Problem	5.29	5.32	6.6	6.11	6.18	Total
Points						

Forest Kobayashi
Math 147
HW 6 Solutions
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5.29 (The Normality Lemma). Let A and B be subsets of a topological space X and let $\{U_i\}_{i\in\mathbb{N}}$ and $\{V_i\}_{i\in\mathbb{N}}$ be two collections of open sets such that
(1) $A \subset \bigcup_{i\in\mathbb{N}} U_i$
(2) $B \subset \bigcup_{i\in\mathbb{N}} V_i$
(3) For each $i \in \mathbb{N}$, $\overline{U_i} \cap B = \varnothing$ and $\overline{V_i} \cap A = \varnothing$.
Then there exist open sets U and V such that $A \subset U$, $B \subset V$, and $U \cap V = \varnothing$.

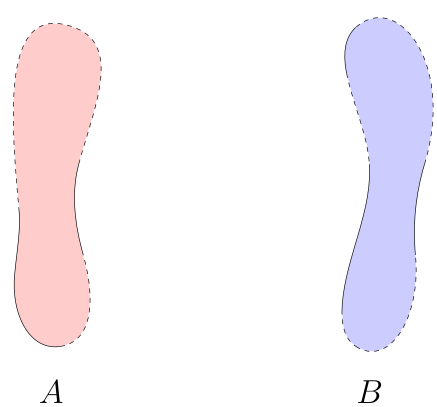
Before a solution, I'll give some intuition on how we might arrive at the candidate U, V that work.

Let's think about what we're given. We have $\{U_i\}_{i\in\mathbb{N}}$, and $\{V_i\}_{i\in\mathbb{N}}$ as defined above, and we want to use them to construct U, V satisfying the given constraints. It seems like it'd be straightforward to satisfy $A \subset U$, $B \subset V$ — they look like they'll probably fall directly out of the conditions. $U \cap V = \varnothing$ is harder, since we're given no direct information about $U_i \cap V_j$. Hence, we'll pick the following as our general approach:

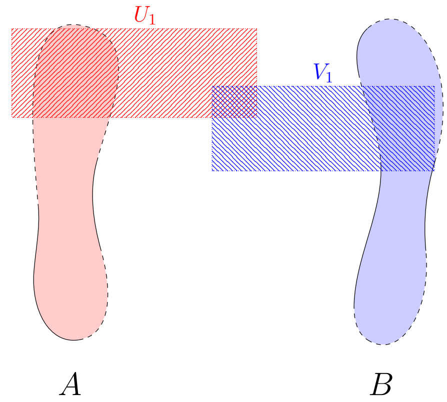
(1) Think about what
$$\tilde{U} = \bigcup_{i\in\mathbb{N}} U_i \qquad \tilde{V} = \bigcup_{i\in\mathbb{N}} V_i$$
look like. In particular, we'll focus on the conditions that break/make $\tilde{U}$ and $\tilde{V}$ *not* work as choices of U, V. Then,

(2) We'll see if we can find a clever way to remove the parts of U_i and V_i that cause problems. If all goes right, we'll find sequences $\{U_i'\}_{i\in\mathbb{N}}$, $\{V_i'\}_{i\in\mathbb{N}}$ whose terms can be unioned to get U, V.

Depict our two sets A, B as follows:



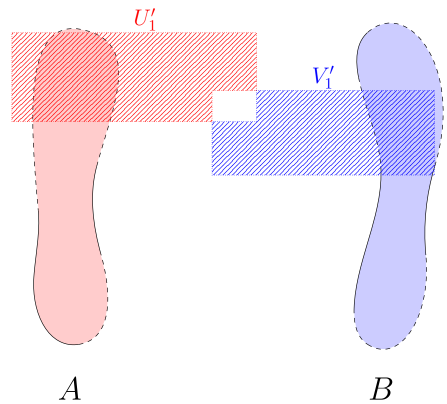
To make the TikZ easier, I'll draw our covers with boxes — but note, in general they could be blobby. Anyways, we now draw U_1, V_1:



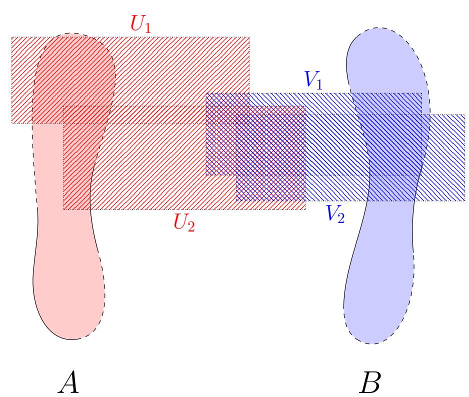
We need to be sure that in our construction of U, we don't include any of the V_i. We want to use the U_i, but as we can see, they might intersect the V_i. The fix? Remove the parts that cause problems. We define

$$U_1' = U_1 - \overline{V_1} \qquad\qquad V_1' = V_1 - \overline{U_1}$$

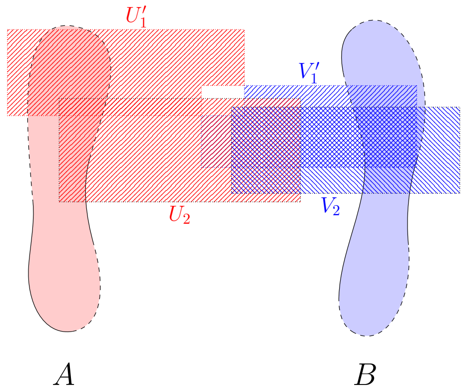
which yields



Now, we consider $n = 2$:



We already know how to rectify U_1, V_1:

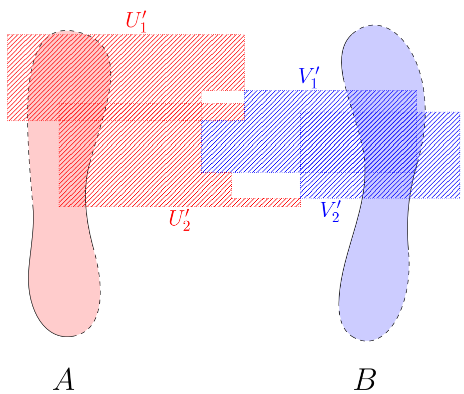


but we still have a problem now. Our addition of U_2, V_2 is complicating the situation! Namely, we have $U_2 \cap V_1' \neq \varnothing$, $U_2 \cap V_2 \neq \varnothing$.[a] We *don't* want to go back and edit our definitions for U_1', V_1' — if we were to take this approach, we'd need to proceed similarly for $n = 2$, $n = 3$, etc. until eventually $U_1' = U_1 - \bigcup_{i\in\mathbb{N}} \overline{V_i}$, which could cause some problems.

Instead, we'll modify U_2 and V_2, leaving U_1' and V_1' untouched. Hence, we let

$$U_2' = U_2 - (\overline{V_1} \cup \overline{V_2}) \qquad V_2' = V_2 - (\overline{U_1} \cup \overline{U_2})$$

which gives us



From which we can see $(U_1' \cup U_2') \cap (V_1' \cup V_2') \neq \varnothing$. Thus, we conjecture that in the general case,

$$U_n' = U_n - \bigcup_{i=1}^{n} \overline{V_i} \qquad V_n' = V_n - \bigcup_{i=1}^{n} \overline{U_i}$$

will work.

[a]Note we could also have the analogous for $V_2 \cap U_1'$ if V_2 were a bit larger.

Solution.

■

5.32. Suppose a space X is regular and has a countable basis. Then X is normal.

Solution. Let A, B be disjoint closed sets. ■

6.6. The space $2^{\mathbb{R}}$ is separable.

Solution.

Let

$$S = \left\{ \bigcup_{i=1}^{n} (p_i, q_i) \;\middle|\; n \in \mathbb{N},\ \text{and } p_i, q_i \in \mathbb{Q} \text{ for each } i \right\}.$$

First, note that the set comprehension above yields at most countably many elements (the set of all finite subsets of a countable set is countable). Hence, S is at most countable.

Now, note that S has a countably infinite subset:

$$S' = \{(0, q) \mid q \in \mathbb{Q}\} \subset S.$$

Thus S is at least countably infinite as well. So $|S| = |\mathbb{N}|$. ✓

Now, let

$$\mathcal{F} = \left\{ f : \mathbb{R} \to \{0, 1\} \;\middle|\; f^{\leftarrow}(\{1\}) \in S \right\}.$$

where $f^{\leftarrow}(\{1\})$ denotes the preimage of $\{1\}$. Also let

Let $U \in \mathcal{T}_{2^{\mathbb{R}}}$ be arbitrary. Then by definition of the product topology,

$$U = \prod_{\xi \in \mathbb{R}} X_\xi$$

where X_ξ is open in $\{0, 1\}$ under the discrete topology, and $X_\xi = \{0, 1\}$ for all but finitely many ξ. Let $\{X_{\xi_i}\}_{i=1}^{n}$ be this finite collection of X_ξ, and for each i, let $Y_i =$ ■

6.11. Every uncountable set in a 2^{nd} countable space has a limit point.

Solution. Let $(X, \mathcal{T})$ be a 2^{nd} countable space with countable basis $\mathcal{B}$, and let $A \subset X$ be uncountable.

Let $a \in A$. Suppose, to obtain a contradiction, that A has no limit points. Then a is an isolated point of A, and by Theorem 3.10, there exists $U_a \in \mathcal{T}$ s.t. $U_a \cap A = \{a\}$. Then by definition of a basis, there exists $B_a \in \mathcal{B}$ such that

$$a \in B_a \subset U_a,$$

and hence

$$B_a \cap A = \{a\}$$

as well. It follows that $\mathcal{B}' = \{B_a\}_{a \in A}$ is an uncountable subset of $\mathcal{B}$, a contradiction.

Thus, A has a limit point. ■

6.18. Suppose x is a limit point of the set A in a 1^{st} countable space X. Then there is a sequence of points $\{a_i\}_{i \in \mathbb{N}}$ that converges to x.

Solution. ■