Problems	5.6(4)	5.11	5.15(no normal)	5.17	5.23	Total
Points						

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HW 5 Solutions
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5.6(4). Show that $\mathbb{R}^2$ with the standard topology is normal.

Solution. First, we introduce some notation.

> **Notational Note:** Let $(X, \mathcal{T})$ be a topological space. Let $x \in X$, and let $Y \subset X$. Then define
> $$d(x, Y) = \inf_{y \in Y} d(x, y)$$
> now the main proof.

Main Proof: Let A, B be disjoint closed subsets of $\mathbb{R}^2$. For each $a \in A$, $b \in B$, let

$$\varepsilon_a = \frac{d(a, B)}{3} \qquad \varepsilon_b = \frac{d(b, A)}{3}$$

and note that by part (1), $\varepsilon_a, \varepsilon_b > 0$. Define

$$U = \bigcup_{a \in A} B_{\varepsilon_a}(a) \qquad V = \bigcup_{b \in B} B_{\varepsilon_b}(b)$$

and observe $U, V \in \mathcal{T}_{\text{std}}$, with $A \subset U$ and $B \subset V$.

Suppose, to obtain a contradiction, that $U \cap V \neq \varnothing$. Let $x \in U \cap V$. Then there exist $a \in A$, $b \in B$ such that $x \in B_{\varepsilon_a}(a) \cap B_{\varepsilon_b}(b)$. It follows that

$$\begin{aligned} d(a, b) &\leq d(a, x) + d(x, b) \\ &\leq \varepsilon_a + \varepsilon_b \end{aligned}$$

WLOG, suppose $\varepsilon_b \leq \varepsilon_a$. Then

$$\begin{aligned} d(a, b) &\leq 2\varepsilon_a \\ &= \frac{2}{3} d(a, B) \end{aligned}$$

a contradiction. Hence $U \cap V = \varnothing$, so $\mathbb{R}^2$ is normal, as desired.

> **Clarifying Note:** Why is this a contradiction? Because
> $$d(a, B) = \inf_{b \in B} d(a, b)$$
> hence for all $b' \in B$, $d(a, B) \leq d(a, b')$. Thus our result would imply
> $$d(a, b) \leq \frac{2}{3} d(a, b),$$
> which holds iff $d(a, b) = 0$, a contradiction.

$\blacksquare$

5.11 (The Incredible Shrinking Theorem). A topological space X is normal if and only if for each pair of open sets U, V such that $U \cup V = X$, there exist open sets U', V' such that $\overline{U'} \subset U$ and $\overline{V'} \subset V$, and $U' \cup V' = X$.

Solution. ■

5.15. A space $(X, \mathcal{T})$ is T_1 if and only if every point in X is a closed set.

Solution. Order topologies are T_1, Hausdorff, and regular. ■

5.17. Let X and Y be regular. Then $X \times Y$ is regular.

Solution. ■

5.23. Let A be a closed subset of a normal space X. Then A is normal when given the relative topology.

Solution. ■