

# Solutions of Test 2 for ISA5305, Spring 2015

10:10-11:50, Wednesday, December 9, 2015

Name : \_\_\_\_\_ SN : \_\_\_\_\_ Gn : \_\_\_\_\_ Index : \_\_\_\_\_

(30%) 1. Fill in the following blanks (no partial credits).

(a) Let  $\{X_i \sim b(5, 0.8), 1 \leq i \leq 10\}$  be a random sample and let  $X = \sum_{i=1}^{10} X_i$ . Then

the p.m.f. of  $X$ ,  $f_X(x) = \underline{C_x^{50}(0.8)^x(0.2)^{50-x}, 0 \leq x \leq 50}$

the moment-generating function  $M_X(t) = \underline{(0.2 + 0.8e^t)^{50}}$

(b) Let  $\{Y_i, 1 \leq i \leq 10\}$  be a random sample of Poisson distribution with variance  $Var(Y_1) = 0.8$  and define  $Y = \sum_{i=1}^{10} Y_i$ . Then

the p.m.f. of  $Y$ ,  $f_Y(y) = \underline{\frac{e^{-8}8^y}{y!}, y = 0, 1, 2, \dots, \infty}$

the moment-generating function  $M_Y(t) = \underline{e^{8(e^t-1)}}$

(c) Define  $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$ , then

$\Gamma(5) = \underline{24}, \Gamma(7) = \underline{720}$

$\Gamma(\frac{1}{2}) = \underline{\sqrt{\pi}}, \Gamma(\frac{5}{2}) = \underline{\frac{3}{4}\sqrt{\pi}}$

(d) Let  $X$  be a random variable having an *exponential* distribution with the variance  $Var(X) = 9$ , then

the p.d.f. of  $X$ ,  $f_X(x) = \underline{\frac{1}{3}e^{-x/3}, x > 0}$

the moment-generating function  $M_X(t) = \underline{\frac{1}{1-3t}, t < \frac{1}{3}}$

(e) Let  $X \sim \chi^2(8)$ , that is,  $X$  has a *Chi-square* distribution with the degrees of freedom 8, then

the p.d.f. of  $X$ ,  $f_X(x) = \underline{\frac{1}{\Gamma(4)2^4}x^3e^{-x/2}, x > 0}$

the moment-generating function  $M_X(t) = \underline{\frac{1}{(1-2t)^4}, t < \frac{1}{2}}$  and the variance  $Var(X) = \underline{16}$

(30%) **2.** Fill the following blanks (no partial credits).

- (a) Let  $\{X_i, 1 \leq i \leq 9\}$  be a random sample of size 9 from  $N(2, 1)$ . Define  $\bar{X} = \frac{1}{9} \sum_{i=1}^9 X_i$  and  $Y = \sum_{i=1}^9 (X_i - 2)^2$ . Then

the moment-generating function  $M_{\bar{X}}(t) = \underline{e^{2t + (t^2/18)}}$

the moment-generating function  $M_Y(t) = \underline{\frac{1}{(1-2t)^{9/2}}}$

- (b) Let  $\{Z_i \sim N(0, 1), 1 \leq i \leq 10\}$  be a random sample of size 10. Define  $V = \sum_{i=1}^{10} Z_i$  and  $W = \sum_{i=1}^{10} Z_i^2$ . Then

the moment-generating function  $M_V(t) = \underline{e^{5t^2}}$

the moment-generating function  $M_W(t) = \underline{\frac{1}{(1-2t)^5}}$

- (c) Let  $\{X_1, X_2, \dots, X_n\}$  be a random sample of size  $n$  from the *exponential distribution* with  $E(X_n) = 3$ . Define  $Y = \sum_{i=1}^n X_i$ . Then

the p.d.f. of  $Y$ ,  $f_Y(y) = \underline{\frac{1}{\Gamma(n)3^n} y^{n-1} e^{-y/3}, y > 0}$

the moment-generating function of  $Y$ ,  $\phi_Y(t) = \underline{\frac{1}{(1-3t)^n}}$

- (d) Let  $Z \sim N(0, 1)$ . Define  $Y = 3Z + 2$ , then

the p.d.f. of  $Y$ ,  $f_Y(y) = \underline{\frac{1}{\sqrt{18\pi}} \exp[-\frac{(y-2)^2}{18}], -\infty < y < \infty}$

the moment-generating function of  $Y$ ,  $\phi_Y(t) = \underline{e^{2t + \frac{9t^2}{2}}}$

- (e) Let  $\{X_i \sim \chi^2(1), 1 \leq i \leq n\}$  be a random sample and define  $W_n = (\sum_{i=1}^n X_i - n)/\sqrt{2n}$ . According to the Central Limit Theorem,

the limiting distribution function of  $W_n$   $\lim_{n \rightarrow \infty} P(W_n \leq w) = \underline{\int_{-\infty}^w \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \Phi(w)}$

the limiting moment-generating function is  $\lim_{n \rightarrow \infty} M_{W_n}(t) = \underline{e^{t^2/2}}$

(10%) **3.** Let the r.v.  $X$  have the p.d.f.  $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$ ,  $0 < x < 1$ , where  $\alpha, \beta > 0$  are known positive integers.

Find  $E[X]$  and  $Var[X]$ .

**Hint:**  $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$  and  $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ .

See Lecture Notes

(15%) 4. Let  $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$  be the order statistics of a random sample  $\{X_1, X_2, \dots, X_n\}$  from the exponential distribution with mean 2.

- (a) Find the probability density function of  $X_{(1)}$ .
- (b) Use the results of **(a)** to find  $E[X_{(1)}]$ .
- (c) Find the probability density function of  $X_{(n)}$ .
- (d) Use the result of **(c)** to find  $E[X_{(n)}]$ .

**See Lecture Notes**

(15%) 5. Let  $X$  have the p.d.f.  $f(x) = \beta x^{\beta-1}$ ,  $0 < x < 1$  for a given  $\beta > 0$ . Define  $Y = -3\beta \ln(X)$ .

- (a) Compute the distribution function of  $Y$ ,  $P(Y \leq y)$ , for  $0 < y < \infty$ .
- (b) Find the moment-generating function  $M_Y(t)$ .
- (c) Find  $E(Y)$  and  $Var(Y)$ .

**See Lecture Notes**