Eigenvalues & Eigenvectors

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix of real numbers.

Definition

A non-zero vector
$$v = \begin{pmatrix} x \\ y \end{pmatrix}$$
is an eigenvector of the matrix A if there exists some real number λ such that
$$Av = \lambda v$$
We call λ the eigenvalue of A corresponding to v.

Example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

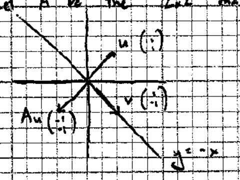
Consider $v = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ then Av

$$\underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} 4 \\ 4 \end{pmatrix}}_{v} = \underbrace{\begin{pmatrix} 12 \\ 12 \end{pmatrix}}_{Av} = \underbrace{3\begin{pmatrix} 4 \\ 4 \end{pmatrix}}_{\lambda v}$$

thus v is an eigenvector of A with eigenvalue $\lambda = 3$.

Example

Let A be the 2×2 matrix of reflection in the line $y = -x$.



$Av = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \lambda v$

$\hookrightarrow \lambda = 1$

$\leftarrow$ every point on the line is an eigenvector (except $0,0$)

$Au = \begin{pmatrix} -1 \\ -1 \end{pmatrix} = -u$, so u is an eigenvector, with eigenvalue $\lambda = -1$

Example

Find a matrix A that has no eigenvectors.

Answer: let A be the matrix of rotation about the origin through an angle θ where $(\theta \neq \pi, 0)$

Then A has no eigenvectors.

Theorem: Cayley-Hamilton theorem

Def^n the polynomial
$$P_A(\lambda) = \det(A - \lambda I)$$
is called the characteristic polynomial of A.

// determinant : $(ad - bc)$

Example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\begin{aligned}
P_A(\lambda) &= \det\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&= \det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} \\
&= (2-\lambda)(2-\lambda) - 1\cdot1 \\
&= \lambda^2 - 4\lambda + 4 - 1 \\
&= \lambda^2 - 4\lambda + 3
\end{aligned}$$

Note:
$$P_A(\lambda) = \lambda^2 - 4\lambda + 3$$
$$P_A(2) = 2^2 - 4(2) + 3 = -1$$

$$\begin{aligned}
P_A(A) &= A^2 - 4A + 3I \\
&= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - 4\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} - \begin{pmatrix} 8 & 4 \\ 4 & 8 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

Cayley-Hamilton Theorem

for any $n\times n$ matrix A, we have
$$P_A(A) = 0I$$