Categorification Schemes for Knot Diagrams

Forest Kobayashi

Francis Su, Advisor

Sam Nelson, Reader

HARVEY MUDD COLLEGE

Department of Mathematics

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Abstract

Mathematical knots are straightforward to define, but very difficult to understand. In particular, it can be very chalenging impossible to determine when two arbitrary knots K_0, K_1 are equivalent. Some algorithms do exist, but most are either extremely inefficient or have no proven time complexity. As such, knot theorists often rely on *knot invariants* to distinguish knots. However, we often don't know where to look for good invariants. One possible reason is that we don't have a good understanding of algebraic structure

come back to this

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Acknowledgments

I am deeply indebted to Edward P. Moore, one my high school English teachers. When I entered his class, I was struggling immensely with academics, math most of all. Throughout these hardships, he never stopped believing in me, instead always pushing me to see in the merit in my ideas, and believe in my capacity to succeed. There is no doubt in my mind that without his guidance, I would not be at Harvey Mudd College today. And I *certainly* would never have had the courage to study math.

Speaking of which: an enormous thank you to Professor Erica Flapan, whose Analysis I class convinced me that I wanted to become a mathematician. And an equally-sized thank you to all of my math professors since — your classes convinced me I wanted to stay one.

Chapter 1

Introduction

All [knot categories] are equal,
but some [knot categories] are
more equal than others.

—*George Orwell, or something*

Perhaps I should start directly with an example, and a preface or something

One of the most fascinating things about knot theory is the disconnect between the relative ease of posing a question, and the near impossibility of finding a rigorous answer for it. Granted, many mathematical fields are like this — but knot theory is somewhat special in the *extremity* of the mismatch. Many of the most fundamental problems in the field can be boiled down to ideas that are accessible to any lay-person, and yet are totally baffling to try and approach mathematically. This sounds a bit over-the-top

My thesis seeks to address this problem. Namely, I am guided by the following broad question: "can we find new frameworks for knot theory that offer a better match between our intuitions and the underlying mathematical framework?" Of particular interest is the problem of *detecting knot equality*, which I discuss below. As we will see, stronger Algebraic structures seem highly desirable here, and that is where I have chosen to focus my efforts.

My exposition in this chapter is structured as follows: first, I offer a big-picture overview of the basic concepts in knot theory from an intuitive standpoint, as well as loose description of the problem I am trying to address in my thesis. As motivation, I take a brief digression into an analogy with $\mathbb{Z}$. Next, I circle back and give more formal definitions of the bread-and-butter objects I'll be using throughout this project, and finish with a summary of some previous results that have guided my work. Then in chapter 2, it's on to my own results.

1.1 Motivation

1.1.1 The Big Picture

PICTURE

Figure 1.1 A joke

In mathematics, a *knot* is an embedding of a circle into $\mathbb{R}^3$.[1] Intuitively, think of taking a rope and twisting it around in space in all sorts of ways, finally fusing the ends together so that we get a closed loop:

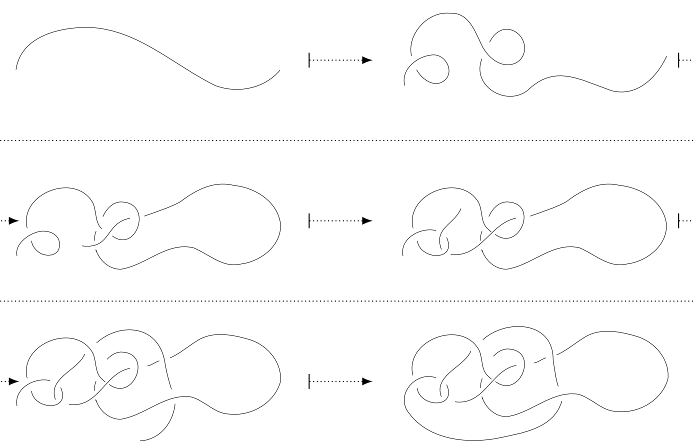


Figure 1.2 Constructing a knot

Note, our loop does not need to have any crossings to be considered a knot — a regular old circle is a perfectly valid knot! We call this the *unknot*, and we'll see that it has some interesting properties later (e.g., it acts like 1 under a knot "multiplication" operation).

[1] Basically, if we zoom in closely enough it looks like a line, and if we walk around in one direction, we get back to where we started.

We say two knots K_1, K_2 are *equivalent* (denoted $K_1 \cong K_2$) if we can deform K_1 into K_2 without cutting the rope and gluing it back together. For example, the left two knots in the diagram below are equivalent, and both are distinct from the knot on the right.[2]

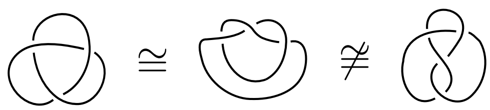
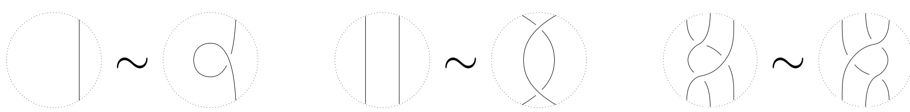


Figure 1.3 Two equivalent knots and one inequivalent one

One of the central questions in knot theory is "given diagrams D_1 and D_2 how do we determine whether they represent the same knot?" If starting from first principles, this question is **HARD** to approach mathematically; it requires a healthy dose of topological machinery to even formulate the question properly. Thankfully, in practice we don't usually need to think about any of this because of a result due to Reidemeister (1927), and independently Alexander and Briggs (1926). In essence, they were able to show that two diagrams D_1, D_2 represent the same knot iff D_1 can be turned into D_2 by a sequence of the following so-called *Reidemeister moves:*



which are creatively referred to (in left-to-right order) as "Reidemeister I," "Reidemeister II," and "Reidemeister III," respectively. Note, while Reidemeister III might look complicated, it's really just saying that we can move a strand over a crossing.

On a theoretical level, this is a very elegant characterization of knot equivalence. However, in practice, the problem is still quite hard. Even if D_1 and D_2 *do* represent the same knots and are both relatively simple, the sequence of moves relating the two can be quite long. It's even worse when $K_1 \ncong K_2$, because then we have to prove a negative result: namely, that there *does not exist* a sequence of Reidemeister moves takes D_1 to D_2! Again, this is usually quite hard. Though Algorithms deciding the problem do exist, they are currently far too inefficient to be practical.Cite lackenby

[2] At this point, it is worth noting that there's a technical distinction between a *diagram* for a knot and the actual knot itself. We'll return to this later when we define things rigorously, but gist is that knots live in $\mathbb{R}^3$ and diagrams are projections onto $\mathbb{R}^2$.

But why? This seems like it should be easy! Our objects are very tangible, the space we're working in ($\mathbb{R}^3$) is well-behaved, and we aren't asking for anything too fancy — just a simple way to determine equality. How can we understand the source of this difficulty? Here, an analogy with something more familiar will be helpful.

1.1.2 An Analogy

Let's say I hand you two integers n, m. Would you be able to tell me if $n = m$? Most likely the answer is yes. For instance, if I said

$$n = 2 \qquad\qquad m = 5,$$

you'd probably be able to distinguish n and m. In particular, they're uh. . . not. . . the same number.

But now let's frame the same problem in a slightly different way. Suppose I had given you something like

$$n = \frac{-1}{3 - \frac{7}{2}} \qquad\qquad m = 5 \cdot 5 - 20$$

instead. Could you still determine if $n = m$? Of course — these equations actually give us the same solutions for n, m as before, we just have to do a little bit of work since the answers are obfuscated by a thin layer of arithmetic. Nevertheless, the problem remains tractable. We simplify the expressions, see that the left gives 2 and the right gives 5, and then we know $2 \neq 5$ and so we can go on our way. Note, this approach continues to work even with extremely messy expressions like

$$n = \frac{(154 - 162) \cdot (-\frac{66}{13})}{6} \cdot \frac{9}{29} - \frac{38}{377}$$

$$m = \frac{\left(\frac{7+90}{36 \cdot 5^2} + 8 \cdot \frac{10}{764}\right) \cdot \frac{1719}{1 - 54 \cdot \frac{3 \cdot 11 \cdot 3}{(63+37)\cdot(35+19)}} - 3 \cdot 9}{7300}.$$

Although computing simplified forms for these expressions is deeply unpleasant, there isn't a big *conceptual* challenge to it. We can feed them into a computer program (e.g. WolframAlpha), determine that $n = 2$ and $m = 5$, and be done.

While simplifying arithmetic expressions might seem like second nature at this point, it's worth taking a moment to pause and think about what's really going on here. The perspective will be helpful in understanding the knot equivalence problem.

Recap. In all of the examples above, we were given two *strings* of symbols encoding arithmetic expressions representing elements of $\mathbb{Z}$. We then applied *simplification rules* to reduce these to canonical forms, here 2 and 5 respectively. There are two essential properties of these simplification rules:

(1) They preseve arithmetic equality. Namely, if string A can be simplified to string B, we know that A and B represent the same number.

(2) They result in canonical representations of our arithmetic expressions. In particular, given any two equivalent expressions A and B, our rules will simplify A and B to the same expression. For instance, if $A =$ "$1 + 2 + 3$" and $B =$ "$10 - 1 - 3$", then our rules will reduce both of these to "6".

In light of these two facts, we then knew that our expressions for n and m could only be equivalent if our canonical representatives ("2" and "5") were identical as strings. Seeing that they were not, we could then conclude that the starting expressions for n and m represented different numbers.

Formally, this can be interpreted as follows (a full treatment is given in the appendix) consider revising; there are some redundancies in this section

Example 1.1 (Equivalence of Arithmetic Expressions)**.** We consider the set of all arithmetic expressions over $\mathbb{Z}$ constructed as follows.

- We select characters from the alphabet $\Sigma = \mathbb{Z} \cup \{+, -, \cdot, \div, (,)\}$.[3]
- Consider the set of all strings formed by concatenating elements of Σ together (e.g., "$1 + (-7)$" and "$(2 \div 4) + (5 \div 2)$"). Discard ones that create nonsensical expressions like "$-)2 + ((\div$". Again, this is done formally in the appendix. Uh you sure about that buddy?
- Say two strings are *equivalent* if they parse to the same values using the standard rules of arithmetic.

Does this get a bit long in the tooth? This perspective gives us an important insight that we probably don't think of on a day-to-day basis: namely, when we say things like $n = 2, m = 5$, we are often really thinking about *equivalence classes* of arithmetic expressions that can be converted

[3] If you prefer fractions to the $\div$ symbol, just think of the two dots in $\div$ as being analogous to the dots in $\langle \cdot, \cdot \rangle$ and then they're notationally equivalent.

into each other using the standard rules of algebra.[4] Hence, when we say things like

$$2 = \frac{-1}{3 - \frac{7}{2}},$$

we really mean

$$2 \in [2] \qquad \text{and} \qquad \frac{-1}{3 - \frac{7}{2}} \in [2].$$

Talk about how we want to apply this same process to knot There are two places where things could go wrong here. First: how do we choose a canonical representative / what do we do if there is no simplification algorithm? And second,

Talk about the importance of canonical representatives, in particular $1.0000\cdots$ and $.9999\cdots$

We offer a brief review of some fundamental concepts from topology. The experienced reader should feel free to skip to the next section.

1.2 Formal Background

Define the diagram of a knot by recalling the distinction between the graph of a function and the function itself. Discuss how the inherently 2D representation is lossy when we compare it with the graph of a single-variable function, which is not lossy. Discuss how this might impact the kind of information we can extract.

Also, maybe talk about tame vs. wild knots?

1.2.1 Knots and Knot Diagrams

As stated earlier, a knot is an embedding of a closed loop into 3-dimensional space. More formally:

Definition 1.1 (Knot)**.** A *knot* is an injective, continuous map $K : S^1 \hookrightarrow \mathbb{R}^3$.

Example 1.2. The following is a knot known as $(7, 2)$

[4]Recall, we denote the equivalence class of x by $[x]$.

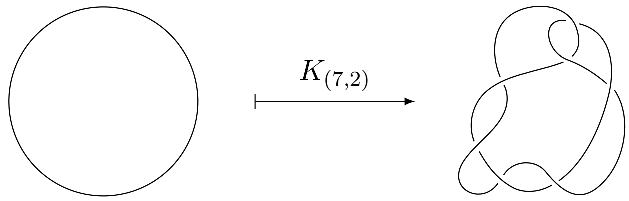


Figure 1.4 The $(7, 2)$ knot.

As we see in the example above, it is often useful to represent knots by 2D diagrams. However, not all diagrams are equivalently useful. For instance, if we were to "line up" all the crossings of the $(7, 2)$ knot like so,

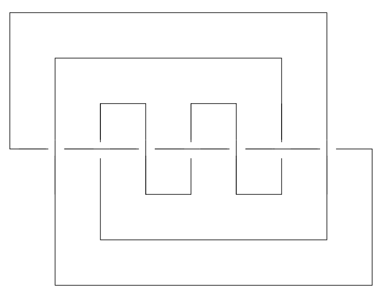


Figure 1.5 Crossings lined up

then if we were to look along the following axis,

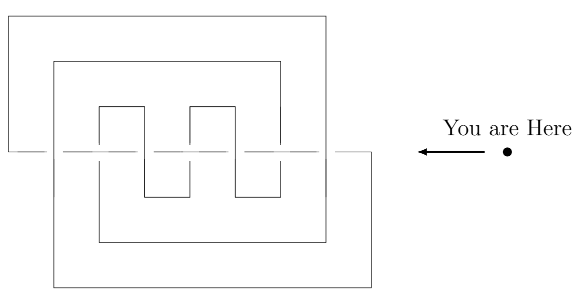


Figure 1.6 New perspective

we might end up seeing something like this:

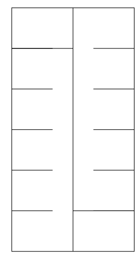


Figure 1.7 Unhelpful sideways view

which is hardly helpful.[5] Hence, we will place some restrictions on what exactly we are allowed to refer to as a *knot diagram.* Making sure these objects are well-behaved will be essential in moving towards a purely combinatorial description of knots.

Definition 1.2 (Knot Diagram)**.** Let K be a knot, and let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be a projection onto a 2-dimensional subspace of $\mathbb{R}^3$. Then we say $D = \pi \circ K(S^1)$ is a *diagram* for K iff D satisfies the following conditions:

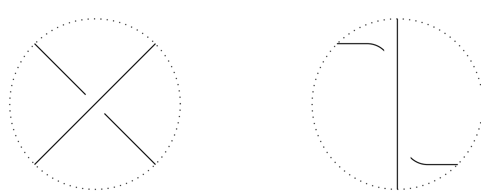
1) D is injective at all but finitely many points $\{y_i\}_{i=1}^n \in \mathbb{R}^2$, called *crossings.*

2) For each of these y_i, there exist exactly two $x \in S^1$ such that $D(x) = y_i$.

3) Each crossing "looks like an X." Formally, there exists $\varepsilon > 0$ such that $B_\varepsilon(y_i) \cap D(S^1)$ is homeomorphic to $\{(x, y) \mid x = 0 \text{ or } y = 0\}$.

4) The diagram contains some information by which we can recover which strand was "on top" at each crossing.

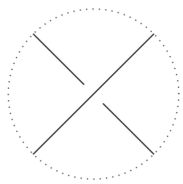
We interpret these properties as follows. Condition (1) requires that strands only cross at *single points* in our diagrams (not entire line segments).



✓ Allowed ✗ Not allowed

[5]Note, we say "might" because there are multiple possible 3D realizations that could yield 1.5 in a 2D projection. But at least one of those would look like 1.7.

Condition (2) requires that we can't have multiple strands crossing at the same point: should these conditions be hyperref'ed



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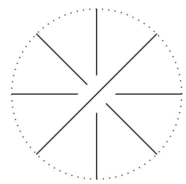

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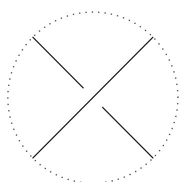
Condition (3) precludes situations like the following:



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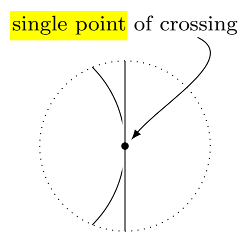


✗ Not allowed

that is, whenever we have a crossing, both of the strands that come in must leave on opposite sides. Finally, condition (4) actually refers to a convention that we have been employing tacitly all along; namely *breaks* in the diagram represent places were crossings occur, and the broken strand is understood to be going "underneath" the unbroken strand.

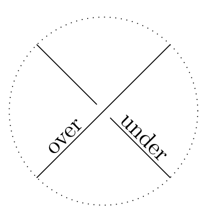


Figure 1.8 Breaks tell us which strand is on top

Remark. At this point, it is worth noting that there is a difference between a *knot* and a *knot diagram*. A *knot* is an abstract function going from S^1 to $\mathbb{R}^3$, while a *knot diagram* is a function going S^1 to $\mathbb{R}^2$. In the section above, we defined knot diagram D in terms of composing a particular projection π

with a knot K. However, we can also think of D as an abstract function in its own right. This gives us the following commutative diagram:

$$\begin{tikzcd} S^1 \arrow[rr, "D"] \arrow[dr, "K"'] & & \mathbb{R}^2 \\ & \mathbb{R}^3 \arrow[ur, "\pi"'] & \end{tikzcd}$$

Figure 1.9 Relationships between D, K, and π

It's worth nothing that this factorization is not unique. In particular, we can have many different K, π pairs that give us the same D:

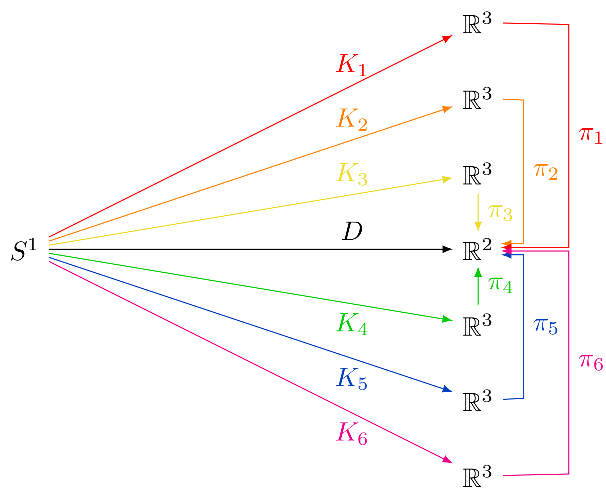


Figure 1.10 Taste the rainbow (Note, this diagram might make no sense)

And in sort of dual vein, different projections π can take the same knot K to many *different* diagrams.

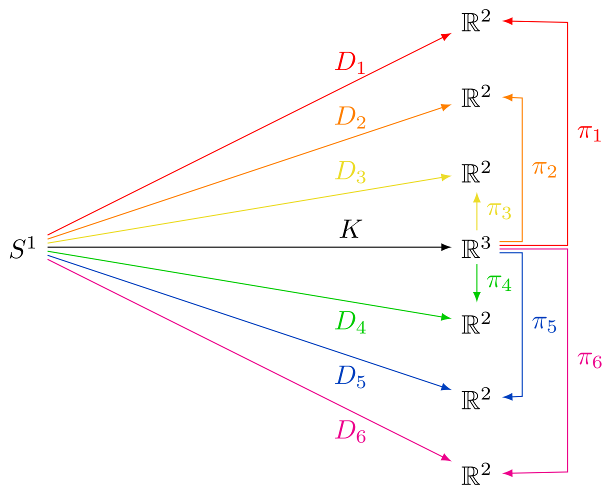


Figure 1.11 A hot mess; this is probably more confusing than helpful

Both of these properties are undesirable. In order to use diagrams to study knots, we must first figure out a way to ensure that they "contain the same information" in some sense.

1.2.2 Ambient Isotopy and Reidemeister Moves

Here's the plan: first, we'll define an equivalence relation on knots. This will come in the form of a concept called *ambient isotopy*. Then, we will define an equivalence relation on knot diagrams. This will come in the form of the *Reidemeister moves*, as discussed in chapter I. Finally, we will discuss *Reidemeister's Theorem*, which more or less states that modding out by these equivalences actually give us isomorphic categories. This seems to be complicated by the fact that the diagram for the countable reidemeister moves I've introduced, the result is not actually equivalent to other diagrams of the unknot…

Chapter 2

Towards Algebraic Structure

2.1 Countable Sequences of Reidemeister Moves

Let K be a knot. We establish some criteria for when we can apply a countable number of Reidemeister moves to K and preserve ambient isotopy. First, we discuss some notational details.

2.1.1 Notation

I need some nice notation for Reidemeister moves. Instead of using greek / roman / hebrew letters, I will elect to employ characters from hiragana (one of the japanese alphabets):

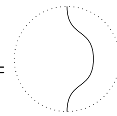
「ひらがな」
(hiragana)

Not only does this avoid confusion (symbols from alphabets like {Greek, English, . . .} are already overloaded with many common mathematical usages), some of the 「ひらがな」 characters really do look a lot like the Reidemeister moves.[1] We use the following convention:

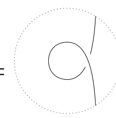
1. Reidemeister "0" moves correspond to just bending a straight line (no

[1] For instructions on how to define macros for these with LaTeX, see https://tex.stackexchange.com/questions/171611/how-to-write-a-single-hiragana-character-in-latex

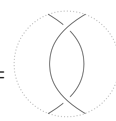
crossings). We denote these with つ (“tsu”)

つ = 

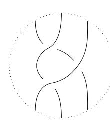
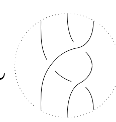
2. Reidemeister I moves correspond go deforming a straight line into a loop. We denote these with の (“no”)

の = 

3. Reidemeister II moves correspond to crossing one strand over another (two crossings). We denote these with ゆ (“yu”).

ゆ = 

4. Reidemeister III moves correspond to sliding a strand past a crossing. We denote these with め (“me”).

め =  $\sim$ 

For Reidemeister I moves, we will use $の_k$ to denote “insert a loop after crossing k.” In the case FIXME: discuss how to choose a canonical start point for the indexing

2.1.2 First up: Reidemeister I

Let S^1 be the circle (unknot). Let $W_\infty : S^1 \hookrightarrow \mathbb{R}^3$ be the an embedding of S^1 as follows.[2]

1) Let $\iota : S^1 \hookrightarrow \mathbb{R}$ be the inclusion map. Note that it is an embedding.

[2]Note, we make a lot of restrictions below to make the construction easier here; we'll relax them later

2) Define a sequence of embeddings $W_n : S^1 \hookrightarrow \mathbb{R}^3$ by iteratively inserting loops into $\iota(S^1)$ using $\circlearrowleft$ moves:

$$
\begin{aligned}
W_0 &= \iota \\
W_1 &= \circlearrowleft_1 \circ \iota \\
W_2 &= \circlearrowleft_2 \circ \circlearrowleft_1 \circ \iota \\
&\vdots \\
W_n &= \left(\bigcirc_{k=1}^{n} \circlearrowleft_k \right) \circ \iota \\
&\vdots
\end{aligned}
$$

subject to the constraints below. As a matter of notation, denote the set of points corresponding to the loop inserted by $\circlearrowleft_k$ by $\mathcal{L}_k$. Then

i) There exists $r \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $W_n(S^1) \subseteq B_r(\mathbf{0})$ (i.e., the W_n are uniformly bounded)
ii) For all $n \in \mathbb{N}$, let $r_n = \frac{1}{2^n}$. Then for $n \geq 1$, there exists $c_n \in \iota(S^1)$ such that $\circlearrowleft_n$ is identity outside of $B_{r_n}(c_n)$ ($\circlearrowleft_n$ only changes things on a ball of radius $1/2^n$ around some point of our starting circle). We call $W_n(S^1) \cap B_{r_n}(c_n)$ the k^{th} *loop*, and denote it $\mathcal{L}_k$.
iii) Finally, the $B_{r_n}(c_n)$ are pairwise disjoint (none of the changes made by the $\circlearrowleft_n$ overlap).

3) With the W_n defined as above, define $W_\infty = \lim_{n\to\infty} W_n$.

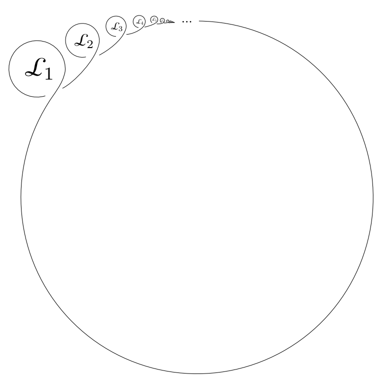


Figure 2.1 An example of a realization of W_∞ ("nonononononono...")

Remark. The picture above might be misleading, since the diameters of the loops do not actually decay as $\frac{1}{2^n}$, and the bounding balls are not actually disjoint. We made this sacrifice so as to afford more legibility to the diagram. wow those $\mathcal{L}_n$ labels are too small

We address the following questions about W_∞:

- Is W_∞ *actually* an embedding of S^1 as we claimed? (Yes)
- In the realization of W_∞ we drew above, the loops appear to bunch up at a point p. Must this be true of any realization of W_∞? (Yes)
- Is W_∞ ambient isotopic to the unknot? (Yes!)

We offer some proofs.

2.1.3 Some proofs

For the following, recall that $f : X \to Y$ is called an *embedding* if f is a homeomorphism from X to $f(X)$ (a *homeomorphism* is a continuous bijection with continuous inverse). First, we have a small lemma.

Lemma 2.1. *For all $k \in \mathbb{N}$ and all $m \geq k$, if $x \in \mathcal{L}_k$, then*

$$W_k(x) = W_m(x).$$

In particular, $W_k(x) = W_\infty(x)$.

Proof. The proof is directly from the definition of the $\mathcal{O}_i$ and the $B_{r_i}(c_i)$. We go through it in detail anyways to get used to the notation.

Recall that we defined

$$\mathcal{L}_k = W_k(S^1) \cap B_{r_k}(c_k).$$

Let $x \in \mathcal{L}_k$ be arbitrary. By construction, we have

$$W_m = \mathcal{O}_m \circ \mathcal{O}_{m-1} \circ \cdots \mathcal{O}_{k+1} \circ W_k.$$

Also by construction, for all $i \in \mathbb{N}$, $\mathcal{O}_i$ is identity outside of $B_{r_i}(c_i)$. It follows that

$$f = \mathcal{O}_m \circ \mathcal{O}_{m-1} \circ \cdots \mathcal{O}_{k+1}$$

is identity outside of

$$U = \bigcup_{j=k+1}^{m} B_{r_j}(c_j).$$

Finally, *again* by construction, all of the $B_{r_i}(c_i)$ are pairwise disjoint. Thus

$$B_{r_k}(c_k) \cap U = \varnothing.$$

Putting it all together: $x \in \mathcal{L}_k$ implies $x \in B_{r_k}(c_k)$ implies $x \notin U$, and so $W_m(x) = f(W_k(x)) = W_k(x)$, as desired. ■

The lemma will aid us in proving the following claim.

Proposition 2.2. *$W_n \to W_\infty$ uniformly.*

Proof. Let $\varepsilon > 0$ be given, and let $N \in \mathbb{N}$ such that

$$\frac{1}{2^N} < \frac{\varepsilon}{2}.$$

Now, let $x \in S^1$ be arbitrary. WTS $\forall n \geq N$, we have $d(W_n(x), W_\infty(x)) < \varepsilon$. To that end, let $n \geq N$ be arbitrary. We have two subcases.

1) Suppose that there exists $k \in \mathbb{N}$ such that $W_\infty(x)\mathcal{L}_k$ (x is sent to a point in a loop). We have two sub-subcases:

   i) Case 1 (the $\mathcal{L}_k$ loop has already been added): Suppose $k \leq n$. Applying the Lemma shows $W_n(x) = W_\infty(x)$, hence $d(W_n(x), W_\infty(x)) = 0 < \varepsilon$, as desired.

   ii) Case 2 (the $\mathcal{L}_k$ loop hasn't been added yet): Suppose $k > n$. Then by definition of the W_n, we have $W_n(x) = \iota(x)$. Now, since $\mathcal{O}_k$ only acts on $B_{r_n}(c_k)$, we have

   $$W_n(x) = \iota(x) \in B_{r_k}(c_k) \qquad \text{and} \qquad W_k(x) = W_\infty(x) \in B_{r_k}(c_k).$$

   Since $k > n \geq N$, we have $r_n = \frac{1}{2^k} < \frac{1}{2^N} < \frac{\varepsilon}{2}$. Hence, by the triangle inequality,

   $$\begin{aligned} d(W_n(x), W_\infty(x)) &\leq d(W_n(x), c_k) + d(c_k, W_\infty(x)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Either way, we see $d(W_n(x), W_\infty(x)) < \varepsilon$, as desired.

2) Suppose there exists no k with $W_\infty(x) \in \mathcal{L}_k$ ($\iota(x)$ is in no loop and hence never got moved). Then $W_\infty(x) = W_n(x) = \iota(x)$. Hence

$$d(W_\infty(x), W_n(x)) = 0 < \varepsilon,$$

as desired.

In either case, we see that for all $x \in S^1$, $d(W_\infty(x), W_n(x)) < \varepsilon$. Thus the W_n converge to $W_\infty(x)$ uniformly! ■

This allows us to prove the following:

Theorem 2.3. *W_∞ is an embedding of S^1.*

Proof. We must show (a) W_∞ is continuous, (b) W_∞ is a bijection between S^1 and $W_\infty(S^1)$, and (c) W_∞^{-1} is continuous on $W_\infty(S^1)$.

1. WTS W_∞ is continuous. To that end, note that each W_n consists of applying a finite sequence of Reidemeister moves to $\iota(S^1)$. By Reidemeister's theorem, it follows that each of the W_n are continuous. Now, since W_∞ is the limit of a uniformly convergent sequence of continuous functions, it follows that $\boxed{W_\infty \text{ is continuous,}}$ as desired.

2. Now, we show W_∞ is a bijection from S^1 to $W_\infty(S^1)$. Surjectivity follows immediately. Injectivity requires a bit more care.

■

Proof. We show W_∞ is a homemorphism onto its image. To that end, we must show W_∞ is continuous and has a continuous inverse.

1. *Proof that W_∞ is continuous:* Note that by Reidemeister's theorem, each of the W_k are homeomorphisms, and hence continuous. We claim that $W_k \to W_\infty$ uniformly.

2. *Proof that W_∞^{-1} is continuous:*

■

We now prove some results motivating the assumptions we made in our construction.

Proposition 2.4. *Let $A(\ell_n)$ be the area bounded by ℓ_n in the diagram. Then if we were to have*

$$\sum_{n=1}^{\infty} A(\ell_n)$$

diverges, then W_∞ is not ambient isotopic to S^1.

Proof. Sketch: if so, then we'd get that the image is not compact, and so that would be sad ∎

Remark. Suppose the loops were allowed to overlap. While this does not inherently break the proof, it does allow us to create some examples where

Proposition 2.5.

Some remarks:

2.1.4 Counterexample: when the loops go in towards the center

Appendix A

Formal Language Theory

A.1 A grammar for arithmetic

Example A.1. In the below, we will define things in a manner reminiscent of a *formal grammar*. If you haven't seen this material before that's totally fine; the idea is just that we want a rigid way to define valid ways of assembling expressions. To do so, we need an *alphabet* of legal characters we're allowed to assemble into expressions (e.g., $1 + 5$ is comprised of the symbols "1", "+", and "5"), as well as rules for how we're allowed to stick the characters together (e.g., "$1 + \div$" doesn't make sense as an expression, so we want to forbid it). We choose the following:

1. Define our *alphabet* $\Sigma = \mathbb{Z} \cup \{+, -, \cdot, \div, (,)\}$. As a fun fact, note that we think of the two dots in the $\div$ symbol as being placeholders (analogous to the dots in notation $\langle \cdot, \cdot \rangle$), then we see $\div$ really is a one-line fraction notation.

2. We now define what we mean by "valid" expressions (this is the part where we remove things like "$1 + \div$"). Let $\Sigma^\star$ be the set of all strings formed by concatenating elements of Σ together ($\star$ is called the *Kleene Star*). We want to restrict our attention to *well-formed formulas* in $\Sigma^\star$ (denoted wwfs) according to the following recursive rules:

   (a) For each $k \in \mathbb{Z}$, "k" is a wff (e.g., the character "2" is perfectly valid on its own as an expression). We call these expressions *atomic*.

   (b) If "a","b" are wffs, then

      i) "$(-(a))$"

ii) "$(a) + (b)$"
iii) "$(a) - (b)$'
iv) "$(a) \cdot (b)$"
v) "$(a) \div (b)$" (when "(b)" $\not\equiv$ "0" in the sense below)

are all wffs. Note, the use of the parens above will allow us to avoid thinking about PEMDAS.

We denote the set of all wffs by $L(\Sigma)$, an abuse of the standard formal grammar notation.

3. We now define the arithmetic rules by which we can modify wwfs to get equivalent wwfs. Let $E \in L(\Sigma)$, and note that in the following, we will use $=$ to denote literal string equality, and $\equiv$ to denote *arithmetic* equality.

(a) $\equiv$ is an equivalence relation

(b) For any $k \in L(\Sigma)$,

i) "(k)" $\equiv (-(-(k)))$. If k is atomic, then $k \equiv (-(-(k)))$ as well.

ii) For all $a \in \mathbb{Z}$, we say the following are equivalent

$$\begin{aligned} \text{“}k\text{”} &\equiv \text{“}((k) + (a)) - (a)\text{”} \\ &\equiv \text{“}((k) - (a)) + (a)\text{”} \\ &\equiv \text{“}(-(a)) + ((k) + (a))\text{”} \\ &\equiv \text{“}(a) - ((a) - (k))\text{”} \end{aligned}$$

iii) For all $a \in \mathbb{Z} \setminus \{0\}$, we say the following are equivalent

$$\begin{aligned} \text{“}k\text{”} &\equiv \text{“}((k) \cdot (a)) \div (a)\text{”} \\ &\equiv \text{“}((k) \div (a)) \cdot (a)\text{”} \\ &\equiv \text{“}(1 \div (a)) \cdot ((k) \cdot (a))\text{”} \end{aligned}$$

iv) 0 has special properties:

$$\begin{aligned} \text{“}k\text{”} &\equiv \text{“}(k) + (0)\text{”} \\ &\equiv \text{“}(0) + (k)\text{”} \\ &\equiv \text{“}(k) - (0)\text{”} \\ &\equiv \text{“}(0) - (-(k))\text{”} \end{aligned}$$

and

$$\begin{aligned}\text{“}0\text{”} &\equiv \text{“}(k)\cdot(0)\text{”}\\ &\equiv \text{“}(0)\cdot(k)\text{”}\end{aligned}$$

v) 1 is also special:

$$\begin{aligned}\text{“}k\text{”} &\equiv \text{“}(1)\cdot(k)\text{”}\\ &\equiv \text{“}(k)\cdot(1)\text{”}\\ &\equiv \text{“}(k)\div(1)\text{”}\end{aligned}$$

and if “k” $\not\equiv$ “0”, then

$$\equiv ((1)\div((1)\div(k)))$$

is this even relevant and so on until the cows come home.

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