

Problems	4.31(F)	4.41	5.1	5.5	5.9	Total
Points						

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Math 147  
HW 4 Solutions  
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**4.31(F).** Consider the following subspace of the lexicographically ordered square:

$$F = \{(x, 1) \mid 0 < x < 1\}.$$

As a set, it is a line. Describe its relative topology, noting any connections to the topologies you have seen already.

**General Note:** Honestly, the edge cases here make this problem a bit of a mess. Feel free to gloss over them by drawing diagrams for each case instead of actually writing things out formally. I've included everything here for the sake of completeness, but you should feel free to omit some parts. Just make it clear to me that you know *why* everything has the form it should! :)

**Notational Notes:** For the purposes of this problem, we will use boldface/angled brackets to denote points, and parentheses for open intervals. E.g.,  $\mathbf{p} = \langle a, b \rangle \in \mathbb{R}^2$ , whereas  $(a, b) \subset \mathbb{R}$ . Note, under this notational scheme, “open intervals” in the lexicographically ordered square will thus be denoted by something like  $(\langle a_0, b_0 \rangle, \langle a_1, b_1 \rangle)$ :

$$(\langle a_0, b_0 \rangle, \langle a_1, b_1 \rangle) = \left\{ \langle x, y \rangle \mid \langle a_0, b_0 \rangle < \langle x, y \rangle < \langle a_1, b_1 \rangle \right\}$$

using square brackets when appropriate. Note that if we have previously defined  $\mathbf{p}_0 = \langle a_0, b_0 \rangle$ ,  $\mathbf{p}_1 = \langle a_1, b_1 \rangle$ , then we would just write this as  $(\mathbf{p}_0, \mathbf{p}_1)$ .

Also, injection will be denoted  $f : A \hookrightarrow B$ , surjection by  $f : A \twoheadrightarrow B$ , and bijection by  $f : A \leftrightarrow B$ .

*Solution.* Let  $(X_{\text{sq}}, \mathcal{T}_{\text{sq}})$  be the lexicographically ordered square, and  $\mathcal{B}_{\text{sq}}$  be the canonical basis for  $\mathcal{T}_{\text{sq}}$ . Let  $\mathcal{T}_F$  be the relative topology on  $F$  inherited from  $X_{\text{sq}}$ . Also let  $(I, \mathcal{T}_I)$  be the interval  $(0, 1)$  together with the subspace topology inherited from  $\mathbb{R}_{\text{LL}}$ . Then we claim  $(F, \mathcal{T}_F)$  is “equivalent” to  $(I, \mathcal{T}_I)$  (denoted  $\mathcal{T}_F \cong \mathcal{T}_I$ ).<sup>1</sup>

*Proof:* By theorem 4.30,

$$\mathcal{B}_F = \{B \cap F \mid B \in \mathcal{B}_{\text{sq}}\}$$

is a basis for  $\mathcal{T}_F$ . Thus, to show  $\mathcal{T}_F \cong \mathcal{T}_I$ , it will suffice to characterize elements of  $\mathcal{B}_F$  (Claim 1) and put them in correspondence with a basis for  $\mathcal{T}_I$  (Claim 2).

*Claim 1:* Elements of  $\mathcal{B}_F$  are of the following forms:<sup>2</sup>

- (1)  $\emptyset$
- (2)  $\{\langle x, 1 \rangle \in F \mid a < x < b\}$  (where  $a, b \in (0, 1)$ )
- (3)  $\{\langle x, 1 \rangle \in F \mid a \leq x < b\}$  (where  $a, b \in (0, 1)$ )

*Proof of Claim 1:* Let  $B_F \in \mathcal{B}_F$  be arbitrary. Then  $\exists B_{\text{sq}} \in \mathcal{B}_{\text{sq}}$  such that  $B_F = B_{\text{sq}} \cap F$ . Let  $\mathbf{0} = \langle 0, 0 \rangle$  and  $\mathbf{1} = \langle 1, 1 \rangle$ . By definition of  $\mathcal{B}_{\text{sq}}$ ,  $B_{\text{sq}}$  is of one of the following forms:

<sup>1</sup>Note: Technically, the formal term to use here is “homeomorphic.” Homeomorphism is the notion of a topological “equivalence,” or in the language of Category Theory, “isomorphism in the category of topological spaces”. This won’t be covered formally until section 8.3, but the idea is that homeomorphisms are bijections preserving open sets. This property of “preserving open sets” will become our topological definition of continuity, and in this context, a homeomorphism will be a continuous function with a continuous inverse.

<sup>2</sup>Note, (2) and (3) can also be written as  $(a, b) \times \{1\}$  and  $[a, b) \times \{1\}$ , respectively

(a)  $\left[ \mathbf{0}, \langle a_1, b_1 \rangle \right),$

(b)  $\left( \langle a_0, b_0 \rangle, \langle a_1, b_1 \rangle \right),$

(c)  $\left( \langle a_0, b_0 \rangle, \mathbf{1} \right]$

We proceed by casework.

(a) Suppose  $B_{\text{sq}} = \left[ \mathbf{0}, \langle a_1, b_1 \rangle \right)$ . Then

$$B_{\text{sq}} \cap F = \left\{ \langle x, 1 \rangle \in X_{\text{sq}} \mid \mathbf{0} < \langle x, 1 \rangle < \langle a_1, b_1 \rangle \right\}.$$

We will consider the cases of  $\langle a_1, b_1 \rangle \leq \langle 0, 1 \rangle$  and  $\langle 0, 1 \rangle < \langle a_1, b_1 \rangle$ . Note that these cases are disjoint and exhaustive.

i) If  $\langle a_1, b_1 \rangle \leq \langle 0, 1 \rangle$ ,  $F \cap B = \emptyset$  (see diagram). This is of form (1).<sup>3</sup>

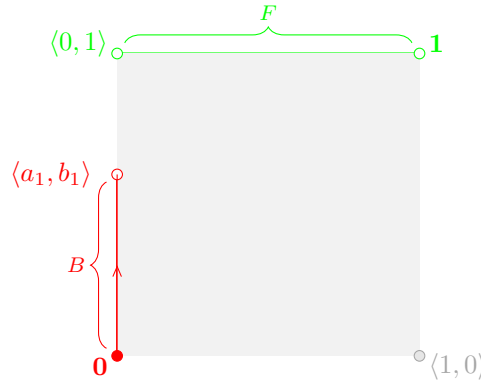


Figure 1:  $F \cap B = \emptyset$ .

ii) Suppose  $\langle 0, 1 \rangle < \langle a_1, b_1 \rangle$ . Then  $F \cap B = \{ \langle x, 1 \rangle \in X_{\text{sq}} \mid x \in (0, a_1) \} = (0, a_1) \times \{1\}$  (see diagram).<sup>4</sup> This is of form (2).

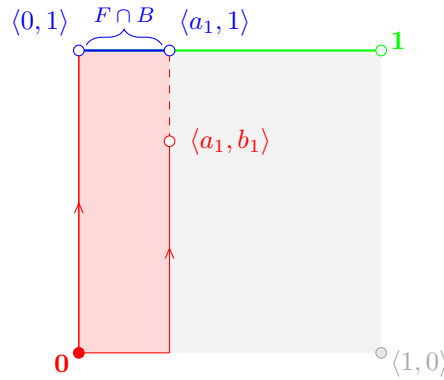


Figure 2:  $F \cap B = (0, a_1) \times \{1\}$ .

<sup>3</sup>Note this holds even if  $\langle a_1, b_1 \rangle = \langle 0, 1 \rangle$ , because the green endpoint is non-inclusive.

<sup>4</sup>Note that  $\langle 0, 1 \rangle \notin F \cap B$  by definition of  $F$ .

hence, if  $B_{\text{sq}}$  is of form (a), then  $B_{\text{sq}} \cap F$  is of the desired form. ✓

- (b) Now, suppose  $B_{\text{sq}} = (\langle a_0, b_0 \rangle, \langle a_1, b_1 \rangle)$ . Omitting the details, we have the following cases: (i)  $a_0 = a_1$ , (ii)  $a_0 \neq a_1$  and  $b_0 = 1$ , and (iii)  $a_0 \neq a_1$  and  $b_0 \neq 1$ .

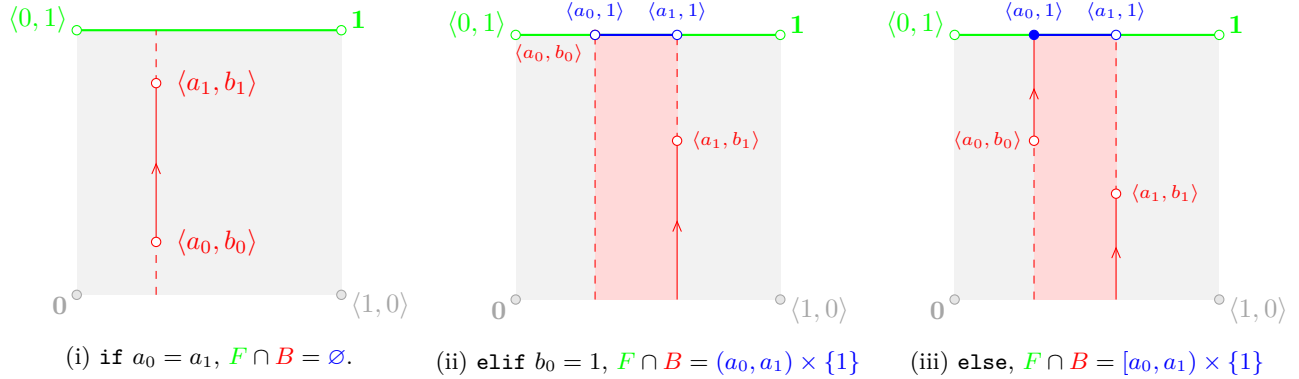


Figure 3: Subcases

these have the forms (1), (2), and (3), respectively.

- (c) Finally, suppose  $B_{\text{sq}}$  has the form  $(\langle a_0, b_0 \rangle, \mathbf{1}]$ . Then we have the following cases: (i)  $a_0 = 1$ , (ii)  $a_0 < 1$  and  $b_0 = 1$ , and (iii)  $a_0 < 1$  and  $b_0 < 1$ . Note that these cases are disjoint and exhaustive.

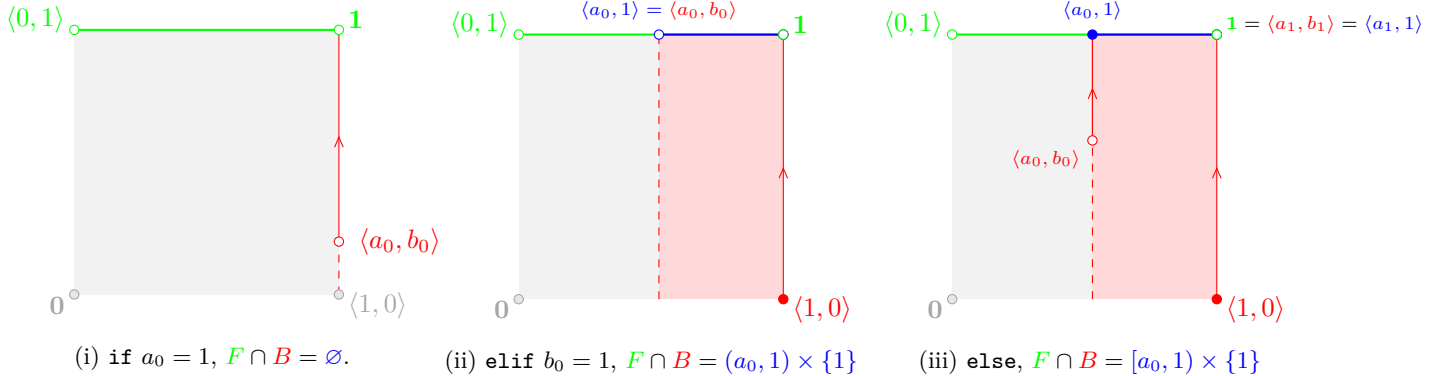


Figure 4: Yet more subcases

which are of the forms (1), (2), and (3), respectively.

Since cases (a) – (c) are exhaustive, This proves the claim. (*Phew...finally!*)

**Note:** you can stop here if you're doing your rewrite, as long as you just make a brief note about the similarity of  $\mathcal{B}_F$  to a basis for  $\mathcal{I}_I$ .

*Claim 2:* We can establish a natural bijection between  $\mathcal{B}_F$  and a basis  $\mathcal{B}_I$  for  $\mathcal{I}_I$ , and extend this to a function  $f : F \hookrightarrow I$  that respects the topologies  $\mathcal{F}_F, \mathcal{I}_I$ .

*Proof of Claim 2:* Let  $\mathcal{B}_I$  be given by

$$\mathcal{B}_I = \{\emptyset\} \cup \{(a, b) \subset (0, 1)\} \cup \{[a, b) \subset (0, 1)\}.$$

Observe that  $\mathcal{B}_I$  is indeed a basis for  $\mathcal{T}_I$ .<sup>5</sup> Thus,  $f : \mathcal{B}_I \hookrightarrow \mathcal{B}_F$  defined by

$$f(B_I) = \begin{cases} \emptyset & \text{if } B_I = \emptyset \\ B_I \times \{1\} & \text{otherwise} \end{cases}$$

is a bijection with inverse given by

$$f^{-1}(B_F) = \pi_x(B_F) \quad (\text{where } \pi_x \text{ is the canonical projection}).$$

We want to show  $\forall U_I \in \mathcal{T}_I$ ,  $f(U_I) \in \mathcal{T}_F$ , and  $\forall V_F \in \mathcal{T}_F$ ,  $f^{-1}(V_F) \in \mathcal{T}_I$ . Let  $U_I \in \mathcal{T}_I$  be arbitrary. Then since  $\mathcal{B}_I$  is a basis, there exists a collection of basis sets  $\{B_{I,\alpha}\}_{\alpha \in \lambda} \subset \mathcal{B}_I$  such that

$$\bigcup_{\alpha \in \lambda} B_{I,\alpha} = U_I.$$

Hence

$$\begin{aligned} f(U_I) &= f\left(\bigcup_{\alpha \in \lambda} B_{I,\alpha}\right) && \text{(substituting for } U_I) \\ &= \bigcup_{\alpha \in \lambda} f(B_{I,\alpha}) && \text{(prop. of } \cup) \end{aligned}$$

For each  $\alpha \in \lambda$ ,  $f(B_{I,\alpha}) \in \mathcal{B}_F$  (by definition of  $f$ ). Hence,  $\{B_{F,\alpha}\}_{\alpha \in \lambda}$  defined by

$$B_{F,\alpha} = f(B_{I,\alpha}) \quad (\alpha \in \lambda)$$

is a subset of  $\mathcal{B}_F$ . It follows that

$$f(U_I) = \bigcup_{\alpha \in \lambda} B_{F,\alpha}$$

is open in  $(F, \mathcal{T}_F)$ . The proof for arbitrary  $V_F \in \mathcal{T}_F$  follows analogously.

Hence,  $f$  respects the topologies on  $F, I$ , so we have

$$(F, \mathcal{T}_F) \cong (I, \mathcal{T}_I)$$

as desired. ■

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<sup>5</sup>In particular, the rightmost term is the collection of standard basis sets for  $\mathcal{T}_I$ , and the left two terms can be obtained by closure under arbitrary union.

**4.41.** Let  $\mathbb{R}^\omega$  be the countable product of copies of  $\mathbb{R}$ . So every point in  $\mathbb{R}^\omega$  is a sequence  $x_1, x_2, x_3, \dots$ . Let  $A \subset \mathbb{R}^\omega$  be the set consisting of all points with only positive coordinates. Show that in the product topology,  $\mathbf{o} = (0, 0, 0, \dots)$  is a limit point of the set  $A$ , and there is a sequence of points in  $A$  converging to  $\mathbf{o}$ . Then show that in the box topology,  $\mathbf{o} = (0, 0, 0, \dots)$  is a limit point of the set  $A$ , but there is no sequence of points in  $A$  converging to  $\mathbf{o}$ .

**Notational Note:** Let  $A$  be a subset of a topological space. Then we will use  $\mathcal{L}(A)$  to denote the limit points of  $A$ .

*Solution.* Let  $\mathcal{T}_{\text{prod}}$  be the product topology, and let  $\mathcal{T}_{\text{box}}$  be the box topology. Denote their corresponding bases by  $\mathcal{B}_{\text{prod}}, \mathcal{B}_{\text{box}}$ . We first show that in either topology,  $\mathbf{o} \in \mathcal{L}(A)$ , and then show the results about sequences.

- (1) The following proof works in both  $\mathcal{T}_{\text{prod}}$  and  $\mathcal{T}_{\text{box}}$ .<sup>6</sup> Hence, let  $\mathcal{T}_\star$  refer to either one of them, and let  $\mathcal{B}_\star$  the corresponding basis.<sup>7</sup>

Let  $U \in \mathcal{T}_\star$  such that  $\mathbf{o} \in U$ . Then since  $\mathcal{B}_\star$  is a basis, there exists  $\{B_\alpha\}_{\alpha \in \lambda} \subset \mathcal{B}_\star$  such that

$$U = \bigcup_{\alpha \in \lambda} B_\alpha.$$

Observe that  $\mathbf{o} \in U$  implies there exists  $\alpha_0 \in \lambda$  such that  $\mathbf{o} \in B_{\alpha_0}$ . By definition of  $\mathcal{B}_\star$ , there exists  $\{U_i\}_{i \in \mathbb{N}} \subset \mathcal{T}_{\text{std}}$  such that

$$B_{\alpha_0} = \prod_{i \in \mathbb{N}} U_i.$$

And so  $\mathbf{o} \in B_{\alpha_0}$  implies  $0 \in U_i$  for each  $i$ . Finally, by definition of  $\mathcal{T}_{\text{std}}$ , there exists  $\{(a_i, b_i)\}_{i \in \mathbb{N}}$  such that

■

<sup>6</sup>This is because  $\mathcal{T}_{\text{box}}$  is *finer* than  $\mathcal{T}_{\text{prod}}$ , and the proof works in  $\mathcal{T}_{\text{box}}$ .

<sup>7</sup>Formally, “let  $\mathcal{T}_\star \in \{\mathcal{T}_{\text{prod}}, \mathcal{T}_{\text{box}}\}$  be arbitrary” — but this level of formality might distract the reader more than it helps, so I omit it.

**5.1.** A space  $(X, \mathcal{T})$  is  $T_1$  if and only if every point in  $X$  is a closed set.

*Solution.*

■

**5.5.** Show that  $\mathbb{R}_{LL}$  is normal.

*Solution.*

■

**5.9.** A topological space  $X$  is normal if and only if for each closed set  $A$  in  $X$  and open set  $U$  containing  $A$  there exists an open set  $V$  such that  $A \subset V$ , and  $\overline{V} \subset U$ .

*Solution.*

■