

Problem	5.6(4)	5.11	5.15(no normal)	5.17	5.23	Total	Forest Kobayashi
Points							Math 147
							HW 5 Solutions
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**5.6(4).** Show that  $\mathbb{R}^2$  with the standard topology is normal.

*Solution.* First, we introduce some notation.

**Notational Note:** Let  $(X, \mathcal{F})$  be a topological space. Let  $x \in X$ , and let  $Y \subset X$ . Then define

$$d(x, Y) = \inf_{y \in Y} d(x, y).$$

*Main Proof:* Let  $A, B$  be disjoint closed subsets of  $\mathbb{R}^2$ . For each  $a \in A, b \in B$ , let

$$\epsilon_a = \frac{d(a, B)}{2} \qquad \epsilon_b = \frac{d(b, A)}{2}$$

and note that by part (1),  $\epsilon_a, \epsilon_b > 0$ . Define

$$U = \bigcup_{a \in A} B_{\epsilon_a}(a) \qquad V = \bigcup_{b \in B} B_{\epsilon_b}(b)$$

and observe  $U, V \in \mathcal{I}_{\text{std}}$ , with  $A \subset U$  and  $B \subset V$ . We want to show  $U \cap V = \emptyset$ .

Suppose, to obtain a contradiction, that  $U \cap V \neq \emptyset$ . Let  $x \in U \cap V$ . Then there exist  $a \in A, b \in B$  such that  $x \in B_{\epsilon_a}(a) \cap B_{\epsilon_b}(b)$ . It follows that

$$\begin{aligned} d(a, b) &\leq d(a, x) + d(x, b) \\ &< \epsilon_a + \epsilon_b \end{aligned} \tag{*}$$

WLOG, suppose  $\epsilon_b \leq \epsilon_a$ . Then

$$\begin{aligned} d(a, b) &< 2\epsilon_a \\ &= d(a, B) \\ &\leq d(a, b) \end{aligned}$$

so  $d(a, b) < d(a, b)$ , a contradiction.<sup>1</sup> Hence,  $U \cap V = \emptyset$ , so  $U, V$  are disjoint open sets containing  $A$  and  $B$  respectively. Since  $A, B$  were arbitrarily chosen, it follows that  $\mathbb{R}^2$  is normal, as desired. ■

<sup>1</sup>The align environment given should be read as “ $d(a, b) < 2\epsilon_a = d(a, B) \leq d(a, b)$ ,” not as  $d(a, b) < 2\epsilon_a, d(a, b) = d(a, B)$ , and so on. Also, here our contradiction is  $d(a, b) < d(a, b)$ , but we could also just skip (\*) and directly contradict the triangle inequality by  $d(a, x) + d(x, b) < d(a, b)$ . I prefer the former, just because it better matches arguments seen in Analysis.

**5.11 (The Incredible Shrinking Theorem).** A topological space  $X$  is normal if and only if for each pair of open sets  $U, V$  such that  $U \cup V = X$ , there exist open sets  $U', V'$  such that  $\overline{U'} \subset U$  and  $\overline{V'} \subset V$ , and  $U' \cup V' = X$ .

I'll provide two solutions: one using Theorem 5.9, another using Theorem 5.10.

*Solution 1:* First, we prove a lemma.<sup>2</sup> The ( $\implies$ ) direction will follow as a corollary.

**Lemma 1.** *Let  $(X, \mathcal{T})$  be normal. Let  $U, V \in \mathcal{T}$  such that  $U \cup V = X$ . Then there exists  $U' \in \mathcal{T}$  such that  $\overline{U'} \subset U$ , and  $U' \cup V = X$ .*

I'll provide two proofs. The first uses theorem 5.9 (and is hence *much* cleaner), while the second uses the definition of normality (and is hence much longer / more involved). I included both, so that people who tried to use normality directly could see how to proceed.

*Proof 1:* Note that  $V^c$  is closed, and  $V^c \subset U$ . Then by theorem 5.9, there exists  $U' \in \mathcal{T}$  such that

$$V^c \subset U' \subset \overline{U'} \subset U.$$

Note that  $V^c \subset U' \implies U' \cup V = X$ . Hence, we have our desired  $U'$ . □

*Proof 2:*  $U, V \in \mathcal{T}$  implies  $U^c$  and  $V^c$  are closed. Observe that

$$\begin{aligned} U^c \cap V^c &= (U \cup V)^c \\ &= \emptyset, \end{aligned}$$

hence  $U^c, V^c$  are disjoint closed sets. Then by definition of normality, there exist disjoint open sets  $U', V'$  such that

$$U^c \subset V' \quad \text{and} \quad V^c \subset U'.$$

Note that  $V^c \subset U' \implies U' \cup V = X$ . It remains to show  $\overline{U'} \subset U$ . Since  $U' \cap V' = \emptyset$  and  $U^c \subset V'$ , we have

$$U' \subset V'^c \subset U$$

and because  $V'^c$  is closed,  $\overline{U'} \subset V'^c$  as well. This proves the claim. □

Now, the main proof.

( $\implies$ ): Suppose  $X$  is normal.

Let  $U, V \in \mathcal{T}$  such that  $U \cup V = X$ . Then by the lemma, there exists  $U' \in \mathcal{T}$  such that  $\overline{U'} \subset U$ , and  $U' \cup V = X$ . Now, applying the lemma to the pair  $(V, U')$ , we obtain the desired  $V'$ . ✓

( $\impliedby$ ): Suppose that  $\forall U, V \in \mathcal{T}$  s.t.  $U \cup V = X$ , there exists  $U', V' \in \mathcal{T}$  s.t.  $\overline{U'} \subset U$ ,  $\overline{V'} \subset V$ , and  $U' \cup V' = X$ . WTS  $X$  is normal. We will apply Theorem 5.9.

Let  $A \subset X$  be an arbitrary closed set, and let  $U \in \mathcal{T}$  such that  $A \subset U$ .<sup>3</sup> Observe that  $A^c$  is open, and  $U^c \subset A^c$ . It follows that  $X = U \cup A^c$ . Then by hypothesis, there exists  $U', V' \in \mathcal{T}$  such that

$$\overline{U'} \subset U \quad \overline{V'} \subset A^c$$

and  $U' \cup V' = X$ . From this it follows that  $(U')^c \subset V'$ , hence

$$(U')^c \subset V' \subset A^c.$$

<sup>2</sup>I'm just proving it as a lemma so that I can offer two proofs. In an actual writeup, I'd just use one of them.

<sup>3</sup>At least one such  $U$  exists, namely  $X$ , hence we can freely declare  $U$  in this manner.

Taking the complement, we have

$$A \subset (V')^c \subset U',$$

and since  $\overline{U'} \subset U$ , this yields

$$A \subset U' \subset \overline{U'} \subset U$$

as desired. Since  $A$  and  $U$  were arbitrarily chosen, Theorem 5.9 implies  $X$  is normal. ✓

■

*Solution 2:* We employ Theorem 5.10.

( $\Rightarrow$ ): Suppose  $X$  is normal. Let  $U, V \in \mathcal{T}$  such that  $U \cup V = X$ . Then  $U^c, V^c$  are closed, and by DeMorgan's Laws,

$$U^c \cap V^c = (U \cup V)^c = \emptyset,$$

hence they are disjoint as well. Then by Theorem 5.10, there exist disjoint  $U_0, V_0 \in \mathcal{T}$  such that

$$U^c \subset U_0 \quad V^c \subset V_0 \quad \overline{U_0} \cap \overline{V_0} = \emptyset.$$

Because  $U_0 \subset \overline{U_0}$  and  $V_0 \subset \overline{V_0}$ , taking complements yields

$$(\overline{U_0})^c \subset (U_0)^c \subset U \quad (\overline{V_0})^c \subset (V_0)^c \subset V \quad (\overline{U_0})^c \cup (\overline{V_0})^c = X.$$

Let  $U' = (\overline{U_0})^c$  and  $V' = (\overline{V_0})^c$ , and note that these are open. Then the above can be reexpressed as

$$U' \subset (U_0)^c \subset U \quad V' \subset (V_0)^c \subset V \quad U' \cup V' = X,$$

and since  $(U_0)^c, (V_0)^c$  are closed, Theorem 3.20 implies

$$U' \subset \overline{U'} \subset (U_0)^c \subset U \quad V' \subset \overline{V'} \subset (V_0)^c \subset V$$

as desired. ✓

( $\Leftarrow$ ): Suppose  $\forall U, V \in \mathcal{T}$  s.t.  $U \cup V = X$ , there exists  $U', V' \in \mathcal{T}$  s.t.  $\overline{U'} \subset U, \overline{V'} \subset V$ , and  $U' \cup V' = X$ . WTS  $X$  is normal.

Let  $A, B$  be arbitrary disjoint closed sets. Then  $U = A^c, V = B^c$  are open, and  $U \cup V = X$  (DeMorgan's Laws).

By hypothesis, there exists  $U', V' \in \mathcal{T}$  such that

$$\overline{U'} \subset U \quad \overline{V'} \subset V \quad U' \cup V' = X.$$

Since  $U' \subset \overline{U'}$  and  $V' \subset \overline{V'}$ , complementation yields

$$U^c \subset (\overline{U'})^c \subset (U')^c \quad V^c \subset (\overline{V'})^c \subset (V')^c \quad (U')^c \cap (V')^c = \emptyset.$$

Finally, substituting  $U^c = A$  and  $V^c = B$ , we see  $(\overline{U'})^c, (\overline{V'})^c$  are disjoint open sets separating  $A$  and  $B$ . Thus  $X$  is normal, as desired. ✓

■

**5.15.** Order topologies are  $T_1$ , Hausdorff, and regular.

*Solution.* Let  $X$  be a totally ordered set, and  $\mathcal{T}$  be the associated order topology. Denote the elements of the canonical basis as follows:

- $(-\infty, a) = \{x \in X \mid x < a\}$
- $(a, \infty) = \{x \in X \mid a < x\}$
- $(a, b) = \{x \in X \mid a < x < b\}$ .

Square brackets will indicate inclusivity, as usual.

**Note.** Although the notation here is almost identical to that of the standard topology on  $\mathbb{R}$ , we need not have  $X = \mathbb{R}$ . In fact,  $X$  is guaranteed to have *no algebraic structure* whatsoever. Be sure to keep this in mind as we proceed!

- (1) We apply Theorem 5.1. Let  $x \in X$  be arbitrary. Then  $(-\infty, x) \cup (x, \infty)$  is open. By complement,  $\{x\}$  is closed, hence  $(X, \mathcal{T})$  is  $T_1$ .

**Remark.** By Theorem 5.7.2, we actually just need to show regularity now that we have  $T_1$ . But in case you'd like to show Hausdorff constructively for extra practice, I've included a proof of Hausdorffness below.

- (2) WLOG, suppose  $x < y$ . We proceed by casework.

- (i) Suppose there exists  $z \in X$  such that  $x < z < y$ . Then  $U = (-\infty, x)$ ,  $V = (z, \infty)$  are disjoint open sets with  $x \in U$ ,  $y \in V$ .
- (ii) Suppose no such  $z$  exists. Then  $U = (-\infty, y)$ ,  $V = (x, \infty)$  are disjoint open sets with  $x \in U$ ,  $y \in V$ .

hence  $(X, \mathcal{T})$  is Hausdorff.

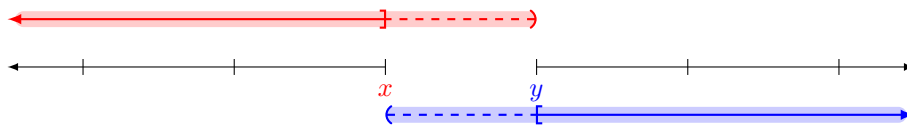


Figure 1: Subcase (ii). Note the gap between  $x$  and  $y$ .

- (3) To show regularity, we will employ Theorem 5.8. But first, a small Lemma.

**Lemma 2.** Let  $(a, b) \subset X$ . Then  $\overline{(a, b)} \subset [a, b]$ .

*Proof:* Note that  $X - [a, b] = (-\infty, a) \cup (b, \infty)$  is open, hence  $[a, b]$  is closed. By Theorem 3.20, we have  $\overline{(a, b)} \subset [a, b]$ .

**Remark.** We actually can't do better than this in the general case (i.e., we need not have  $\overline{(a, b)} = [a, b]$ ). For example, you can find subspaces of the lexicographically ordered square that refuse to play nice. Also, if  $X$  is a discrete set (such as  $\mathbb{N}$  or  $\mathbb{Z}$ ), plenty of counterexamples exist.

Let  $x \in X$  be arbitrary, and let  $U \in \mathcal{T}$  such that  $x \in U$ . Then there exist  $a, b \in X \cup \{-\infty, \infty\}$  such that

$$x \in (a, b) \subset U.^4$$

**Claim:** There exists  $(a', b') \subset (a, b)$  such that  $x \in (a', b') \subset \overline{(a', b')} \subset (a, b)$ .

<sup>4</sup>Note, this is just a concise way of declaring a basic open set.

**Proof of Claim:** When typing this up, I found a slightly cleaner version of the argument I was using at <http://web.math.ku.dk/~moller/e02/3gt/opg/S31.pdf>, and have modified my proof accordingly.

Let  $A = (a, x)$ , and  $B = (x, b)$ . Then we have four subcases.

i) Suppose that  $A, B = \emptyset$ . Then  $(a, b) = \{x\}$ , which is clopen. Hence take  $(a', b') = (a, b)$ , and the claim holds. ✓

ii) Suppose  $A = \emptyset$  and  $B \neq \emptyset$ , and let  $b' \in B$ . Then let  $a' = a$ , and note  $(a', b') = [x, b')$ . Hence  $x \in (a', b')$ , and

$$\overline{(a', b')} = \overline{[x, b')} \subset [x, b'] \subset (a, b)$$

so the claim holds. ✓

iii) Suppose  $A \neq \emptyset$  and  $B = \emptyset$ . Analogously to the above, we let  $a' \in A$  and  $b' = b$ , which yields

$$\overline{(a', b')} = \overline{(a', x]} \subset [a', x] \subset (a, b)$$

as desired. ✓

iv) Suppose  $A \neq \emptyset \neq B$ . Then let  $a' \in A$ ,  $b' \in B$ . It follows that

$$x \in (a', b') \subset \overline{(a', b')} \subset [a', b'] \subset (a, b)$$

as desired. ✓

since these cases are exhaustive, this proves the claim. Then by Theorem 5.8,  $X$  is regular, as desired.

■

**5.17.** Let  $X$  and  $Y$  be regular. Then  $X \times Y$  is regular.

*Solution.* We prove a lemma.

**Lemma 3.** Let  $A \subset X$ ,  $B \subset Y$ . Then  $\overline{A \times B} = \overline{A} \times \overline{B}$ .

We use the notation  $\mathcal{L}(S)$  to denote the limit points of a set  $S$ . Here're a few proofs:

*Proof 1:* The claim is equivalent to

$$p \in \overline{A \times B} \iff p \in \overline{A} \times \overline{B}.$$

We proceed by contrapositive. That is,

$$p \notin \overline{A \times B} \iff p \notin \overline{A} \times \overline{B}.^a$$

We prove both directions simultaneously.<sup>b</sup> The following are equivalent:

- (1)  $p = (p_x, p_y) \notin \overline{A \times B}$
- (2) There exists  $U \in \mathcal{T}_{\text{prod}}$  s.t.  $p \in U$  and  $(U - \{p\}) \cap (A \times B) = \emptyset$
- (3) For  $U$  quantified as above, there exists  $B = U_A \times U_B \in \mathcal{B}_{\text{prod}}$  such that  $p \in B$ , and

$$\begin{aligned} \emptyset &= B \cap (A \times B) \\ &= (U_A \times U_B) \cap (A \times B) \\ &= (U_A \cap A) \times (U_B \cap B). \end{aligned}$$

- (4) There exists  $U_A \in \mathcal{T}_A$ ,  $U_B \in \mathcal{T}_B$  such that at least one of  $(U_A \cap A)$ ,  $(U_B \cap B)$  is empty
- (5) At least one of  $p_x \notin \overline{A}$ ,  $p_y \notin \overline{B}$  is true
- (6)  $p \notin \overline{A} \times \overline{B}$

□

<sup>a</sup>Note, this is just saying that the set complements are equal.

<sup>b</sup>This introduces a mess with variable quantifications, but hopefully the argument makes sense

*Proof 2:* We prove the claim directly.

( $\subseteq$ ) : Let  $p = (p_A, p_B) \in \overline{A \times B}$ . We want to show  $p_A \in \overline{A}$  or  $p_B \in \overline{B}$ .<sup>a</sup>

- (1) Suppose  $p \in A \times B$ . Then we're done. ✓
- (2) Now, suppose  $p \in \mathcal{L}(A \times B)$ . Let  $U_A \in \mathcal{T}_A$  and  $U_B \in \mathcal{T}_B$  be arbitrarily chosen. Then  $U = U_A \times U_B \in \mathcal{T}_{\text{prod}}$ , and hence

$$(U - \{p\}) \cap (A \times B) \neq \emptyset$$

Thus, let  $q = (q_A, q_B) \in (U - \{p\}) \cap (A \times B)$ . It follows that

$$\pi_x(q) \in (\pi_x(U - \{p\}) \cap \pi_x(A \times B))$$

or equivalently,

$$q_A \in (U_A - \{p_A\}) \cap A$$

and similarly,  $q_B \in (U_B - \{p_B\}) \cap B$ . Hence,  $\overline{A \times B} \subset \overline{A} \times \overline{B}$ . ✓

( $\supseteq$ ) : The reverse direction essentially consists of reversing the steps above. Note, you need to consider both of the cases  $p \in \mathcal{L}(A) \times B$  and  $p \in A \times \mathcal{L}(B)$ . □

<sup>a</sup>The or here is inclusive.

*Proof 3:* We prove the two directions separately.

$(\subseteq)$  : The product of closed sets is closed.<sup>a</sup> Since  $A \times B \subset \overline{A} \times \overline{B}$ , Theorem 3.20 implies

$$\overline{A \times B} \subset \overline{\overline{A} \times \overline{B}} = \overline{A} \times \overline{B}.$$

$(\supseteq)$  : Use either of the arguments above. □

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<sup>a</sup>As justification, note that  $(\overline{A})^c \times (\overline{B})^c$  is a product of open sets, and is thus open in  $\mathcal{T}_{\text{prod}}$ , hence  $\overline{A \times B}$  is closed.

We proceed by Theorem 5.8.

Let  $p \in X \times Y$  be arbitrary, and let  $U \in \mathcal{T}_{\text{prod}}$  such that  $p \in U$ . Then by definition of  $\mathcal{T}_{\text{prod}}$ , there exist  $U_X \in \mathcal{T}_X$ ,  $U_Y \in \mathcal{T}_Y$  such that

$$p \in U_X \times U_Y \subset U.$$

Since  $X, Y$  are regular, there exist  $V_X \in \mathcal{T}_X$ ,  $V_Y \in \mathcal{T}_Y$  such that

$$\pi_x(p) \in V_X \subset \overline{V_X} \subset B_X \quad \text{and} \quad \pi_y(p) \in V_Y \subset \overline{V_Y} \subset B_Y.$$

Thus  $p \in V_X \times V_Y$ , which is open in  $\mathcal{T}_{\text{prod}}$ . Then

$$p \in V_X \times V_Y \subset \overline{V_X \times V_Y} = \overline{V_X} \times \overline{V_Y} \subset B_X \times B_Y \subset U$$

so by Theorem 5.8,  $X \times Y$  is regular. ■

**5.23.** Let  $A$  be a closed subset of a normal space  $X$ . Then  $A$  is normal when given the relative topology.

*Solution.* Let  $\mathcal{T}_X$  be the topology on  $X$ , and  $\mathcal{C}_X$  be the set of closed sets in  $(X, \mathcal{T})$ . Define  $\mathcal{T}_A, \mathcal{C}_A$  analogously for the relative topology on  $A$ .

Let  $B, C \in \mathcal{C}_A$  be disjoint. Then by Theorem 4.28, there exist  $B', C' \in \mathcal{C}_X$  such that

$$B = B' \cap A \qquad C = C' \cap A.$$

Then since  $\mathcal{C}_X$  is closed under arbitrary intersection, it follows that  $B, C$  are closed in  $(X, \mathcal{T})$  as well.<sup>5</sup> Then by normality, there exist disjoint  $U, V \in \mathcal{T}_X$  such that  $B \subset U$  and  $C \subset V$ . Observe

$$B = (B \cap A) \subset (U \cap A) \qquad C = (C \cap A) \subset (V \cap A),$$

and by definition,  $(U \cap A), (V \cap A)$  are open sets in  $(A, \mathcal{T}_A)$ . Since  $U \cap V = \emptyset$ , we have

$$(U \cap A) \cap (V \cap A) = \emptyset$$

as well, thus we have found disjoint open sets separating  $A, B$ . Hence,  $A$  is normal with the relative topology, as desired. ■

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<sup>5</sup>OK, I know I defined  $\mathcal{C}_X$  above, but I was worried that all the script  $C$ 's flying around were getting confusing!