MA 190

Sun Tzu - Chinese Remainder Theorem

Problem - Woman carrying eggs
→ Knocked over by horse, horse crushes eggs.
rider offers to pay for the eggs
lady doesn't remember how many eggs she had, however she does remember that when she:
→ took them out 13 at a time, 3 left over.
→ took them out 14 at a time, 6 left over.
→ took them out 15 at a time, 9 left over.

What is the least number of eggs that she could've had in the basket?

Chinese remainder Theorem
→ find the smallest non-ve number x such that the following equations hold simultaneously:

$$\left.\begin{aligned} x &\equiv 3 \bmod 13 \\ x &\equiv 6 \bmod 14 \\ x &\equiv 9 \bmod 15 \end{aligned}\right\} (*)$$

→ there exists this number x

PROOF

let $a \equiv 14^{-1} \bmod 13$ — coprime → gcd is 1

$b \equiv 15^{-1} \bmod 13$

Wild guess:
take X as $3 \cdot 14 \cdot a \cdot 15 \cdot b$
→ $3 \cdot 1 \cdot 1 = 3$ → (because $a \cdot a^{-1} = 1$)

$X \equiv 3 \bmod 13$

$X \equiv 0 \bmod 14$ (because multiple of 14)

$X \equiv 0 \bmod 15$ (because multiple of 15)

let $c \equiv 13^{-1} \bmod 14$
$d \equiv 15^{-1} \bmod 14$

let $e \equiv 13^{-1} \bmod 15$
$f \equiv 14^{-1} \bmod 15$

Second guess:
take Y as $6 \cdot 13 \cdot c \cdot 15 \cdot d$

$Y \equiv 0 \bmod 13$

$Y \equiv 6 \bmod 14$

$Y \equiv 0 \bmod 15$

Third guess:
take Z as $9 \cdot 13 \cdot e \cdot 14 \cdot f$

$Z \equiv 0 \bmod 13$

$Z \equiv 0 \bmod 14$

$Z \equiv 9 \bmod 15$

Now take
$x = X + Y + Z$

$$\left.\begin{aligned} x &\equiv X + Y + Z \bmod 13 \equiv 3+0+0 \equiv 3 \\ x &\equiv X + Y + Z \bmod 14 \equiv 0+6+0 \equiv 6 \\ x &\equiv X + Y + Z \bmod 15 \equiv 0+0+9 \equiv 9 \end{aligned}\right\} \text{satisfies}$$

$a \equiv 14^{-1} \bmod 13$
$a \equiv 1$

$b \equiv 15^{-1} \bmod 13$
$b \equiv 7$

$c \equiv 13^{-1} \bmod 14 \equiv (-1)^{-1} \bmod 14$
$c \equiv 13$

$d \equiv 15^{-1} \bmod 14$
$d \equiv 1$

$e \equiv 13^{-1} \bmod 15$
$e \equiv 7$

$f \equiv 14^{-1} \bmod 15$
$f \equiv 14$

$$\left.\begin{aligned} X &= 3 \cdot 14 \cdot a \cdot 15 \cdot b = 3 \cdot 14 \cdot 1 \cdot 15 \cdot 7 \\ Y &= 6 \cdot 13 \cdot c \cdot 15 \cdot d = 6 \cdot 13 \cdot 13 \cdot 15 \cdot 1 \\ Z &= 9 \cdot 13 \cdot e \cdot 14 \cdot f = 9 \cdot 13 \cdot 7 \cdot 14 \cdot 14 \end{aligned}\right\} x = X + Y + Z \rightarrow \text{a number that satisfies}$$

MA 130 Looking for least number, so

$x \bmod 13 \cdot 14 \cdot 15$

Find smallest answer.

(2,694 is smallest)
& find workings.

$x \equiv 180,144 \bmod 13 \cdot 14 \cdot 5 \equiv 2694 \bmod 13 \cdot 14 \cdot 15$, 2694 is the smallest integer answer for simultaneous equations (*)

The method works for any system

$$\left.\begin{aligned} x &\equiv a \bmod l \\ x &\equiv b \bmod m \\ x &\equiv c \bmod n \end{aligned}\right\} *$$

where l, m, and n are all coprime.

Such a system (*) has a solution