Matrix Inverses

$\rightarrow$ $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix}$ $\quad B = \begin{pmatrix} 10 & -12 & 5 \\ -3 & 3 & -1 \\ -1 & 2 & -1 \end{pmatrix}$ $\quad AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$A^{-1} = B$

- who cares?
- how did I get B from A?
  - ↳ tomorrow: why does it work

$$\left.\begin{aligned} x + 2y + 3z &= 1 \\ 2x + 5y + 5z &= 2 \\ 3x + 8y + 6z &= 3 \end{aligned}\right\} (*)$$

(*) means simultaneous, here.

the system of equations (*) can be rewritten as

$$\overset{A}{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

to find $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ we could multiply both sides times A^{-1}, to get

$$A^{-1} A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 & -12 & 5 \\ -3 & 3 & -1 \\ -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

hence, $x = 1$, $y = 0$, $z = 0$

Where did B come from?

Gauss-Jordan method for finding the inverse of an $n \times n$ matrix A.

$$\underset{n \times 2n}{(A \,\vdots\, I)} \xrightarrow[\text{row operations}]{\text{elementary}} (I \,\vdots\, B) \qquad \text{where } B = A^{-1}$$

Elementary Row Operations

1) $R_i \longleftarrow R_i + \lambda R_j \qquad (i \neq j,\ \lambda \in \mathbb{R})$

2) $R_i \longleftarrow \lambda R_i \qquad (\lambda \neq 0)$

3) $R_i \longleftrightarrow R_j \qquad (i \neq j)$

Example

Find A^{-1} where $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix}$

$$A \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 5 & 0 & 1 & 0 \\ 3 & 8 & 6 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 2 & -3 & -3 & 0 & 1 \end{array}\right) \quad \begin{aligned} R_2 &\longleftarrow R_2 - 2R_1 \\ R_3 &\longleftarrow R_3 - 3R_1 \end{aligned}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right) \quad R_3 \longleftarrow R_3 - 2R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & -6 & 3 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right) \quad \begin{aligned} R_1 &\longleftarrow R_1 - 3R_3 \\ R_2 &\longleftarrow R_2 + R_3 \\ R_3 &\longleftarrow -R_3 \end{aligned}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -12 & 5 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right) \quad R_1 \longleftarrow R_1 - 2R_2$$

$\underbrace{}_{B \,=\, A^{-1}}$

Why does the Gauss-Jordan Method succeed in finding the inverse of an invertible square matrix?

Row Operation I

e.g.

$$\underset{A}{\begin{pmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \\ 7 & 1 & -2 \end{pmatrix}} \quad R_2 \longleftarrow R_2 + 3R_1 \quad : \underset{B}{\begin{pmatrix} 1 & 3 & 5 \\ 5 & 5 & 21 \\ 7 & 1 & -2 \end{pmatrix}}$$

note:

$$\rightarrow \underset{E}{\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \underset{A}{\begin{pmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \\ 7 & 1 & -2 \end{pmatrix}} = \underset{B}{\begin{pmatrix} 1 & 3 & 5 \\ 5 & 5 & 21 \\ 7 & 1 & -2 \end{pmatrix}}$$

$3 \quad 1 \quad 0 \rightarrow 3R_1 + 1R_2 + 0R_3$

/* Row operations are matrix multiplication */

Row Operation II

$$\underset{A}{\begin{pmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \\ 7 & 1 & -2 \end{pmatrix}} \quad R_3 \longleftarrow 4R_3 \quad : \underset{B}{\begin{pmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \\ 28 & 4 & -8 \end{pmatrix}}$$

→ any row operation can be replaced by premultiplication by a square matrix E

$$\underset{E}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}} \underset{A}{\begin{pmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \\ 7 & 1 & 2 \end{pmatrix}} = \underset{B}{\begin{pmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \\ 28 & 4 & -8 \end{pmatrix}}$$

Row Operation III

$$\underset{A}{\begin{pmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \\ 7 & 1 & -2 \end{pmatrix}} \quad R_1 \updownarrow R_3 \quad : \underset{B}{\begin{pmatrix} 7 & 1 & -2 \\ 2 & -4 & 6 \\ 1 & 3 & 5 \end{pmatrix}}$$

$$\underset{E}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}} \underset{A}{\begin{pmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \\ 7 & 1 & -2 \end{pmatrix}} = \underset{B}{\begin{pmatrix} 7 & 1 & -2 \\ 2 & -4 & 6 \\ 1 & 3 & 5 \end{pmatrix}}$$

Back to Gauss-Jordan..

if $(A : I) \xrightarrow[\text{operations}]{\text{row}} (I : B)$

then there are square matrices

$E_1, E_2, \ldots, E_K$ such that

$$E_K \cdots E_3 E_2 E_1 A = I$$

thus

$$(E_K \cdots E_3 E_2 E_1) A A^{-1} = I A^{-1}$$

$$(E_K \cdots E_3 E_2 E_1) I = I A^{-1}$$ → This proves, (kind of.) that the gauss-jordan method works.

PROBLEM: A factory requires energy, steel + labour to manufacture machines of types A, B, and C.

Resources	A	B	C	weekly amounts
energy	2 MWh	3 MWh	2 MWh	100 MWh
steel	1 tonne	1 tonne	4 tonnes	70 tonnes
labour	20 hrs	10 hrs	10 hrs	500 hrs

What production figures ensure that all resources are used?

Soln

Suppose we manufacture x units of type A, y units of type B, and z units of type C, such that all resources are used.

$$\left.\begin{array}{l} 2x + 3y + 2z = 100 \\ x + y + 4z = 70 \\ 20x + 10y + 10z = 500 \end{array}\right\} (*) \qquad \begin{pmatrix} 2 & 3 & 2 \\ 1 & 1 & 4 \\ 20 & 10 & 10 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 100 \\ 70 \\ 500 \end{pmatrix}$$

$$R_3 \leftarrow \tfrac{1}{10} R_3 \quad \begin{pmatrix} 2 & 3 & 2 \\ 1 & 1 & 4 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 100 \\ 70 \\ 50 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 4 \\ 2 & 3 & 2 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 70 \\ 100 \\ 50 \end{pmatrix} \quad R_1 \leftrightarrow R_2$$

$$\begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & -1 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 70 \\ -40 \\ -90 \end{pmatrix} \quad \begin{array}{l} R_2 \longleftarrow R_2 - 2R_1 \\ R_3 \longleftarrow R_3 - 2R_1 \end{array}$$

Cont.

$$\begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & 0 & -13 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 70 \\ -40 \\ -130 \end{pmatrix}$$

$$\left.\begin{aligned} x + y + 4z &= 70 \\ y - 6z &= -40 \\ -13z &= -130 \end{aligned}\right\} - \begin{bmatrix} x = 10 \\ y = 20 \\ z = 10 \end{bmatrix} - (*) \text{ Solved.}$$

Row Operations

$$\text{pivot} \rightarrow \begin{cases} 2x + 3y + 2z = 100 \\ x + y + 4z = 70 \\ 20x + 10y + 10z = 500 \end{cases}$$

} system of linear equations

The above system is equivalent to the following system

$$\begin{bmatrix} R_2 \longleftarrow R_2 - \frac{1}{2}R_1 \\ R_3 \longleftarrow R_3 - 10R_1 \end{bmatrix}$$

$$\begin{cases} 2x + 3y + 2z = 100 \\ \text{pivot} \longrightarrow -\frac{1}{2}y + 3z = 20 \\ -20y + -10z = -500 \end{cases}$$

} second system

The second system is equivalent to the following system

$$\begin{bmatrix} R_3 \longleftarrow R_3 - 40R_2 \end{bmatrix}$$

$$2x + 3y + 2z = 100$$

$$-\frac{1}{2}y + 3z = 20$$

$$-130z = -1300 \longrightarrow z = 10$$

// Back substitution

$$z = 10 \rightarrow y = 20 \rightarrow x = 10$$

<u>Terminology</u>

2 is the <u>pivot</u> in the 1st stage

$-\frac{1}{2}$ is the pivot of the 2nd stage

// equivalent = yield the same values for variables

1st step: take multiples of 1st row from all subsequent equations

↳ how many is determined by the pivot.

2nd step: repeat for second line, and all subsequent, until a singular variable is easily identifiable

// Gaussian elimination

↳ Q:

Does this procedure always work?

A:

No

it doesn't work if a pivot ends up at 0.

→ some systems have no solutions.