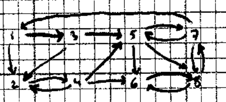
Google

A list of words
→ eigenvalues, rabbits, golden ratio, page rank
results in a few pages most likely of interest being listed.

The webpages containing the words can be represented as a diagram of nodes (one per webpage) and arrows, showing links between those webpages.

Sample diagram



When listing pages google first assigns a number I_n to each page P_n. This I_n is the "importance" of page P_n. Google then lists the most important pages first.

miaow

take $I_1 \approx I_7/3$ // because P_7 links to P_1 and two others

$$\left.\begin{aligned}
I_2 &\approx I_1/2 + I_3/2 + I_4/3 \\
I_3 &= I_1/2 \\
I_4 &= I_2/1 \\
I_5 &= I_3/2 + I_4/3 + I_7/3 \\
I_6 &= I_4/3 + I_5/3 + I_8/2 \\
I_7 &= I_5/3 + I_8/2 \\
I_8 &= I_5/3 + I_6 + I_7/3
\end{aligned}\right\} \text{System of linear equations}$$

How do we determine I_n?

Let's express the system using matrices.

yooo, it's that Keanu reeves movie!

// sparsity

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 0 \\ 1/2 & 0 & 1/2 & 1/3 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/3 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1/3 & 1 & 1/3 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \\ I_8 \end{pmatrix}}_{v} = \underbrace{\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \\ I_8 \end{pmatrix}}_{v}$$

// v is an eigenvector with eigenvalue $\lambda = 1$

We can find (using a computer) that an eigenvector with eigenvalue $\lambda = 1$ for A is

$$v = \begin{pmatrix} 0.0600 \\ 0.0675 \\ 0.0300 \\ 0.0675 \\ 0.0975 \\ 0.2025 \\ 0.1800 \\ 0.2950 \end{pmatrix}$$

$I_8 > I_6 > I_7 > I_5 > I_2, I_4 > I_1 > I_3$

But: how do we calculate the eigenvectors of a square matrix A?

Rabbits

F_n : number of rabbits after n months

$F_0 = 1, \quad F_1 = 1, \quad F_2 = 2, \quad F_3 = 3, \quad F_4 = 5,$ etc.

where

$$\boxed{F_{n+1} = F_n + F_{n-1}}$$

Aim:
Find an explicit formula for F_n in terms of n.

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}$$

$$\left\{ \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} F_1 \\ F_0 \end{pmatrix} \right\} \; (*)$$

// how do we calculate the n^{th} power of a matrix?
→ eigenvalues

THEOREM

if a 2×2 matrix A has eigenvalues λ_1, λ_2 with corresponding eigenvectors v_1 and v_2,
and if ~~A is invertible~~

$$T = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}$$ is invertible (has a nonzero determinant)

then

$$T^{-1} A T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

// note: squaring the matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^2 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2^2 & 0 \\ 0 & 3^2 \end{pmatrix}$$

→ for diagonal matrices

PROOF

$$T^{-1} A T = T^{-1} A \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}$$

$$= T^{-1} \begin{pmatrix} | & | \\ Av_1 & Av_2 \\ | & | \end{pmatrix}$$

$$= T^{-1} \begin{pmatrix} | & | \\ \lambda_1 v_1 & \lambda_2 v_2 \\ | & | \end{pmatrix}$$ // because $Av = \lambda v$ if eigenvectors

$$= T^{-1} \underbrace{\begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}}_{T} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$ // $T^{-1} \cdot T = I$

$$= I \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

if $T^{-1} A T = \underbrace{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}}_{D}$

then $A = T D T^{-1}$

So $A^n = (T D T^{-1})^n = (TDT^{-1})(TDT^{-1})(TDT^{-1}) \ldots (TDT^{-1})$ // $T^{-1} T = I$

$$\boxed{A^n = T D^n T^{-1}} \; (**)$$

$$A^n = T \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} T^{-1}$$

EXAMPLE:

$A \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$P_A(\lambda) = \det(A - \lambda I)$
$= (1-\lambda)(-\lambda) - 1$
$= \lambda^2 - \lambda - 1$
roots are $\dfrac{1 \pm \sqrt{1+4}}{2}$

golden ratio $= \varphi$

eigenvalues of A: $\lambda_1 = \dfrac{1+\sqrt{5}}{2}$, $\lambda_2 = \dfrac{1-\sqrt{5}}{2}$ ($\bar{\varphi}$)

Eigenvectors:
$v_1 = \begin{pmatrix} \varphi \\ 1 \end{pmatrix}$
$v_2 = \begin{pmatrix} 1 \\ -\varphi \end{pmatrix}$

$T = \begin{pmatrix} \varphi & 1 \\ 1 & -\varphi \end{pmatrix}$

from (**)

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = T \begin{pmatrix} \varphi^n & 0 \\ 0 & \bar{\varphi}^n \end{pmatrix} T^{-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$$

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \left(\frac{1}{\sqrt{5}} \varphi^n - \frac{1}{\sqrt{5}} \bar{\varphi}^n \right)$$

$\varphi = \frac{1+\sqrt{5}}{2} = 1.618033\ldots$ // the golden ratio.

Problem

The population of teenagers in a village is infected with a virus. Each week, 20% of the healthy teens test positive, and 30% of the infected teens become healthy.

There are 500 teens in the village, of which 100 are initially infected.

DETERMINE the number of infected teenagers after 1, 2, 3, ... weeks
& INVESTIGATE what happens in the long term.

No of healthy teens at week n : x_n
No of infected teens at week n : y_n

$x_0 = 400$
$y_0 = 100$

... the cat wants the dagger ...

$$\left.\begin{aligned} x_n &= 0.8\,x_{n-1} + 0.3\,y_{n-1} \\ y_n &= 0.2\,x_{n-1} + 0.7\,y_{n-1} \end{aligned}\right\} (*)$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \underbrace{\begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}}_{A} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} \quad (+)$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \begin{pmatrix} 400 \\ 100 \end{pmatrix} = \begin{pmatrix} 350 \\ 150 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \begin{pmatrix} 350 \\ 150 \end{pmatrix} = \begin{pmatrix} 325 \\ 175 \end{pmatrix}$$

Recall:

$$T^{-1} A T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

if T is an invertible 2x2 matrix, whose columns are eigenvectors of A with corresponding eigenvalues λ_1, λ_2.

// note: lecture last week, finding eigenvalues & eigenvectors

$$A = T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} T^{-1} \longrightarrow A^n = T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} T^{-1} T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} T^{-1} \ldots$$

$$\longrightarrow A^n = T \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} T^{-1} \quad (**)$$

let's find the eigenvalues of A

$$\begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \longrightarrow \det(A - \lambda I) = 0$$

$$\det \begin{pmatrix} 0.8-\lambda & 0.3 \\ 0.2 & 0.7-\lambda \end{pmatrix} = 0$$

$$(0.8-\lambda)(0.7-\lambda) - 0.3 \cdot 0.2 = 0$$

$$\lambda^2 - 1.5\lambda + .56 - 0.06 = 0$$

$$\lambda^2 - 1.5\lambda + .5 = 0$$

$$(\lambda - .5)(\lambda - 1) = 0$$

$$\lambda_1 = \tfrac{1}{2} \quad \lambda_2 = 1$$

*↳ What happens in the long run? from + and **

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = T \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} T^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

or:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = T \begin{pmatrix} 1 & 0 \\ 0 & (\frac{1}{2})^n \end{pmatrix} T^{-1} \begin{pmatrix} 400 \\ 100 \end{pmatrix}$$

For large n, $(\frac{1}{2})^n \approx 0$ // stabilises

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A \underbrace{A^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}}_{\substack{\text{eigenvector of } A \\ \text{with eigenvalue} \\ \lambda = 1}} = A^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Eigenvector of A with $\lambda = 1$:

$$\longrightarrow (A - \lambda I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{pmatrix} \begin{pmatrix} 300 \\ 200 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

↳ long run answer.