (2011):

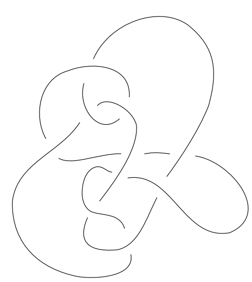
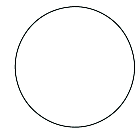


Figure 2.4 A “hard” unknot



Although perhaps not immediately obvious, this is an unknot. To see this, note that the strand going horizontally across the middle of the knot can be pulled up behind the arc at the top, tucked down through the same arc, and then out through the middle arc, at which point one can apply a Reidemeister II move followed by a reidemeister I move to undo the knot. For a more detailed view using only Reidemeister moves, see Kauffman and Lambropoulou (2011).

One can verify that there are no Reidemeister I or Reidemeister II moves that we can perform without adding more crossings to the diagram. What's more, there are actually *no* Reidemeister III moves available at all. Thus, this is a knot diagram that has to be made more complicated before it can be reduced, similarly to the fractions in $\mathbb{Q}$.

However, there is a key distinction between these two cases. With the fractions in $\mathbb{Q}$, it's always fairly obvious what the next step should be. We scan left-to-right looking for two fully-simplified sub-expressions we can combine together; if they're both fractions, then we convert everything to a common denominator. Next, we combine the numerators, and then finally we simplify. While seemingly not as efficient as parsing arithmetic expressions in $\mathbb{Z}$, we can still develop a greedy algorithm to attack the problem.

By contrast, there seems to be no obvious strategy to determine which strands we should move first on the unknot shown above — or at least, there isn't one that scales well to large knots.[10] Hence the simplification problem for knots seems a bit "harder."

Now, regarding (2) ("even if there were such a simplification algorithm, this still wouldn't immediately give us a canonical representative."). The problem here is that in $\mathbb{Z}$ and $\mathbb{Q}$, the normal forms yielded by our simplification algorithms ended up coinciding with our canonical forms. Thus, after simplifying completely, we could simply check for literal equality. This is not always the case with Knots.

[10] This doesn't mean one does not exist — but to the author's knowledge, nobody has found one yet.

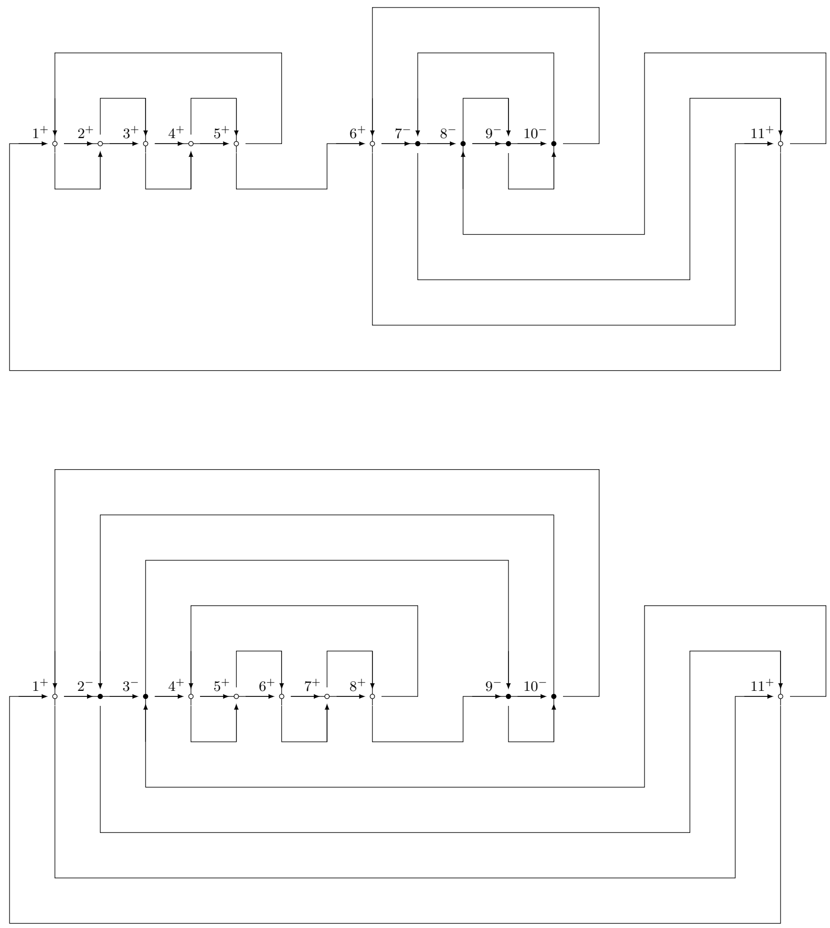
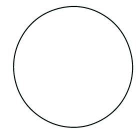


Figure 2.5 Two drawings of the same knot.

The reader can verify that these diagrams contain the same number of crossings. It is harder to see why they are equivalent, though the way we've drawn them might be suggestive of an argument.[11] It is known that the diagrams above are in normal form (in that they achieve the minimal crossing number), although we will not discuss the details in this document. In any case, we see that for knots, ⟦normal form⟧ $\not\Rightarrow$ ⟦canonical form⟧.

Taken in tandem, these problems make approaching the knot equivalence problem quite challenging. However, there's one saving grace: if we only want

[11] If the reader would like to give this a shot, we heavily encourage a more hand-wavey proof that does *not* attempt to employ the Reidemeister moves.



coarse, back-of-the-envelope heuristics for determining whether two knots are "probably" equivalent, then we can actually employ strategies similar to those used in $\mathbb{Z}, \mathbb{Q}$. This is the motivation for *invariants*.

2.4 Invariants, Briefly.

Hey you — **Think fast!** In 10 seconds or less, which of the following are true?[12]

1. $5\left(3^3 \cdot 11\right)^2 = 2(72 + 33 - 8)$
2. $-\dfrac{2}{\left(\sqrt{47} + \frac{1}{47}\right)^3} = 47 - \dfrac{1}{47^2}$
3. $3x^4 + (x+3)(x^2+2x+2) + \dfrac{2}{3}(x - x^2) = 2\left(x^4 + \dfrac{3}{2}x(x^2 - 3x)\right) + 3x$

Ok, hopefully that was far too little time to actually figure them all out. As it turns out, they are all false. Here are some short arguments for why:

1. Note that all the factors on the left side (5, 3, 11) are odd, while the right has a leading factor of 2. So they're not equal.
2. Observe that the left side is negative, while the right side is positive (since $47 > 1 > \frac{1}{47^2}$).
3. This one takes more time to verify, but the leading coefficients don't match (3 and 2, respectively). Also, the right side has no constant term, whereas the left side does.

Implicitly, each of these are using an *invariant* of arithmetic strings. For 1., we know that if two expressions are equal, then if one is even, the other must be even too. For 2., we are using the fact that equivalent expressions must have the same sign. And for 3., we use an invariant of polynomials — namely that when simplified, their coefficients must match up.

We can define similar invariants for many other problems. It's worth noting that these don't always look like simple arithmetic properties, as demonstrated by the following example:

Example 2.1. Consider a 7×7 chessboard with a knight placed on each square. Is there a way to move each night exactly once such that after each knight has been moved, no knight has left the board, and no knights have doubled up on the same square?

[12]. . . I'm counting.

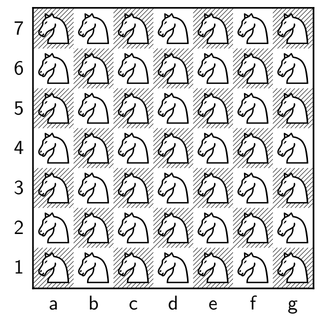


Figure 2.6 Diagram of the Situation

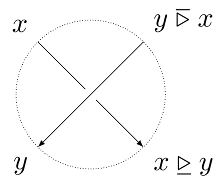
There is an extremely short solution. To emphasize just *how* short, we've included a redacted version of a short proof by one J. Yossarian, just to taunt the reader (an unredacted version can be found in Appendix C.1, Page 221)

> *Claim:* [redacted]
>
> *Proof:* [redacted] are [redacted] and [redacted]. [redacted] [redacted] a [redacted] an [redacted]. [redacted] [redacted] the [redacted], [redacted]'[redacted] and [redacted], [redacted] a [redacted]. [redacted]

Likewise, many of the invariants we employ in Knot Theory are hard to connect to explicit algebraic structures we understand. Here are some high-level examples — we won't be studying invariants directly, so don't worry too much about the details, the point is just to know that they exist.

Example 2.2 (Biquandles)**.** Let X be a set and K be a knot represented by some diagram D. Label ("color") each arc in D by an element of X. Then, define two binary operations $\underline{\triangleright}, \overline{\triangleright}$ (read "under" and "over," respectively) that describe how our labels change when strands cross.[13]

[13]Technically the words here are in the wrong order — we aren't actually guaranteed that such operations exist if we begin with an *arbitrary* coloring. But this is meant to capture the high-level overview of what the biquandle axioms seek to do, so we'll leave it like this for today.



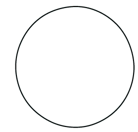


Figure 2.7 Example of a colored crossing

By translating the Reidemeister moves into algebraic axioms for $\trianglerighteq, \mathbin{\overline{\triangleright}}$, we can turn "coloring by X" into a knot invariant:[14]

1. For all $x \in X$, $x \trianglerighteq x = x \mathbin{\overline{\triangleright}} x$,
2. For all $x, y \in X$, the maps $\alpha_y, \beta_y : X \to X$ and $S : X \times X \to X \times X$ defined by
$$\alpha_y(x) = x \mathbin{\overline{\triangleright}} y, \quad \beta_y(x) = x \mathbin{\overline{\triangleright}} y \quad \text{and} \quad S(x,y) = (y \mathbin{\overline{\triangleright}} x, x \trianglerighteq y)$$
are all invertible, and
3. For all $x, y, z \in X$, we have the following *exchange laws*:
$$\begin{aligned}(x \trianglerighteq y) \trianglerighteq (z \trianglerighteq y) &= (x \trianglerighteq z) \trianglerighteq (y \mathbin{\overline{\triangleright}} z)\\ (x \trianglerighteq y) \mathbin{\overline{\triangleright}} (z \trianglerighteq y) &= (x \mathbin{\overline{\triangleright}} z) \trianglerighteq (y \mathbin{\overline{\triangleright}} z)\\ (x \mathbin{\overline{\triangleright}} y) \mathbin{\overline{\triangleright}} (z \mathbin{\overline{\triangleright}} y) &= (x \mathbin{\overline{\triangleright}} z) \mathbin{\overline{\triangleright}} (y \trianglerighteq z)\end{aligned}$$

In this case we call $(X, \trianglerighteq, \mathbin{\overline{\triangleright}})$ a *biquandle*. $\Diamond$

Biquandles are an example of *coloring invariants*, which are currently a popular area of study. Another example of a well-known class of knot invariants is the family of *knot polynomials*. Knot polynomials are generally agreed to be some of our most powerful invariants, combining ease of computation with relatively good performance in distinguishing between knots. An example is the celebrated *Jones polynomial*, which can be recursively computed from a knot diagram using the *Kauffman Bracket*:

Example 2.3 (Jones Polynomial)**.** Define a map [ ] on formal sums of knot diagrams as follows:

[14] The biquandle axioms can look somewhat intimidating at first, but with the right diagrams, one can see that they are direct translations of the reidemeister moves. We haven't included such diagrams since again, we won't really be working with biquandles, but for more we encourage the reader to reference Nelson and Elhamdadi (2015), which we found to be an excellent resource.

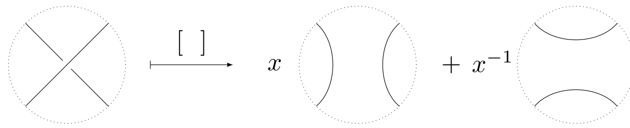


Figure 2.8 Kauffman Bracket

where the diagram is unchanged outside of the dotted neighborhood. After applying the bracket map until there are no crossings remaining, we're left with a formal sum whose terms are diagrams of some number of unlinked unknots. For each such term, apply the following conversion:

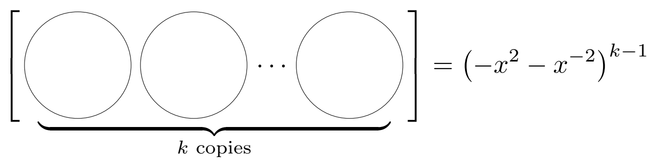


Figure 2.9 Converting unknots to polynomial terms

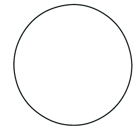
Now every term is a Laurent polynomial in x. Combine them all and and simplify to yield the *Jones* polynomial.[15] $\diamondsuit$

The Jones Polynomial has many remarkable properties; the reader is encouraged to poke around through some of the literature on it.

2.4.1 Fantastic Invariants & Where to Find Them

The section above is meant to highlight two main points: First, invariants are helpful. We did not discuss explicit performance metrics, but it is safe to say that employing them is generally much easier than doing a brute-force search with Reidmeister moves. Second, powerful invariants can look very different from each other. Biquandles and the Jones polynomial both operate on diagrams, but do so through very different mechanisms. This can make it tricky to identify where we should look *next* for new invariants, since it can be hard to interpret exactly what structure they are preserving. What other strategies can we try?

[15] This process is easier to follow by looking at some examples; the reader is encouraged to do so. This is also covered in Nelson and Elhamdadi (2015); an approach that combines knot polynomials *with* coloring invariants like biquandles can be seen in Nelson et al. (2017).



There are two main ways to approach this problem. First, we can try and generalize the strategies that we know work to see if we can get modified versions that perform better. A lot of interesting work goes on here currently, including some of our own results on more general methods for combining coloring invariants with knot polynomials (Kobayashi and Nelson (2019)). However, there is also a more direct strategy—if we can find ways to describe algebraic structure in the Knot category more explicitly, we might be able to understand invariants from a more functorial perspective. Currently, this is generally avoided, since the structure of the Knot category is not very well-understood. However, there is one interesting approach that has yielded some preliminary results recently, which is to make use of unknotting moves.

Unknotting moves are operations on a diagram that can be used to reduce arbitrary knots to the circle (an example is being allowed to “flip” which strand is on top at any given crossing). Observe that for an arbitrary knot K, playing a sequence of unknotting moves for K in reverse gives us a way to create K out of an unknotted circle. Hence, there is a sense in which unknotting moves encode the information of how to build a knot, and so it seems plausible that examining them could give us insights into our invariants.

The literature contains precedent for this idea; for instance, it has been shown that an unknotting move called the *delta move* affects some polynomial invariants in predictable ways (see Kanenobu and Nikkuni (2005), Ganzell (2014)). However, there has been little analysis of the effects of other kinds of unknotting moves, the effects of performing multiple moves in succession, and whether the results found can be generalized to other kinds of invariants. So, in general, the idea remains largely unexplored.

Our motivating goal for this project was to try and attack the problem of using unknotting moves to define a group-like structure on knots. The hope is that if we can do so, it will make it easier to search for powerful invariants by looking at something more homomorphism flavored.

Chapter 3

Knots and Knot diagrams

and an ocean tumbled by with a private boat for Max
and he sailed off through night and day
and in and out of weeks
and about over a year
to where the wild things are.

—Maurice Sendak, *Where the Wild Things Are*

The material here is targeted at readers with a basic background in topology. Experts are advised to skim (or even skip).

Knots are fundamentally 3D objects. Studying 3D objects can be hard, so we'd much rather work with 2D representations if possible. *Knot diagrams* facilitate this process. The idea is that placing strict requirements on what we call a "knot diagram" allows us to translate results about these 2D objects to ones for the actual knots. It's important to be very explicit about how we do this, as the restrictions we place on our diagrams will dictate what families of knots we can study with them. The goal for this chapter is to give the reader intuition for this process.

Our exposition is structured as follows: First, we give the formal definition of a knot and discuss some of its equivalent versions (Pages 32 to 36). We also offer some intuition for why each requirement in the definition is necessary. Next, we introduce the standard equivalence relation on knots, which is known as *ambient isotopy* (Pages 36 to 38). In doing so, we first examine two other definitions of equivalence that seem reasonable but turn out to be flawed (*homeomorphism* and *non-ambient isotopy*, respectively). Finally, we define *knot categories* and their associated *diagram categories*. This offers a nice segue into the next chapter, which discusses the most common knot category/diagram category pair: *polygonal knots* and *regular diagrams*.

3.1 Definition of a Knot

We have to define knots before defining knot diagrams. But to make our the definition tangible, we need to draw pictures. Hence, we'll have to make use of knot diagrams before we've actually defined them. The reader shouldn't worry about this, since it turns out most of the things we'll need to address in our formal treatment of knot diagrams are edge cases.

Recall, the intuitive description of a knot that we gave earlier (Fig. 2.2 on Page 16) involved taking a rope, twisting it around in space, and then fusing the ends to get a closed loop. Importantly, the strand remained "unbroken" (in the sense that there were no cuts in the rope anywhere), and never passed through itself. This is encoded in the following definition.

Definition 3.1 (Topological Knot)**.** Let $(X, \mathcal{T})$ be a topological space. Then a *knot* is a continuous map $K : [0, 1] \to X$ such that K is injective on $[0, 1)$, and $K(0) = K(1)$. $\diamondsuit$

We'll discuss the importance of the choice of codomain $(X, \mathcal{T})$ later on. But first, to build intuition, we'll restrict our analysis to the case where $(X, \mathcal{T})$ is $\mathbb{R}^3$ with the standard topology (denoted $\mathcal{T}_{\text{std}}$). This gives us the field of *classical knot theory*, whose objects of study are *classical knots*.[1]

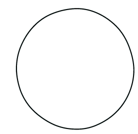
3.1.1 Classical Knots

Most of the intuition from $\mathbb{R}^3$ generalizes to the other choices of $(X, \mathcal{T})$ we'll be interested in. We just chose to examine the special case of *classical knots* first to decrease the number of moving parts in the definition.

Definition 3.2 (Classical Knot)**.** A *classical knot* is a topological knot where $(X, \mathcal{T})$ is $(\mathbb{R}^3, \mathcal{T}_{\text{std}})$. $\diamondsuit$

Aside. The choice $[0, 1]$ is arbitrary; we could modify our definition to use any interval $[a, b]$ and still get a topologically equivalent framework. Similarly, choosing K to be injective on $[0, 1)$ vs $(0, 1]$ is also arbitrary since we have $K(0) = K(1)$. We'll remove these annoyances later with an equivalent definition in terms of S^1, but first, let's talk about what this definition is doing.

[1] Some authors choose to use S^3 instead of $\mathbb{R}^3$ because S^3 is compact. We won't worry about this distinction today.



First, the codomain is $\mathbb{R}^3$, so we get an object that lives in 3D space like we want. Second, we do indeed get a "loop;" we can think of the the domain $[0, 1]$ as parameterizing how far along a 1D path we are.[2] As an example, if we had $K(t) = (\cos(2\pi t), \sin(2\pi t), 0)$, then looking down along the z axis and playing time forwards, we would see something like the following:

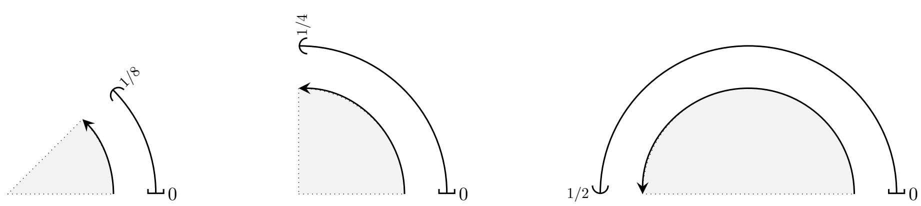


Figure 3.1 Analogy for the domain

Here, the inner ring shows $K(t)$ itself, while the outer ring is just showing us a reminder of what the corresponding values of t are.

Next, note that requiring K be *continuous* and $K(0) = K(1)$ prevents us from getting any "breaks" in the strand (a rope wouldn't make a good knot if it had a cut in it); see Fig. 3.2 below. Similarly, the injectivity condition prevents us from getting self-intersections in our rope; see Fig. 3.3 below.

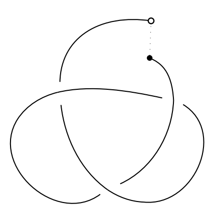


Figure 3.2 A "knot" with a break in it

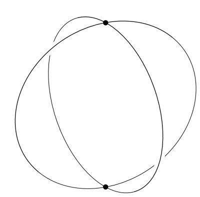


Figure 3.3 A "knot" with points of self-intersection (black)

To see that the "knot" in Fig. 3.3 really *can* be realized as a continuous function $K : [0, 1] \to \mathbb{R}^3$ (and thus the injectivity condition is strictly necessary), consider the following construction:

[2] You might wonder if space-filling curves would cause problems here. It turns out they won't, since we would need to loosen one of {injectivity, continuity} to get one.

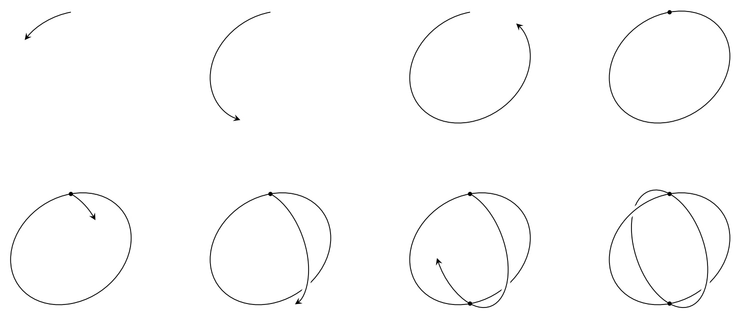


Figure 3.4 Example of how we'd construct the self-intersecting "knot"

Hence we see we need all of the pieces of the definition given.

3.1.2 Equivalent Definition in terms of S^1

Okay: Definition 3.2 is fine and intuitive, but as alluded to before, it can be written more succinctly. In particular, the whole "injective on $[0, 1)$ and $K(0) = K(1)$ except you could also use $(0, 1]$" part can be done away with by gluing the ends of $[0, 1]$ together to get a circle. This motivates the second definition, which is more commonly used.

Definition 3.3 (Classical Knot, redux)**.** A *knot* is an injective, continuous map $K : S^1 \hookrightarrow \mathbb{R}^3$. ◊

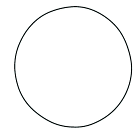
Note, since we already identify 0 and 1 in S^1, we don't have to state the extra injectivity conditions. We'll use this definition throughout the rest of this document.

It is straightforward to show that this is equivalent to the previous definition. Here's the strategy: let $K : [0, 1] \to \mathbb{R}^3$ be a knot (in the sense of the first definition), and let π be the quotient map $\pi : [0, 1] \to [0, 1]/\{0 \sim 1\}$. Then show that there exists a unique continuous $K' : S^1 \hookrightarrow \mathbb{R}^3$ such that the following diagram commutes:

$$\begin{tikzcd}
{[0,1]} \arrow[rr, "K"] \arrow[dr, "\pi"'] & & \mathbb{R}^3 \\
& S^1 \arrow[ur, "K'"'] &
\end{tikzcd}$$

Figure 3.5 Commutative diagram

This is straightforward enough that it might seem silly to state explicitly, but it is cool to see it in action. Check out https://youtu.be/7cDroN4n8EQ for an example animation.



Example 3.1. The following is a knot known as $(7, 2)$.

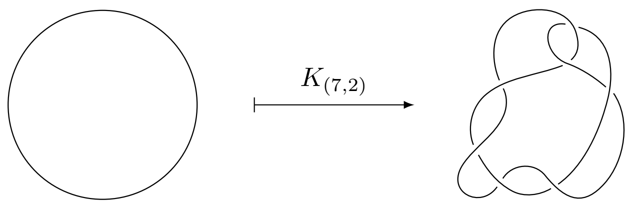


Figure 3.6 The $(7, 2)$ knot

$\diamondsuit$

The name $(7, 2)$ convention comes from the fact that 7 is the minimum number of crossings we can have in a diagram for the knot. The 2 is just a label dictated by convention so that we know *which* of the 7-crossing knots we're dealing with.[3]

There is a sense in which knots act like homeomorphims. In particular, for a knot K, the *image* of S^1 under K is still a closed loop (and in fact, this loop is homeomorphic to S^1). But since K is not a bijection (K is not not onto $\mathbb{R}^3$), we can't call it a homeomorphism. We can, however, call it an embedding.

Definition 3.4 (Embedding)**.** Let $(X, \mathcal{T})$, $(Y, \mathcal{S})$ be topological spaces, and let $f : X \to Y$. Then we say f is an *embedding* iff f is a homeomorphism between X and $f(X)$, where $f(X)$ is given the subspace topology from Y. $\diamondsuit$

Proposition 3.1. *Let $K : S^1 \hookrightarrow \mathbb{R}^3$ be a knot. Then K is an* embedding.

Proof. We want to show K is a homeomorphism between X and $K(X)$. Note, any injective function is bijective with its image, so K is a bijection between X and $K(x)$. Now, the definition of a knot guarantees K is continuous. Given these conditions, one can show that it suffices to prove K is a closed map (i.e. image of a closed set is closed).

Let $A \subseteq S^1$ be closed. Since S^1 is compact, it follows that A is compact. Since K is continuous, it follows that $K(A)$ is compact. Now, since $\mathbb{R}^3$ is

[3] For a gallery of the so-called *prime* knots up to a given crossing number, see http://katlas.math.toronto.edu/wiki/The_Rolfsen_Knot_Table

Hausdorff, all compact sets are closed. Hence $K(A)$ is closed, so K is a closed map, and thus a homeomorphism (as desired). $\square$

3.1.3 Other Codomains

We want to take a moment to stress that the choice of codomain is a hugely important factor in determining the behavior of our knots. Knottedness (as a general phenomenon) *fundamentally* arises out from taking a small space and trying to find ways to fit into a slightly larger one. Here, we've taken S^1 (a 1-manifold) and stuck it into $\mathbb{R}^3$, but one might wonder whether we'd get different theories if we tried, say, $\mathbb{R}^2$, $\mathbb{R}^4$, or $\mathbb{R}^n$.

It turns out that we do. In both $\mathbb{R}^2$ and $\mathbb{R}^4$, *all* knots turn out to be equivalent to each other. The intuition is a bit different in $\mathbb{R}^2$ than in $\mathbb{R}^4$, however. We'll examine the $\mathbb{R}^2$ case in more detail during Chapter 8, but the loose idea is that $\mathbb{R}^2$ "confines" our embedded copy of S^1 too tightly to allow it to get all tangled up (indeed, the string can't cross itself when restricted to $\mathbb{R}^2$). In $\mathbb{R}^4$, we see the opposite phenomenon. Because we have a whole extra spatial dimension to work with, the string can never quite get "stuck" on itself in $\mathbb{R}^4$ — there's always a sneaky way to get around the barrier.[4]

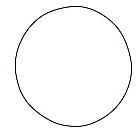
Interestingly, things of this flavor turn out to be a fairly general phenomenon — given a "tame" n-manifold N and an m-manifold M (with some additional niceness constraints), if $n - m \neq 2$, the embedded copy of M in N is often easy to unknot. However, when we have equality ($n = m - 2$), things can get hairier. The interested reader is encouraged to read more in Daverman and Venema (2009), but be warned, the content is very prerequisite-heavy.

Anyways, in light of the trivial behavior in $\mathbb{R}^2$ and $\mathbb{R}^4$, the only other codomains we'll look at here will be *thickened orientable surfaces*. These yield *virtual knot theory*, which we will discuss in Section 4.3. Until then the reader should just assume we're always thinking about knots from $S^1 \hookrightarrow \mathbb{R}^3$.

3.2 Knot Equivalence

We'll now go about formalizing what it means for two knots K_0, K_1 (oriented or unoriented) to be *the same*. On an intuitive level, we want our definition to capture the idea that if we were to tie a rope up in the shapes of K_0, K_1, then K_0 would be "equivalent to" K_1 iff by playing with the rope for long

[4]We have some slides posted at https://bedmathandbeyond.xyz/files/kaestner-brackets.pdf with a small visualization, but the reader should look elsewhere for a detailed explanation.



enough, we could find a way to make K_0 look exactly like K_1 (without just cutting the string apart and gluing it back together). We seek a way to make this idea formal. We'll discuss two definitions that turn out to fail before introducing the correct one.

Attempt 1 (Homeomorphism)**.** Let K_0, K_1 be knots. Then we say K_0 and K_1 are *equivalent* iff there exists a homeomorphism between their images. $\Diamond$

This does not work. In particular, since K_0, K_1 are embeddings, their images are always homeomorphic to S^1, and thus to each other. So this would make *all* knots equal, which we don't want to have happen. Here's a much better idea: what if we defined K_0, K_1 to be equivalent if we can find a continuous "path" through intermediate knots that get us from K_0 to K_1? This is captured by *isotopy*:

Attempt 2 (Isotopy)**.** Let $K_0, K_1 : S^1 \hookrightarrow \mathbb{R}^3$ be knots. Then we define an *isotopy* from K_0 to K_1 to be a function $H : [0,1] \times X \to Y$ such that

1. For all $x \in X$, $t \in [0,1]$, the map $H_t(x) = H(t,x)$ is an embedding,
2. For all $x \in X$, we have $H(0,x) = K_0(x)$ and $H(1,x) = K_1(x)$, and
3. H is continuous. $\Diamond$

Note, condition (1) guarantees that we take a "path" through other embeddings. Condition (2) requires that this "path" start at f and end at g. Condition (3) guarantees that the path doesn't have any "jumps."

This almost works, but there's a small hiccup that breaks this definition as well. The reader might take a moment to try and identify the issue. Here's some food for thought: Homeomorphism put all knots into the same equivalence class. Does isotopy do the same? Or does it distinguish some families of knots? Which ones?

As it turns out, this definition also puts all knots in the same equivalence class. Here's some intuition: Let $x \in X$ be fixed. Because the continuity condition on H only requires $\forall \varepsilon > 0$, there exists $\delta > 0$ such that $d(K(t,x), K(t+\delta,x))$, we can get two arbitrary embeddings K_0, K_1 to be equivalent to each other by just "shrinking all the differences down to an arbitrarily small region" An example for the trefoil (known as *Bachelor's unknotting*) is shown below.

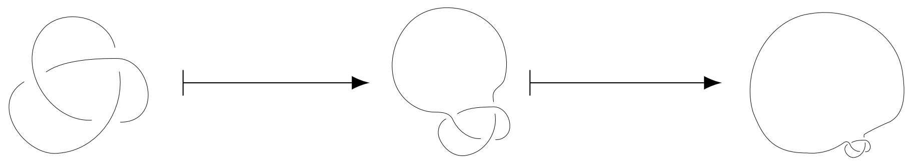


Figure 3.7 Bachelor's unknotting

By shrinking down the region in which the crossings occur, we have found an isotopy removing all of the crossings of the trefoil. Note, we could even play it in reverse, and spontaneously generate a trefoil out of an unknot! Clearly this is undesirable. But we are on the right track. It turns out the "correct" definition will be very similar. Instead of performing an isotopy on the embeddings themselves, we perform it on the ambient space. This is known as *ambient isotopy*.

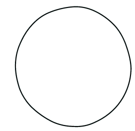
Definition 3.5 (Ambient Isotopy)**.** Let K_0, K_1 be knots. Then we say K_0 is *ambient isotopic* or *equivalent* to K_1 iff there exists a map $H : [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$ such that

1. For all $\mathbf{x} \in \mathbb{R}^3$, $t \in [0,1]$, the map $H_t(\mathbf{x}) = H(\mathbf{x}, t)$ is a homeomorphism,
2. For all $s \in S^1$, $H(K_0(s), 0) = K_0(s)$ and $H(K_0(s), 1) = K_1(s)$, and
3. H is continuous. ◊

This turns out to be the "correct" notion of equivalence, and with it, we can begin to partition our knots into equivalence classes. However, it can often be unwieldy to work with ambient isotopies directly, as topological embeddings can get quite messy. As such we often restrict ourselves to a family of knots where ambient isotopy can be characterized by the *Reidemeister moves*, which affords us a more combinatorial approach to equivalence. This nice family is known as the collection of *tame knots*, which are defined in terms of the even simpler *polygonal knots*. A more careful look at the definitions of tameness will be given in Chapter 6.

3.3 Tame and Polygonal Knots

In this section, we will (very briefly) examine the "simplest" possible knots, *polygonal knots*. If the reader would like a deeper examination of the topic, they are encouraged to take a look at Burde and Zieschang (2003) and Crowell and Fox (1963).



Polygonal knots form the backbone for modern combinatorial knot theory. The idea is that since they are formed out of finite unions of straight line segments, they can be studied combinatorially, which makes them nice. What's a little less obvious is that we can actually extend some of the resulting ideas to *any* knot ambient isotopic to a polygonal knot. Indeed, essential results such as *Reidemeister's Theorem* rely on our knots of interest being ambient isotopic to polygonal knots.

Definition 3.6 (Polygonal Knot)**.** A *polygonal knot* $\mathsf{K} : S^1 \to \mathbb{R}^3$ is a knot comprised of a finite union of straight line segments.[5] $\Diamond$

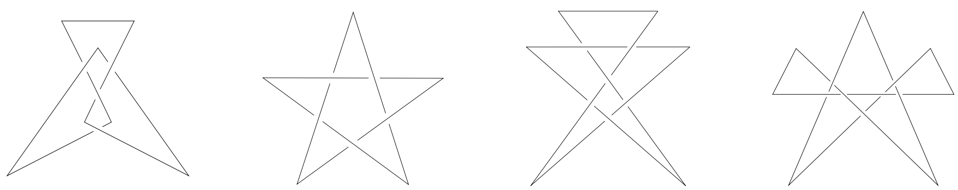


Figure 3.8 Examples of some polygonal knots

It might be helpful to think of polygonal knots in terms of finite sequences of vertices. This really hammers home the idea that polygonal knots can be encoded with finite information.

Example 3.2. We would write the following knot as $\mathsf{K} = (\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3, \mathsf{v}_4, \mathsf{v}_5, \mathsf{v}_6, \mathsf{v}_7)$, where each $\mathsf{v}_\mathsf{i} \in \mathbb{R}^3$.

[5]Note that we will also try to use the *sans-serif* font to distinguish polygonal knots from their topological counterparts, since sans-serif letters are straight and angular.

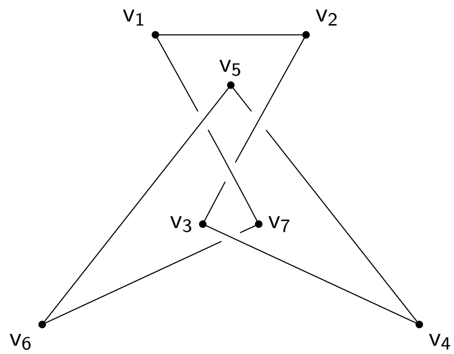


Figure 3.9 Polygonal $(4,1)$ knot with vertices labeled

$\Diamond$

The promised combinatorial niceness comes from the following theorem:

Theorem 3.2. *Let* $\mathsf{K}_0, \mathsf{K}_1 : S^1 \hookrightarrow \mathbb{R}^3$ *be polygonal knots. Then* $\mathsf{K}_0 \cong \mathsf{K}_1$ *iff there exists an ambient isotopy* $\mathsf{F} : [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$ *with* $\mathsf{F}(1, \mathsf{K}_0) = \mathsf{K}_1$ *that can be realized entirely through a finite sequence of the two* elementary moves *(note, in the below, the dashed lines are draw somewhat arbitrarily — we don't really care what the knot looks like away from our local picture):*

1) Subdivision:

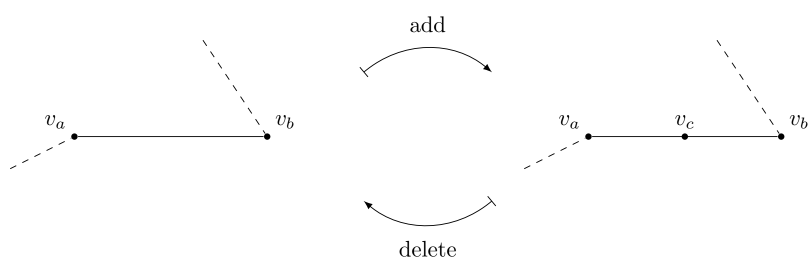


Figure 3.10 Elementary move 1: adding or deleting a vertex

2) Kinking (note, we require that no line segment passes through the triangle in $\mathbb{R}^3$ *with vertices* v_a, v_b, v_c*):*

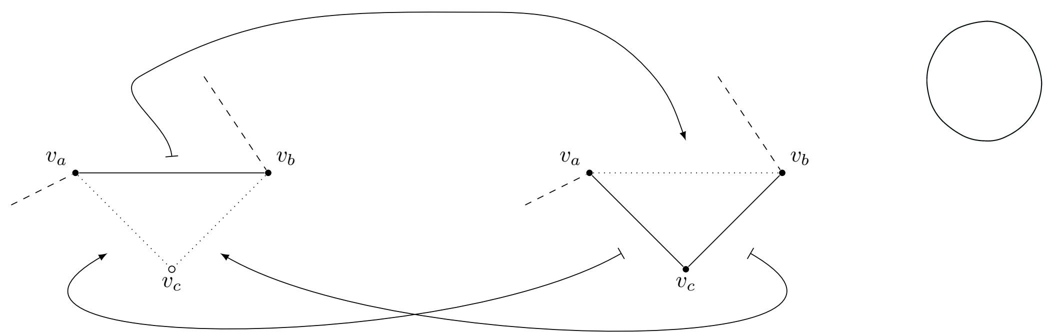


Figure 3.11 Elementary move 2: swapping edges on a triangle

Remark. It's worth emphasizing that although we've drawn the moves above to appear planar, they're really happening in $\mathbb{R}^3$. So we *could* have all sorts of things happening above or below our region of interest. We just need to require that there are no obstacles in performing the kink move.

We will not offer a proof of the theorem, since working with ambient isotopy directly turns out to be somewhat of a headache, and the proof won't be terribly informative for the contexts we *do* interact with ambient isotopy in later on. The point is just to illustrate how simple ambient isotopy is for polygonal knots.

We now define *tame knots* from polygonal knots.

Definition 3.7 (Tame Knot)**.** Let $K : S^1 \hookrightarrow \mathbb{R}^3$ be a knot. Suppose that there exists a polygonal knot $\mathsf{K} : S^1 \hookrightarrow \mathbb{R}^3$ such that $K \cong \mathsf{K}$. Then we say *K is a tame knot.* $\Diamond$

We call knots that are not tame *wild* knots. Little is known about wild knots; they are the focus of Part III.

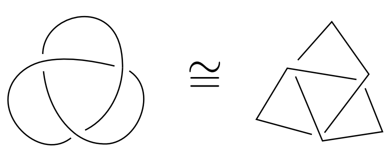


Figure 3.12 Example of a tame knot

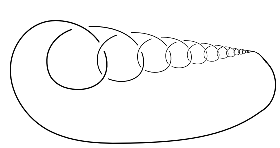


Figure 3.13 Example of a wild knot (the loops repeat infinitely)

The next step in constructing our theory is to state Reidemeister's theorem for tame knots, which will allow us to turn the 3D moves in Theorem 3.2 into purely 2D moves on diagrams. In order for this to work, we need our diagrams to give us enough information to reconstruct everything that's going on in 3D up to ambient isotopy.

3.4 Knot Diagrams

When first learning about functions $f : \mathbb{R} \to \mathbb{R}$ (e.g., in a Calculus I course), we are often taught to think of *functions* and *graphs of functions* interchangeably. For instance, if we were given

$$f(x) = -\frac{x^2}{4} + 2x,$$

we'd probably imagine a picture similar to the following:

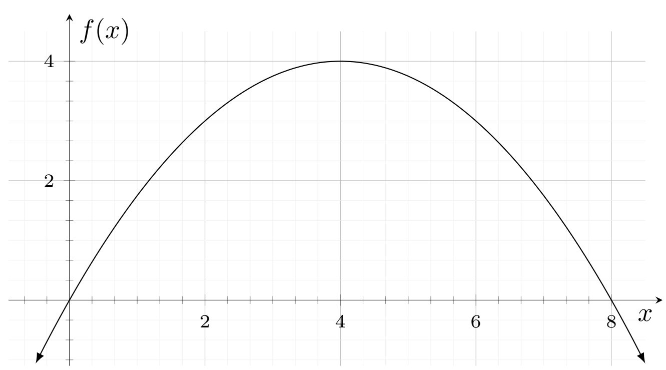


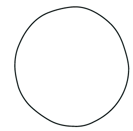
Figure 3.14 Plot of $f(x) = -\frac{x^2}{4} + 2x$

Hence, we might be mystified to see a definition like the following in an Analysis I book.

Definition 3.8 (Graph of a Function)**.** Let $f : X \to Y$. The *graph* of f is defined to be

$$G(f) = \{(x, f(x)) \mid x \in X\}.$$

Note $G(f) \subset X \times Y$, and when $X = Y = \mathbb{R}$, we often represent $G(f)$ visually by plots like in Fig. 3.14. $\Diamond$



This is confusing. Why have we made this extra definition? Does its presence mean there's a flaw in thinking of a function and its graph interchangeably?

Well, sort of, but they don't really cause problems for $f : \mathbb{R} \to \mathbb{R}$. First, note that in modern set theory, the definition of "Graph of a Function" we gave above is actually how *functions themselves* are defined formally. Nonetheless, there are some caveats, e.g.

- Our picture here doesn't actually include the full domain of the function, because our drawing space is finite.

- For multivariable functions, e.g. $f : \mathbb{R}^2 \to \mathbb{R}$, it is often no longer possible to give an injective 2D representation of $G(f)$ without making some concessions. In particular, we are constrained to represent $G(f)$ on a 2D canvas, so we can't just plot a point at every (x, y, z) pair.

We'll focus on the latter for today. How do we represent 3D objects in 2D? Often, we try to find clever ways of including extra information in our diagrammatic representations such that we can recover some of the information we lose in projection. In the particular case of $f : \mathbb{R}^2 \to \mathbb{R}$, we have various tools at our disposal, such as making use of *color*, *grid lines* and *contours*.

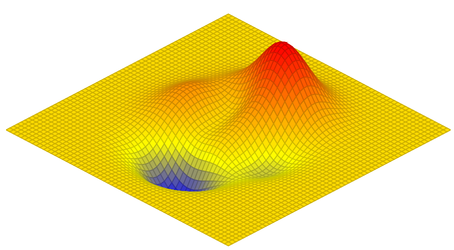


Figure 3.15 A surface

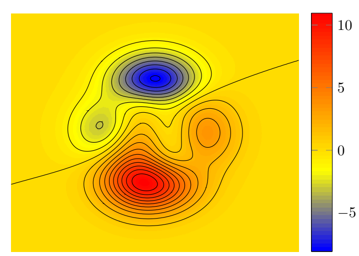


Figure 3.16 Contour plot of the same surface

Using these tools, we can create 2D representations from which we can (more or less) reconstruct the 3D picture we started with. We will employ a similar idea in dealing with knots, although this turns out to be a bit of a delicate matter. In particular, because we want to make diagrams our fundamental object of study, in order to maintain rigor we must be *very exacting* in understanding *what* our diagrams represent, and *how* we can pull that back to a result about our actual knot in 3D. $\Diamond$

We motivate this with the following examples.

3.4.1 What makes a "helpful" knot diagram

As we have seen, it is often useful to represent knots by 2D diagrams. However, not all diagrams are equivalently helpful. For instance, if we were to "line up" all the crossings of the $(7, 2)$ knot like so,

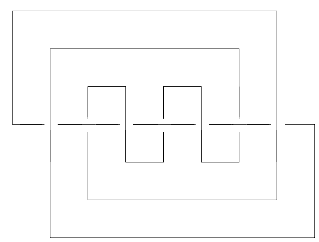


Figure 3.17 $(7, 2)$ with crossings lined up

then if we were to look along the following axis,

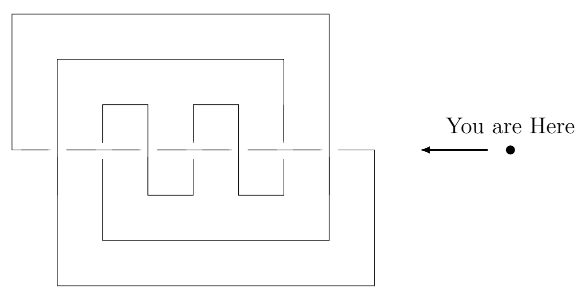


Figure 3.18 New perspective

we might end up seeing something like this:

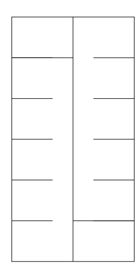
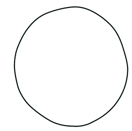


Figure 3.19 Unhelpful sideways view

which is hardly helpful.[6] Hence, we will place some restrictions on what exactly we are allowed to refer to as a *knot diagram*. The standard set of rules here are for what are known as *regular diagrams*.



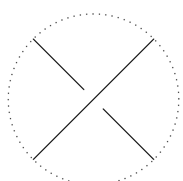
Definition 3.9 (Regular Knot Diagram)**.** Let K be a knot, and let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be a projection onto a 2-dimensional subspace of $\mathbb{R}^3$. Then we say $D = \pi \circ K(S^1)$ is a *diagram* for K iff D satisfies the following conditions:

1) D is injective at all but finitely many points $\{y_i\}_{i=1}^n \in \mathbb{R}^2$, called *crossings*.
2) For each of these y_i, there exist exactly two $x \in S^1$ such that $D(x) = y_i$.
3) Each crossing "looks like an X." Formally, there exists $\varepsilon > 0$ such that $B_\varepsilon(y_i) \cap D(S^1)$ is homeomorphic to $\{(x, y) \mid x = 0 \text{ or } y = 0\}$.
4) The diagram contains some information by which we can recover which strand was "on top" at each crossing.

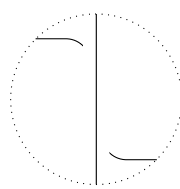
$\Diamond$

Note. If K is oriented, we also require the diagram to include information that allows us to recover the orientation.

We interpret these properties as follows. Condition (1) requires that strands only cross at *single points* in our diagrams (not entire line segments).

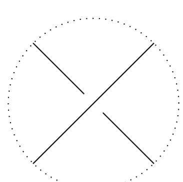


✓ Allowed



✗ Not allowed

Condition (2) requires that we can't have multiple strands crossing at the same point:



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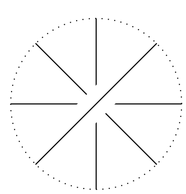
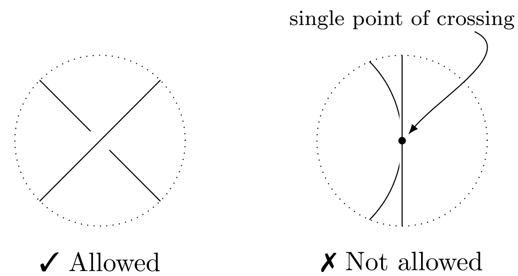


✗ Not allowed

[6]Note, we say "might" because there are multiple possible 3D realizations that could yield Fig. 3.17 in a 2D projection. But at least one of those would look like Fig. 3.19.

Condition (3) precludes situations like the following:



that is, we don't allow crossings to take the form of "tangencies;" both of the strands that come in must leave on opposite sides. Finally, condition (4) actually refers to a convention that we have been employing tacitly all along; namely *breaks* in the diagram represent places were crossings occur, and the broken strand is understood to be going "underneath" the unbroken strand.

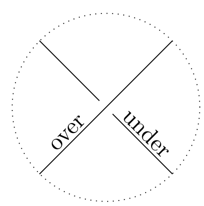


Figure 3.20 Breaks tell us which strand is on top

Now, we have the following important theorem:

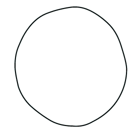
Theorem 3.3. *Let $K : S^1 \hookrightarrow \mathbb{R}^3$ be a knot. Then K is tame iff there exists a regular diagram D of K.*

Proof. Left as an exercise. *Hint:* For the forward direction, first, prove a lemma showing that for conditions (2), (3), and (4), we can deform the space ever so slightly to fix all of our issues. For condition (1), do the same and then use the finite polygonal representation to get injectivity at all but finitely many points.

There are a number of ways to do the back direction. Use the diagram to get your polygonal representation. ☐

Later, it will be helpful to draw diagrams of *wild* knots as well as tame knots, hence we make the definition below. To the best of my knowledge, this is not something that has been treated before.

Definition 3.10 (Regular Diagrams for Wild Knots)**.** Identical to the definition for a *regular knot diagram*, except we allow injectivity to fail at a countable collection of points. ◊



In general, all knots and their diagrams are assumed to be *tame* unless otherwise stated.

Remark. At this point, let's take some time again to highlight the difference between a *knot* and a *regular diagram*. A *knot* is an abstract function going from S^1 to $\mathbb{R}^3$, while a *regular diagram* is a function going S^1 to $\mathbb{R}^2$. In the section above, we defined knot diagram D in terms of composing a particular projection π with a knot K. However, we can also think of D as an abstract function in its own right. This gives us the following commutative diagram:

$$\begin{tikzcd} S^1 \arrow[rr, "D"] \arrow[dr, "K"'] & & \mathbb{R}^2 \\ & \mathbb{R}^3 \arrow[ur, "\pi"'] & \end{tikzcd}$$

Figure 3.21 Relationships between D, K, and π

It's worth nothing that this factorization is not unique! In particular, we can have many different K, π pairs that give us the same D:

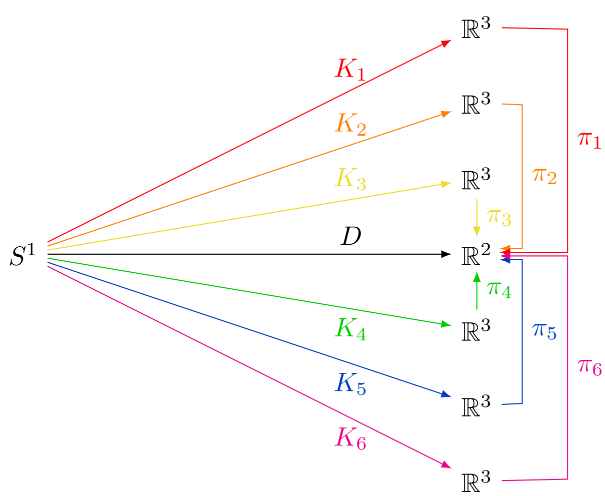


Figure 3.22 Taste the rainbow!

I feel like having multiple copies of $\mathbb{R}^3$ might be an abuse of commutative diagram notation (I'm sorry to any Category Theory enthusiasts out there), but hopefully it gets the point across. Anyways, in sort of dual vein, different projections π can take the same knot K to many *different* diagrams.

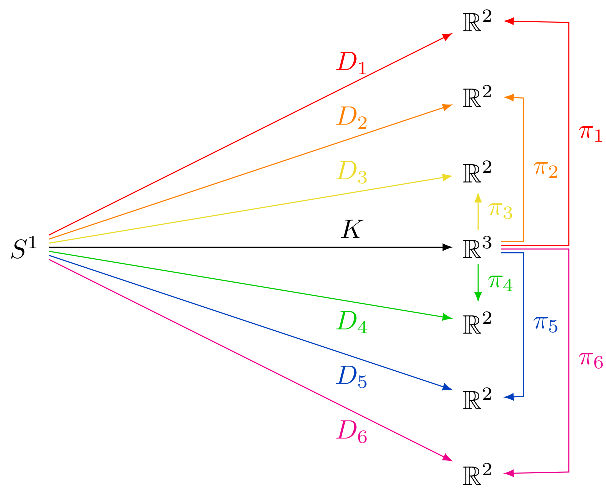
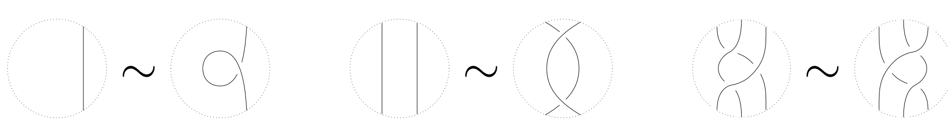


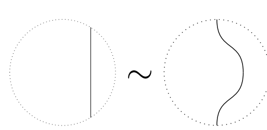
Figure 3.23 Another colorful diagram

Both of these properties are undesirable. In order to use diagrams to study knots, we must figure out a way to ensure that they "contain the same information" in some sense. In particular, we want to define an *equivalence* relation on the category of regular diagrams such that equivalence classes of diagrams correspond exactly to equivalence classes of knots under ambient isotopy. This is what *Reidemeister's Theorem* allows.

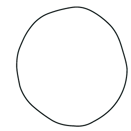
Theorem 3.4 (Reidemeister, 1927)**.** *Let $K_0, K_1 : S^1 \hookrightarrow \mathbb{R}^3$ be tame knots, and let D_0, D_1 be regular diagrams for K_0, K_1. Then $K_0 \cong K_1$ iff D_0 can be turned into D_1 by applying a finite sequence of the following moves:*



together with "planar isotopy:"



i.e., we can arbitrarily reshape a strand in a neighborhood provided we don't change the endpoints or the crossing information.



Note, if we can locally straighten out K_0, K_1 to look like polygonal knots, the theorem follows as a straightforward corollary to Theorem 3.2. This is the point that most proofs of Reidemeister's theorem start from (see Prasolov et al. (1997), for instance). However, getting there is not entirely trivial, as topological embeddings can be extremely poorly-behaved. For instance, our knot could look like the Weierstrass[7] function from the side, but a plain straight line from the top.

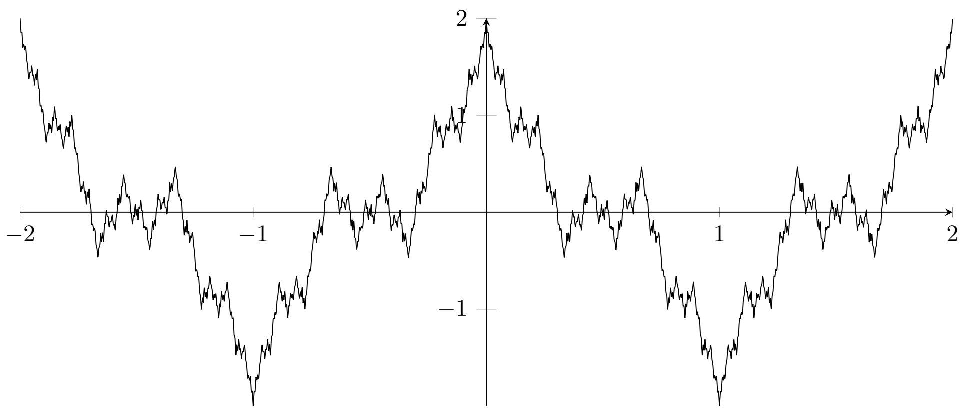


Figure 3.24 Side-view: Weierstrass function

Figure 3.25 Top-down-view: Just a line

We address this rigorously in Part III once we have the machinery to discuss lifting ambient isotopies from $\mathbb{R}^2$ to $\mathbb{R}^3$, but until then, we hope the high-level ideas have been communicated to the reader satisfactorily.

3.5 Orientation

The final concept we'll add to our knots is *orientation*. Note, if we pick an arbitrary point $s \in S^1$, there are two options for how to transverse the curve before ending up back where we started. We call these *orientations*.

[7] Pronunciation guide: [ˈvaɪɐʃtʁaːs]

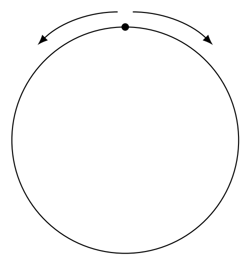


Figure 3.26 The two possible choices of orientation on S^1

We can think of the orientation as giving us a canonical ordering of the elements of S^1.[8] A choice of orientation on S^1 similarly induces an orientation on our knots, which we denote by arrows in the diagrams.

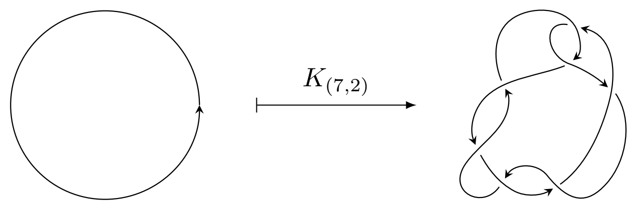


Figure 3.27 An oriented $(7, 2)$

An interesting consequence of this fact is that crossings are now chiral, coming in two non-rotationally-equivalent flavors. We refer to them as *positive* and *negative* respectively, according to the right hand rule.[9] This is in contrast to the case with unoriented knots, where all crossings are rotationally equivalent.

[8] The interested reader should read more about *cyclic orders*. An interesting note is that unlike the traditional $<$ relation we're used to in $\mathbb{R}$, cyclic orders are not binary relations. It doesn't really make sense to say "$s_1 < s_2$" in S^1, since if we simply continue out from s_2 far enough we'll get back to s_1, so it would seem $s_2 < s_1$ as well. The solution is to employ a *ternary* relation. That is, we define an order-like relation by saying $s_1 < s_2 < s_3$ iff we encounter s_1, s_2, s_3 sequentially while traversing S^1 in some direction, and s_2 and s_3 occur before we see s_1 again.

[9] Point your index finger along the overstrand, and middle finger along the understrand. If your thumb points up, the crossing is positive, otherwise it's negative.

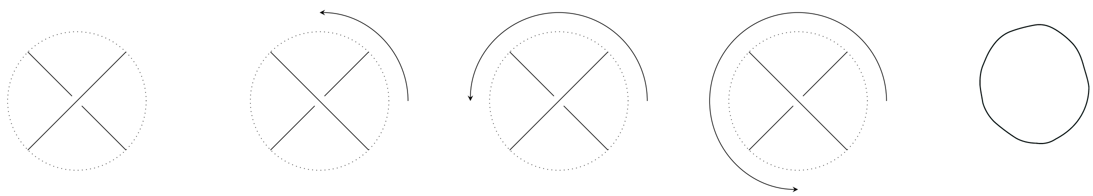


Figure 3.28 Rotating an unoriented crossing $3\times$

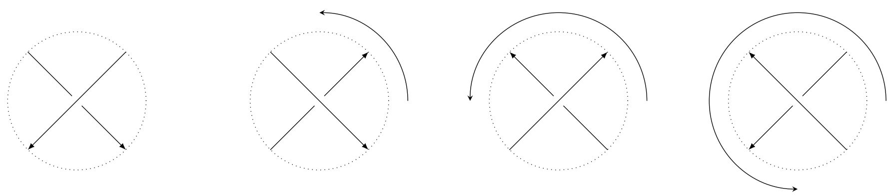


Figure 3.29 Rotating a positive crossing $3\times$

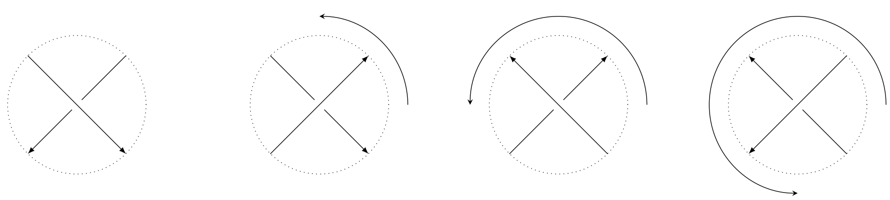


Figure 3.30 Rotating a negative crossing $3\times$

As one can see, there is no way to convert a positive crossing to a negative crossing without performing a reflection on the dotted neighborhood.

Orientation is important because it's necessary for making certain operations well-defined. An example is the *connected sum*, which we define in loose terms below.

Definition 3.11 (Connected Sum)**.** Let K_0, K_1 be oriented knots, with associated diagrams D_0, D_1. Then we define the *connected sum* of K_0, K_1 by slicing D_0, D_1 along arcs that only intersect each at two points, and gluing the ends together in a way that's consistent with the orientation. $\Diamond$

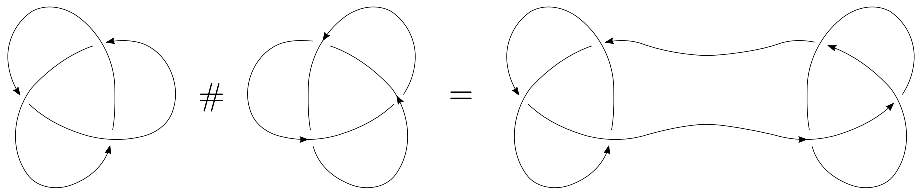
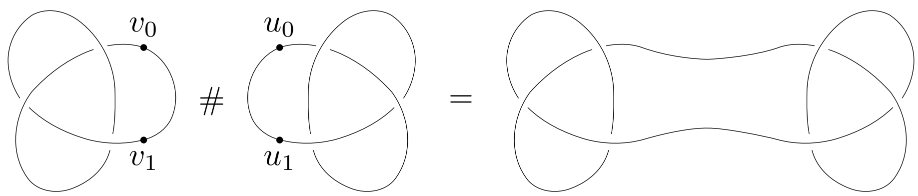


Figure 3.31 Example of the connected sum

Without orientation, it wouldn't be clear where to glue the knots back together after cutting them. In particular: dropping the orientation for now, we can see that we formed Fig. 3.31 by attaching u_0 to v_0 in the diagram below.



However, without orientation to keep us in check, we could just as well have connected (v_0, u_1) and (u_0, v_1). This yields two different knots — the former, a square knot, and the latter, a granny knot. Hence, orientation is important!

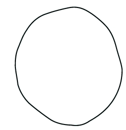
This about wraps it up for the background. We close with some remarks about the connected sum and how it ties in with our desire to find more algebraic descriptions of knot structure.

Proposition 3.5. *The connected sum is commutative and well-defined for tame knots.*

We'll give a sketch. If being fully rigorous, a lot of the details here can be a pain to write out explicitly, but conceptually they are fairly straightforward.

Sketch. Given K_0, K_1, we'll show $K_0 \# K_1$ can be ambient isotopied to $K_1 \# K_0$, with arbitrary choice of cut points.

Use ambient isotopy to turn $K_0 \# K_1$ into a polygonal knot. Because it has finitely-many strands, the it doesn't get infinitely bunched up at any point. This intuition translates to the existence of a $\varepsilon > 0$ such that for any point x_0 on $\mathsf{K}_0 \# \mathsf{K}_1$, $\overline{B_\varepsilon(\mathsf{x}_0)}$ intersects $\mathsf{K}_0 \# \mathsf{K}_1$ at exactly two points.



Use ambient isotopy to shrink the K_1 portion down until it's bounded in $B_\varepsilon(\mathsf{x}_1)$ for some x_1 on the K_1 portion of $\mathsf{K}_0 \# \mathsf{K}_1$.[10] This allows us to move the K_1 around arbitrarily inside of $\mathsf{K}_0 \# \mathsf{K}_1$. Position K_1 so that it is on the desired strand of K_0 for the connected sum $\mathsf{K}_1 \# K_0$. Unshrink K_1, and then shrink K_0 and apply the same process to it. □

The connected sum enjoys the following properties that we will not prove here.

Proposition 3.6.

1. *# defines a* monoid *on tame knots, with the unknot being the identity element.*

2. *Every tame knot has a unique prime factorization under #.*

3. *The unknot cannot be written as the connected sum of two non-trivial knots.*

The unique prime factorization part is particularly interesting, since it's reminiscent of multiplication in $\mathbb{Z}$. We'd be curious to see whether the permutation representations we introduce in the next chapter can be used to generate an operation analogous to addition in $\mathbb{Z}$, as this would create ring structure on tame knots.

[10] Note that this must be done so that the K_0 portion remains unmodified.

Part II

Combinatorial Representations

Chapter 4

The Gauss Code

> And when he came to the place where the wild things are
> they roared their terrible roars and gnashed their terrible teeth
> and rolled their terrible eyes and showed their terrible claws
>
> —Maurice Sendak, *Where the Wild Things Are*

Working with diagrams and Reidemeister moves is significantly better than working with 3D objects, but it still has its limitations. What we really want is a way to represent knots so that we (or, more likely, a computer) can apply algebraic techniques to this representation to get us results about knots themselves. This is what is offered by a combinatorial representation. Essentially, we boil the information in our diagram down to a finite string in such a way that we can recover the original diagram losslessly.[1] Then, by translating the Reidemeister moves to permutations on this string, we can view the study of knots in a purely combinatorial way.

4.1 Definitions

One of the most well-known combinatorial representations is the *signed Gauss code*:

Definition 4.1 (Signed Gauss Code)**.** Let K be an oriented knot represented as a regular diagram. Suppose K has n crossings. Then we encode K in a string コ[2] of symbols by the following scheme:

[1] At least, up to planar isotopy.

[2] Katakana *ko*, pronounced [ko]. Chosen for the Katakana transcription of “code,” 「コード」。

1. Pick some starting point p_0 on K, and a direction along which to transverse K.
2. Starting at p_0, begin transversing K. Label new crossings as with $1, \dots, n$ in the order that they're encountered. Each crossing should be visited exactly twice; we only label a crossing the first time.
3. Whenever we encounter a crossing, we record three pieces of information:
   (a) The crossing label,
   (b) whether we're on the under/overstrand, and
   (c) the sign of the crossing.

   We write this compactly by k_x^ϵ, where $k \in \{1, \dots, n\}$ is the label we've assigned our crossing, $x \in \{u, o\}$ denotes whether we're on the **u**nderstrand or **o**verstrand, and $\epsilon \in \{+, -\}$ denotes the sign of k.

We call the resulting string of $2n$ characters the *signed Gauss code*.[3] $\lozenge$

In the following, it will often be helpful to visualize signed Gauss codes by *linear diagrams*. Much of the work on the drawing program[4] (and on finding convenient graph encodings) builds on earlier work done jointly with Jonathan Hayase.

Definition 4.2 (Linear Presentation)**.** Let $\sqsupset$ be a signed Gauss code for an n-crossing knot diagram D. Then a *linear presentation* for $\sqsupset$ (also called a *linear diagram*) is a knot diagram satisfying the following rules:

1. All of the arcs are drawn on grid lines (straight up/down, left/right),
2. All of the crossing points $1, \dots, n$ are colinear, and
3. Each crossing k is drawn such that the 〔end of the arc corresponding to k's first appearance in the signed Gauss code〕 is horizontal. $\lozenge$

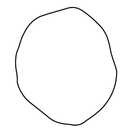
We now record some properties of the signed Gauss code.

Proposition 4.1. *The signed Gauss code has the following properties:*

[3] Note, this definition includes some non-standard normalizations, such as labelling the crossings in order, and recording the handedness of each crossing *both* times it is encountered (instead of just the second). These conventions are just to make the correspondence between diagram and code clearer, and to give our drawing program a simpler input. However, it is completely equivalent to the definition of *signed Gauss code* given elsewhere.

[4] See https://github.com/PythonNut/linear-presentation. The project is currently a work-in-progress.

(I) The substring consisting of ⟦the first appearance of each crossing in the signed Gauss code⟧ is strictly increasing. E.g., in the (11, 2) example we have



$1_u^-, 2_o^-, 3_o^-, 4_u^-, 5_o^+, 6_u^+, 7_o^-, 3_u^-, 8_o^+, 5_u^+, 6_o^+, 9_o^+, 10_u^+, 8_u^+, 4_o^-, 7_u^-, 11_u^-, 1_o^-, 2_u^-, 10_o^+, 9_u^+, 11_o^-.$

(II) Two diagrams are planar *isotopic iff they have the same Gauss code (up to the choice of basepoint).*[5]

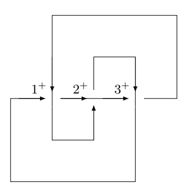
(III) Given the signed Gauss code $\sqsupset$ *for an* n*-crossing knot diagram, for all* $k = 1, \ldots, n$*, there is an even number of characters in* $\sqsupset$ *occurring between* k_u *and* k_o*.*

(I) follows immediately from the definition. (II) follows from the fact that planar isotopies preserve crossings. For (III), a short proof can be given by noting that the portion of the code between k_u and k_o (in either direction) defines a closed curve γ in the plane. By the Jordan curve theorem, any curve that enters the region must come back out. This intuition can be used to generate a short proof.

We now provide some examples of signed Gauss codes and associated linear presentations. Part (3) of Example 4.1 might be the most helpful in understanding part (3) of Definition 4.2.

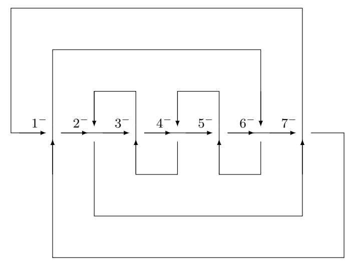
Example 4.1.

1. The signed Gauss code $1_u^+, 2_o^+, 3_u^+, 1_o^+, 2_u^+, 3_o^+$ corresponds to a diagram for the trefoil:

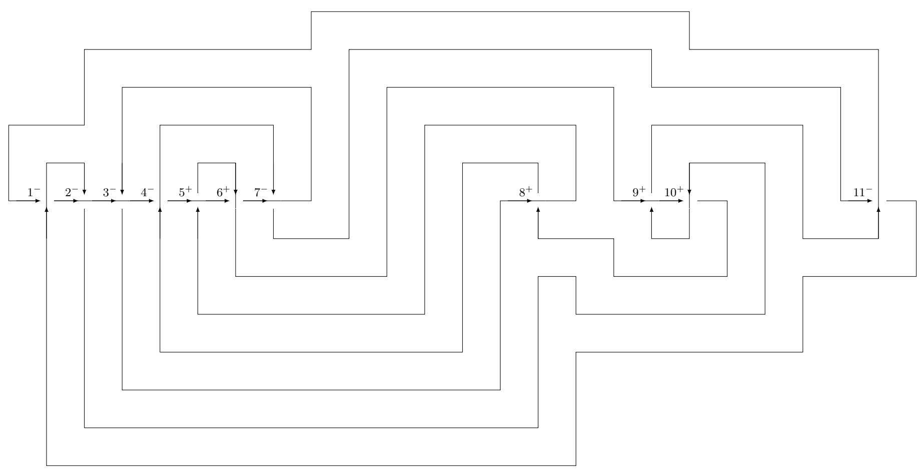


2. The signed Gauss code $1_u^-, 2_o^-, 3_u^-, 4_o^-, 5_u^-, 6_o^-, 7_u^-, 1_o^-, 6_u^-, 5_o^-, 4_u^-, 3_o^-, 2_u^-, 7_o^-$ corresponds to a diagram for the $(7, 2)$ knot:

[5]The choice of basepoint corresponds to a cyclic permutation on the Gauss code followed by a relabeling.



3. The signed Gauss code $1_u^-, 2_o^-, 3_o^-, 4_u^-, 5_o^+, 6_u^+, 7_o^-, 3_u^-, 8_o^+, 5_u^+, 6_o^+, 9_o^+, 10_u^+, 8_u^+, 4_o^-, 7_u^-, 11_u^-, 1_o^-, 2_u^-, 10_o^+, 9_u^+, 11_o^-$ corresponds to the following diagram of the $(11, 2)$ knot:

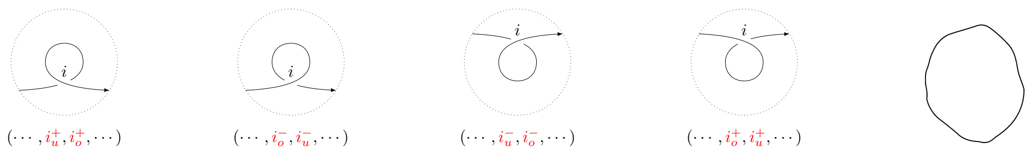


Note that 8's first appearance in the signed Gauss code comes in 8_o^+, hence 8 is drawn with the overstrand horizontal. This is what is meant by part 3 of Definition 4.2. ◊

We now examine how the Reidemeister moves can be translated to the language of the signed Gauss code.

4.2 Gauss Codes and Reidemeister Moves

In the context of signed Gauss codes, Reidemeister I corresponds to insertion/deletion of an adjacent pair:



$(\cdots, i_u^+, i_o^+, \cdots)$ $(\cdots, i_o^-, i_u^-, \cdots)$ $(\cdots, i_u^-, i_o^-, \cdots)$ $(\cdots, i_o^+, i_u^+, \cdots)$

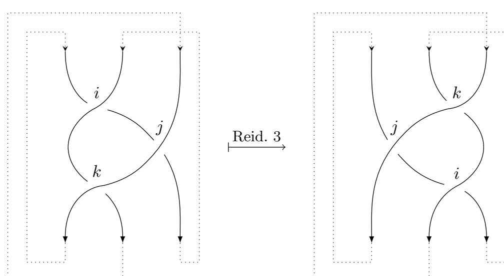
Figure 4.1 signed Gauss code Reidemeister I

Reidemeister II corresponds to insertion/deletion of a pair of pairs, albeit with some constraints based on the diagram (see Note):

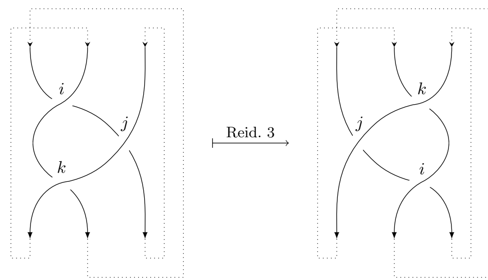
$(\cdots, i_u^+, j_u^-, \cdots, i_o^+, j_o^-, \cdots)$ $(\cdots, i_o^-, j_o^+, \cdots, i_u^-, j_u^+, \cdots)$ $(\cdots, i_u^-, j_u^+, \cdots, j_o^+, i_o^-, \cdots)$ $(\cdots, i_o^+, j_o^-, \cdots, j_u^-, i_u^+, \cdots)$

Figure 4.2 signed Gauss code Reidemeister II

Reidemeister III corresponds to swapping three pairs. We have multiple cases depending on the orientation of the strands; we'll just show two here:



$$(\cdots, i_u^+, j_u^+, \cdots, i_o^+, k_u^+, \cdots, j_o^+, k_o^+, \cdots)$$
$$(\cdots, j_u^+, i_u^+, \cdots, k_u^+, i_o^+, \cdots, k_o^+, j_o^+, \cdots)$$



$$(\cdots, i_u^+, j_u^+, \cdots, j_o^+, k_o^+, \cdots, i_o^+, k_u^+, \cdots)$$

$$(\cdots, j_u^+, i_u^+, \cdots, k_o^+, j_o^+, \cdots, k_u^+, i_o^+, \cdots)$$

Figure 4.3 Signed Gauss Code Reidemeister III

Note. Given two arcs in a knot diagram, it's not always true that we can perform a Reidemeister II move right away. For instance, consider the $(5, 1)$ knot given by $1_u^+, 2_o^+, 3_u^+, 4_o^+, 5_u^+, 1_o^+, 2_u^+, 3_o^+, 4_u^+, 5_o^+$:

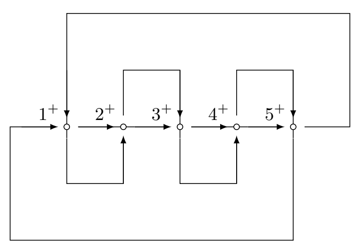


Figure 4.4 A $(5, 1)$ knot

Suppose we wanted to perform a Reidemeister II move crossing the $(2_u^+, 3_o^+)$ arc over the $(5_o^+, 1_u^+)$ arc.

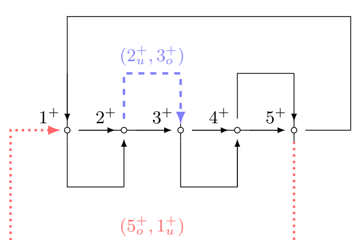
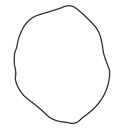


Figure 4.5 A $(5, 1)$ Knot.

If we *only* look at the signed Gauss code rules, this reasonable. We just add two new crossings, 6, 7 as follows: $1^+_u, 2^+_o, 6^-_o\, 7^+_o, 3^+_u, 4^+_o, 5^+_u, 1^+_o, 2^+_u, 3^+_o, 4^+_u, 5^+_o, 7^+_u, 6^-_u$, which would give us something like this in the new diagram:



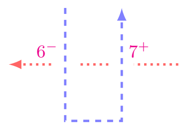


Figure 4.6 Local View of the Magically-Inserted Crossings

However, dutiful examination of Fig. 4.5 reveals that Fig. 4.6 doesn't make sense. In particular, there's no way to add such crossings without first introducing *additional* crossings just to get the two strands to be adjacent. When we discuss virtual knots later we will do away with these petty worldly concerns. For now, however, it is a problem we need to address.

To describe things formally, we want to re-interpret the signed Gauss code (and thus the associated knot diagram) as describing a combinatorial embedding of a planar graph.[6] Then, we can just say that we're allowed to do Reidemeister II moves whenever the edges represented by two arcs are in the same face of the graph.

Some care here is required. We can't just treat all of the crossings as vertices in the new graph and be done with it, because (a) this makes it unclear what we mean when we draw two different identical-looking edges between the same pair of vertices, and (b) even then, we wouldn't be guaranteed a unique planar embedding of the graph, which would mean we wouldn't have a well-defined notion of "face" (see Fig. 4.7).

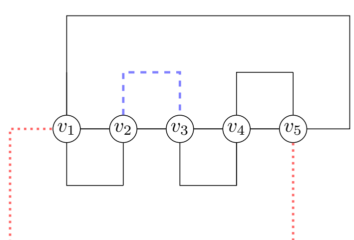


Figure 4.7 Naively treating each crossing as a vertex

[6] For those familiar with *rotation systems*, the signed Gauss code is really just defining one for the kind of 4-valent planar graph shown in Fig. 4.7.

Point (a) above is referring to the ambiguity in how we're meant to distinguish between pairs of edges line like the two from $v_2 \to v_3$ in the figure above. Point (b) refers to the fact that even if we *were* to allow multiple such edges (maybe by labeling the edges to distinguish them), then both Fig. 4.7 and Fig. 4.8 are isomorphic as graphs, but the faces in each are not the same.

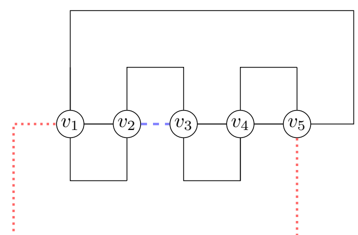


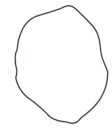
Figure 4.8 A planar graph isomorphic to that in Fig. 4.7. Note, the two edges between v_2, v_3 have been exchanged.

There are multiple ways to resolve this problem; the one listed below is particularly convenient when trying to programmatically reconstruct diagrams from signed Gauss codes.Note, the graph below will "forget" the sign of the crossings in the knot, but since this turns out to be very simple to recover from the Gauss code, we ignore it for now.

Definition 4.3 (Diagram Graph)**.** Given a signed Gauss code $\beth$ for an n-crossing knot diagram D, define a planar (undirected) graph representation $G = (V, E)$ of the knot as follows:

(1) For each $k = 1, \dots, n$, define five vertices k_u^{in}, k_u^{out}, k_o^{in}, k_o^{out}, and k^{mid}. Add them all to V.

(2) For each k as above: add edges from k^{mid} to each of k_u^{in}, k_u^{out}, k_o^{in}, and k_o^{out} to E. Also add the edges $\{k_u^{\text{in}}, k_o^{\text{out}}\}$, $\{k_o^{\text{out}}, k_u^{\text{out}}\}$, $\{k_u^{\text{out}}, k_o^{\text{in}}\}$, $\{k_o^{\text{in}}, k_u^{\text{in}}\}$ to E. See Fig. 4.9, Fig. 4.10 for a diagram.



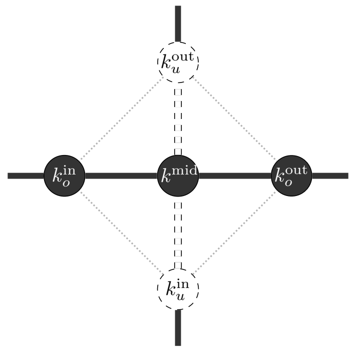


Figure 4.9 Local view of a positive crossing.

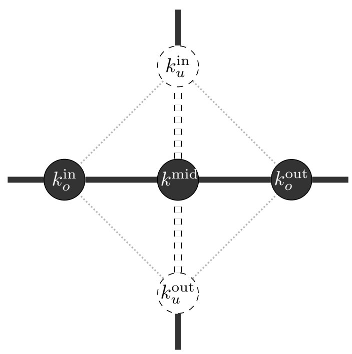


Figure 4.10 Local view of a negative crossing. Note, the k_u's have been swapped relative to Fig. 4.9.

We have drawn the edges $\{k_u^{\mathrm{in}}, k\}$ and $\{k, k_u^{\mathrm{out}}\}$ dashed to emphasize that they correspond to understrands in the knot diagram D. The dotted edges are dotted to emphasize that they do not appear in the original diagram at all.

(3) For each consecutive pair of symbols $\left(i_{x_i}^{\epsilon_i}, j_{x_j}^{\epsilon_j}\right)$ in $\sqsupset$, add the edge $\{i_{x_i}, j_{x_j}\}$ to E.

(4) The steps above are sufficient in the case of minimal-crossing diagrams for prime knots (the reader might try and verify this after reading the proof of Theorem 4.7); however, we need to do a bit more for the general case. We'll give further exposition on this point *after* we finish stating the definition.

For each $k = 1, \dots, n$, let ℓ be the crossing that comes directly after k's first appearance in the signed Gauss code. Then define u_k, u_ℓ and v_k, v_ℓ such that *if* the signed Gauss code were re-indexed to start at k, then the *right* vertex of k (see Figs. 4.9 and 4.10) would be v_k, the *top* vertex of ℓ would be v_ℓ, the *bottom* vertex of k would be u_k, and the *left* crossing ℓ would be u_ℓ. Explicitly, the casework is as follows.

- (k_u^+): $v_k = k_u^{\text{out}}$, $u_k = k_o^{\text{out}}$
  - (ℓ_u^+): $v_\ell = \ell_o^{\text{in}}$, $u_\ell = \ell_u^{\text{in}}$
  - (ℓ_u^-): $v_\ell = \ell_o^{\text{out}}$, $u_\ell = \ell_u^{\text{in}}$
  - (ℓ_o^+): $v_\ell = \ell_u^{\text{out}}$, $u_\ell = \ell_o^{\text{in}}$
  - (ℓ_o^-): $v_\ell = \ell_u^{\text{in}}$, $u_\ell = \ell_o^{\text{in}}$
- (k_u^-): $v_k = k_u^{\text{out}}$, $u_k = k_o^{\text{in}}$
  - (ℓ_u^+): $v_\ell = \ell_o^{\text{in}}$, $u_\ell = \ell_u^{\text{in}}$
  - (ℓ_u^-): $v_\ell = \ell_o^{\text{out}}$, $u_\ell = \ell_u^{\text{in}}$
  - (ℓ_o^+): $v_\ell = \ell_u^{\text{out}}$, $u_\ell = \ell_o^{\text{in}}$
  - (ℓ_o^-): $v_\ell = \ell_u^{\text{in}}$, $u_\ell = \ell_o^{\text{in}}$
- (k_o^+): $v_k = k_o^{\text{out}}$, $u_k = k_u^{\text{in}}$
  - (ℓ_u^+): $v_\ell = \ell_o^{\text{in}}$, $u_\ell = \ell_u^{\text{in}}$
  - (ℓ_u^-): $v_\ell = \ell_o^{\text{out}}$, $u_\ell = \ell_u^{\text{in}}$
  - (ℓ_o^+): $v_\ell = \ell_u^{\text{out}}$, $u_\ell = \ell_o^{\text{in}}$
  - (ℓ_o^-): $v_\ell = \ell_u^{\text{in}}$, $u_\ell = \ell_o^{\text{in}}$
- (k_o^-): $v_k = k_o^{\text{out}}$, $u_k = k_u^{\text{out}}$
  - (ℓ_u^+): $v_\ell = \ell_o^{\text{in}}$, $u_\ell = \ell_u^{\text{in}}$
  - (ℓ_u^-): $v_\ell = \ell_o^{\text{out}}$, $u_\ell = \ell_u^{\text{in}}$
  - (ℓ_o^+): $v_\ell = \ell_u^{\text{out}}$, $u_\ell = \ell_o^{\text{in}}$
  - (ℓ_o^-): $v_\ell = \ell_u^{\text{in}}$, $u_\ell = \ell_o^{\text{in}}$

Finally, add the edges $\{u_k, u_\ell\}$, $\{v_k, v_\ell\}$ to E.

Then we define the resulting $G = (V, E)$ to be the *diagram graph*. $\Diamond$

See Fig. 4.11 on page Page 67 for an example of the full diagram graph for the $(5, 1)$ knot. Note, part (4) of Definition 4.3 is meant to address situations like the following: Consider the segment contained in the dashed region of Fig. 4.11.[7]

[7]For reference, the signed Gauss code for this diagram is given by $1_u^+, 2_o^+, 3_u^+, 4_u^+, 5_o^+, 6_u^+, 4_o^+, 5_u^+, 6_o^+, 1_o^+, 2_u^+, 3_o^+$.

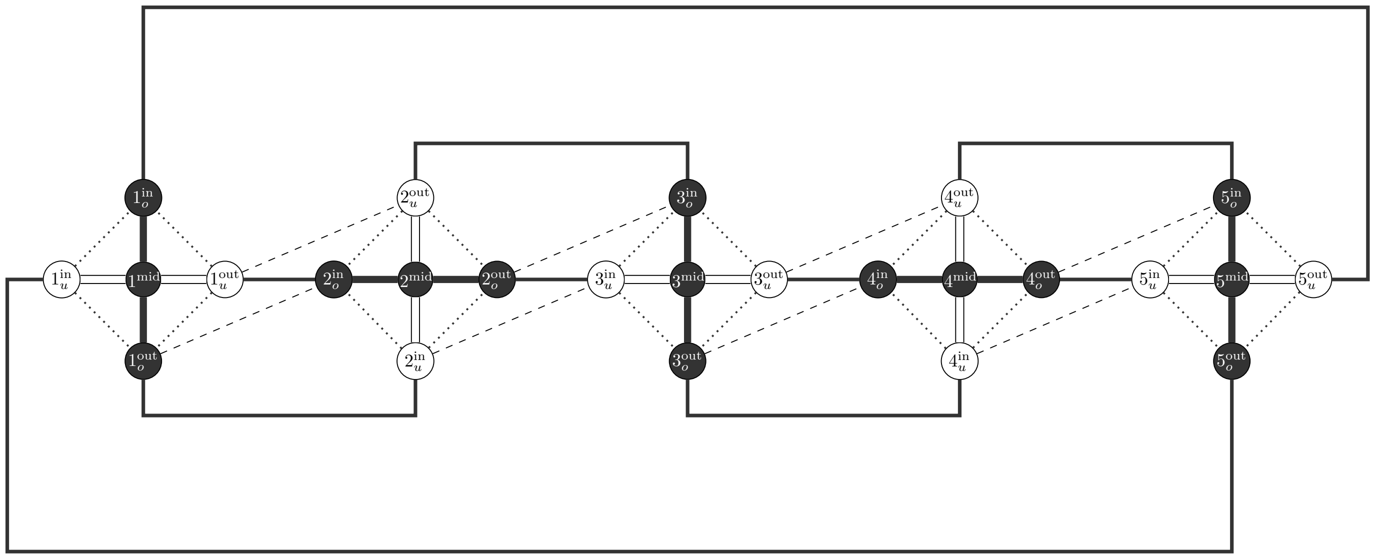


Figure 4.11 Example of the diagram graph for the diagram of $(5,1)$ shown in Fig. 4.5. The dashed edges are the ones added in in step (4) of Definition 4.3. See below for more.

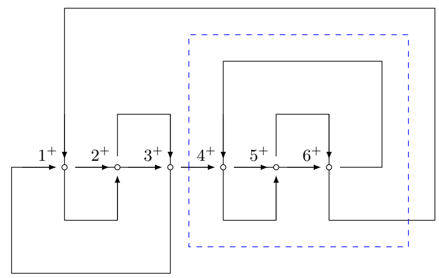


Figure 4.12 An example knot.

Suppose we were to omit the edges from step (4) of Definition 4.3. Then observe that, holding the rest of the graph constant, mirroring the region yields a valid graph isomorphism (Fig. 4.13).

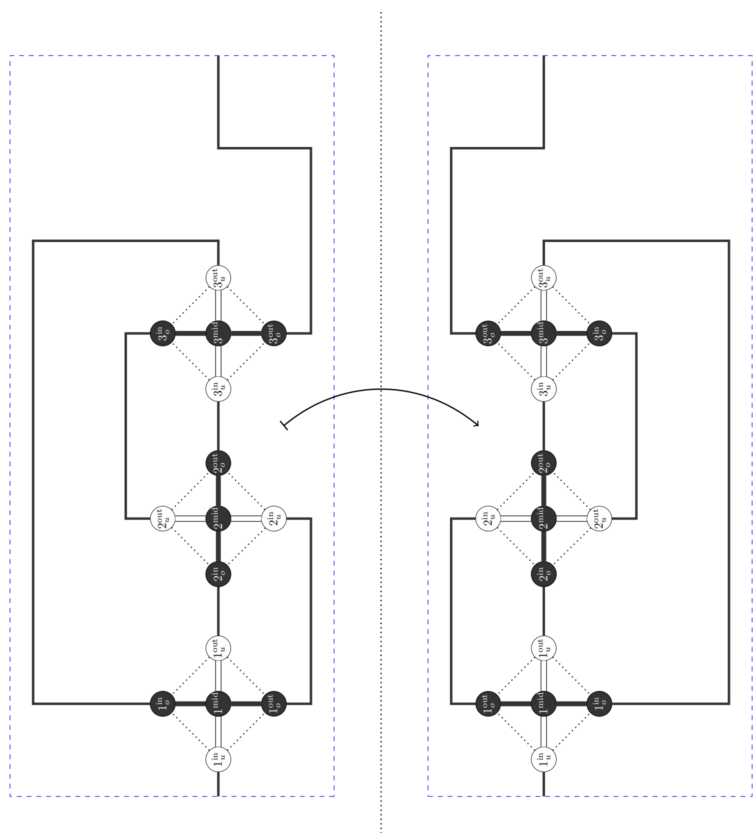
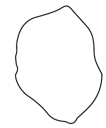
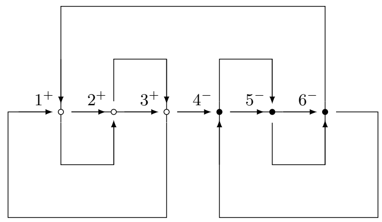


Figure 4.13 A valid graph isomorphism.



However, this would change our graph from representing the knot in Fig. 4.12 to representing the following:



As the two are not even ambient isotopic, this is undesirable. Thankfully, part (4) of Definition 4.3 prevents this by adding auxiliary edges as follows. Consider the following neighborhood of the original knot:

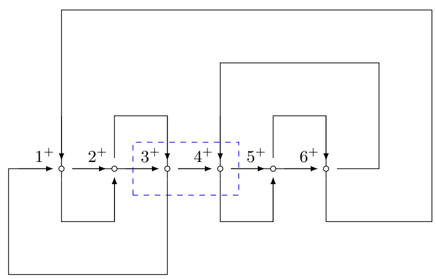


Figure 4.14 A neighborhood of crossings 3 and 4

In the knot graph *with* the edges added by part (4), the neighborhood appears as follows:

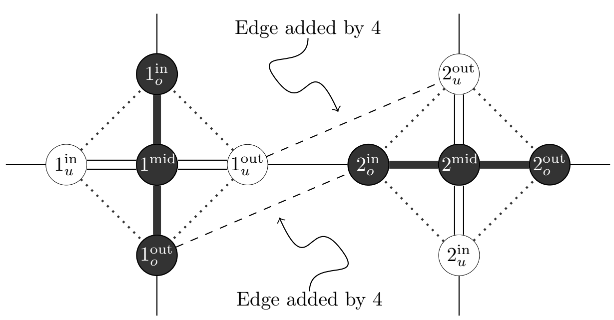


Figure 4.15 Zoomed view of the neighborhood of Fig. 4.14 in the diagram graph

We now show that the diagram graph has a unique planar embedding. To that end, we'll make heavy use of the following theorem of Whitney.[8]

Theorem 4.2 (Whitney)**.** *Let G be a simple graph planar graph. Suppose G is* 3*-connected. Then G has a unique*[9] *planar embedding.*

There are three hypotheses we need to verify: G is simple, G is planar, and G is 3-connected.[10] Simplicity follows by construction. Planarity and 3-connectedness are slightly more involved, so we'll treat them separately in the below. In both cases, the following vocabulary will be helpful.

Definition 4.4 (Crossing subgraph, underlying crossing, and representing vertices)**.** Let $\sqsupset$ be a signed Gauss code for a regular diagram D, and let $G = (V, E)$ be the associated diagram graph. Let $v = k_x^\square \in V$, where $x \in \{u, o, \ \}$ and $\square \in \{\text{in}, \text{mid}, \text{out}\}$.[11] Let G_k be given by $V_k = \left\{k^{\text{mid}}, k_u^{\text{in}}, k_u^{\text{out}}, k_o^{\text{in}}, k_o^{\text{out}}\right\}$ and all their edges.[12] Then we call

- G_k the *crossing k subgraph* of G,
- k the *underlying crossing* of v, and
- V_k the set of *representing vertices* of k. ◊

Ok — first, we show the diagram graph is planar.

4.2.1 The Diagram Graph is Planar

Recall, a *subdivision* of a graph $G = (V, E)$ is obtained by inserting vertices into the middle of edges of E.[13] The following is a well-known result.[14]

Proposition 4.3. *Let $G = (V, E)$ be an arbitrary graph. Then G is planar iff every subdivision of G is planar.*

The following proposition is also straightforward to prove. We will actually only need the special case where we are connecting v_0 and v_2 (where v_0, v_1, v_2

[8] For a nice treatment of this theorem, see Bondy and Murty (2008). The theorem itself is stated as THEOREM 10.28 on pg. 267.

[9] Here, "unique" means that any other embedding of G has the same set of faces (considered as sets of edges).

[10] Recall, a graph is called *k-connected* if it has more than k vertices, and removing

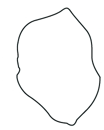
[11] Note, x is the blank character only when we have k^{mid}

[12] Explicitly, $E_k = \{\{v_1, v_2\} \in E \mid v_1, v_2 \in V'\}$.

[13] Explicitly: let $e = \{v_1, v_2\} \in E$. Then (a) define a new vertex v' and add it to V, (b) delete the edge $\{v_1, v_2\}$ from E, and (c) add $\{v_1, v'\}$, $\{v', v_2\}$ to E.

[14] See PROPOSITION 10.3 on pg. 246 of Bondy and Murty (2008).

are consecutive vertices in a face), but the proof is completely identical, so we kept the more general version.



Proposition 4.4. *Let $G = (V, E)$ be a planar graph. Consider an arbitrary planar embedding of G (denote it $\mathcal{E}$), and let v_0, v_1 be two vertices on some face F of $\mathcal{E}(G)$. Then $G' = (V, E \cup \{\{v_0, v_1\}\})$ is planar.*

Sketch. We proceed by construction. By definition of a face, F is a connected subset of $\mathbb{R}^2$ such that $F \cap \mathcal{E}(G) = \varnothing$ and $\partial\overline{F} \subseteq \mathcal{E}(G)$.

Observe that closure preserves connectedness, hence $\overline{F}$ is connected. Since we're working in $\mathbb{R}^2$, connected implies path connected, so $\overline{F}$ is path connected. Thus there exists a path $\gamma : [0, 1] \hookrightarrow F$ with $\gamma(0) = \mathcal{E}(v_0)$ and $\gamma(1) = \mathcal{E}(v_1)$. We define use this to define the embedding of G' by[15]

$$\mathcal{E}'(\square) = \begin{cases} \mathcal{E}(\square) & \text{if } \square \in V \text{ or } \square \in E, \\ \overrightarrow{\gamma}([0,1]) & \text{if } \square = \{v_0, v_1\} \end{cases}$$

Since $F \cap \mathcal{E}(G) = \varnothing$, it follows that $\overrightarrow{\gamma}([0, 1]) \cap \mathcal{E}(G) = \varnothing$. Hence the new edge doesn't cross any of the embeddings of the previous edges. Thus $\mathcal{E}'$ is a planar embedding too. $\square$

Now we can show that the diagram graph is planar.

Proposition 4.5 (Planarity of Diagram Graph)**.** *Let $\sqsupset$ be the signed Gauss code for an n-crossing knot diagram D, and let $G = (V, E)$ be the associated diagram graph. Then G is planar.*

Proof. Note that by definition of a regular diagram, the non-simple graph $G_0 = (V_0, E_0)$ defined by

1. V_0 is the set of crossing points of D (labeled $1, \ldots, n$), and
2. $E_0 = \{\{u_0, v_0\} \in V_0 \mid u_0, v_0 \text{ are the endpoints of a semiarc of } D\}$

is planar (see Fig. 4.7).[16] Observe that taking $k^{\text{mid}} = k$ and then performing step (1), the first part of step (2), and step (3) of Definition 4.3 corresponds

[15] Recall that we use $\overrightarrow{f}(A)$ to denote "the image of A under f."

[16] Sketch: The knot diagram trivially gives an embedding of the graph, since by definition crossings only occur at points of V_0, so we can just take the semi-arcs to be embeddings of the edges, and all the boxes get ticked.

to taking a subdivision of G_0.[17] In particular, we have subdivided each edge $\{v_i, v_j\}$ twice by inserting an "out" vertex and an "in" vertex, assigned to v_i, v_j as dictated by the signed Gauss code. Thus, by Proposition 4.3, the resulting graph $G_1 = (V_1, E_1)$ is planar.

Now observe that by Proposition 4.4, performing the second part of step (2) (adding the rims to each of the crossing k subgraphs) yields a new planar graph G_2.[18] A similar argument can be applied to show that step (4) corresponds to applying Proposition 4.4 to G_2, hence the resulting graph G_3 is planar as well. Since G_3 is the diagram graph itself, we now have the desired result. □

Now, we show the diagram graph is 3-connected.

4.2.2 The Diagram Graph is 3-Connected

We start by proving that each crossing k subgraph is 3-connected, then we use this to argue the case for the diagram graph.

Lemma 4.6. *Let $\sqsupset$ be a signed Gauss code for a regular n-crossing diagram D, and let $G = (V, E)$ be the associated diagram graph. Then for all $k = 1, \ldots, n$, the crossing k subgraph is 3-connected.*

Proof. Note that for all k, by construction, G_k is isomorphic to the wheel graph W_4. One can do a brute-force check to verify W_4 is 3-connected[19] □

Now, we attack the diagram graph.

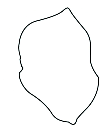
Theorem 4.7. *Given a signed gauss code $\sqsupset$ for an n-crossing knot diagram D, there exists a unique planar embedding of the associated diagram graph $G = (V, E)$.*

---

[17] These steps were: 〔step (1)〕 adding the representative vertices, 〔first part of (2)〕 adding all the spokes from k^{mid} to the other representative vertices, and 〔step (3)〕 connecting the rims of the crossing k subgraphs together as dictated by the signed Gauss code.

[18] If being extremely rigorous, one would need to argue that none of these added edges can ever cross *each other*. This is straightforward: the crossing k subgraphs are each planar (since they're isomorphic to W_4), and we haven't added edges anywhere else.

[19] A straightforward way is to apply Menger's theorem and argue that between any two vertices, there exist 3 internally vertex-disjoint paths (fancy way of saying paths that are disjoint except the endpoints). One can use the symmetry of W_4 to reduce the argument to just 3 cases: (1) two adjacent vertices on the rim, (2) two opposite vertices on the rim, and (3) the center vertex with any other vertex.



Proof. Again, by Theorem 4.2, we want to show that G is 3-connected. To that end, consider an arbitrary cut (S, T) of G, and let the cut-set be E_{cut}.[20] We want to show $|E_{\text{cut}}| \geq 3$. We have two subcases.

1. Suppose E_{cut} contains an edge from one of the crossing k subgraphs, denote it G_k. Then S, T partition G_k, and by Lemma 4.6, we have at least 3 edges in E_{cut}, as desired.

2. Now suppose E_{cut} does not contain an edge from any of the crossing k subgraphs. Then S, T partition the crossing k subgraphs. By the pigeonhole principle, there exists some k such that if ℓ is the crossing that comes directly after k's first appearance in the signed Gauss code, the representing vertices V_k, V_ℓ satisfy $V_k \subseteq S$, and $V_\ell \subseteq T$.[21]

   Recall the definitions of u_k, u_ℓ, v_k, v_ℓ from Definition 4.3 step (4). Note that u_k, u_ℓ and v_k, v_ℓ are distinct. Thus the edges $\{u_k, u_\ell\}$, $\{v_k, v_\ell\}$, $\{v_k, u_\ell\}$ are all distinct. The first two are added in step (4), the latter in step (3). Thus $|E_{\text{cut}}| \geq 3$, as desired.

Since these cases are exhaustive, we have $|E_{\text{cut}}| \geq 3$, as desired. □

Huzzah! Now, we can *finally* go back and clear up the problems with the Gauss code Reidemeister II move. In particular, we have

Proposition 4.8. *We can perform a Reidemeister II move on two semi-arcs of a regular diagram D iff they are part of a shared face in the diagram graph G.*

The proof follows directly from the construction.

4.2.3 Some More Notes on the Diagram Graph

With the (admittedly significant) planarity constraints on the diagram graph, we can encode our knots losslessly up to a choice of chiraliy, as described in the following proposition.

[20]Recall, a cut is a partition of V into two disjoint sets S, $T \neq \varnothing$, such that the subgraphs $G_S = (S, E\mid_S)$, $G_T = (T, E\mid_T)$ (where $E\mid_S$, $E\mid_T$ denote is the original edge set restricted to S, T respectively) are connected. The set of edges bridging G_S, G_T in G is defined to be the *cut-set*, denoted $E_{\text{cut}} = \{\{s,t\} \in E \mid s \in S, t \in T\}$.

[21]To argue this: first note that the case where there's only 1 crossing subgraph is covered by the first case. Hence, without loss of generality $n \geq 2$. Now observe that there are n such k, ℓ pairs, and only two sets to partition them across.

Proposition 4.9. *For knots K_0, K_1 with diagrams D_0, D_1, the associated diagram graphs G_0, G_1 are isomorphic iff D_0 and D_1 are equivalent by planar isotopy together with (at most) a single reflection.*

Again, this follows more or less by construction. The upshot is that we can now fully encode knot equivalence by moves on the signed Gauss code subject to planarity constraints.[22]

Proposition 4.10. *Given: two sets V, E representing a diagram graph G, there exists a greedy algorithm for enumerating the faces of the G that runs in $O(n)$ time (where $n = |V|$).*

Proof. Initialize an empty list of faces. Subdivide each edge in the graph into two directed half-edges with opposite orientations ($O(n)$). Note of these, there are only $4n$ edges that we're actually interested in (the half-edges representing arcs in the diagram). Also observe that each half-edge is contained in exactly 1 face of G.

Initialize a counter to 0 to keep track of how many half edges we've seen so we know when to stop. Choose an arbitrary starting edge, and perform an $O(1)$ check to make sure it's not the interior half-edge for one of the vertex-k subgraphs. If it is, then it's another $O(1)$ operation to find a correct starting point (this can be done by adding casework to navigate through the vertex k-subgraph).

Once at a valid starting point, follow the graph counterclockwise around the boundary of each face, popping half-edges off as they are encountered. Observe that throughout this process, the decision of "which edge to traverse next" can always be made in $O(1)$ time because the rim vertices of the vertex k subgraphs are canonically ordered.[23] □

Of course, we rarely need a full list of the faces, but enumerating one face vs. all of the faces turns out to be a difference of a constant in the average case if we use a strategy like the one above.

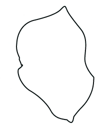
Proposition 4.11. *Determining whether we can perform a Reidemeister II move is $O(n)$ in number of crossings.*

Proof. Use an algorithm similar to the above. □

[22] Alternatively, we can encode the Gauss code Reidemeister moves directly in moves on the diagram graph.

[23] In terms of implementation, we can do something simple like representing each vertex by a tuple (v, label) where label encodes the under/over/in/out/mid information, or by representing the vertices as integers where $k^{\text{mid}} \mapsto 5k$, $k_u^{\text{out}} \mapsto 5k+1$, etc., and then using the remainder mod 5 to calculate the label.

4.3 Virtual Knots



While the diagram graph defined in the previous chapter is a nice computational tool for manipulating knots, from a more theoretical standpoint, it leaves quite a bit to be desired.

Ultimately, our purpose with the signed Gauss code was to try and identify a cleaner algebraic way of understanding knot theory in terms of purely combinatorial structure. Unfortunately, the planarity restrictions we had to place on Reidemeister II make it hard to see this panning out. What are we to do? We have a few options:

1) Abandon the signed Gauss code approach and search for something more fruitful,

2) Double down on it and work at building up a rich theory around the diagram graph, or

3) Extend our interpretation of "knot" to a new context where we don't have to worry about planarity concerns at all.

We'll choose option (3), which will lead us to the field of *virtual knot theory* (first introduced by Kauffman (1998)). The idea is to make signed Gauss codes our fundamental object of study, without including any concerns about planarity conditions. We'll give two small pieces of motivation before getting into the thick of it. One draws an analogy with defining the complex numbers, the other with drawing planar graphs on surfaces other than $\mathbb{R}^2$.

Question 1 (Motivation)**.** Is it possible to find an x such that $x^2 = -1$?

Answer 1. No. For any $x \in \mathbb{R}$ we have $x^2 \geq 0$. Hence such an x doesn't exist. □

But of course, we could use complex numbers instead to make things work out.

Answer 2. Define a new symbol i such that $i^2 = -1$. Then this gives the desired x.[a] □

[a]Or, for a more algebraic point of view, instead of thinking "$i^2 = -1$" we can view $\mathbb{C}$ as working in $\mathbb{R}[i]/\langle i^2 + 1 \rangle$. Same end result, but the explicit focus on modding out by an ideal might carry a different flavor for some people.

While we often take the complex numbers for granted, it's important to recognize that at first glance defining $i^2 = -1$ might seem like nonsense —

or at the very least, a cop-out. But after years of careful study, we now know that $\mathbb{C}$ can be very intuitive, and in many ways is actually nicer than $\mathbb{R}$. For instance, every polynomial in $\mathbb{C}$ of degree n has n roots in $\mathbb{C}$. The same property is not enjoyed by the real numbers.

This is analogous to the relationship between *classical knots* (the things we have been referring to as "knots" up to this point) and *virtual knots*. With virtual knots, every Gauss code has a corresponding knot. With *classical* knots, we can sometimes get Gauss codes that are comparable to $x^2 = -1$ — unless we extend our scope, it seems like there's no way to make sense of the statement. The extension comes from loosening the planarity constraints on the signed Gauss code. How do we interpret the result? To start, consider the following question.

Question 2 (Motivation)**.** Is it possible to draw the $K_{3,3}$ graph without any edge crossings?

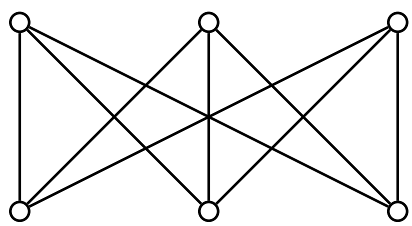


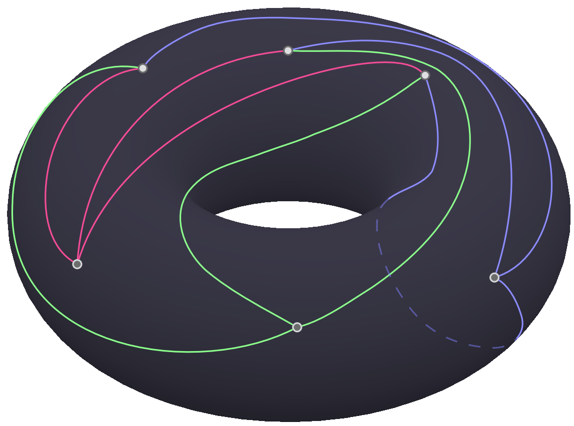
Figure 4.16 The $K_{3,3}$ graph.

We'll examine two answers.

Answer 1. No. One can show that in a planar graph, if there are no cycles of length 3, then $|E| \leq 2|V| - 4$. For $K_{3,3}$, one can verify that $|E| = 9$ and $|V = 6|$. But $9 \not\leq 2 \cdot 6 - 4 = 8$, so $K_{3,3}$ is non-planar. Hence it cannot be drawn without edge crossings. □

Nice! Rigorous, sensible, and to-the-point. Here's another answer.

Answer 2. Yeah just draw it on a donut.



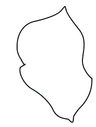


Figure 4.17 $K_{3,3}$ on a torus. Apologies for the strange color theme; due to the author's partial colorblindness, more standard palettes were hard to work with.

□

4.3.1 Definitions

We define (oriented) *virtual knots* in terms of signed Gauss codes.

Definition 4.5 (Virtual Knot)**.** A *virtual knot* is a string $\sqsupset$ of $2n$ characters such that for each $k = 1, \dots, n$, for some choice of $\epsilon_k \in \{+, -\}$, the symbols $k_u^{\epsilon_k}$, $k_o^{\epsilon_k}$ each appear in $\sqsupset$ exactly once. ◊

Compare with Definition 4.1. Here, instead of defining Gauss codes in terms of knots, we've defined knots in terms of Gauss codes. Returning to our analogy with $\mathbb{C}$ and $\mathbb{R}$, this would be like defining $\mathbb{C}$ as "the set of all roots of single-variable polynomials with real coefficients."[24]

We define equivalence for virtual knots as follows:

Definition 4.6 (Equivalence of Virtual Knots)**.** We say two virtual knots K_1, K_2 are *equivalent* if K_1 can be transformed into K_2 by a finite sequence of Gauss code Reidemeister moves (this time without the planarity constraints on Reidemeister II). ◊

How do we interpret virtual knots and their equivalence geometrically?

[24]To see this, note that for any $z \in \mathbb{C}$, $(x - z) \cdot (x - \overline{z})$ has real coefficients.

Treating this question in full would take us beyond the scope of our purposes today, but we will give a high-level overview.[25] Our first order of business is to define virtual knot diagrams.

Definition 4.7 (Virtual knot diagram)**.** A *virtual knot diagram* is defined identically to the classical case, only now we include special *virtual* crossings that we insert whenever we're forced to violate planarity to connect strands together. Virtual crossings are denoted by circles, as in Fig. 4.18. ◊

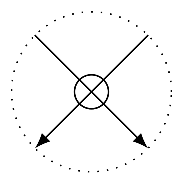


Figure 4.18 A virtual crossing

Note. Virtual crossings do not appear in the signed Gauss code. This is because there's a sense in which they're not really "there" — rather, they're an artifact of our projection into $\mathbb{R}^2$. We'll expand on this below.

Example 4.2. Consider the virtual knot given by $1_o^+ 2_o^+ 1_u^+ 2_u^+$. If we were to try and interpret this as a classical knot, we'd run into some problems:

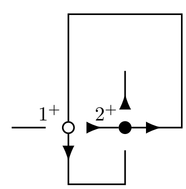
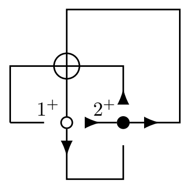


Figure 4.19 Now what?

Note, there's no way to connect the strand to crossing 2 in the desired manner without introducing a *new* crossing along the $(2_o^+, 1_o^+)$ semiarc. Hence, we employ a virtual crossing, which allows us to complete the diagram without problems.

[25] For more, the reader is encouraged to investigate Scott Carter et al. (2002)



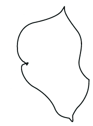


Figure 4.20 Virtual diagram

◊

How do we interpret the resulting "knot" as an embedding? A hint comes from **Motivating Question 2**. As we saw there, it's still possible to draw $K_{3,3}$ without having edges cross each other if we work on a torus. Recalling the connection between signed Gauss codes and planar graphs that we established through the diagram graph, it seems reasonable to think of virtual knots as knots that we draw on thickened surfaces.[26] In this context, we see that virtual crossings don't really represent *real* crossings of the strands in our knot. Rather, they're artifacts of our 2D representations.

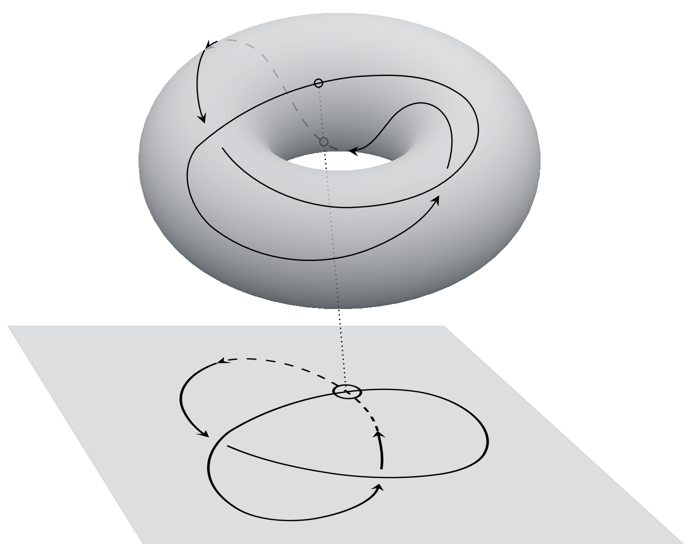


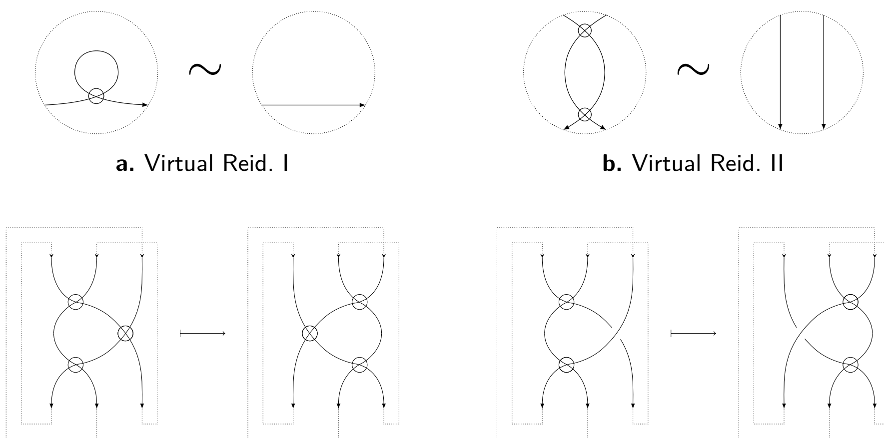
Figure 4.21 An example of how we might obtain something like Fig. 4.20.

Some care has to be taken in reworking our interpretation of what it means for knots to be "equivalent" in this new context — e.g., it seems

[26] We need $[-\varepsilon, \varepsilon]$ of wiggle room to let the strands pass over/under each other.

we might be able to get two inequivalent unknots by drawing closed loops longitudinally / latitudinally on the surface. We will not worry too much about this today; we're only interested in the combinatorial aspects. For more, the reader is encouraged to look at Scott Carter et al. (2002).

By carefully studying Fig. 4.21, the reader might realize that performing Reidemeister moves on the surface of the torus can actually introduce extra virtual crossings into our 2D projection.[27] This suggests that the purely diagrammatic Reidemeister moves are insufficient for manipulating virtual diagrams.[28] Indeed this is the case. To work purely in terms of diagrams, it's necessary to introduce an expanded moveset, which is given by adding the following four operations:[29]



a. Virtual Reid. I **b.** Virtual Reid. II

c. Virtual Reid. III **d.** Mixed Reid. III

Figure 4.22 The virtual moves. Note that in the cases of Virtual Reid. II and Virtual/Mixed Reid. III, we should really be displaying analogues for *all* of the classical moves (i.e., the different possible combinations of connectivities/orientations)

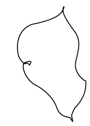
Together with the 3 standard diagrammatic Reidemeister moves, these are sufficient to fully encode equivalence of virtual knot diagrams.

[27] This can also be derived entirely from looking at the signed Gauss code Reidemeister moves without planarity constraints.

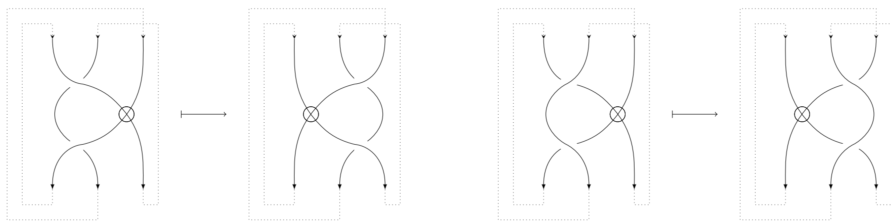
[28] Of course, by definition the signed Gauss code Reidemeister moves still suffice for virtual equivalence.

[29] Note that as with the Reidemeister moves, one should actually include all possible combinations of orientations / connectivities on the diagrams above.

4.4 A Very Brief Note on Unknotting Moves



It is very important to note that although they might look similar to the other virtual diagram moves, the following are *not* valid local modifications for virtual knot diagrams.[30]



a. Forbidden Move I **b.** Forbidden Move II

Figure 4.23 The Forbidden Moves

In fact, if we were to allow this move together with the others, it would be sufficient to unknot *any* knot, virtual or classical! See here[31] for an excellent step-by-step demonstration of unknotting the trefoil using these moves.

The forbidden moves are examples of *unknotting move*. These are typically defined in terms of modification rules for diagrams, but we'll define them below in terms of Gauss codes, since we're trying to move in more combinatorial directions.

Definition 4.8 (Unknotting Moves & Unknotting Sequence)**.** An *unknotting move* is a modification rule for the Gauss code such that, together with the Gauss code Reidemeister moves, the operation is sufficient to reduce any Gauss code to one for the unknot (e.g., the empty string).

Such a sequence of Gauss codes + Reidemeister moves is called a *unknotting sequence*. $\diamondsuit$

Example 4.3. The forbidden moves displayed above act on the Gauss code by exchanges of the form

$$(\ldots, i, \ldots, i, j, \ldots, j, \ldots) \mapsto (\ldots, i, \ldots, j, i, \ldots, j, \ldots)$$

together with the appropriate choices of $\{u, o\}$ and $\{+, -\}$ (note, the middle j, i always have the same u/o label). Adding in the versions with other connectivities

[30]Once more, one should really be considering all possible orientations / connectivities for these moves.

[31]In case the link doesn't work, the url is https://www1.cmc.edu/pages/faculty/VNelson/unknottingthetrefoil.html.

and orientations gets us moves of the form

$$(\dots, j, \dots, i, j, \dots, i, \dots) \mapsto (\dots, j, \dots, j, i, \dots, i, \dots),$$

again with the corresponding choices of $\{u, o\}$ and $\{+, -\}$. Hence we see that these moves allow for essentially arbitrary permutations of the Gauss code. $\lozenge$

One of the reasons unknotting moves are fascinating is because it seems they could offer us a nice way to define explicit algebraic structure on knots. For instance, consider the following: let K be a knot given by some Gauss code $\sqsupset_K$, and let $\sigma_1, \sigma_2, \dots, \sigma_n$ be an unknotting sequence for $\sqsupset_K$. If we use $\sqsupset_{\bigcirc}$ to denote the Gauss code for the unknot, we think of this process as follows:

$$(\sigma_n \circ \sigma_{n-1} \circ \cdots \circ \sigma_2 \circ \sigma_1)(\sqsupset_K) = \sqsupset_{\bigcirc}.$$

But then note, since the σ_i all have well-defined notions of "inverses," we can also write

$$\begin{aligned} \sqsupset_K &= (\sigma_n \circ \sigma_{n-1} \circ \cdots \circ \sigma_2 \circ \sigma_1)^{-1}(\sqsupset_K) \\ &= \left(\sigma_1^{-1} \circ \sigma_2^{-1} \circ \cdots \circ \sigma_{n-1}^{-1} \circ \sigma_n^{-1}\right)(\sqsupset_K). \end{aligned}$$

That is, we can express K in terms of a "recipe" for building $\sqsupset_K$ from the unknot! This idea becomes even more tantalizing when we see just how simple these moves can be:

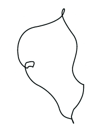
Proposition 4.12. *Let K be a* classical *knot represented by a Gauss code $\sqsupset_K$. Then being allowed to exchange $k_u^{\epsilon_k}$, $k_o^{\epsilon_k}$ arbitrarily yields an unknotting move.*

Sketch. Note, this corresponds to switching which strand is on top vs. on bottom at any given crossing. Using this, one can switch the crossings until $\sqsupset_K$ represents a knot that is always passing "underneath" itself.[32] A simple pigeonhole principle argument can be applied to show that there must always exist a simplifying Reidemeister I or Reidemeister II move in this situation. $\square$

Remark. Using this, one can show that the move defined by "flipping k consecutive crossings" is an unknotting move as well. The idea is that we can derive the single-crossing flip by simply inserting $k-1$ Reidemeister I loops into $\sqsupset_K$ after some crossing, flipping all k of these, and then removing the $k-1$ Reidemeister I loops.

[32] In terms of the Gauss code: If one considers the sub-string consisting of the first time we encounter each crossing k, all of them would be labeled with overcrossings.

Note too that the single-crossing-flip can be used to derive the k-crossing version. In this sense, the two are "equivalent." For more on these ideas, see Nakanishi (1994).



One might note that the unknotting moves we've discussed so far appear to have a very "symmetric group" flavor to them. Indeed, both of our examples seem to be encoding transpositions of some flavor, which we might recall are the generators for S_n. We spend the next chapter describing one way of making this explicit. Before that, we have a brief discussion of how *if* we were able to make this unknotting move representation algebraically well-defined in an efficiently-computable way, then we would be able to reduce knot equivalence to the unknot detection problem.

4.4.1 Addendum: Reducing Knot Equivalence to the Unknotting Problem

Currently, we do not have an NP solution for determining knot equivalence (Lackenby (2016)). However, we do have an NP solution for determining if a given knot is unknotted (Lackenby (2013)). *If* something like the following were true, then we would be able to explicitly construct a reduction showing knot equivalence is NP. We should be very clear that we expect Problem 4.1 is almost certainly impossible, but we have no proof of this. We remain hopeful that a related approach could prove fruitful (or at the very least, interesting), which is why we have chosen not to omit its mention.

Problem 4.1. Let K_0, K_1 be tame knots represented by Gauss codes $\sqsupset_{K_0}$, $\sqsupset_{K_1}$. Given an unknotting sequence Σ_{K_0} for K_0, does there exist a polynomial-time algorithm for converting Σ_{K_0} to Σ_{K_1}, such that Σ_{K_1} unknots K_1 iff $K_0 \cong K_1$? $\diamondsuit$

We actually need a second proposition as well, but it is straightforward in the case of classical knots. We have not considered the virtual case, but we imagine an analogous version holds.

Proposition 4.13. *Given an n-crossing classical knot K represented by a Gauss code $\sqsupset_K$, there exists an $O(n)$ time algorithm for computing an unknotting sequence reducing $\sqsupset_K$ to a (non-simplified) Gauss code for the unknot.*

Proof. One can simply traverse $\sqsupset_K$, checking at each crossing $k_{x_k}^{\epsilon_k}$ whether we have encountered k in the Gauss code already or not. If we haven't, then we apply a crossing-flip move if needed to ensure $x_k = o$. Else, we continue. Per the argument in Proposition 4.12, the resulting Gauss code represents an unknot. $\square$

This gives us the following.

Proposition 4.14. *If Problem 4.1 is true, then the (classical) knot equivalence problem is polynomial-time reducible to the unknotting problem, and hence NP.*

Sketch. Given K_0, compute Σ_{K_0}. Apply Problem 4.1 to yield Σ_{K_1}. Apply Σ_{K_1} to K_1. Then the result is an unknot iff $K_0 \cong K_1$, hence we have a polynomial-time reduction to unknot detection. □

As we've said, Problem 4.1 seems like far too much to hope for. However, other approaches can yield similar reductions, and we wonder whether there's one that could yield a feasible approach to demonstrating knot equivalence is NP. For now, however, this seems out of reach.

We'll now switch our focus to examining an algebraic formalism for knots that encodes the idea of "unknotting moves *generating* knots, together with Reidemeister equivalence." This is the focus of Chapter 5.

Chapter 5

Connections to S_n

till Max said "BE STILL!"
and tamed them with the magic trick
of staring into all their yellow eyes without blinking once
and they were frightened and called him the most wild thing of all
and made him king of all wild things.

—Maurice Sendak, *Where the Wild Things Are*

In leaving off the last chapter, we mentioned that unknotting moves appear reminiscent of actions of the symmetric group (in that they involve permuting the orderings of our characters in the Gauss code). In particular, we saw that the forbidden moves allow us to perform near-arbitrary exchanges of adjacent characters in the Gauss code. We also saw that unknotting moves allow us to think about a Gauss code $\sqsupset_K$ as being "generated" by applying an unknotting sequence in reverse to the code for the unknot, $\sqsupset_{\bigcirc}$. In this chapter we want to try and make this correspondence more explicit by finding a way to view knots as a byproduct of group-like structure.

For our particular approach, we will describe knots in terms of elements of the Symmetric group for a countable set acting on a string of characters, which will take the form of a standard Gauss code for the unknot. We should stress that equivalence will *not* translate into a clean set of rules here — in fact, we have not yet found an explicit formalism for incorporating it — nevertheless in computational searches, the formalism seems to yield interesting behavior.

The main challenge for the project comes from the fact that knots can have different numbers of crossings. Hence it's not clear what we're supposed to do if somebody hands us a permutation encoding the move "flip crossing

17" when all we have is a trefoil. Choosing crossings arbitrarily is no good in this situation, since flipping different crossings in a knot generally yields different effects. For example, twist knots can be unknotted with a single move by flipping one of the crossings in the clasp, while flipping one of the crossings in the twisted portion generally just reduces the number of crossings by 2.[1]

One solution is to use the fact that Reidemeister I and II moves *insert* or *delete* elements from our string to add /remove extra crossings until we can interpret the result. But this would make our rules convoluted to say the least, and would raise all sorts of questions about how/where we should be inserting crossings to get the expressions to match. Hence this is undesirable as well.

What we'd really like is a way to view everything — Reidemeister moves, unknotting moves, and all the moves in between — in terms of permutations, where nothing is ever "added" or "deleted."The solution we propose is to choose our canonical unknot string to be one that already contains every possible crossing. In this light, we can view Reidemeister I and II as just moving these crossings to new locations.

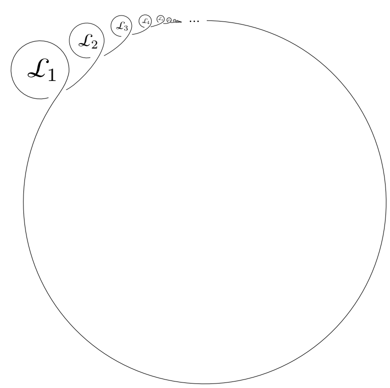
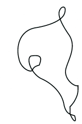


Figure 5.1 An unknot that contains countably many crossings.

It's not immediately clear that this is something we can *do* without breaking ambient isotopy, since the result doesn't immediately appear tame. It turns out that it works, and justifying this is the first order of business for this chapter.

[1]The exception being the trefoil and figure-eight knots.

5.1 Countable Reidmeister I Moves



In addition to helping us establish connections to permutations, the theorems in this section will also form the inspiration for our examination of general topological embeddings in Part III. The proofs below will not use uniform convergence (later ones will), and we won't go as deep in investigating the broader implications of our results so as to avoid going too far afield.[2] We begin with a straightforward lemma.

Lemma 5.1. *Let $\{V_n\}_{n\in\mathbb{N}}$ be a collection of closed, pairwise-disjoint subsets of $\mathbb{R}^3$. For all $n \in \mathbb{N}$, let $f_n : \mathbb{R}^3 \to \mathbb{R}^3$ be a homeomorphism such that f_n is identity on $\mathbb{R}^3 \setminus V_n^\circ$. Suppose too that*

$$\lim_{n\to\infty} \operatorname{diam}(V_n) = 0.$$

Then f defined by

$$f(x) = \begin{cases} f_1(x) & \textit{if } x \in V_1 \\ f_2(x) & \textit{if } x \in V_2 \\ \quad\vdots & \\ f_n(x) & \textit{if } x \in V_n \\ \quad\vdots & \\ \quad x & \textit{if } x \textit{ is not in any } V_n \end{cases}$$

is a homeomorphism.

Proof. First, we show f is a bijection by defining an inverse. Let

$$g(x) = \begin{cases} f_1^{-1}(x) & \text{if } x \in V_1 \\ f_2^{-1}(x) & \text{if } x \in V_2 \\ \quad\vdots & \\ f_n^{-1}(x) & \text{if } x \in V_n \\ \quad\vdots & \\ \quad x & \text{if } x \text{ is not in any } V_n. \end{cases}$$

[2]The uniform convergence framework is more powerful, but it also is *much easier* to accidentally misuse. Essentially, it will allow us to drop the "disjointness" conditions in our theorems below, but this will sometimes causes us to lose bijectivity on the ambient space and thus break our ambient isotopies.

Let $x \in \mathbb{R}^3$ be arbitrary; we want to show $g(f(x)) = x$. We have two simple subcases.

1) Suppose that for some $n \in \mathbb{N}$, we have $x \in V_n$. Then $g(f(x)) = g(f_n(x)) = f_n^{-1}(f(x)) = x$.

2) Suppose that there exists no $n \in \mathbb{N}$ with $x \in V_n$. Then $g(f(x)) = g(x) = x$.

Hence $g = f^{-1}$, so f is a bijection (as desired).

Now, we want to show f is continuous. There are any number of ways to do this; we choose a more topologically-flavored argument. Observe that for all $n \in \mathbb{N}$, $f|_{V_n} = f_n|_{V_n}$, which is continuous. Hence f is continuous on each V_n. One can then apply a sequential continuity argument to show f is continuous on $X_0 = \overline{\bigcup_{n\in\mathbb{N}} V_n}$.[3]

Now observe that $X_q = \mathbb{R}^3 \setminus \bigcup_{n\in\mathbb{N}} V_n^\circ$ is closed, and that $f|_{X_1} = \text{Identity}|_{X_1}$, which is continuous. Thus f is continuous on X_1. Now, since $X_0 \cup X_1 = \mathbb{R}^3$ and f is continuous on both, the gluing lemma gives us that f is continuous on $\mathbb{R}^3$.

Applying an identical argument for f^{-1} shows it is continuous as well. Hence f is a homeomorphism. $\square$

Remark. By taking all but finitely many of the f_n to be identity, we get the same result for finite collections of homeomorphisms.

Remark. One can drop the $\text{diam}(V_n) \to 0$ condition by assuming the collection of V_n is *locally-finite* instead, but this precludes us from packing countably-many V_n's into a compact space, so we won't use it.

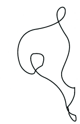
We have one more elementary lemma before we prove the analogue for ambient isotopy.

Lemma 5.2. *Let* $K_0 : [0,1] \times \mathbb{R}^3 \hookrightarrow \mathbb{R}^3$*, and let* $F : [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$*. Then* $K_1 : S^1 \hookrightarrow \mathbb{R}^3$ *defined by*

$$K_1(s) = F(1, K_0(s))$$

is an embedding.

[3] The only sneaky case comes in noticing that $x \in X_0$ does not require $x \in V_n$ for some n. To address it: note that the only other case is if x is a limit point of $\bigcup_{n\in\mathbb{N}}$. Then note that because $\text{diam}(V_n) \to 0$, one can apply a sequential continuity argument to obtain the desired result.



Proof. By definition of an ambient isotopy, $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(x) = F(1, x)$ is a homeomorphism. By definition of an embedding, K_0 is a homeomorphism onto its image, hence $f \circ K_0 = K_1$ is too. Thus K_1 is an embedding. $\square$

We now prove our first main result, which is essentially an extension of Lemma 5.1 above to ambient isotopy.

Theorem 5.3 (Countable Concatenations of Ambient Isotopies)**.** *Let $(K_n)_{n\in\mathbb{N}}$ be a collection of embeddings from $S^1 \hookrightarrow \mathbb{R}^3$. Now, suppose that for all $n \in \mathbb{N}$ there exists a closed set V_n such that*

1. *$K_n \cong K_{n+1}$ by an ambient isotopy $F_n : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}^3$ such that F_n is identity on $[0, 1] \times (\mathbb{R}^3 \setminus V_n^\circ)$,*
2. *For all $n \neq m$, $V_n \cap V_m = \varnothing$, and*
3. *We have*
$$\lim_{n\to\infty} \operatorname{diam}(V_n) = 0.$$

Then $K_{\lim} = \lim_{n\to\infty} K_n$ is an embedding, and $K_1 \cong K_{\lim}$.

Proof. In light of Lemma 5.2, that $K_{\lim}$ is an embedding will follow from our construction of an ambient isotopy from K_0 to $K_{\lim}$.

To that end, define $F : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}^3$ by

$$F(t, x) = \begin{cases} F_1(t, x) & \text{if } x \in V_1, \\ F_2(t, x) & \text{if } x \in V_2, \\ \quad\vdots & \\ F_{n-1}(t, x) & \text{if } x \in V_{n-1}, \\ F_n(t, x) & \text{if } x \in V_n, \\ \quad\vdots & \\ \quad x & \text{otherwise.} \end{cases}$$

We want to show F is an ambient isotopy.[4]

Observe that for all $x \in \mathbb{R}^3$, $F(0, x) = x$, and for all $s \in S^1$, $F(1, K_1(s)) = K_{\lim}(s)$ (by construction). Now observe that by definition of the an ambient

[4] To that end, recall that we need to show (1) $F(0, x) = x$, (2) For all $s \in S^1$, $F(1, K_1(s)) = K_{\lim}(s)$, (3) For all $t \in [0, 1]$, $F(t, \cdot)$ is a homeomorphism, and (4) F is continuous.

isotopy, for all $t \in [0,1]$ and $n \in \mathbb{N}$, $f_{t,n}(x) = F_n(t,x)$ is a homeomorphism. Now, since for all $t \in [0,1]$ we have

$$F(t,x) = \begin{cases} F_1(t,x) & \text{if } x \in V_1, \\ F_2(t,x) & \text{if } x \in V_2, \\ \quad\vdots & \\ F_{n-1}(t,x) & \text{if } x \in V_{n-1}, \\ F_n(t,x) & \text{if } x \in V_n, \\ \quad\vdots & \\ \quad x & \text{otherwise} \end{cases}$$

$$= \begin{cases} f_{t,1}(x) & \text{if } x \in V_1, \\ f_{t,2}(x) & \text{if } x \in V_2, \\ \quad\vdots & \\ f_{t,n-1}(x) & \text{if } x \in V_{n-1}, \\ f_{t,n}(x) & \text{if } x \in V_n, \\ \quad\vdots & \\ \quad x & \text{otherwise,} \end{cases}$$

Lemma 5.1 implies $F(t,\cdot)$ is a homeomorphism. Hence, it just remains to show that F is continuous. This follows from the gluing lemma.

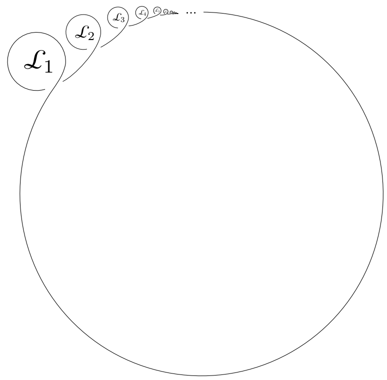
Note that for all $n \in \mathbb{N}$, $A_n = [0,1] \times V_n$ is a closed set with $F|_{A_n} = F_n|_{A_n}$. Hence F is continuous on A_n. Defining $X_0 = \overline{\bigcup_{n\in\mathbb{N}} A_n}$, we can use a similar sequential continuity argument as in Lemma 5.1 to show F is continuous on X_0.[5]

Similarly, observe that $X_1 = \mathbb{R}^3 \setminus \bigcup_{n\in\mathbb{N}} V_n^\circ$ is closed, and $F|_{X_1} = \text{Identity}|_{X_1}$, hence F is continuous here as well.

It follows that F is continuous, hence F is an ambient isotopy. $\square$

Proposition 5.4. *The countable Reidemeister I move example we described above can be realized as the kind of ambient isotopy described in Theorem 5.3.*

[5] Again, the key comes from $\operatorname{diam}(V_n) \to 0$ implying sequential continuity.



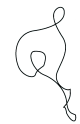


Figure 5.2 A countable sequence of Reidemeister I moves, for the 3rd time.

Sketch. Without loss of generality, suppose the diagram in Fig. 5.2 represents a projection onto the xy plane. Observe that none of the loops in the diagram overlap, so we can find closed sets U_n separating each of them when viewed as subsets of $\mathbb{R}^2$. Taking $V_n = U_n \times \mathbb{R}$ for each n, we get the desired closed subsets of $\mathbb{R}^3$. One can then use Reidemeister's theorem to argue that we can find ambient isotopies F_n representing the process of 〔inserting a single loop in the strand in V_n while keeping the boundary fixed〕. Applying Theorem 5.3, we obtain the desired result. $\square$

Example 5.1. In a similar manner, one can also employ this result to show that cases like the following are possible:

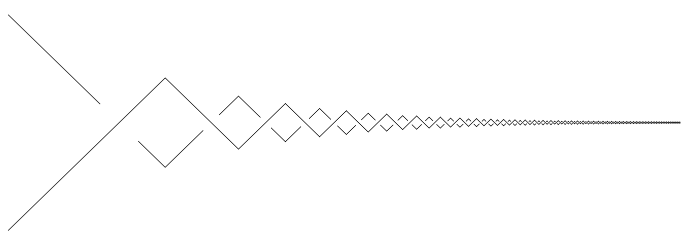


Figure 5.3 Countable Reidemeister II

However, some care is required. Note that it's not immediately obvious how we could define the disjoint closed sets V_n — in fact, it seems impossible. The trick is to do it in parts. Suppose we start with a caret-shaped arc, like the following:

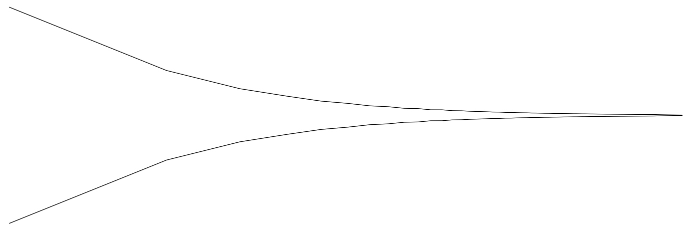


Figure 5.4 A caret-shaped arc

Insert half of the moves as follows (note, the drawing software distorted the figure slightly, so the correspondence with Fig. 5.3 might be a bit hard to see):

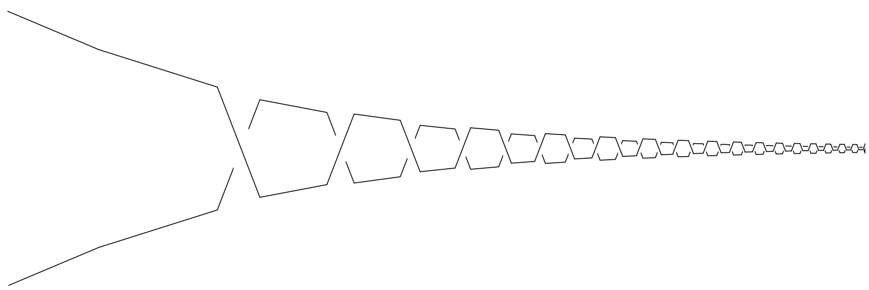


Figure 5.5 Half of the Reidemeister II moves

This can be achieved by taking the V_n's to be the dotted regions shown in the below.

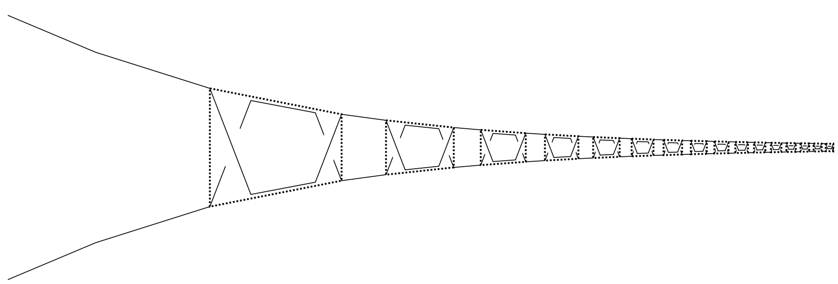
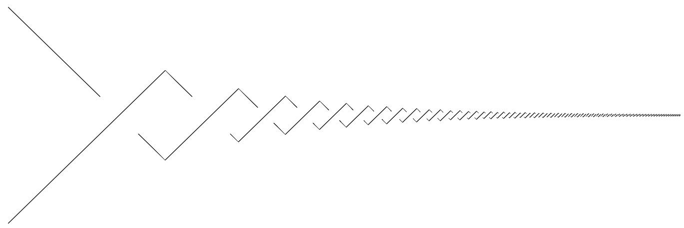
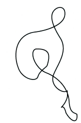


Figure 5.6 The closed neighborhoods, shown with dotted lines

After this, one can apply a similar technique to the "flat" strands to yield a figure like Fig. 5.3. $\Diamond$

Remark. Although the result might look very similar Fig. 5.3, to the best of our knowledge, the following figure *cannot* be obtained from Theorem 5.3.





For this, we need the uniform convergence form of Theorem 5.3, which we prove in Theorem 7.4.

5.1.1 An Alternative Argument: A C^1 Parameterization

In case the arguments above are not convincing, we have constructed an explicit parameterization of an arc with a diagram similar to that of Fig. 5.2 that is C^1 embedded in $\mathbb{R}^3$.[6] The idea behind the parameterization is as follows: take a big circle and a small circle, where the center of the small circle lies on the circumference of the big circle, and the plane of the small circle is normal to the tangent vector of the big circle. Let θ denote the polar angle of the big circle, and r its radius. As $\theta \to 0$, we'll make the radius of the small circle decay like $r\theta^3$, while watching a point on its circumference oscillate with frequency $\propto \frac{\omega}{\theta}$. C^1-ness will then follow from the C^1-ness of the function

$$g(\theta) = \begin{cases} r\theta^3 \sin\left(\frac{\omega}{\theta}\right) & \theta \neq 0 \\ 0 & \theta = 0 \end{cases} \tag{5.1}$$

for $r, \omega > 0$.

Proposition 5.5. *Consider $f : [0, \pi/3] \to \mathbb{R}^3$ defined[7] by: $f(0) = 0$, and for all $\theta \in (0, \pi/3]$,*

$$f(\theta) = \underbrace{\begin{bmatrix} 0 & \frac{\cos(\theta)}{r} & 0 \\ 0 & \frac{\sin(\theta)}{r} & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \theta^3 \sin\left(\frac{\omega}{\theta}\right) \\ \theta^3 \cos\left(\frac{\omega}{\theta}\right) \\ 0 \end{bmatrix}}_{\boldsymbol{v}} + \underbrace{\begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \\ 0 \end{bmatrix}}_{\mathbf{v}_0}$$

[6]This is proven in Crowell and Fox (1963) to be sufficient to guarantee tameness — see Appendix A for more details, as well as some other examples.

[7]We choose the domain $[0, \pi/3]$ just because it looks nice with the rest of the parameters we chose when plotting. This quirk can be removed by inserting constants in various places.

$$= \begin{bmatrix} \frac{t^3\cos(t)\cos(\omega/t)}{r} + r\cos(\theta) \\ \frac{t^3\cos(t)\sin(\omega/t)}{r} + r\sin(\theta) \\ t^3\sin(\omega/t) \end{bmatrix}$$

Then f is C^1 on $(0, \pi/3)$, and $\lim_{\theta\to 0^+} f'(\theta) = (0, r, 0)$.[8]

Remark. In the equations above, M rotates us into the orthogonal frame for the big circle's tangent vector (here, the big circle is parameterized to lie in the xy-plane), $\mathbf{v}$ represents the point on the circumference of the small circle, and $\mathbf{v}_0$ shifts the result so that the center of the small circle would lie on the circumference of the big circle.

Proof. One can argue that $\lim_{\theta\to 0} f'(\theta)$ exists by noting that the function $g(\theta)$ defined in Eq. (5.1) is C^1, and then using the product rule, etc. Feeding it into `Mathematica` also works.

It follows that $f(\theta)$ can be extended to a C^1 function $f : S^1 \hookrightarrow \mathbb{R}^3$. $\square$

We now provide some plots. Code for an interactive 3D version using `Julia` can be found in Appendix C.2

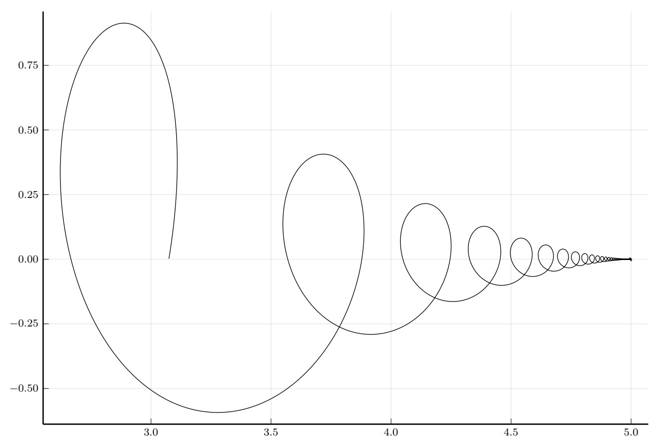
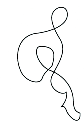


Figure 5.7 The parameterized function with $r = 5$, $\omega = 2\pi^2$, here projected onto the xz plane.

[8] In fact $\lim_{\theta\to 0} f'(\theta) = (0, r, 0)$ as well (not just from the 0^+ side), we just included right-handed limit to emphasize that we've only defined f on $[0, \pi/3]$. On that note, $\lim_{\theta\to\pi/3^-} f'(\theta)$ exists as well, but that's not really a surprise — the only place things could really go wrong are at $\theta = 0$.



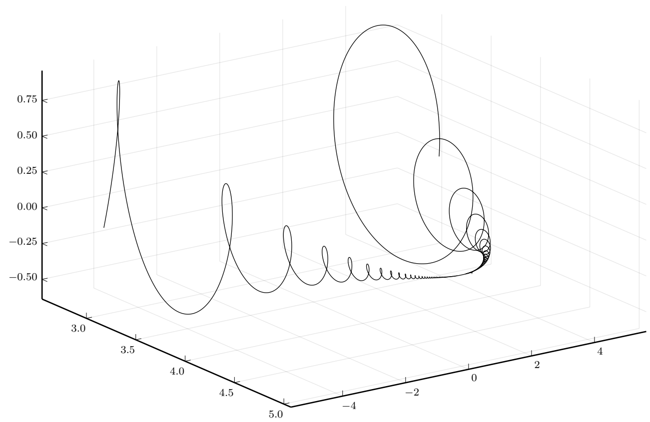


Figure 5.8 A 3D view of a version defined over $[-\pi/3, \pi/3]$.

Note, if we were to use θ^2 instead of θ^3, we'd lose the C^1 condition, but it'd still yield some pretty plots.

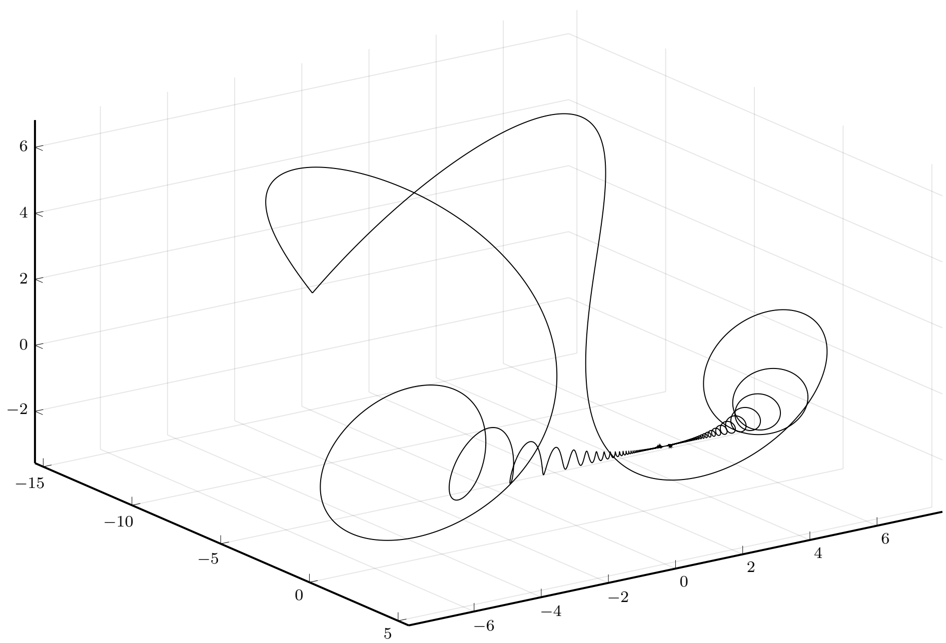


Figure 5.9 Using θ^2 instead

So, to recap: we've shown that under certain mild conditions, we can apply countably-many Reidemeister moves to our knots and still yield tame embeddings. We use this to create our desired representation in terms of S_n.

5.2 Using Countable-Crossings to Define an Action

The effects of the Proposition 5.4 can be summarized as "we can create many bounded, tame arcs that have infinitely many crossings." We apply this to create our combinatorial representation. Note that in our creation of a "standard unknot code," we're imagining a slightly modified version of Proposition 5.4 in which we insert *both* a $+$ and a $-$ version of crossing k using Reidemeister I moves. While it might seem strange to have *two* crossings labeled by k in our Gauss codes, this will turn out to be more natural for our purposes. Further, we'll see that only one can appear nontrivially (i.e., not locally removable by a Reidemeister I move) at any given time.

In deciding how to add both of these $+/-$ characters, we've elected to define the standard unknot in a way that the process corresponds to the insertion of a framed Reidemeister I move for each k.

Definition 5.1 (Standard Gauss sequence for the unknot)**.** The *standard Gauss sequence for the unknot*, denoted $\sqsupset_{\bigcirc}$, is defined by

$$\begin{aligned}\sqsupset_{\bigcirc} &= \mathop{\#}_{k=1}^{\infty}(k_u^+, k_o^+, k_o^-, k_u^-) \\ &= 1_u^+, 1_o^+, 1_o^-, 1_u^-, 2_u^+, 2_o^+, 2_o^-, 2_u^-, \dots, n_u^+, n_o^+, n_o^-, n_u^-, \dots \qquad \Diamond\end{aligned}$$

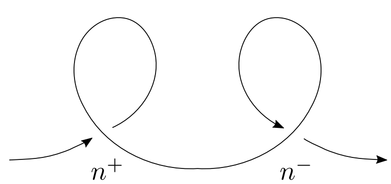
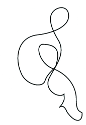


Figure 5.10 An example of one of the inserted blocks

Question 1. Is this the most natural choice of canonical representative for $\bigcirc$? After reading the below and seeing the way we try and use $\sqsupset_{\bigcirc}$ in our Symmetric group formalism, the reader is encouraged to ponder this question. The author would be excited to hear any insights! $\Diamond$



We use this to define the standard Gauss sequence for a knot K in a way that will be germane to viewing it in terms of permutations. The idea is very simple (we're basically just inserting the symbols that are missing from $\sqsupset_{\bigcirc}$ in $\sqsupset_K$ in the places they would be in $\sqsupset_{\bigcirc}$), but the notation in the below is almost laughably verbose. This is because we were trying to define things as precisely as possible to make translation into computer programs easier. We encourage the reader to reference the examples if the definition is hard to parse.

Definition 5.2 (Standard Gauss Sequence)**.** Let K be an n-crossing knot given by some signed Gauss code $\sqsupset_K = k^{\epsilon_1}_{1,x_1}, k^{\epsilon_2}_{2,x_2}, \dots, k^{\epsilon_{2n}}_{2n,x_{2n}}$. Note, in $\sqsupset_K$ we have indexed ϵ_i by $i = 1, \dots, 2n$ to describe the sign of the 〔crossing partially represented by the i^{th} symbol in K〕.[9] We'll define a related set of symbols (here labeled as $\neg\mu_j$ over $j = 1, \dots, n$) by "$\neg\mu_j$ is the *opposite* of the sign of crossing j in K." Again, to be extra clear, note the difference in indexing — i^{th} symbol, j^{th} crossing. The symbol μ is chosen just to avoid confusion with the ϵ's; the $\neg$ is to emphasize the fact that we want the opposite of the sign of crossing k.

We'll also define symbols α_j, β_j by

- If $\neg\mu_j = -$, then $\alpha_j = o$, $\beta_j = u$.
- If $\neg\mu_j = +$, then $\alpha_j = u$, $\beta_j = o$.

Use these to construct a sequence of n 4-symbol blocks b_j as follows:

- If $\neg\mu_j = -$, then b_j is given by

$$b_j = k^{\epsilon_{2j-1}}_{2j-1,x_{2j-1}} k^{\epsilon_{2j}}_{2j,x_{2j}} j^{\neg\mu_j}_{\alpha_j} j^{\neg\mu_j}_{\beta_j},$$

  and

- If $\neg\mu_j = +$, then b_j is defined by

$$b_j = j^{\neg\mu_j}_{\alpha_j} j^{\neg\mu_j}_{\beta_j} k^{\epsilon_{2j-1}}_{2j-1,x_{2j-1}} k^{\epsilon_{2j}}_{2j,x_{2j}}.$$

Then we overwrite the definition of $\sqsupset_K$ to be a string of $4n$ symbols constructed by concatenating the b_j,

$$\sqsupset^{\text{fin}}_K = \mathop{\#\!\!\#}_{j=1}^{n} b_j.$$

We call this the *finite portion* of the *standard Gauss sequence representation*. The full *standard Gauss sequence representation* is given by

$$\sqsupset_K = \sqsupset^{\text{fin}}_K \mathbin{\#\!\!\#} \left(\mathop{\#\!\!\#}_{k=2n+1}^{\infty} k^+_u, k^+_o, k^-_o, k^-_u \right). \qquad \lozenge$$

[9] "Partially" because each crossing is encoded by two symbols in $\sqsupset_K$

From now on, when we write $\sqsupset_K$, unless otherwise stated, we are thinking of the standard Gauss sequence representation. The following proposition asserts that $\sqsupset_K$ is realizable by ambient isotopy (up to relabeling to ensure both the $+$ and $-$ copies of crossing k are included).

Proposition 5.6. *Let K be an n-crossing knot. Then the underlying geometric representative for K is ambient isotopic to one realizing the standard Gauss sequence for K (with some minor relabeling to ensure both k^+, k^- appear).*

Proof. Immediate from the construction of $\sqsupset_K$. Insert the finitely-many Reidemeister I moves as dictated by the $j_{\alpha_j}^{\neg\mu_j}$, $j_{\beta_j}^{\neg\mu_j}$ to construct $\sqsupset_K^{\text{fin}}$, and then apply Theorem 5.3 to yield the infinite tail. □

We now provide some examples of the $\sqsupset_K^{\text{fin}}$, and show how they compare to equivalent-length standard unknot sequences.

Note. IMPORTANT! We *highly* recommend the reader try interacting with some of the concepts below programmatically. If desired, `sage`[10] code can be found open-source here[11] on Github.

At the time of writing, the code is mainly in the prototyping phase, hence documentation is not extensive, and we have not done thorough bug-testing. However, basic functionality appears to be quite solid. If the reader has any questions and/or encounters a bug, we heavily encourage them to submit an issue report[12], or to contact the author at `fkobayashi@g.hmc.edu`.

Example 5.2 (Standard Gauss Sequences)**.** In the following, we'll highlight the newly-inserted portions gray.

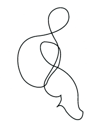
- Consider the trefoil given by $1_u^+, 2_o^+, 3_u^+, 1_o^+, 2_u^+, 3_o^+$. Then $\sqsupset_{(3,1)}^{\text{fin}}$ is given by

$$\sqsupset_{(3,1)}^{\text{fin}} = 1_u^+, 2_o^+, 1_o^-, 1_u^-, 3_u^+, 1_o^+, 2_o^-, 2_u^-, 2_u^+, 3_o^+, 3_o^-, 3_u^-.$$

  Compare this to the finite part of the standard unknot sequence of the same length. We've grayed out the same portions in the $\sqsupset_\bigcirc$ as in $\sqsupset_{(3,1)}^{\text{fin}}$ to focus the reader's eyes on the differences.

$$\sqsupset_\bigcirc = 1_u^+, 1_o^+, 1_o^-, 1_u^-, 2_u^+, 2_o^+, 2_o^-, 2_u^-, 3_u^+, 3_o^+, 3_o^-, 3_u^-$$
$$\sqsupset_{(3,1)}^{\text{fin}} = 1_u^+, 2_o^+, 1_o^-, 1_u^-, 3_u^+, 1_o^+, 2_o^-, 2_u^-, 2_u^+, 3_o^+, 3_o^-, 3_u^-$$

[10] If the link doesn't work: https://www.sagemath.org/
[11] If the link doesn't work: https://github.com/redpanda1234/permutation-knots
[12] If the link doesn't work: https://github.com/redpanda1234/permutation-knots/issues



- The purpose of the definition of the b_j's is more apparent in knots where not every crossing is the same sign. Consider the figure eight knot given by Gauss code $1_u^-, 2_o^-, 3_u^+, 4_o^+, 2_u^-, 1_o^-, 4_u^+, 3_o^+$. Then compare $\sqsupset_{\bigcirc}$ and $\sqsupset_{(4,1)}^{\text{fin}}$:

$$\sqsupset_{\bigcirc} = 1_u^+1_o^+, 1_o^-1_u^-, 2_u^+2_o^+, 2_o^-2_u^-, 3_u^+3_o^+, 3_o^-3_u^-, 4_u^+4_o^+, 4_o^-4_u^-$$
$$\sqsupset_{(4,1)}^{\text{fin}} = 1_u^+1_o^+, 1_u^-2_o^-, 2_u^+2_o^+, 3_u^+4_o^+, 2_u^-1_o^-, 3_o^-3_u^-, 4_u^+3_o^+, 4_o^-4_u^-$$

- Finally, we give the example of the $(6,1)$ knot given by

$$1_u^+, 2_o^+, 3_u^-, 4_o^-, 5_u^-, 6_o^-, 2_u^+, 1_o^+, 6_u^-, 5_o^-, 4_u^-, 3_o^-.$$

  The codes are

$$\sqsupset_{\bigcirc} = 1_u^+1_o^+, 1_o^-1_u^-, 2_u^+2_o^+, 2_o^-2_u^-, 3_u^+3_o^+, 3_o^-3_u^-, 4_u^+4_o^+, 4_o^-4_u^-, 5_u^+5_o^+, 5_o^-5_u^-, 6_u^+6_o^+, 6_o^-6_u^-$$
$$\sqsupset_{(6,1)}^{\text{fin}} = 1_u^+2_o^+, 1_o^-1_u^-, 3_u^-4_o^-, 2_o^-2_u^-, 3_u^+3_o^+, 5_u^-6_o^-, 4_u^+4_o^+, 2_u^+1_o^+, 5_u^+5_o^+, 6_u^-5_o^-, 6_u^+6_o^+, 4_u^-3_o^-$$

$\Diamond$

The point of the above is that we can think of signed Gauss codes as infinite sequences where finitely many of the terms have been derranged. Using this, we can begin to formally establish connections between knot equivalence and the finitary symmetric group on a countable set. Recall the following:

Definition 5.3 (Group Action)**.** Let $(G, \cdot)$ be a group with identity element e, and let S be a set. Let $\varphi : G \times S \to S$ such that for all $s \in S$,

1. $\varphi(e, s) = s$, and
2. For all $g, h \in G$,
$$\varphi(g \cdot h, e) = \varphi(g, \varphi(h, e)) \qquad \Diamond$$

Definition 5.4 (Finitary Symmetric Group)**.** Let $\mathcal{N}$ be a countable set. Then the set $S_{\aleph_0}$ of all bijections $f : \mathcal{N} \to \mathcal{N}$ forms a group under function composition. Consider the subgroup S_{fin} defined by

$$S_{\text{fin}} = \{\sigma \in S_{\aleph_0} \mid \sigma \text{ fixes all but finitely many } n \in \mathcal{N}\}.$$

We call S_{fin} the *finitary symmetric group* on $\mathcal{N}$. $\Diamond$

Given the definitions above, we'll usually think about $\mathcal{N}$ as being a collection of the characters

$$\mathcal{N} = \bigcup_{k \in \mathbb{N}} \{k_u^+, k_o^+, k_o^-, k_u^-\},$$

in which case we denote $\mathcal{N}$ by $\mathcal{N}_{\text{str}}$. However, we'll note that there's another interesting set of characters we can use that yields some nice properties.

Definition 5.5 ($\mathbb{Z}[i]$ Representation of Gauss sequences)**.** For the sake of notational compactness, let $\mathcal{N}_{\text{g.int}} = \mathbb{Z} \cup i\mathbb{Z}$ (i.e., the set of all purely real or purely imaginary elements of $\mathbb{Z}[i]$).[13] Define a bijection $f : \mathcal{N}_{\text{str}} \to \mathcal{N}_{\text{g.int}}$ by

$$\begin{aligned} f(k_u^+) &= -k & \qquad f(k_u^-) &= -k \cdot i \\ f(k_o^+) &= k & \qquad f(k_o^-) &= k \cdot i \end{aligned}$$

observe that f is indeed a valid bijection. Hence, using $\mathcal{N}_{\text{g.int}}$ as our set of label characters is fully equivalent to using $\mathcal{N}_{\text{str}}$. $\diamondsuit$

Observe that multiplying $\sqsupset_K^{\text{fin}}$ by i gives the obverse, while multiplying by $i^2 = -1$ gives the reverse.

Independent of whichever set we choose to be our alphabet for forming the Gauss code strings, we have the following proposition.

Proposition 5.7. *For each knot K represented by a* standard *signed Gauss sequence $\sqsupset_K$, $\sqsupset_K$ can be realized as the action of an element $\sigma_K \in S_{\text{fin}}$ on the standard signed Gauss sequence for the unknot, $\sqsupset_{\bigcirc}$.*

Proof. By construction of the standard Gauss sequence, writing the $\sqsupset_{\bigcirc}$ and $\sqsupset_K$ on two adjacent lines, one can immediately see the correspondence with the two-line notation for σ_K:

$$\sqsupset_{\bigcirc} \xmapsto{\begin{pmatrix} 1_u^+ & 1_o^+ & 1_o^- & \cdots & n_u^- \\ \sigma_K(1_u^+) & \sigma_K(1_o^+) & \sigma_K(1_o^-) & \cdots & \sigma_K(n_u^-) \end{pmatrix}} \sqsupset_K.$$

Note that exactly one of the pairs $(1_u^+, 1_o^+)$, $(1_o^-, 1_u^-)$ is fixed by σ_K. $\square$

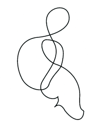
From this, we can calculate the cycle representation of our permutations. Full tables for the cycle representation of knots up to 8 crossings are given in Appendix C.3 in Table C.1 (using the $\mathcal{N}_{\text{g.int}}$ notation) and in Table C.2 (using the $\mathcal{N}_{\text{str}}$ representation). Again, the code we used to work with these can be found on Github.[14].

5.2.1 The Problem with Equivalence

Interpreting *knot equivalence* in this context gets a bit thorny, and it seems that there's likely no way to make it compatible with the group structure

[13] "g. int" is used to suggest "Gaussian integers."

[14] Once more, if the link doesn't work: https://github.com/redpanda1234/permutation-knots



(although we have not explicitly constructed counterexamples). On the one hand, viewed in abstraction, the Reidemeister moves can clearly be defined in terms of elements $\sigma \in S_{\text{fin}}$, as detailed by the following proposition.

Proposition 5.8. *Let の, ゆ, and め denote the Reidemeister I, II, and III moves, respectively.*[15] *Then the collections of all valid の, ゆ and め moves (denoted* $S_{\text{の}}$*,* $S_{\text{ゆ}}$*, and* $S_{\text{め}}$*) form countable subsets of* S_{fin}*.*

Proof. We just consider the case of の; the others are identical. Let K_0, K_1 be knots with Gauss sequences コ_{K_0}, コ_{K_1}, such that K_0 is related to K_1 by applying a single Reidemeister move. By Proposition 5.7, we can represent them by permutations $\sigma_{K_0}, \sigma_{K_1} \in S_{\text{fin}}$ such that

$$\text{コ}_{K_0} = \sigma_{K_0} \cdot \text{コ}_{\bigcirc} \qquad \text{and} \qquad \text{コ}_{K_1} = \sigma_{K_1} \cdot \text{コ}_{\bigcirc}.$$

Hence

$$\begin{aligned} \sigma_{K_0}^{-1} \cdot \text{コ}_{K_0} &= \text{コ}_{\bigcirc} \\ &= \sigma_{K_1}^{-1} \cdot \text{コ}_{K_1}, \end{aligned}$$

and thus we have

$$\sigma_{K_1}\sigma_{K_0}^{-1}\text{コ}_{K_0} = \text{コ}_{K_1}.$$

Then define $\sigma_{\text{の},K_0\to K_1}$ by

$$\sigma_{\text{の},K_0\to K_1} = \sigma_{K_1}\sigma_{K_0}^{-1},$$

and observe that it is an element of S_{fin} realizing the desired の move. By considering all such K_0, K_1, we can show that we get countably many such σ that act nontrivially on the unknot in distinct ways, which proves the claim. $\square$

On the other hand, it's worth noting that applying $\sigma_{\text{の},K_0\to K_1}$ to the Gauss sequence for some other knot K_2 might not yield a の-type move, and similarly with ゆ and め. In fact, it's unclear whether fully unrestricted use of such moves might allow for arbitrary permutations of the Gauss codes, and thus unknotting operations. We summarize this in the following question:

[15] These characters are Hiragana *no* ([no]), *yu* ([jɯ]), and *me* ([め]). We choose them for their visual similarity to diagrammatic representations of our Reidemeister moves.

Question 2. Is $\langle S_{\text{の}} \rangle = \langle S_{\text{ゆ}} \rangle = \langle S_{\text{め}} \rangle = S_{\text{fin}}$? Since it suffices to check whether or not each of the moves can be used to define arbitrary transpositions, this seems straightforward to check, but we haven't had time to examine it yet. It's worth noting that one should be extra careful to note that the definition of the Gauss code Reidemeister moves is encoded in terms of one-line notation, *not* cycle notation.

If the equality does not hold for at least one of the moves (suppose it's $S_{\text{の}}$), then we get a subgroup. What does it look like? What do the cosets look like? To what extent do they correspond with equivalence classes of diagrams under Reidemeister I? $\Diamond$

This is symptomatic of the bigger problem with trying to use S_{fin} to discuss knot equivalence: Essentially, no matter what we do, determining which $\sigma \in S_{\text{fin}}$ we're allowed to apply to a knot K seems to require specific knowledge about K itself. Here are some descriptions of why. In all of the following, let K be an n-crossing knot with Gauss sequence $\sqsupset_K$.

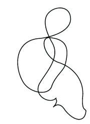
- Say we want to perform a Reidemeister I move between crossings i and j. This can be accomplished by a $\sigma_{\text{の}}$ move that essentially shifts the entire portion of $\sqsupset_K$ from j onwards to the right, while moving the corresponding $(n+1^{\epsilon_{n+1}}_{x_n+1}, n+1^{\neg\epsilon_{n+1}}_{\neg x_n+1})$ block to appear right after i, and then relabeling the $n+1-j$ crossings we've just modified so that they're in increasing order once again. The problem here is that if we try and apply this move to another knot, say one with $m > n$ crossings. Suddenly, it's not clear that the permutations we used to define the above can be used, since the $n+1^{\text{st}}$ crossing might be involved in a nontrivial portion of the knot.

- Reidemeister II suffers from similar concerns.

- For Reidemeister III, the permutations $\sigma_{\text{め}}$ employed are comparatively simple to define, but we have to be careful about when exactly we can use them. In particular, the constraint that we have to start with a Gauss code of a certain form in order to apply the move is troubling.

- Even removing the dependence on the choice of basepoint seems non-trivial, since this would require permuting *only* the portions of $\sqsupset_K^{\text{fin}}$ that did not arise from the $j_{\alpha_j}^{\neg mu_j}$, $j_{\beta_j}^{\neg\mu_j}$. Again, this would require specific knowledge of K.

There are a number of possible directions for future work that could address these issues.



1. Can we find a way to interpret the signed Gauss sequence in a way that's more labeling-agnostic? For instance, if we were to insert a countable number of "blank" characters between each of the non-trivial symbols in $\sqsupset_K$ (i.e., the non $j_{\alpha_j}^{\neg\mu_j}$, etc. ones), then we could resolve the problem of Reidmeister I and Reidmeister II by just thinking about exchanging the labels for the new crossings we're inserting with an appropriate choice of blank characters.

   One way to achieve something like this would be to think of $\sqsupset_{\bigcirc}$ as actually representing all of $\mathbb{Q}$, together with appropriate $+/-$ and u/o symbols. Then, if we want to insert a crossing between crossings 1 and crossing 2, instead of doing the shifting and relabeling process we described for Reidemeister I, we could think the operation as inserting a crossing $\frac{1}{2}$. Of course, this would require us to create a knot that has a crossing everywhere on an embedded copy of $\mathbb{Q}$ in S^1, which it seems we can do using Theorem 7.4. The particular construction we'd use solves the problem of Reidmeister I, but still leaves the issues of Reidemeister II and Reidemeister III. We wonder whether using this, there's a way to construct a standard unknot that is "universal" for Reidmeister I *and* II moves simultaneously, in the sense that our $\sqsupset_{\bigcirc}$ is for Reidemeister I.

2. Even if it turns out Reidemeister equivalence is just too much to ask for from a group-like structure, it seems that the formalism described above certainly yields a nice framework for categorification of knots. For instance, one could create a subcategory for 〔each Reidemeister equivalence class of knots〕 in which the objects are sequences in one of the $\mathcal{N}$, and the morphisms are the $\sigma \in S_{\text{fin}}$ that correspond to valid Reidmeister moves. In this context, we could view *unknotting operations* in terms between these subcategories. We'd be interested to see if interesting structure can arise out of this perspective.

The point is that equivalence seems challenging to incorporate, especially given the difficulty in working with the formalism above without some guiding conjectures. Hence, we leave this as a direction for future work, and shift instead to summarizing some of our computational results.

5.3 Computational Observations & Directions for Future Work

As we've stated twice before, source code for the following can be found on Github.[16].

Based on our `sage` computations, we've noticed the following patterns.[17]

- For any two knots K_0, K_1 with up to 8 crossings, when the products $\sigma_{K_0}\sigma_{K_1}$ and $\sigma_{K_1}\sigma_{K_0}$ act on $\sqsupset_{\bigcirc}$, it appears that (after relabeling) the result is a prime knot.[18] We would like to see if this can be verified theoretically, and if not, whether it still tends to hold "most" of the time as $n \to \infty$. This seems very unlikely to us, but we'd be willing to be surprised.

- For classical knots K_0, K_1, it is possible for $\sigma_{K_0}\sigma_{K_1}$ to act on $\sqsupset_{\bigcirc}$ to yield a virtual knot. For example, $(\sigma_{(6,3)}\sigma_{(6,2)}) \cdot \sqsupset_{\bigcirc}$ yields a virtual trefoil with code $4_o^-, 6_o^+, 4_u^-, 6_u^+$. By contrast, $(\sigma_{(6,2)}\sigma_{(6,3)}) \cdot \sqsupset_{\bigcirc}$

- We computed the length of the Reidemeister I / II reduced forms of the Gauss codes resulting from all pairwise multiplications up to 8 crossings (see Table C.3 for explicit numbers, and Fig. C.1 for a heatmap). In general, it seems the heatmap is approximately symmetric. We wonder whether this can be explained purely in terms of the permutations.

- We have also made a log-scale plot of the orders of the groups that are generated pairs of the σ_K (see Fig. C.2). We wonder whether there's any correlation with Fig. C.1. We'd also be curious to see whether the patches that appear have interpretable meaning.

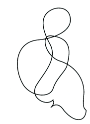
We'd be interested in seeing further work in understanding these observations. In addition to the above, we have the following questions for further work:

- How do we interpret the multiplication operation in S_{fin} topologically? One hiccup is that if crossing k appears with two different signs in σ_{K_0}, σ_{K_1}, then we have to relabel in the product to interpret the result. But

[16] https://github.com/redpanda1234/permutation-knots

[17] Note, whenever we say "up to 8 crossings," that does not mean the bound is sharp; rather, it is just that we have not tried higher crossing-number knots.

[18] We should note that our primality checker was not terribly sophisticated, hence there could be a flaw in it. Essentially, the program just recursively removed all possible Reidemeister I and Reidemeister II moves and then checked the result for an obvious point to cut the Gauss code.



putting that aside, the primality phenomenon described above seems very intriguing, and we would like to see diagrammatic interpretations of what's going on.

- We can represent the σ_K by permutation matrices that have been padded with countable sequences of 0's. As $n \to \infty$, how are the non-zero parts of these matrices distributed in the space of $n \times n$ matrices?

- We described earlier how we can think of Reidemeister moves as describing subsets of S_{fin}. We reiterate the questions we had then: Do these subsets generate S_{fin} in full? If not, they form subgroups; what do the cosets look like? So on and so forth.

Part III

Wild Knots

Chapter 6

Tame & Wild Knots

"And now," cried Max, "let the wild rumpus start!"

—Maurice Sendak, *Where the Wild Things Are*

The goal of this chapter is to define the basic concepts of *tameness* and *wildness*. Both of these are defined in multiple ways throughout the literature, and for the uninitiated, it can be non-obvious how to reconcile the different characterizations. We've endeavored to collect some of the most common definitions we've seen, and show that they are equivalent.[1] One should note that in general, the equivalence of these definitions *does not necessarily generalize* to higher-dimensional cases, e.g. surfaces embedded in $\mathbb{R}^4$.

For our "starting-point" definitions, we'll use those given in Daverman and Venema (2009). These match the definitions used when studying embeddings of general m-manifolds into n-manifolds.

Note that beyond this chapter, not much of this material to come will be called upon explicitly. Nevertheless, we have felt it is important to include for two reasons:

1. Given how ubiquitous the PL category is, we found it hard to find references for whether fundamental results like "ambient orientation-preserving homeomorphism guarantees ambient isotopy" are valid in the full Topological category. It appears much of this information has become mathematical folklore, which can make it difficult for non-experts to approach questions of interest about wild knots. We hope this helps to address part of that problem.

[1] We will not include proofs for *all* of the equivalences, but for the two most common definitions, we've included references to full proofs whenever we omit the details.

2. Second, it provides a good opportunity to ease ourselves into thinking about ambient isotopy without immediately rushing to Reidemeister's theorem, as this will not be available to us once we move into the Topological category.

Prerequisites & Further Reading

Discussing Tame and Wild knots requires some knowledge of PL topology. We've included a (very) short crash course in Appendix B, and a more philosophical discussion in Section 6.1. The former essentially consists of a short collection of definitions and propositions about affine and convex sets, ending with the definition of a simplicial complex. If the reader would like a more detailed reference work, we have some suggestions.

- The standard reference for the topic appears to be Rourke and Sanderson (1982). Although we have not personally used this text, it seems to be well-regarded (although we gather that PL topology has generally fallen out of favor).

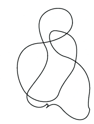
- Starbird and Su (2019) offers an excellent inquiry-based-learning (IBL) approach accessible to an undergraduate. We would highly recommend this text for learning the basics of the material; however, as the IBL approach means essentially all proofs are left to the reader, those looking for a quick-and-easy reference work might consider other sources.

- Bryant (2001) gives a very readable birds-eye view of PL topology that we found compelling, even though diagrams are relatively few and far between.

- Sakai (2013) includes comprehensive exposition on PL topology through simplicial complexes, without assuming local finiteness. We did find the notation and writing style a bit too terse for our tastes (especially given the relatively few diagrams), but this could be a matter of personal preference.

For further reading on the topic of tame/wild embeddings in general, we found Daverman and Venema (2009) to be an excellent resource with clear exposition and fantastic illustrations. However, it assumes a high level of familiarity with prerequisite material, and hence might inaccessible for most undergraduates.

Anyways, for our first section, we'll give a brief discussion of the main categories used when studying knots. This is not an area of expertise for the author, so we would welcome any insights, helpful examples, or analogies the reader has to offer.



6.1 A Word on the Topological, PL, and C^∞ Categories

First, the philosophical motivation. Wild knots can offer seemingly pathological behavior. For instance, as we show later in Example 7.2, the following arc is impossible to untie:

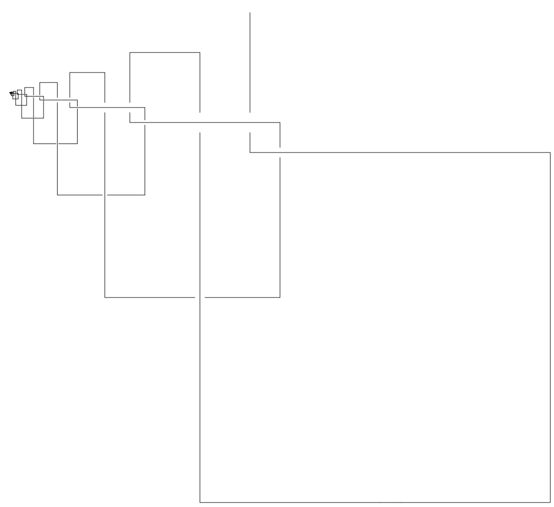


Figure 6.1 A wild arc of Fox and Artin (1948)

This counterintuitive property is often listed as one of the reasons why we omit wild knots from our study of knot theory. Actually, as we'll show (see Fig. 7.8 and the associated discussion), the reason this arc is impossible to unknot is quite intuitive: If we try to remove each of the stitches in succession, then at every step we end up dragging some points in the ambient space *through* the subsequent loop. As we continue to untie, these same points (together with friends they pick along on the way) get dragged down further and further towards the wild point, and in the limit, all of them converge. Hence we lose bijectivity. Not so scary and mysterious after all!

But in any case: Historically, wild knots have been hard to approach.

Hence knot theory is often restricted to "nicer" contexts where describing the behavior of our knots (and their relationships to each other) can be reduced to finite, combinatorial means. In the PL category for instance, we require our knots to decompose into finitely many linear pieces, and also require any *modifications* we'd like to make to them to be encoded by maps on the ambient space that decompose similarly (see Theorem 3.2). Then, because linear-flavored objects can be described with finite information,[2] this ends up giving us a context in which essentially *all* of our questions about knots can be simplified into finite problems (again, this was the exact idea behind Part II).

The point to highlight in the above is that in making a choice of a particular category to work in, we are requiring both our *objects* (knots embedded into $\mathbb{R}^3$) and our *morphisms* (functions deforming $\mathbb{R}^3$) to have a particular kind of structure. By choosing our restrictions judiciously, we can strike a nice balance between generality and tidiness in the theories we construct.

As seen through the existence of wild knots in the Topological category, working in different categories can yield very different results in the theory. Hence it is important for us to always clarify exactly which category we're assuming, so that readers have a sense for the scope of our results. The goal of this section is to remind us of this fact, and of some basic properties of each of the common categories, which we list below.

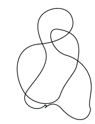
- The *Topological* category.
  - Objects: Embeddings $K : S^1 \hookrightarrow \mathbb{R}^3$ (or S^3)
  - Morphisms: Ambient isotopies $F : [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$.[3]
  - Notes: Most general context to work in, but allows "pathological" behavior.
- The *PL* category.
  - Objects: PL Embeddings $K : S^1 \hookrightarrow \mathbb{R}^3$
  - Morphisms: PL Ambient isotopies $F : [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$.[4]

[2] E.g., given the endpoints of a line segment, any point p in the middle can be described by a single parameter encoding what percentage of the way from end 1 to end 2 p is

[3] Equivalently, ambient orientation-preserving homeomorphisms (see 6.3.2)

[4] Equivalently, ambient orientation-preserving PL homeomorphisms (see 6.3.2)

- Notes: Rarely worked with directly, but forms the backbone of almost all modern combinatorial knot theory. Underpins Reidemeister's theorem.



- The C^k category ($k \geq 1$).
  - Objects: k-times continuously differentiable embeddings $K : S^1 \hookrightarrow \mathbb{R}^3$.
  - Morphisms: k-times continuously differentiable ambient isotopies $F : [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$.
  - Notes: Uncommon. By Theorem A.1 in Appendix A (together with an argument that every tame knot can be realized by a C^1 representative), this is more-or-less isomorphic to the PL Category. However it does allow for knots whose diagrams in $\mathbb{R}^2$ are *feral* (see Section 8.2).
- The C^∞ category, also commonly referred to as the *Smooth* category (although we *vehemently* protest this choice of phrasing, since we've found authors are not always consistent on whether "smooth" means "C^1" or "C^∞," especially between fields).
  - Objects: Infinitely-differentiable embeddings $K : S^1 \hookrightarrow \mathbb{R}^3$.
  - Morphisms: Infinitely-differentiable ambient isotopies $F : [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$.
  - Notes: Also known to be "equivalent" to the PL category (in the sense that every C^∞ knot is topologically ambient-isotopic to a PL knot, and vice versa).

With these distinctions in mind, we now move into a discussion of various *tameness* properties, which essentially describe whether an embedding can be encoded by a PL one.

6.2 What it Means to be Tame, Wild, and (Locally) Flat

We'll begin by clarifying some vocabulary that caused us some confusion in trying to understand the characterization of tame vs. wild knots. As discussed in the preface, there are many definitions given for tameness.[5] The examples we gave were the following:

> **Common Definition 1.** We say a knot $K : S^1 \hookrightarrow \mathbb{R}^3$ is *tame* iff it is ambient isotopic to a polygonal knot. $\Diamond$
>
> **Common Definition 2.** We say a knot $K : S^1 \hookrightarrow \mathbb{R}^3$ is *tame* iff for every point x on K, there exists a neighborhood U_x such that the pair $(U_x, K \cap U_x) \cong$ the standard $(\text{ball}, \text{diameter})$ pair. $\Diamond$
>
> **Common Definition 3.** We say a knot $K : S^1 \hookrightarrow \mathbb{R}^3$ is *tame* iff it can be thickened to an embedding of a solid torus. $\Diamond$
>
> **Common Definition 4.** We say a knot $K : S^1 \hookrightarrow \mathbb{R}^3$ is *tame* iff it has a diagram with finitely many crossings. $\Diamond$

Of these, Common Definition 1 is the closest to the definition given in Daverman and Venema (2009) for tameness in the context of embeddings of general manifolds. Common Definition 2 corresponds to the definition of *local flatness*, but that turns out to coincide with tameness in the case of embeddings $S^1 \hookrightarrow \mathbb{R}^3$. With some added clarification on what "thickened" means in Common Definition 3, we were able to find a convincing argument for the $(\Longrightarrow)$ direction, but have not been able to find a similar one for the $(\Longleftarrow)$ direction. Finally, we believe Common Definition 4 is misleading, and hence should be avoided.

6.2.1 Basic Definition

Almost all of our definitions will match those given in the introduction in Daverman and Venema (2009). Note, they define *equivalence of embeddings* in terms of ambient homeomorphisms, not ambient isotopies. In Section 6.3, we show that as far as tameness vs. wildness is concerned, "ambient homeomorphism" can be replaced by "ambient orientation-preserving homeomorphism" in the above, at which point the correspondence with ambient isotopy can be shown. We provide references for proofs in each of the Topological, PL, and C^1 categories.

Definition 6.1 (Ambient homeomorphic)**.** Let $(X, \mathcal{T})$, $(Y, \mathcal{S})$ be topological spaces. Let $\iota_0, \iota_1 : X \hookrightarrow Y$ be embeddings.[6] Then ι_0, ι_1 are said to be *ambient homeomorphic*

[5] We reproduce these below, but if the reader would like to see them in their original context, they were Common Definitions 1-4 in the Preface.

[6] Recall, this just means ι_0, ι_1 are homeomorphisms onto their images.

if there exists a homeomorphism $f : Y \to Y$ such that $f \circ \iota_0 = \iota_1$. ◊

Note, taking $t = 1$ in the definition of ambient isotopy directly implies the existence of an ambient homeomorphism. It is the converse we are uncertain about (although we expect it to be true).

Definition 6.2 (Tame & Wild Embeddings)**.** Let X be a polyhedron and let Y be a PL manifold. Then we say $K : X \hookrightarrow Y$ is a *tame embedding* iff it is ambient homeomorphic to a PL embedding. An embedding that is *not* tame is called *wild.* ◊

This is slightly different from the definition of tame & wild *subsets*, which we alert the reader to now:

Definition 6.3 (Tame & Wild Subsets)**.** Let Y be a PL manifold, and let $A \subseteq Y$ be closed. Then A is said to be *tame* iff there exists an ambient homeomorphism $f : Y \to Y$ such that $f(A)$ is a subpolyhedron of Y. A is said to be *wild* if it is ambient homeomorphic to a simplicial complex but not tame. ◊

The correspondence can be established through the following proposition, which is stated informally in Daverman and Venema (2009).

Proposition 6.1. *Let Y be a PL manifold and let $A \subseteq Y$ be closed. Use ι to denote the inclusion $\iota : A \hookrightarrow Y$. Then A is* tame as a subset *iff there exists a polyhedron X and a homeomorphism $f : X \to A$ such that $\iota \circ f : X \to Y$ is a tame embedding.*

Lastly, we describe two local properties embeddings can have: *local flatness*, and *local tameness.*

Definition 6.4 (Local Flatness at a Point)**.** Let X, Y be m and n-manifolds respectively, with $m < n$. Let $K : X \hookrightarrow Y$ be an embedding, and let $x \in X$. Then we say K is *locally flat* at x iff there exists an open set $U_x \in Y$ s.t. $K(x) \in U_x$ and there exists a homeomorphism $f : U_x \to \mathbb{R}^n$ such that $\overrightarrow{f}(U_x \cap K(X)) = \mathbb{R}^m$.[7] ◊

Note, with all variables as above, sometimes we'll also say f is *locally flat* at $f(x)$.

Definition 6.5 (Local Flatness of an Embedding)**.** With all variables quantified as above, then if K is locally flat for all $x \in X$, then we call K *locally flat.* ◊

[7]Long way of saying $(U_x, U_x \cap K(X)) \cong (\mathbb{R}^n, \mathbb{R}^m)$. We just chose the former to avoid having to introduce new notation.

Note, the definition of local flatness is just encoding the idea that we can "straighten out" our embedded copy of X in Y around each point in the image. We can define *local tameness* in a similar manner.

Definition 6.6 (Local Tameness of an Embedding)**.** Let X be a polyhedron with a fixed triangulation, and let N be a *topological* n-manifold. Let $K : X \hookrightarrow N$ be an embedding. Then K is said to be *locally tame* iff for every $x \in X$, there exists [a PL neighborhood $U_x \subseteq N$ s.t. $K(x) \in U_x$] and a homeomorphism $f_x : U_x \to \mathbb{R}^n$ such that

$$f_x \circ K|_{\overleftarrow{K}(U_x)}$$

is PL with respect to the above fixed triangulation of X. ◊

Of course, for our purposes today we'll primarily be interested in the case of embeddings $K : S^1 \hookrightarrow \mathbb{R}^3$. Can the definitions above be made to match Common Definitions 1-4 in these cases? The following gives the affirmative for 1 and 2, but we will need to wait until later to determine the answer for 3 and 4.

6.3 Correspondence with Common Def. 1

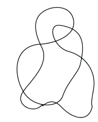
There are two main sticking points in establishing the correspondence here. First, can replace "homeomorphism" with "orientation-preserving homeomorphism" in our definitions for tameness / wildness / local flatness and get the same theory? And second, is the existence of an orientation-preserving homeomorphism equivalent to the existence of an ambient isotopy? We show the affirmative for the former in $\mathbb{R}^n$, and direct the reader to existing proofs for the latter in $\mathbb{R}^3$.

6.3.1 Homeomorphism vs. Orientation-Preserving Homeomorphism

Showing the equivalence of "homeomorphism" and "orientation-preserving homeomorphism" in $\mathbb{R}^n$ is straightforward. In each of the PL, C^1, and Topological categories, every homeomorphism must be either orientation-preserving or orientation-reversing, hence we can make a non-orientation-preserving homeomorphism into an orientation-preserving one by simply applying a flip as necessary.

Theorem 6.2. *Every homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is either* orientation-preserving *or* orientation-reversing.

Sketches. We give a sketch for a proof in each category.



- (PL) Given a fixed triangulation K of $\mathbb{R}^n$ and an n-simplex $\triangle^n \in K$, note that any PL homeomorphsim either fixes or reverses the orientation of $\triangle^n$. Now, the orientation of $\triangle^n$ induces a canonical orientation on each of its $(n-1)$-faces $\triangle^{n-1}$. This gives us a unique compatible orientation on each of the other simplices $\triangle^{n'}$ that have $\triangle^{n-1}$ as a face. One can then show this extends to a unique compatible orientation for all of K.

- (C^1)[8] Observe that in the C^1 category, the derivative matrix

$$Df(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) \cdots\cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) \cdots\cdots & \frac{\partial f_2}{\partial x_n}(x) \\ \vdots & \vdots \quad \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(x) & \frac{\partial f_n}{\partial x_2}(x) \cdots\cdots & \frac{\partial f_n}{\partial x_n}(x) \end{bmatrix}$$

  exists, is continuous in $x \in \mathbb{R}^n$, and invertible everywhere (f is C^1 with C^1 inverse). Now, note that for all $x \in \mathbb{R}^n$, the local orientation of $\overrightarrow{f}(\mathbb{R}^n)$ is given by $\mathrm{sgn}(\det(Df(x)))$. One can show that $\det(Df(x))$ is continuous in x. Now, observe that $\mathbb{R}^n$ is path-connected (hence $\overrightarrow{f}(\mathbb{R}^n)$ is as well), and consider an arbitrary path $\gamma : [0,1] \hookrightarrow \mathbb{R}^n$. Observe that $\det(Df(\gamma(t))) \neq 0$ for all t ($Df(x)$ is invertible everywhere), hence by continuity, the sign of $Df(x)$ is the same at the endpoints $\gamma(0)$, $\gamma(1)$. Since γ was arbitrarily chosen, it follows that $\mathrm{sgn}(Df(x))$ is constant over $\mathbb{R}^n$. Hence we get a unique orientation over all of $\overrightarrow{f}(\mathbb{R}^n)$.

- (Topological) This is given in Crowell and Fox (1963) on pg. 8, although the matter is essentially definitional. We've reproduced a slightly more detailed version below for completeness.

  First, we consider homeomorphisms of S^n. Let $g : S^n \to S^n$ be a homeomorphism. Observe that the induced map $g_* : H_n(S^n) \to H_n(S^n)$ is an isomorphism. Since $H_n(S^n) = \mathbb{Z}$, there are only two choices for g_*, determined by whether $g_*(1) = 1$ or $g_*(1) = -1$. In the first case, we call g *orientation-preserving*, and in the second, *orientation-reversing*.

  Recall that we can think of S^n as a one-point compactification of $\mathbb{R}^n$ by $S^n \cong \mathbb{R}^n \cup \{\infty\}$. Observe that in this light, our original homeomorphism

[8] See http://www.math.columbia.edu/~faulk/Lecture6.pdf for a bit more detail.

$f : \mathbb{R}^n \to \mathbb{R}^n$ has a unique extension to $f : \mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\}$ by taking

$$f(x) = \begin{cases} f(x) & x \in \mathbb{R}^n \\ \infty & x = \infty. \end{cases}$$

hence f is either orientation-reversing or orientation-preserving.

Again, this is essentially definitional. The harder part is show that this is consistent with both of the definitions above. We won't do that. ☐

As a simple corollary we have the desired equivalence of "homeomorphism" and "orientation-preserving homeomorphism" in the definitions of tameness, wildness, and locally flat.

Corollary 6.3. *An embedding $K : X \hookrightarrow \mathbb{R}^n$ is tame iff there exists an orientation-preserving ambient homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f \circ K$ is a PL embedding. Analogously for local flatness.*

Proof.

($\Rightarrow$) : Suppose we have an orientation-preserving ambient homeomorphism satisfying the desired properties. Then take this to be our ambient homeomorphism and we're done.

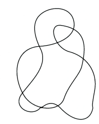
($\Leftarrow$) : Suppose we have a homeomorphism satisfying the desired properties. If it's orientation-preserving, then we're done. If is not, then by Theorem 6.2, it is orientation-reversing. Hence, compose it with

$$g = \begin{bmatrix} -1 & 0 & 0 \cdots\cdots 0 \\ 0 & 1 & 0 \cdots\cdots 0 \\ 0 & 0 & 1 \cdots\cdots 0 \\ \vdots & \vdots & \vdots \quad \ddots \quad \vdots \\ 0 & 0 & 0 \cdots\cdots 1 \end{bmatrix}.$$

And observe that the result is an orientation-preserving homeomorphism, and that we can take a subdivision of our triangulation on $\mathbb{R}^n$ to show that we get a PL embedding, as desired. ☐

Hence, as far as tameness, wildness, and local flatness are concerned, "homeomorphism" and "orientation-preserving homeomorphism" are equivalent.

6.3.2 Orientation-Preserving Homeomorphism vs. Ambient Isotopy



Now, we collect some references establishing the equivalence of orientation-preserving homeomorphism and ambient isotopy in each of the three categories.

1. *The Topological Category:* Here, the desired result is provided by Fisher (1960) in Theorem 4.

   > **Definition 6.7** (Deformation (Fisher)). A *deformation* of a manifold M is a homeomorphism $f : M \to M$ such that f is isotopic to the identity. $\Diamond$
   >
   > **Theorem 6.4** (Fisher, Theorem 4). *If $X = M$ is an n-manifold, then every f in the group $G^0(M)$ of homeomorphisms of M is a deformation of M.*

2. *The PL Category:* One can find the desired proof in Burde and Zieschang (2003). The numbering below follows that of the second edition.

   > **Theorem 6.5** (Burde & Zieschang, Proposition 1.10). *Let K_0, $\mathsf{K}_1 : S^1 \hookrightarrow S^3$ be PL. Then there exists a PL orientation-preserving homeomorphism $f : S^3 \to S^3$ such that $f \circ \mathsf{K}_0 = \mathsf{K}_1$ iff there exists a PL ambient isotopy taking K_0 to K_1.*
   >
   > **Corollary 6.6** (Burde & Zieschang, Corollary 3.16). *If two tame knots are topologically equivalent*[a] *then they are PL equivalent.*
   >
   > [a] I.e., by topological ambient isotopy / toplogical orientation-preserving homeomorphism

3. *The C^1 Category:* A proof of the following is provided in Milnor and Weaver (1997) as Lemma 2 in §6 (pg. 34).

   > **Lemma 6.7** (Milnor, Lemma 2). *Any orientation-preserving diffeomorphism f of $\mathbb{R}^m$ is C^1 isotopic to the identity.*

We also collect two helpful online threads the reader can investigate if interested.

- See Hatcher (2014)'s post on MathOverflow for a general discussion of references for proofs in each category.

- See user98602 (2015)'s post on StackExchange for a proof of the PL case. Note, user whippedcream (2016)'s post in the same thread includes the argument for the C^1 category given in Milnor and Weaver (1997) with a few more details filled in.

In any case, we have our desired result:

Corollary 6.8. *Let $K : S^1 \hookrightarrow \mathbb{R}^3$. Then K is ambient homeomorphic to a PL knot iff K is ambient isotopic to a PL knot. In particular, Definition 6.2 and Common Definition 1 are equivalent.*

6.4 Correspondence with Common Def. 2

In the following, we examine the correspondence between Definition 6.2 and Common Definition 2 (this was "tame" iff "locally flat"). We prove that for embeddings $K : S^1 \hookrightarrow \mathbb{R}^3$, "tame implies locally flat." We outsource most of the "locally flat implies tame" proof to Bing (1954). Because the "tame implies locally flat" argument is somewhat trivial, we'll spice it up a bit by using the opportunity to demonstrate a technique for explicitly constructing ambient isotopies.

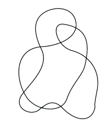
Because it is generally *not* true that tameness is equivalent to local flatness for arbitrary embeddings of m-manifolds in n-manifolds, we would like to discourage the use of Common Definition 2 as the "starting-point" definition of tameness in knot theory. A few more notes, which we summarize from Daverman and Venema (2009):

- It is true that in general, if M, N are PL m, n manifolds with M tamely embedded in N, then if $n - m \neq 2$, M is locally flat in N. (Daverman and Venema (2009), Theorem 1.2.1)

- With all variables as above, in the case $n - m = 2$, M *is* locally flat at a dense set of points.

- Local flatness generally does not imply tameness.

With these facts in mind, we move on to establishing that tameness and local flatness *do* coincide in the case of $S^1 \hookrightarrow \mathbb{R}^3$. First, we note that the definition for local flatness (Definition 6.4) can be reworded slightly. The claim is trivial; we are including for the sake of being very explicit.



Lemma 6.9. *The* $(U_x, U_x \cap K(X)) \xmapsto{f} (\mathbb{R}^n, \mathbb{R}^m)$ *condition in the definition of local flatness can be replaced with* $(U_x, U_x \cap K(X)) \xmapsto{f}$ *(n-ball, sub-m-ball) where* ∂*(sub-m-ball)* $\subseteq \partial$*(n-ball).*

Proof. Follows directly from the fact that n-ball's are homeomorphic to $\mathbb{R}^n$. $\square$

We'll often use this definition above when discussing local flatness.

Lemma 6.10. *Let* $K_0, K_1 : S^1 \hookrightarrow \mathbb{R}^3$*. Suppose there exists a homeomorphism* $f : \mathbb{R}^3 \to \mathbb{R}^3$ *such that* $f \circ K_0 = K_1$*. Then* K_0 *is locally flat iff* K_1 *is.*

Proof.

($\Rightarrow$) : Suppose K_1 is locally flat. The proof that K_0 is as well is essentially identical to the one below.

($\Leftarrow$) : Suppose K_0 is locally flat. We want to show K_1 is. To that end, let $p \in \overrightarrow{K_1}(S^1)$. Then $f^{-1}(p) \in \overrightarrow{K_0}(S^1)$. Define $q = f^{-1}(p)$.

Since K_0 is locally, flat there exists an open set $U_q \subseteq \mathbb{R}^3$ and a homeomorphism $f_q : \mathbb{R}^3 \to \mathbb{R}^3$ such that

(a) $q \in U_q$, and

(b) $(U_q, \overrightarrow{K_0}(S^1) \cap U_q) \xmapsto{f_q}$ (3-ball, 3-ball diameter).

Define $U_p = \overrightarrow{f}(U_q)$, and note that U_p is open with $p \in U_p$. Now, observe that $f_p : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f_p = f_q \circ f^{-1}$$

is a homeomorphism satisfying

$$(U_p, \overleftarrow{K_1}(S^1) \cap U_p) \xmapsto{f_p} \text{(3-ball, 3-ball diameter)}.$$

Hence K_1 is locally flat at p. Since p was arbitrarily chosen, it follows that K_1 is locally flat. $\square$

We use this in the following proposition, which is given as an exercise in Daverman and Venema (2009).

Proposition 6.11 (Daverman and Venema, Exercise 1.2.3)**.** *Every tame 1-sphere in* $\mathbb{R}^3$ *is locally flat.*

Before we give the proof, an important note:

Note. The argument below is *comically* overkill, but we've included it because the general technique of "parameterizing a closed region by lines connecting 〔points on the boundary〕 to 〔points on some 1-manifold we want to deform〕" is what underpins a lot of our ambient isotopy proofs later.[9] The idea is that we can take the boundary to be fixed, after which *all* of the interior points must "follow" the points on the 1-manifold we're deforming.

Because the explicit construction required for this technique is often so tedious, we will usually choose to omit it in our proofs and speak in only vague terms instead. This might make it a challenge for the reader to fill in the gaps on their own, so we've chosen to showcase the technique in detail here to expose the key ideas in a more approachable context. Hopefully this will help with the readability the proofs to come.

Proof. By Lemma 6.10, it suffices to show polygonal knots are locally flat. Hence, let $\mathsf{K} : S^1 \hookrightarrow \mathbb{R}^3$ be polygonal, and let $x \in \overrightarrow{\mathsf{K}}(S^1)$. Also let E denote the set of all straight edges in $\overrightarrow{\mathsf{K}}(S^1)$; by definition, E is finite. We proceed with two subcases.

1) Suppose x is an interior point of some $\mathsf{E}_x \in E$ (i.e., x is not a vertex). Then let

$$\varepsilon = \min_{\substack{\mathsf{E} \in E \\ \mathsf{E} \neq \mathsf{E}_x}} d(x, \mathsf{E}),$$

and observe $\varepsilon > 0$. Then $(B_\varepsilon(x), B_\varepsilon(x) \cap \mathsf{K})$ is trivially homeomorphic to the (3-ball, 3-ball diameter) pair.

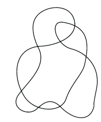
2) Now, suppose x is one of the vertices of K (i.e., one of the points where we join two polygonal segments). We construct the desired ambient homeomorphism explicitly.

   Let $\mathsf{E}_{x,1}, \mathsf{E}_{x,2} \in E$ be the distinct edges with $\mathsf{E}_{x,1} \cap \mathsf{E}_{x,2} = \{x\}$. Let

$$\varepsilon = \frac{1}{2} \min_{\substack{\mathsf{E} \in E \\ \mathsf{E} \neq \mathsf{E}_{x,1} \\ \mathsf{E} \neq \mathsf{E}_{x,2}}} d(x, \mathsf{E}),$$

   and observe $\varepsilon > 0$. Note that since the other endpoints of $\mathsf{E}_{x,1}$, $\mathsf{E}_{x,2}$ correspond to the start of new strands, $\varepsilon < \frac{1}{2}\text{length}(\mathsf{E}_{x,1})$, $\frac{1}{2}\text{length}(\mathsf{E}_{x,2})$. We draw a diagram in the plane of $\mathsf{E}_{x,1}$, $\mathsf{E}_{x,2}$ below.

[9] E.g., this appears in the proof of Proposition 9.4 when we make reference to Proposition B.8.



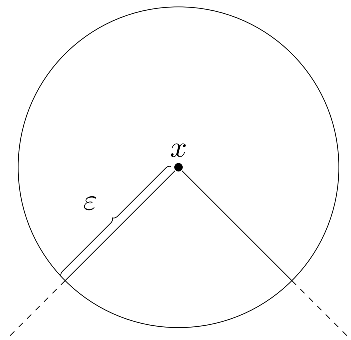


Figure 6.2 x, shown together with a 2D cross section of the closed ball $\overline{B_\varepsilon(x)}$ and the two line segments $\mathsf{E}_{x,1}$, $\mathsf{E}_{x,2}$

Note that $\mathsf{E}_{x,1}$, $\mathsf{E}_{x,2}$ partition this cross-section of $B_\varepsilon(x)$ into two regions R_1, R_2, each bounded by $\mathsf{E}_{x,1} \cup \mathsf{E}_{x,2}$ a circular arc (which we'll call A_1, A_2 respectively).

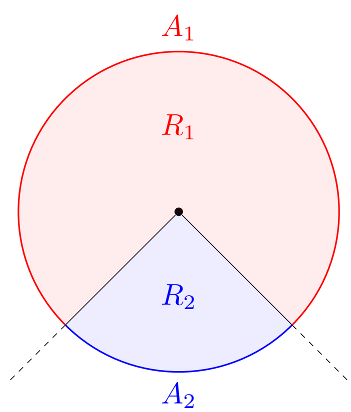


Figure 6.3 The diagram with R_1, R_2, A_1, A_2 labeled.

By some trig, one can find explicit parameterizations of lines linking each point of A_1 to a point of A_2 such that none of the lines cross.

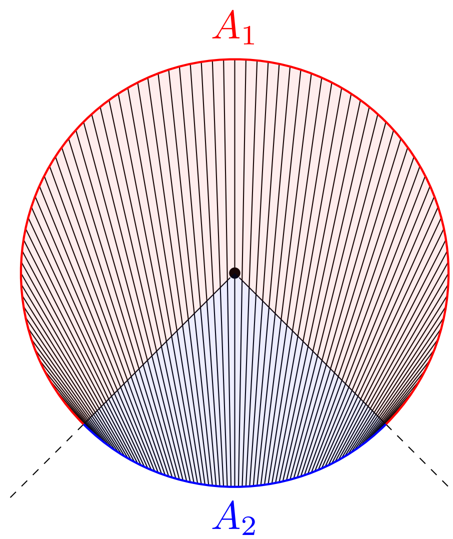


Figure 6.4 An example of the lines

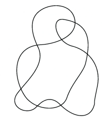
Note that each of these lines intersect $\mathsf{E}_{x,1} \cup \mathsf{E}_{x,2}$ at a unique point. Using this, we can parameterize every point p in the region R_1° as follows (and analogously for each $q \in R_2^\circ$):

(a) Let ℓ_p be the unique line through p (guaranteed uniqueness since none of the lines cross).

(b) Let c_p be the unique point of $A_1 \cap \ell_p$, and let e_p be the unique point of $(\mathsf{E}_{x,1} \cup \mathsf{E}_{x,2}) \cap \ell_p$. Then observe that there exists a unique $\lambda \in [0, 1]$ such that[10]

$$p = \lambda c_p + (1 - \lambda) e_p.$$

[10]Note, we're really just writing p as a convex combination of c_p and e_p.



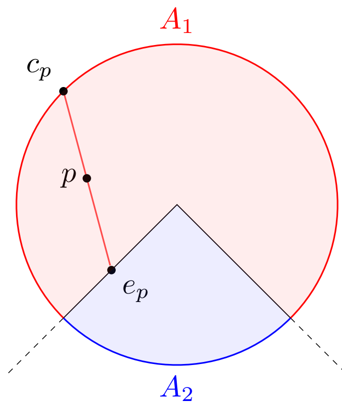


Figure 6.5 An example of c_p, e_p, with ℓ_p shown in red

This gives us lines like the following:

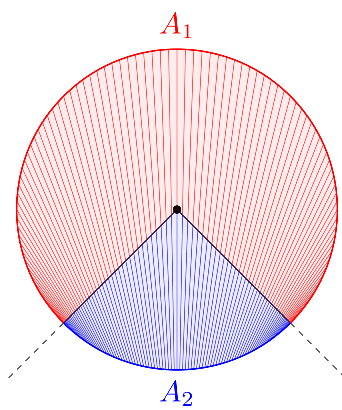


Figure 6.6 The lines after being split over $\mathsf{E}_{x,1} \cup \mathsf{E}_{x,2}$

Now, observe that if we fix $A_1 \cup A_2$, then the position of any point in $B_\varepsilon(x) \cap \operatorname{aff}(\mathsf{E}_{x,1} \cup \mathsf{E}_{x,2})$ is fully determined by the position of the corresponding e_p.[11]

One can perform a similar parameterization to make each of the e_p's themselves depend only on the position of our vertex x (in this case, the two points of $\overline{B_\varepsilon(x)} \cap (\mathsf{E}_{x,1} \cup \mathsf{E}_{x,2})$ play the role of the c_p's, and x plays the role of the e_p's). Together with the above, this makes the

[11] The aff here is just used to restrict $B_\varepsilon(x)$ to the $\mathsf{E}_{x,1} \cup \mathsf{E}_{x,2}$ plane.

position of *every* point in $B_\varepsilon(x) \cap \operatorname{aff}(\mathsf{E}_{x,1} \cup \mathsf{E}_{x,2})$ dependent only on the position of x.

Finally, we extend this to all of B_ε. There are two ways to do this. One is to "rotate" around the axis passing through the midpoints of A_1 and A_2, applying the same parameterization in each of the rotated sections. This gives us a cone-shaped blue region in 3D, and the complementary-shaped red region. The second is as follows: LOG, suppose that $\operatorname{aff}(\mathsf{E}_{x,1} \cup \mathsf{E}_{x,2})$ is parallel to the xy plane. Then "extend" the lines in the z direction by performing an identical parameterization to the above in each parallel cross section. Note, as we go further up/down in z, the c_p get closer to x.

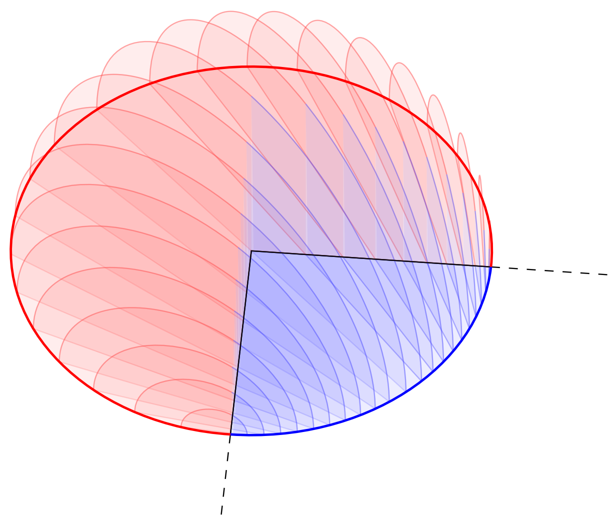
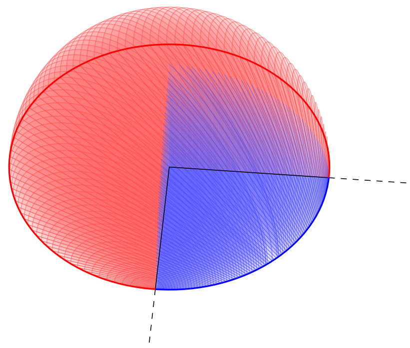


Figure 6.7 The $z > 0$ portion of the sets defined by "extending" our lines in the z direction within $B_\varepsilon(x)$.



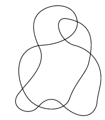


Figure 6.8 A denser version of the same plot.

Now, consider the map that takes x to the midpoint of the secant line between the ends of $\mathsf{E}_{x,1}$, $\mathsf{E}_{x,2}$, with all the other points in $B_\varepsilon(x)$ following x by way of their parameterizations.[12]

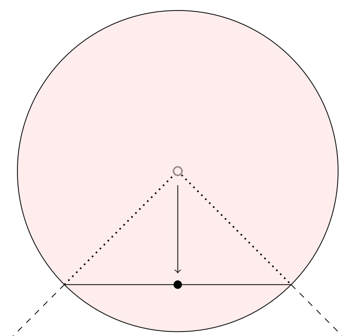


Figure 6.9 Taking x to the secant line

One can verify that this yields a homeomorphism (in fact, a PL homeomorphism) on $\overline{B_\varepsilon(x)}$ fixing the boundary. At last, one can take an open sub-ball around the shifted x and show that it yields a (3-ball, 3-ball diameter) pair, as desired.

In either case, we see K is locally flat at x. By Lemma 6.10, it follows that any tame knot is locally flat. ☐

[12]Note, we can turn this into an ambient isotopy directly by sliding our points along these lines.

Again, the proof is overkill. The point is just that one should think of this technique of "parameterizing points in some neighborhood by connecting them with curves to ⌈anchor⌋ points" whenever we mention it in later.

Anyways — Proposition 6.11 gives us "tame implies locally flat." The reverse argument follows from the lemma below, together with a theorem of Bing (1954).

Lemma 6.12. *Let $K : S^1 \hookrightarrow \mathbb{R}^3$ be an embedding. Then if K locally flat, then K is locally tame.*

See Definition 6.4, Definition 6.6 for a refresher on the definitions for locally flat and locally tame.

Proof. Let $p \in \overrightarrow{K}(S^1)$ be arbitrary. Since K is locally flat, by Lemma 6.9, there exists a homeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ and an open set U_p such that

1) $\overrightarrow{f}(U_p) =$ standard 3-ball, and
2) $\overrightarrow{f}\left(U_p \cap \overrightarrow{K}(S^1)\right) =$ standard 3-ball diameter.

Note that since U_p is homeomorphic to the standard 3-ball, then there exists a homeomorphism $g_0 : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\overrightarrow{g_0}(U_p)$ is a 3-simplex. Similarly, one can construct an ambient homeomorphism $g_1 : \mathbb{R}^3 \to \mathbb{R}^3$ taking the (3-ball, 3-ball diameter) pair to a (polyhedron, 1-chain) pair. Taking

$$h = g_1 \circ f \circ g_0^{-1}$$

and taking the appropriate subdivisions gives us the PL homeomorphism required in the definition of local tameness. □

Now, we have the following proposition of Bing (1954).

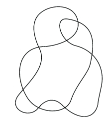
Theorem 6.13 (Bing, Theorem 9)**.** *Each locally tame closed subset K of a triangulated 3-manifold with boundary is tame.*

Proof. This is done in Bing (1954); see the paper for details on definitions and/or the proof. □

Applying the result to $\mathbb{R}^3$ and using Proposition 6.1 yields the desired result. Chaining the statements above together gives us

Corollary 6.14. *Let $K : S^1 \hookrightarrow \mathbb{R}^3$ be an embedding. Then K is tame iff K is locally flat. In particular, Definition 6.2 and Common Definition 2 are equivalent.*

6.5 Correspondence with Common Def. 3



We would like to slightly protest the use of Common Definition Common Definition 3. In particular, we feel the choice of the word "thickened" is imprecise enough to be misleading. For instance, as we have argued, the knots shown in Appendix A are tame (since they are C^1), but it isn't immediately obvious how to "thicken" any of them to a torus.[13] In fact, we can construct simpler examples of objects for which this confusion might arise. Consider the following knot $K : S^1 \hookrightarrow \mathbb{R}^3$:

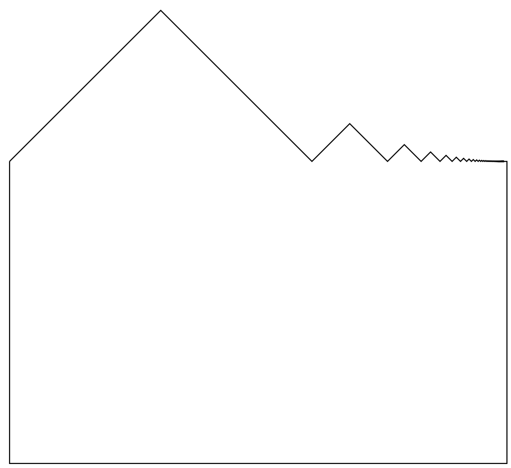


Figure 6.10 A very excitable plane curve

Using a later result (Proposition 9.4[14]), we can show that this is indeed a tame knot. So, what happens if we try to "thicken" it to an embedding of the torus? Here, we'll interpret "thicken" to mean "find a simple rule to associate each point p of this curve with a 2-disk $D_p \subseteq \mathbb{R}^3$ such that p is the center of D_p."

If we try this with most polygonal knots K, we can just think of following an algorithm like the one below:

1) Let E denote the set of all the straight edges in $\overrightarrow{\mathsf{K}}(S^1)$. Let $r > 0$ be

[13]It certainly appears that it's not always possible to find a tubular neighborhood of our embedding, but we could be wrong.

[14]This proposition basically just confirms that planar isotopy is legal even for general topological embeddings

given by

$$r = \min_{\substack{\mathsf{E}_1, \mathsf{E}_2 \in E \\ \mathsf{E}_1 \cap \mathsf{E}_2 = \varnothing}} d(\mathsf{E}_1, \mathsf{E}_2).$$

2) For each vertex x of K joining edges E_1, E_2, extend a line ℓ_x of length $2r$ through x such that

   (a) x is the midpoint of ℓ_x, and

   (b) In the plane $\operatorname{aff}(\mathsf{E}_1, \mathsf{E}_2)$, ℓ_x bisects $\angle \mathsf{E}_1 \mathsf{E}_2$.

3) Connect all the endpoints for consecutive ℓ_x's together in a way that "follows" the shape of K (apologies for the lack of diagram here). This effectively gives us a "ribbon" tracing out the shape of K.

4) Finally, construct our embedded torus by "sliding" disks along the body of the knot, such that the boundary of each disk is always "flush" with the ribbon. This gives us an embedding of the torus.

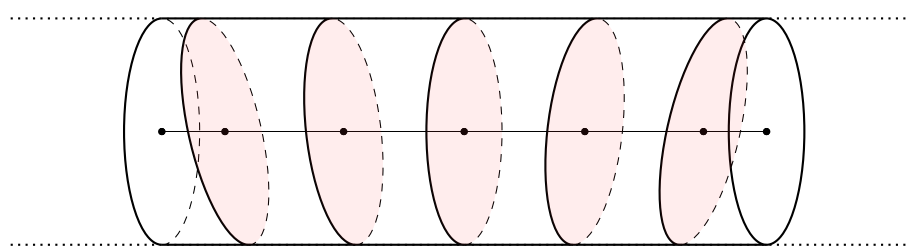
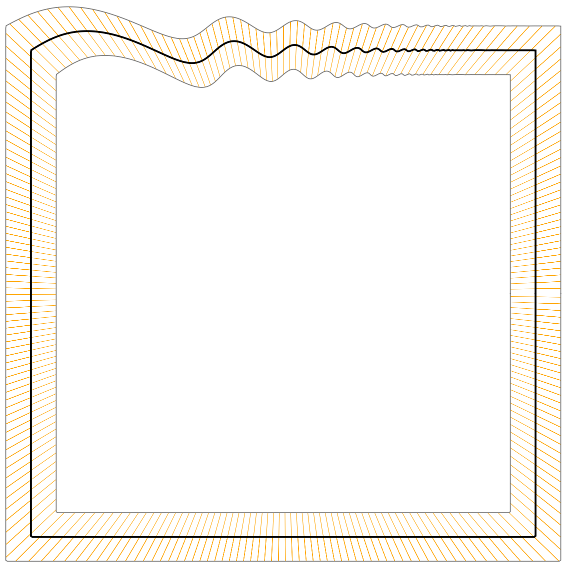


Figure 6.11 An example of sliding the disks along.

This works out well-enough for most knots. Indeed, it seems straightforward to extend this idea to all piecewise C^1 knots by employing *parallel curves:*[15]

[15]See https://en.wikipedia.org/wiki/Parallel_curve, for example.



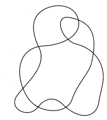


Figure 6.12 An example for a piecewise C^1 knot. Note, this figure does not display *actual* parallel curves, but rather just a slightly-shrunk and slightly-enlarged version of the square.

But again, it's not clear how to translate this to the context of our knot in Fig. 6.10, because we can't define r.

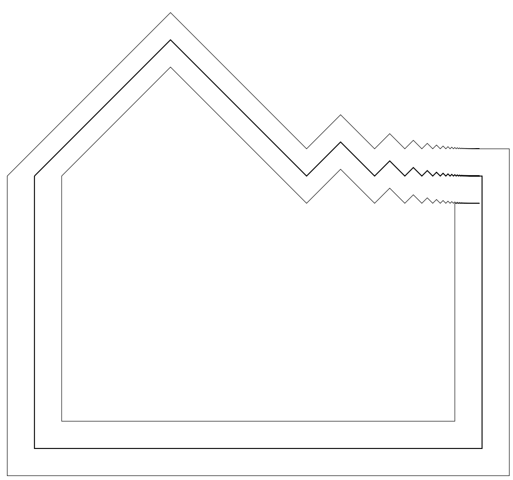


Figure 6.13 An attempt to at a similar approach for the curve.

It doesn't work at the limit point of our sawtooth pattern. Fair enough, the approach was quite naive. After some consternation, we find that in this case, "shrinking" and "enlarging" the diagram can get us what we want.

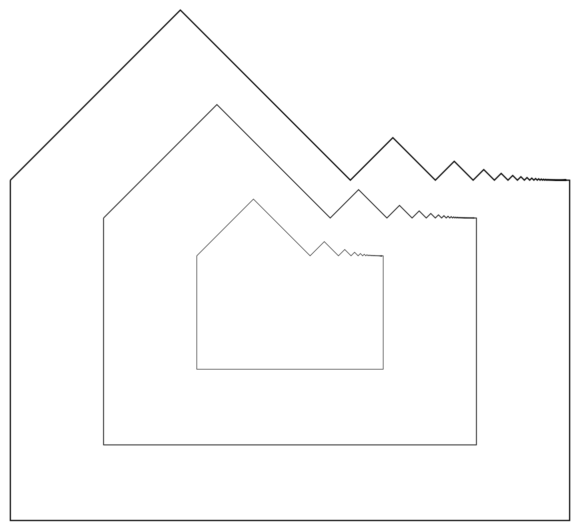


Figure 6.14 The desired embedding of the torus.

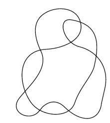
It's not immediately clear to us how to make this construction more general, and we have not been able to find a proof or a more precise statement of the claim. Hence, we propose a formal statement in the below. We'll label it as a conjecture (even though it's almost surely been proven before) because we have not worked out the full details ourselves, due to time constraints. Our description uses the definition of a *lift*, which we state for purely comic effect.

Definition 6.8. Let $C = (\mathrm{ob}(C), \mathrm{hom}(C))$ be a category. Let $X, Y, Z \in \mathrm{ob}(C)$, and let $f \in \mathrm{hom}(X, Y)$, $g \in \mathrm{hom}(Z, Y)$, and $h \in \mathrm{hom}(X, Z)$ such that $f = g \circ h$. Then we say that h is a *lift* of f, or that f *factors through* h. $\Diamond$

$$\begin{tikzcd} & Z \arrow[d, "g"] \\ X \arrow[ur, "h"] \arrow[r, "f"'] & Y \end{tikzcd}$$

Figure 6.15 A commutative diagram for Definition 6.8

Conjecture 1. *A knot $K : S^1 \hookrightarrow \mathbb{R}^3$ is tame iff there exists an embedding $K_{\text{tor}} : S^1 \times \mathbb{D}^2 \hookrightarrow \mathbb{R}^3$ such that the embedding $\iota : S^1 \hookrightarrow S^1 \times \mathbb{D}^2$ given by*



$$\iota(s) = (s, \mathbf{0})$$

(where $\mathbf{0}$ is the center of $\mathbb{D}^2$)[16] *satisfies $K = K_{\text{tor}} \circ \iota$.*

$$\begin{tikzcd} & S^1 \times \mathbb{D}^2 \arrow[d, "K_{\text{tor}}"] \\ S^1 \arrow[ur, "L"] \arrow[r, "K"'] & \mathbb{R}^3 \end{tikzcd}$$

Figure 6.16 A commutative diagram for Conjecture 1

It's worth mentioning that we have recently found a discussion of this proposition in a MathOverflow post by user Lilalas (2019); however, a proof of the hard portion (existence of K_{tor} implies K tame) was not included. We offer a tentative sketch of a possible approach, but we offer no guarantees on correctness.

Possible Sketch. Let $K : S^1 \hookrightarrow \mathbb{R}^3$ be an embedding.

($\Rightarrow$) : Suppose K lifts to an embedding $K_{\text{tor}} : S^1 \times \mathbb{D}^2 \hookrightarrow \mathbb{R}^3$ by the map $\iota : S^1 \hookrightarrow S^1 \times \mathbb{D}^2$. Note, for all $s \in S^1$, the image $D_s = \overrightarrow{K_{\text{tor}}}(s \times \mathbb{D}^2) \subseteq \mathbb{R}^3$ is homeomorphic to $\mathbb{D}^2$, hence $\operatorname{diam}(D_s) > 0$. Also observe that for all $t \in S^1$ with $t \neq s$, $t \notin D_s$.[17]

Show that in a non-trivially knotted region of K, the diameters $\operatorname{diam}(D_s)$ are bounded by the distance between strands participating in a crossing. Then, suppose (to obtain a contradiction) that K were wild. Show that the distance between crossing strands goes to 0 as we approach the wild point, and obtain the contradiction.

($\Leftarrow$) : For the reverse proof, use the fact that K is tame to obtain a homeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\mathsf{K} = f \circ K$ is polygonal. Construct the tubular neighborhood N_{tube} of $\overrightarrow{\mathsf{K}}(S^1)$ using the

[16] Really, any point in $\left(\mathbb{D}^2\right)^\circ$ works.
[17] Even stronger, $D_t \cap D_s = \varnothing$

algorithm described above (Page 129). Note, since f is a homeomorphism, $\overleftarrow{f}(N_{\text{tube}})$ is homeomorphic to the torus $S^1 \times \mathbb{D}^2$. Define K_{tor} in terms of $\overleftarrow{f}(N_{\text{tube}})$, and show that it satisfies the desired properties. $\boxed{?}$

In any case: While this definition might have some theoretical properties, it is hard to find a reference for, and seems to be easy to accidentally misapply if care is not taken to be rigorous (e.g., again, we imagine most people employing this definition would guess that Fig. A.2 is wild at first glance).

Question 3. What does the torus in Fig. A.2 look like? We've managed to make some for earlier examples; e.g., an xy plane projection of Fig. 5.8 yields a picture that looks qualitatively similar to the following, hence we can draw a tubular neighborhood without too much trouble:

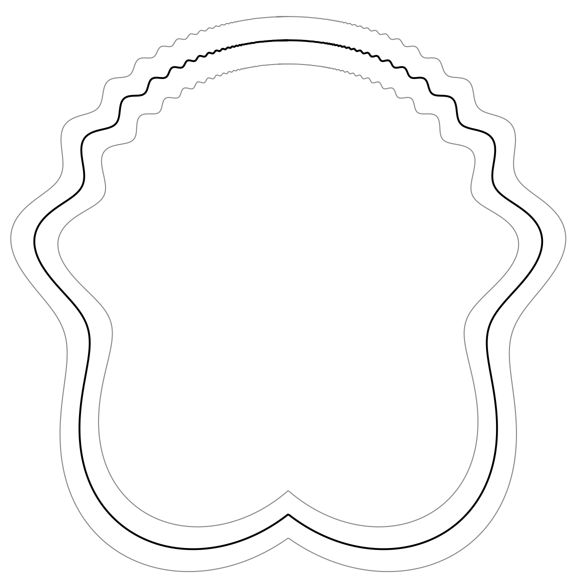
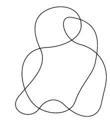


Figure 6.17 A rough "flavor" for what the xy-plane projection of Fig. 5.8 looks like. Note, this is actually a different (nicer-looking) function, but the idea is the same.

We'd be interested in seeing more plots of the associated embedded toruses for feral knots. ♢

6.6 Common Def. 4



This section is very short: Given the abundance of examples of tame knots with countably-many crossings that we have demonstrated, we discourage the use of Common Definition 4. We wonder whether the definition given there was supposed to be "for every tame knot $K : S^1 \hookrightarrow \mathbb{R}^3$, there exists a projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ such that $\pi(K)$ has only finitely many crossings."

We are skeptical of this claim as well — it's not obvious what there is to stop us from taking 6 arcs shaped like the one in Fig. A.2 and orienting them so that the "core" of each of the spirals (formally, the line normal to the plane in Fig. A.3 passing through the center of the spiral) lies along one of the x, y, or z axes (each axis gets two spirals, one pointing in the negative direction, the other pointing in the positive direction). It seems that no projection of such a knot would yield only finitely-many crossings, but we'd be willing to be surprised.

6.7 Conclusion

In this chapter, we've introduced several common definitions for tameness/wildness/local flatness for knots, and established correspondence (or lack thereof) with the definitions given in Daverman and Venema (2009). We also gave a brief discussion of the differences between the Topological, PL, C^k, and C^∞ categories. The remainder of this document will be concerned with building tools for working hands-on with ambient isotopy in the Topological category. Our goal is ultimately to lay the groundwork for characterizing ambient isotopy between embeddings that can be represented by a countable union of polygonal segments. To that end, we first turn our attention to some machinery based in the tools of real analysis.

Chapter 7

Machinery

"Now stop!" Max said and sent the wild things off to bed
without their supper. And Max the king of all wild things was lonely
and wanted to be where someone loved him best of all.

—Maurice Sendak, *Where the Wild Things Are*

In this chapter, we develop two pieces of machinery that are extremely useful in working with general topological ambient isotopies. The first is a *uniform convergence* version of Theorem 5.3, which has the benefit of not requiring disjointness for our V_n. This is a double-edged sword: On the one hand, it affords us greater latitude in the kinds of situations where we can apply it, but on the other, it is much easier to accidentally misuse.

The second is *separation of strands*, which essentially allows us to partition our knot into segments that are all "isolated" from each other except at their ends (where they share mutual points with the neighboring strands).[1] This will allow us to work "locally" on our knots by keeping our ambient isotopies contained to regions where they can only affect one part, which will prove indispensable in the subsequent chapters.

7.1 Uniform Convergence and Ambient Isotopy

The following theorems will play a central role in allowing us to prove all arcs are ambient isotopic in $\mathbb{R}^2$. The idea is to use uniform convergence to concatenate countably-many ambient isotopies together without breaking

[1]Note, we will not make any guarantees about how complicated the segments themselves look, only that they are separated from each other.

any of our continuity concerns. We first recall some definitions from Analysis, then we prove the ambient isotopy uniform convergence theorem.

Definition 7.1 (Uniform Convergence)**.** Let (E, d), (E', d') be metric spaces, and for all $n \in \mathbb{N}$, let $f_n : E \to E'$. Also let $f : E \to E'$. Then we say $f_n \to f$ *uniformly* iff for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n > N$, then $\forall x \in E$,

$$|f(x) - f_n(x)| < \varepsilon.$$

We will sometimes denote uniform convergence by $f_n \xrightarrow{u} f$. ◊

Proposition 7.1. *For all $n \in \mathbb{N}$, let $f_n : E \to E'$ be a continuous function. Now, let $f : E \to E'$, and suppose that $f_n \xrightarrow{u} f$. Then f is continuous.*

The proof is an $\frac{\varepsilon}{3}$ argument and can be found in any standard analysis text.

Corollary 7.2. *Let (E, d), (E', d') be metric spaces, and for all $n \in \mathbb{N}$, let $f_n : E \to E'$ such that f_n is a homeomorphism. Let $f : E \to E'$ be a bijection and suppose that*

1. *$f_n \xrightarrow{u} f$, and*
2. *$f_n^{-1} \xrightarrow{u} f^{-1}$*

Then f is a homeomorphism.

Proof. Continuity of f, f^{-1} follows directly from Proposition 7.1. Bijectivity gives us a homeomorphism. □

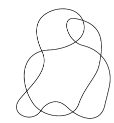
We have the following straightforward lemma.

Lemma 7.3. *Let (E, d) be a metric space. For all $k \in \mathbb{N}$, let $V_k \subseteq E$, and let $f_k : E \to E$ be a homeomorphism such that f_k is identity on $E \setminus V_k$. Define $f : E \to E$ by*

$$\begin{aligned} f &= \lim_{n\to\infty} \bigcirc_{k=1}^{n} f_k \\ &= \lim_{n\to\infty} (f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1). \end{aligned}$$

Then provided

1. *f is a bijection,*



2. *The* g_n^{-1} *defined by* $g_n^{-1} = f_1^{-1} \circ \cdots \circ f_{n-1}^{-1} \circ f_n^{-1}$, *satisfy* $g_n^{-1} \xrightarrow{u} f^{-1}$, *and*

3. *The* V_k *satisfy*
$$\lim_{n\to\infty} \operatorname{diam}\left(\bigcup_{k=n}^{\infty} V_k\right) = 0,$$

Then f *is a homeomorphism.*

Remark. Note, if E is a compact Hausdorff space, we can drop the condition on the g_n^{-1}'s, since a continuous bijection from a compact space to a Hausdorff space is always a homeomorphism. In particular, if $\bigcup_{k=1}^{\infty} V_k$ is bounded in some compact set, then it suffices to prove f is continuous. We stated the more general version of the result here, but from now on, we will generally assume that we have this simplifying condition.

Proof. For all $n \in \mathbb{N}$, define
$$g_n = \bigcirc_{k=1}^{n} f_k,$$
and note that it is a homeomorphism with inverse
$$\begin{aligned} g_n^{-1} &= \bigcirc_{k=1}^{n} f_{n+1-k}^{-1} \\ &= f_1^{-1} \circ f_2^{-1} \circ \cdots \circ f_{n-1}^{-1} \circ f_n^{-1}. \end{aligned}$$

Since we assume bijectivity of f and $g_n^{-1} \to f^{-1}$ (thus f^{-1} is continuous) by hypothesis, it suffices to show that f is continuous. To that end, we want to show $g_n \xrightarrow{u} f$. Let $\varepsilon > 0$ be given. Note that because
$$\lim_{n\to\infty} \operatorname{diam}\left(\bigcup_{k=n}^{\infty} V_k\right) = 0,$$
there exists $n_0 \in \mathbb{N}$ such that $n > n_0$ implies
$$\operatorname{diam}\left(\bigcup_{k=n}^{\infty} V_k\right) < \varepsilon.$$

Hence, let $n > n_0$ be fixed, and let U_n denote
$$U_n = \bigcup_{k=n}^{\infty} V_k$$

Observe that by definition, $f = g_n$ on $E \setminus U_n$, hence they are trivially ε-close on $E \setminus U_n$.

For showing ε-closeness on the U_n, note that $\overrightarrow{f}(U_n) = \overrightarrow{g_n}(U_n)$. Thus because $\operatorname{diam}(U_n) < \varepsilon$, it follows that f, g_n are ε-close on U_n as well. Hence $g_n \xrightarrow{u} f$, so f is continuous.

By Corollary 7.2, it follows that f is a homeomorphism. □

This extends naturally to the following version for ambient isotopies. The statement might appear a bit complicated, but it's really just saying that whenever we're given a collection of ambient isotopies that are restricted to acting on sets whose radii decay sufficiently quickly, we can play countably many of them back-to-back in finite time and yield an ambient isotopy.

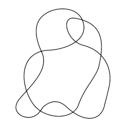
Theorem 7.4 (Ambient Isotopy and Uniform Convergence)**.** *Let (E, d) be a metric space. For all $k \in \mathbb{N}$, let $V_k \subseteq E$, and let $F_k : [0,1] \times E \to E$ be an ambient isotopy such that F_k is identity on $[0,1] \times (E \setminus V_k)$. For all such k, define $f_k(x) : E \to E$ by $f_k(x) = F_k(1, x)$; note that by definition of ambient isotopy, $f_k(x)$ is a homeomorphism. Then define $F : [0,1] \times E \to E$ as follows: For all $(t, x) \in [0,1] \times E$, let*

$$F(t,x) = \begin{cases} F_1(2t, x) & \text{if } t \in \left[0, \frac{1}{2}\right] \\ F_2(4(t-1/2), f_1(x)) & \text{if } t \in \left(\frac{1}{2}, \frac{3}{4}\right] \\ F_3(8(t-3/4), f_2(f_1(x))) & \text{if } t \in \left(\frac{3}{4}, \frac{7}{8}\right] \\ \vdots & \\ F_n\left(2^n\left(t - 1 + \frac{1}{2^{n-1}}\right), \bigcirc_{k=1}^{n-1} f_k(x)\right) & \text{if } t \in \left(1 - \frac{1}{2^{n-1}}, 1 - \frac{1}{2^n}\right] \\ \vdots & \\ \bigcirc_{k=1}^{\infty} f_k(x) & \text{if } t = 1. \end{cases}$$

Then provided

1. *$F(1, \cdot)$ is a bijection,*
2. *$\bigcup_{k=1}^{\infty} V_k$ is bounded in a compact set, and*
3. *The V_k satisfy*
$$\lim_{n\to\infty} \operatorname{diam}\left(\bigcup_{k=n}^{\infty} V_k\right) = 0,$$

then F is an ambient isotopy.



Proof. Note that 〔$\bigcup_{k=1}^{\infty} V_k$ is bounded in a compact set〕 allows us to apply the simplifying condition for Lemma 7.3.

1. To show $F(0,\cdot)$ is identity: Observe $F(0,x) = F_1(0,x) = x$, since $F_1(0,x)$ is an ambient isotopy (and hence identity for $t = 0$).

2. To show $F(t,x)$ is a homeomorphism for each $t \in [0,1)$: If $t \neq 1$, then there exists $n \in \mathbb{N}$ such that $t \in \left(1 - \frac{1}{2^{n-1}}, 1 = \frac{1}{2^n}\right]$. Hence
$$F(t,x) = F_n\left(2^n\left(t - 1 + \frac{1}{2^{n-1}}\right), \bigcirc_{k=1}^{n-1} f_k(x)\right),$$
which is a finite composition of homeomorphisms, and thus a homeomorphism.[2]

3. To show that $F(1,\cdot)$ is a homeomorphism: Observe that $F(1,\cdot)$ is a bijection, f_k is constant on $E \backslash V_k$ for each k, and $\lim_{n\to\infty} \operatorname{diam}(\bigcup_{k=n}^{\infty} V_k) = 0$. Thus by Lemma 7.3, $F(1,\cdot)$ is a homeomorphism.

4. Finally, to show that F is continuous: Define a sequence of ambient isotopies $G_n : [0,1] \times E \to E$ such that
$$G_n(t,x) = \begin{cases} F_1(2t,x) & \text{if } t \in \left[0, \frac{1}{2}\right] \\ F_2(4(t-1/2), f_1(x)) & \text{if } t \in \left[\frac{1}{2}, \frac{3}{4}\right] \\ F_3(8(t-3/4), f_2(f_1(x))) & \text{if } t \in \left[\frac{3}{4}, \frac{7}{8}\right] \\ \vdots & \\ F_n\left(2^n\left(t-1+\frac{1}{2^{n-1}}\right), \bigcirc_{k=1}^{n-1} f_k(x)\right) & \text{if } t \in \left[1 - \frac{1}{2^{n-1}}, 1 - \frac{1}{2^n}\right] \\ \bigcirc_{k=1}^{n} f_k(x) & \text{otherwise.} \end{cases}$$
Observe that G_n is a piecewise function whose pieces are each continuous. By the gluing lemma, it suffices to show G_n is continuous at the points where the pieces meet. This follows by observing that for each $k \in \{1,\ldots,n-1\}$, $F_k(1,x) = F_{k+1}(0,x)$, and $F_n(1, \bigcirc_{k=1}^{n-1} f_k(x)) = f_n \circ \bigcirc_{k=1}^{n-1} f_k(x) = \bigcirc_{k=1}^{n} f_k(x)$.

   We want to show $G_n \xrightarrow{u} F$. By construction, $G_n(t,x) = F(t,x)$ everywhere except perhaps for (t,x) in
$$U_n = \left[1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}\right] \times \left(\bigcup_{k=n}^{n} V_k\right).$$

[2]"Finite composition of homeomorphisms" follows because for all $t' \in [0,1]$, $F_n(t',x)$ and $\bigcirc_{k=1}^{n-1} f_k(x)$ are both homeomorphisms.

But observe that for all $(t,x) \in U_n$, we have $G_n(t,x),\ F(t,x) \in U_n$, hence as $n \to \infty$

$$d(G_n(t,x),\ F(t,x)) < \operatorname{diam}(U_n) < \varepsilon.$$

It follows that $G_n \xrightarrow{u} F$. By Proposition 7.1, it follows that F is continuous.

Thus F is an ambient isotopy. $\square$

This is a very powerful technique, and it allows us to prove that all sorts of wacky curves are secretly ambient isotopic. For instance, consider the following example:

Example 7.1. The following arc can be unknotted by an ambient isotopy $F : [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$ keeping the endpoints fixed.

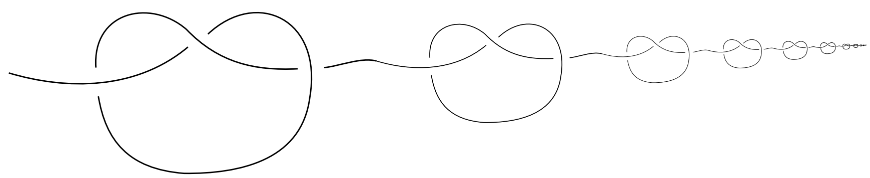
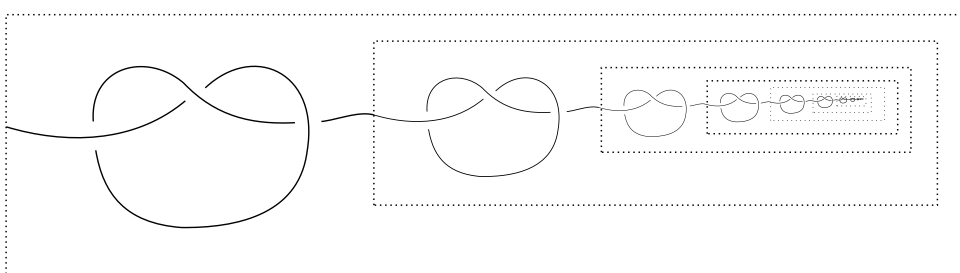
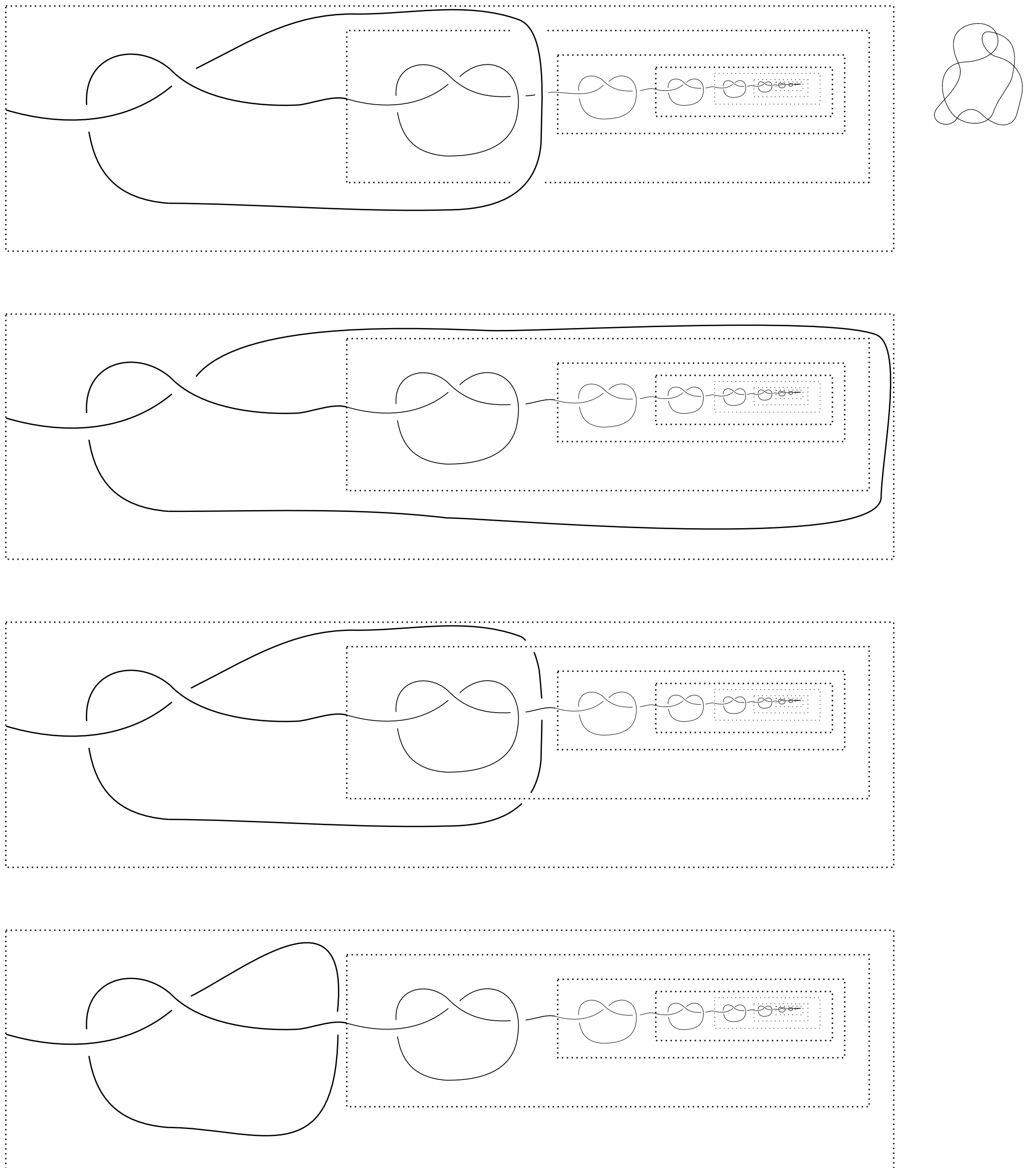


Figure 7.1 A very wild-looking tame arc

The trick is to define the V_k to be a sequence of nested boxes as follows:



The reader should imagine these as properly nested rectangular prisms in $\mathbb{R}^3$. For each of the V_k, define an ambient isotopy F_k performing the following trick:



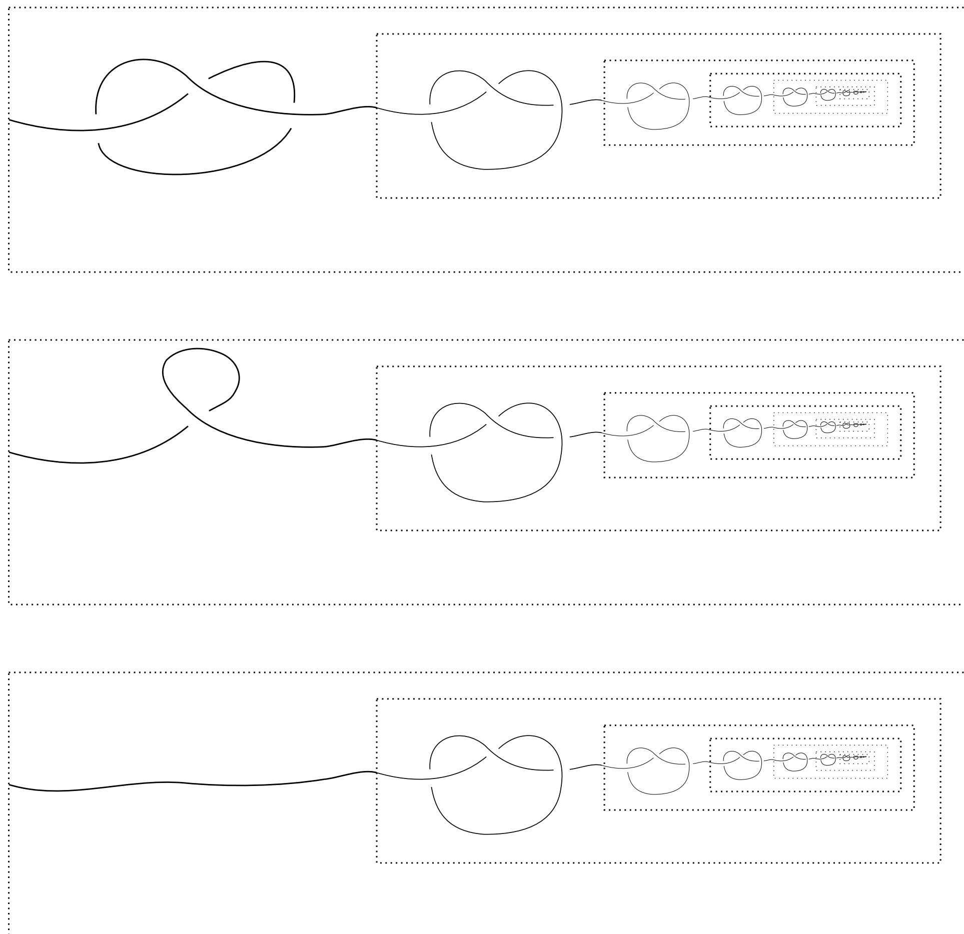


Figure 7.2 An example F_n.

Observe that we can now apply Theorem 7.4 to show that the arc can be untangled in finite time. By contrast, we *cannot* apply a similar argument to even a slightly-extended version of this arc:

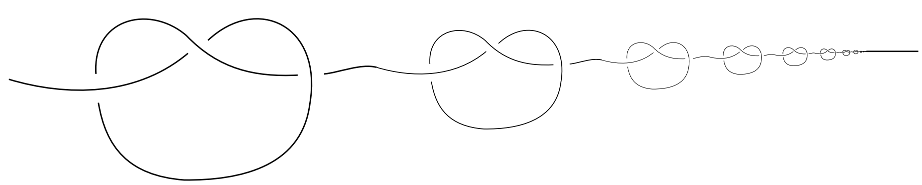


Figure 7.3 A case where we can't define the V_n such that $\lim_{n\to\infty} \operatorname{diam}(\bigcup_n V_n) \neq 0$

◊

We have to be extremely careful when applying this argument, as it is very easy to subtly fail the hypotheses without noticing. For instance:

Example 7.2 (Fox-Artin Curve)**.** Consider the following arc of Fox and Artin (1948).

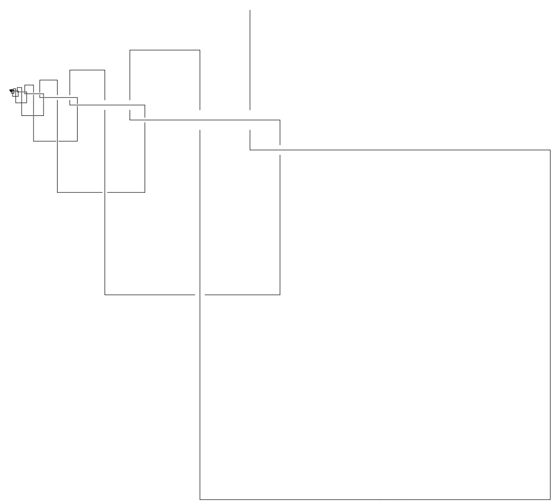


Figure 7.4 A non-example

At first, it might appear reasonable to use Theorem 7.4 to “undo” each of the loops one-by-one. Certainly, one can define a sequence of boxes that satisfy both the 〔$\bigcup_{k=1}^{\infty} V_k$ is bounded in a compact set〕 and 〔$\lim_{n\to\infty} \bigcup_{n\in\mathbb{N}} V_n = 0$〕 conditions:

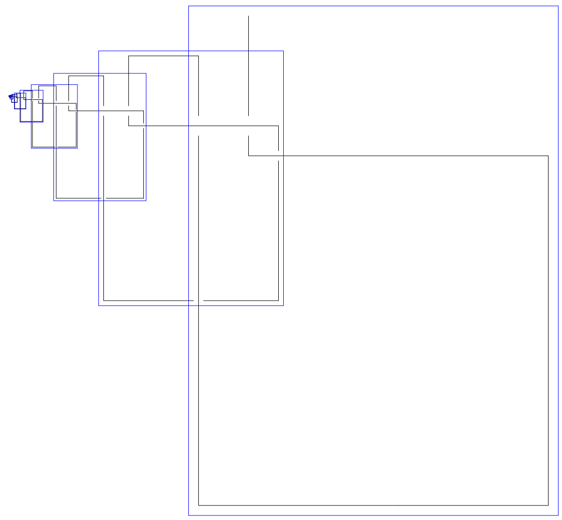


Figure 7.5 Some proposed V_n

And indeed, one can find a sequence of ambient isotopies that take us arbitrarily close to the unknotted strand. Yet, we can *prove* that this is actually a wild arc, using the invariant introduced by Fox and Artin (1948).

Theorem 7.5 (Fox-Artin[3] Tameness Invariant)**.** *Suppose that $\gamma : [0,1] \hookrightarrow \mathbb{R}^3$ is tamely embedded. Use $\mathcal{S}$ to denote the image*

$$\mathcal{S} = \overrightarrow{\gamma}([0,1]),$$

and let $p = \gamma(0)$. Now, let $\{V_n\}_{n=1}^{\infty}$ be a properly-nested sequence of nested closed sets such that for each n, $p \in V_n^{\circ}$, and further

$$\{p\} = \bigcap_{n=1}^{\infty} V_n.$$

Then there exists $N \in \mathbb{N}$ such that the homomorphism i_*

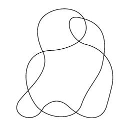
$$\pi_1(V_N - \mathcal{S}) \xrightarrow{i_*} \pi_1(V_1 - \mathcal{S})$$

induced by $i : V_N - \mathcal{S} \hookrightarrow V_1 - \mathcal{S}$ is trivial.[4]

We will try to be very thorough in the proof for those who might have seen the fundamental group before, but not had a lot of experience with it. This will likely

[3] Pronunciation Guide: [ˈaɐ̯tiːn]

[4] For those who might not have seen it before: Here, π_1 denotes the fundamental group.



strike others as a bit overkill. The version in Fox & Artin's paper is much terser, and can be found on page 7.[5]

Proof. If γ is tame, by definition, there exists a closed set $U \subseteq \mathbb{R}^3$ and a homeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that we have

$$(U, \mathcal{S}) \overset{f}{\cong} (\text{3-ball},\ \text{3-ball diameter}).$$

Without loss of generality, we suppose that for all $n \in \mathbb{N}$, $V_n \subseteq U$. Now, let $p' = f(p)$, $\mathcal{S}' = \overrightarrow{f}(\mathcal{S})$ (note this is the diameter), and let let $(V_n')_{n\in\mathbb{N}}$ be given by $V_n' = \overrightarrow{f}(V_n)$. Note that since f is a homeomorphism, the V_n' satisfy all the properties of the V_n. In particular, $p' \in V_1'^{\circ}$. Thus there exists $\varepsilon > 0$ such that

$$B_\varepsilon(p') \subseteq V_1'.$$

Also not that $\{p'\} = \bigcap_{n=1}^{\infty} V_n'$ implies that for some $N \in \mathbb{N}$, we have

$$V_N' \subseteq B_\varepsilon(p') \subseteq V_1'$$

and thus

$$[V'_N - \mathcal{S}'] \subseteq [B_\varepsilon(p') - \mathcal{S}'] \subseteq [V'_1 - \mathcal{S}']. \tag{7.1}$$

Choose some point $q \in W_N$ to serve as the base point for the fundamental groups of the sets in Eq. (7.1). Then observe that a

1. The inclusion $i : V_N' - S' \hookrightarrow V_1' - S'$ is trivially given by composing $i_1 : V_N' - S' \hookrightarrow B_\varepsilon(p')$ and $i_2 : B_\varepsilon(p') \hookrightarrow V_1' - S'$

$$\begin{tikzcd}
V_N' - \mathcal{S}' \arrow[rr, hook, "i"] \arrow[dr, hook, "i_1"'] & & V_1' - S' \\
& B_\varepsilon(p') \arrow[ur, hook, "i_2"'] &
\end{tikzcd}$$

Figure 7.6 Self-indulgent commutative diagram

2. $\pi_1(B_\varepsilon(p') - \mathcal{S}')$ is trivial (since $B_\varepsilon(p') - \mathcal{S}'$ is an open ball with a radius removed), and hence
3. The induced map $i_{2,*} : \pi_1(B_\varepsilon(p') - \mathcal{S}') \hookrightarrow \pi_1(V_1' - \mathcal{S}')$ is trivial, and hence because $i_* = i_{2,*} \circ i_{1,*}$, i_* is trivial.

[5]In case the page numbering has gotten messed up, the paragraph begins with "To show that Y is wildly imbedded [sic] we first develop a necessary condition for an imbedding of an arc to be tame."

$$
\begin{tikzcd}
\pi_1(V'_N - \mathcal{S}') \arrow[rr, hook, "i_*"] \arrow[dr, hook, "i_{*,1}"'] & & \pi_1(V'_1 - S') \\
& \pi_1(B_\varepsilon(p')) \arrow[ur, hook, "i_{*,2}"'] &
\end{tikzcd}
$$

Figure 7.7 Another self-indulgent commutative diagram

Now, since homeomorphism preserves the fundamental group, taking $V_N = \overleftarrow{f}(V'_N)$ yields the desired result. $\Diamond$

One can verify that the V_n's we drew in Fig. 7.5 don't satisfy this property. What's the problem?

The answer is that we lose bijectivity at $t = 1$ — not on the curve itself, but rather on the ambient space. There's no way to undo the curve shown in Fig. 7.5 without dragging infinitely-many points down into the wild point. To visualize why, imagine passing a solid torus vertically down through the first big loop.

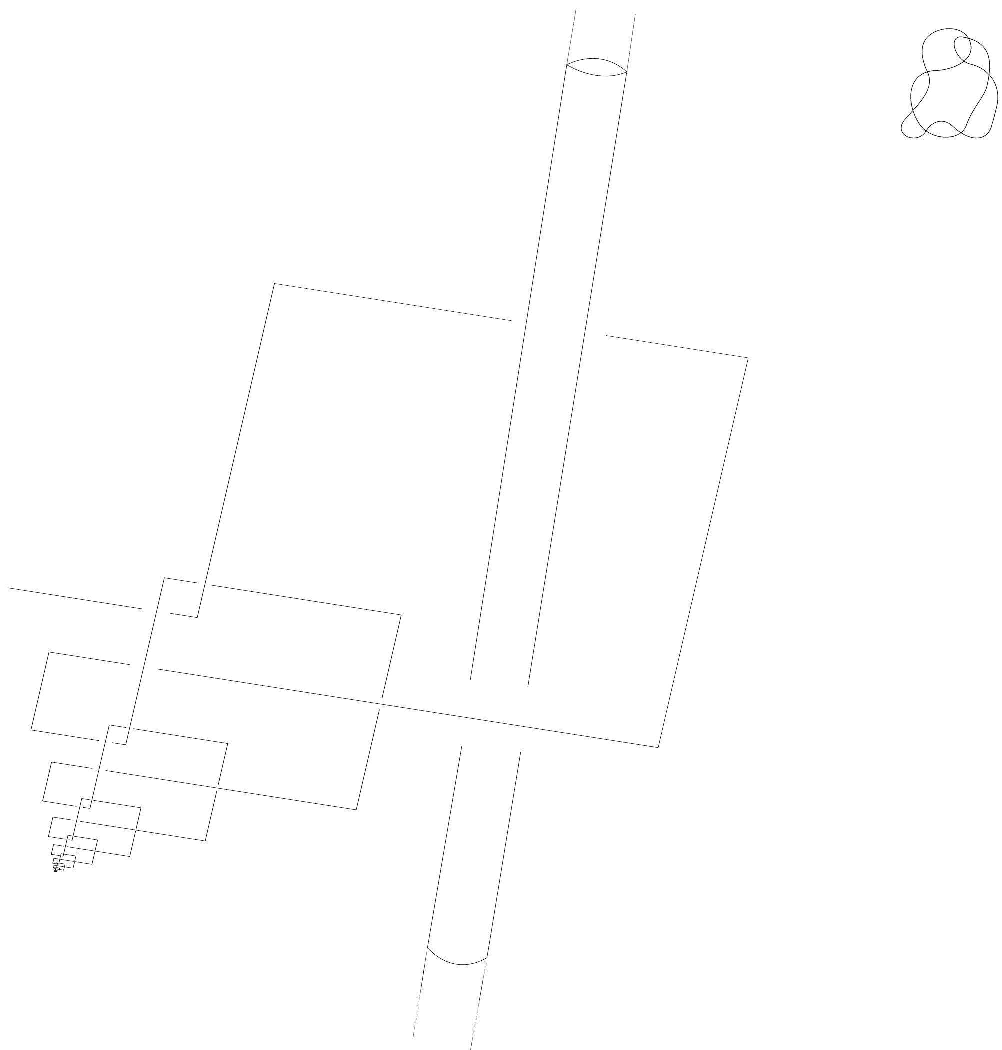


Figure 7.8 A 3D perspective.

Observe that as we remove the first loop, we end up pulling the cylinder *through* the second loop. Note, we're going to replace the torus with a line so as to make the diagrams easier to create. The same argument works either way; we felt the torus was more intuitive.

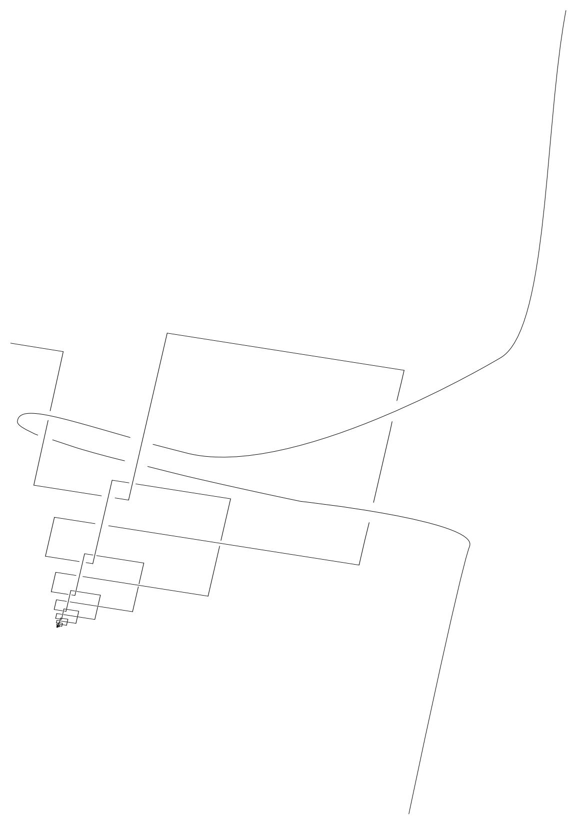


Figure 7.9 Pulling out a stitch.

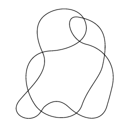
As we iterate this process further, note that the string/torus will get dragged further and further downwards until finally converging to the wild point. In fact, the same result holds for a torus passing through *any* such loop, so performing our proposed ambient isotopy would actually collapse infinitely-many points of the ambient space down to the wild point. Hence, this approach is no good, as it violates bijectivity.[6] ◊

Note. While we explicitly drew the torus/line in the figures above, note that we're not actually thinking about embedding a torus into $\mathbb{R}^3$. Rather, it's just a helpful visualization to see how the ambient space *must* get bent as we perform our proposed sequence of ambient isotopies.

Remark. One might wonder what would happen if instead we were allowed to move the loose end and pull *it* through the stitches one-by-one. Note that doing so would require moving said strand arbitrarily close to the wild point; hence, it would

[6] An alternative argument is to show the "undoing" maps don't converge uniformly.

be lost in the limit.



The preceding two examples are meant to convey that we must proceed with caution when dealing with Theorem 7.4. Example 7.1 shows that it's easy to take examples that work and have slight extensions break them. Theorem 7.4 shows us that it's easy to be deceived by our pictures; we must be careful to be very rigorous when entering the realm of general topological embeddings.

Now, we turn to our second tool: separating strands.

7.2 Separating Strands

7.2.1 A Note on Notation in S^1

From now on, we'll often have to work directly with sequences, open sets, etc. in S^1. To that end we'll want to have some nice notational shorthand to avoid having to say things like "let $s_0, s_1, s_3 \in S^1$ such that, if thought of as an embedded $\mathbb{R}^2$, we would encounter s_1 before s_2 when traveling counterclockwise around S^1 starting at $s_0 \ldots$" Instead, we'll write this with symbols.

- By $s_0 \prec^{\circlearrowleft} s_1 \prec^{\circlearrowleft} \cdots \prec^{\circlearrowleft} s_n$, we mean that we'd encounter s_0 before s_1 before ... before s_n *before* encountering s_0 a second time, if traversing around S^1 counterclockwise. Note, in displaystyle, $\prec^{\circlearrowleft}$ looks like the following:
$$s_0 \overset{\circlearrowleft}{\prec} s_1 \overset{\circlearrowleft}{\prec} \cdots \overset{\circlearrowleft}{\prec} s_n$$
In analogy with $<$, $\leq$, the $\preceq^{\circlearrowleft}$ symbol allows the possibility of equality.

- By $(s_0, s_1)_{\circlearrowleft}$, $[s_0, s_1]_{\circlearrowleft}$, we mean
$$(s_0, s_1)_{\circlearrowleft} = \left\{ s \in S^1 \;\middle|\; s_0 \overset{\circlearrowleft}{\prec} s \overset{\circlearrowleft}{\prec} s_1 \right\} \qquad [s_0, s_1]_{\circlearrowleft} = \left\{ s \in S^1 \;\middle|\; s_0 \overset{\circlearrowleft}{\preceq} s \overset{\circlearrowleft}{\preceq} s_1 \right\}$$
and analogously for $(s_0, s_1]_{\circlearrowleft}$, $[s_0, s_1)_{\circlearrowleft}$. We'll denote the topology generated by the $(s_0, s_1)_{\circlearrowleft}$ as $\mathcal{T}_{\circlearrowleft}$.

- Often, we'll think of S^1 as a metric space. We'll denote the "ball of radius epsilon around s" by $B_\varepsilon^{\circlearrowleft}(s)$.

- In general, we will try to refer to elements of S^1 by s, t, etc., but do note that t is also used for the time variable in ambient isotopies.

Note, despite the notation above looking evocative of an orientation, we are not actually thinking about S^1 as endowed with one. It just so happens that in the below, we will often need to think about the "next" element of a sequence in S^1, but again this will be detached from thinking of S^1 as oriented.

All of the theorems we show below work in $\mathbb{R}^2$ as well as $\mathbb{R}^3$, which we can see by taking K to be restricted to a 2D affine subset of $\mathbb{R}^3$.

Proposition 7.6. *Let $K : S^1 \hookrightarrow \mathbb{R}^3$ be an embedding, and let $I_0 = [s_0, t_0]_{\circlearrowleft}$ and $I_1 = [s_1, t_1]_{\circlearrowleft}$ be disjoint closed arcs of S^1. For the sake of notational compactness, define $Y_0 = \overrightarrow{K}(A_0)$ and $Y_1 = \overrightarrow{K}(A_1)$. Then*

$$\inf_{\substack{p_0 \in Y_0 \\ p_1 \in Y_1}} d(p_1, p_2) > 0.$$

Proof. Since K is an embedding, it is a homeomorphism onto its image. Now, since I_0, I_1 are disjoint compact sets, it follows that their images under K are as well. Hence Y_0, Y_1 are disjoint compact sets. From this point, the claim can be proven by one of the following methods:

1. (With Analysis) Let $f : Y_0 \to \mathbb{R}$ be defined as follows: for all $y_0 \in Y_0$, we have
$$f(y_0) = \inf_{y_1 \in Y_1} d(y_0, y_1).$$
Continuity of f follows directly from the triangle inequality. Now, since Y_0, Y_1 are disjoiont, $f(y_0) > 0$ for all $y_0 \in Y_0$. Since Y_0 is compact, f attains $\inf_{y_0 \in Y_0} f(y_0)$ on Y_0, which yields the desired result.

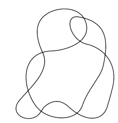
2. (With Topology) $\mathbb{R}^3$ is Hausdorff; apply the fact that disjoint compact sets in a Hausdorff space can be separated by open sets. After a little finagling the proof should pop out. □

Corollary 7.7. *The same result holds if we replace "disjoint closed arcs" in Proposition 7.6 with "arbitrary sets that have disjoint closures."*

Proof. The exact same proof as above works. We just chose to state the specific case in Proposition 7.6 first because it's more directly intuitive. □

The following theorem essentially says that we can separate strands in a knot by closed, cone-shaped neighborhoods that only possibly intersect at endpoints. Note, we'll index almost all variables with 0 subscripts to make it easier to refer to the analogues for the other strand described at the end of the statement.

Theorem 7.8. *Let $K : S^1 \to \mathbb{R}^3$ be a knot, and let $[s_0, t_0]_{\circlearrowleft} \subseteq S^1$ with $s_0 \neq t_0$. Then there exists a nonempty, open, connected set $U_0 \subseteq \mathbb{R}^3$ such that*



1. $\overrightarrow{K}((s_0, t_0)_{\circlearrowleft}) \subsetneq U_0$,
2. $\overrightarrow{K}([s_0, t_0]_{\circlearrowleft}) \subseteq \overline{U_0}$, *and*
3. $\overrightarrow{K}([s_0, t_0]^c_{\circlearrowleft}) \cap \overline{U_0} = \varnothing$.

Further, given another disjoint arc $[s_1, t_1]_{\circlearrowleft}$, the associated U_1 satisfies $\overline{U_0} \cap \overline{U_1} = \varnothing$.

Proof. Use I_0 to denote $[s_0, t_0]_{\circlearrowleft}$, and let its midpoint be denoted m_0. We first define a collection of "nested" subarcs of I_0 centered at m_0.[7]

Let ℓ be the length of I_0. Define $\varepsilon_{0,\max} = \frac{\ell}{2}$, and $E_0 \subseteq \mathbb{R}$ by

$$E_0 = (0, \varepsilon_{0,\max}).$$

For all $\varepsilon_0 \in E$, let

$$I_{\varepsilon_0} = B^{\circlearrowleft}_{\varepsilon_0}(m_0),$$

and observe that for all $\varepsilon_0' < \varepsilon_0$, we have

$$I_{\varepsilon_0'} \subsetneq I_{\varepsilon_0} \subsetneq I_0. \tag{7.2}$$

These are the desired "nested" subarcs of I_0.[8] Also note that taking closures in Eq. (7.2) preserves all of the containments.

Now: Consider the complement I_0^c in S^1, and note that it is an open arc of S^1. Also note that for all $\varepsilon_0 \in E_0$, $I_\varepsilon \subsetneq I_0$ implies

$$\overline{I_{\varepsilon_0}} \cap \overline{I_0^c} = \varnothing,$$

hence by Corollary 7.7, δ_{ε_0} given by

$$\delta_{\varepsilon_0} = d(\overrightarrow{K}(I_0^c), \overrightarrow{K}(I_{\varepsilon_0}))$$

satisfies $\delta_{\varepsilon_0} > 0$. We use these δ_{ε_0}'s to construct the desired open neighborhood U_0.

[7] We put "nested" in scare quotes because we'll actually have uncountably many subarcs in this collection, so there's no canonical notion of "successor" or "predecessor."

[8] For some intuition, note that as $\varepsilon_0 \nearrow \varepsilon_{0,\max}$, we have $I_{\varepsilon_0} \nearrow I_0$.

For each $\varepsilon \in E$, let $U_{\varepsilon_0} \subseteq \mathbb{R}^3$ be defined by

$$U_{\varepsilon_0} = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; d(\boldsymbol{x}, \overrightarrow{K}(\overline{I_{\varepsilon_0}})) < \frac{\delta_{\varepsilon_0}}{3} \right\}.$$

Note that each of these U_{ε_0} are open and nonempty.[9] Also note that by definition of δ_{ε_0}, for all $\varepsilon_0 \in E_0$, we have

$$U_{\varepsilon_0} \cap I_0^c = \varnothing.$$

Now, let

$$U_0 = \bigcup_{\varepsilon_0 \in E_0} U_{\varepsilon_0}.$$

Observe that U_0 is nonempty, open, and connected.[10] Also note that

1. $\overrightarrow{K}\big((s_0, t_0)_{\circlearrowleft}\big) \subsetneq U_0$,

2. $\overrightarrow{K}\big([s_0, t_0]_{\circlearrowleft}\big) \subsetneq \overline{U_0}$, and

3. By the distributive property,

$$\begin{aligned}
\overrightarrow{K}(I_0^c) \cap U_0 &= \overrightarrow{K}(I_0^c) \cap \left(\bigcup_{\varepsilon_0 \in E_0} U_{\varepsilon_0} \right) \\
&= \bigcup_{\varepsilon_0 \in E_0} \left(\overrightarrow{K}(I_0^c) \cap U_{\varepsilon_0} \right) \\
&= \varnothing.
\end{aligned}$$

Since $s_0, t_0 \notin I_0^c$ it follows that $\overrightarrow{K}(I_0^c) \cap \overline{U_0} = \varnothing$ as well, as desired.

This completes the proof of the claimed properties of U_0; it remains to show that given another arc $I_1 = [s_1, t_1]_{\circlearrowleft}$ with $I_0 \cap I_1 = \varnothing$, the associated U_1 satisfies $\overline{U_0} \cap \overline{U_1} = \varnothing$.

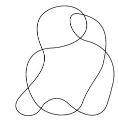
We proceed by contradiction. Suppose that $\overline{U_0} \cap \overline{U_1} \neq \varnothing$. We show that the points of intersection must be at the ends of the arcs $\overrightarrow{K}(I_0)$, $\overrightarrow{K}(I_1)$.

To see this, note that by definition, for all $\varepsilon_0 \in E_0$,

$$d(\overrightarrow{K}(I_{\varepsilon_0}),\ \overrightarrow{K}(I_1)) \geq \delta_{\varepsilon_0}.$$

[9]We need the $\frac{1}{3}$ for the $\overline{U_0} \cap \overline{U_1} = \varnothing$ claim.

[10]Connectedness follows from being a union of pairwise non-disjoint connected sets.



and similarly, for all $\varepsilon_1 \in E_1$,

$$d(\overrightarrow{K}(I_0),\ \overrightarrow{K}(I_{\varepsilon_1})) \geq \delta_{\varepsilon_1}.$$

Because the $U_{\varepsilon_0}, U_{\varepsilon_1}$, were defined in in terms of $\frac{\delta_{\varepsilon_i}}{3}$, one can use a simple triangle inequality argument to show that for all $\varepsilon_0 \in E_0, \varepsilon_1 \in E_1$, we have

$$d(U_{\varepsilon_0}, U_{\varepsilon_1}) > \frac{1}{3} \min\{\delta_{\varepsilon_0}, \delta_{\varepsilon_1}\}.$$

It follows that if $d(U_0, U_1) = 0$, this must occur when one of the δ's is $\underline{0}$ (i.e., $\varepsilon_0 = \varepsilon_{0,\max}$ or $\varepsilon_1 = \varepsilon_{1,\max}$, and hence $I_{\varepsilon_0} = I_0$, $I_{\varepsilon_1} = I_1$). This implies $\overrightarrow{K}(I_0)$, $\overrightarrow{K}(I_1)$ share an endpoint, a contradiction (囧) — the arcs were assumed to be disjoint. $\square$

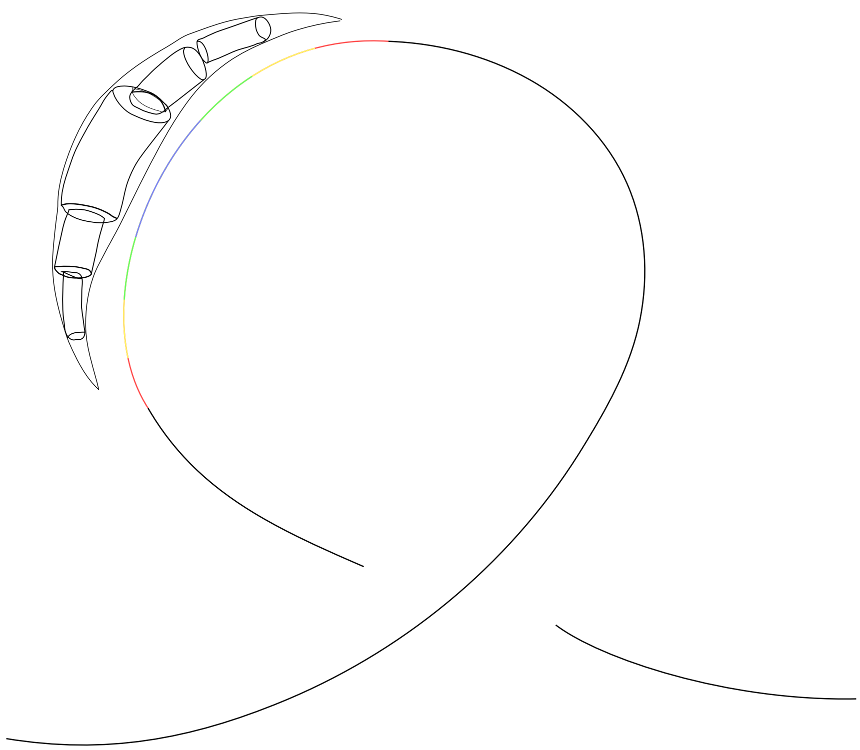


Figure 7.10 Example of V and the V_ε

We have a number of remarks to make about this theorem.

Remark. Note that in general, as $\varepsilon_0 \nearrow \varepsilon_{0,\max}$, we get $\delta_{\varepsilon_0} \searrow 0$. Further note that $\delta_{\varepsilon_0} : (0, \varepsilon_0) \to \mathbb{R}$ (viewed as a function of ε_0) is continuous and can be continuously extended to a compact domain $[0, \varepsilon_{0,\max}]$ by taking $\delta_{\varepsilon_0}(0) = d\big(m_0, \overline{A^c}\big)$ and $\delta_{\varepsilon_0}(\varepsilon_{0,\max}) = 0$. Hence δ_{ε_0} is in fact uniformly continuous.

Remark. Note that because we chose to define the I_{ε_0}'s in terms of the m_0, if the parameterization of K places $K(m_0)$ closer to one of the endpoints $K(s_0)$ or $K(t_0)$, then we'll get a much thinner & narrower U_0 than the one shown in Fig. 7.10. There are ways to change the proof above that avoid this issue, but we stuck with this version for simplicity.

On a similar note, observe that the theorem makes no claims about *substrands* of the strand I_0 being separated from each other. Indeed, K could even be everywhere wild on I_0! The point is just that we can isolate the behavior of K on I_0 from the behavior of K on $S^1 \setminus I_0$, provided we keep our endpoints fixed.

We have the following question:

Question 4. Is the U_0 defined in Theorem 7.8 always homeomorphic to $\mathbb{R}^3$? $\lozenge$

If we restrict ourselves to the analogous theorem for $\mathbb{R}^2$, a "yes" follows from the Theorem 8.1 (Jordan-Schoenflies; see next section). In $\mathbb{R}^3$, we don't have such a result; in particular, we can find pathological objects like the *Alexander horned sphere* that don't partition our space nicely into two topological 3-balls. Is it possible to find a knot for which the U_0 displays this kind of behavior? If not, this would provide us a nice correspondence with *local flatness*.

Chapter 8

Ambient Isotopy in $\mathbb{R}^2$

Then all around from away across the world
he smelled good things to eat
so he gave up being king of where the wild things are.
But the wild things cried, "Oh please don't go –
we'll eat you up — we love you so!"
And Max said, "No!"

—Maurice Sendak, *Where the Wild Things Are*

We will now examine how ambient isotopy behaves in $\mathbb{R}^2$. Our main goal is to show that in $\mathbb{R}^2$, *all* embeddings are ambient isotopic. The motivation is to use this result to show that in $\mathbb{R}^3$, for two curves γ_1, γ_2 embedded in a compact neighborhood $V \subseteq \mathbb{R}^3$ such that γ_1, γ_2 share the same endpoints, then if γ_1, γ_2 have no crossing points in the diagram given by some projection map π, then γ_1, γ_2 are ambient isotopic by an ambient isotopy fixing ∂V. This will be useful later when we attack the problem of determining which wild knots can be represented by a countable union of polygonal segments.

To that end, we must first discuss the extent to which we can separate distinct strands of a knot by closed neighborhoods. This will allow us to modify our knots locally by considering ambient isotopies in these neighborhoods that fix the boundaries. Then, we will be able to apply Theorem 7.4 or Theorem 5.3 to combine all of these into a *single* ambient isotopy achieving our desired effects.

8.1 All Embeddings in $\mathbb{R}^2$ are Ambient Isotopic

We begin by recalling the Jordan-Schoenflies theorem:[1]

Theorem 8.1 (Jordan-Schoenflies)**.** *Let $\gamma : S^1 \to \mathbb{R}^2$ be a simple closed curve, and let $\iota : S^1 \hookrightarrow \mathbb{R}^2$ be the standard inclusion of S^1 as the unit circle. Then there exists a homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f \circ \gamma = \iota$.*

Note, "simple closed curve" is equivalent to "embedding." We just chose "simple closed curve" to match the language commonly used when stating the theorem. Anyways, the following is an immediate corollary:

Corollary 8.2 (All Embeddings in $\mathbb{R}^2$ are Ambient Homeomorphic.)**.** *Let $K_0, K_1 : S^1 \hookrightarrow \mathbb{R}^2$. Then K_0 and K_1 are ambient homeomorphic.*

Proof. By Jordan-Schoenflies, there exist ambient homeomorphisms $f_0, f_1 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f_0 \circ K_0 = \iota =$ and $f_1 \circ K_1 = \iota$. Hence

$$\begin{aligned} (f_1^{-1} \circ f_0) \circ K_0 &= f_1^{-1} \circ (f_0 \circ K_0) \\ &= f_1^{-1} \circ \iota \\ &= f_1^{-1} \circ (f_1 \circ K_1) \\ &= K_1. \end{aligned}$$

□

We now show the main result: In $\mathbb{R}^2$, ambient orientation-preserving homeomorphism is equivalent to ambient isotopy. Again, we know this holds in the PL case, but were unable to find a proof for the topological case. First, we have the following:

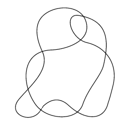
Lemma 8.3. *Let $K_0, K_1 : S^1 \hookrightarrow \mathbb{R}^2$ be simple polygonal curves. Then K_0, K_1 are ambient isotopic by some $F : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ such that for some compact set V, F is identity on $[0,1] \times (\mathbb{R}^2 \setminus V^\circ)$.*

Proof. This follows as a consequence of an argument similar to those given in Appendix B.4. In particular, see Corollary B.10 □

Our goal is to take finer and finer polygonal approximation of K, using Lemma 8.3 to argue ambient isotopy with the unit circle, and taking a limit to obtain the result. There is a problem that we need to be mindful of, though: If we aren't careful, naively connecting subsequent points together with straight lines will introduce crossings, which would prevent us from

[1] Pronunciation guide: [ʒɔʀdɑ̃] and [ˈʃøːnfliːs], respectively.

applying Lemma 8.3. Our strategy will be to use Theorem 7.8 to separate the strands of our knot from each other, and then use the lemma below to connect the endpoints of these strands by an ε-close polygonal approximation.



Lemma 8.4. *Let $X \subseteq \mathbb{R}^n$ be open and connected. Then X is polygonally path-connected.*[2]

Proof. Let $x_0 \in X$ be arbitrarily chosen. We want to show that x_0 is polygonally path connected to every other point of X. To that end, define C_{poly} by

$$C_{\text{poly}} = \{x \in X \mid \text{ there exists a finite polygonal path in } X \text{ from } x_0 \text{ to } x\}.$$

Note that C_{poly}, C^c_{poly} partition X. We claim C_{poly} is clopen.

- (C_{poly} is open): Let $x_1 \in C_{\text{poly}}$ be arbitrarily chosen. Then by definition, there exists a polygonal path $P_{0,1}$ from x_0 to x_1. We'll use this later.

  Now, since X is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(x_1) \subseteq X$. We want to show $B_\varepsilon(x_1) \subseteq C_{\text{poly}}$. To that end, let $x_2 \in B_\varepsilon(x_1)$ be arbitrary, and bserve that the straight line from x_1 to x_2 is a polygonal path in $B_\varepsilon(x_1)$ (and hence in X as well). Dxenote it by $P_{1,2}$. Then concatenating $P_{0,1}$ and $P_{1,2}$ yields another finite polygonal path, hence $x_2 \in C_{\text{poly}}$.

  Since x_2 was arbitrarily chosen, it follows that $B_\varepsilon(x_1) \subseteq C_{\text{poly}}$, as desired.

- (C_{poly} is closed): Here, we want to show C^c_{poly} is open. We proceed by contradiction.

  To that end, let $x_1 \in C^c_{\text{poly}}$ be arbitrarily chosen. Suppose, to obtain a contradiction, that $\forall \varepsilon > 0$, there exists $x_\varepsilon \in B_\varepsilon(x_1)$ such that $x_\varepsilon \notin C^c_{\text{poly}}$. Then $x_\varepsilon \in C_{\text{poly}}$. Thus there exists a path $P_{0,\varepsilon}$ from x_0 to x_ε. We have two subcases.

  1. Suppose that the straight line path $P_{\varepsilon,1}$ from x_ε to x_1 were contained in X. Then concatenating $P_{0,\varepsilon}$ and $P_{\varepsilon,1}$ would yield a finite polygonal path from x_0 to x_1, 囧.
  2. Suppose that the straight line path $P_{\varepsilon,1}$ from x_ε to x_1 were not contained in X. Then $B_\varepsilon(x_1) \nsubseteq X$. Since $\varepsilon > 0$ was arbitrarily chosen, it follows that there exists no $\varepsilon > 0$ such that $B_\varepsilon(x_1) \subseteq X$. 囧, $x_1 \in X$, and X is open.

[2] I.e., every two points are connected by a finite sequence of line segments.

In either case we obtain a contradiction. Hence, C_{poly}^c is open.

It follows that C_{poly} is clopen.

Finally, observe that C_{poly} is nonempty, since for any $x \in X$, taking ε sufficiently small yields x is polygonally path connected to any $x' \in B_\varepsilon(x)$. The result follows. □

Remark. Note, we don't make any mention of whether the path above is guaranteed to be simple. In fact, we can guarantee this: since our path is a finite union of polygonal line segments, it intersects itself at most finitely many times (if two segments are fully parallel, that case can be treated by taking the union of the two and performing the following procedure). When that occurs, take a subdivision of the path at the points of intersection and trim out the extraneous loops. This gives a simple polygonal path.

The theorem now follows fairly easily.

Theorem 8.5. *Let $\iota : S^1 \hookrightarrow \mathbb{R}^2$ be the standard embedding of S^1 as the unit circle, and let $K : S^1 \hookrightarrow \mathbb{R}^2$ be arbitrary. Then ι, K are ambient isotopic.*

Proof. We'll construct an ambient isotopy from ι to K by applying a uniform convergence argument à la Theorem 7.4. First, select 3 distinct points $s_0 \prec^{\circlearrowleft} s_1 \prec^{\circlearrowleft} s_2 \in S^1$.

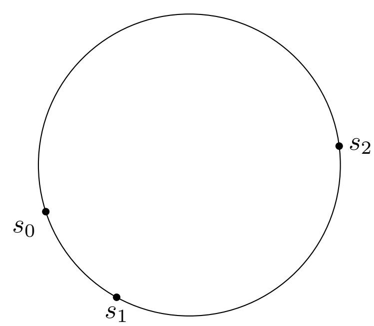
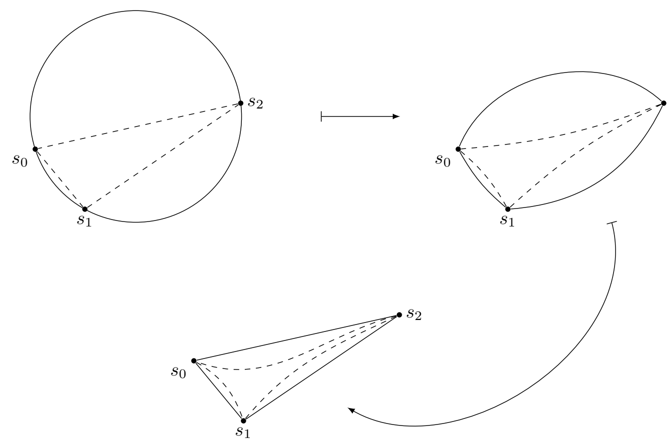


Figure 8.1 Example s_0, s_1, s_2

Apply an ambient isotopy F_0 to turn ι into a triangle with these points as the vertices.[3]

[3] See Appendix B.4 for some inspiration on how to define this extremely rigorously.



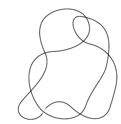


Figure 8.2 Turning ι into a triangle. The strange layout is just to get the diagram to fit on the page. Note, the dashed lines represent the boundary of the original triangle $\triangle\iota(s_0)\iota(s_1)\iota(s_2)$

We now define a uniformly convergent sequence of polygonal curves converging to our embedding.

For each $n \in \mathbb{N}$, let $\varepsilon_n = \frac{1}{2^n}$, and note that $\varepsilon_n > 0$. Observe that $K : S^1 \hookrightarrow \mathbb{R}^2$ is a continuous function between metric spaces and K has a compact domain. Then by the Heine-Cantor theorem we have that K is uniformly continuous, hence there exists $\delta_n > 0$ such that for arbitrarily chosen[4]

$$s_{0_n} \overset{\circlearrowleft}{\prec} s_{1_n} \overset{\circlearrowleft}{\prec} \cdots \overset{\circlearrowleft}{\prec} s_{k_n},$$

provided that 〔for all $i = 1, \ldots, k_n - 1$ we have $d(s_{i_n}, s_{i+1_n}) < \delta_n$〕, it follows that 〔for all such i we have $d(K(s_{i_n}), K(s_{i+1_n})) < \varepsilon_n$ as well〕.

For each $n \in \mathbb{N}$, define P_n to be the common refinement of the partition defined above with P_{n-1}. Observe that P_n gives us a collection of closed intervals $[s_{i_n}, s_{i+1_n}]_{\circlearrowleft}$ such that the interiors are pairwise disjoint; denote these by I_{i_n}. Observe that by Theorem 7.8, there exists an open neighborhood U_{i_n} such that U_{i_n} is open, nonempty, and connected, and $\overrightarrow{K}(I_{i_n})$ intersects $\overline{U_{i_n}}$ only at the end points $K(s_{i_n})$, $K(s_{i+1_n})$. Applying Lemma 8.4, we can connect the endpoints by a finite polygonal path $P_{i_n,i+1_n}$.

Further, observe that by the construction of U_{i_n}, we have $d(P_{i_n,i+1_n}, \overrightarrow{K}(I_{i_n})) < \varepsilon_n$. Now, define a polygonal embedding K_n by linking together all of the P_{i_n}.

[4]The double subscripts here are just meant to communicate that "s_{0_n}" refers to a different element for each n. Hopefully it's not too confusing.

Observe that the result is a simple, closed, polygonal curve.[5] By Lemma 8.3, it follows that all of the K_n are ambient isotopic to each other by ambient isotopies $F_n : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$. For each $n = 0, 1, \ldots$, define $f_n : \mathbb{R}^2 \to \mathbb{R}^2$ to be the homeomorphism given by $f_n(x) = F_n(1, x)$. Then define an ambient isotopy $F : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$F(t,x) = \begin{cases} F_0(2t, x) & \text{if } t \in \left[0, \frac{1}{2}\right] \\ F_1(4(t-1/2), f_0(x)) & \text{if } t \in \left(\frac{1}{2}, \frac{3}{4}\right] \\ F_2(8(t-3/4), f_1(f_2(x))) & \text{if } t \in \left(\frac{3}{4}, \frac{7}{8}\right] \\ \vdots & \\ F_n\left(2^{n+1}\left(t - 1 + \frac{1}{2^n}\right), \bigcirc_{k=0}^{n-1} f_k(x)\right) & \text{if } t \in \left(1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}\right] \\ \vdots & \end{cases}$$

After arguing bijectivity, one can then apply a uniform convergence argument similar to that in the proof of Theorem 7.4 to the sequence of F_n's to show$F$ is an ambient isotopy. Note that by construction of the F_n, we have $F(1, \iota(s)) = K(s)$ for all $s \in S^1$. Hence ι and K are ambient isotopic, as desired! $\square$

The following is immediate.

Corollary 8.6. *Let $K_0, K_1 : S^1 \hookrightarrow \mathbb{R}^2$ be arbitrary topological embeddings. Then K_0 and K_1 are ambient isotopic.*

Proof. K_0 and K_1 are both ambient isotopic to ι and ambient isotopy is an equivalence relation (just apply one at $2\times$ speed and then then the inverse of the other also at $2\times$ speed). $\square$

Before moving on, it's worth mentioning that a very similar proof to the above yields the following version for arcs:

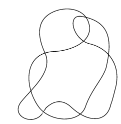
Theorem 8.7. *Let $\gamma_0, \gamma_1 : [0,1] \hookrightarrow \mathbb{R}^2$ be embeddings such that $\gamma_0(0) = \gamma_1(0)$ and $\gamma_1(0) = \gamma_1(1)$. Also let $V \subseteq \mathbb{R}^2$ be closed such that*

$$\partial V \cap \overrightarrow{\gamma_0}([0,1]) = \{\gamma_0(0), \gamma_0(1)\} = \partial V \cap \overrightarrow{\gamma_1}([0,1])$$

Then there exists an ambient isotopy $F : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ from γ_0 to γ_1 such that F is identity on $[0,1] \times (\mathbb{R}^2 - V^\circ)$.

[5] "Simple" follows because the $\overline{U_{i_n}}$ are all disjoint except perhaps at the endpoints. The endpoints are unchanged by applying Lemma 8.4.

In particular, we can locally modify curves in a way that doesn't affect the rest.



Sketch. Use a similar polygonal approximation argument as in the above. To get F identity on $[0,1] \times (\mathbb{R}^2 - V^\circ)$, use the fact that $\partial V \cap \overrightarrow{K_0}([0,1]) = \{K_0(0), K_0(1)\} = \partial V \cap \overrightarrow{K_1}([0,1])$ to find appropriate partitions of $[0,1]$ such that the approximation gets finer as we approach $[0,1]$. $\square$

8.2 Feral Points

Observe that in Theorem 8.5, we have placed *no conditions* on K_0, K_1 other than that they are topological embeddings. Hence, even cases like the following example are covered!

Example 8.1. Consider a curve defined piecewise by the following dynamics.

$$f(r,\theta) = \begin{cases} \dot{r} = -r^2 \\ \dot{\theta} = -1. \end{cases} \tag{8.1}$$

This gives us solutions of the form

$$\begin{cases} r(t) = \frac{1}{t+\frac{1}{r_0}} \\ \theta(t) = -t + \theta_0. \end{cases} \tag{8.2}$$

Consider a solution given by $\theta_0 = 0$ and $r_0 > 0$. Then converting to Cartesian coordinates by $x(t) = r(t)\cos(\theta(t))$ and $y(t) = r(t)\sin(\theta(t))$, we have

$$\begin{cases} x(t) = \frac{\cos(t)}{t+\frac{t}{r_0}} \\ y(t) = -\frac{\sin(t)}{t+\frac{t}{r_0}} \end{cases}$$

Note, we used sin odd, cos even to get the $-$ signs out of the arguments. Extend $x(t), y(t)$ to $t = \infty$ by taking $x(\infty) = \dot{x}(\infty) = 0$, $y(\infty) = \dot{y}(\infty) = 0$. As justification: From Eq. (8.1), one can see that $t = \infty$ gives a fixed point in cartesian coordinates (since $\dot{r} = 0$), and by Eq. (8.2), we can see that it indeed occurs at $(0,0)$, hence this continuous (and in fact smooth) on $\mathbb{R}^{\geq 0} \cup \{\infty\}$.[6] We now have a curve that looks something like the following, depending on our choice of r_0:[7]

[6]The topology we choose on $\mathbb{R}^{\geq 0} \cup \{\infty\}$ is defined by U is open iff $U \in \mathcal{T}_{\text{std}}$ or $U = V \cup \{\infty\}$ where $V \in \mathcal{T}_{\text{std}}$ is unbounded.

[7]Smaller r_0 gives us loops that look more circular, since $\dot{r}$ is nonlinear in r.

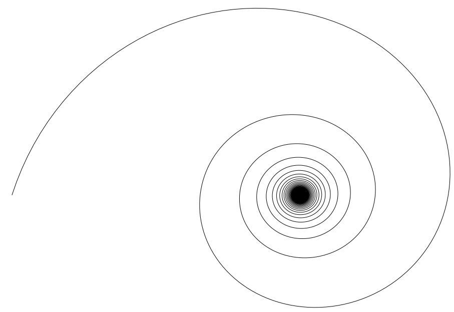


Figure 8.3 A spiral.

We want to reparameterize to $\tau \in [0,1]$. Observe that $t : [0,1] \to \mathbb{R}^{\geq 0} \cup \{\infty\}$ given by $\tau(t) = \tan\left(\frac{\pi}{2} t\right)$ is continuous and monotonic, hence

$$\gamma(\tau) = \begin{cases} x(\tau) = x(t(\tau)) \\ y(\tau) = y(t(\tau)) \end{cases}$$

gives a valid reparameterization of the original trajectory. Now, we seek to make γ closed. To do so, consider two solutions γ_0, γ_1 with slightly different initial radii r_0, r_1. This gives us something like the following.

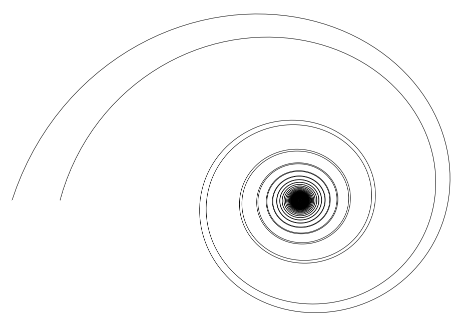


Figure 8.4 γ_0 and γ_1.

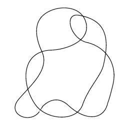
One can use the $\dot{r}$ equation in Eq. (8.1) to argue that γ_0, γ_1 only intersect at $t = 1$. Now, define $\gamma_{0,1}(t)$ by

$$\gamma_{0,1}(t) = \begin{cases} \gamma_0(2t) & t \in \left[0, \frac{1}{2}\right) \\ (0,0) & t = \frac{1}{2} \\ \gamma_1(2-2t) & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

It follows that $\gamma_{0,1}(t)$ is continuous.[8] We now extend $\gamma_{0,1}$ to a fully closed curve

[8]Unfortunately, it is not smooth; $\dot{x}(t), \dot{y}(t)$ oscillate unboundedly. To get a smooth (but

by giving it a π rotation plus a translation, and concatenating with the curve in Fig. 8.4.



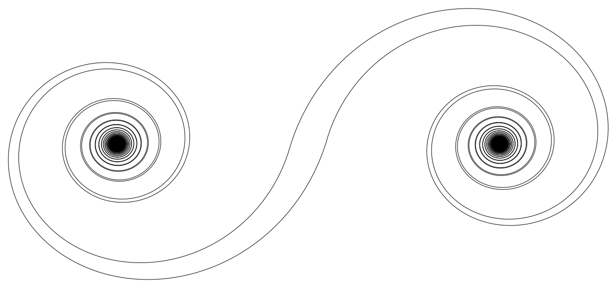


Figure 8.5 A "Wild-Looking" Curve in $\mathbb{R}^2$

Again: as can be seen from Fig. 8.4, one can define this curve explicitly by applying a rotation to $\gamma_{0,1}(t)$, applying an appropriate shift, and adjusting the time scales to get agreement on the two boundary points. From Theorem 8.5, it follows that this curve is ambient isotopic to the unit circle! $\diamondsuit$

There's something strange going on at the centers of the spirals. Namely, any line passing through them intersects the boundary of the curve infinitely many times. For lack of a better name, we'll call such points *feral* points.[9] We'll draw a distinction here between *wiggly feral points* and *swirly feral points* in the definitions below. Note, in $\mathbb{R}^2$, it might not be immediately obvious why we would want to think of *wiggly feral points* as pathological at all, since they seem very easy to control. However, as we'll see in $\mathbb{R}^3$, points that look like *swirly feral points* when we take a projection into $\mathbb{R}^2$ might turn out to be just *wiggly feral points* in another.

less pretty looking) curve, one can replace the $\dot{r} = -r^2$ equation with $\dot{r} = -r$ to yield a logarithmic spiral, after which the construction above gives a smooth curve. If doing this, there are a number of ways to close the two open ends together: e.g., one could extend the tangent line for the inner curve outwards until it's possible to close with the outer curve using a semi-circular cap. This would yield a C^1 embedding, but not a C^2 one.

[9] *Feral* is chosen for its meaning as "having escaped from domestication and become wild" (Merriam-Webster, https://www.merriam-webster.com/dictionary/feral). If a name for this concept already exists, or if the reader has a better suggestion for a name, please contact the author at `fkobayashi@g.hmc.edu`.

Definition 8.1 (Wiggly Feral Point)**.** Let $K : S^1 \hookrightarrow \mathbb{R}^2$ be an embedding. Let $s \in S^1$, and let $p = K(s)$. Reparameterize K to be in polar form $r_p(t), \theta_p(t)$ (for $t \in S^1$) such that the pole is at p. Then if there exist sequences $(s_n)_{n\in\mathbb{N}}, (t_n)_{n\in\mathbb{N}}$ such that

1. For all $n \in \mathbb{N}$, $s_n \prec^{\circlearrowleft} t_n \prec^{\circlearrowleft} s_{n+1}$, and
2. At least one of the following holds:
   (a) For all $n \in \mathbb{N}$,
   $$r_p(t_n) > \max\{r_p(s_n), r_p(s_{n+1})\},$$
   or
   (b) For all $n \in \mathbb{N}$,
   $$\theta_p(t_n) > \max\{\theta_p(s_n), \theta_p(s_{n+1})\}.$$

Then we call p a *wiggly feral point* (sometimes just *wiggly point*) of K. We say that K is *wiggly feral* or *wiggly* at p. $\Diamond$

Remark. This definition is meant to encode the idea that the graph of K "oscillates" infinitely often as we approach p. Note, this is quite different form our example in Fig. 8.5, where we actually have $\theta \nearrow \infty$. By contrast, *wiggly feral points* are meant to capture behavior like the following:

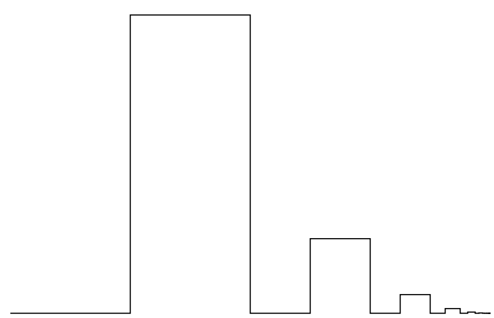
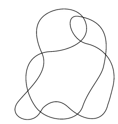


Figure 8.6 An arc with a wiggly feral point.



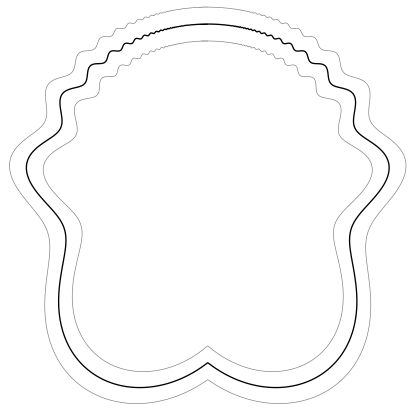


Figure 8.7 A closed curve in $\mathbb{R}^2$ with a wiggly feral point. We've drawn a tubular neighborhood around it just for pizzazz.

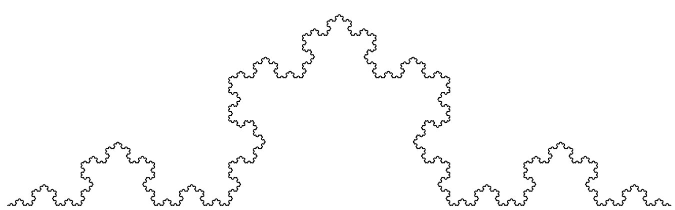


Figure 8.8 An everywhere-wiggly arc. Essentially every non-self-intersecting fractal is everywhere wiggly.

The case where $\theta \to \infty$ is covered by *swirly feral points.*

Definition 8.2 (Swirly Feral Point)**.** Let $K : S^1 \hookrightarrow \mathbb{R}^2$ be an embedding. Let $s \in S^1$ be arbitrarily chosen, and let $p = K(s)$. Reparameterize K to be in polar form $r_p(t), \theta_p(t)$ (for $t \in S^1$) such that the pole is at p. Then if $\theta_p(t) \to \pm\infty$ as $t \to s$, we say p is a *swirly feral point* (sometimes just *swirly point*) of K. We say K is *swirly feral* or sometimes just *swirly* at p. $\Diamond$

Sometimes it will be helpful to think of the swirly feral points without needing to reparameterize. The following proposition gives one way of doing so.

Proposition 8.8. *Let $K : S^1 \hookrightarrow \mathbb{R}^2$ be an embedding. Let $s \in S^1$ be arbitrarily chosen, and let $p = K(s)$. Consider an arbitrary infinite sequence* *of nested arcs I_n:*

$$\underbrace{[s_1, t_1]_{\circlearrowleft}}_{I_1} \subseteq \underbrace{[s_2, t_2]_{\circlearrowleft}}_{I_2} \subseteq \cdots \subseteq \underbrace{[s_n, t_n]_{\circlearrowleft}}_{I_n} \subseteq \cdots .$$

For each $n \in \mathbb{N}$, define γ_n to be the (possibly non-simple) curve defined by taking $\overrightarrow{K}(I_n)$ and connecting $K(s_i)$ and $K(t_i)$ with a straight line. Use w_n to denote the winding number of γ_n about p. Then p is a swirly feral point *of K iff there exists such a sequence of I_n's such that*

$$\lim_{n\to\infty} w_n = \pm\infty.$$

Remark. One should note that we require *nested* arcs in the proposition above. The reason might not be immediately obvious: One can imagine a variety of swirly-looking curves that yield visually-similar graphs to Fig. 8.5 but always "double back" before w_n can get unbounded (we apologize for the lack of diagrams here). Why don't we want to include those? As it turns out, they are actually easier to treat, and are really better thought of as wiggly feral points.

One might wonder whether feral points have to obey some kind of discreteness properties. This appears not to be the case, but we have not investigated the matter in great detail. As a first example, note that Fig. 8.8 is actually *everywhere* wiggly, hence we certainly can't expect discreteness from wiggly feral points.[10] For swirly-feral points, the situation is a bit trickier, but we propose the following counterexample to the claim "swirly feral points must be topologically discrete."

Counterexample 8.2. Let K be defined iteratively from K_n as follows.

1. Let K_0 be the spiral embedding from Example 8.1.

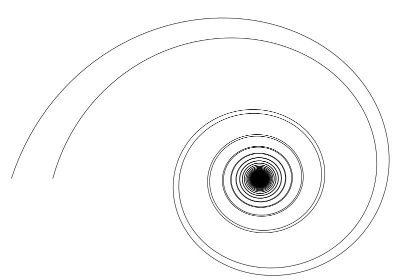
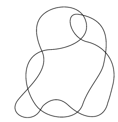


Figure 8.9 The spiral, redux

[10]For an even more pathologically-flavored example, one might consider an *Osgood curve*, which can actually have positive measure in $\mathbb{R}^2$.



2. Draw a line through the center of the spiral:

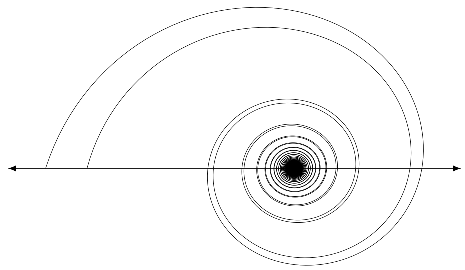


Figure 8.10 A line through the centers

3. Define K_i as follows: At each point of intersection x_i of the line with the spiral, $\overrightarrow{K}(S^1)$ locally looks like an arc. Hence, remove a small portion from K, and replace it with a shrunk-down copy of the spiral.[11]

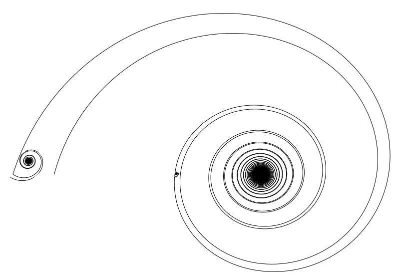


Figure 8.11 A crude diagram for the replacement process. Unfortunately, `pgfplots`'s drawing capabilities gave out on us past the second replacment.

By applying uniform convergence, one can show that the limit of the K_i is continuous.

This gives an infinite sequence of swirly points converging to another swirly point. Hence we see the swirly points are not topologically discrete.

We imagine that a similar construction can be used in a fractal-like construction to yield a curve that is swirly feral on an uncountable set. We leave some questions for future work:

Question 5. Let $K : S^1 \hookrightarrow \mathbb{R}^3$. Just how "not discrete" can the swirly feral points of K be? E.g.,

[11]To guarantee the spiral will fit, we can apply Theorem 7.8.

1) Can the swirly points of K be perfect?[12]
2) Can the swirly points of K be dense in $\overrightarrow{K}(S^1)$?
3) Can the swirly points of K have positive measure in $\overrightarrow{K}(S^1)$? (Here, we're viewing S^1 as $[0, 2\pi]/\{0 \sim 2\pi\}$ inheriting the Lebesgue measure on $[0, 1]$).
4) Can K be swirly almost everywhere?
5) Can K be everywhere swirly feral? $\Diamond$

We have the following conjecture:

Conjecture 2. *The answers to the first three above are "yes," "yes," "yes," "no," and "no." Here's some of our intuition:*

1) *This seems fairly straightforward to accomplish by constructing a K that maps a Cantor set to swirly feral points (e.g., identify the two loose ends of each sub-spiral with the endpoints of a Cantor set in S^1 and prove the correspondence).*
2) *Here, one might try a construction that iterates through $\{2\pi t \mid t \in \mathbb{Q} \cap [0, 1]\}$ where the radii of our inserted sub-spirals decay according to the denominator of t.*[13]
3) *Here, one might try the same argument as in (1) but with the Smith-Volterra-Cantor set.*
4) *Our intuition is a bit shaky on these last two, but it seems that any construction for an "almost everywhere swirly feral embedding" would require the swirls to get small so quickly that in the limit, there's no winding about our points of interest. We'd be interested to see an example, though.*
5) *If almost everywhere swirly feral fails, then so does everywhere swirly feral.*

This concludes our discussion of the 2D case. We now move into 3D, where all the interesting pathologies lie.

[12] Recall, perfect means "closed and has no isolated points."
[13] This is basically Thomae's function but swirly.

Chapter 9

Moving into $\mathbb{R}^3$

> The wild things roared their terrible roars and gnashed their terrible teeth
> and rolled their terrible eyes and showed their terrible claws
>
> —Maurice Sendak, *Where the Wild Things Are*

In this chapter, we partially extend the machinery from the previous chapter for use in $\mathbb{R}^3$. The idea is that away from crossing points, arcs in our diagrams in $\mathbb{R}^2$ can just be treated with Theorem 8.7, and this lifts to an ambient isotopy in $\mathbb{R}^3$ that only affects one strand. We will use this in developing a simple condition on our diagrams that guarantees we can represent the corresponding embeddings by countable unions of polygonal segments — namely, that the set of crossing points are topologically discrete. In addition to tame arcs, certain classes of wild arcs (e.g., the Fox-Artin curve in Example 7.2) will fall into this category. This sets the stage for future work in which we hope to use Theorem 7.4 to show an analogue of Reidemeister's theorem holds. We present a sketch of how we might plan to attack the problem, however, we have not confirmed the details.

We begin by defining a slightly-relaxed version of regular diagrams, namely *discrete diagrams*. These are identical to regular diagrams, except we allow the set of crossing points to be *discrete* instead of restricting them to be *finite*. Note, the discreteness condition allows at most countably-many crossings (Proposition 9.1).

In Section 9.1 and Section 9.2, we then develop the two pieces of machinery needed to analyze such diagrams. Section 9.1 is concerned with demonstrating that planar isotopy is *always* valid, even without PL constraints. This allows us to control the behavior of our knots away from crossings, and the proofs are fairly straightforward in light of Chapter 8. In Section 9.2, we do the complimentary analysis, showing we can control the behavior of our knots near crossing points; in particular, we can represent every crossing by two perpendicular line segments.[1]

Finally, we put these pieces together to yield the promised proof that a discrete diagram is sufficient to guarantee a representation as a countable union of polygonal segments. Anyways, the definition.

Definition 9.1 (Discrete Diagram)**.** Let $K : S^1 \hookrightarrow \mathbb{R}^3$ be an embedding. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$, and let D be the diagram of K given by π. Then we say D is a *discrete* diagram iff

1. (No multi-crossings): For all $p \in D$, $|\overleftarrow{\pi}(p)| \leq 2$
2. (All crossings are transverse): The diagram is locally X-shaped at points of crossing, and
3. (Crossings are discrete): The set $\mathcal{C}$ of crossing points of D is topologically discrete in $\mathbb{R}^2$. ◊

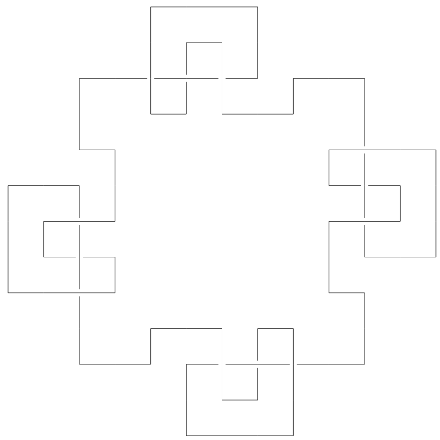
We have the following proposition:

Proposition 9.1. *If D is a discrete diagram, then the set of crossing points $\mathcal{C}$ is at most countable.*

Proof. $\mathcal{C}$ is a subspace of $\mathbb{R}^2$, which is second-countable. It follows $\mathcal{C}$ is second-countable. Since a discrete space is second-countable iff it is countable, we have the desired result. □

One might wonder what a *non*-discrete diagram might look like, and whether it is even possible to obtain one. Indeed we can — *everywhere-wild* knots are examples of embeddings for which no diagram is discrete. We provide an example construction below (imagine taking the limit as $n \to \infty$), but for more on this subject, one should reference Shilepsky (1974) and Bothe (1981).

[1] Of course, this might seem obvious, but a rigorous argument turns out to be highly technical given that we are *only* assuming the crossings are discrete (and not that our knot is tame).



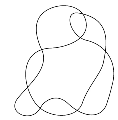


Figure 9.1 Step 1 in the construction of an everywhere-wild knot

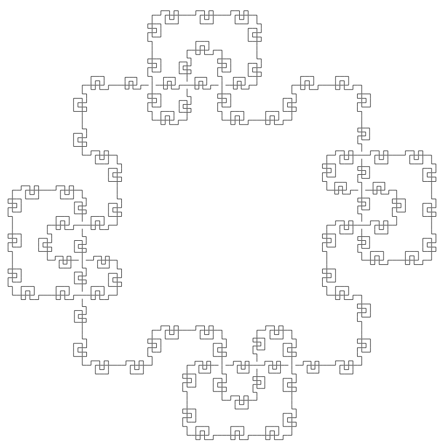


Figure 9.2 Step 2 in the construction of an everywhere-wild knot

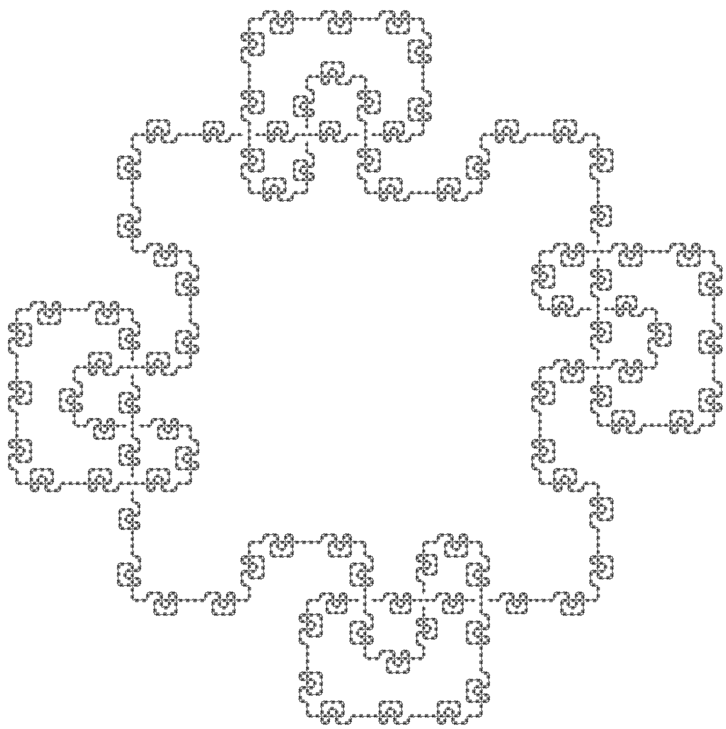


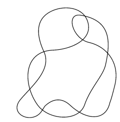
Figure 9.3 Step 3 in the construction of an everywhere-wild knot

9.1 Controlling Behavior Away from Crossings

We now prove lemmas that allow us to perform planar isotopies in the toplogical category. The first says that we can polygonalize our diagrams away from crossing points.

Lemma 9.2. *Let $K : S^1 \hookrightarrow \mathbb{R}^3$ be an embedding with diagram D and associated projection π. Let $[s_0, t_0]_{\circlearrowleft} \subseteq S^1$ such that $A = \overrightarrow{K}([s_0, t_0]_{\circlearrowleft})$ participates in no crossings in D. Then there exists an ambient isotopy $F : [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$ taking A to a curve P such that $\overrightarrow{\pi}(P)$ is polygonal.*

Sketch. Use Theorem 7.8 to find a closed neighborhood $V \subseteq \mathbb{R}^3$ separating A from other strands (except at the endpoints). Use $\overrightarrow{\pi}(V)$ to apply Theorem 8.7 to the diagram to yield an ambient isotopy $G : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ yielding the desired polygonal curve in $\mathbb{R}^2$. Without loss of generality, suppose that π projects onto the xy plane; then we can lift G an ambient isotopy in $\mathbb{R}^3$ by the following:



$$F(t,(x,y,z)) = \begin{bmatrix} G_x(t,(x,y)) \\ G_y(t,(x,y)) \\ z \end{bmatrix}$$

Here, G_x, G_y denote the x and y components of G, respectively. That F is an ambient isotopy on $\mathbb{R}^3$ can be argued from the fact that G is on $\mathbb{R}^2$. $\square$

Corollary 9.3. *With all variables quantified as above, there exists an ambient isotopy $F : [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$ taking A to a curve P such that P is polygonal in $\mathbb{R}^3$.*

Note that this is stronger than the preceding claim, which only guaranteed $\overrightarrow{\pi}(P)$ is polygonal.

Sketch. First, apply Lemma 9.2. Because the curve is now represented as a *finite* collection of polygonal segments P, one can argue that it's possible to perturb π slightly (such that it now includes a portion of the view along the z direction) without introducing any new crossing points. Applying the result again to this new curve yields a result that is polygonal in $\mathbb{R}^3$ (since its x, y, and z components are now all linear). $\square$

Now, we can show that given a crossing-less strand in a knot, we can locally "straighten it out" to some extent. The removal of feral points is important in that it allows us to create a "nice" neighborhood of our curve in the below.

Proposition 9.4. *Let $K : S^1 \hookrightarrow \mathbb{R}^3$ be an embedding with diagram D and associated projection π. Let $[s_0,t_0]_\circlearrowleft \subseteq S^1$ such that $\overrightarrow{K}([s_0,t_0]_\circlearrowleft)$ does not participate in a crossing in D. Now, let $\gamma : [s_0,t_0]_\circlearrowleft \hookrightarrow \mathbb{R}^3$ be a curve such that*

1. *(Same endpoings in $\mathbb{R}^3$): $\gamma(s_0) = K(s_0)$ and $\gamma(t_0) = K(t_0)$, and also*
2. *(Same diagram in $\mathbb{R}^2$): On all of $[s_0,t_0]_\circlearrowleft$, $\pi \circ \gamma = \pi \circ K$.*

Then if we define

$$K_\gamma(s) = \begin{cases} K(s) & s \notin [s_0,t_0]_\circlearrowleft \\ \gamma(s) & s \in [s_0,t_0]_\circlearrowleft, \end{cases}$$

then $K \cong K_\gamma$.

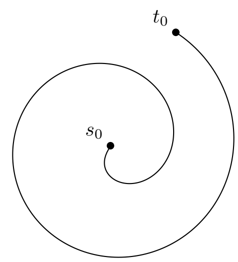


Figure 9.4 An example of A.

Proof. Let $A_K = \overrightarrow{K}([s_0, t_0]_{\circlearrowleft})$, and let $A_\gamma = \overrightarrow{K}([s_0, t_0]_{\circlearrowleft})$. Also let $C = \overrightarrow{\pi}(A_K) = \overrightarrow{\pi}(A_\gamma)$. Observe that the inverse image

$$M = \overleftarrow{\pi}(C)$$

gives us an embedding of a closed subset $S \subseteq \mathbb{R}^2$ into $\mathbb{R}^3$:

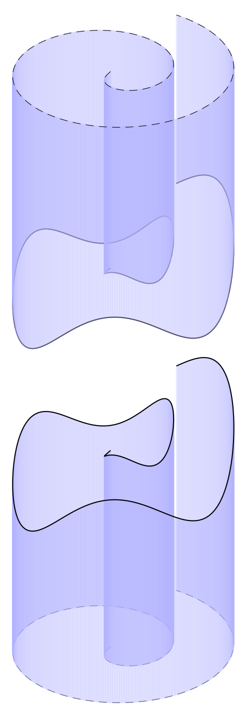
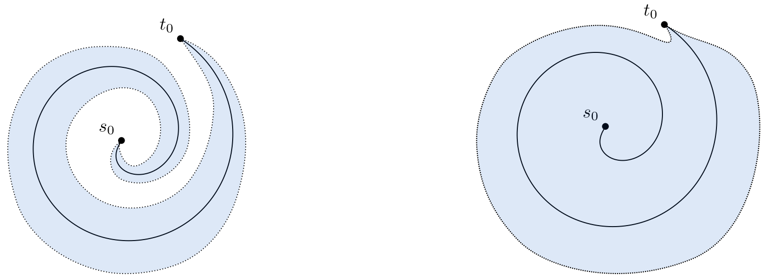


Figure 9.5 An example of the embedded S (here shown in blue), shown split into two halves to make the diagram easier to interpret. Note, viewing from the top down yields Fig. 9.4

Now, let $V \subseteq \mathbb{R}^2$ be the neighborhood guaranteed by Theorem 7.8. We want to find a way to guarantee that ∂V gives us two curves that "trace" the shape of C, instead of kind of "enveloping" it (see Fig. 9.6).



a. ✓ Good **b.** ✗ Bad

Figure 9.6 Examples of good and bad V's.

To that end, one can apply Lemma 9.2 to yield an ambient isotopy F_P giving a polygonal representation P_K of A_K, and argue that we can guarantee the existence of such a V in that case. Note that by the construction in Lemma 9.2, applying F_P also yields a polygonal representative P_γ of A_γ. In any case, the inverse image of the diagram for A_K gives us a polygonal version of Fig. 9.5, denote it S_P. Observe that one can now apply Theorem 8.7 to S_P to yield an ambient isotopy $G : [0,1] \times S_P \to S_P$ taking P_K to P_γ. By using a similar argument to Proposition B.8 (Appendix B.4), one can then extend this to an ambient isotopy $H : [0,1] \times (V \times \mathbb{R}) \to \mathbb{R}^3$ that fixes ∂V.[2] Finally, defining $F : [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$ by

$$F(t,x) = \begin{cases} F_P(3t, x) & t \in \left[0, \frac{1}{3}\right] \\ H(3t-1, F_P(1,x)) & t \in \left[\frac{1}{3}, \frac{2}{3}\right] \\ F_P(3-3t, H(1, F_P(1,x))) & t \in \left[\frac{2}{3}, 1\right] \end{cases}$$

yields an ambient isotopy taking A_K to A_γ while keeping the endpoints fixed, and not moving any of the other strands of K. It follows that F defines an ambient isotopy from K to K_γ. □

[2]In 2D, the lines are defined by putting one anchor point in ∂V, and making the other a point on the diagram for P_K. The properties we need are maintained when we consider the analogues in $\mathbb{R}^3$.

9.2 Controlling Behavior Near Crossings

In the following lemma we show that given any knot K with a discrete diagram, we can "straighten out" the strands near each crossing c_i in the sense that we can find a cylinder around $\overleftarrow{\pi}(\{c_i\})$ which K intersects at only four points. In particular, we can do this in a way that doesn't move any of the strands outside of our cylinder.

Theorem 9.5 (Cleaning up near crossings)**.** *Let $K : S^1 \to \mathbb{R}^3$ be a knot. Suppose that K admits a discrete diagram D, and let c_i be an arbitrary crossing of D. Then there exists* [$\varepsilon, \nu > 0$ *with* $\nu < \varepsilon$] *such that there exists an ambient isotopy $F : [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$ with the following properties:*

(i) *F is identity outside of $[0,1] \times \overleftarrow{\pi}(B_\varepsilon(c_i))$, and*

(ii) $|F(1,K) \cap \partial[\overleftarrow{\pi}(B_\nu(c_i))]| = 4$.

Note. Observe that the two lines above employ radii for the balls. The outer one will serve as a sort of "buffer zone" we can push things into in case we find some extra strands floating around.

Remark. At first it might seem like it'd be sufficient to just shrink the radius until we're sufficiently close to the crossing. However, it turns out this can fail for cases like the following where a feral point occurs at the crossing.

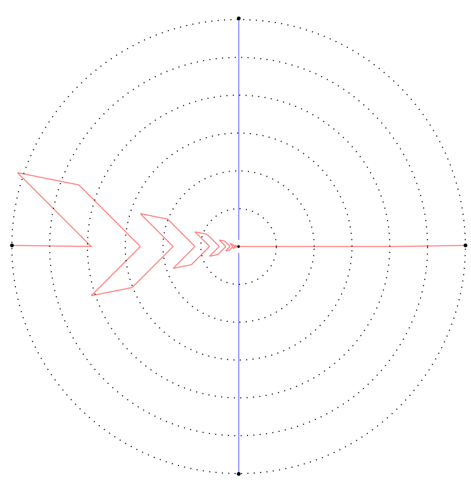
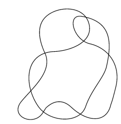


Figure 9.7 An example of a crossing and a B_ε where $\forall \eta \in (0, \varepsilon)$, K intersects $B_\eta(c_i)$ more than 4 times.

Proof. By definition of a discrete diagram, c_i is an isolated point of the set of crossings, $\mathcal{C}$. Hence, there exists $\varepsilon > 0$ such that $B_\varepsilon(c_i) \cap \mathcal{C} = \{c_i\}$.



In general, we can have many arcs of K contained in $B_\varepsilon(c_i)$; however, the definition of a discrete diagram stipulates only *two* such arcs will be involved in the crossing itself (see Fig. 9.8).

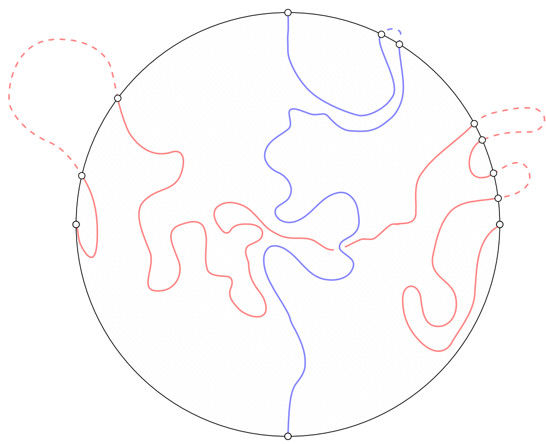


Figure 9.8 Example of $B_\varepsilon(c_i)$. The dashed lines show places where the strands poke outside $B_\varepsilon(c_i)$. Note, if being pedantic, the ball should be technically be centered on c_i.

We'll take a somewhat funky notation here and use $\overleftarrow{A}$ to denote the preimage of the arcs in Fig. 9.8.

$$\overleftarrow{A} = K(S^1) \cap \overleftarrow{\pi}(B_\varepsilon(c_i)).$$

We can visualize $\overleftarrow{A}$ by something like the following:

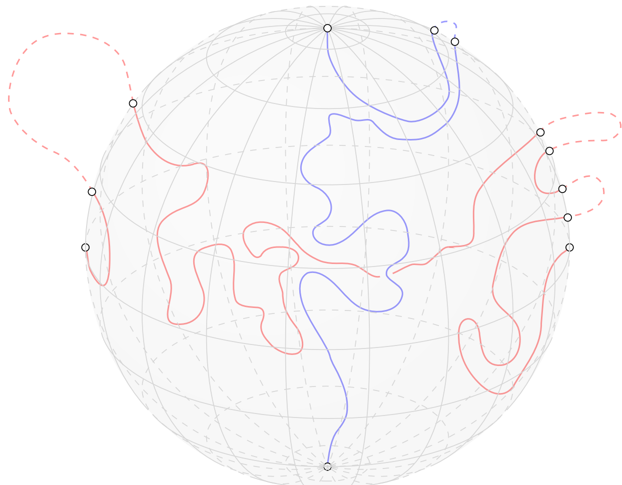


Figure 9.9 Example $\overleftarrow{A}$ in $\mathbb{R}^3$, with a 3-ball drawn to help communicate that we're no longer in $\mathbb{R}^2$. Note that the dashed lines are *not* included in $\overleftarrow{A}$.

Without loss of generality, suppose our projection π was onto the xy plane. We want to show that $\overleftarrow{A}$ is bounded in the z direction. To that end, note that K is a continuous function from a compact space, hence it is bounded. This gives us $\overleftarrow{A}$ is bounded, hence there exists $r_\varepsilon > 0$ such that $\overleftarrow{A} \subseteq \overline{B_\varepsilon(c_i)} \times [-r_\varepsilon, r_\varepsilon]$. Call this cylinder C_ε.

$$C_\varepsilon = \overline{B_\varepsilon(c_i)} \times [-r, r].$$

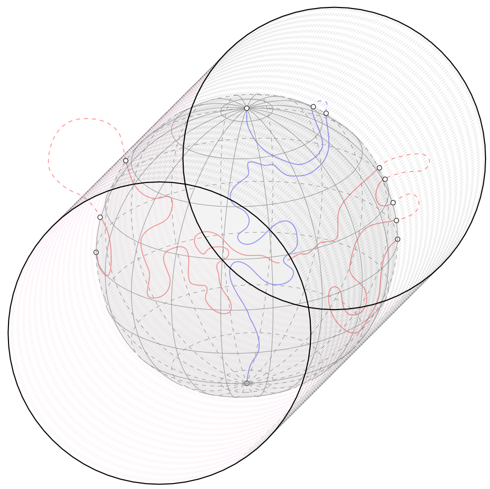


Figure 9.10 An example drawing for C_ε

We now have a claim. Essentially, it states that if we look at a sub-cylinder C_η that's small enough, there are only finitely many points where our strands intersect the boundary. This will be important when we construct an ambient isotopy to remove them later. Note, there is nothing special about using c_i defined in the following claim. The same should hold for any point in $\mathbb{R}^3$.

Claim 1: There exists $\eta, r_\eta > 0$ such that $\eta < \varepsilon$, $r_\eta < r_\varepsilon$, and C_η defined by

$$C_\eta = \overline{B_\eta(c_i)} \times [-r_\eta, r_\eta].$$

is such that $\partial C_\eta \cap \overleftarrow{A}$ is finite.

Proof of Claim 1: We show existence for η; the existence of r_η follows similarly.

Suppose, to obtain a contradiction, that no such η exists. One can show that this requires the strand to oscillate infinitely (think wiggly-feral point), per the following sketch: The only other way the claim could occur would be if for each η, there were a strand going around part of the circumference of $B_\eta(c_i)$. One can show that each such strand has nonzero length, and that this causes problems with the fact that there are infinitely-many η (we'd end up filling a portion of the region).

Let $\eta_0, \eta_1 \in (0, \varepsilon]$ with $\eta_0 \neq \eta_1$; without loss of generality suppose $\eta_0 < \eta_1$. By hypothesis, infinitely many arcs $\{A_j\}_{j \in J}$ of $\overleftarrow{A}$ intersect both of the cylinders.

$$C_{\eta_0} = B_{\varepsilon_0}(c_i) \times [-r_\varepsilon, r_\varepsilon] \qquad\qquad C_1 = B_{\eta_1}(c_i) \times [-r_\varepsilon, r_\varepsilon].$$

This might look something like the following:

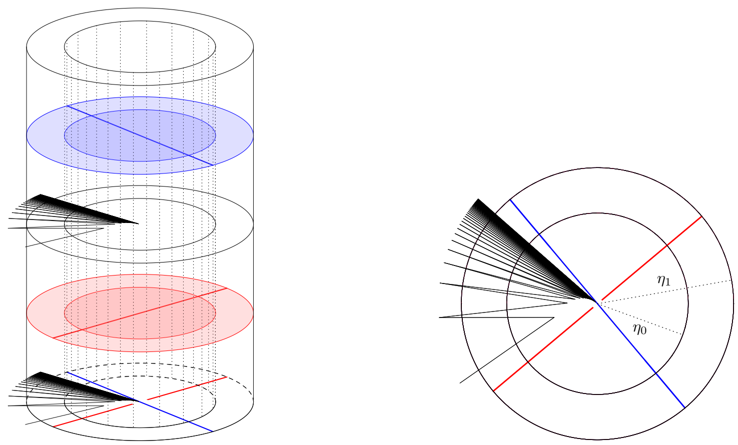


Figure 9.11 An example visualization in 3D showing the top strand of the crossing, another arc, and the bottom strand of the crossing, each separated vertically. The projection obtained is shown at the bottom, and the corresponding top-down view displayed on the right.

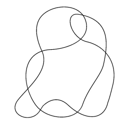
We want to show this breaks continuity of K. To that end, we'll employ sequential continuity. First, we define a collection of sub-strands that live in the annulus between η_0 and η_1: For all $j \in J$, let

$$B_j = \begin{cases} A_j \cap \overline{C_{\eta_1} - C_{\eta_0}} & \text{if } |A_j \cap C_{\eta_0}| = 1 = |A_j \cap C_{\eta_1}|, \text{ and} \\ \varnothing & \text{otherwise.} \end{cases}$$

Note that the B_j omit the arcs that just oscillate around one of ∂C_{η_0}, ∂C_{η_1} without ever going over to the other.

To ensure the B_j are actually valid arcs, we must subdivide them into their maximal connected components and remove all of the empty entries. Reindex the B_j accordingly to yield $\{B_\ell\}_{\ell \in L}$. Then define sets $\{O_\ell\}_{\ell \in L}$, $\{I_\ell\}_{\ell \in L}$ (O for "outer", I for "inner") by for all $\ell \in L$,

$$O_\ell = B_\ell \cap \partial C_{\eta_1} \qquad \text{and} \qquad I_\ell = B_\ell \cap \partial C_{\eta_0}.$$



Let $(O_n)n \in \mathbb{N}$ be an arbitrary sequence of distinct O_ℓ's. Note that C_{η_1} is compact, thus we can apply Bolzano-Weierstrass to $\{O_n\}_{n\in\mathbb{N}}$ to obtain a convergent subsequence $\{O_{n_k}\}_{k\in\mathbb{N}}$. Now, observe that because the O's are endpoints of arcs bridging ∂C_{η_0}, C_{η_1}, each O_{n_k} has a corresponding $I_{n_k} \in \partial C_{\eta_0}$. Apply Bolzano-Weierstrass to $\{I_{n_k}\}_{k\in\mathbb{N}}$ to get *another* convergent subsequence, call it $(I_{n_m})_{m\in\mathbb{N}}$. Note that restricting the O's to $(O_{n_m})_{m\in\mathbb{N}}$ still yields a convergent sequence.

Now, recall that because K is an embedding, it is a homeomorphism onto its image. Thus, image and preimage under K preserve convergence, and so taking the elementwise inverse of the O_{n_m}, I_{n_m} gives us sequences in

$$\{\mathcal{O}_{n_m}\}_{m\in\mathbb{N}} \qquad \text{and} \qquad \{\mathcal{I}_{n_m}\}_{m\in\mathbb{N}}$$

in S^1. One can show that they must converge to the same $s \in S^1$ as follows: Observe each arc A_{n_m} linking O_{n_m}, I_{n_m} corresponds to an interval of one of the forms

$$[\mathcal{I}_{n_m}, \mathcal{O}_{n_m}]_{\circlearrowleft} \qquad \text{or} \qquad [\mathcal{O}_{n_m}]_{\circlearrowleft}, \mathcal{I}_{n_m}.$$

Since S^1 is compact, in order for these intervals to be disjoint (required since the images are disjoint) the lengths must go to 0 as $m \to \infty$. Thus $\mathcal{O}_{\ell_m} \to \mathcal{I}_{\ell_m}$.

But note, taking the image once more gets us that $K(\mathcal{O}_{\ell_m}) = O_{\ell_m} \not\to I_{\ell_m} = K(\mathcal{I}_{\ell_m})$ (since the O_{ℓ_m}, I_{ℓ_m} are separated by $\eta_1 - \eta_0$). But this is a contradiction (囧), K preserves convergence. Thus, there must exist $\eta > 0$ such that only finitely many of the arcs of $\overleftarrow{A}$ intersect $B_\eta(c_i) \times \mathbb{R}$. Applying a similar argument for r_η gives us the existence of the desired C_η, which proves the claim. $\square$

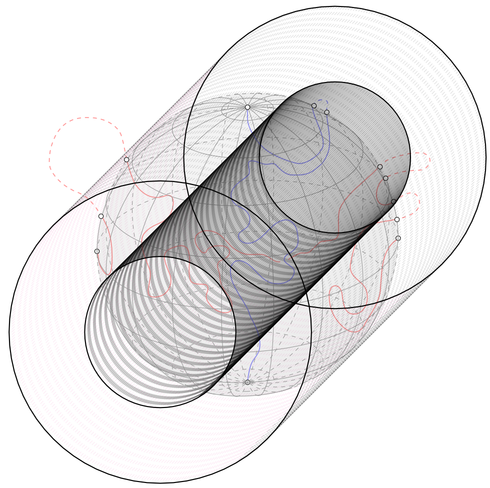


Figure 9.12 An attempted drawing of C_ε and C_η.

Remark. Note that in the particular example shown in Fig. 9.12, we could have just taken $\eta = \varepsilon$. An example of when we can't is when one of the strands travels in a circular arc partway around ∂C_ε.

OK: By the claim, there exists η, $r_\eta > 0$ such that

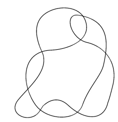
$$\eta \in (0, \varepsilon) \qquad \text{and} \qquad r_\eta \in (0, r_\varepsilon)$$

and $\overleftarrow{A}$ intersects $\partial C_\eta = B_\eta(c_i) \times [r_\eta, r_\eta]$ at finitely-many points. Apply the claim a second time to define a proper sub-cylinder C_ν by $C_\nu = B_\nu(c_i) \times [-r_\nu, r_\nu]$ (where $\nu < \eta$ $r_\nu < r_\eta$) such that $\overleftarrow{A}$ only intersects C_ν at finitely-many points.

The next claim essentially states that we can remove [the strands that don't participate in our crossing] from C_ν by pushing them out into C_η. Note, the fact that we have only *finitely* many points of intersection helps in avoiding an appeal to Theorem 7.4.[3]

> **Claim 2:** Let S_o, S_u be the strands corresponding to the overstrand and understrand (respectively) in Fig. 9.8. Let $\{S_j\}_{j=1}^n$ be the collection of arcs of $\overleftarrow{A}$ that intersect C_η but *do not* participate in the crossing. Then we can find an ambient isotopy $F : [0,1] \times C_\eta \to C_\eta$ such that

[3]There is a more direct proof using the techniques of Appendix B.4, but it's horrible to write out formally.



1. F keeps 〔both the boundary of C_η *and* the crossing strands S_o, S_u〕 fixed, and
2. For all $x \in S_j$, $F(1,x) \in C_\eta \setminus C_\nu$ (F moves all the other strands outside C_ν).

Proof of Claim 2: Observe that for each $j = 1, \ldots, n$, ∂T_j yields a crossing-less curve. Thus, one can apply Proposition 9.4 to find ambient isotopies moving each into $C_\eta \setminus C_\nu^\circ$, keeping the rest of the region fixed. Since we have only finitely-many strands to remove, concatenating them yields another ambient isotopy. $\square$

One way to visualize the net effect of this process is as follows. Note that the crossing strands (marked with the black dots in Fig. 9.13) partition $B_\eta(c_i)$ into four disjoint regions R_1, R_2, R_3, R_4 that contact $\partial B_\nu(c_i)$. Let M_1, M_2, M_3, M_4 be defined by for each index $\mu \in \{1,2,3,4\}$, $M_\mu = \partial R_\mu$.

Note: In the figures below, we accidentally re-drew $B_\nu(c_i)$ as $B_\varepsilon(c_i)$, and $B_\eta(c_i)$ as some ball containing B_ε. We apologize for this inconsistency; unfortunately, we currently do not have enough time to go back and fix them.

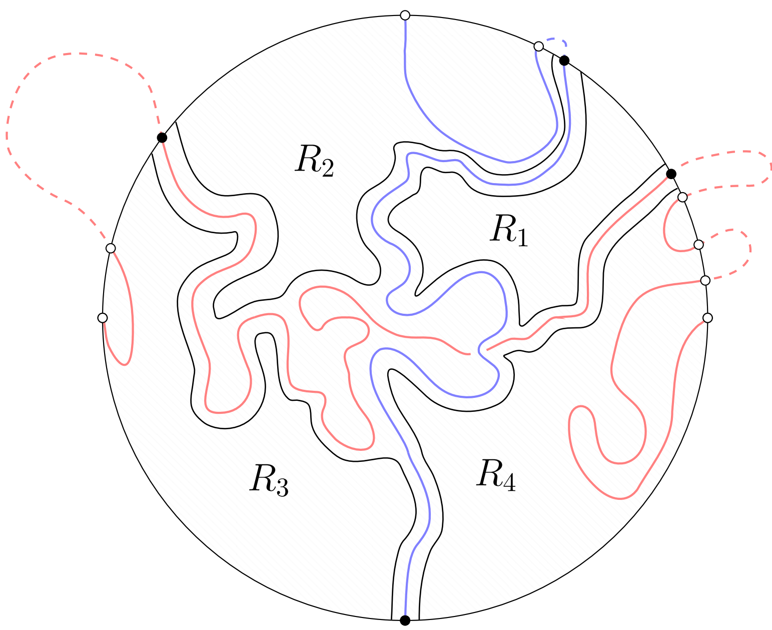
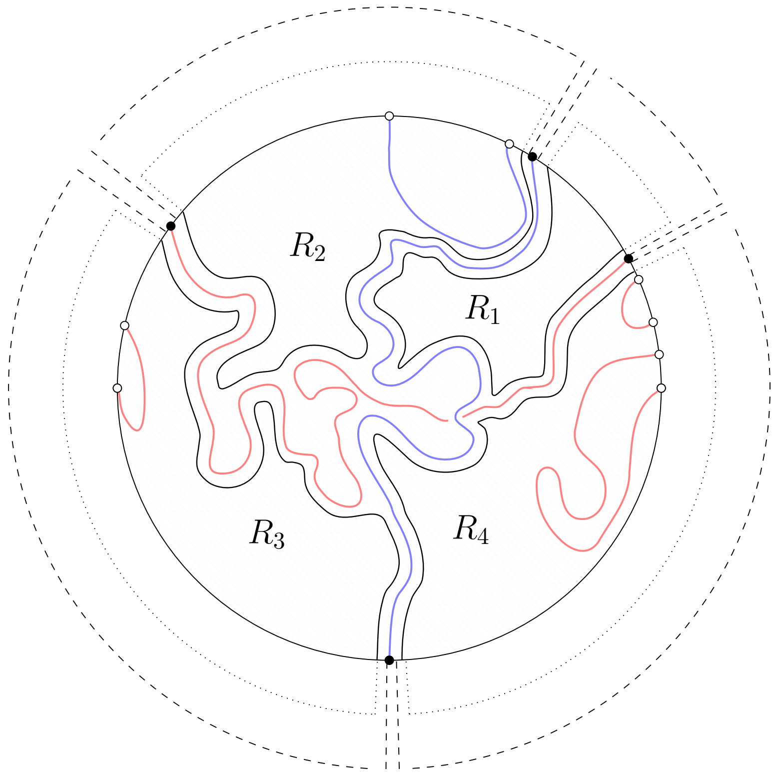


Figure 9.13 The regions R_1, R_2, R_3, R_4, and (approximations of) the corresponding M_1, M_2, M_3, M_4's in black. Not to scale.

Then we "push" $\partial R_\mu \cap \partial \overline{B_\nu(c_i)}$ outwards to the dotted curves shown below, and $\partial R_\mu \cap B_\nu(c_i)$ out to the circumference. The dashed lines (representing $\partial B_\eta(c_i)$) remain fixed.



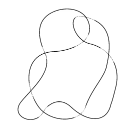
Finally: observe that by construction,

1. $C_\nu \subsetneq C_\eta \subsetneq C_\varepsilon$,
2. $|F(1,K) \cap \partial[\overleftarrow{\pi}(B_\nu(c_i))]| = |F(1,K) \cap \partial C_\nu| = |\{\text{endpoints of } S_o, S_u\}| = 4$, and
3. ∂C_ε remains fixed.

As desired. $\square$

Corollary 9.6 (K can be polygonalized near crossings)**.** *Let K, c_i, C_ν, C_ε be as defined in the preceding theorem/proof. Then there exists an ambient isotopy $F : [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$ such that*

1. *F is identity outside of $[0,1] \times C_\varepsilon$, and*



2. *$F(1,K) \cap C_\varepsilon$ is polygonal.*

Sketch. By the Theorem 9.5, we can find an ambient isotopy removing all non-crossing strands from C_ν while keeping ∂C_ε and 〔the crossing strands themselves〕 fixed. Apply this ambient isotopy from $t = 0$ to $t = 1/2$. Then,

- Observe that there are no other strands in $\overleftarrow{\pi}(B_\nu(c_i))$ "above" S_o or "below" S_u.
- Use this to define an ambient isotopy flattening out S_o in the z direction (so that it's only non-polygonal in x and y), while translating it half of the way to the top circular cap of C_ν. Apply a similar ambient isotopy flattening S_u out in z and moving it half of the way to the bottom circular cap.[4]
- Use a proof similar to that of Proposition 9.4 (taking a smaller sub-cylinder if necessary) to turn the crossing strands into straight lines while keeping ∂C_ν fixed.

This yields the desired result. □

Finally, this concludes the analysis of behavior near crossing points. We are now ready to put it all together.

9.3 The Conclusion

Together, these give us the following theorem:

Theorem 9.7 (Discrete Diagram Implies Countably Polygonal)**.** *Let $K : S^1 \hookrightarrow \mathbb{R}^3$ be a knot. Then if K admits a discrete diagram D, then K is ambient isotopic to a knot comprised of (at most) a countable union of polygonal segments.*

Proof. Apply Theorem 9.5 to triangulate K at each of the crossing points. Note that this implicitly defines a collection of pairwise-disjoint arcs connecting each of the resulting C_ν together. Applying Corollary 9.3, we can polygonalize each of these connecting strands with a finite collection of line segments.

[4] Although drawn for a different purpose, Fig. 9.11 displays this idea nicely the blue/red circles. Note, there we drew the crossing strands as straight lines in xy as well as in z; we have not done this yet in our construction.

Finally, observe that for each crossing, this gives us a finite collection of corresponding polygonal segments. By Proposition 9.1, we have at most countably-many crossings in the diagram, hence the total number of polygonal segments employed is at most countable! □

The following appears to be a reasonable extension of the claim.

Conjecture 3. *If the set of wild points are topologically discrete, then the same result as above holds.*

The techniques above can be used to obtain a variety of other expected results. E.g.,

Proposition 9.8. *Let $K_0, K_1 : S^1 \hookrightarrow \mathbb{R}^3$. Suppose that the diagrams for K_0, K_1 are related by planar isotopy. Then $K_0 \cong K_1$.*

In particular:

Corollary 9.9. *Let $K_0, K_1 : S^1 \hookrightarrow \mathbb{R}^3$ such that K_0, K_1 share the same discrete diagram D. Then $K_0 \cong K_1$.*

We conclude this section with a discussion of directions for future work.

9.3.1 A Generalization of Reidemeister's Theorem?

The usual proof of Reidemeister's theorem relies on using polygonalizations to reduce the problem of ambient isotopy to the elementary moves. Now that we have a locally-polygonal representation for some wild knots, can we prove an analogous theorem for countable sequences of moves? E.g., something like

Conjecture 4. *Let $K_0, K_1 : S^1 \hookrightarrow \mathbb{R}^3$ be knots with discrete diagrams. Suppose that $K_0 \cong K_1$. Then there exists a sequence of Reidemeister moves satisfying the hypotheses of Theorem 7.4 taking K_0 to K_1.*

One plan of attack might be as follows:

1. Theorem 7.4 (or alternatively, Theorem 5.3) gives us an "if" direction (provided we're careful to mind bijectivity in the Theorem 7.4 case). Hence, the main plan of attack would be to find an only-if proof.

2. One idea could be as follows: First, show that given an ambient isotopy $F : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}^3$ taking K_0 to K_1, we can realize the same effects on K_0, K_1 by restricting F to a compact subspace V. Then use this to argue that F restricted to $F : [0, 1] \times V \to V$ is uniformly continuous. Hence, we can partition $[0, 1] \times V$ into a finite collection of small sub-regions such that F is guaranteed to vary by less than ε on them.

3. First, note that the local sets we constructed in the proof in the previous section (e.g., the C_ν, or the sets guaranteed by Theorem 7.8) can be made arbitrarily small. Use this to argue that our choices of polygonal representatives can be made continuously dependent in some sense on the input knots themselves.

4. Then, show that in light of the above, away from wild points, we can realize the effects of the ambient isotopy by local polygonal moves. Near the wild points, keep the effect of the given F.

5. Define a uniformly convergent sequence of ambient isotopies "shrinking" the region where we keep the effects of F down to the wild points. Finally, apply Theorem 7.4 to yield the result.

Chapter 10

Conclusion & Directions for a Category

> but Max stepped into his private boat and waved good-bye
> and sailed back over a year
> and in and out of weeks
> and through a day
> and into the night of his very own room
> where he found his supper waiting for him
> and it was still hot.
>
> —Maurice Sendak, *Where the Wild Things Are*

What's the big takeaway from this project? What's the summary?

Sometimes we can reduce the behavior of objects in a messier knot category to that of ones in a cleaner sub-category. For instance, in studying tame knots, our motivation was to noting that all knots that are topologically ambient isotopic to a polygonal knot inherit the nice structure of the PL category.

However, as we saw in Chapter 5, it seems that broadening our perspective to the context of 〔knots that can be described by *discrete* (but possibly infinite) packets of information〕 could be a more natural place to study tame knots. What tools do we have for doing so? The work of Chapter 7 gave us two key strategies: Namely, being able to separate strands from each other (although not necessarily from themselves; that requires the discreteness conditions), and applying uniform convergence / bijectivity arguments to guarantee ambient isotopy under limiting conditions.

In Chapter 9, we used these tools to formally prove that if we loosen the "finite crossings" condition on regular diagrams to just "discrete crossings," we get a formalism that is tied to a particular class of well-behaved wild knots in a way that is directly analogous to regular diagrams and tame knots. In particular, regular diagrams (finite crossings) are the tools for studying polygonal knots (finite unions of line segments), while discrete diagrams (possibly countably many crossings) seem to offer potential for studying "discrete" knots (possibly countable unions of line segments). In a qualitative sense, this makes these "discrete" knots (even the wild ones) feel much more similar to *tame* knots than to the pathological *everywhere-wild* knots.

We conclude this report with a discussion of possible approaches to making this "qualitative" similarity more rigorous. Our main focus is on how we might generalize our definition of simplicial complexes to this new context; this would represent a new category living in between PL and TOP. Again, we would like to emphasize that this is not the author's area of specialty (really, few things are), so we would appreciate receiving any feedback the reader has to offer.

10.1 A Countably PL Category?

Recall the following definition for a locally-finite simplicial complex:

Definition 10.1 (Simplicial Complex)**.** A *simplicial complex* is a collection $K = \{\sigma_i\}$ of linear simplices such that:

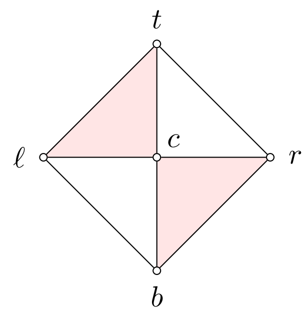
1. For all $\sigma \in K$, if τ is a face of σ, then $\tau \in K$.
2. For all $\sigma_1, \sigma_2 \in K$, if $\tau = \sigma_1 \cap \sigma_2 \neq \varnothing$, then τ is a face of both σ_1 and σ_2. $\Diamond$

We also have

Definition 10.2. Let K be a simplicial complex. Then we define the *underlying space* of K (denoted $|K|$) by

$$|K| = \bigcup_{\sigma \in K} \sigma. \qquad \Diamond$$

Example 10.1. Consider the following simplicial complex:



The simplices are as follows (Note, we'll use $\langle v_1, v_2, \ldots, v_n \rangle$ to denote "the convex hull of $\{v_1, v_2, \ldots, v_n\}$").

1. The 0-simplices: $K^0 = \{\langle t \rangle, \langle \ell \rangle, \langle b \rangle, \langle r \rangle, \langle c \rangle\}$
2. The 1-simplices: $K^1 = \{\langle t, \ell \rangle, \langle \ell, b \rangle, \langle b, r \rangle, \langle r, t \rangle, \langle t, c \rangle, \langle \ell, c \rangle, \langle b, c \rangle, \langle r, c \rangle\}$
3. The 2-simplices: $K^2 = \{\langle t, \ell, c \rangle, \langle b, r, c \rangle\}$.

One can verify the axioms are satisfied. The underlying space is just all of the stuff except for the interior of the two white simplices, $\langle t, r, c \rangle$ and $\langle b, \ell, c \rangle$. $\Diamond$

Now another definition.

Definition 10.3 (Locally finite)**.** Let K be a simplicial complex. Then we say K is *locally finite* if for all 0-simplices $\sigma \in K$, σ is a vertex of only finitely-many simplices of K. $\Diamond$

If one is not careful, this definition for locally-finite can create some unexpected collections being referred to as "simplicial complexes." Of course, the weak topology is meant to prevent silly situations such as these, but we'll pretend it doesn't exist for the time being. This allows us to construct some PL-flavored structures that seem germane to examining wild knots in a way that's agnostic to the pathology at the wild point.

Example 10.2. Consider a proposed simplicial complex defined as follows.

1. For all $n \in \mathbb{N}$, let I_n be defined by

$$I_n = \left[\frac{1}{2^n}, \ \frac{1}{2^{n-1}}\right]$$

Observe these are all 1-simplices.

2. Also define

$$a_n = \left\{\frac{1}{2^n}\right\} \qquad b_n = \left\{\frac{1}{2^{n-1}}\right\}$$

and

$$I_\infty = \{0\}.$$

Observe these are all 0-simplices. Also note that $\partial I_n = a_n \cup b_n$.

Let $S = I_\infty \cup \bigcup_{n\in\mathbb{N}} \{I_n, a_n, b_n\}$ (just scooping up all the items defined above). Is this a valid simplicial complex? Is it locally finite? ◊

Sketch. We break the statement into parts. The results are straightforward, we have just sought to be exhaustive for completeness.

Claim 1: The collection S defined in the question is a valid simplicial complex.

Proof of Claim 1:

1. First, we verify that for all $\sigma \in K$, if τ is a face of σ, then $\tau \in K$. Let $\sigma \in K$ be arbitrary. We have two sub-cases.

   1) Suppose σ is a 0-simplex. Then σ is its only face, so the claim holds.
   2) Suppose σ is a 1-simplex. Then by construction, there exists $n \in \mathbb{N}$ such that $\sigma = \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]$. Hence the faces of σ are $a_n = \left\{\frac{1}{2^n}\right\}$ and $b_n = \left\{\frac{1}{2^{n-1}}\right\}$, both of which are elements of S (by construction).

   Hence we see that the first condition is satisfied.

2. Let $\sigma_1, \sigma_2 \in K$ be arbitrarily chosen. We want to show that if $\tau = \sigma_1 \cap \sigma_2 \neq \varnothing$, then τ is a face of both σ_1 and σ_2.

   First, note that if $\sigma_1 = \sigma_2$, then the claim is trivial. Hence suppose $\sigma_1 \neq \sigma_2$. We proceed by casework.

   1) Suppose σ_1, σ_2 are both 0-simplices. Then since $\sigma_1 \neq \sigma_2$, we have $\sigma_1 \cap \sigma_2 = \varnothing$, so the claim is satisfied vacuously.
   2) Now suppose one of σ_1, σ_2 is I_∞. Note that for any $n \geq 0$, I_∞ is disjoint from *each* of a_n, b_n, and I_n. Hence $\sigma_1 \cap \sigma_2 = \varnothing$, so in this case the claim is also satisfied vacuously.
   3) Now suppose at least one of σ_1, σ_2 is not a 0-simplex, and that neither σ_1, σ_2 is I_∞.[a] One can verify the following:

i) Suppose one of σ_1, σ_2 is a 0-simplex and the other is a 1-simplex. Without loss of generality, let σ_1 be the 1-simplex and σ_2 be the 0-simplex. Then σ_1 is of the form $I_n = \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right] = \langle a_n \cup b_n \rangle$ for some n (recall, a_n, b_n were defined as 1-element sets). Then $\sigma_1 \cap \sigma_2 \neq \varnothing$ iff σ_2 is one of a_n or b_n, at which point the claim follows by construction of S.

ii) Suppose σ_1, σ_2 are of the form I_n, I_m respectively, and without loss of generality, suppose $n > m$. Then observe we have $I_n \cap I_m \neq \varnothing$ iff $n = m + 1$. In this case, we see $I_n \cap I_m = \left\{\frac{1}{2^{n-1}}\right\} = \left\{\frac{1}{2^m}\right\}$, which is an element of S by construction.

In any case, we see the condition is satisfied.

It follows that S is a valid simplicial complex.

[a]This is wordy, but we're just taking the complement of the two cases we've already done.

Claim 2: S is locally finite.
Proof of Claim: By a similar argument to the above, one can verify that

1. For all $n \in \mathbb{N}$, we have $I_\infty \cap a_n = I_\infty \cap b_n = I_\infty \cap I_n$. Hence I_∞ is a vertex of exactly one simplex, namely itself.
2. For all $n \in \mathbb{N}$, a_n is a vertex of I_n and I_{n+1}. Similarly, for $n > 2$, b_n is a vertex of I_{n-1}, I_n. For $n = 1$, b_1 is a vertex of I_1 and that's it.

In any case, we see that the simplicial complex is locally finite.

Claim 3: $|S| = [0, 1]$.
Proof of Claim 3: Let $x \in [0, 1]$ be arbitrary. We want to show $\exists \sigma \in K$ such that $x \in \sigma$. We proceed by casework.

1. Suppose $x = 0$. Then $x \in I_\infty$, as desired.
2. Suppose that $x \neq 0$. Then there exists $n \in \mathbb{N}$ such that

$$\frac{1}{2^n} \leq x \leq \frac{1}{2^{n-1}},$$

from which we have $x \in I_n$.[a]

In either case, it follows that there exists $\sigma \in K$ such that $x \in \sigma$. Hence $[0,1] \subseteq \bigcup_{\sigma \in K} \sigma$. Now observe that by construction, for all $\sigma \in K$, $\sigma \subseteq [0,1]$. Hence $\bigcup_{\sigma \in K} \sigma = [0,1]$.

Hence we have $|S| = [0,1]$, as desired.

[a] To be extremely explicit: one can take $\log_2$ of each term in the inequality above; since $\log_2$ is monotonic, the ordering is preserved. Then, one can use the Archimedean property of the reals to obtain the existence of the desired n.

In any case, we see that according to all of our definitions, this is a honest-to-goodness locally-finite simplicial complex. $\square$

As we mentioned way back in Chapter 1, one can construct similar simplicial complexes in $\mathbb{R}^n$. For instance, consider the following collection in $\mathbb{R}^2$:

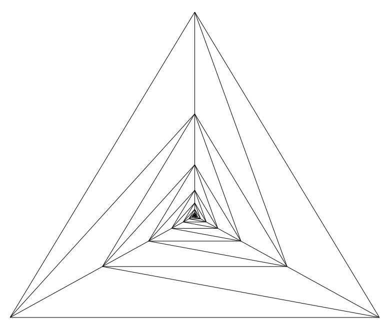


Figure 10.1 A "locally-finite" simplicial complex.

Analogous objects in $\mathbb{R}^3$ seem like they could be very apt for describing "discrete" wild knots, especially in light of Theorem 9.7. Of course, we must be careful not to go *too* far in this direction; for instance, we would not want to include objects like the following:

Example 10.3 (Non-example)**.** Let $K = \{\sigma_i\}$ be a collection of linear simplices defined by

$$K = \{\{x\} \mid x \in \mathbb{R}\}. \qquad \Diamond$$

This particular behavior could be constrained by requiring that K be countable, but that would still allow for complexes like the following:

Example 10.4 (Non-example)**.** Let $K = \{\sigma_i\}$ be a collection of linear simplices defined by

$$K = \{\{x\} \mid x \in \mathbb{Q}\}. \qquad \Diamond$$

Inspired by these examples (but also by wanting to prevent the non-examples given above), we imagine we could achieve the desired effects by looking at something like the following (note, we encourage the reader to improve upon our naming conventions):

Definition 10.4 (A proposed definition)**.** Let $K = \{\sigma_i\}$ be a simplicial complex. Define K to be *almost locally finite* if it is locally finite at all but finitely-many 0-simplices. $\Diamond$

Or maybe something like the following:

Definition 10.5 (Another proposed definition)**.** Let $K = \{\sigma_i\}$ be a locally-finite simplicial complex. Define a *feral point* of K to be a 0-simplex $\triangle^0 \in K$ such that for all $\varepsilon > 0$, there exists infinitely-many $\sigma_i \in K$ such that $B_\varepsilon(\triangle^0) \cap \sigma_i \neq \varnothing$. Then endow K with the subspace topology, and call K *countably-PL* if there are only finitely-many feral points. $\Diamond$

We'd be interested in seeing if any of these proposed definitions provide us *objects* for a new category. Morphisms would be similar to PL maps, only now with extra requirements to respect the "feral" structure, e.g. feral points map to feral points, plus maybe some analogue to the star condition at these points.

We hope that future work in this direction might help to shed light on deeper structure to knots. We are particularly excited about the implications of Chapter 5, and wonder whether they can be generalized to *everywhere-wild* knots. In particular, to study these "discrete" knots we used symmetry groups of discrete sets. We wonder whether continuous symmetry groups could be used to study wild knots, which (in some sense) have "continuous" collections of crossings (although it seems we lose transverseness).

We'd also be interested in seeing deeper study of feral points, particularly diagrammatic invariants for distinguishing them from wild points. Lastly, we'd like to see a careful proof of an analogue to Reidemeister's theorem in this new context, as well the development of a new, slightly-stronger category in which to study our embeddings.

Part IV

Appendix

Appendix A

A Gallery of Some 3D Representations of Feral Knots

The following theorem is proven in Crowell and Fox (1963) (Appendix I, pg. 147).

Theorem A.1. *Let K be a knot in $\mathbb{R}^3$ which is rectifiable and which is given as the image of a periodic vector-valued function $p(s) = (x(s), y(s), z(s))$ of arc length s whose derivative $p'(s) = (x'(s), y'(s), z'(s))$ exists and is continuous for all s. Then K is tame.*

From what we can tell, the rectifiability condition is actually superfluous, as it follows directly from the C^1 hypothesis. C^1 implies bounded variation, which in turn implies rectifiability. We imagine Crowell and Fox (1963) were simply trying to keep the prerequisite knowledge to a minimum. In any case, we restate the result as follows.

Theorem A.2. *Let $K : S^1 \hookrightarrow \mathbb{R}^3$. Then if K is C^1, K is tame.*

We'll cite this freely in the below.

A.1 Countable Twists

Example A.1 (Countable Twists)**.**

- Description: A countable # of twists

- Example countable polygonal representation

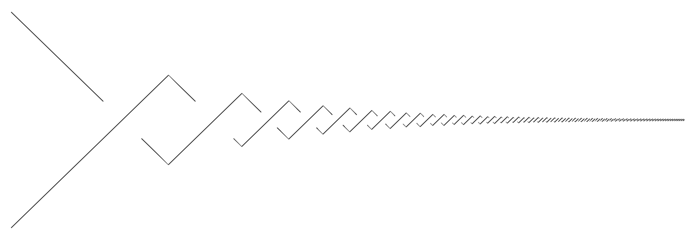


Figure A.1 Countable Polygonal Representation

- Proof of tameness, method 1: Use Reidemeister I moves and Theorem 7.4
- Proof of tameness, method 2: Explicit, C^1 3D parameterization as an arc: $f : [-\pi, \pi] \hookrightarrow \mathbb{R}^3$, parameters r, ω.

$$f(\theta) = \begin{cases} \begin{bmatrix} r\theta^3 \cos\left(\frac{\omega}{\theta}\right) \\ r\theta^3 \sin\left(\frac{\omega}{\theta}\right) \\ \theta^2 \end{bmatrix} & \theta \geq 0 \\ -f(-\theta) & \theta < 0 \end{cases}$$

C^1: all of $(-\pi, \pi)$. Can be extended past $\pm\pi$ easily.

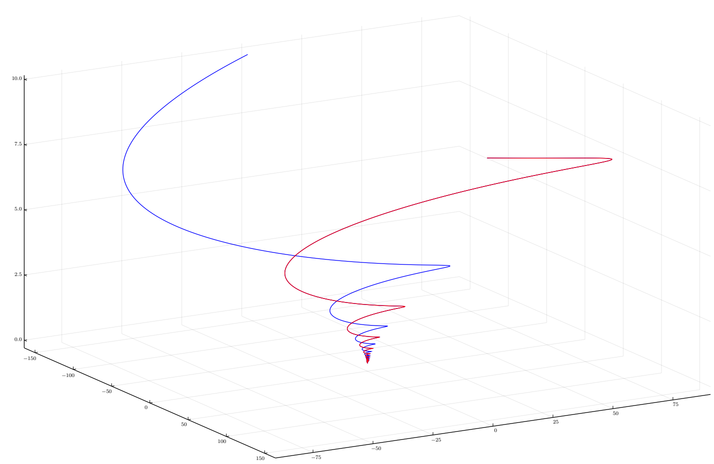


Figure A.2 Perspective view

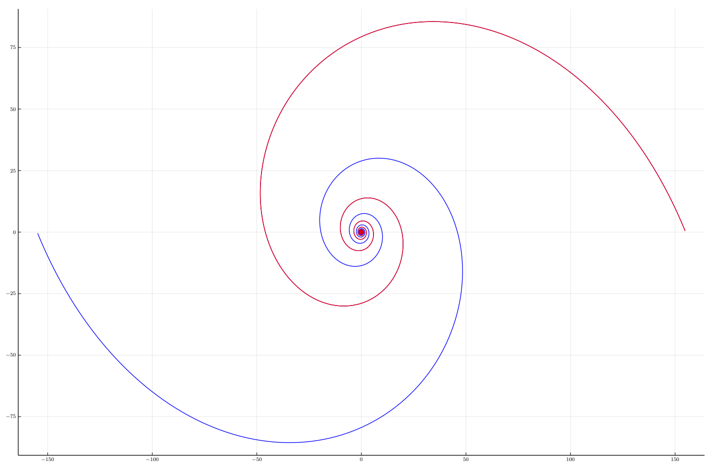


Figure A.3 Top view

- Proof of tameness, method 3: Employ the top-down view and apply Proposition 9.4.

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A.2 Countable Reidemeister II

Example A.2 (Countable Reidemeister II)**.**

- Description: A countable # of Reidemeister II's
- Example countable polygonal representation

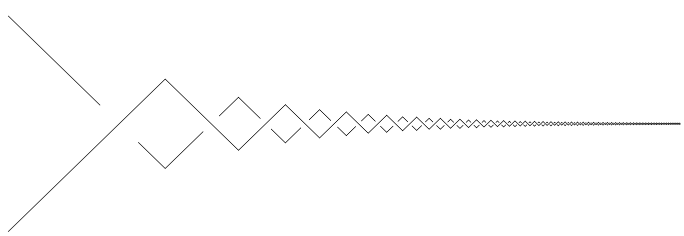

Figure A.4 Countable Polygonal Representation

- Proof of tameness, method 1: Use Reidemeister II moves and Theorem 7.4, or alternatively, using Theorem 5.3 in two steps.

- Proof of tameness, method 2: Explicit, C^1 3D parameterization as an arc: $f : [-\pi, \pi] \hookrightarrow \mathbb{R}^3$, parameters r, ω.

$$f(\theta) = \begin{cases} \begin{bmatrix} r\theta^3 \cos\left(\frac{\omega}{\theta}\right) \\ \theta^2\theta^2 \end{bmatrix} & \theta \geq 0 \\ \begin{bmatrix} -r\theta^3 \cos\left(\frac{\omega}{\theta}\right) \\ -\theta^2\theta^2 \end{bmatrix} & \theta < 0 \end{cases}$$

C^1: everywhere on $(-\pi, \pi)$. Can be extended past $\pm\pi$ easily.

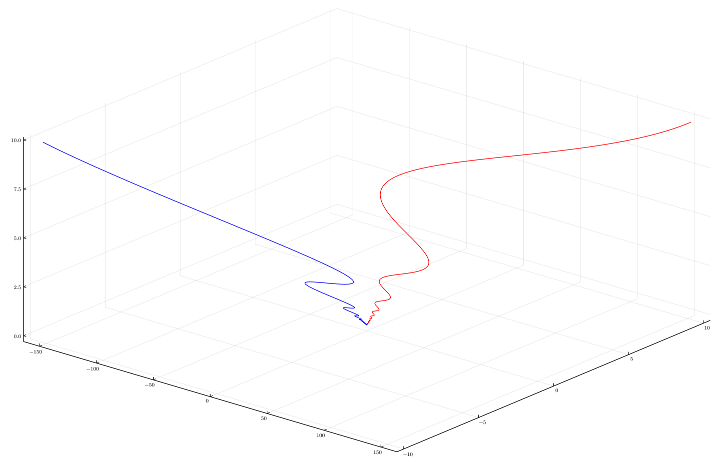

Figure A.5 Perspective view

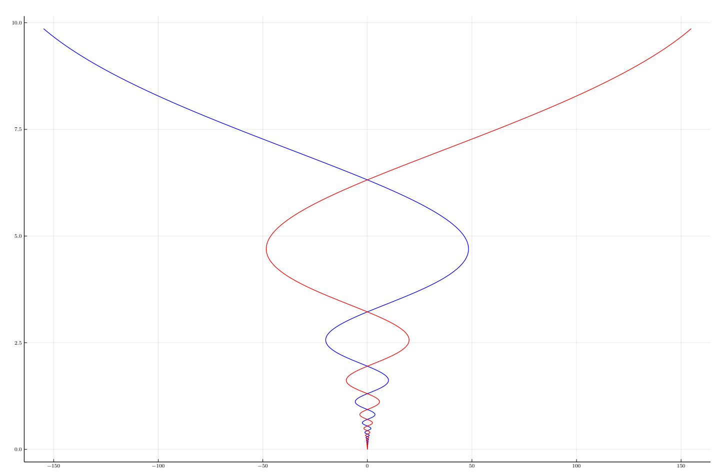

Figure A.6 Side view

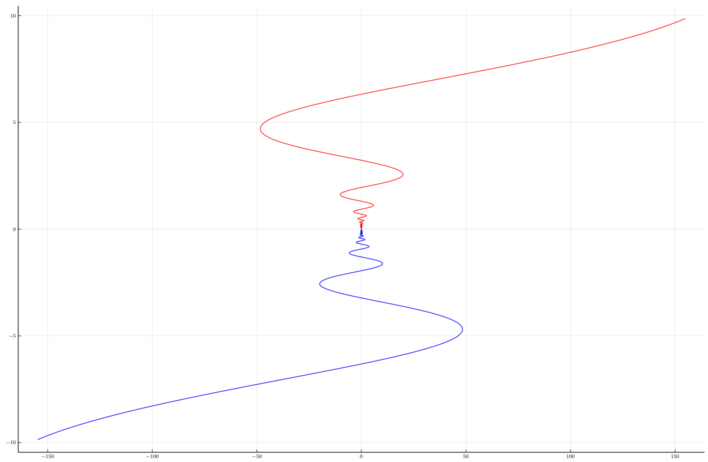

Figure A.7 Top view

- Proof of tameness, method 3: Employ the top-down view and apply Proposition 9.4.

◊

Appendix B

A Crash Course in PL Topology

> What is the big-picture idea of Linear Algebra? Reduce infinite spaces to finite descriptions. That is it—*this* is the meaning of a basis!
>
> —Weiqing Gu

For our purposes today, we can think of Piecewise-Linear Topology as "the business of trying to study continuous topological objects by approximating them with linear (more accurately, affine) structures." Because such structures can be given finite descriptions, there is a sense in which this will bring a more "discrete" flavor to our topological objects. Of particular importance will be the concept of a *simplicial complex*, what it means for one to be *locally finite*, and how we can use this to build up our understanding of knots. We'll begin by discussing some vocabulary.

B.1 Affine Sets

Really we'll only be interested in convex sets for today. But to define them, we need the concept of *affine independence*, so it seemed reasonable to do a brief primer on *affine* topics while we're at it.

Everything in the below should be general over any Banach space with field $\mathbb{R}$, but for today, we'll only consider $\mathbb{R}^n$.

Definition B.1 (Affine Set). Let $F \subseteq \mathbb{R}^n$. Then F is called a *affine set* (also sometimes called a *flat*, but we won't use that here) if for all $x, y \in F$ and $t \in \mathbb{R}$, we have

$$[1-t]x + ty \in F.$$

Geometrically, "every line through two points in F is itself a subset F." $\Diamond$

Note, this is essentially just a linear subspace that we allow to be shifted so as to not include the 0 vector. For instance:

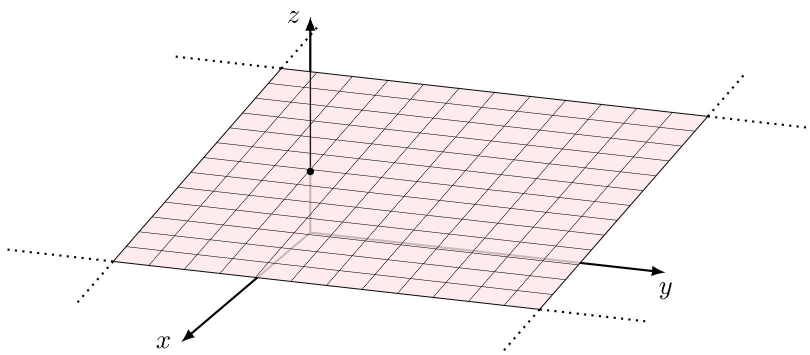

Figure B.1 An affine set. The black circle indicates a point of intersection.

is an affine set. This is encoded in the following proposition.

Proposition B.1. *Let $F \subseteq \mathbb{R}^n$ be an affine set. Then for arbitrary $x \in F$, $F - x = \{v - x \mid v \in F\}$ is a linear subspace of $\mathbb{R}^n$.*

As an example, imagine shifting the plane in Fig. B.1 down to the origin.

Just as we have a notion linear independence for vector spaces, we have affine independence for affine sets.

Definition B.2 (Affine Independence)**.** Let $\{v_i\}_{i=1}^n \subseteq \mathbb{R}^n$. Then we say $\{v_i\}$ are *affinely-independent* iff for any $j = 1, \ldots, n$, the set

$$\mathcal{B}_j = \{v_i - v_j \mid i \neq j\}$$

is linearly independent. $\Diamond$

Note again, by Proposition B.1, this is just saying we have a shifted linear subspace. We now define an analog to the *span* of a linearly independent set.

Definition B.3 (Affine Hull)**.** Let $A \subset \mathbb{R}^n$ be arbitrary. Then the *affine hull* of A,

denoted $\mathrm{aff}(A)$, is given by

$$\mathrm{aff}(A) = \left\{ \sum_{i=1}^{k} t_i x_i \;\middle|\; k \in \mathbb{N}^{>0},\ \text{each } t_i \in \mathbb{R},\ x_i \in A,\ \text{and } \sum_{i=1}^{k} t_i = 1 \right\}. \qquad \Diamond$$

Note, we require $k > 0$ so that we don't accidentally pick up the 0 vector. One can use Proposition B.1 to verify the following:

Proposition B.2. *Let $X \subseteq \mathbb{R}^n$, and let $v \in F$. Then $\mathrm{aff}(X) - v = \mathrm{span}(X - v)$.*

This formalizes our intuition that the affine hull is essentially a shifted version of linear span. Finally, we introduce the analog to linear transformations, namely *affine maps*. These are essentially just linear transformations followed by shifts; we'll use them later to define PL maps.

Definition B.4 (Affine Map)**.** Let $n, m \in \mathbb{N}$, and let $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$ be affine sets.[1] Then $f : A \to B$ is said to be affine iff for all $x, y \in A$ and all $t \in \mathbb{R}$, we have

$$f([1-t]x + ty) = [1-t]f(x) + tf(y). \qquad \Diamond$$

Note, it was important to require both A and B to be affine in order to guarantee we don't escape the domain / codomain in the equation above. Again, we can reduce this to more familiar terms with the following proposition.[2]

Proposition B.3. *Let $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$ be affine sets, and let $f : A \to B$. Then f is affine iff for all $v_0 \in A$, the function $f_{v_0} : [F - x_0] \to [B - f(x_0)]$ defined by*

$$f^{v_0}(v) = f(v + v_0) - f(v_0)$$

is linear.

In interpreting the above, it might be helpful to note that by Proposition B.1, $F - x_0$, $B - f(x_0)$ are linear subspaces of $\mathbb{R}^n, \mathbb{R}^m$ respectively. Also note that on the right-hand-side, we need the $+v_0$ in the $f(v + v_0)$ to ensure that we are feeding f something that really is in its domain.

Alright! This gives us everything we need to discuss convex sets and simplicial complexes.

[1] We make no assumptions about the relative sizes of n, m.

[2] This basically says "affine maps can be thought of as shifting the affine set to the origin, applying a linear transformation, and then shifting it somewhere else."

B.2 Convex Sets, Simplices, & Cells

Convex sets are (more or less) restricted versions of affine sets. Recall, in defining affine sets, we were essentially requiring that any point p on a line passing through points x, y of our space A had to also be an element of A. For convex sets we'll do something similar, only instead of a *line* passing through x, y, we look at *line segments* with endpoints x, y. This is encoded in the following definition.

Definition B.5 (Convex Sets)**.** Let $X \subseteq \mathbb{R}^n$. Then we say X is *convex* iff for all $x, y \in X$ and $t \in [0, 1]$, we have

$$[1-t]x + ty \in X. \qquad \Diamond$$

Note, affine sets are trivially convex. Just as we defined the affine hull to mimic span for affine sets, we'll define the convex hull to mimic span for convex sets.

Definition B.6 (Convex Hull)**.** Let $A \subseteq \mathbb{R}^n$. Then the *convex hull* of A is given by

$$\langle A \rangle = \left\{ \sum_{i=1}^{k} t_i x_i \;\middle|\; t_1 + \cdots + t_k = 1, \text{ and } [\text{for all } i = 1, \ldots, k,\ t_i \geq 0 \text{ and } x_i \in A] \right\}. \quad \Diamond$$

B.3 Simplices & Cells and their Complexes

Note, in this section, we mostly follow the exposition in J. L. Bryant's *Piecewise Linear Topology*, which we found the most readable of the sources we referenced.[3]

Definition B.7 (Simplex)**.** Let $\{v_i\}_{i=1}^n$ be affinely independent in $\mathbb{R}^m$. Then the convex hull $\sigma = \langle \{v_i\}_{i=1}^n \rangle$ is called an $n-1$ *simplex.* $\Diamond$

Definition B.8 (Face)**.** Let $\sigma = \langle v_i \rangle_{i=1}^n$. Then for all subsets $J \subseteq \{1, \ldots, n\}$, we call $\tau = \langle v_j \rangle_{j \in J}$ a *face* of σ. We will often denote this by $\tau < \sigma$. $\Diamond$

Definition B.9 (Convex Linear Cell)**.** $A \subseteq \mathbb{R}^m$ is called a *convex linear cell* iff there exist $v_1, \ldots, v_k$ (*not* necessarily affinely independent) such that

$$A = \langle \{v_i\}_{i=1}^k \rangle.$$

If n is the size of a maximal affinely independent subset of A, we say A is a *convex linear* $n-1$*-cell.* $\Diamond$

[3] Can be found here: https://www.maths.ed.ac.uk/~v1ranick/papers/pltop.pdf

Remark. Note, every convex linear n-cell A can be written as

$$A = \bigcup_{i=1}^{\ell} \sigma_i$$

where the σ_i are n-simplices without any "gaps" between them.

Definition B.10 (Simplicial Complex)**.** Let $K = \{\sigma_i\}$, where the σ_i are all k-simplices in $\mathbb{R}^n$. Then we call K a *simplicial complex* iff

(1) (**Closure under faces**): For all $\sigma \in K$, for all faces $\tau \subseteq \sigma$, we have $\tau \in K$ as well.

(2) (**Intersection only along faces**): For all $\sigma, \tau \in K$, if $\sigma \cap \tau \neq \varnothing$, then $\sigma \cap \tau$ is a face of both σ and τ. $\diamondsuit$

As an example, the following is a simplicial complex defined by taking the standard tetrahedron and including each of its faces:

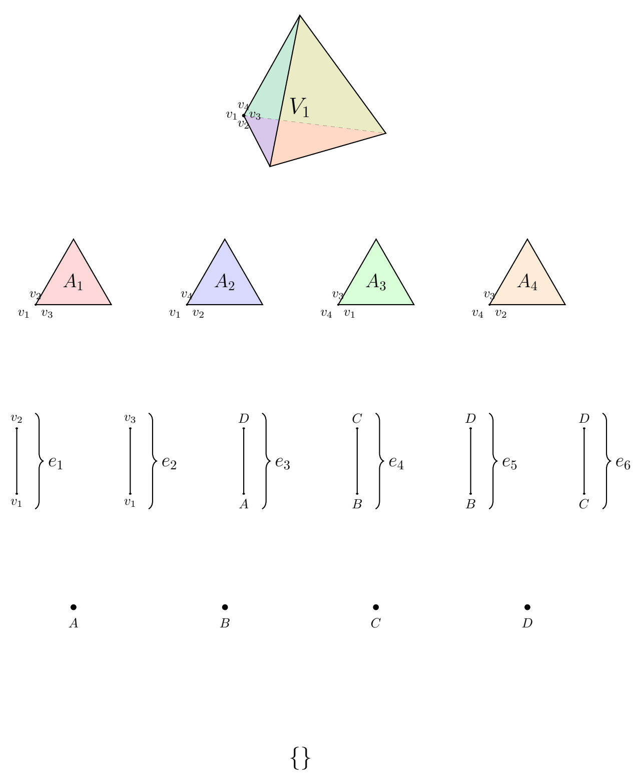

Figure B.2 Example simplicial complex

In particular, $K = \{V_1, A_1, A_2, A_3, A_4, e_1, e_2, e_3, e_4, e_5, e_6, A, B, C, D, \{\}\}$. The following proposed "complex" fails condition (2):

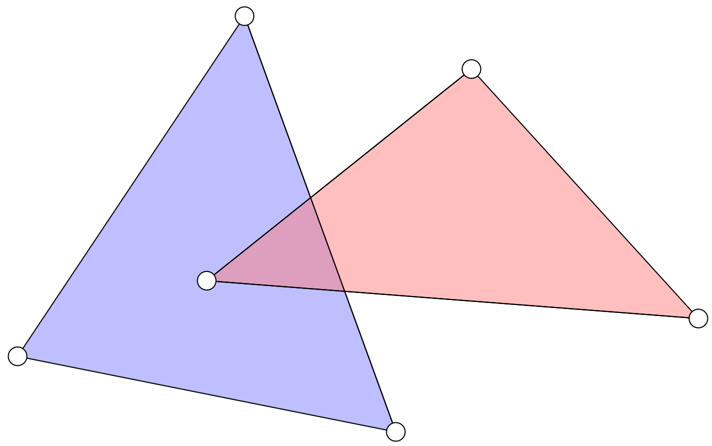

Figure B.3 An example of a collection K that fails condition (2).

We can perform a resolution by subdividing faces until the condition is regained:

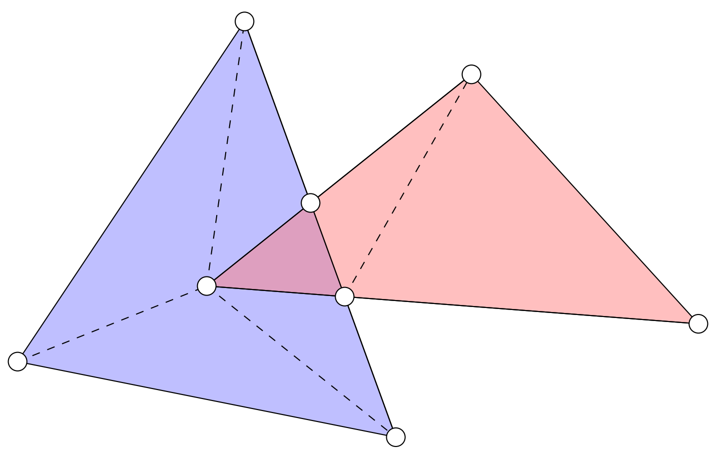

Figure B.4 Subdividing to resolve the problem.

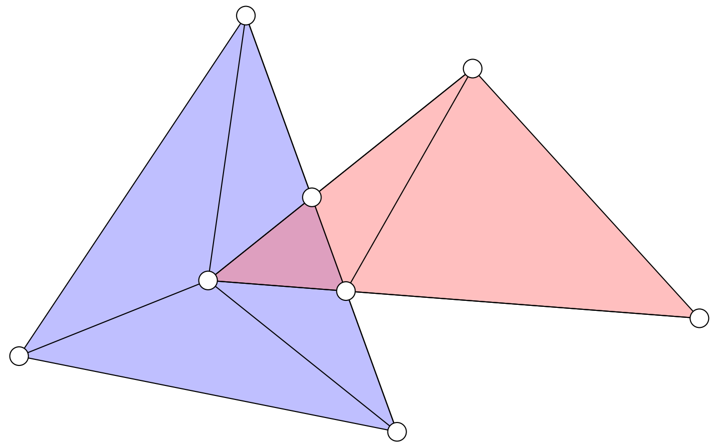

Figure B.5 The new complex.

Definition B.11 (Polyhedron)**.** Let $K = \{\sigma_i\}$ be a simplicial complex. Then the *polyhedron* of K (also called the *underlying space* of K), denoted $|K|$, is defined by

$$|K| = \bigcup_{\sigma \in K} \sigma. \qquad \lozenge$$

We think of $|K|$ as being endowed with the *weak* topology, which (given some niceness constraints) coincides with the subspace topology.

Definition B.12 (Weak topology)**.** Let K be a simplicial complex. Define the *weak topology* on $|K|$ as follows:

> Let $U \subseteq |K|$ be arbitrary. Then we say U is open iff for all $\sigma_i \in K$, $U \cap \sigma_i$ is open in σ_i. $\lozenge$

The appropriate niceness constraint is provided by *local finiteness*. This is defined slightly differently in different texts.

Definition B.13 (Locally Finiteness)**.** Let $K = \{\sigma_i\}$ be a simplicial complex. Then we say K is *locally finite* if for all 0-simplices $v \in K$, there exists $\varepsilon > 0$ such that there exists only finitely-many σ_i for which

$$\sigma_i \cap B_\varepsilon(v) = \varnothing.$$

$\lozenge$

Note, other sources might define local finiteness in a different manner. Namely,

Definition B.14 (Local Finiteness (Alt.))**.** Let K be a simplicial complex. Then we say K is *locally finite* if for all 0-simplices $v \in K$, there exists only finitely-many σ_i such that v is a vertex of σ_i. $\Diamond$

As far as we can tell, these conditions are not equivalent, but we could be missing something here. For instance, under Definition B.14, what's to stop us from something like the following?

> **Non-example:** Let $K = \{\sigma_i\}$ be defined by
>
> $$K = \{\{x\} \mid x \in [0, 1]\}.$$

Naively, this appears to satisfy all of the axioms for a simplicial complex. Closure under faces is trivially satisfied. And all of our simplices are disjoint, so "intersection only along faces" is satisfied as well. But under Definition B.13, this is *not* locally finite, while it is under Definition B.14.

Anyways, we have the following proposition, which we do not prove.

Proposition B.4. *Let K be a simplicial complex that is locally finite in the sense of Definition B.13. Then $A \subseteq |K|$ is closed in $\mathcal{T}_{\text{weak}}$ iff it is closed in $|K|$ with the subspace topology on $\mathbb{R}^n$ (where n is the maximal dimension of a simplex in the complex).*

Definition B.15 (Subcomplex)**.** Let K be a simplicial complex. Then $L \subseteq K$ is said to be a *subcomplex* if it is a simplicial complex. $\Diamond$

Definition B.16 (Boundary subcomplex)**.** Let K be a simplicial complex. Then for all $\sigma \in K$, we define the *boundary subcomplex* of σ by

$$\dot{\sigma} = \{\tau \mid \tau \neq \sigma\}. \qquad \Diamond$$

Definition B.17 (Interior)**.** The *interior* of σ is defined by

$$\mathring{\sigma} = \sigma - |\dot{\sigma}|. \qquad \Diamond$$

Definition B.18 (Subdivision)**.** Let K_1, K_2 be simplicial complexes. Then we say K_1 is a *subdivision* of K_2 iff

1. $|K_1| = |K_2|$, and
2. For all $\sigma \in K_1$, we have $\sigma \in K_2$ as well.

Often, subdivision is denoted by $\prec$. $\Diamond$

We now define sensible morphisms for our complexes.

Definition B.19 (Simplicial Map)**.** Let K, L be simplicial complexes, and let $f : |K| \to |L|$. Then we call f a *simplicial map* iff for all $\sigma \in K$, we have $f(\sigma) \in L$ (with the added condition that the vertices of σ *must* be mapped to the vertices of $f(\sigma)$). $\Diamond$

Proposition B.5. *A simplicial map is fully determined by where it sends each of the vertices (0-simplices) of K.*

Definition B.20 (Piecewise Linear)**.** Let K, L be simplicial complexes. Then a function $f : |K| \to |L|$ is said to be *piecewise-linear* (or *PL*) iff there exist subdivisions $K' \prec K$, $L' \prec L$ such that $f : K' \to L'$ is simplicial. $\Diamond$

Definition B.21 (PL Homeomorphism)**.** Let K, L be simplicial complexes. Then we say $f : |K| \to |L|$ is a *PL homeomorphism* iff there exist subdivisions $K' \prec K$, $L' \prec L$ such that f bijective *and* simplicial. $\Diamond$

We collect some propositions.

Proposition B.6. *Let K be a simplicial complex, and let $A \subseteq |K|$ be compact. Then there exists a finite subcomplex $L < K$ such that $A \subseteq |L|$.*

Proof. See Hatcher[4], Appendix A, Proposition A.1. $\square$

This gives us that knots are polygonal.

Corollary B.7. *Let $K : S^1 \hookrightarrow \mathbb{R}^3$ be a PL embedding. Then K is polygonal.*

Sketch. Note, K is continuous and S^1 is compact, hence $\overrightarrow{K}(S^1)$ is compact. By Proposition B.6, it follows that $\overrightarrow{K}(S^1)$ consists of a finite union of polygonal segments. $\square$

Hopefully this has given the reader a bit of a sense for the basics of PL topology. For more, we encourage referencing the works listed in Chapter 6.

B.4 Some Examples of Rigorously-Constructing Ambient Isotopies

Rigorously constructing ambient isotopies can be tedious and unpleasant. However, given that our focus is on trying to provide rigorous foundations for working with *topological* ambient isotopies, we'll include some notes on this process below. As with the rest of the material in the appendix, this is strictly optional.

[4] http://pi.math.cornell.edu/~hatcher/AT/ATapp.pdf

Proposition B.8. *Let $\triangle_0$ be an n-simplex, and let $x_0 \in \triangle_0^\circ$ be arbitrary. Denote the barycenter of $\triangle_0$ by c_0. Then there exists an ambient isotopy $\mathcal{C}_{x_0}$: $[0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathcal{C}_{x_0}$ is identity outside of $\triangle_0^\circ$ and $\mathcal{C}_{x_0}(1, x_0) = c_0$.*

Proof. Let $p \in \triangle_0^\circ$ be arbitrary. Then there exists a unique "anchor point" $a_p \in \partial\triangle_0$ such that there exists $s_p \in [0,1]$ such that $p = s_p \cdot x_0 + (1 - s_p) \cdot a_p$. Note, this is equivalent to saying that a_p, p, x_0 are *colinear* and either $p = a_p$, $p = x_0$, or p is *strictly* between a_p and x_0. These lines are drawn in *light orange* in Fig. B.6.

Claim: Let $x(t) = t \cdot c_0 + (1-t)x_0$ (note, this is the straight-line homotopy from x_0 to c_0). Then the desired ambient isotopy $\mathcal{C}_{x_0}$ is given by

$$\mathcal{C}_{x_0}(t, p) = s_p \cdot x(t) + (1 - s_p) \cdot a_p. \qquad (\forall p \in \triangle_0)$$

See Fig. B.6 for an illustration.

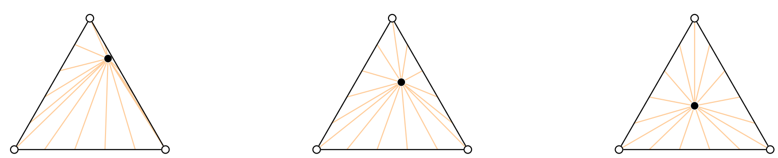

Figure B.6 The desired ambient isotopy

Proof of Claim: First, we show (a) that the proposed $\mathcal{C}_{x_0}$ doesn't send points outside of $\triangle_0$, (b) that we have identity on the boundary, and (c) that we indeed have an ambient isotopy.

1. We want to show $\mathcal{C}_{x_0}$ doesn't send points outside of $\triangle_0$. Note, $x(t) \in \triangle_0$ for all t. Also note that $s_p + (1 - s_p) = 1$ always. Since $a_p \in \triangle_0$ as well, this shows $\mathcal{C}_{x_0}$ is a convex combination of points in $\triangle_0$. Since $\triangle_0$ is convex, it follows that $\mathcal{C}_{x_0}(t, p) \in \triangle$ for all $t \in [0,1]$, $p \in \triangle_0$. ✓

2. We want to show $\mathcal{C}_{x_0}$ is identity on the boundary. Note that if $p \in \partial\triangle_0$, then $p = a_p$ and hence $s_p = 0$. It follows that $\mathcal{C}_{x_0}(t, \cdot)$ is identity on $\partial\triangle_0$. ✓

3. To show that $\mathcal{C}_{x_0}(t, x)$ is an ambient isotopy, we must demonstrate that it is identity at $t = 0$, that the image of $\triangle_0$ at $t = 1$ is the desired modified version, that $\mathcal{C}_{x_0}(t, \cdot)$ is a homeomorphism for each t, and finally, that $\mathcal{C}_{x_0}$ is continuous overall.

   (i) The desired properties at $t = 0, 1$ follow immediately from the definition.

(ii) We want to show that $\mathcal{C}_{x_0}(t,\cdot)$ is a homeomorphism for each t. Hence, let $t_0 \in [0,1]$ be arbitrary. Write $x(t_0) = x_0 + \delta(t_0)$ (from the definition of $x(t)$ above, we have $\delta(t_0) = t_0(c_0 - x_0)$). Now, note that

$$\begin{aligned}\mathcal{C}_{x_0}(t_0,p) &= s_p \cdot (x_0 + t_0(c_0 - x_0)) + (1 - s_p)a_p \\ &= \Big(s_p x_0 + (1 - s_p)a_p\Big) + s_p t_0(c_0 - x_0) \\ &= p + s_p t_0(c_0 - x_0).\end{aligned}$$

Ok now the ε-δ part: Let $\varepsilon > 0$ be given. Through some fairly unpleasant trig, one can find a $\delta_0 > 0$ such that $q \in B_{\delta_0}(p)$ implies the angle $\angle p x_0 q$ is arbitrarily small. Then one can then show that the rays $\overrightarrow{x_0 p}$ and $\overrightarrow{x_0 q}$ intersect $\partial\triangle_0$ at points that are arbitrarily close. From this, we can constraint $|s_p - s_q| < \varepsilon$, and use this to show continuity.

To see that $C_{x_0}(t_0,\cdot)$ has a continuous inverse, observe that p and $C_{x_0}(t_0,p)$ have the same anchor point a_p and displacement parameter s_p. Thus, define

$$\mathcal{C}_{x_0}^{-1}(t_0,p) = s_p x(t_0) - s_p t_0(c_0 - x_0),$$

and note that this is indeed a well-defined inverse, and the same continuity argument as above demonstrates that it is continuous.

(iii) One can then directly verify that $\mathcal{C}_{x_0}$ is continuous. $\square$

The second lemma is similar. We omit some of the details of the proof since they are similar to that given above.

Proposition B.9. *Let $\triangle_0, \triangle_1$ be n-simplices that share an $n-1$ face, and let c_0, c_1 be the barycenters of $\triangle_0, \triangle_1$ respectively. Then there exists an ambient isotopy $\mathcal{S}_{0,1} : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathcal{S}_{0,1}$ is fixed on $\mathbb{R}^n - (\triangle_0 \cup \triangle_1)^\circ$, and $\mathcal{S}_{0,1}(1, c_0) = c_1$.*

Proof. The same proof as above works when $\triangle_0, \triangle_1$ are regular (see Fig. B.7). However, when $\triangle_0, \triangle_1$ are *not* regular, we have some additional things to worry about. In particular, $\triangle_0 \cup \triangle_1$ might not be convex, so we can't just do a "straight-line" ambient isotopy.

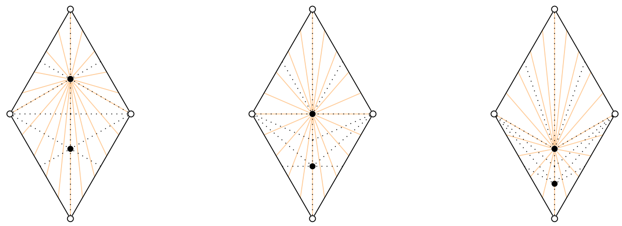

Figure B.7 The desired isotopy when the simplices are regular

If $\triangle_0 \cup \triangle_1$ is not convex, we must be a little more creative. In particular, we see that the ambient isotopy above (Fig. B.7) isn't actually the most natural choice. Instead, consider the following (see Fig. B.8 for illustration):

1) Let $\text{—}_{0,1} = \{u_1 \cdots u_n\}$ denote the shared $n-1$ face of $\triangle_0$ and $\triangle_1$. Let v_0 and v_1 be the vertices of $\triangle_0$ and $\triangle_1$ (respectively) such that $v_0, v_1 \notin \text{—}_{0,1}$.

2) Then note: for all $x_0 \in \triangle_0^\circ$, there exists a unique $y \in \text{—}_{0,1}$ such that x_0 is collinear with v_0, y. Similarly for $x_1 \in \triangle_1^\circ$.

3) Now, for each $y \in \text{—}_{0,1}$, we join these lines with the following parameterization:

$$Y(s) = \begin{cases} 2s \cdot y & + (1-2s) \cdot v_0 \quad s \in \left[0, \frac{1}{2}\right] \\ (2s-1) \cdot v_1 + (2-2s) \cdot y & \quad s \in \left(\frac{1}{2}, 1\right] \end{cases}$$

the two cases here are illustrated by the *orange* and *blue* lines in Fig. B.8. Note that $Y\left(\frac{1}{2}\right) = y$.

4) Now we define our ambient isotopy $\mathcal{S}_{0,1}$. First, we need to look at the kinked line $Y_c(s)$ the barycenters of $\triangle_0, \triangle_1$ live on. The corresponding s_0, s_1 will be necessary in defining $\mathcal{S}_{0,1}$.

Let's do it. Let c_0, c_1 be the barycenters of $\triangle_0, \triangle_1$ respectively. We want to show they really *do* lie on the same Y. To that end, note that the line through v_0, c_0 to $\text{—}_{0,1}$ and the line from v_1, c_1 to $\text{—}_{0,1}$ both end at the barycenter of $\text{—}_{0,1}$ (to verify this, write everything in barycentric coordinates). Hence, the corresponding Y (denote it Y_c) satisfies $c_0, c_1 \in Y_c([0,1])$. Thus there exists s_0, s_1 such that $Y_c(s_0) = c_0$, and $Y_c(s_1) = c_1$. In particular, one can show that $s_0 = \frac{1}{3}$, $s_1 = \frac{2}{3}$ independent of the choice of $\triangle_0, \triangle_1$.

Now we can define $\mathcal{S}_{0,1}$ itself. To do so, we'll introduce a fairly complicated formula (which exposes more information about how we derived the formula) and then show that it simplifies from a complete mess to something very simple. Let $x \in (\triangle_0 \cup \triangle_1)^\circ$ be arbitrary. Write it as $Y(s_0)$ for some Y as described above in step 3). We define $\mathcal{S}_{0,1}$ by

$$\mathcal{S}_{0,1}(t,x) = \mathcal{S}_{0,1}(t,Y(s)) = \begin{cases} Y(t[s_1 - s_0 + s] + [1-t]s) & s \in \left[0, \frac{1}{2}\right] \\ Y(\text{some mystery function!}) & s \in \left[\frac{1}{2}, 1\right], \end{cases}$$

where we leave the reader to puzzle out what the mystery function is.

It follows that $\mathcal{S}_{0,1}$ is the desired ambient isotopy.

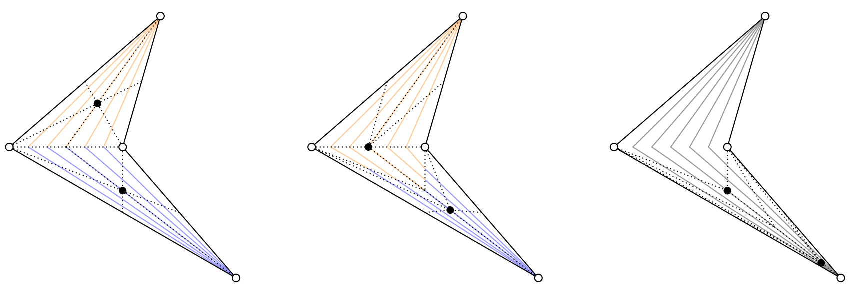

Figure B.8 An example where $\triangle_0 \cup \triangle_1$ is not convex (see **Important Note** below).

Important Note: Not all of the dotted lines are drawn in the final diagram, and I've removed the color-coding. I'm not terribly pleased with this, but it was getting to be too troublesome to do all of the trig arithmetic in TikZ to justify continuing onwards. The point is that one should imagine extending the reasoning of the first two diagrams, and observe that the barycenter gets where it needs to go. □

Corollary B.10. *Let $\triangle_0, \triangle_1, \ldots, \triangle_k$ be n-simplices such that for each $i = 0, \ldots, k-1$, $\triangle_i$ and $\triangle_{i+1}$ share an $n-1$-face. Let $x_0 \in \triangle_0$ and $x_1 \in \triangle_k$. Then there exists an ambient isotopy $H_{x_0,x_1} : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ such that H is fixed outside of $\bigcup_{i=0}^k \triangle_k^\circ$, i.e.*

$$H(t,x) = x \qquad \text{whenever} \qquad x \in \mathbb{R}^n - \bigcup_{i=0}^{k} \triangle_k^\circ,$$

and $H(1,x_0) = x_1$.

Proof. Note, by Proposition B.8, there exists an ambient isotopy $\mathcal{C}_{x_0} : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ that leaves $\mathbb{R}^n - \triangle_0^\circ$ fixed, and takes x_0 to the barycenter c_0 of $\triangle_0$.

Now, by Proposition B.9, there exists a sequence of ambient isotopies $\mathcal{S}_{0,1}$, $\mathcal{S}_{1,2}$, ..., $\mathcal{S}_{k-1,k}$ taking the barycenter of $\triangle_0$ to that of $\triangle_1$, the barycenter of $\triangle_1$ to $\triangle_2$, and so on. For each $i \in \{0, \ldots, k-2\}$, let $f_i(x) = \mathcal{S}_{i,i+1}(1, x)$ It follows that $\mathcal{S}_{0,k}$ defined by

$$\mathcal{S}_{0,k} = \begin{cases} \mathcal{S}_{0,1}(k \cdot t, x) & t \in \left[0, \frac{1}{k}\right] \\ \mathcal{S}_{1,2}((k \cdot t) - 1, f_0(x)) & t \in \left[\frac{1}{k}, \frac{2}{k}\right] \\ \vdots & \\ \mathcal{S}_{i,i+1}\Big((k \cdot t) - i, \bigcirc_{j=1}^{i-1} f_j(x)\Big) & t \in \left[\frac{i}{k}, \frac{i+1}{k}\right] \\ \vdots & \\ \mathcal{S}_{k-1,k}\Big((k \cdot t) - (k-1), \bigcirc_{j=1}^{k-2} f_j(x)\Big) & t \in \left[\frac{k-1}{k}, 1\right] \end{cases}$$

can be shown to be a valid ambient isotopy.[5] □

[5] Just apply the gluing lemma.

Appendix C

Misc

C.1 Solution to the Chessboard Problem

Solution. **Claim:** No.

Proof: Observe that there are 25 black squares and 24 white squares. Also note that every legal move takes a knight to an opposite-color square. Hence after moving all the knights, we'd have 25 on black and 24 on white, which is a contradiction. ♘

C.2 `Julia` Code for the Countable Reidemeister I Example

```
using Plots
using Colors
plotlyjs()

# r₁ is the big r for the circle this is based on.
#
# This function just computes the equation seen in our explicit
# parameterization of the countable reidemeister I knot. Note that
# this uses the form defined in terms of the matrix-vector equation,
# since we found it more tangible when prototyping.
function to_cartesian(θ; r₁=3)
    # This is the special "f(0)=0" case in our piecewise definition of
    # our function.
    if θ == 0
        return [r₁,0,0]
    end
```

```
    # Compute the radius of the smaller circle
    r₂ = abs(θ)^3
    x = r₁*cos(θ)
    y = r₁*sin(θ)

    # The radial vector is orthogonal to the tangent vector, hence
    # we can use it as one of the basis vectors for the orthogonal
    # frame.
    vperp = [x, y, 0]
    vperp /= sqrt((x^2 + y^2)) # Normalize it

    # Since the big circle lies in the xy-plane, the z direction is
    # also always one of the basis vectors for the orthogonal plane.
    zperp = [0,0,1]

    scale=2*π^2
    if θ > 0
        v_to_transform = [r₂*sin(scale/θ), r₂*cos(scale/θ), 0.0]
    else
        v_to_transform = [r₂*sin(-1*scale/θ), r₂*cos(scale/θ), 0.0]
    end

    # Rotate into the orthogonal plane
    R_mat = [zperp vperp [0.0; 0.0; 0.0]]
    v_out = (R_mat * v_to_transform) + [x,y,0]
    return v_out
end

step = .0001
θs = 0:step:π/2
θ₂s = reverse(-1 .* θs)


xyz_vals = to_cartesian.(θs)
xyz_opp = to_cartesian.(θ₂s) # The mirror image
append!(xyz_opp, xyz_vals)

xs = [p[1] for p in xyz_opp]
ys = [p[2] for p in xyz_opp]
zs = [p[3] for p in xyz_opp]

lcol= colorant"#D6D7D9"
bcol= colorant"#1D252C"

plot(camera=(60, 25), size=(1500,1000),
     legend=false, background_color=bcol,
     color = lcol)#, grid=false)#, axis=false, ticks=false)

plot!(xs, ys, zs, color=lcol, lw=.5)
```

C.3 Combinatorial Representations

C.3.1 Cycle representations

Index	Cycle representation	Order
$(3,1)$	$(-3,-2)(1,2)$	2
$(4,1)$	$(-3,-2i,4,3,i,-i,2i)$	7
$(5,1)$	$(-5,-4,-2,-3)(1,2,4,3)$	4
$(5,2)$	$(-5,-4,-2,-3)(1,2,4)$	12
$(6,1)$	$(-2,-3i,6i,-4i,1,2,4i)(-6i,3i,-5i,5i)$	28
$(6,2)$	$(-6,-2i,4i)(-5i,5i,-4i,3i)(-3i,1,2i)$	12
$(6,3)$	$(-6,4i,6)(-5,5,-3i)(-1,1,-2i,-4i)(2i,3i)$	12
$(6,4)$	$(-6,-5)(-3,-2)(1,2)(4,5)$	2
$(6,5)$	$(-3,-2)(-5i,-4i)(5i,6i)(1,2)$	2
$(7,1)$	$(-7,-6,-4)(-5,-2,-3)(1,2,4)(3,6,5)$	3
$(7,2)$	$(-7i,2i)(-6i,4i,-4i,5i)(-5i,7i,-3i,3i,-2i,i,-i,6i)$	8
$(7,3)$	$(-7,-6,-4,-5,-2,-3)(1,2,4)(3,6,5)$	6
$(7,4)$	$(-7,-6,-2,-3,-5,-4)(1,2,4)(3,6,7,5)$	12
$(7,5)$	$(-7i,6i,-4i,3i,-5i,7i)(-6i,5i)(-3i,i,-i,2i)(-2i,4i)$	12
$(7,6)$	$(-3,-5i,i,-i,2i)(-7i,7i,-2i,4,3,6i,-6i,5i)$	40
$(7,7)$	$(-6,-4)(-3,-2i,4,7,6,3,5i,-5i,i,-i,2i)$	22
$(8,1)$	$(-4,-5i,8i,-6i,3,i,-i,2i,-3,-2i,4,6i)(-8i,5i,-7i,7i)$	12
$(8,2)$	$(-2,3i,5i,2,-4i,-8i,-7i,-5i,-1,1)(-6i,-3i)(4i,7i)(6i,8i)$	10
$(8,3)$	$(-4,-7i,7i,-8i,5i)(-3,-5i,3,6i,-2)(-6i,1,2,4,8i)$	5
$(8,4)$	$(-5,-8i,8i,-7i,6i,-4,-6i,3,i,-i,2,4,7i,-2,-3)$	15
$(8,5)$	$(-8,-4,-7)(-5,-2i,4,8,5,i,-i,2i,-3)(3,6,7)$	9
$(8,6)$	$(-2,3i,5i,2,-4i,-8i,-5i,-1,1)(-7i,-3i,-6i)(4i,7i,6i,8i)$	36
$(8,7)$	$(-8,-7,-5)(-6,-2i,i,-i,2i,-3i,5,4,7,6,3i,-4)$	12
$(8,8)$	$(-8,7,-1,1,-2,3,-6i,-3,5i,8)(-7,2,-4i)$	30
$(8,9)$	$(-8,-6,-4i,8,7,5,3i,-5,-2i,4i,-7)(-3i,6,i,-i,2i)$	55
$(8,10)$	$(-5,-7i,7i,-6i,8i,-8i,5,3,1,2,4,6i,-4,-2,-3)$	15
$(8,11)$	$(-2,3i,5i,2,-4i,-8i,-7i,-3i,-6i,-5i,-1,1)(4i,7i)(6i,8i)$	12
$(8,12)$	$(-3,-5i,8i,-6i,1,2,4,3,6i,-2)(-8i,5i,-7i,7i)$	20
$(8,13)$	$(-7,4,-3,5,-i,-2i,-4,7)(-6,2i,3)(-5,8i,6,-8i)$	24
$(8,14)$	$(-2,-3i,6i)(-8i,3i,-5i,8i)(-7i,7i,-6i,1,2,4i)(-4i,5i)$	12
$(8,15)$	$(-8i,-5i,-6i,-i,-2i,-4i,-7i)(2i,3i,5i,8i,4i,6i)$	42
$(8,16)$	$(-6,8i,6,-i,-2,3)(-3,5i,4i,2,-4i,-7i)$	6
$(8,17)$	$(-6,-8i,8i,-7i,3,6,i,-i,2i,-3,-5,-4,-2i,4,7i)$	15
$(8,18)$	$(-5,-4,-2i,4,7,3i)(-8i,5,8i)(-6i,1,2i,-3i,6i)$	30
$(8,19)$	$(-8,-7,6,-5,4,-6)(-4,-2,-3,5,7,3,1,2)$	24
$(8,20)$	$(-4,6i,8i,7i,-2i)(-3,-5i,-7i,-6i,5i,4)(2i,3,1)$	30
$(8,21)$	$(-6,8i,6,7i,-2i,-4i,1,2i,-3i)(-7i,3i,5i,4i)$	36

Table C.1 $n_{\mathrm{g.int}}$ cycle presentations for knots up to 8 crossings

Index	Cycle representation	Order
$(3,1)$	$(3^+_u, 2^+_u)(1^+_o, 2^+_o)$	2
$(4,1)$	$(3^+_u, 2^-_u, 4^+_o, 3^+_o, 1^-_o, 1^-_u, 2^-_o)$	7
$(5,1)$	$(5^+_u, 4^+_u, 2^+_u, 3^+_u)(1^+_o, 2^+_o, 4^+_o, 3^+_o)$	4
$(5,2)$	$(5^+_u, 4^+_u, 2^+_u, 3^+_u)(1^+_o, 2^+_o, 4^+_o)$	12
$(6,1)$	$(2^+_u, 3^-_u, 6^-_o, 4^-_u, 1^+_o, 2^+_o, 4^-_o)(6^-_u, 3^-_o, 5^-_u, 5^-_o)$	28
$(6,2)$	$(6^+_u, 2^-_u, 4^-_o)(5^-_u, 5^-_o, 4^-_u, 3^-_o)(3^-_u, 1^+_o, 2^-_o)$	12
$(6,3)$	$(6^+_u, 4^-_o, 6^+_o)(5^+_u, 5^+_o, 3^-_u)(1^+_u, 1^+_o, 2^-_u, 4^-_u)(2^-_o, 3^-_o)$	12
$(6,4)$	$(6^+_u, 5^+_u)(3^+_u, 2^+_u)(1^+_o, 2^+_o)(4^+_o, 5^+_o)$	2
$(6,5)$	$(3^+_u, 2^+_u)(5^-_u, 4^-_u)(5^-_o, 6^-_o)(1^+_o, 2^+_o)$	2
$(7,1)$	$(7^+_u, 6^+_u, 4^+_u)(5^+_u, 2^+_u, 3^+_u)(1^+_o, 2^+_o, 4^+_o)(3^+_o, 6^+_o, 5^+_o)$	3
$(7,2)$	$(7^-_u, 2^-_o)(6^-_u, 4^-_o, 4^-_u, 5^-_o)(5^-_u, 7^-_o, 3^-_u, 3^-_o, 2^-_u, 1^-_o, 1^-_u, 6^-_o)$	8
$(7,3)$	$(7^+_u, 6^+_u, 4^+_u, 5^+_u, 2^+_u, 3^+_u)(1^+_o, 2^+_o, 4^+_o)(3^+_o, 6^+_o, 5^+_o)$	6
$(7,4)$	$(7^+_u, 6^+_u, 2^+_u, 3^+_u, 5^+_u, 4^+_u)(1^+_o, 2^+_o, 4^+_o)(3^+_o, 6^+_o, 7^+_o, 5^+_o)$	12
$(7,5)$	$(7^-_u, 6^-_o, 4^-_u, 3^-_o, 5^-_u, 7^-_o)(6^-_u, 5^-_o)(3^-_u, 1^-_o, 1^-_u, 2^-_o)(2^-_u, 4^-_o)$	12
$(7,6)$	$(3^+_u, 5^-_u, 1^-_o, 1^-_u, 2^-_o)(7^-_u, 7^-_o, 2^-_u, 4^+_o, 3^+_o, 6^-_o, 6^-_u, 5^-_o)$	40
$(7,7)$	$(6^+_u, 4^+_u)(3^+_u, 2^-_u, 4^+_o, 7^+_o, 6^+_o, 3^+_o, 5^-_o, 5^-_u, 1^-_o, 1^-_u, 2^-_o)$	22
$(8,1)$	$(4^+_u, 5^-_u, 8^-_o, 6^-_u, 3^+_o, 1^-_o, 1^-_u, 2^-_o, 3^+_u, 2^-_u, 4^+_o, 6^-_o)(8^-_u, 5^-_o, 7^-_u, 7^-_o)$	12
$(8,2)$	$(2^+_u, 3^-_o, 5^-_o, 2^+_o, 4^-_u, 8^-_u, 7^-_u, 5^-_u, 1^+_u, 1^+_o)(6^-_u, 3^-_u)(4^-_o, 7^-_o)(6^-_o, 8^-_o)$	10
$(8,3)$	$(4^+_u, 7^-_u, 7^-_o, 8^-_u, 5^-_o)(3^+_u, 5^-_u, 3^+_o, 6^-_o, 2^+_u)(6^-_u, 1^+_o, 2^+_o, 4^+_o, 8^-_o)$	5
$(8,4)$	$(5^+_u, 8^-_u, 8^-_o, 7^-_u, 6^-_o, 4^+_u, 6^-_u, 3^+_o, 1^-_o, 1^-_u, 2^+_o, 4^+_o, 7^-_o, 2^+_u, 3^+_u)$	15
$(8,5)$	$(8^+_u, 4^+_u, 7^+_u)(5^+_u, 2^-_u, 4^+_o, 8^+_o, 5^+_o, 1^-_o, 1^-_u, 2^-_o, 3^+_u)(3^+_o, 6^+_o, 7^+_o)$	9
$(8,6)$	$(2^+_u, 3^-_o, 5^-_o, 2^+_o, 4^-_u, 8^-_u, 5^-_u, 1^+_u, 1^+_o)(7^-_u, 3^-_u, 6^-_u)(4^-_o, 7^-_o, 6^-_o, 8^-_o)$	36
$(8,7)$	$(8^+_u, 7^+_u, 5^+_u)(6^+_u, 2^-_u, 1^-_o, 1^-_u, 2^-_o, 3^-_u, 5^+_o, 4^+_o, 7^+_o, 6^+_o, 3^-_o, 4^+_u)$	12
$(8,8)$	$(8^+_u, 7^+_o, 1^+_u, 1^+_o, 2^+_u, 3^+_o, 6^-_u, 3^+_u, 5^-_o, 8^+_o)(7^+_u, 2^+_o, 4^-_u)$	30
$(8,9)$	$(8^+_u, 6^+_u, 4^-_u, 8^+_o, 7^+_o, 5^+_o, 3^-_o, 5^+_u, 2^-_u, 4^-_o, 7^+_u)(3^-_u, 6^+_o, 1^-_o, 1^-_u, 2^-_o)$	55
$(8,10)$	$(5^+_u, 7^-_u, 7^-_o, 6^-_u, 8^-_o, 8^-_u, 5^+_o, 3^+_o, 1^+_o, 2^+_o, 4^+_o, 6^-_o, 4^+_u, 2^+_u, 3^+_u)$	15
$(8,11)$	$(2^+_u, 3^-_o, 5^-_o, 2^+_o, 4^-_u, 8^-_u, 7^-_u, 3^-_u, 6^-_u, 5^-_u, 1^+_u, 1^+_o)(4^-_o, 7^-_o)(6^-_o, 8^-_o)$	12
$(8,12)$	$(3^+_u, 5^-_u, 8^-_o, 6^-_u, 1^+_o, 2^+_o, 4^+_o, 3^+_o, 6^-_o, 2^+_u)(8^-_u, 5^-_o, 7^-_u, 7^-_o)$	20
$(8,13)$	$(7^+_u, 4^+_o, 3^+_u, 5^+_o, 1^-_u, 2^-_u, 4^+_u, 7^+_o)(6^+_u, 2^-_o, 3^+_o)(5^+_u, 8^-_o, 6^+_o, 8^-_u)$	24
$(8,14)$	$(2^+_u, 3^-_u, 6^-_o)(8^-_u, 3^-_o, 5^-_u, 8^-_o)(7^-_u, 7^-_o, 6^-_u, 1^+_o, 2^+_o, 4^-_o)(4^-_u, 5^-_o)$	12
$(8,15)$	$(8^-_u, 5^-_u, 6^-_u, 1^-_u, 2^-_u, 4^-_u, 7^-_u)(2^-_o, 3^-_o, 5^-_o, 8^-_o, 4^-_o, 6^-_o)$	42
$(8,16)$	$(6^+_u, 8^-_o, 6^+_o, 1^-_u, 2^+_u, 3^+_o)(3^+_u, 5^-_o, 4^-_o, 2^+_o, 4^-_u, 7^-_u)$	6
$(8,17)$	$(6^+_u, 8^-_u, 8^-_o, 7^-_u, 3^+_o, 6^+_o, 1^-_o, 1^-_u, 2^-_o, 3^+_u, 5^+_u, 4^+_u, 2^-_u, 4^+_o, 7^-_o)$	15
$(8,18)$	$(5^+_u, 4^+_u, 2^-_u, 4^+_o, 7^+_o, 3^-_o)(8^-_u, 5^+_o, 8^-_o)(6^-_u, 1^+_o, 2^-_o, 3^-_u, 6^-_o)$	30
$(8,19)$	$(8^+_u, 7^+_u, 6^+_o, 5^+_u, 4^+_o, 6^+_u)(4^+_u, 2^+_u, 3^+_u, 5^+_o, 7^+_o, 3^+_o, 1^+_o, 2^+_o)$	24
$(8,20)$	$(4^+_u, 6^-_o, 8^-_o, 7^-_o, 2^-_u)(3^+_u, 5^-_u, 7^-_u, 6^-_u, 5^-_o, 4^+_o)(2^-_o, 3^+_o, 1^+_o)$	30
$(8,21)$	$(6^+_u, 8^-_o, 6^+_o, 7^-_o, 2^-_u, 4^-_u, 1^+_o, 2^-_o, 3^-_u)(7^-_u, 3^-_o, 5^-_o, 4^-_o)$	36

Table C.2 n_{str} cycle presentations for knots up to 8 crossings

C.3.2 Multiplication Tables

	(3, 1)	(4, 1)	(5, 1)	(5, 2)	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(7, 1)	(7, 2)	(7, 3)	(7, 4)	(7, 5)	(7, 6)	(7, 7)	(8, 1)	(8, 2)	(8, 3)	(8, 4)	(8, 5)	(8, 6)	(8, 7)	(8, 8)	(8, 9)	(8, 10)	(8, 11)	(8, 12)	(8, 13)	(8, 14)	(8, 15)	(8, 16)	(8, 17)	(8, 18)	(8, 19)	(8, 20)	(8, 21)
(3, 1)	0	5	5	3	7	8	8	3	3	7	10	7	7	10	9	8	9	7	8	9	10	7	11	8	11	8	7	8	10	9	11	9	10	10	8	9	10
(4, 1)	6	4	7	7	10	4	9	8	9	9	8	9	9	5	7	6	8	12	10	7	8	12	6	11	6	10	12	10	8	12	10	10	8	8	10	9	9
(5, 1)	3	5	5	3	9	10	7	3	6	6	12	6	7	12	9	9	10	9	9	9	10	9	11	10	12	8	9	9	9	11	13	11	9	9	8	9	12
(5, 2)	3	5	4	3	8	10	7	3	6	7	12	7	7	12	9	9	10	11	9	9	10	11	11	10	12	6	11	9	9	11	13	11	9	8	7	9	12
(6, 1)	4	10	7	9	6	8	6	7	4	9	9	9	11	9	11	12	12	8	8	12	14	8	11	9	12	11	8	9	14	7	10	11	14	11	12	11	10
(6, 2)	8	9	9	9	8	3	7	9	9	11	9	11	11	9	9	8	12	10	8	14	12	10	11	8	10	12	10	12	12	9	9	11	12	10	11	10	7
(6, 3)	8	9	9	9	6	2	6	9	9	10	9	10	10	7	10	11	9	9	13	13	11	9	7	10	9	10	9	13	9	10	10	7	11	8	11	12	9
(6, 4)	3	7	6	3	10	10	7	0	6	7	13	7	7	13	11	9	11	10	10	10	10	10	11	7	12	9	10	10	9	12	14	7	10	11	8	11	8
(6, 5)	3	8	8	6	7	9	10	6	0	4	9	4	6	10	10	8	8	6	7	5	13	6	14	8	13	8	6	9	13	9	10	10	7	12	11	10	11
(7, 1)	5	7	7	5	10	11	10	3	2	7	14	7	7	14	11	8	12	11	11	11	10	11	10	10	12	10	11	11	7	13	13	10	10	10	8	11	13
(7, 2)	10	5	12	12	9	9	7	13	8	14	3	14	14	6	5	9	7	10	10	12	11	10	11	12	11	13	10	11	11	9	5	11	8	12	11	11	10
(7, 3)	5	7	7	5	10	11	7	3	2	6	14	6	7	14	11	8	12	11	11	11	10	11	11	11	12	10	11	11	7	13	13	10	11	11	5	11	13
(7, 4)	5	7	6	3	10	11	10	4	5	7	14	7	7	14	11	8	12	13	10	11	10	13	10	10	12	9	13	11	7	13	15	12	10	8	3	11	13
(7, 5)	10	6	12	12	9	9	10	13	10	14	6	14	14	5	4	5	4	10	8	10	11	10	12	12	11	13	9	11	9	10	4	8	5	12	15	11	8
(7, 6)	9	7	10	10	11	8	10	11	9	12	9	12	12	7	6	9	8	12	6	8	11	12	10	10	11	11	12	5	11	7	8	11	10	8	13	8	7
(7, 7)	9	7	10	10	12	7	11	10	11	9	11	8	8	7	7	5	10	10	12	5	9	10	8	11	7	13	10	12	9	14	12	11	10	10	11	10	9
(8, 1)	10	8	11	11	12	7	9	12	10	11	8	11	11	5	7	9	8	12	10	8	10	12	10	9	10	11	12	10	11	5	10	11	8	10	14	9	10
(8, 2)	7	12	11	11	8	8	9	11	5	13	10	13	13	9	12	9	12	7	9	11	16	8	6	9	14	10	7	10	15	7	9	11	14	10	14	11	7
(8, 3)	6	8	9	7	10	12	11	8	6	11	12	11	11	12	9	11	9	8	8	10	12	8	15	6	16	9	7	8	12	10	12	11	10	12	12	8	12
(8, 4)	8	6	9	8	12	14	13	9	8	9	12	9	9	10	6	6	5	10	4	8	8	10	7	13	12	9	10	6	10	12	12	11	5	11	9	10	12
(8, 5)	10	8	10	10	14	10	11	8	13	10	12	10	10	11	11	9	12	16	13	9	8	16	5	13	6	13	16	14	8	16	12	13	9	5	10	12	12
(8, 6)	5	12	11	11	8	8	9	11	4	13	10	13	13	9	12	9	12	8	9	11	16	8	6	9	12	10	8	10	15	7	9	11	14	9	14	11	9
(8, 7)	11	7	11	11	13	6	10	9	14	11	12	11	11	11	10	8	10	11	15	9	7	9	7	11	7	14	9	15	9	15	12	12	8	11	11	14	10
(8, 8)	6	11	10	9	9	7	10	3	5	11	12	11	11	10	10	10	9	9	5	13	13	9	14	7	13	8	8	7	12	7	13	12	15	11	11	11	10
(8, 9)	11	8	12	12	8	10	9	11	13	12	11	11	11	11	11	8	12	14	16	12	6	14	6	12	8	15	14	16	10	14	11	13	10	11	10	11	10
(8, 10)	7	8	8	8	11	13	8	8	6	9	13	8	9	13	10	11	8	11	7	9	11	11	14	8	15	7	11	7	11	11	13	12	8	10	11	9	11
(8, 11)	7	12	11	11	8	8	9	11	5	13	10	13	13	9	12	9	12	8	9	11	16	8	5	9	12	10	8	10	15	6	9	11	14	9	14	10	9
(8, 12)	7	8	9	9	10	12	11	9	7	11	12	11	11	12	9	11	9	10	8	10	14	10	15	6	16	9	10	8	12	10	12	11	10	12	12	8	12
(8, 13)	10	8	10	10	14	12	11	8	13	10	7	9	9	9	11	8	11	15	13	7	8	15	9	14	9	12	15	13	7	15	13	12	9	10	10	11	13
(8, 14)	7	12	11	11	8	10	5	10	7	13	10	13	13	9	11	14	11	8	8	12	16	7	15	6	12	11	8	9	15	8	8	11	4	11	14	11	10
(8, 15)	11	10	13	13	9	10	11	14	10	13	7	13	15	8	8	9	9	8	12	11	12	9	6	13	9	11	8	12	13	8	7	11	12	11	16	11	10
(8, 16)	8	10	11	10	9	9	12	7	8	9	5	10	12	8	11	11	11	9	9	9	13	7	12	12	13	11	9	11	12	9	9	8	11	15	13	10	10
(8, 17)	10	8	10	10	14	10	11	8	11	11	12	11	11	10	10	10	10	14	12	8	9	14	9	13	10	11	14	12	4	14	12	11	8	10	12	11	10
(8, 18)	10	10	10	10	11	11	8	11	12	10	8	11	11	11	12	11	9	10	8	12	12	10	10	10	12	8	10	8	6	10	12	15	12	7	12	9	12
(8, 19)	7	9	8	8	12	10	10	3	10	7	11	7	7	15	11	10	13	14	10	8	5	14	4	11	12	11	14	9	10	14	16	13	7	12	8	13	14
(8, 20)	9	9	10	10	11	10	5	11	8	12	11	10	12	11	7	9	9	11	8	11	13	11	14	6	11	10	11	8	12	11	11	12	8	10	13	7	10
(8, 21)	10	11	12	12	7	8	9	9	10	12	8	12	13	8	12	7	12	10	12	12	14	10	8	13	12	13	10	12	13	9	9	11	12	12	14	10	8

Table C.3 Multiplication table giving the lengths of all of the product knots in Reidemeister I/II reduced form

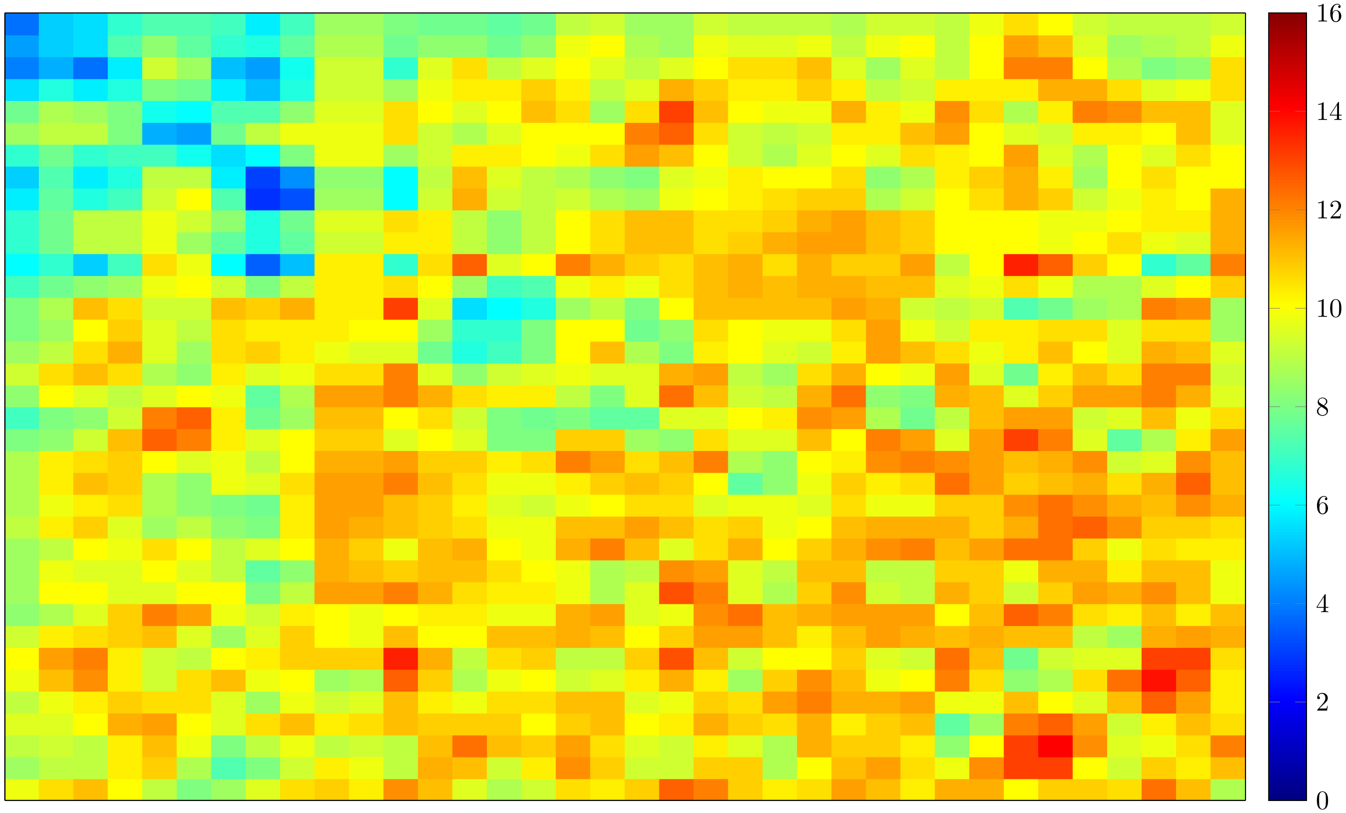

Figure C.1 Heatmap for the multiplication table. Observe that the table is *not* symmetric.

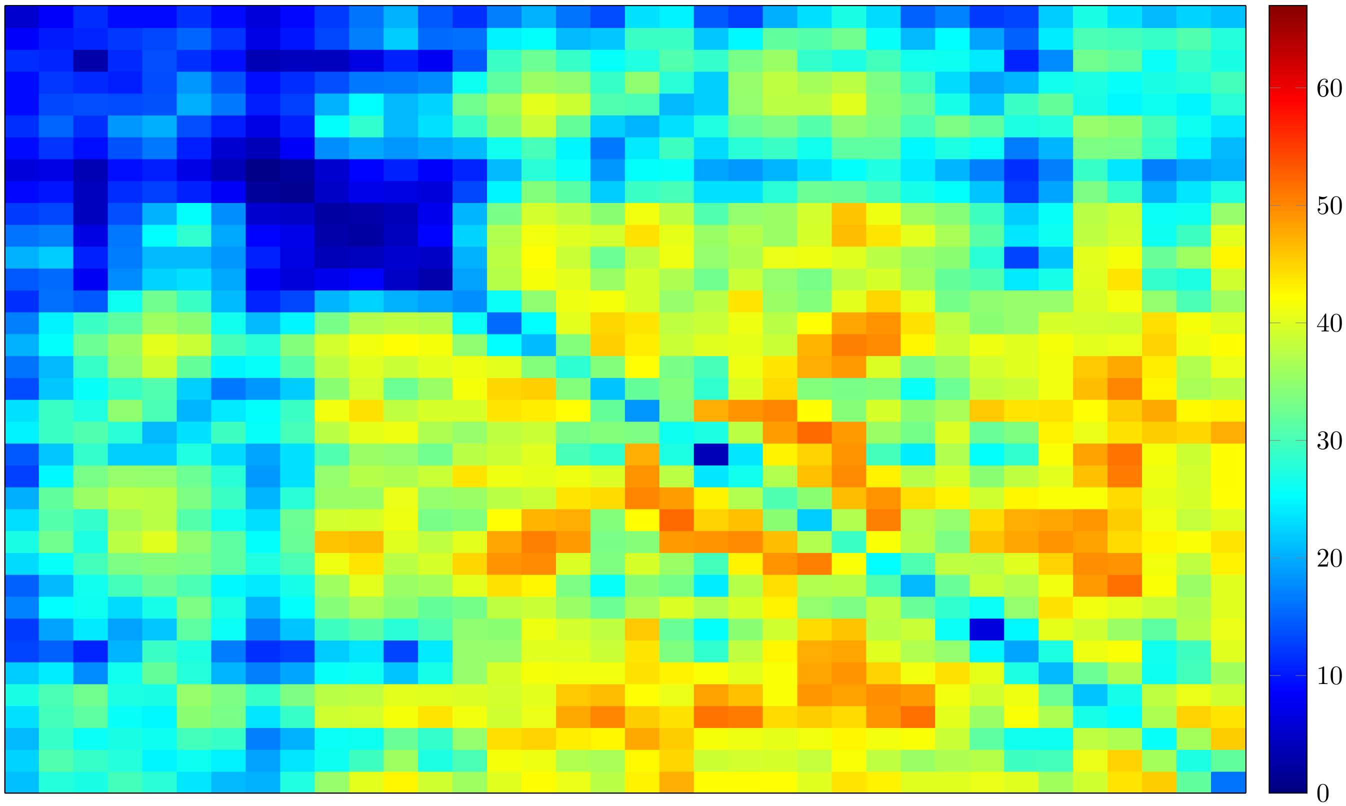

Figure C.2 Log-scale heatmap showing the order of the group generated by pairs of our knot cycles. This one is symmetric.

Bibliography

Alexander, J. W. 1923. On the Deformation of an n Cell. *Proc Natl Acad Sci USA* 9(12):406. doi:10.1073/pnas.9.12.406.

Alexander, J. W., and G. B. Briggs. 1926. On Types of Knotted Curves. *Annals of Mathematics* 28(1/4):562–586. doi:10.2307/1968399.

Bing, R. H. 1954. Locally tame sets are tame. *Annals of Mathematics* 59(1):145–158. URL http://www.jstor.org/stable/1969836.

Bondy, Adrian, and M. Ram Murty. 2008. *Graph Theory | Adrian Bondy | Springer*. Springer-Verlag London. URL https://www.springer.com/gp/book/9781846289699.

Bothe, H. 1981. Homogeneously wild curves and infinite knot products. *Fundamenta Mathematicae* 113(2):91–111. URL http://eudml.org/doc/211288.

Bryant, John L. 2001. Chapter 5 - Piecewise Linear Topology. *Handbook of Geometric Topology* 219–259. doi:10.1016/B978-044482432-5/50006-8.

Burde, G., and H. Zieschang. 2003. *Knots*. De Gruyter studies in mathematics, Walter de Gruyter. URL https://books.google.com/books?id=DJHI7DpgIbIC.

Crowell, R. H., and R. H. Fox. 1963. *Introduction to Knot Theory | R. H. Crowell | Springer*. Springer-Verlag New York. doi:10.1007/978-1-4612-9935-6.

Daverman, Robert J., and Gerard A. Venema. 2009. *Embeddings in Manifolds*, *Graduate Studies in Mathematics*, vol. 106. American Mathematical Society.

Fisher, Gordon M. 1960. On the group of all homeomorphisms of a manifold. *Transactions of the American Mathematical Society* 97(2):193–212. URL http://www.jstor.org/stable/1993298.

Fox, Ralph H., and Emil Artin. 1948. Some wild cells and spheres in three-dimensional space. *Annals of Mathematics* 49(4):979–990. URL http://www.jstor.org/stable/1969408.

Ganzell, Sandy. 2014. Local moves and restrictions on the Jones polynomial. *J Knot Theory Ramifications* 23(02):1450,011. doi:10.1142/S0218216514500114.

Hass, Joel, Jeffrey C. Lagarias, and Nicholas Pippenger. 1998. The Computational Complexity of Knot and Link Problems. *arXiv* URL https://arxiv.org/abs/math/9807016. math/9807016.

Hatcher, Allen. 2014. Reference for a fact (?) on homeomorphic knot complements. MathOverflow. URL https://mathoverflow.net/q/185803. https://mathoverflow.net/q/185803.

Kanenobu, Taizo, and Ryo Nikkuni. 2005. Delta move and polynomial invariants of links. *Topology and its Applications* 146-147:91–104. doi:10.1016/j.topol.2003.02.003.

Kauffman, Louis H. 1998. Virtual Knot Theory. *arXiv* URL https://arxiv.org/abs/math/9811028. math/9811028.

———. 2012. INTRODUCTION TO VIRTUAL KNOT THEORY. *J Knot Theory Ramifications* 21(13):1240,007. doi:10.1142/S021821651240007X.

Kauffman, Louis H., and Sofia Lambropoulou. 2011. HARD UNKNOTS AND COLLAPSING TANGLES | Introductory Lectures on Knot Theory. *WORLD SCIENTIFIC* 46:187–247. doi:10.1142/9789814313001_0009.

Kobayashi, Forest, and Sam Nelson. 2019. Kaestner brackets. 1909.09920.

Kojman, M., M. A. Perles, and S. Shelah. 1990. Sets in a euclidean space which are not a countable union of convex subsets. *Israel J Math* 70(3):313–342. doi:10.1007/BF02801467.

Lackenby, Marc. 2013. A polynomial upper bound on Reidemeister moves. *arXiv* URL https://arxiv.org/abs/1302.0180. 1302.0180.

———. 2016. Elementary knot theory. *arXiv* URL https://arxiv.org/abs/1604.03778. 1604.03778.

Lilalas. 2019. Reference request: A knot is tame if and only if it has a tubular neighbourhood. MathOverflow. URL https://mathoverflow.net/q/335888. URL:https://mathoverflow.net/q/335888 (version: 2019-07-11), https://mathoverflow.net/q/335888.

Matthes, Alain. 2012. How can I draw an arc from point A -> B on a 3D sphere in TikZ? URL https://tex.stackexchange.com/a/49589.

Milnor, J., and D.W. Weaver. 1997. *Topology from the Differentiable Viewpoint*. Princeton Landmarks in Mathematics and Physics, Princeton University Press. URL https://books.google.com/books?id=BaQYYJp84cYC.

Moise, Edwin E. 1954. Affine structures in 3-manifolds viii. invariance of the knot-types; local tame imbedding. *Annals of Mathematics* 59(1):159–170. URL http://www.jstor.org/stable/1969837.

Nakanishi, Yasutaka. 1994. ON GENERALIZED UNKNOTTING OPERATIONS. *J Knot Theory Ramifications* 03(02):197–209. doi:10.1142/S0218216594000150.

Nelson, S., and N. Oyamaguchi. 2017. Trace diagrams and biquandle brackets. *Int J Math* 28(14):1750,104. doi:10.1142/S0129167X1750104X.

Nelson, Sam, and Mohamed Elhamdadi. 2015. *Quandles: An Introduction to the Algebra of Knots*. Providence, Rhode Island: American Mathematical Society.

Nelson, Sam, Michael E. Orrison, and Veronica Rivera. 2017. Quantum enhancements and biquandle brackets. *J Knot Theory Ramifications* 26(05):1750,034. doi:10.1142/S0218216517500341.

Prasolov, V.V., V.V. Prasolov, and A.B. Sossinsky. 1997. *Knots, links, braids and 3-manifolds: an introduction to the new invariants in low-dimensional topology*. Translations of mathematical monographs, American Mathematical Society. URL https://books.google.com/books?id=dFjvAAAAMAAJ.

Reidemeister, Kurt. 1927. Elementare Begründung der Knotentheorie. *Abh Math Semin Univ Hamburg* 5(1):24–32. doi:10.1007/BF02952507.

Rourke, Colin P., and Brian J. Sanderson. 1982. *Introduction to Piecewise-Linear Topology.* Springer, Berlin, Heidelberg. doi:10.1007/978-3-642-81735-9.

Sakai, Katsuro. 2013. *Geometric Aspects of General Topology | Katsuro Sakai | Springer.* Springer Japan. doi:10.1007/978-4-431-54397-8.

Scott Carter, J., Seiichi Kamada, and Masahico Saito. 2002. STABLE EQUIVALENCE OF KNOTS ON SURFACES AND VIRTUAL KNOT COBORDISMS. *J Knot Theory Ramifications* 11(03):311–322. doi:10.1142/S0218216502001639.

Shilepsky, Arnold C. 1974. Homogeneity and extension properties of embeddings of s1 in e3. *Transactions of the American Mathematical Society* 195:265–276. URL http://www.jstor.org/stable/1996730.

Starbird, Michael, and Francis Su. 2019. *Topology Through Inquiry.* Providence, Rhode Island: MAA Press.

user98602. 2015. Equivalence of knots: ambient isotopy vs. homeomorphism. Mathematics Stack Exchange. URL https://math.stackexchange.com/q/1312016. https://math.stackexchange.com/q/1312016.

whippedcream. 2016. Equivalence of knots: ambient isotopy vs. homeomorphism. Mathematics Stack Exchange. URL https://math.stackexchange.com/q/1857823. URL:https://math.stackexchange.com/q/1857823 (version: 2016-07-13), https://math.stackexchange.com/q/1857823.