Problem	5.29	5.32	6.6	6.11	6.18	Total
Points						

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Math 147
HW 6 Solutions
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5.29 (The Normality Lemma). Let A and B be subsets of a topological space X and let $\{U_i\}_{i\in\mathbb{N}}$ and $\{V_i\}_{i\in\mathbb{N}}$ be two collections of open sets such that
(1) $A \subset \bigcup_{i\in\mathbb{N}} U_i$
(2) $B \subset \bigcup_{i\in\mathbb{N}} V_i$
(3) For each $i \in \mathbb{N}$, $\overline{U_i} \cap B = \varnothing$ and $\overline{V_i} \cap A = \varnothing$.
Then there exist open sets U and V such that $A \subset U$, $B \subset V$, and $U \cap V = \varnothing$.

Before a solution, I'll give some intuition on how we might arrive at the candidate U, V that work.

Let's think about what we're given. We have $\{U_i\}_{i\in\mathbb{N}}$, and $\{V_i\}_{i\in\mathbb{N}}$ as defined above, and we want to use them to construct U, V satisfying the given constraints. It seems like it'd be straightforward to satisfy $A \subset U$, $B \subset V$ — they look like they'll probably fall directly out of the conditions. $U \cap V = \varnothing$ is harder, since we're given no direct information about $U_i \cap V_j$. Hence, we'll pick the following as our general approach:

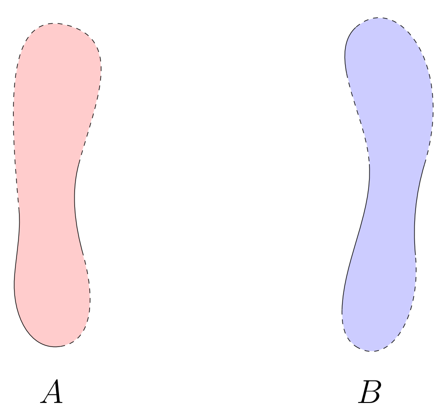
(1) Think about what

$$\tilde{U} = \bigcup_{i\in\mathbb{N}} U_i \qquad \tilde{V} = \bigcup_{i\in\mathbb{N}} V_i$$

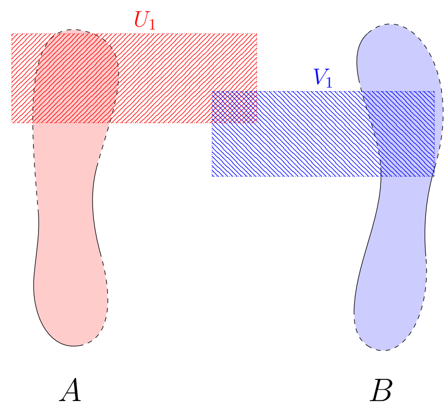
look like. In particular, we'll focus on the conditions that break/make $\tilde{U}$ and $\tilde{V}$ *not* work as choices of U, V. Then,

(2) We'll see if we can find a clever way to remove the parts of U_i and V_i that cause problems. If all goes right, we'll find sequences $\{U_i'\}_{i\in\mathbb{N}}$, $\{V_i'\}_{i\in\mathbb{N}}$ whose terms can be unioned to get U, V.

Depict our two sets A, B as follows:



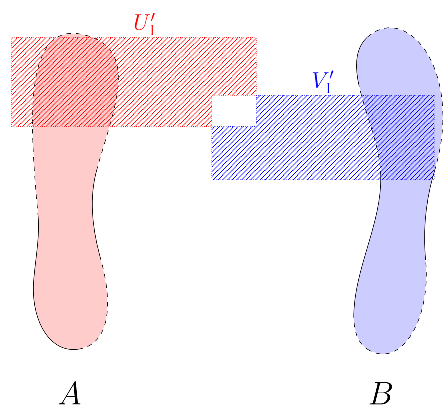
To make the TikZ easier, I'll draw our covers with boxes — but note, in general they could be blobby. Anyways, we now draw U_1, V_1:



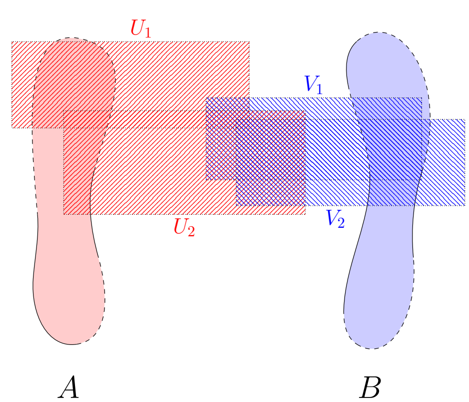
We need to be sure that in our construction of U, we don't include any of the V_i. We want to use the U_i, but as we can see, they might intersect the V_i. The fix? Remove the parts that cause problems. We define

$$U_1' = U_1 - \overline{V_1} \qquad\qquad V_1' = V_1 - \overline{U_1}$$

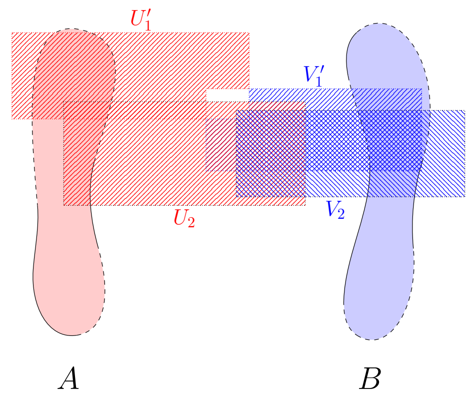
which yields



Now, we consider $n = 2$:



We already know how to rectify U_1, V_1:

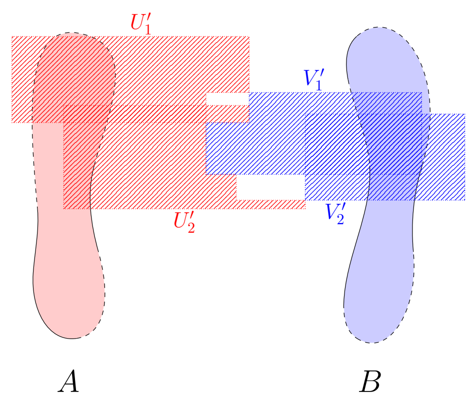


but we still have a problem now. Our addition of U_2, V_2 is complicating the situation! Namely, we have $U_2 \cap V_1' \neq \varnothing$, $U_2 \cap V_2 \neq \varnothing$.[a] We *don't* want to go back and edit our definitions for U_1', V_1' — if we were to take this approach, we'd need to proceed similarly for $n = 2$, $n = 3$, etc. until eventually $U_1' = U_1 - \bigcup_{i \in \mathbb{N}} \overline{V_i}$, which could cause some problems.

Instead, we'll modify U_2 and V_2, leaving U_1' and V_1' untouched. Hence, we let

$$U_2' = U_2 - (\overline{V_1} \cup \overline{V_2}) \qquad V_2' = V_2 - (\overline{U_1} \cup \overline{U_2})$$

which gives us



From which we can see $(U_1' \cup U_2') \cap (V_1' \cup V_2') \neq \varnothing$. Thus, we conjecture that in the general case,

$$U_n' = U_n - \bigcup_{i=1}^{n} \overline{V_i} \qquad V_n' = V_n - \bigcup_{i=1}^{n} \overline{U_i}$$

will work.

[a] Note we could also have the analogous for $V_2 \cap U_1'$ if V_2 were a bit larger.

Solution.

■

5.32. Suppose a space X is regular and has a countable basis. Then X is normal.

Solution. Let A, B be disjoint closed sets. ■

6.6. The space $2^{\mathbb{R}}$ is separable.

Solution. We first find a candidate countable dense subset. As a candidate, we consider the set of all finite unions of intervals with rational endpoints:

> Let
> $$S = \left\{ \bigcup_{i=1}^{n} (p_i, q_i) \;\middle|\; n \in \mathbb{N},\ \text{and } p_i, q_i \in \mathbb{Q} \text{ for each } i \right\}.$$
> First, note that the set comprehension above is at most countable (the set of all finite subsets of a countable set is countable). Hence, S is at most countable as well.
>
> Now, note that S has a countably infinite subset:
> $$S' = \{(0, q) \mid q \in \mathbb{Q}\} \subset S.$$
> Thus S is at least countable as well, and so $|S| = |\mathbb{N}|$. ✓

We want to show $\overline{S} = 2^{\mathbb{R}}$. First, we recall the definition of the topology on $2^{\mathbb{R}}$:

> Consider $\{0,1\}^{\mathbb{R}}$, where $\{0,1\}$ has the discrete topology. By definition of the product topology, basic open sets in $\{0,1\}^{\mathbb{R}}$ are of the form
> $$B = \prod_{\alpha \in \mathbb{R}} X_\alpha$$
> where only finitely many of the $X_\alpha \neq \{0,1\}$.

A "brief" aside about infinite cartesian products (encouraged if you're confused):

> **Note.** What does this mean? Since our indexing set is $\mathbb{R}$, we can't interpret this cartesian product in terms of infinite lists like
> $$B = \left\{ (x_1, x_2, \ldots) \;\middle|\; x_\alpha \in X_\alpha \text{ for all } \alpha \in \mathbb{R} \right\}$$
> because the real numbers aren't countable. Hence, we need a more general notion of associating an index to an element of a collection. The solution is to define a choice function. In looking at our naïve list above, we might realize that a solution is to think of an infinite "list" as a function
> $$f(i) = x_i.$$
> that is, given a number i, f gives us something that we'll call the i^{th} element of the list.
>
> This makes more sense if we look at some finite/countable examples. Let's say I have the list $(1, 4, 2)$. 1 is the first element, 4, is the second element, and 2 is the third element. Then I can think of this list in terms of the following function:
> $$f(1) = 1$$
> $$f(2) = 4$$
> $$f(3) = 2$$
> Observe f associates 1 to the first element, 2 to the second element, and 3 to the third element. In this way, it encodes the exact same information as $(1, 4, 2)$.
>
> For an infinite example, consider some sequence $\{x_i\}_{i \in \mathbb{N}}$. Then we can think of $\{x_i\}_{i \in \mathbb{N}}$ as

a function $f : \mathbb{N} \to \{\text{all of the } x_i\text{'s}\}$ defined by

$$\begin{aligned} f(1) &= x_1 \\ f(2) &= x_2 \\ f(3) &= x_3 \\ &\vdots \\ f(i) &= x_i \\ &\vdots \end{aligned}$$

For countable lists, this perspective is honestly much more trouble than it's worth. But what it *does* do is play nicely with uncountable collections! Let's take an arbitrary element of B, call it x. Then although we *can't* express x by a list

$$x = (x_\alpha)_{\alpha \in \mathbb{R}} = (x_{\alpha_1}, x_{\alpha_2}, \ldots)$$

since (again) the real numbers can't be written out sequentially, if we can define a function

$$f(\alpha) = x_\alpha,$$

then this will do the job for us.

So how do we know we *can* define such a function? Well, we don't. As it turns out, the existence of such an f for infinite indexing sets is provably *independent* from Zermelo–Fraenkel set theory. Note, "independent" does not mean "unreasonable" or "inconsistent" — rather, it means that we can neither prove nor refute the existence of f using the standard axioms of set theory!

For this reason, if we want to talk about f, our only option is to assert its existence axiomatically. This is the famous *axiom of choice*:

Axiom 1 (Choice)**.** Let $\{X_\alpha\}_{\alpha \in \lambda}$ be a set of non-empty sets indexed by λ. Then there is a function

$$f : \lambda \to \bigcup_{\alpha \in \lambda} X_\alpha$$

such that for each $\alpha \in \lambda$, $f(\alpha)$ is an element of X_α. □

We can think of f as defining the elements of a "list" where the α^{th} element is chosen from X_α. In this sense, we can formulate infinite cartesian products as the set of all such f:

$$\prod_{\alpha \in \lambda} X_\alpha = \left\{ f : \lambda \to \bigcup_{\alpha \in \lambda} X_\alpha \;\middle|\; f(\alpha) \in X_\alpha \text{ for all } \alpha \in \lambda \right\}.$$

In this light, we can think of $\{0,1\}^{\mathbb{R}}$ as

$$\{0,1\}^{\mathbb{R}} = \prod_{\alpha \in \mathbb{R}} \{0,1\}$$

and we consider elements of $\{0,1\}^{\mathbb{R}}$ as functions

$$f : \mathbb{R} \to \{0,1\}.$$

That is, each $x \in \mathbb{R}$ is associated either to 0 or 1. In terms of the product topology, we see open sets in $\{0,1\}^{\mathbb{R}}$

■

6.11. Every uncountable set in a 2^{nd} countable space has a limit point.

Solution. Let $(X, \mathscr{T})$ be a 2^{nd} countable space with countable basis $\mathscr{B}$, and let $A \subset X$ be uncountable.

Let $a \in A$. Suppose, to obtain a contradiction, that A has no limit points. Then a is an isolated point of A, and by Theorem 3.10, there exists $U_a \in \mathscr{T}$ s.t. $U_a \cap A = \{a\}$. Then by definition of a basis, there exists $B_a \in \mathscr{B}$ such that

$$a \in B_a \subset U_a,$$

and hence

$$B_a \cap A = \{a\}$$

as well. It follows that $\mathscr{B}' = \{B_a\}_{a \in A}$ is an uncountable subset of $\mathscr{B}$, a contradiction.

Thus, A has a limit point. ■

6.18. Suppose x is a limit point of the set A in a 1^{st} countable space X. Then there is a sequence of points $\{a_i\}_{i\in\mathbb{N}}$ that converges to x.

Solution. Since X is 1^{st} countable, Theorem 6.15 implies that there exists a nested countable neighborhood basis

$$\mathcal{B}_x = \{B_i\}_{i\in\mathbb{N}}$$

for x. Now, define $\{x_i\}_{i\in\mathbb{N}}$ by $x_i \in B_i$ for each i.[1]

Claim: $x_i \to x$.

Proof of Claim: Let $U \in \mathcal{T}$ such that $x \in U$. Then by definition of a countable neighborhood basis, there exists $N \in \mathbb{N}$ such that

$$x \in B_N \subset U.$$

Since $\mathcal{B}_x$ is nested, for all $n > N$ we have

$$B_n \subset B_N \subset U$$

and since $x_n \in B_n$, this implies $x_n \in U$. Then by definition, $x_n \to x$. ■

[1] In the interests of being exhaustively rigorous here, note that each B_i is nonempty (because $x \in B_i$), hence this declaration is valid.