Pairing Model

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$\hat{H}_0$commutation relations

Show that the unperturbed Hamiltonian commutes with the spin projection $\hat{S}_z$:

$$
\begin{aligned}
\left[H_0, \hat{S}_z\right] &= \\
&= \left(\sum_{p\sigma}(p-1)a^\dagger_{p\sigma}a_{p\sigma}\right)\left(\frac{1}{2}\sum_{p'\sigma'}\sigma' a^\dagger_{p'\sigma'}a_{p'\sigma'}\right) - \\
&\quad \left(\frac{1}{2}\sum_{p''\sigma''}\sigma'' a^\dagger_{p''\sigma''}a_{p''\sigma''}\right)\left(\sum_{p'''\sigma'''}(p'''-1)a^\dagger_{p'''\sigma'''}a_{p'''\sigma'''}\right) \\
&= \frac{1}{2}\sum_{p\sigma p'\sigma'}(p-1)\sigma'\hat{n}_{p\sigma}\hat{n}_{p'\sigma'} - \frac{1}{2}\sum_{p''\sigma''p'''\sigma'''}(p'''-1)\sigma''\hat{n}_{p''\sigma''}\hat{n}_{p'''\sigma'''}
\end{aligned}
$$

In this form it is apparent that these two sums are identical and cancel to give $[\hat{H}_0, \hat{S}_z] = 0$.
Show that the unperturbed Hamiltonian commutes with the total spin squared operator $\hat{S}^2$:

$$
\begin{aligned}
\left[\hat{H}_0, \hat{S}^2\right] &= \left[\hat{H}_0, \hat{S}_z^2\right] + \frac{1}{2}\left[\hat{H}_0, (\hat{S}_+\hat{S}_- + \hat{S}_-\hat{S}_+)\right] && (1) \\
&= \left[\hat{H}_0, \hat{S}_z\right]\hat{S}_z + \hat{S}_z\left[\hat{H}_0, \hat{S}_z\right] + \frac{1}{2}\left[\hat{H}_0, (\hat{S}_+\hat{S}_- + \hat{S}_-\hat{S}_+)\right] && (2) \\
&= \frac{1}{2}\left(\left[\hat{H}_0, \hat{S}_+\right]\hat{S}_- + \hat{S}_+\left[\hat{H}_0, \hat{S}_-\right] + \left[\hat{H}_0, \hat{S}_-\right]\hat{S}_+ + \hat{S}_-\left[\hat{H}_0, \hat{S}_+\right]\right) && (3)
\end{aligned}
$$

Now consider the commutators $\left[\hat{H}_0, \hat{S}_+\right]$ and $\left[\hat{H}_0, \hat{S}_-\right]$.

$$
\left[\hat{H}_0, \hat{S}_+\right] = \left(\sum_{p\sigma}(p-1)a^\dagger_{p\sigma}a_{p\sigma}\right)\left(\sum_{p'}a^\dagger_{p'+}a_{p'-}\right) - \left(\sum_{p''}a^\dagger_{p''+}a_{p''-}\right)\left(\sum_{p'''\sigma'''}(p'''-1)a^\dagger_{p'''\sigma'''}a_{p'''\sigma'''}\right) \tag{4}
$$

Using the anti-commutation relations of the creation and annihilation operators, we can show that the first product of sums in eq. 4 contains the second and therefore, $\left[\hat{H}_0, \hat{S}_+\right] = 0$. Focusing on the first product of sums in eq. 4:

$$\sum_{p\sigma p'}(p-1)\hat{a}^\dagger_{p\sigma}\hat{a}_{p\sigma}\hat{a}^\dagger_{p'+}\hat{a}_{p'-} \tag{5}$$

$$= \sum_{p\sigma p'}(p-1)\hat{a}^\dagger_{p\sigma}\left(\delta_{pp'}\delta_{\sigma+} - \hat{a}^\dagger_{p'+}\hat{a}_{p\sigma}\right)\hat{a}_{p'-} \tag{6}$$

$$= \sum_{p\sigma p'}(p-1)\left(\delta_{pp'}\delta_{\sigma+}\hat{a}^\dagger_{p\sigma}\hat{a}_{p'-} - \hat{a}^\dagger_{p\sigma}\hat{a}^\dagger_{p'+}\hat{a}_{p\sigma}\hat{a}_{p'-}\right) \tag{7}$$

$$= \sum_{p\sigma p'}(p-1)\left(\delta_{pp'}\delta_{\sigma+}\hat{a}^\dagger_{p\sigma}\hat{a}_{p'-} - \hat{a}^\dagger_{p'+}\hat{a}^\dagger_{p\sigma}\hat{a}_{p'-}\hat{a}_{p\sigma}\right) \tag{8}$$

$$= \sum_{p\sigma p'}(p-1)\left(\delta_{pp'}\delta_{\sigma+}\hat{a}^\dagger_{p\sigma}\hat{a}_{p'-} - \hat{a}^\dagger_{p'+}\left(\delta_{pp'}\delta_{\sigma-} - \hat{a}_{p'-}\hat{a}^\dagger_{p\sigma}\right)\hat{a}_{p\sigma}\right) \tag{9}$$

$$= \sum_{p\sigma p'}(p-1)\left(\delta_{pp'}\delta_{\sigma+}\hat{a}^\dagger_{p\sigma}\hat{a}_{p'-} - \delta_{pp'}\delta_{\sigma-}\hat{a}^\dagger_{p'+}\hat{a}_{p\sigma} + \hat{a}^\dagger_{p'+}\hat{a}_{p'-}\hat{a}^\dagger_{p\sigma}\hat{a}_{p\sigma}\right) \tag{10}$$

$$= \sum_{p\sigma p'}(p-1)\left(\hat{a}^\dagger_{p'+}\hat{a}_{p'-}\hat{a}^\dagger_{p\sigma}\hat{a}_{p\sigma}\right) \tag{11}$$

Comparing the results of this calculation with the second product of sums in eq. 4, which, reordering indicies, we can write as

$$\begin{aligned}&\sum_{p''p'''\sigma}(p'''-1)\left(\hat{a}^\dagger_{p''+}\hat{a}_{p''-}\hat{a}^\dagger_{p'''\sigma'''}\hat{a}_{p'''\sigma'''}\right)\\ &= \sum_{p\sigma p'}(p'-1)\left(\hat{a}^\dagger_{p+}\hat{a}_{p-}\hat{a}^\dagger_{p'\sigma'}\hat{a}_{p'\sigma'}\right)\end{aligned}$$

we notice that the two terms cancel leaving $\left[\hat{H}_0, \hat{S}_+\right] = 0$.

Using the same procedure, we can show that $\left[\hat{H}_0, \hat{S}_-\right] = 0$. With these results, we find that $\left[\hat{H}_0, \hat{S}^2\right] = 0$.

$\hat{V}$ commutation relations

Show that $\hat{V}$ commutes with $\hat{S}_z$:

$$\begin{aligned}\left[\hat{V}, \hat{S}_z\right] &= \left(\sum_{qs}(-g)\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+}\right)\left(\frac{1}{2}\sum_{p\sigma}\sigma a^\dagger_{p\sigma}a_{p\sigma}\right)\\ &\quad - \left(\frac{1}{2}\sum_{p'\sigma'}\sigma' a^\dagger_{p'\sigma'}a_{p'\sigma'}\right)\left(\sum_{q's'}(-g)\hat{a}^\dagger_{q'+}\hat{a}^\dagger_{q'-}\hat{a}_{s'-}\hat{a}_{s'+}\right)\end{aligned} \tag{12}$$

The first term may be re-written:

$$\frac{(-g)}{2}\sum_{qsp\sigma}\sigma\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+}\hat{a}^\dagger_{p\sigma}\hat{a}_{p\sigma}$$

$$\frac{(-g)}{2}\sum_{qsp\sigma}\sigma\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s-}\left(\delta_{sp}\delta_{+\sigma}-\hat{a}^\dagger_{p\sigma}\hat{a}_{s+}\right)\hat{a}_{p\sigma}$$

$$\frac{(-g)}{2}\sum_{qsp\sigma}\left(\delta_{sp}\delta_{+\sigma}\sigma\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{p\sigma}-\sigma\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}^\dagger_{p\sigma}\hat{a}_{s+}\hat{a}_{p\sigma}\right)$$

$$\frac{(-g)}{2}\sum_{qsp\sigma}\left(\delta_{sp}\delta_{+\sigma}\sigma\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{p\sigma}-\sigma\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\left(\delta_{sp}\delta_{-\sigma}-\hat{a}^\dagger_{p\sigma}\hat{a}_{s-}\right)\hat{a}_{s+}\hat{a}_{p\sigma}\right)$$

$$\frac{(-g)}{2}\sum_{qsp\sigma}\left(\delta_{sp}\delta_{+\sigma}\sigma\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{p\sigma}-\delta_{sp}\delta_{-\sigma}\sigma\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s+}\hat{a}_{p\sigma}+\sigma\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}^\dagger_{p\sigma}\hat{a}_{s-}\hat{a}_{s+}\hat{a}_{p\sigma}\right)$$

$$\frac{(-g)}{2}\sum_{qsp\sigma}\left((+)\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+}-(-)\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s+}\hat{a}_{s-}+\sigma\hat{a}^\dagger_{p\sigma}\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{p\sigma}\hat{a}_{s-}\hat{a}_{s+}\right)$$

$$\frac{(-g)}{2}\sum_{qsp\sigma}\left(\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+}+\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s+}\hat{a}_{s-}+\sigma\hat{a}^\dagger_{p\sigma}\hat{a}^\dagger_{q+}\left(\delta_{qp}\delta_{-\sigma}-\hat{a}_{p\sigma}\hat{a}^\dagger_{q-}\right)\hat{a}_{s-}\hat{a}_{s+}\right)$$

$$\frac{(-g)}{2}\sum_{qsp\sigma}\left(\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+}+\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s+}\hat{a}_{s-}+\delta_{qp}\delta_{-\sigma}\sigma\hat{a}^\dagger_{p\sigma}\hat{a}^\dagger_{q+}\hat{a}_{s-}\hat{a}_{s+}-\sigma\hat{a}^\dagger_{p\sigma}\hat{a}^\dagger_{q+}\hat{a}_{p\sigma}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+}\right)$$

$$\frac{(-g)}{2}\sum_{qsp\sigma}\left(\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+}+\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s+}\hat{a}_{s-}+(-)\hat{a}^\dagger_{q-}\hat{a}^\dagger_{q+}\hat{a}_{s-}\hat{a}_{s+}-\sigma\hat{a}^\dagger_{p\sigma}\hat{a}^\dagger_{q+}\hat{a}_{p\sigma}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+}\right)$$

$$\frac{(-g)}{2}\sum_{qsp\sigma}\left(\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+}+\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s+}\hat{a}_{s-}-\hat{a}^\dagger_{q-}\hat{a}^\dagger_{q+}\hat{a}_{s-}\hat{a}_{s+}-\sigma\hat{a}^\dagger_{p\sigma}\left(\delta_{qp}\delta_{+\sigma}-\hat{a}_{p\sigma}\hat{a}^\dagger_{q+}\right)\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+}\right)$$

$$\frac{(-g)}{2}\sum_{qsp\sigma}\left(\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+}+\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s+}\hat{a}_{s-}-\hat{a}^\dagger_{q-}\hat{a}^\dagger_{q+}\hat{a}_{s-}\hat{a}_{s+}-\delta_{qp}\delta_{+\sigma}\sigma\hat{a}^\dagger_{p\sigma}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+}+\sigma\hat{a}^\dagger_{p\sigma}\hat{a}_{p\sigma}\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+}\right)$$

$$\frac{(-g)}{2}\sum_{qsp\sigma}\left(\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+}+\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s+}\hat{a}_{s-}-\hat{a}^\dagger_{q-}\hat{a}^\dagger_{q+}\hat{a}_{s-}\hat{a}_{s+}-\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+}+\sigma\hat{a}^\dagger_{p\sigma}\hat{a}_{p\sigma}\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+}\right)$$

$$\frac{(-g)}{2}\sum_{qsp\sigma}\left(\sigma\hat{a}^\dagger_{p\sigma}\hat{a}_{p\sigma}\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+}\right)$$

Comparing this result to the second product of sums in eq. 12, which may be re-written:

$$\frac{+g}{2}\sum_{p\sigma qs}\sigma\hat{a}^\dagger_{p\sigma}\hat{a}_{p\sigma}\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+}$$

the terms cancel so that $\left[\hat{V},\hat{S}_z\right]=0$.

Show that $\hat{V}$ commutes with $\hat{S}^2$:

$$\begin{aligned}\left[\hat{V},\hat{S}^2\right] &= \left[\hat{V},\hat{S}_z^2\right] + \frac{1}{2}\left(\left[\hat{V},\hat{S}_+\hat{S}_-\right] + \left[\hat{V},\hat{S}_-\hat{S}_+\right]\right)\\ &= \hat{S}_z\left[\hat{V},\hat{S}_z\right] + \left[\hat{V},\hat{S}_z\right]\hat{S}_z + \frac{1}{2}\left(\hat{S}_+\left[\hat{V},\hat{S}_-\right] + \left[\hat{V},\hat{S}_+\right]\hat{S}_- + \hat{S}_-\left[\hat{V},\hat{S}_+\right] + \left[\hat{V},\hat{S}_-\right]\hat{S}_+\right)\\ &= \frac{1}{2}\left(\hat{S}_+\left[\hat{V},\hat{S}_-\right] + \left[\hat{V},\hat{S}_+\right]\hat{S}_- + \hat{S}_-\left[\hat{V},\hat{S}_+\right] + \left[\hat{V},\hat{S}_-\right]\hat{S}_+\right)\end{aligned}$$

Writing out $\left[\hat{V},\hat{S}_-\right]$:

$$\begin{aligned}\left[\hat{V},\hat{S}_-\right] = &\left(\sum_{qs}(-g)\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+}\right)\left(\sum_m a^\dagger_{m-}a_{m+}\right)\\ &- \left(\sum_{m'} a^\dagger_{m'-}a_{m'+}\right)\left(\sum_{q's'}(-g)\hat{a}^\dagger_{q'+}\hat{a}^\dagger_{q'-}\hat{a}_{s'-}\hat{a}_{s'+}\right)\end{aligned} \tag{13}$$

Re-writing the first prduct of sums using the anti-commutation relations and noting that in 18 and 22 the first terms give zero for the fermionic case since $\hat{a}_{s+}\ \hat{a}_{s+}$ would act to remove a particle from an empty state and $\hat{a}^\dagger_{q-}\ \hat{a}^\dagger_{q-}$ would act to add a particle to an occupied state.

$$(-g)\sum_{qsm}\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+}\hat{a}^\dagger_{m-}\hat{a}_{m+} \tag{14}$$

$$(-g)\sum_{qsm}\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}^\dagger_{m-}\hat{a}_{m+}\hat{a}_{s+} \tag{15}$$

$$(-g)\sum_{qsm}\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\left(\delta_{sm} - \hat{a}^\dagger_{m-}\hat{a}_{s-}\right)\hat{a}_{m+}\hat{a}_{s+} \tag{16}$$

$$(-g)\sum_{qsm}\delta_{sm}\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{m+}\hat{a}_{s+} - \hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}^\dagger_{m-}\hat{a}_{s-}\hat{a}_{m+}\hat{a}_{s+} \tag{17}$$

$$(-g)\sum_{qsm}\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s+}\hat{a}_{s+} - \hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}^\dagger_{m-}\hat{a}_{s-}\hat{a}_{m+}\hat{a}_{s+} \tag{18}$$

$$(-g)\sum_{qsm} -\hat{a}^\dagger_{m-}\hat{a}^\dagger_{q+}\hat{a}_{m+}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+} \tag{19}$$

$$(-g)\sum_{qsm} -\hat{a}^\dagger_{m-}\left(\delta_{qm} - \hat{a}_{m+}\hat{a}^\dagger_{q+}\right)\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+} \tag{20}$$

$$(-g)\sum_{qsm} -\delta_{qm}\hat{a}^\dagger_{m-}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+} + \hat{a}^\dagger_{m-}\hat{a}_{m+}\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+} \tag{21}$$

$$(-g)\sum_{qsm} -\hat{a}^\dagger_{q-}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+} + \hat{a}^\dagger_{m-}\hat{a}_{m+}\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+} \tag{22}$$

$$(-g)\sum_{qsm}\hat{a}^\dagger_{m-}\hat{a}_{m+}\hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\hat{a}_{s-}\hat{a}_{s+} \tag{23}$$

Comparing the result in 23 to the second product of sums in eq. 13, re-written here as

$$g \sum_{mqs} \hat{a}_{m-}^\dagger \hat{a}_{m+}^\dagger \hat{a}_{q+}^\dagger \hat{a}_{q-}^\dagger \hat{a}_{s-} \hat{a}_{s+}$$

we see that $\left[\hat{V}, \hat{S}_-\right] = 0$. Using the same procedure, we can show that $\left[\hat{V}, \hat{S}_+\right] = 0$ and conclude that $\left[\hat{V}, \hat{S}^2\right] = 0$.

Pair creation and annihilation operators

We consider the system with total spin $S = 0$ (no broken pairs) and define the pair creation and annihilation operators.

$$\begin{aligned} \hat{P}_p^+ &= \hat{a}_{p+}^\dagger \hat{a}_{p-}^\dagger \\ \hat{P}_p^- &= \hat{a}_{p-} \hat{a}_{p+} \end{aligned}$$

and the full Hamiltonian

$$\hat{H} = \sum_{p\sigma} (p-1) \hat{a}_{p\sigma}^\dagger \hat{a}_{p\sigma} - g \sum_{qs} \hat{P}_q^+ \hat{P}_s^- \tag{24}$$

Show that $\hat{H}$ commutes with the product of the pair creation and pair annihilation operators:

$$\left[\hat{H}, \hat{P}_r^+ \hat{P}_r^-\right] = \left[\hat{H}_0, \hat{P}_r^+ \hat{P}_r^-\right] + \left[\hat{V}, \hat{P}_r^+ \hat{P}_r^-\right] \tag{25}$$

Taking the first commutator in eq. 25:

$$\left[\hat{H}_0, \hat{P}_r^+ \hat{P}_r^-\right] = \left(\sum_{p\sigma} (p-1) a_{p\sigma}^\dagger a_{p\sigma}\right) \left(\hat{a}_{r+}^\dagger \hat{a}_{r-}^\dagger \hat{a}_{r-} \hat{a}_{r+}\right) - \left(\hat{a}_{r+}^\dagger \hat{a}_{r-}^\dagger \hat{a}_{r-} \hat{a}_{r+}\right) \left(\sum_{p\sigma} (p-1) a_{p\sigma}^\dagger a_{p\sigma}\right)$$

Re-writing the first product of sums to have the same form as the second product of sums leaves no remaining terms (26). Thus $\left[\hat{H}_0, \hat{P}_r^+ \hat{P}_r^-\right] = 0$.

$$
\begin{aligned}
&\left(\sum_{p\sigma}(p-1)a^\dagger_{p\sigma}a_{p\sigma}\right)\left(\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}\right)\\
&=\sum_{p\sigma}(p-1)\hat{a}^\dagger_{p\sigma}\hat{a}_{p\sigma}\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}\\
&=\sum_{p\sigma}(p-1)\hat{a}^\dagger_{p\sigma}\left(\delta_{pr}\delta_{\sigma+}-\hat{a}^\dagger_{r+}\hat{a}_{p\sigma}\right)\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}\\
&=\sum_{p\sigma}(p-1)\left(\delta_{pr}\delta_{\sigma+}\hat{a}^\dagger_{p\sigma}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}-\hat{a}^\dagger_{p\sigma}\hat{a}^\dagger_{r+}\hat{a}_{p\sigma}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}\right)\\
&=\sum_{p\sigma}(p-1)\left(\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}-\hat{a}^\dagger_{p\sigma}\hat{a}^\dagger_{r+}\hat{a}_{p\sigma}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}\right)\\
&=\sum_{p\sigma}(p-1)\left(\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}+\hat{a}^\dagger_{r+}\hat{a}^\dagger_{p\sigma}\hat{a}_{p\sigma}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}\right)\\
&=\sum_{p\sigma}(p-1)\left(\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}+\hat{a}^\dagger_{r+}\hat{a}^\dagger_{p\sigma}\left(\delta_{pr}\delta_{\sigma-}-\hat{a}^\dagger_{r-}\hat{a}_{p\sigma}\right)\hat{a}_{r-}\hat{a}_{r+}\right)\\
&=\sum_{p\sigma}(p-1)\left(\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}+\delta_{pr}\delta_{\sigma-}\hat{a}^\dagger_{r+}\hat{a}^\dagger_{p\sigma}\hat{a}_{r-}\hat{a}_{r+}-\hat{a}^\dagger_{r+}\hat{a}^\dagger_{p\sigma}\hat{a}^\dagger_{r-}\hat{a}_{p\sigma}\hat{a}_{r-}\hat{a}_{r+}\right)\\
&=\sum_{p\sigma}(p-1)\left(\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}+\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}+\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}^\dagger_{p\sigma}\hat{a}_{r-}\hat{a}_{r+}\hat{a}_{p\sigma}\right)\\
&=\sum_{p\sigma}(p-1)\left(\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}+\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}+\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\left(\delta_{pr}\delta_{\sigma-}-\hat{a}_{r-}\hat{a}^\dagger_{p\sigma}\right)\hat{a}_{r+}\hat{a}_{p\sigma}\right)\\
&=\sum_{p\sigma}(p-1)\left(\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}+\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}+\delta_{pr}\delta_{\sigma-}\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r+}\hat{a}_{p\sigma}-\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}^\dagger_{p\sigma}\hat{a}_{r+}\hat{a}_{p\sigma}\right)\\
&=\sum_{p\sigma}(p-1)\left(\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}+\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}+\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r+}\hat{a}_{r-}-\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}^\dagger_{p\sigma}\hat{a}_{r+}\hat{a}_{p\sigma}\right)\\
&=\sum_{p\sigma}(p-1)\left(\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}-\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}^\dagger_{p\sigma}\hat{a}_{r+}\hat{a}_{p\sigma}\right)\\
&=\sum_{p\sigma}(p-1)\left(\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}-\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\left(\delta_{pr}\delta_{\sigma+}-\hat{a}_{r+}\hat{a}^\dagger_{p\sigma}\right)\hat{a}_{p\sigma}\right)\\
&=\sum_{p\sigma}(p-1)\left(\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}-\delta_{pr}\delta_{\sigma+}\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{p\sigma}+\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}\hat{a}^\dagger_{p\sigma}\hat{a}_{p\sigma}\right)\\
&=\sum_{p\sigma}(p-1)\left(\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}-\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}+\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}\hat{a}^\dagger_{p\sigma}\hat{a}_{p\sigma}\right)\\
&=\sum_{p\sigma}(p-1)\left(\hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\hat{a}_{r-}\hat{a}_{r+}\hat{a}^\dagger_{p\sigma}\hat{a}_{p\sigma}\right)
\end{aligned}
\tag{26}
$$

Working out the second commutator from eq. 25:

$$
\begin{aligned}
\left[\hat{V},\hat{P}^+_r\hat{P}^-_t\right] &= (-g)\sum_{qs}\left[\hat{P}^+_q\hat{P}^-_s,\hat{P}^+_r\hat{P}^-_r\right] \\
&= \hat{P}^+_q\left[\hat{P}^-_s,\hat{P}^+_r\right]\hat{P}^-_r + \hat{P}^+_q\hat{P}^+_r\left[\hat{P}^-_s,\hat{P}^-_r\right] + \left[\hat{P}^+_q,\hat{P}^+_r\right]\hat{P}^-_s\hat{P}^-_r + \hat{P}^+_r\left[\hat{P}^+_q,\hat{P}^-_r\right]\hat{P}^-_s
\end{aligned} \tag{27}
$$

The commutation relations between pair creation and annihilation operators were found to be:

$$
\left[\hat{P}^+_p,\hat{P}^-_q\right] = \hat{a}^\dagger_{p+}\hat{a}^\dagger_{p-}\hat{a}_{q-}\hat{a}_{q+} - \hat{a}_{q-}\hat{a}_{q+}\hat{a}^\dagger_{p+}\hat{a}^\dagger_{p-}
$$

Re-writing the first term

$$
\begin{aligned}
&\hat{a}^\dagger_{p+}\hat{a}^\dagger_{p-}\hat{a}_{q-}\hat{a}_{q+} \\
&\hat{a}^\dagger_{p+}\left(\delta_{pq} - \hat{a}_{q-}\hat{a}^\dagger_{p-}\right)\hat{a}_{q+} \\
&\hat{a}^\dagger_{p+}\hat{a}_{p+} - \hat{a}^\dagger_{p+}\hat{a}_{q-}\hat{a}^\dagger_{p-}\hat{a}_{q+} \\
&\hat{a}^\dagger_{p+}\hat{a}_{p+} - \hat{a}_{q-}\hat{a}^\dagger_{p+}\hat{a}_{q+}\hat{a}^\dagger_{p-} \\
&\hat{a}^\dagger_{p+}\hat{a}_{p+} - \hat{a}_{q-}\left(\delta_{pq} - \hat{a}_{q+}\hat{a}^\dagger_{p+}\right)\hat{a}^\dagger_{p-} \\
&\hat{a}^\dagger_{p+}\hat{a}_{p+} - \hat{a}_{p-}\hat{a}^\dagger_{p-} + \hat{a}_{q-}\hat{a}_{q+}\hat{a}^\dagger_{p+}\hat{a}^\dagger_{p-}
\end{aligned}
$$

cancels the second leaving

$$
\left[\hat{P}^+_p,\hat{P}^-_q\right] = \hat{a}^\dagger_{p+}\hat{a}_{p+} - \hat{a}_{p-}\hat{a}^\dagger_{p-}
$$

Generalizing the form for the commutation relation between $\hat{P}^+_p$ and $\hat{P}^-_q$

$$
\begin{aligned}
&\left[\hat{P}^\pm_p,\hat{P}^\mp_q\right] = \pm(\hat{a}^\dagger_{p+}\hat{a}_{p+} - \hat{a}_{p-}\hat{a}^\dagger_{p-}) \\
&\left[\hat{P}^\pm_p,\hat{P}^\pm_q\right] = 0
\end{aligned} \tag{28}
$$

These results reduce the four terms in eq. 27 to two, which can be expanded taking into account that the pair creation and annihilation operators act to select terms with their specific state index from the sum over all states.

$$
\begin{aligned}
&\hat{P}^+_q\left[\hat{P}^-_s,\hat{P}^+_r\right]\hat{P}^-_r + \hat{P}^+_q\hat{P}^+_r\left[\hat{P}^-_s,\hat{P}^-_r\right] + \left[\hat{P}^+_q,\hat{P}^+_r\right]\hat{P}^-_s\hat{P}^-_r + \hat{P}^+_r\left[\hat{P}^+_q,\hat{P}^-_r\right]\hat{P}^-_s \\
&= \hat{P}^+_q\left[\hat{P}^-_s,\hat{P}^+_r\right]\hat{P}^-_r + \hat{P}^+_r\left[\hat{P}^+_q,\hat{P}^-_r\right]\hat{P}^-_s \\
&= \hat{a}^\dagger_{q+}\hat{a}^\dagger_{q-}\left(\hat{a}_{r-}\hat{a}^\dagger_{r-} - \hat{a}^\dagger_{r+}\hat{a}_{r+}\right)\hat{a}_{r-}\hat{a}_{r+} + \hat{a}^\dagger_{r+}\hat{a}^\dagger_{r-}\left(\hat{a}^\dagger_{r+}\hat{a}_{r-} - \hat{a}_{r-}\hat{a}^\dagger_{r-}\right)\hat{a}_{s-}\hat{a}_{s+} \\
&= 0
\end{aligned}
$$

since the first term contains two $\hat{a}_{r-}$ operators, the second term contains two $\hat{a}_{r+}$ operators, the third contains two $\hat{a}_{r+}^{\dagger}$ operators and the fourth contains two $\hat{a}_{r+}^{\dagger}$ operators, therefore, action of $\left[\hat{V}, \hat{P}_r^{+}\hat{P}_t^{-}\right]$ on any fermionic state produces 0. Combining this result with that of eq. 26 shows that the Hamiltonian commutes with the pair creation and annihilation operators.

The Hamiltonian Matrix

Restricting the effective Hilbert space to only the two lowest single-particle states (they are doubly degenerate) and considering only two particles, we can construct a Hamiltonian matrix in the basis of 2-particle Slater determinants (eq. 29).

$$\begin{aligned} \left|\Phi_1^{\mathrm{SD}}\right\rangle &= \hat{a}_{1+}^{\dagger}\hat{a}_{1-}^{\dagger}\left|0\right\rangle \\ \left|\Phi_2^{\mathrm{SD}}\right\rangle &= \hat{a}_{2+}^{\dagger}\hat{a}_{2-}^{\dagger}\left|0\right\rangle \end{aligned} \tag{29}$$

Furthermore, our system is considered to have no broken pairs and total spin $S = 0$. The matrix elements of the Hamiltonian can be calculated separately for the one- and two-body parts using Wick's theorem. For example, the one-body part of the first matrix element:

$$\begin{aligned} &\left\langle 0\right| \hat{a}_{1-}\hat{a}_{1+}\left(\sum_{p\sigma}(p-1)a_{p\sigma}^{\dagger}a_{p\sigma}\right)\hat{a}_{1+}^{\dagger}\hat{a}_{1-}^{\dagger}\left|0\right\rangle \\ &= \left\langle 0\right| \hat{a}_{1-}\hat{a}_{1+}\left(0\hat{a}_{1+}^{\dagger}\hat{a}_{1+} + 0\hat{a}_{1-}^{\dagger}\hat{a}_{1-} + 1\hat{a}_{2+}^{\dagger}\hat{a}_{2+} + 1\hat{a}_{2-}^{\dagger}\hat{a}_{2-}\right)\hat{a}_{1+}^{\dagger}\hat{a}_{1-}^{\dagger}\left|0\right\rangle \\ &= \left\langle 0\right| \hat{a}_{1-}\hat{a}_{1+}\left(0\hat{a}_{1+}^{\dagger}\hat{a}_{1+} + 0\hat{a}_{1-}^{\dagger}\hat{a}_{1-}\right)\hat{a}_{1+}^{\dagger}\hat{a}_{1-}^{\dagger}\left|0\right\rangle \\ &= 0 \end{aligned}$$

The two body part of the first matrix element:

$$\begin{aligned} &\left\langle 0\right| \hat{a}_{1-}\hat{a}_{1+}\left(\sum_{pq}(-g)\hat{a}_{p+}^{\dagger}\hat{a}_{p-}^{\dagger}\hat{a}_{q-}\hat{a}_{q+}\right)\hat{a}_{1+}^{\dagger}\hat{a}_{1-}^{\dagger}\left|0\right\rangle \\ &= \left\langle 0\right| \hat{a}_{1-}\hat{a}_{1+}(-g)\left(\hat{a}_{1+}^{\dagger}\hat{a}_{1-}^{\dagger}\hat{a}_{1-}\hat{a}_{1+} + \hat{a}_{1+}^{\dagger}\hat{a}_{1-}^{\dagger}\hat{a}_{2-}\hat{a}_{2+} + \hat{a}_{2+}^{\dagger}\hat{a}_{2-}^{\dagger}\hat{a}_{1-}\hat{a}_{1+} + \hat{a}_{2+}^{\dagger}\hat{a}_{2-}^{\dagger}\hat{a}_{2-}\hat{a}_{2+}\right)\hat{a}_{1+}^{\dagger}\hat{a}_{1-}^{\dagger}\left|0\right\rangle \\ &= \left\langle 0\right| \hat{a}_{1-}\hat{a}_{1+}(-g)\left(\hat{a}_{1+}^{\dagger}\hat{a}_{1-}^{\dagger}\hat{a}_{1-}\hat{a}_{1+}\right)\hat{a}_{1+}^{\dagger}\hat{a}_{1-}^{\dagger}\left|0\right\rangle \\ &= -g \end{aligned}$$

The second matrix element:

$$\begin{aligned} &\left\langle 0\right| \hat{a}_{1-}\hat{a}_{1+}\left(\sum_{p\sigma}(p-1)a_{p\sigma}^{\dagger}a_{p\sigma}\right)\hat{a}_{2+}^{\dagger}\hat{a}_{2-}^{\dagger}\left|0\right\rangle \\ &= \left\langle 0\right| \hat{a}_{1-}\hat{a}_{1+}\left(0\hat{a}_{1+}^{\dagger}\hat{a}_{1+} + 0\hat{a}_{1-}^{\dagger}\hat{a}_{1-} + 1\hat{a}_{2+}^{\dagger}\hat{a}_{2+} + 1\hat{a}_{2-}^{\dagger}\hat{a}_{2-}\right)\hat{a}_{2+}^{\dagger}\hat{a}_{2-}^{\dagger}\left|0\right\rangle \\ &= 0 \end{aligned}$$

$$
\begin{aligned}
&\langle 0|\, \hat{a}_{1-}\hat{a}_{1+} \left(\sum_{pq} (-g) \hat{a}^\dagger_{p+}\hat{a}^\dagger_{p-}\hat{a}_{q-}\hat{a}_{q+} \right) \hat{a}^\dagger_{2+}\hat{a}^\dagger_{2-}\, |0\rangle \\
&= \langle 0|\, \hat{a}_{1-}\hat{a}_{1+}(-g) \left(\hat{a}^\dagger_{1+}\hat{a}^\dagger_{1-}\hat{a}_{1-}\hat{a}_{1+} + \hat{a}^\dagger_{1+}\hat{a}^\dagger_{1-}\hat{a}_{2-}\hat{a}_{2+} + \hat{a}^\dagger_{2+}\hat{a}^\dagger_{2-}\hat{a}_{1-}\hat{a}_{1+} + \hat{a}^\dagger_{2+}\hat{a}^\dagger_{2-}\hat{a}_{2-}\hat{a}_{2+} \right) \hat{a}^\dagger_{2+}\hat{a}^\dagger_{2-}\, |0\rangle \\
&= \langle 0|\, \hat{a}_{1-}\hat{a}_{1+}(-g) \left(\hat{a}^\dagger_{1+}\hat{a}^\dagger_{1-}\hat{a}_{2-}\hat{a}_{2+} \right) \hat{a}^\dagger_{2+}\hat{a}^\dagger_{2-}\, |0\rangle \\
&= -g
\end{aligned}
$$

The full 2×2 matrix where d is the level spacing

$$
H = \begin{pmatrix} -g & -g \\ -g & 2d - g \end{pmatrix} \tag{30}
$$

This matrix is diagonalized to yield eigenenergies (with $d = 1$)

$$
E = 1 - g \pm \sqrt{1 + g^2} \tag{31}
$$

The ground state energy, $E_0 = 1 - g - \sqrt{1 + g^2}$, as g goes from -1 to 1 goes from ~ 0.5 to ~ -1.5. This suggests that when the two-body interaction is attractive (g positive), the one pair two level system has at least one bound state. Our shell model code (see Table 1) reproduces exactly the analytic result.

When we have four levels and four particles, the number of basis states is $\binom{4}{4/2} = 6$, and the 6 Slater determinants are

$$
\begin{aligned}
&\hat{a}^\dagger_{1+}\hat{a}^\dagger_{1-}\hat{a}^\dagger_{2+}\hat{a}^\dagger_{2-}\, |0\rangle \\
&\hat{a}^\dagger_{1+}\hat{a}^\dagger_{1-}\hat{a}^\dagger_{3+}\hat{a}^\dagger_{3-}\, |0\rangle \\
&\hat{a}^\dagger_{1+}\hat{a}^\dagger_{1-}\hat{a}^\dagger_{4+}\hat{a}^\dagger_{4-}\, |0\rangle \\
&\hat{a}^\dagger_{2+}\hat{a}^\dagger_{2-}\hat{a}^\dagger_{3+}\hat{a}^\dagger_{3-}\, |0\rangle \\
&\hat{a}^\dagger_{2+}\hat{a}^\dagger_{2-}\hat{a}^\dagger_{4+}\hat{a}^\dagger_{4-}\, |0\rangle \\
&\hat{a}^\dagger_{3+}\hat{a}^\dagger_{3-}\hat{a}^\dagger_{4+}\hat{a}^\dagger_{4-}\, |0\rangle
\end{aligned} \tag{32}
$$

Action of the Hamiltonian on each Slater determinant results in

$$
\begin{aligned}
\hat{H}\, |1 \pm 2\pm\rangle &= (2 - 2g)\, |1 \pm 2\pm\rangle - g\, |1 \pm 3\pm\rangle - g\, |1 \pm 4\pm\rangle - g\, |2 \pm 3\pm\rangle - g\, |2 \pm 4\pm\rangle + 0\, |3 \pm 4\pm\rangle \\
\hat{H}\, |1 \pm 3\pm\rangle &= g\, |1 \pm 2\pm\rangle + (4 - 2g)\, |1 \pm 3\pm\rangle - g\, |1 \pm 4\pm\rangle - g\, |2 \pm 3\pm\rangle + 0\, |2 \pm 4\pm\rangle - g\, |3 \pm 4\pm\rangle \\
\hat{H}\, |1 \pm 4\pm\rangle &= -g\, |1 \pm 2\pm\rangle - g\, |1 \pm 3\pm\rangle + (6 - 2g)\, |1 \pm 4\pm\rangle - 0\, |2 \pm 3\pm\rangle - g\, |2 \pm 4\pm\rangle - g\, |3 \pm 4\pm\rangle \\
\hat{H}\, |2 \pm 3\pm\rangle &= -g\, |1 \pm 2\pm\rangle - g\, |1 \pm 3\pm\rangle - g\, |1 \pm 4\pm\rangle + (6 - 2g)\, |2 \pm 3\pm\rangle - g\, |2 \pm 4\pm\rangle - g\, |3 \pm 4\pm\rangle \\
\hat{H}\, |2 \pm 4\pm\rangle &= -g\, |1 \pm 2\pm\rangle + 0\, |1 \pm 3\pm\rangle - g\, |1 \pm 4\pm\rangle - g\, |2 \pm 3\pm\rangle + (8 - 2g)\, |2 \pm 4\pm\rangle - g\, |3 \pm 4\pm\rangle \\
\hat{H}\, |3 \pm 4\pm\rangle &= 0\, |1 \pm 2\pm\rangle - g\, |1 \pm 3\pm\rangle - g\, |1 \pm 4\pm\rangle - g\, |2 \pm 3\pm\rangle - g\, |2 \pm 4\pm\rangle + (10 - 2g)\, |3 \pm 4\pm\rangle
\end{aligned}
$$

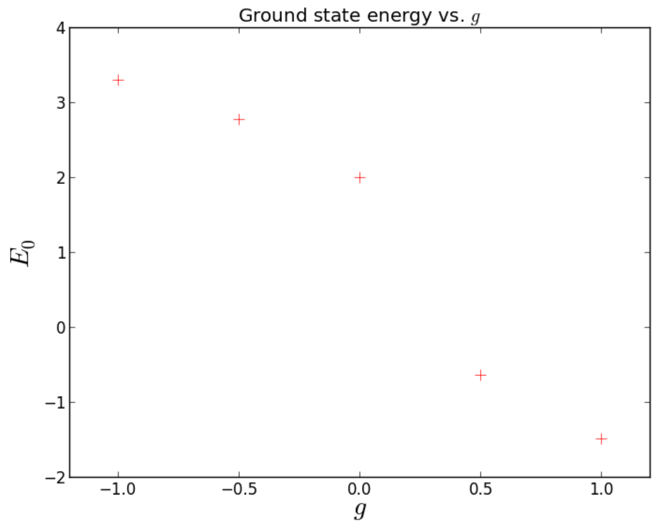

Figure 1: Ground state energies for five different values of g - calculated by diagonalizing the 6×6 Hamiltonian matrix of eq. 33. These results match our code's output given in Table 2.

where a short-hand notation $|p \pm q\pm\rangle$ is employed to represent a Slater determinant with one pair in level p and one in level q.
So the Hamiltonian matrix is

$$\begin{bmatrix} 2-2g & -g & -g & -g & -g & 0 \\ -g & 4-2g & -g & -g & 0 & -g \\ -g & -g & 6-2g & 0 & -g & -g \\ -g & -g & 0 & 6-2g & -g & -g \\ -g & 0 & -g & -g & 8-2g & -g \\ 0 & -g & -g & -g & -g & 10-2g \end{bmatrix} \tag{33}$$

We diagonalize this matrix numerically for $g = \{-1.0, -0.5, 0.0, 0.5, 1.0\}$ and ensure that our code's output, listed in Table 2, matches the results. This test ensures that our code correctly generates the Hamiltonian matrix. Similarly to the one pair two level case, we note that as g becomes negative (as the two-body interaction becomes repulsive), the system becomes unbound (see Figure 1).

g	E_{gs}
-1.0	0.59
-0.5	0.38
0.0	0.0
0.5	-0.62
1.0	-1.41

Table 1: Ground state energies as a function of interaction strength g for the case of one pair in 2 single particle levels.

g	E_{gs}
-1.0	3.30
-0.5	2.78
0.0	2.0
0.5	-0.64
1.0	-1.49

Table 2: Ground state energies as a function of interaction strength g for the case of two pairs in four levels.

g	E_{gs}(2,2)	E_{gs}(6,6)	E_{gs}(8,8)
-1.0	0	0	0
-0.5	0	0	0
0.0	0	0	0
0.5	-1	-6	-10
1.0	-2	-12	-20

Table 3: Ground state energies $E_{gs}(N_l, N_{part})$ for the case with the single-particle Hamiltonian removed. The energies are shown as a function of interaction strength g for the case of N_{part} particles in $N_l = \frac{\Omega}{2}$ single particle "levels."

When we remove the single particle piece of the Hamiltonian and keep only the two particle interaction, all particles are brought down to the same degenerate energy level, with degeneracy Ω and ground state energy

$$E_0 = -\frac{g}{4} n(\Omega - n + 2) \tag{34}$$

Our code reproduces the analytic result eq. 34 for $g > 0$ (an attractive two-body interaction). When g becomes negative, eq. 34 gives the highest possible energy level due to the two-body interaction and the ground state energy becomes 0.

As we then vary the interaction strength g we observe the following pattern: In the negative g case, the interaction is repulsive, and so the best the system can do is break even. In this case, the ground state energy is zero, regardless of g. When g becomes positive, however, the interaction is attractive, and as g becomes more and more positive, the interaction becomes increasingly attractive and the system correspondingly becomes more and more bound. We illustrate this for several cases in Table 3.

```
import numpy as np
from decimal import Decimal
import os
from scipy.misc import comb
from numpy import linalg as LA
import itertools

# bitCount() counts the number of bits set (not an optimal function)

def bitCount(int_type):
    """ Count bits set in integer """
    count = 0
    while(int_type):
        int_type &= int_type - 1
        count += 1
    return(count)

# allperms() generates a .jpg file displaying a composite image of all
# the WORST hairstyles from the '80s
#
# actually it generates all possible <Nbits> words as a list of strings

def allperms(Nbits):
  result = ["".join(seq) for seq in itertools.product("01", repeat=Nbits)]
  return result



def slaterdet(numLevels, numPairs):
# file = open("slater_dets.list", "w")
# file.write("This is my output! You have " + str(numParticles) + " particles! \n")
  # This is where you can do your calculation. You can nest the file.write line inside a loop, if you need to
# nLvls=numStates/2
# nPairs=numParticles/2

  numSDs=int(round(comb(numLevels, numPairs, exact=False)))
  SD = np.zeros([numSDs,numLevels])

  ap=allperms(numLevels)
  apmax=len(ap)
  # loop over generated list of <nLvls>-bit words and select ones with <nPairs> bits set
  # i.e. eliminate states that don't have the specified number of pairs
  idxSD=0
  idxLVL=0
  for i in range(apmax):
    state=int(ap[i],2)
    # if word has right number of pairs
    # we add a new SD
    if (bitCount(state) == numPairs):
      idxLVL=0
      for j in range(numLevels):
        SD[idxSD][idxLVL]=ap[i][j]
        idxLVL+=1
```

```
            idxSD+=1



#    file.close()
    return SD

def hamiltonian(numLevels, numPairs, dim_H, G,dE):
    # The following lines should give you a NumPy array, where each row corresponds to a Slater determinant
#    SDarray = np.genfromtxt('slater_dets.list-sample1')

    SDarray = slaterdet(numLevels, numPairs)
    b = slaterdet(numLevels, numPairs)
    print 'My array of Slater determinants is \n', SDarray

    # This next line should let you check that the data was imported correctly
    print 'It contains', np.size(SDarray,0), 'Slater determinants constructed from', np.size(SDarray,1), 'single particle states.'

    H = np.zeros([dim_H, dim_H])
    ne=0
    for i in range(dim_H):
            for q in range(numLevels):
                if SDarray[i][q]==1:
                    for p in range(numLevels):
                        if SDarray[i][p]==0:
                            b[i][q]=0
                            b[i][p]=1
                            for j in range(dim_H):
                                if np.array_equal(SDarray[j],b[i]):
#           print "SDarray,b: ", SDarray[j],b[i]
                                    H[i][j]=H[i][j]-G
                            b[i][p]=0
                            b[i][q]=1
        print(H*G)

        for i in range(dim_H):
            for j in range(numLevels):
                if SDarray[i][j]==1:
                    ne=ne+(numLevels-j-1)
            H[i][i]=H[i][i]+2*ne*dE-G*numPairs
            ne=0
        print H

    return H

if __name__ == '__main__':
    Nlevels = int(raw_input('Number of Levels = '))
#    Nstates = 2*Nlevels
    Npairs = int(raw_input('Number of Pairs = '))
    size_H = comb(Nlevels, Npairs, exact=False)
#    g=1
    dE=1 #0
#    h = hamiltonian(Nlevels, Npairs, size_H, g,dE)
#    EigValues, EigVectors = LA.eigh(h)
```

```
# permute = EigValues.argsort()
# print '\n For g =', g, 'the eigenvalues of the Hamiltonian are \n', EigValues[
    permute]
# print 'and the corresponding eigenvectors are \n', EigVectors[:,permute], '\n'
  for g in np.linspace(-1, 1, num=5):
#   slaterdet(Nstates, Nparticles)
#   h = hamiltonian(Nstates, Nparticles, size_H, g)
#   EigValues, EigVectors = LA.eigh(h)

    h = hamiltonian(Nlevels, Npairs, size_H, g,dE)
    EigValues, EigVectors = LA.eigh(h)

    permute = EigValues.argsort()
    print '\n For g =', g, 'the eigenvalues of the Hamiltonian are \n', EigValues[
   permute]
    print 'and the corresponding eigenvectors are \n', EigVectors[:,permute], '\n'
```

shell_model.py