Clock arithmetic is very much like school arithmetic, in particular:

i) $a + b = b + a$ // commutative

i') $a \cdot b = b \cdot a$

ii) $(a + b) + c = a + (b + c)$ // associative

ii') $(ab)c = a(bc)$

iii) $a(b + c) = ab + ac$ // distributive

What about an arithmetic system where (i') fails?

Matrix Arithmetic

A matrix is (~~that film with Keanu Reeves~~) an array of numbers.

↳ arranged neatly into rows & columns.
↳ each row and column have the same length.

Examples

$\begin{pmatrix} 1 & 2 & -4 \\ 3 & 1 & 6 \end{pmatrix}$ 2×3 matrix

$\begin{pmatrix} 1 & \sqrt{2} \\ \pi & 3 \end{pmatrix}$ 2×2 matrix

$\begin{pmatrix} 1 & 2 & -4 & \pi & e \end{pmatrix}$ 1×5 matrix (row vector)

$\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ 3×1 matrix (column vector)

Two $m \times n$ matrices are added by adding corresponding entries, and the result is an $m \times n$ matrix

↳ only matrices of the same dimensions can be added.

$$\begin{pmatrix} 5 & 7 & 13 \\ 1 & -2 & 4 \end{pmatrix} + \begin{pmatrix} -1 & -2 & 1 \\ 0 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 14 \\ 1 & 1 & 9 \end{pmatrix}$$

Given an $m \times n$ matrix A, we write $-A$ to denote the matrix obtained from A by multiplying every entry by -1.

$A \begin{pmatrix} -1 & 3 \\ -4 & 2 \end{pmatrix}$ $-A \begin{pmatrix} 1 & -3 \\ 4 & -2 \end{pmatrix}$

// Note: any matrix A added to matrix $-A$,
// every entry will be equal to 0. (and still has the same dimensions)

↳ zero matrix

Multiplying Matrices

→ Multiplying a row vector ($1 \times n$ matrix) by a column vector ($n \times 1$ matrix)

let $R = (a_1, a_2, \dots, a_n)$ be a row vector of length n, and let $C = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ be a column vector of length n.

We define $\left[RC = a_1b_1 + a_2b_2 + a_2b_3 + \dots + a_nb_n \right]$

Any matrix

$A = \begin{pmatrix} 1 & — & 2 \\ 3 & — & 4 \end{pmatrix}$ → can be regarded as a collection of rows.

$B = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 3 & 3 \\ 0 & 2 & 4 \end{pmatrix}$ → can be regarded as a collection of columns

LET

A be an $m \times n$ matrix, and let B be an $n \times p$ matrix.

We define multiplication as follows. ↳ note: R_1 is the same length as C_1 (n)

$$\underbrace{\begin{pmatrix} — R_1 — \\ — R_2 — \\ \vdots \\ — R_m — \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} | & | & & | \\ C_1 & C_2 & \cdots & C_p \\ | & | & & | \end{pmatrix}}_{B}$$

$$= \underbrace{\begin{pmatrix} R_1C_1 & R_1C_2 & R_1C_3 & R_1C_4 & \cdots & R_1C_p \\ R_2C_1 & R_2C_2 & R_2C_3 & R_2C_4 & \cdots & R_2C_p \\ R_3C_1 & R_3C_2 & R_3C_3 & R_3C_4 & \cdots & R_3C_p \\ R_4C_1 & R_4C_2 & R_4C_3 & R_4C_4 & \cdots & R_4C_p \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ R_mC_1 & R_mC_2 & R_mC_3 & R_mC_4 & \cdots & R_mC_p \end{pmatrix}}_{AB}$$

→ an $m \times p$ matrix

x99 (if colourblind)

boo

13 boo (triskaidekaphobic)

Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 3 & 3 \\ 0 & 2 & 0 \end{pmatrix}$$

$$R_1C_1 = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = 1 \qquad R_1C_2 = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = 14 \qquad R_1C_3 = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = 7$$

$$AB = \begin{pmatrix} 1 & 14 & 7 \\ 1 & 35 & 19 \end{pmatrix}$$

Scalar Multiplication

If A is a matrix, and K is a number, we let KA denote the matrix gotten from A by multiplying each entry in A by K.

Identity Matrices

Consider

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A = \begin{pmatrix} 2 & 5 \\ 6 & 7 \end{pmatrix} \qquad IA = \begin{pmatrix} 2 & 5 \\ 6 & 7 \end{pmatrix} = A = \begin{pmatrix} 2 & 6 \\ 6 & 7 \end{pmatrix} = AI$$

$IA = A = AI$

↳ true for any matrix A

In general let I denote an $n \times n$ matrix, with each diagonal entry equal to 1, and every other entry equal to zero.

e.g. $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ → we call I an identity matrix.

SQUARE MATRIX: rows are the same length as columns ($n \times n$ matrix)

DIAGONAL ENTRY 1,1 2,2 3,3 ... etc. positions.

Inverse of a Matrix

Defn [Let A be an $n \times n$ matrix. The inverse of A is an $n \times n$ matrix B such that $AB = I = BA$]

Consider a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = ad-bc\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

↳ scalar multiplication

→ PROPOSITION:

Given a matrix A of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have $A^{-1} = (ad-bc)^{-1}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

provided $(ad-bc)$ is invertible.