group project ma190

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1 Shuffling Cards Like Never Before

At one stage or another, we've all felt like we might get lucky playing cards. When you're missing one card to complete the perfect hand, you feel like nothing can stop you except some serious bad luck. But is it really bad luck if it is the far more likely outcome? What are the odds of the next card you pick up being that one card? Well, as it turns out, the odds are very, very low.

In mathematics, a factorial is the product of an integer and all the integers below it (Oxford Languages). It is denoted by the starting integer and the symbol "!".

This system is used abundantly in the area of probability, where the number of different ways a set of items can be arranged is equal to the factorial of the number of items in the set. It works something like this:

If there are 5 possible items you could choose for the first position, and you randomly choose one, then there are 4 remaining for the next one, and so on until there is only one possible choice. If you were to write every different way the items could be arranged, you would find that there are 120 of them. It can then be observed that,

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

.

So, if there were a set of 5 items, they could be lined up in 5! (or 120) ways. Thus, in a standard deck of 52 cards, there are 52! ways of arranging them. This sounds fine until you realise just how large this number is. Remember,

$$52! = 52 \times 51 \times 51 \times \cdots \times 2 \times 1$$

This number gets very, very large very, very fast. In fact, it comes out to roughly

80658175170943878571660636856403766975289505440883277824000000000000

Or more conveniently,

$$8.0658 \times 10^{67}$$

... that is quite large... but how large?

Well, consider the fact that so far, only about 436,117,076,640,000,000 seconds (https://81018.com/universeclock/) have passed since the universe was created, you can immediately see that it is unlikely a single arrangement of a deck of cards has ever been repeated.

But that number of seconds is still hard to visualise, so let me explain it to you via an example created by Scott Czepiel.

First, set a timer to 8.0658×10^{67} seconds. Then start by standing on the equator. Every 1 billion years, take 1 step. When you finally complete the journey all the way around, take 1 drop out of the Pacific Ocean. Continue repeating this process until the ocean is empty, and then place 1 piece of paper flat on the ground. After refilling the ocean, repeat the entire process again until the stack of paper reaches the sun.

If you look at the timer, the left three most digits haven't changed.

It would take doing this entire process about another 3000 times in order to finish off the clock to 0.

So, next time you're feeling lucky in a round of cards, don't place that bet.

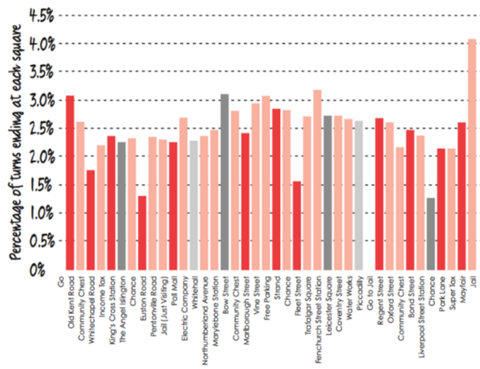
2 Mastering Monopoly with Matrices

As we approach the Christmas holidays, it's time to remember that Monopoly exists, and then realise why you tried so hard to forget it. The classic board game can be truly brutal, but what if there was a way you could harness probability and statistics to come up with a strategy that will allow you to win every single time? Well, believe it or not, there is.

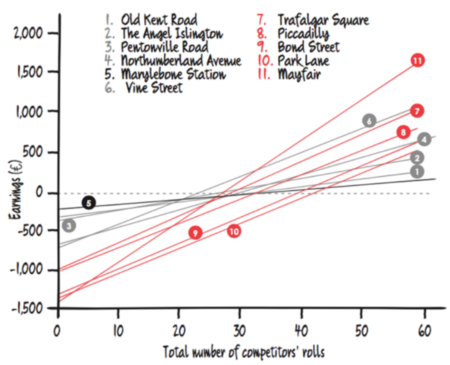
It can be worked out that over the course of playing infinite games of monopoly, it becomes far more likely that you will land on some squares as opposed to others. This is important because it means you can predict which properties will have the most visitors, and thus generate the highest income.

The mathematician Hannah Fry did this by using a mathematical system known as Markov Chains. Brilliant.org defines Markov chains as "a mathematical system that experiences transitions from one state to another according to certain probabilistic rules". In other words, it is a network of probabilities that can be used to predict future probabilities based off nothing but the systems current state, calculated using complex matrix multiplication.

Fry began by calculating the probability of going from any one square on the monopoly board to any other one, which resulted in a massive table of probabilities. This table was then used as the basis for a countably infinite Markov chain, which yielded the result of the percentage likelihood of landing on any square relative to all the others, which can be seen in the following graph:

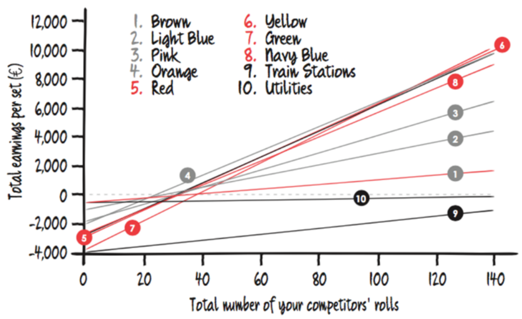


With this data, Fry then factored in the money you make off of each square, calculating the expected return in investment and producing the following plot:



This graph clearly points to Mayfair being the best square in the game; not only does it quickly regain its own value, it goes on to earn far more than any other square.

But this isn't the end of the story when it comes to monopoly. As the name suggests, the game is all about buying sets of properties, and when this was factored in, the results changed drastically:



Now, the red, green and yellow sets are actually the best sets in the game, although admittedly followed closely by the navy-blue set.

However, these graphs are measured in terms of competitor roles. This means that the more competitors you have, the more relevant and decisive they become.

Although this is the end of the story in terms of numbers, I want to point out another interesting discovery which came from this research. The mathematician and stand-up comedian Matt Parker also attempted to tackle this problem, but in a very different and less elegant way. He programmed a computer to play imaginary games of monopoly in order to brute force the probabilities of landing on each square.

This is fascinating to me, because Parker managed to produce the exact same probabilities as Fry, by using a completely different, but more realistic method. For me, this just goes to prove that mathematics, although often seemingly abstract, is rooted in reality and is simply an efficient way of tackling real world problems.

3 Combinatoric Nonsense

So, 52! ways to order a deck of cards, but what we care about is getting the cards that we need. Now, we all know that the only sensible way to do this is to cheat, however, not everyone is a skilled cardsharp, so the rest of us have to make do with luck.

As mathematicians, we approach this by finding the probability of any given outcome, but that starts to get weird, because depending on how you define a problem, you can come up with a vastly different probability.

For example;
You're playing a game with a standard 52-card deck, and you've been dealt a hand of 2 cards. To win, you need to have *two aces* in your hand. Here's the kicker - you already know what one of the cards is.
Let's look at two scenarios:

1. What's the probability that you have two aces, given that *your first card is an ace?*

2. What's the probability that you have two aces, given that *your first card is the ace of spades?*

Firstly, let's look at:

3.1 Scenario 1

We want to find the probability P of having two aces, given that we have at least one ace. (For simplicity, we can assume that the order of the cards doesn't matter) We can write that as $P(\textit{both aces} \mid \textit{have ace})$

Since these two probabilities are clearly not independent (you have to have at least one ace to have both aces), we can use this formula:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

writing it as

$$\frac{P(\textit{both aces} \cap \textit{have an ace})}{P(\textit{have an ace})}$$

Now, since having two aces means you have at least one ace, the intersection above is redundant (i.e. two aces is a subset of at least one ace). So that gives us

$$\frac{P(\textit{both aces})}{P(\textit{have an ace})}$$

So, how do we write this out in numbers? Well, the numerator, $P(both\ aces)$ can be written as

$$\frac{\binom{4}{2}}{\binom{52}{2}}$$

as it's the number of possible combinations of aces over the number of total possible combinations of cards, and the denominator we could write as the complement of having at least one ace; in other words, the probability that neither card is an ace. That would look something like this:

$$1 - \frac{\binom{48}{2}}{\binom{52}{2}}$$

Therefore our final equation for scenario 1 looks like this:

$$\frac{\frac{\binom{4}{2}}{\binom{52}{2}}}{1 - \frac{\binom{48}{2}}{\binom{52}{2}}} = \frac{1}{33}$$

Now it's time for

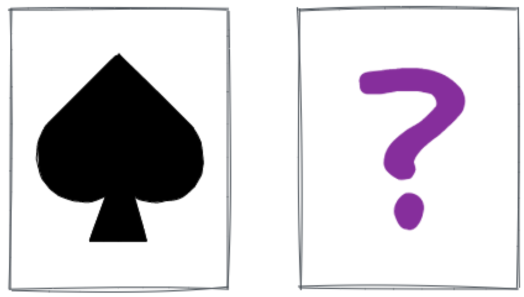
3.2 Scenario 2

Now, our second scenario is, logically, the same. It should have the same answer, right? Well, let's see.

$$P(both\ aces|ace\ of\ spades)$$

We could do something similar to scenario 1, and compute the probability using the formula from above, or we could think about it in a much simpler way.

So, in our hand we have two cards, the ace of spades, and something else, some mystery card.



Now, this mystery card could be anything, and is *equally likely to be any card*, other than the ace of spades. Therefore, we can just write the answer down straight away:

$$\frac{3}{51} = \frac{1}{17}$$

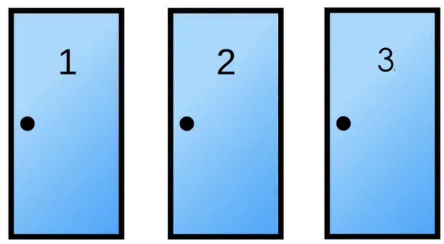
That's about twice as likely as in the first scenario, just by specifying what suit we have. This is a great example of the subtlety of conditional probability. In this example, the discrepancy comes from the difference between "at least one" ace, and *one* ace of spades.

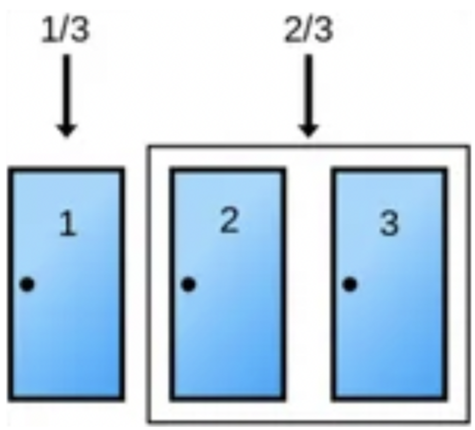
4 The Monty Hall Problem

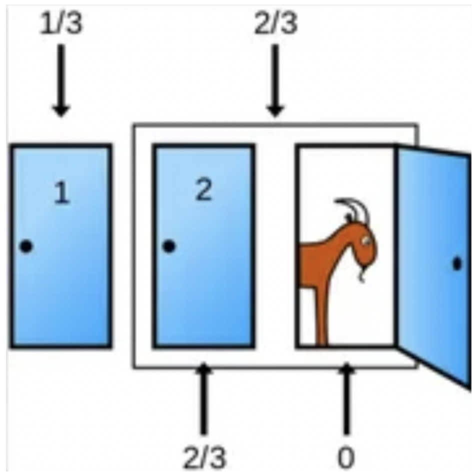
The Monty Hall problem is a brain teaser, in the form of a probability puzzle, based on the American television game show Let's Make a Deal and named after its original host, Monty Hall.

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car, behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

The answer may surprise you but it is in fact to your advantage to switch choice. When you made the choice, there was a one in 3 chance the car was behind the door you chose and a 2 in 3 chance the car was behind one of the doors you didn't choose. This probability does not change after the host opens one of the unchosen doors. When the host reveals that one of the unchosen doors does not have a car behind it, the 2 in 3 chance of the car being behind one of the unchosen doors rests on the unchosen and unrevealed door, as opposed to the 1 in 3 chance of the car being behind the door you chose initially.







Behind door 1	Behind door 2	Behind door 3	Result if you stay at door 1	Result if you switch doors
Goat	Goat	Car	Goat	Car
Goat	Car	Goat	Goat	Car
Car	Goat	Goat	Car	Goat

From the table above 2 out of the 3 times the player decides to switch doors the player wins the car

4.1 Bayes' theorem

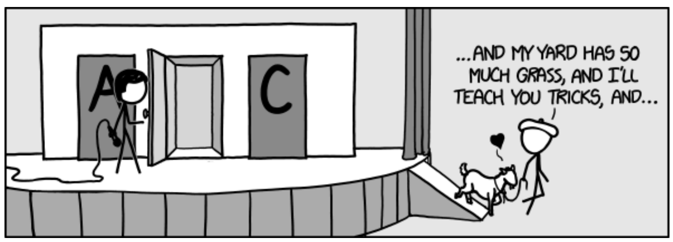
$$P(H|E) = \frac{P(E|H)}{P(E)} \tag{1}$$

$$P(H|E) = \frac{P(E|H) \times P(H)}{P(E|H) \times P(H + P(E|notH) \times P(notH))} \tag{2}$$

P(H) is the prior probability that door 1 has a car behind it, without knowing about the door that Monty reveals. This 1/3 P(notH) is the probability that we did not pick the door with the car behind it. Since the door either has the car behind it or not, P(notH) = 1 - P(H) = 2/3 P(EH) is the probability that Monty shows a door with a goat behind it, given that there is a car behind door 1. Since Monty always shows a door with a goat, this is equal to 1. P(E — notH) is the probability that Monty shows the goat, given that there is a goat behind door 1. Again, since Monty always shows a door with a goat, this is equal to 1.

$$P(H|E) = \frac{(1) \times (1/3)}{(1 \times (1/3) + 1 \times (2/3))} = \frac{(1/3)}{1} = \frac{1}{3}$$

(3)



Of course, none of this matters if you like goats.

source: https://xkcd.com/1282/