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Topology
HW 1 Comments
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1 Important Notes:

Before we get to the main stuff, a few meta-notes about the grading:

(1) **Important:** If anything I've said here seems to conflict with what Prof. Su has told you in class, let one of us know so that we can talk about it during our weekly grading meeting.

(2) If any of my comments are ever unclear and/or you think I've made a mistake, you should contact me directly at fkobayashi@g.hmc.edu, and I'll try to get back to you as soon as I can! Note, I probably won't be receive your message if you send it via facebook messenger (since I don't use facebook often), so don't send things there.

(3) The tentative grading scheme is as follows:

- Each problem is worth 5 points.
- 4 points go to the correctness of your argument.
- 1 point will go to the clarity of your writing. Since this is likely something that wasn't emphasized in your previous courses, points will not be deducted here until roughly the fourth problem set.
- If you score lower than a 4/5 on any particular problem, you can resubmit a rewrite once (but only once) for an opportunity for full credit!
- If you accidentally forgot to attach a problem and/or you submitted a proof of the wrong problem (`#relatable`), talk to Prof. Su first — I can't award any points here until he gives the OK, but chances are he'll let you resubmit to me for credit `:)`

The details might change a bit as we test this out, but the main thing to note is that the clarity of your writing will be directly factored into your grade. See section 2.1 below for the rationale.

(4) Here's how the comments work. Each will consist of an **A**, **B**, or **C**, followed by some $\mathbf{n} \in \mathbb{N}$. The categories are as follows:

- **An** comments are about style/formatting standards for mathematical writing. This is something we've decided to emphasize, so consider skimming them each week even if you didn't receive a comment!
- Labels of the form **Bn** will refer to comments about the correctness of your proof.
- Labels of the form **Cn** are usually just some comments on notational standards / LaTeX tips (if applicable). You might not see these every week.

When grading your homeworks, I tried to stay consistent with the comment labeling scheme given above. However, I do make mistakes sometimes, so if something looks funny, don't hesitate to send me an email!

(5) This week, I accidentally did all of the grading in pencil, so it might be hard to read some of my comments. Sorry about this! In the future, I will try to adhere to the following scheme:

- Strikethrough in pencil means that I'm offering a suggested edit to your phrasing (per the writing style standards below). It does not necessarily mean that your argument is incorrect! Also, usually when I do this, I'll offer a reference to a grading comment or give a suggested replacement phrasing.
- Red pen will be used for comments on correctness.
- Anything else we add will be specified in the future.

2 General comments and standards:

2.1 Why the emphasis on writing style?

Well, to quote Prof. Su's handout on good mathematical writing,

> "Learning to communicate effectively is not just a service to your audience; it is also an exercise in clarifying and structuring your own thinking."

This is important! Improving your communication skills doesn't just benefit your reader, it also benefits *you*. If your proof is very terse and clearly written, you'll have a much easier time understanding the structure of your argument, and will also have an easier time identifying potential flaws in your reasoning. So, without further ado, here are the standards I'll be applying to your homeworks throughout the semester.

2.2 Document formatting

- Please leave ample room for comments on your problem sets! This can mean larger margins, whitespace after each solution, or both! You don't have to put each problem on its own page, but it'd be helpful to have at least `\vspace{5cm}` or so `:)`.
- Please do not box your responses to problems! This makes it hard to fit my comments in.
- Try to leave a little bit of space at the top of the first page of your pset in which I can tally points
- Only applicable for people who handwrite their psets: Since I'd like to use pencil to suggest edits (pencil allows me to write smaller notes and also makes it easier for you to read the original version of your proof), it'd be awesome if I could ask you to your writeups in pen on white unruled paper. If you'd really prefer not to though (e.g., because pencil makes it easier to rewrite things), that's totally reasonable! In that case, just try to skip every other line so that I have room to write stuff.

2.3 Organizing your reasoning

TL;DR: you want all of the logic in your proof to be ordered linearly from left to right, top to bottom. Important structure in your arguments should be highlighted visually through the use of headers, line breaks, and indentation.

Basically, you want to make it as easy as possible for your reader to understand your proof on a first reading. Having very linear arguments is helpful in this respect. However, even if your writing is perfect, the *reader* might make make a mistake, such as misremembering how the definition of an important variable. When this occurs, they'll need to go back and re-read an earlier section, so you want to make it as easy as possible for them to find what they're looking for. Visual cues are extremely helpful here! And since we're in the context of homework and psets (instead of the kind of formal mathematical writing you'd find in a journal), we can choose to employ them!

Now, for the long version:

2.3.1 Notes on linear arguments:

As a general rule, any information your reader needs to understand a given section should be stated as close before where it's needed as possible. Both "close" and "before" are important here! Try not to structure arguments in such a way that requires a `#throwback` to something from three paragraphs earlier, and try to *never* cite information that comes later.

These ideas are applicable at all levels of structure in your proof:

(1) Paragraphs: each paragraph should follow logically from the previous ones. Paragraphs should never contain a reference to an argument made in a *subsequent* paragraph. As an example, suppose we're trying to show $A \implies D$. Then we should not do the following:

> *Proof.* We have A. Thus, [...], and so B. As we show in the subsequent section, $B \implies C$, and thus we have [...], hence D.
>
> It remains to show $B \implies C$. Suppose B, then [...], and so C. Thus the proof is complete. ■

If the reason you've split up the proof of $B \implies C$ is because the argument is long and confusing, then here are two remedies you can try:

> *Proof.* First, we prove a small lemma.
>
> > **Lemma 1.** $B \implies C$.
> >
> > *Proof of Lemma.* Observe that B implies [...], thus C. □
>
> **Main Proof:** We have A. Thus, [...], and so B. By the lemma, $B \implies C$, and so C. Then [...], hence D. Thus $A \implies D$, as desired. ■

Note, even though we've put the proof of $B \implies C$ earlier on, this argument is still linear. Your reader can verify the proof of the lemma on its own first, and once they're satisfied it's correct, they can freely apply it in the proof body. If you'd like the proof of $B \implies C$" to remain within the main body of your argument, you can frame it as a claim:

> We have A. Thus [...] B.
>
> **Claim:** $B \implies C$.
>
> **Proof of Claim:** Since B, note [...], therefore C. ✓
>
> Thus C, and so [...], therefore D.

which is also fine.

(2) Sentences *within* your paragraphs should be arranged so as to minimize analogous issues, and

(3) So too with the actual clauses within your sentences. Quick and dirty rules:

- Variables should always be defined *before* they're used
- Quantification of variables should adhere to the rules described in **A2**
- In terms of in-sentence logic, try not to phrase things as "B, because $A \implies B$, and we have A." It's usually better ot say something like "observe that A. We have $A \implies B$, so B," or something to that effect.

2.3.2 Visual cues — whitespace

You can actually improve the readability of your arguments substantially just by making judicious use of whitespace! Some tips:

(1) If the main idea of your proof is $A \implies B \implies C \implies D$, then there should be line breaks between your treatment of $A \implies B$, $B \implies C$, and $C \implies D$:

> *Proof.* Suppose A. Then [...], so B.
>
> Now, observe that B implies [...], thus C.
>
> Finally, we want to show $C \implies D$. By definition of C, we have [...], hence D.
>
> So $A \implies D$, as desired. ■

Note, if the proof of each step is only a few lines long, then this isn't explicitly necessary. Use your own judgment, and I'll give you feedback if I think you should split up your argument more `:)`

(2) If you introduce a lemma, it might be easier to follow your argument if both the statement and proof of the lemma are indented (or otherwise visually distinguished) relative to the main proof body. I typically use a leftbar environment (sorry the double leftbar looks weird here — imagine the outer one doesn't exist), but you don't have to do this:

> *Proof.* First, we introduce a small lemma.
>
> > **Lemma 2.** *Hello world! I'm a lemma.*
> >
> > *Proof of Lemma.* Proof of lemma goes here. Note, I like to use □ instead of ■ for the QED symbol for lemmas and things, so as to avoid the reader confusing the "end of lemma proof" with the "end of main proof." □
>
> Now, we proceed to the main proof. Suppose A. Then [...], and so B. By the lemma, $B \implies C$, and [...], thus D.

A leftbar environment can be defined with the `mdframed` package as follows:

```
\newmdenv[
  skipabove=5, % whitespace before
  skipbelow=5, % whitespace after
  innerleftmargin = .5em, % indentation of text relative to the bar itself
  innerrightmargin = 0pt, % we don't need to shorten the RHS
  innertopmargin = .5em,  % vertical space between top of leftbar and text
  innerbottommargin = .5em, % ibid but buttom of leftbar
  leftmargin = 2em, % indentation of environment relative to the text
  rightmargin = 0em,
  linewidth = 2pt, % width of the bar; use pt to make independent of font
  topline = false, % no frame at top
  rightline = false, % no frame at right
  bottomline = false % no frame at bottom
  ]{leftbar}
```

And you can use the leftbar environment like so:

```
\begin{leftbar}
  This text is in a leftbar environment
\end{leftbar}
```

2.3.3 Visual cues — common proof types

For multi-part proofs, it's super helpful to use visual cues instead of verbal ones to indicate what each component is. I'll give examples below, and TeX code for each one. For each of the following, note that using ✓is optional, *unless* you're making all of your proofs without the use of these headers (discouraged).

(1) To show $A \iff B$, you should format your proof as follows:

> *Proof.*
>
> $(\Rightarrow):$ Suppose A. Then [… lots of math happening here! …], hence B. ✓
>
> $(\Leftarrow):$ Suppose B. Then [… also lots of math happening here! …], therefore A.
>
> Thus, $A \iff B$. ■

Since syntactically, the "forward" direction is technically $B \implies A$ ("A if B"), you should be sure to include a "suppose A" after the ($\implies$)!

For our purposes, this is easier to read than something like this:

> *Proof.* First, we show the forward direction. Suppose A. Then [. . .], hence B. Now, we show the backward direction. Suppose B. Then [. . .] A. Since we have $A \implies B$ and $B \implies A$, it follows that $A \iff B$.

For those of you using LaTeX, to make the first proof, I define the following environment (I believe this requires the `enumitem` package):

```
% The indentation for customized iff labels really grinds my gears.
% Hence, a new environment to make everything right in the world
% again.
\newcommand*{\iffenum}[1]{\expandafter\@iffenum\csname c@#1\endcsname}
\newcommand*{\@iffenum}[1]{%
  \ifcase#1\or{$(\Rightarrow):$}\or{$(\Leftarrow):$}%
  \else\@ctrerr\fi}

\AddEnumerateCounter{\iffenum}{\@iffenum}{$(\Rightarrow):$}
\newenvironment{iffproof}{%
  \begin{enumerate}[label=\iffenum*, leftmargin=4em]%
  }{\end{enumerate}}
```

With this, creating the iff portion of the proof above was as simple as

```
\begin{iffproof}
  \item Suppose $A$. Then [\ldots lots of math happening here!\ \ldots], hence
    $B$. \cmark
  \item Suppose $B$. Then [\ldots also lots of math happening here! \ldots],
    therefore $A$.
\end{iffproof}
```

Where `\cmark` requires the `pifont` package and is defined by `\newcommand{\cmark}{\text{\ding{51}}}`.

(2) Proofs of set equality are almost exactly the same. Let A and B be sets. Then you should format your proof like

> *Proof.*
>
> $(\subseteq):$ Let $x \in A$ be given. Then [. . .] $x \in B$, so $A \subset B$. ✓
>
> $(\supseteq):$ Let $x \in B$ be given. Then [. . .] $x \in A$, so $B \subset A$. ✓
>
> thus $A = B$. ■

Similarly to the `iffproof` environment, I have a `seteqproof` environment defined as follows (feel free to set the environment name to something more ergonomic, though):

```
% Similarly, but for subset supset proofs
\newcommand*{\seteqenum}[1]{\expandafter\@seteqenum\csname c@#1\endcsname}
\newcommand*{\@seteqenum}[1]{%
  \ifcase#1\or {$(\subset):$}\or {$(\supset):$}\else\@ctrerr\fi
}
\AddEnumerateCounter{\seteqenum}{\@seteqenum}{$(\subset):$}
\newenvironment{seteqproof}{%
  \begin{enumerate}[label=\seteqenum*, leftmargin=4em]%
  }{\end{enumerate}}
```

(3) When doing a proof with casework, you should format this as follows:

> Suppose A. We want to show B. We have the following cases:
>
> (1) Suppose (case 1). Then [. . .] B. ✓
>
> (2) Suppose (case 2). Then we have the following subcases:
>
> i) Suppose (subcase i). Then [. . .] B. ✓
>
> ii) Suppose (subcase ii). Then [. . .] B. ✓
>
> Since these cases are exhaustive, B. ✓
>
> (3) Suppose (case 3). Then [. . .] B. ✓
>
> Since these cases are exhaustive we have $A \implies B$, as desired. ■

The LaTeX code used above again requires the `enumitem` package, and is as follows:

```
\begin{enumerate}[label=(\arabic*)]
  \item Suppose (case 1). Then [\ldots] $B$. \cmark
  \item Suppose (case 2). Then we have the following subcases:
    \begin{enumerate}[label=\roman*)]
      \item Suppose (subcase i). Then [\ldots] $B$. \cmark
      \item Suppose (subcase ii). Then [\ldots] $B$. \cmark
    \end{enumerate}
    Since these cases are exhaustive, $B$. \cmark
  \item Suppose (case 3). Then [\ldots] $B$. \cmark
\end{enumerate}
```

By contrast, you should not do the following:

> Suppose A. We want to show B. First, if (case 1), then [. . .] B. B also follows from (case 2), because if (subcase i), then [. . .] and so B, and if (subcase ii), then [. . .] and so B as well, so in either case, B. If (case 3), [. . .] B. So no matter what, B. Therefore $A \implies B$.

A few more points: when doing proofs by casework, you should *almost always try to make your cases disjoint.* That is, (case 1) should imply ((not case 2) and (not case 3)), and so on. In the rare event that this is not the most natural way to write your proof, then you should *take special care to highlight to your reader that your cases are non-disjoint* — and be sure to add in an extra sentence at the end justifying why they're exhaustive!

Also, if you have a situation where (case 2) implies some claim C, you should *not* start your treatment of (case 2) with "Suppose C." You should start with "Suppose (case 2). Then C, and so [. . .] thus B." This makes it significantly easier for your reader to follow!

(4) Inductive proofs should look something like

> *Proof.* We proceed by induction.
>
> **Base Case:** Let $k = k_0$ (where $k_0 \in \mathbb{N}$ stands for the base case here in this example). Observe that [...], thus (statement P) holds for $k = k_0$. ✓
>
> **Inductive Hypothesis:** Now, suppose that (statement P) holds for some $k \in \mathbb{N}$ with $k \geq k_0$.
>
> **Inductive Step:** We want to show (statement P) holds for $k + 1$. Note that [...], hence (statement P) holds for $k + 1$. ✓
>
> Thus, by the principle of mathematical induction, (statement P) holds for all $k \in \mathbb{N}$ s.t. $k \geq k_0$.

although you have more leeway here.

(5) Proofs by contradiction should *always* start with an explicit "suppose, to obtain a contradiction, that $\neg B$."

2.4 Terseness is next to godliness

Ok, so this might seem hypocritical of me after throwing multiple pages of formatting guidelines at you, but you should strive to make your writing terse! In general, try to give your reader *exactly* as much information as they need — no more, no less. Sometimes, this means omitting the details for non-essential steps, and focusing instead on writing in-depth about the non-trivial portions of your proof. Other times, it means rewriting a long constructive proof if you realize contradiction is more efficient. For some perspective, each of the problems from this week were possible to prove in under ≈10 lines, while staying perfectly rigorous!

Here are some DOs and DONTs (this isn't very comprehensive currently; I'll update it more next week. For the time being, please re-read section 2 of Prof. Su's "good mathematical writing" document, particularly the part on deciding what's important to say, avoiding red herrings, stepping back to simplify, and refining repeatedly).

DOs

(1) Try to remove as much redundancy as possible from your writing! That is, don't repeat yourself, and avoid saying the same thing multiple times in a row just with slightly different wording. For an example of what *not* to do, look at the previous sentence (heh)

(2) Define variables if you find yourself referring the same quantity more than once or twice. E.g., if you're doing something involving limit points, you'll probably talk a lot about $(U - \{p\}) \cap A$. As such, you might consider saying "let $Y = (U - \{p\}) \cap A$" so that you don't have to rewrite the expression repeatedly and/or make significant use of "the aforementioned intersection."

(3) Ok, this one requires a bit of qualification, but *do* feel free to use symbols and shorthand in your writing *IF* doing so is clearer and more efficient than English.

Ok, yes, I know we've all been taught to avoid shorthand like the plague — but it's not *all* bad if used responsibly! The issue is more that often when people start using symbols, they start *overusing* symbols. Still, it exists for a reason, so don't discard its usefulness completely.

There are some situations in which I think shorthand is actually universally superior to English. For instance, I believe you should almost always use "$\in$" instead of "contained in," particulary since "contained in" is easy to confuse with $\subset$. As an example, I encountered some sentences like this when grading:

> Every non-empty set in the indiscrete topology is open.

On a first reading, it's not clear that the author means "let $\mathcal{T}$ be the indiscrete topology on a set X. Then every element of $\mathcal{T}$ is an open set." Instead, it sounds like they're saying "under the indiscrete topology, every $U \subseteq X$ is open," which is generally false. Thus, shorthand and symbols sometimes patch the ambiguities left by English language.

Here's a (non-exhaustive) list of notation I think you should usually use over their English counterparts:

- $|X|$ (instead of "cardinality of X" — definitely don't say "the size of X.")
- $\in, \notin$
- $\subset, \supset, \not\subset, \not\supset$
- $<, \leq, =, \geq, >$, and $\neq$
- $\cap$ (often $\cup$ as well, but sometimes it's easier to say "together with" so that you don't have to define extra sets, as in the definition of closure)
- Set complement (whichever of A^c, $X - A$ you prefer — each has its merits; A^c is often easier to read, but leaves ambiguity as to what the space we're complementing within is)
- Arithmetic operations and stuff
- Homeomorphism and equivalence relations ($\cong, \sim, \equiv$)

Sometimes, the decision for whether or not you should use shorthand depends on whether you want your reader to slow down when parsing the current section of your proof. E.g., if you're just trying to give the reader a summary of what to expect from the following paragraph, it might be ok to say "we want to show $\forall x \in U$, there exists $U_x \in \mathcal{T}$ s.t. $x \in U_x$" before you actually dive into proving it. If, however, you're at a really subtle point in the argument, you might consider writing things out with more English to force your reader to slow down. Here is a (again, non-exhaustive list) of symbols that fall under this "sometimes" category:

- $\exists, \forall$
- $\implies$, $\impliedby$, $\iff$ (although honestly I kind of dislike $\impliedby$)
- WTS (for "want to show" — be aware, most people intensely dislike this), and st or s.t. (for "such that")
- $(\Rightarrow\Leftarrow)$ (contradiction)
- $\exists!$ (use this one sparingly).

If you do choose to make use of the symbols above, make sure you don't *always* use them. Again, they should be primarily to modulate my reading speed. If you want to try out using them, I'll let you know if I think it's getting excessive.

Lastly here're some things you should really try to avoid:

- Most formal logic symbols (e.g., $\wedge, \vee, \neg, \therefore, \because$)
- Not much else comes to mind right now, but I'll add more in the future if this changes.

DONTs

(1) Don't feel like you need to include non-essential steps, especially if they distract from the overall structure of your argument. Taking an example from this pset, I would actually encourage you *not* to bother with proving that the union of two finite sets is finite. In the future, you can simply assert it!

Remember your audience: 131 is a prerequisite for this class, so you should assume that the reader can fill in some of these gaps. If you have something that feels like it's in an awkward middle ground, when in doubt, leave a proof in a footnote / endnote!

(2) Similarly, don't feel that it's necessary to re-prove results from previous classes unless the problem explicitly requires it. As an example, since most people have taken Analysis (I think), you can simply assert that open balls are open in the standard topology on $\mathbb{R}^n$. If you'd like to be thorough, you can leave a citation for a theorem in a parenthetical, or again in a footnote. If you just want the extra practice though, you can include a proof as a lemma or something!

(3) Speak with a lot of double negatives. Instead of saying "V can't possibly contain a point U doesn't," say "Every point of V is an element of U," or better yet, "for all $x \in V$, $x \in U$."

(4) Don't include red herrings in your proofs! I.e., details / variables that you introduce but never use.

(5) Don't include more than one proof of the same fact! If you have a second proof that's less intuitive but that you're really excited about, feel free to include it as a footnote or after the end of your first proof. But you should not put both of them in the main body.

Ok, onwards to the grading comments.

3 Writing Style Comments

A1. This comment is about proper meta-discourse. When trying to signal to your reader what strategies you'll pursue for your proof, you should keep the following considerations in mind:

- You don't need to say things things like "the way to show $A = B$ is to show $A \subset B$ and $B \subset A$." Similarly with "to prove $A \iff B$, we must prove $A \implies B$ and $B \implies A$." While these are great if writing for a less experienced audience, 147 assumes at least two prior proof-based courses, so your reader should be familiar with these tactics :)

  So, provided it's clear *what* you're trying to prove in these cases, you should just use the formatting guidelines listed in the **Visual cues** section.

- Also, when doing proof by casework, you should use the formatting guidelines listed in **Visual cues**. As stated there, you should always try to make it *as clear as possible* that your cases are exhaustive. Don't do a proof by casework inline, unless it's really short.

- Before beginning a proof by contradiction, remember to be sure to indicate to your reader that that's what you're doing by including a "Suppose, to obtain a contradiction."

A2. Properly quantifying variables is extremely important. Here are the rules (loosely adapted from Prof. Flapan's guidelines — pay *particular* attention to the scoping rules):

(1) Variables defined in the problem statement do not need to be re-defined. E.g., if the problem statement says "prove property P for a topological space $(X, \mathcal{T})$," you do not need to start your proof with "Let $(X, \mathcal{T})$ be a topological space."

(2) Other than that, you should *always* define a variable before you use it. Variable definitions should come at the start of the sentence, and you should give a domain explicitly. E.g., for some property P satisfied over a set X, say "$\forall x \in X,\ P(x)$" instead of "$P(x) \forall x \in X$" (of course, there are some exceptions — but this is a good starting point). We should *never* say "we have $P(x)$ for all x" without saying what x is.

(3) "Let" declares a variable until the objects the definition depends on go out of scope, or until the variable gets redefined. Hence, if you start a proof about a topological space $(X, \mathcal{T})$ with "Let $x \in X$," then you can freely refer to x until the end of the problem, unless $(X, \mathcal{T})$ gets redefined somewhere in between. Similarly when defining variables in casework.

(4) "For all," "$\forall$," and "if" only declare a variable until the end of the sentence. If you want to use the variable later, you should use "let."

This might not be a canonical rule, but we're going to do things this way in 147 so that you can use $\forall$ declarations when you need an unimportant dummy variable, e.g. if you're trying to highlight a property of a set using its elements, but don't need to use these element later.

(5) Given a set X, if you want to prove a claim for all $x \in X$ then you must start your proof with "let $x \in X$ be given" or "let $x \in X$ be arbitrary."

(6) "There exists" and "$\exists$" declare a variable if it does not depend on a $\forall$ or an "if" statement earlier in the sentence. Example:

> Consider $\mathbb{Z}$. $\forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}$ s.t. $k = 2n$. Now, take $u = 3 + k$.

what is u supposed to be? It's unclear. The way we've defined it makes it look like a single value, but really it's referring to a whole class of values contingent on the particular choice of n. By contrast, saying

> Let $n \in \mathbb{Z}$ be given. Then $\exists k \in \mathbb{Z}$ s.t. $k = 2n$. Now, take $u = 3 + k$.

is fine, because we know exactly what u is supposed to be, since we're assuming we've been given n.

The one exception I'll make is for things like

> Let $(X, \mathcal{T})$ be a topological space, and suppose U is open. Then $\forall x \in X$, there exists $U_x \in \mathcal{T}$ such that $x \in U_x$, and $U_x \subset X$. Observe that
>
> $$\bigcup_{x \in X} U_x = U,$$
>
> hence U is open.

because we're implicitly defining a choice function here, so things work out.

A3. Try to avoid giving red herrings and/or unnecessary levels of detail. As an example, I saw a lot of people saying things like

> "Let λ be a (possibly uncountable) indexing set, and let $\{U_\alpha\}_{\alpha\in\lambda}$ be a collection of open sets indexed by λ."

Here, I think it's less confusing to say "Let λ be an arbitrary indexing set" instead of "let λ be a (possibly uncountable) indexing set." What's the difference? Formally, not much — "possibly uncountable" is equivalent to "countable or uncountable," which is the same as "arbitrary." But this is exactly the problem!

If you say "possibly uncountable" instead of "arbitrary," your reader might think "well, *possibly uncountable* is equivalent to *arbitrary*, so the only reason the author would pick the former over the latter would be if it's going to be important later that λ might not be countable." Then, if no such issue arises, they'll be confused! Hence, only choose the "possibly uncountable" phrasing if it's really relevant. This was a particular example, but try to apply similar reasoning to all of your proofs!
Of course ultimately, what is "necessary" vs. "unnecessary" to show can be subjective, so just follow your gut and I'll give you feedback on whether or not I agree :)

A4. Your writing should never include phrases like "obviously," "clearly," or "trivially." The rationale is that these words offer no positive content to your reader. Consider the following scenarios:

(1) Suppose that you're saying "clearly, $A \implies B$" without actually proving it. Then
- In the event that your reader agrees this fact *is* self-evident, then you didn't need to say "clearly" at all the first place.
- If your reader does *not* think this fact is self-evident, then you risk either making them think you're being lazy, or (if your reader is looking to your writing for guidance) maybe even making them feel inadequate.

Neither of these is beneficial, so consider using one of the following strategies instead:
- If the claim is sufficiently besides the point, you can simply assert it. E.g., "A is infinite, while B is finite, hence $A \not\subset B$." You don't have to prove it, but you still shouldn't say "clearly $A \not\subset B$."
- If it seems appropriate, you could give a brief hint of how one might show the claim, e.g. "one can use induction to show $A \implies B$." This gives your reader an idea of what direction to look in, and also doesn't sound as abrasive.

(2) Suppose that you're saying "clearly" to make a conclusion (e.g., "Note that [. . .] thus, we clearly have A"). In this case as well, the word adds no positive content — either your preceding argument was well-written and sound (in which case the claim should be clear independent of whether you assert it is), or it wasn't, in which case saying "clearly" doesn't actually change anything.

In summary, use neutral words like "observe" or "note" instead of "clearly."

A5. If you find yourself repeatedly saying phrases like "some point in" or "the same intersection as before" to refer to the same object, you should check to if your proof reads better if you define this quantity as a variable. Examples:

> Suppose $(A \cap B) \cup C \neq \varnothing$. Then there is some point that is in $(A \cap B)$ or C. It follows that there is some point not contained in the complement of this set above, that is, there is some point in $X - ((A \cap B) \cup C)$.

It's often easier for a reader to follow something like this:

> Let $Y = (A \cap B) \cup C$, and suppose $Y \neq \varnothing$. Then there exists $y \in Y$, and $y \in (A \cap B)$ or $y \in C$. $y \notin X - Y$.

It's also shorter!

A6. Try not to be redundant! Writing things like

> "Consider $(A \cap B)^C$, that is, the complement of the intersection of U and V"

can often clutter your proofs unnecessarily. Keeping it simple will make it easier to read!

A7. When trying to say $A \implies B$, A, hence B, don't say "B, because $A \implies B$, and A." Instead, say something like "A, and $A \implies B$, hence B." If you're concerned that the direction you're going in seems unmotivated unless the reader knows what you're trying to do, you can try adding a WTS: "We want to show B. We have A. But note, $A \implies B$, hence B."

In general, justification should come *before* your result. An exception is when you're making a parenthetical citation of a theorem — in this case, it's ok to say something like "$(A \cap B)^c = A^c \cup B^c$ (DeMorgan's Laws)," instead of "by DeMorgan's Laws, $(A \cap B)^c = A^c \cup B^c$." However, the second form is usually still preferred.

A8. Don't use English in situations where symbols are more efficient and unambiguous. I saw lots of proofs that said things like

> "The complement of the intersection of the two sets U and V is equivalent to the set that is the complement of the union of U and V, which is true by DeMorgan's Laws," or

> "Since points either live in the set $X - A$ or in the set A, then because every neighborhood is not a subset of $X - A$, there must be a point contained in A in every neighborhood of p, by properties of sets."

These are completely correct! However, phrasing it in this way could make it harder for a reader to follow what you're saying. The *length* of the sentence makes it easy to get lost partway through and forget what's going on, especially these come in the middle of a larger proof. By contrast, things like

- "By DeMorgan's Laws, $X - (U \cap V) = (X - U) \cup (X - V)$," or
- "Observe that $(U \cap V)^C = U^C \cup V^C$ (DeMorgan's Laws)"

are actually much easier for your reader to parse. See the DOs and DONTs stuff in §2.4 for more!

A9. Here are some general comments about vocabulary in math writing:

(a) Avoid writing very forcefully (proof by intimidation is not allowed)! That is, try not to say things in such a way that it sounds like the reader is being pressured into believing you. Examples:

i) Everything in **A4**

ii) "We must have [...]" and "it must be [...]"

iii) "It is a fact that [...]"

iv) "Surely, it follows that [...]"

v) "We will show that [...]"

vi) "We know [...]"

vii) "Thus, we have shown that [...]"

One of the great things about math is that, as a reader, you are never under any obligation to believe the writer. At every step of the argument, you are equipped to fact-check them, construct counterexamples, and generally be a nuiscance. And as a writer, you want to encourage this! Remember, your goal is to *be* right, not to sound right. Hence, you don't want to use phrasing that makes the reader feel like they're being biased. Use neutral, objective phrasing, like

i) See **A4**

ii) "Observe that [...]"

iii) "By definition, [...]"

iv) "It follows that [...]"

v) "We want to show that [...]"

vi) "We have [...]"

vii) "Thus, [...]"

and strive to construct a proof such that by the end, your reader will be totally convinced of your correctness, all of their own accord.

(b) This might be a bit too nitpicky, but be cognizant of what different words connote.

- "Assume" vs. "suppose:" this might just be me, but in my eyes, these words connote two very different things. In my experience "assume" is usually used when you are imposing a simplifying constraint on a problem — e.g., "assume $\sin(\theta) = \theta$" — while "suppose" is used for proving statements of the form "if A then B."

  I think the difference is really one of hypotheticals. To me "assume" makes it sound like we're saying we're fixing a concrete interpretation to the predicate A, and saying "A is true, [...], thus B is also true," instead of "*in the event* that A is true, it would also follow that B is true." Thus I prefer "suppose" for suppositions.

- To connect steps in an argument, I try to sample from the following pool:
  - Inference / premise introduction: "observe," "note," "notice," "we have,"
  - Implications: "thus," "it follows," "hence," "and so," "then," "whereby," "observe," "note," "and so," "whence," etc.
  - Implied by: "since," "because"
  - Introducing an unexpected step, or achieving a contradiction: "but," "however"
  - Final step: add "therefore," to the list under "implications" (although feel free to break this guideline and use "therefore" wherever you please)

  And don't really use much else. Notice how few "implied by" vocab words are listed — see **A7**!

- Don't say things like "The set A has a limit point"

4 Correctness

4.1 Execercise 3.5

B1. When verifying the finite complement topology, note that $U \in \mathscr{T}$ need not imply $X - U$ is finite — we could have $U = \varnothing$, whence $X - U = X$.

5 Comments on TeX / Notation

C1. You might consider using $\mathscr{T}$ instead of $\mathcal{T}$ to refer to a topology, to be consistent with Prof. Su's book:

```
% Use a better script font
\usepackage[mathscr]{euscript}
\newcommand*{\ms}[1]{\ensuremath{\mathscr{#1}}}
```

which you can then call simply by `\ms T`.

C2. I saw a lot of people using $B(p, r)$ (or was it $B(r, p)$?) to denote "the ball of radius ϵ centered about point p." As far as I can tell, this is non-standard notation; I think most people use $B_r(p)$ instead.