| Problem | 5.29 | 5.32 | 6.6 | 6.11 | 6.18 | Total |
|---|---|---|---|---|---|---|
| Points | | | | | | |

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Math 147
HW 6 Solutions
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**5.29 (The Normality Lemma).** Let $A$ and $B$ be subsets of a topological space $X$ and let $\{U_i\}_{i\in\mathbb{N}}$ and $\{V_i\}_{i\in\mathbb{N}}$ be two collections of open sets such that
(1) $A \subset \bigcup_{i\in\mathbb{N}} U_i$
(2) $B \subset \bigcup_{i\in\mathbb{N}} V_i$
(3) For each $i \in \mathbb{N}$, $\overline{U_i} \cap B = \varnothing$ and $\overline{V_i} \cap A = \varnothing$.
Then there exist open sets $U$ and $V$ such that $A \subset U$, $B \subset V$, and $U \cap V = \varnothing$.

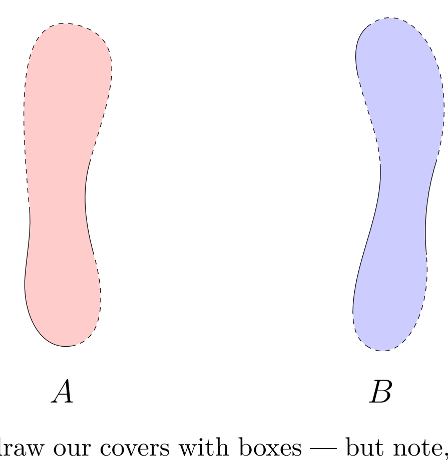
Before a solution, I'll give some intuition on how we might arrive at the candidate $U, V$ that work.

Let's think about what we're given. We have $\{U_i\}_{i\in\mathbb{N}}$, and $\{V_i\}_{i\in\mathbb{N}}$ as defined above, and we want to use them to construct $U, V$ satisfying the given constraints. It seems like it'd be straightforward to satisfy $A \subset U$, $B \subset V$ — they look like they'll probably fall directly out of the conditions. $U \cap V = \varnothing$ is harder, since we're given no direct information about $U_i \cap V_j$. Hence, we'll pick the following as our general approach:

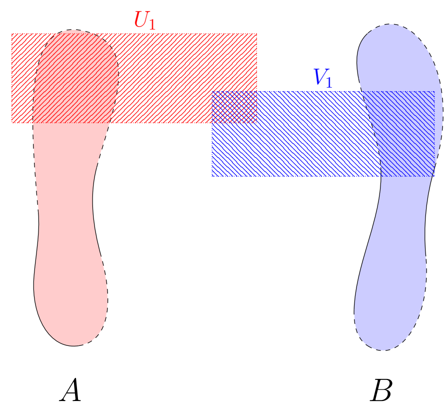
(1) Think about what
$$\tilde{U} = \bigcup_{i\in\mathbb{N}} U_i \qquad \tilde{V} = \bigcup_{i\in\mathbb{N}} V_i$$
look like. In particular, we'll focus on the conditions that break/make $\tilde{U}$ and $\tilde{V}$ *not* work as choices of $U, V$. Then,

(2) We'll see if we can find a clever way to remove the parts of $U_i$ and $V_i$ that cause problems. If all goes right, we'll find sequences $\{U_i'\}_{i\in\mathbb{N}}$, $\{V_i'\}_{i\in\mathbb{N}}$ whose terms can be unioned to get $U, V$.

Depict our two sets $A, B$ as follows:



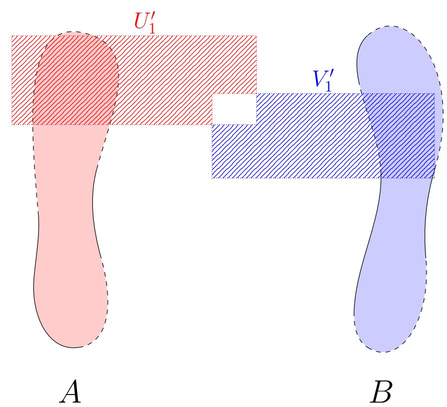

To make the TikZ easier, I'll draw our covers with boxes — but note, in general they could be blobby. Anyways, we now draw $U_1, V_1$:



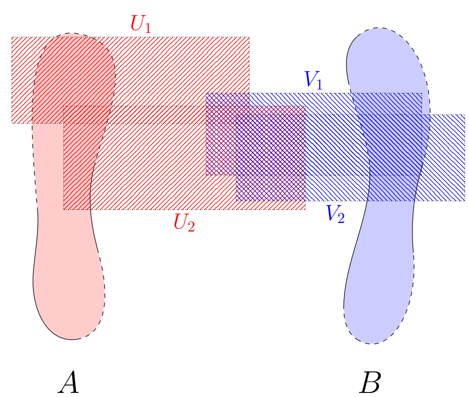

We need to be sure that in our construction of $U$, we don't include any of the $V_i$. We want to use the $U_i$, but as we can see, they might intersect the $V_i$. The fix? Remove the parts that cause problems. We define

$$U_1' = U_1 - \overline{V_1} \qquad\qquad V_1' = V_1 - \overline{U_1}$$

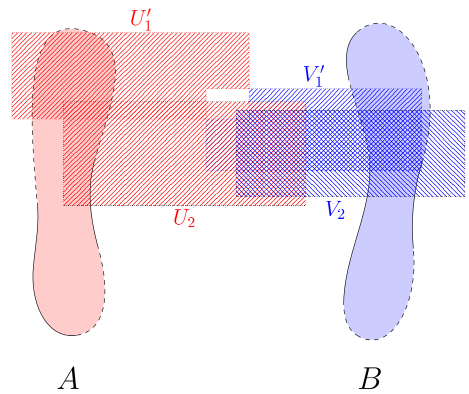
which yields




Now, we consider $n = 2$:




We already know how to rectify $U_1, V_1$:

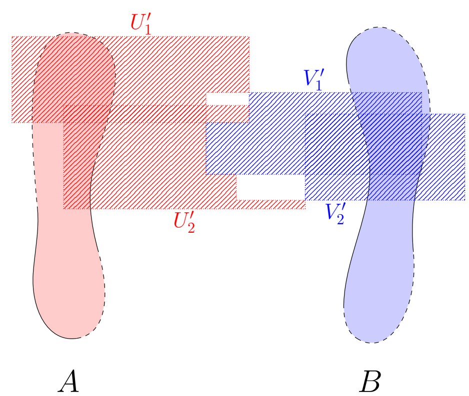



but we still have a problem now. Our addition of $U_2, V_2$ is complicating the situation! Namely, we have $U_2 \cap V_1' \neq \varnothing$, $U_2 \cap V_2 \neq \varnothing$.[a] We *don't* want to go back and edit our definitions for $U_1', V_1'$ — if we were to take this approach, we'd need to proceed similarly for $n = 2$, $n = 3$, etc. until eventually $U_1' = U_1 - \bigcup_{i \in \mathbb{N}} \overline{V_i}$, which could cause some problems.

Instead, we'll modify $U_2$ and $V_2$, leaving $U_1'$ and $V_1'$ untouched. Hence, we let

$$U_2' = U_2 - (\overline{V_1} \cup \overline{V_2}) \qquad V_2' = V_2 - (\overline{U_1} \cup \overline{U_2})$$

which gives us




From which we can see $(U_1' \cup U_2') \cap (V_1' \cup V_2') \neq \varnothing$. Thus, we conjecture that in the general case,

$$U_n' = U_n - \bigcup_{i=1}^{n} \overline{V_i} \qquad V_n' = V_n - \bigcup_{i=1}^{n} \overline{U_i}$$

will work.

[a] Note we could also have the analogous for $V_2 \cap U_1'$ if $V_2$ were a bit larger.

*Solution.*

■

**5.32.** Suppose a space $X$ is regular and has a countable basis. Then $X$ is normal.

*Solution.* Let $A, B$ be disjoint closed sets. ■

**6.6.** The space $2^{\mathbb{R}}$ is separable.

*Solution.* ■

**6.11.** Every uncountable set in a $2^{\text{nd}}$ countable space has a limit point.

*Solution.* ■

**6.18.** Suppose $x$ is a limit point of the set $A$ in a $1^{\text{st}}$ countable space $X$. Then there is a sequence of points $\{a_i\}_{i \in \mathbb{N}}$ that converges to $x$.

*Solution.* $\blacksquare$