Determinants

- ♣ Definition of det(A), $|A|$, where $A \in R^{n\times n}$
- ♣ Cofactor and minor at (i, j)-position of A
- ♣ Properties of determinants
- ♣ Some examples
- ♣ Applications
  - □ Check linear independence of matrix column vectors
  - □ The computation of A^{-1}
  - □ The solution of $A\mathbf{x} = \mathbf{b}$
  - □ Area of parallelogram, volume of parallelepiped

Even and Odd Permutations

Definition: A *permutation* of integers $1, 2, \cdots, n$ is an ordered list of these n integers, for examples,

$$(1,2,3) \quad (2,3,1) \quad (3,1,2)$$

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A permutation of $1, 2, \cdots, n$ is called *even* or *odd* according to whether the number of inversions of natural order $1, 2, \cdots, n$ that are present in the permutation is even or odd respectively.

Let $A \in R^{n\times n}$, $det \ : \ R^{n\times n} \ \rightarrow \ R$ is a function defined as

$$det(A) = \sum_{\sigma} (-1)^{sign(\sigma)} \prod_{i=1}^{n} a_{i,\sigma(i)}$$

where σ is a permutation of $1, 2, \cdots, n$ and

$$sign(\sigma) = \begin{cases} 0 & if\ \sigma\ is\ even \\ 1 & if\ \sigma\ is\ odd \end{cases}$$

Formulas of det(A) for $A \in R^{2\times2},\ R^{3\times3}$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

Then

$$det(A) = a_{11}a_{22} - a_{21}a_{12}$$

$$\begin{aligned} det(C) &= 3\cdot4\cdot2 + 2\cdot2\cdot0 + 1\cdot1\cdot1 \\ &- 1\cdot4\cdot0 - 3\cdot1\cdot2 - 2\cdot1\cdot2 \\ &= 15 \end{aligned}$$

$$\begin{aligned} det(B) &= b_{11}b_{22}b_{33} + b_{12}b_{23}b_{31} + b_{21}b_{32}b_{13} \\ &- b_{31}b_{22}b_{13} - b_{12}b_{21}b_{33} - b_{11}b_{32}b_{23} \end{aligned}$$

where

$$\sigma_1 = (1,2,3) \quad \sigma_2 = (2,3,1) \quad \sigma_3 = (3,1,2)$$

$$\sigma_4 = (3,2,1) \quad \sigma_5 = (2,1,3) \quad \sigma_6 = (1,3,2)$$

Minors and Cofactors

Theorem: Let $A = [a_{ij}] \in R^{n \times n}$. Then

(a) $det(A) = \sum_{k=1}^{n} a_{ik} A_{ik}$ (expansion by row i)

(b) $det(A) = \sum_{k=1}^{n} a_{kj} A_{kj}$ (expansion by column j)

where the (i,k) cofactor $A_{ik} = (-1)^{i+k} det(M_{ik})$ of A, and the (i,k) minor $det(M_{ik}),\ where\ M_{ik} \in R^{(n-1)\times(n-1)}$, is defined to be the determinant of the submatrix of A by deleting the $i-th$ row and $k-th$ column from A.

♣ *Properties of Determinants*

(a) $det(A^t) = det(A)$

(b) $det(D) = \prod_{i=1}^{n} d_{ii}$, where $D = [d_{ij}]$ is a diagonal matrix

(c) The determinant changes sign when two rows are exchanged

(d) If two rows of A are identical, then $det(A) = 0$

(e) The elementary operation of subtracting a multiple of one row from another row leaves the determinant unchanged

(f) If A has a zero row, then $det(A) = 0$

(g) If A is either upper-Δ or lower-Δ, then $det(A) = \prod_{i=1}^{n} a_{ii}$

(h) If A is singular then det(A)=0. If A is invertible, $det(A) \neq 0$

(i) If $A, B \in R^{n \times n}$, then $det(AB) = det(A)det(B)$

(j) If Q is orthogonal, then det(Q) equals 1 or -1

The Computation of A^{-1} Using Determinant

$$det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \ \forall\ 1 \le i \le n$$

Then

$$Aadj(A) = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & \cdot & \cdot & A_{n1} \\ A_{12} & A_{22} & \cdot & \cdot & A_{n2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{1n} & A_{2n} & \cdot & \cdot & A_{nn} \end{bmatrix} = det(A)I_n$$

Note that

$$A^{-1} = [b_{ij}] \quad \Rightarrow \quad b_{ij} = \frac{A_{ji}}{det(A)} = \frac{(-1)^{j+i}}{det(A)}det(M_{ji})$$

♣ The solution of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{det(A)}adj(A)\mathbf{b}$

♣ Cramer's Rule

The $j - th$ component of $\mathbf{x} = A^{-1}\mathbf{b}$ is $x_j = \frac{det(B_j)}{det(A)}$, where

$$B_j = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_{j-1}, \mathbf{b}, \mathbf{a}_{j+1}, \cdots, \mathbf{a}_n]$$

Note that B_j matrix is formed by replacing the $j - th$ column of A by the column vector $\mathbf{b}$

♣ Use Gauss-Jordan method and Cramer's rule to compute A^{-1}

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{5}\begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}$$

Area of Parallelogram, Volume of Parallelepiped

Suppose $\mathbf{a}_1, \mathbf{a}_2 \in R^2$ make an acute angle θ ($\theta < 90^{\text{deg}}$, then the area of parallelogram enclosed by $\mathbf{a}_1, \mathbf{a}_2$ and their parallel vectors equals

$$\|\mathbf{a}_1\|_2 \|\mathbf{a}_2\|_2 \sin\theta = det([\mathbf{a}_1, \mathbf{a}_2]) = a_{11}a_{22} - a_{21}a_{12}$$

The volume of parallelepiped spanned by vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ equals

$$|\langle \mathbf{a} \times \mathbf{b}, \mathbf{c} \rangle| = det([\mathbf{a}, \mathbf{b}, \mathbf{c}])$$

Exercises

- Find $det(A)$, where $A = [a_{ij}]$ with $a_{ij} = i + j$

- Find $det(T)$, where $T = [t_{ij}]$ with $t_{ij} = \begin{cases} 1 & if\ i = j\ or\ i = j+1 \\ -1 & if\ i = j-1 \\ 0 & otherwise \end{cases}$

- Find $det(M_n)$, where $M_n = [\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{x}, \cdots, \mathbf{e}_{n-1}, \mathbf{e}_n]$, i.e., the $k-th$ column of the identity matrix I_n is replaced by $\mathbf{x}$.

- Find $det(H_{\mathbf{u}})$, where $H_{\mathbf{u}} = I - 2\mathbf{u}\mathbf{u}^t$ with $\|\mathbf{u}\|_2 = 1$.

- Let

$$A = \begin{bmatrix} A_k & B \\ O & C_{n-k} \end{bmatrix}, \quad K = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}, \ j = \sqrt{-1}$$

(a) Find det(A), det(K), det(F), and λ such that $det(\lambda I - K) = 0$

(b) Find A^{-1}, K^{-1}, F^{-1}, H^{-1}

♣ *Examples*

$$A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

□ $det(A_2) = 3$, $det(A_3) = 4$, $det(A_4) = 5$, $det(A_n) = n + 1$

Note that A_n is a tridiagonal matrix with $det(A_n) = 2det(A_{n-1}) - det(A_{n-2})$ for $n \geq 3$