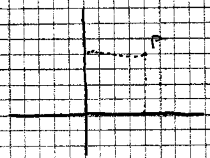
Geometry of Matrices → Linear Transformations of the Plane

$\mathbb{R}^2$ denotes the xy plane



A point P can be represented by a pair of real numbers (x, y)

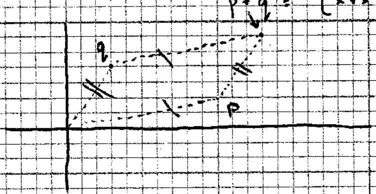
A transformation of the plane is a function

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

which inputs a point $P = (x,y)$, and outputs a point $T(P) = T(x,y)$

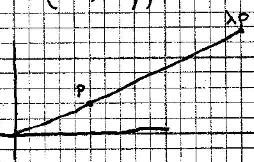
We can add two points $p(x,y)$ and $q(x', y')$ using matrix addition.

$$p+q = (x+x', \ y+y')$$



We can multiply a point $p(x,y)$ by a scalar $\lambda \in \mathbb{R}$, using scalar multiplication of matrices

$$\lambda P = (\lambda x, \ \lambda y)$$



// λ = lambda

Definition:

A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be linear if

1) $T(p+q) = T(p) + T(q)$

2) $T(\lambda p) = \lambda \cdot T(p)$

for all $p, q \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$

Example:

Consider the transformation

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x,y) \longmapsto (2x+4y, \ 6x+8y)$$

For instance

$T(1,2) = (10, 22)$

$T(-1,2) = (6, 10)$

Is T linear?

Consider $p(x,y)$ and $q(x',y')$

$$\begin{aligned} T(p+q) &= T(x+x', \ y+y') \\ &= (2(x+x') + 4(y+y'), \ 6(x+x') + 8(y+y')) \\ &= ((2x+4y) + (2x'+4y'), \ (6x+8y) + (6x'+8y')) \\ &= (2x+4y, \ 6x+8y) + (2x'+4y', \ 6x'+8y') \end{aligned}$$

// write as the sum of two points

$$= T(p) + T(q)$$

← condition 1 proven

Consider

$$\begin{aligned} T(\lambda p) &= T(\lambda x, \lambda y) \\ &= (2\lambda x + 4\lambda y, \ 6\lambda x + 8\lambda y) \\ &= \lambda(2x+4y, \ 6x+8y) \end{aligned}$$

// scalar multiplication of matrices

$$= \lambda T(p)$$

← condition 2 proven

Therefore, T is linear.

Consider

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x, y) \mapsto (x^2, y^2)$$

$T(2, -3) = (4, 9)$

<u>is T linear?</u>

if you want to prove that something is linear, you need to prove for arbitrary points.

if you want to disprove, you just need one counterexample.

Consider

$p = (5, 7)$ $\quad$ $T(\lambda p) = T(10, 14) = (100, 196)$

$\lambda = 2$ $\quad$ $\lambda T(p) = 2(25, 49) = (50, 98)$

$(100, 196) \neq (50, 98)$

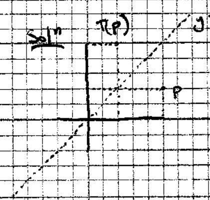
$T(\lambda p) \neq \lambda T(p) \rightarrow \therefore T$ is not linear.

Example

let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation of the plane, reflecting in a line $y = x$.

is T linear?

<u>Sol^n</u>



$p : (x, y)$

$T(p) : (y, x)$

linear?

Consider two points $p, q = (x, y), (x' + y')$

$$T(p + q) = T(x + x', y + y')$$
$$= (y + y', x + x')$$
$$= (y, x) + (y' + x)$$
$$= T(p) + T(q) \quad \checkmark$$

$$T(\lambda p) = T(\lambda x + \lambda y)$$
$$= (\lambda y + \lambda x)$$
$$= \lambda(y + x)$$
$$= \lambda T(p) \quad \checkmark$$

<u>Linear</u>