Problem	5.6(4)	5.11	5.15(no normal)	5.17	5.23	Total
Points						

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Math 147
HW 5 Solutions
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5.6(4). Show that $\mathbb{R}^2$ with the standard topology is normal.

Solution. First, we introduce some notation.

> **Notational Note:** Let $(X, \mathcal{T})$ be a topological space. Let $x \in X$, and let $Y \subset X$. Then define
> $$d(x, Y) = \inf_{y \in Y} d(x, y).$$

Main Proof: Let A, B be disjoint closed subsets of $\mathbb{R}^2$. For each $a \in A$, $b \in B$, let

$$\epsilon_a = \frac{d(a, B)}{2} \qquad \epsilon_b = \frac{d(b, A)}{2}$$

and note that by part (1), $\epsilon_a, \epsilon_b > 0$. Define

$$U = \bigcup_{a \in A} B_{\epsilon_a}(a) \qquad V = \bigcup_{b \in B} B_{\epsilon_b}(b)$$

and observe $U, V \in \mathcal{T}_{\text{std}}$, with $A \subset U$ and $B \subset V$. We want to show $U \cap V = \varnothing$.

Suppose, to obtain a contradiction, that $U \cap V \neq \varnothing$. Let $x \in U \cap V$. Then there exist $a \in A$, $b \in B$ such that $x \in B_{\epsilon_a}(a) \cap B_{\epsilon_b}(b)$. It follows that

$$\begin{aligned} d(a, b) &\leq d(a, x) + d(x, b) \qquad (\star) \\ &< \epsilon_a + \epsilon_b \end{aligned}$$

WLOG, suppose $\epsilon_b \leq \epsilon_a$. Then

$$\begin{aligned} d(a, b) &< 2\epsilon_a \\ &= d(a, B) \\ &\leq d(a, b) \end{aligned}$$

so $d(a, b) < d(a, b)$, a contradiction.[1] Hence, $U \cap V = \varnothing$, so U, V are disjoint open sets containing A and B respectively. Since A, B were arbitrarily chosen, it follows that $\mathbb{R}^2$ is normal, as desired.

> **Clarifying Note:** How did we get the final inequality? Well, note that
> $$d(a, B) = \inf_{b \in B} d(a, b)$$
> hence for all $b' \in B$, $d(a, B) \leq d(a, b')$. Since $d(a, B) > 0$, it follows that for all $b' \in B$,
> $$d(a, B) \leq d(a, b').$$

■

[1] The align environment given should be read as "$d(a, b) < 2\epsilon_a = d(a, B) \leq d(a, b)$," not as $d(a, b) < 2\epsilon_a$, $d(a, b) = d(a, B)$, and so on. Also, here our contradiction is $d(a, b) < d(a, b)$, but we could also just skip $(\star)$ and directly contradict the triangle inequality by $d(a, x) + d(x, b) < d(a, b)$. I prefer the former, just because it better matches arguments seen in Analysis.

5.11 (The Incredible Shrinking Theorem). A topological space X is normal if and only if for each pair of open sets U, V such that $U \cup V = X$, there exist open sets U', V' such that $\overline{U'} \subset U$ and $\overline{V'} \subset V$, and $U' \cup V' = X$.

Solution. First, we prove a lemma.[2] The ($\implies$) direction will follow as a corollary.

Lemma 1. *Let $(X, \mathcal{T})$ be normal. Let $U, V \in \mathcal{T}$ such that $U \cup V = X$. Then there exists $U' \in \mathcal{T}$ such that $\overline{U'} \subset U$, and $U' \cup V = X$.*

I'll provide two proofs. The first uses theorem 5.9 (and is hence *much* cleaner), while the second uses the definition of normality (and is hence much longer / more involed). I included both, so that people who tried to use normality could see how to proceed.

Proof 1: Note that V^c is closed, and $V^c \subset U$. Then by theorem 5.9, there exists $U' \in \mathcal{T}$ such that

$$V^c \subset U' \subset \overline{U'} \subset U.$$

Note that $V^c \subset U' \implies U' \cup V = X$. Hence, we have our desired U'. □

Proof 2: $U, V \in \mathcal{T}$ implies U^c and V^c are closed. Observe that

$$\begin{aligned} U^c \cap V^c &= (U \cup V)^c \\ &= \varnothing, \end{aligned}$$

hence U^c, V^c are disjoint closed sets. Then by definition of normality, there exist disjoint open sets U', V' such that

$$U^c \subset V' \qquad \text{and} \qquad V^c \subset U'.$$

Note that $V^c \subset U' \implies U' \cup V = X$. It remains to show $\overline{U'} \subset U$. Since $U' \cap V' = \varnothing$ and $U^c \subset V'$, we have

$$U' \subset V'^c \subset U$$

and because V'^c is closed, $\overline{U'} \subset V'^c$ as well. This proves the claim. □

Now, the main proof.

($\Rightarrow$) : Suppose X is normal.

Let $U, V \in \mathcal{T}$ such that $U \cup V = X$. Then by the lemma, there exists $U' \in \mathcal{T}$ such that $\overline{U'} \subset U$, and $U' \cup V = X$. Now, applying the lemma to the pair (V, U'), we obtain the desired V'. ✓

($\Leftarrow$) : Suppose that $\forall U, V \in \mathcal{T}$ s.t. $U \cup V = X$, there exists U', V' s.t. $\overline{U'} \subset U$, $\overline{V'} \subset V$, and $U' \cup V' = X$. WTS X is normal. We will apply Theorem 5.9.

Let $A \subset X$ be an arbitrary closed set, and let $U \in \mathcal{T}$ such that $A \subset U$.[3] Observe that A^c is open, and $U^c \subset A^c$. It follows that $X = U \cup A^c$. Then by hypothesis, there exists $U', V' \in \mathcal{T}$ such that

$$\overline{U'} \subset U \qquad\qquad \overline{V'} \subset A^c$$

and $U' \cup V' = X$. Observe that this last condition implies $(U')^c \subset V'$, hence we have

$$(U')^c \subset V' \subset \overline{V'} \subset A^c$$

Taking the complement and employing $U' \subset U$, we have

$$A \subset (\overline{V'})^c \subset (V')^c \subset U' \subset U.$$

[2] I'm just proving it as a lemma so that I can offer two proofs. In an actual writeup, I'd just use one of them.
[3] At least one such U exists, namely X, hence we can freely declare U in this manner.

Finally, since $\overline{U'} \subset U$, we have

$$A \subset U' \subset \overline{U'} \subset U$$

as desired. Since A and U were arbitrarily chosen, Theorem 5.9 implies X is normal.

$\blacksquare$

5.15. Order topologies are T_1, Hausdorff, and regular.

Solution. Let X be a totally ordered set, and $\mathscr{T}$ be the associated order topology. Denote the elements of the canonical basis as follows:

- $(-\infty, a) = \{x \in X \mid x < a\}$
- $(a, \infty) = \{x \in X \mid a < x\}$
- $(a, b) = \{x \in X \mid a < x < b\}$.

Square brackets will indicate inclusivity, as usual.

Note. Although the notation here is almost identical to that of the standard topology on $\mathbb{R}$, we need not have $X = \mathbb{R}$. In fact, X is guaranteed to have *no algebraic structure* whatsoever. Be sure to keep this in mind as we proceed!

(1) We apply Theorem 5.1. Let $x \in X$ be arbitrary. Then $(-\infty, x) \cup (x, \infty)$ is open. By complement, $\{x\}$ is closed, hence $(X, \mathscr{T})$ is T_1.

(2) WLOG, suppose $x < y$. We proceed by casework.

   (i) Suppose there exists $z \in X$ such that $x < z < y$. Then $U = (-\infty, x)$, $V = (z, \infty)$ are disjoint open sets with $x \in U$, $y \in V$.

   (ii) Suppose no such z exists. Then $U = (-\infty, y)$, $V = (x, \infty)$ are disjoint open sets with $x \in U$, $y \in V$.

   hence $(X, \mathscr{T})$ is Hausdorff.

   Remark. By Theorem 5.7.2, we actually just need to show regularity now that we have T_1. But in case you'd like to show Hausdorff constructively for extra practice, this is one way you might do it.

   Also, here's a graphic of tenuous worth to "help" illustrate subcase (ii):

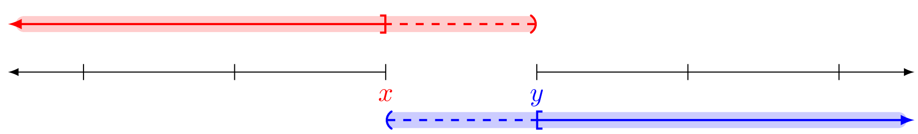

Figure 1: Subcase (ii)

   Note the gap between x and y. U is the top interval, V is the bottom one.

(3) To show regularity, we will employ Theorem 5.8. But first, a small Lemma.

> **Lemma 2.** *Let $(a, b) \subset X$. Then $\overline{(a, b)} \subset [a, b]$.*
>
> *Proof:* Note that $X - \underline{[a, b]} = (-\infty, a) \cup (b, \infty)$ is open, hence $[a, b]$ is closed. By Theorem 3.20, we have $\overline{(a, b)} \subset [a, b]$.

   Remark. We actually can't do better than this in the general case. For example, you can find subspaces of the lexicographically ordered square that refuse to play nice. Also, if X is a discrete set (such as $\mathbb{N}$ or $\mathbb{Z}$), plenty of counterexamples exist.

   Let $x \in X$ be arbitrary. Let $U \in \mathscr{T}$ such that $x \in U$. Then there exist $a, b \in X \cup \{-\infty, \infty\}$ such that

$$x \in (a, b) \subset U.^{4}$$

[4] Note, this is just a concise way of declaring a basic open set.

Claim: There exists $(a', b') \subset (a, b)$ such that $x \in (a', b') \subset \overline{(a', b')} \subset (a, b)$.

Proof of Claim: When typing this up, I found a slightly cleaner version of the argument I was using at `http://web.math.ku.dk/~moller/e02/3gt/opg/S31.pdf`, and have modified my proof accordingly.

Let $A = (a, x)$, and $B = (x, b)$. Then we have four subcases.

i) Suppose that $A, B = \varnothing$. Then $(a, b) = \{x\}$, which is clopen. Hence take $(a', b') = (a, b)$, and the claim holds. ✓

ii) Suppose $A = \varnothing$ and $B \neq \varnothing$, and let $b' \in B$. Then let $a' = a$, and note $(a', b') = [x, b')$. Hence $x \in (a', b')$, and

$$\overline{(a', b')} = \overline{[x, b')} \subset [x, b'] \subset (a, b)$$

so the claim holds. ✓

iii) Supposet $A \neq \varnothing$ and $B = \varnothing$. The proof is analogous to the above. ✓

iv) Suppose $A \neq \varnothing \neq B$. Then let $a' \in A$, $b' \in B$. It follows that

$$x \in (a', b') \subset \overline{(a', b')} \subset [a', b'] \subset (a, b)$$

as desired. ✓

since these cases are exhaustive, we see X is regular, as desired.

■

5.17. Let X and Y be regular. Then $X \times Y$ is regular.

Solution. We prove a lemma.

Lemma 3. *Let $A \subset X$, $B \subset Y$. Then $\overline{A \times B} = \overline{A} \times \overline{B}$.*

We use the notation $\mathcal{L}(S)$ to denote the limit points of a set S. Here're a few proofs:

Proof 1: The claim is equivalent to

$$p \in \overline{A \times B} \iff p \in \overline{A} \times \overline{B}.$$

We proceed by contrapositive. That is,

$$p \notin \overline{A \times B} \iff p \notin \overline{A} \times \overline{B}.^{a}$$

We prove both directions simultaneously.[b] The following are equivalent:

(1) $p = (p_x, p_y) \notin \overline{A \times B}$
(2) There exists $U \in \mathcal{T}_{\text{prod}}$ s.t. $p \in U$ and $(U - \{p\}) \cap (A \times B) = \varnothing$
(3) For U quantified as above, there exists $B = U_A \times U_B \in \mathcal{B}_{\text{prod}}$ such that $p \in B$, and

$$\begin{aligned} \varnothing &= B \cap (A \times B) \\ &= (U_A \times U_B) \cap (A \times B) \\ &= (U_A \cap A) \times (U_B \cap B). \end{aligned}$$

(4) There exists $U_A \in \mathcal{T}_A$, $U_B \in \mathcal{T}_B$ such that at least one of $(U_A \cap A)$, $(U_B \cap B)$ is empty
(5) At least one of $p_x \notin \overline{A}$, $p_y \notin \overline{B}$ is true
(6) $p \notin \overline{A} \times \overline{B}$

□

[a] Note, this is just saying that the set complements are equal.
[b] This introduces a mess with variable quantifications, but hopefully the argument makes sense

Proof 2: We prove the claim directly.

($\subseteq$) : Let $p = (p_A, p_B) \in \overline{A \times B}$. We want to show $p_A \in \overline{A}$ or $p_B \in \overline{B}$.[a]

(1) Suppose $p \in A \times B$. Then we're done. ✓
(2) Now, suppose $p \in \mathcal{L}(A \times B)$. Let $U_A \in \mathcal{T}_A$ and $U_B \in \mathcal{T}_B$ be arbitrarily chosen. Then $U = U_A \times U_B \in \mathcal{T}_{\text{prod}}$, and hence

$$(U - \{p\}) \cap (A \times B) \neq \varnothing$$

Thus, let $q = (q_A, q_B) \in (U - \{p\}) \cap (A \times B)$. It follows that

$$\pi_x(q) \in (\pi_x(U - \{p\}) \cap \pi_x(A \times B))$$

or equivalently,

$$q_A \in (U_A - \{p_A\}) \cap A$$

and similarly, $q_B \in (U_B - \{p_B\}) \cap B$. Hence, $\overline{A \times B} \subset \overline{A} \times \overline{B}$. ✓

($\supseteq$) : The reverse direction essentially consists of reversing the steps above. Note, you need to consider both of the cases $p \in \mathcal{L}(A) \times B$ and $p \in A \times \mathcal{L}(B)$.

□

[a] The or here is inclusive.

> *Proof 3:* We prove the two directions separately.
>
> $(\subseteq)$: The product of closed sets is closed.[a] Since $A \times B \subset \overline{A} \times \overline{B}$, Theorem 3.20 implies
>
> $$\overline{A \times B} \subset \overline{\overline{A} \times \overline{B}} = \overline{A} \times \overline{B}.$$
>
> $(\supseteq)$: Use either of the arguments above. □
>
> [a] As justification, note that $(\overline{A})^c \times (\overline{B})^c$ is a product of open sets, and is thus open in $\mathcal{T}_{\text{prod}}$, hence $\overline{A} \times \overline{B}$ is closed.

We proceed by Theorem 5.8.

Let $p \in X \times Y$ be arbitrary, and let $U \in \mathcal{T}_{\text{prod}}$ such that $p \in U$. Then by definition of $\mathcal{T}_{\text{prod}}$, there exist $U_X \in \mathcal{T}_X$, $U_Y \in \mathcal{T}_Y$ such that

$$p \in U_X \times U_Y \subset U.$$

Since X, Y are regular, there exist $V_X \in \mathcal{T}_X$, $V_Y \in \mathcal{T}_Y$ such that

$$\pi_x(p) \in V_X \subset \overline{V_X} \subset B_X \qquad \text{and} \qquad \pi_y(p) \in V_Y \subset \overline{V_Y} \subset B_Y.$$

Thus $p \in V_X \times V_Y$, which is open in $\mathcal{T}_{\text{prod}}$. Then

$$p \in V_X \times V_Y \subset \overline{V_X \times V_Y} = \overline{V_X} \times \overline{V_Y} \subset B_X \times B_Y \subset U$$

so by Theorem 5.8, $X \times Y$ is regular. ■

5.23. Let A be a closed subset of a normal space X. Then A is normal when given the relative topology.

Solution. Let $\mathcal{T}_X$ be the topology on X, and $\mathcal{C}_X$ be the set of closed sets in $(X, \mathcal{T})$. Similarly, let $\mathcal{T}_A, \mathcal{C}_A$.

Let $B, C \in \mathcal{C}_A$ be disjoint. Then by Theorem 4.28, there exist $B', C' \in \mathcal{C}_X$ such that

$$B = B' \cap A \qquad\qquad C = C' \cap A.$$

Then since $\mathcal{C}_X$ is closed under arbitrary intersection, it follows that B, C are closed in $(X, \mathcal{T})$.[5] Hence by normality, there exist disjoint $U, V \in \mathcal{T}_X$ such that $B \subset U$, and $C \subset V$. Note that

$$B = (B \cap A) \subset U \cap A \qquad\qquad C = (C \cap A) \subset V \cap A,$$

and $U \cap A$, $V \cap A$ are open sets in $(A, \mathcal{T}_A)$. Since they are also disjoint, we see B, C are separated by disjoint open sets in $(A, \mathcal{T}_A)$.

Finally, since B, C were arbitrarily chosen, it follows that A is normal with the relative topology.
■

[5] OK, I know I defined $\mathcal{C}_X$ above, but I was worried that all the script C's flying around were getting confusing!