# Linear Algebra

## Ideas and Applications

FOURTH EDITION

RICHARD C. PENNEY

with website

WILEY

# LINEAR ALGEBRA

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## Ideas and Applications

Fourth Edition

**RICHARD C. PENNEY**
Purdue University

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# PREFACE

I wrote this book because I have a deep conviction that mathematics is about ideas, not just formulas and algorithms, and not just theorems and proofs. The text covers the material usually found in a one or two semester linear algebra class. It is written, however, from the point of view that knowing *why* is just as important as knowing *how*.

To ensure that the readers see not only why a given fact is true, but also why it is important, I have included a number of the beautiful applications of linear algebra.

Most of my students seem to like this emphasis. For many, mathematics has always been a body of facts to be blindly accepted and used. The notion that they personally can decide mathematical truth or falsehood comes as a revelation. Promoting this level of understanding is the goal of this text.

RICHARD C. PENNEY

*West Lafayette, Indiana*
*Updated with October, 2015*

# FEATURES OF THE TEXT

**Parallel Structure** Most linear algebra texts begin with a long, basically computational, unit devoted to solving systems of equations and to matrix algebra and determinants. Students find this fairly easy and even somewhat familiar. But, after a third or more of the class has gone by peacefully, the boom falls. Suddenly, the students are asked to absorb abstract concept after abstract concept, one following on the heels of the other. They see little relationship between these concepts and the first part of the course or, for that matter, anything else they have ever studied. By the time the abstractions can be related to the first part of the course, many students are so lost that they neither see nor appreciate the connection.

This text is different. We have adopted a parallel mode of development in which the abstract concepts are introduced right from the beginning, along with the computational. Each abstraction is used to shed light on the computations. In this way, the students see the abstract part of the text as a natural outgrowth of the computational part. This is not the "mention it early but use it late" approach adopted by some texts. Once a concept such as linear independence or spanning is introduced, it becomes part of the vocabulary to be used frequently and repeatedly throughout the rest of the text.

The advantages of this kind of approach are immense. The parallel development allows us to introduce the abstractions at a slower pace, giving students a whole semester to absorb what was formerly compressed into two-thirds of a semester. Students have time to fully absorb each new concept before taking on another. Since the concepts are utilized as they are introduced, the students see *why* each concept is necessary. The relation between theory and application is clear and immediate.

**Gradual Development of Vector Spaces** One special feature of this text is its treatment of the concept of vector space. Most modern texts tend to introduce this concept fairly late. We introduce it early because we need it early. Initially, however, we do not develop it in any depth. Rather, we slowly expand the reader's understanding by introducing new ideas as they are needed.

This approach has worked extremely well for us. When we used more traditional texts, we found ourselves spending endless amounts of time trying to explain what a vector space is. Students felt bewildered and confused, not seeing any point to what they were learning. With the gradual approach, on the other hand, the question of what a vector space is hardly arises. *With this approach, the vector space concept seems to cause little difficulty for the students.*

**Treatment of Proofs** It is essential that students learn to read and produce proofs. Proofs serve both to validate the results and to explain why they are true. For many students, however, linear algebra is their first proof-based course. They come to the subject with neither the ability to read proofs nor an appreciation for their importance.

Many introductory linear algebra texts adopt a formal "definition-theorem-proof" format. In such a treatment, a student who has not yet developed the ability to read abstract mathematics can perceive both the statements of the theorems and their proofs (not to mention the definitions) as meaningless abstractions. They wind up reading only the examples in the hope of finding "patterns" that they can imitate to complete the assignments. In the end, such students wind up only mastering the computational techniques, since this is the only part of the course that has any meaning for them. In essence, we have taught them to be nothing more than slow, inaccurate computers.

Our point of view is different. *This text is meant to be read by the student – all of it!* We always work from the concrete to the abstract, never the opposite. We also make full use of geometric reasoning, where appropriate. We try to explain "analytically, algebraically, and geometrically." We use carefully chosen examples to motivate both the definitions and theorems. Often, the essence of the proof is already contained in the example. Despite this, we give complete and rigorous *student-readable* proofs of most results.

**Conceptual Exercises** Most texts at this level have exercises of two types: proofs and computations. We certainly do have a number of proofs and we definitely have lots of computations. The vast majority of the exercises are, however, "conceptual, but not theoretical." That is, *each exercise asks an explicit, concrete question which requires the student to think conceptually in order to provide an answer.* Such questions are both more concrete and more manageable than proofs and thus are much better at demonstrating the concepts. They do not require that the student already have facility with abstractions. Rather, they act as a bridge between the abstract proofs and the explicit computations.

**Applications Sections Doable as Self-Study** Applications can add depth and meaning to the study of linear algebra. Unfortunately, just covering the "essential" topics in the typical first course in linear algebra leaves little time for additional material, such as applications.

Many of our sections are followed by one or more application sections that use the material just studied. This material is designed to be read unaided by the student and thus may be assigned as outside reading. As an aid to this, we have provided two levels of exercises: self-study questions and exercises. The self-study questions are designed to be answerable with a minimal investment of time by anyone who has carefully read and digested the relevant material. The exercises require more thought and a greater depth of understanding. They would typically be used in parallel with classroom discussions.

We feel that, in general, there is great value in providing material that the students are responsible for learning on their own. Learning to read mathematics is the first step in learning to do mathematics. Furthermore, there is no way that we can ever teach everything the students need to know; we cannot even predict what they need to know. Ultimately, the most valuable skill we teach is the ability to teach oneself. The applications form a perfect vehicle for this in that an imperfect mastery of any given application will not impede the student's understanding of linear algebra.

**Early Orthogonality Option** We have designed the text so that the chapter on orthogonality, with the exception of the last three sections, may be done immediately following Chapter 3 rather than after the section on eigenvalues.

**True-False Questions** We have included true-false questions for most sections.

**Chapter Summaries** At the end of each chapter there is a chapter summary that brings together major points from the chapter so students can get an overview of what they just learned.

**Student Tested** This text has been used over a period of years by numerous instructors at both Purdue University and other universities nationwide. We have incorporated comments from instructors, reviewers, and (most important) students.

**Technology** Most sections of the text include a selection of computer exercises under the heading Computer Projects. Each exercise is specific to its section and is designed to support and extend the concepts discussed in that section.

These exercises have a special feature: they are designed to be "freestanding." In principle, the instructor should not need to spend any class time at all discussing computing. Everything most students need to know is right there. In the text, the discussion is based on MATLAB®.

**Meets LACSG Recommendations** The Linear Algebra Curriculum Study Group (LACSG) recommended that the first class in linear algebra be a "student-oriented" class that considers the "client disciplines" and that makes use of technology. The above comments make it clear that this text meets these recommendations. The LACSG also recommended that the first class be "matrix-oriented." We emphasize matrices throughout.

# ACKNOWLEDGMENTS

The author would like to thank all of students, teachers, and reviewers of the text who have made comments, many of which have had significant impact on the final form of the fourth edition. In particular, he would like to thank Bill Dunbar and his students, Ken Rubenstein, and Steven Givant for their particularly useful input.

# ABOUT THE COMPANION WEBSITE

This book is accompanied by a companion website:

http://www.wiley.com/go/penney/linearalgebra

The website includes:

- Instructors' Solutions Manual
- Figures

# CHAPTER 1

# SYSTEMS OF LINEAR EQUATIONS

## 1.1 THE VECTOR SPACE OF $m \times n$ MATRICES

It is difficult to go through life without seeing matrices. For example, the 2014 annual report of Acme Squidget might contain the Table 1.1, which shows how much profit (in millions of dollars) each branch made from the sale of each of the company's three varieties of squidgets in 2014.

**TABLE 1.1 Profits: 2014**

| | ***Red*** | ***Blue*** | ***Green*** | ***Total*** |
|---|---|---|---|---|
| Kokomo | 11.4 | 5.7 | 6.3 | 23.4 |
| Philly | 9.1 | 6.7 | 5.5 | 21.3 |
| Oakland | 14.3 | 6.2 | 5.0 | 25.5 |
| Atlanta | 10.0 | 7.1 | 5.7 | 22.8 |
| Total | 44.8 | 25.7 | 22.5 | 93.0 |

If we were to enter this data into a computer, we might enter it as a rectangular array without labels. Such an array is called a **matrix**. The Acme profits for 2014 would be described by the following matrix. This matrix is a $5 \times 4$ matrix (read "five by four") in that it has five rows and four columns. We would also say that its "size" is $5 \times 4$. In general, a matrix has **size** $m \times n$ if it has $m$ rows and $n$ columns.

*Linear Algebra: Ideas and Applications*, Fourth Edition. Richard C. Penney.

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**Definition 1.1** *The set of all $m \times n$ matrices is denoted $M(m, n)$.*

$$P = \begin{bmatrix} 11.4 & 5.7 & 6.3 & 23.4 \\ 9.1 & 6.7 & 5.5 & 21.3 \\ 14.3 & 6.2 & 5.0 & 25.5 \\ 10.0 & 7.1 & 5.7 & 22.8 \\ 44.8 & 25.7 & 22.5 & 93.0 \end{bmatrix}$$

Each row of an $m \times n$ matrix may be thought of as a $1 \times n$ matrix. The rows are numbered from top to bottom. Thus, the second row of the Acme profit matrix is the $1 \times 4$ matrix

$$[9.1, 6.7, 5.5, 21.3]$$

This matrix would be called the "profit vector" for the Philly branch. (In general, any matrix with only one row is called a **row vector**. For the sake of legibility, we usually separate the entries in row vectors by commas, as above.)

Similarly, a matrix with only one column is called a **column vector**. The columns are numbered from left to right. Thus, the third column of the Acme profit matrix is the column vector

$$\begin{bmatrix} 6.3 \\ 5.5 \\ 5.0 \\ 5.7 \\ 22.5 \end{bmatrix}$$

This matrix is the "green squidget profit vector."

If $A_1, A_2, \ldots, A_n$ is a sequence of $m \times 1$ column vectors, then the $m \times n$ matrix $A$ that has the $A_i$ as columns is denoted

$$A = [A_1, A_2, \ldots, A_n]$$

Similarly, if $B_1, B_2, \ldots, B_m$ is a sequence of $1 \times n$ row vectors, then the $m \times n$ matrix $B$ that has the $B_i$ as rows is denoted

$$B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{bmatrix}$$

In general, if a matrix is denoted by an uppercase letter, such as $A$, then the entry in the $i$th row and $j$th column may be denoted by either $A_{ij}$ or $a_{ij}$, using the corresponding lowercase letter. We shall refer to $a_{ij}$ as the "$(i,j)$ entry of $A$." For example, for the matrix $P$ above, the $(2,3)$ entry is $p_{23} = 5.5$. Note that the row number comes first. Thus, the most general $2 \times 3$ matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

We will also occasionally write "$A = [a_{ij}]$," meaning that "$A$ is the matrix whose $(i,j)$ entry is $a_{ij}$."

At times, we want to take data from two tables, manipulate it in some manner, and display it in a third table. For example, suppose that we want to study the performance of each division of Acme Squidget over the two-year period, 2013–2014. We go back to the 2013 annual report, finding the 2013 profit matrix to be

$$Q = \begin{bmatrix} 11.0 & 5.5 & 6.1 & 22.6 \\ 9.0 & 6.3 & 5.3 & 20.6 \\ 14.1 & 5.9 & 4.9 & 24.9 \\ 9.7 & 7.0 & 5.8 & 22.5 \\ 43.8 & 24.7 & 22.1 & 90.6 \end{bmatrix}$$

If we want the totals for the two-year period, we simply add the entries of this matrix to the corresponding entries from the 2014 profit matrix. Thus, for example, over the two-year period, the Kokomo division made $5.5 + 5.7 = 11.2$ million dollars from selling blue squidgets. Totaling each pair of entries, we find the two-year profit matrix to be

$$T = \begin{bmatrix} 22.4 & 11.2 & 12.4 & 46.0 \\ 18.1 & 13.0 & 10.8 & 41.9 \\ 28.4 & 12.1 & 9.9 & 50.4 \\ 19.7 & 14.1 & 11.5 & 45.3 \\ 88.6 & 50.4 & 44.6 & 183.6 \end{bmatrix}$$

In matrix notation, we indicate that $T$ was obtained by summing corresponding entries of $Q$ and $P$ by writing

$$T = Q + P$$

In general, if $A$ and $B$ are $m \times n$ matrices, then $A + B$ is the $m \times n$ matrix defined by the formula

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

For example

$$\begin{bmatrix} 2 & 3 & 0 \\ 1 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 1 \\ 1 & 5 & 0 \end{bmatrix}$$

Addition of matrices of different sizes is not defined.

What if, instead of totals for each division and each product, we wanted two-year *averages*? We would simply multiply each entry of $T = P + Q$ by $\frac{1}{2}$. The notation for this is "$\frac{1}{2}T$." Specifically,

$$\frac{1}{2}T = \begin{bmatrix} 11.20 & 05.60 & 06.20 & 23.00 \\ 09.05 & 06.50 & 05.40 & 20.95 \\ 14.20 & 06.05 & 04.95 & 25.20 \\ 09.85 & 07.05 & 05.75 & 22.65 \\ 44.30 & 25.20 & 22.30 & 91.80 \end{bmatrix}$$

In general, if $c$ is a number and $A = [a_{ij}]$ is an $m \times n$ matrix, we define

$$cA = c[a_{ij}] = [ca_{ij}] = [a_{ij}]c = Ac \tag{1.1}$$

Hence,

$$2\begin{bmatrix} 2 & 3 & 0 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 0 \\ 2 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 0 & -1 \end{bmatrix}2$$

There is also a notion of subtraction of $m \times n$ matrices. In general, if $A$ and $B$ are $m \times n$ matrices, then we define $A - B$ to be the $m \times n$ matrix defined by the formula

$$A - B = [a_{ij}] - [b_{ij}] = [a_{ij} - b_{ij}]$$

Thus,

$$\begin{bmatrix} 3 & 5 & 1 \\ 1 & 5 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

In linear algebra, the terms **scalar** and **number** mean essentially the same thing. Thus, multiplying a matrix by a real number is often called **scalar multiplication.**

## The Space $\mathbb{R}^n$

We may think of a $2 \times 1$ column vector $X = \begin{bmatrix} x \\ y \end{bmatrix}$ as representing the point in the plane with coordinates $(x, y)$ as in Figure 1.1. We may also think of $X$ as representing

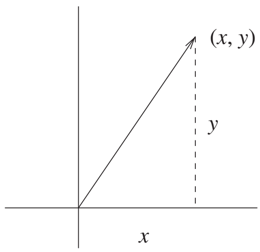



**FIGURE 1.1** Coordinates in $\mathbb{R}^2$.

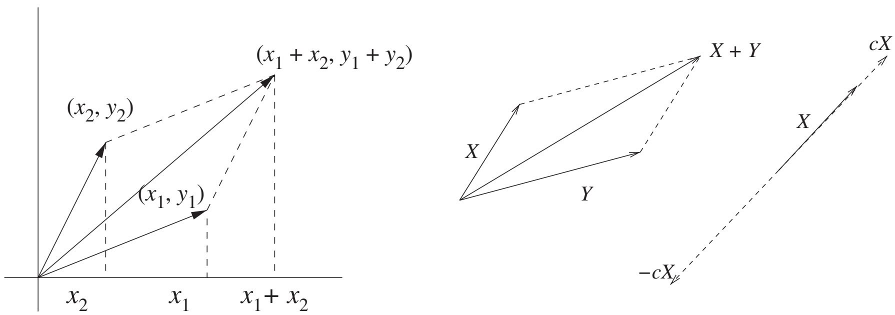



**FIGURE 1.2** Vector algebra.

the vector from the point $(0, 0)$ to $(x, y)$—that is, as an arrow drawn from $(0, 0)$ to $(x, y)$. We will usually denote the set of $2 \times 1$ matrices by $\mathbb{R}^2$ when thought of as points in two-dimensional space.

Like matrices, we can add pairs of vectors and multiply vectors by scalars. Specifically, if $X$ and $Y$ are vectors with the same initial point, then $X + Y$ is the diagonal of the parallelogram with sides $X$ and $Y$ beginning at the same initial point (Figure 1.2, right). For a positive scalar $c$, $cX$ is the vector with the same direction as that of $X$, but with magnitude expanded (or contracted) by a factor of $c$.

Figure 1.2 on the left shows that when two elements of $\mathbb{R}^2$ are added, the corresponding vectors add as well. Similarly, multiplication of an element of $\mathbb{R}^2$ by a scalar corresponds to multiplication of the corresponding vector by the same scalar. If $c < 0$, the direction of the vector is reversed and the vector is then expanded or contracted by a factor of $-c$ (Figure 1.2, right).

## EXAMPLE 1.1

Compute the sum of the vectors represented by $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and draw a diagram illustrating your computation.

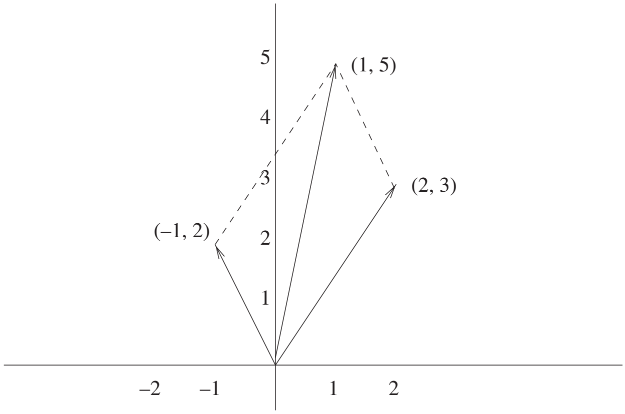



**FIGURE 1.3** Example 1.1.

*Solution.* The sum is computed as follows:

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1+2 \\ 2+3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

The vectors (along with their sum) are plotted in Figure 1.3.

Similarly, we may think of the $3 \times 1$ matrix

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

as representing either the point $(x, y, z)$ in three-dimensional space or the vector from $(0, 0, 0)$ to $(x, y, z)$ as in Figure 1.4. Matrix addition and scalar multiplication are describable as vector addition just as in two dimensions. We will usually denote the set of $3 \times 1$ matrices by $\mathbb{R}^3$ when thought of as points in three-dimensional space.

What about $n \times 1$ matrices? Even though we cannot visualize $n$ dimensions, we still envision $n \times 1$ matrices as somehow representing points in $n$ dimensional space. The set of $n \times 1$ matrices will be denoted as $\mathbb{R}^n$ when thought of in this way.

**Definition 1.2** *$\mathbb{R}^n$ is the set of all $n \times 1$ matrices.*

### Linear Combinations and Linear Dependence

We can use our Acme Squidget profit matrices to demonstrate one of the most important concepts in linear algebra. Consider the last column of the 2014 profit

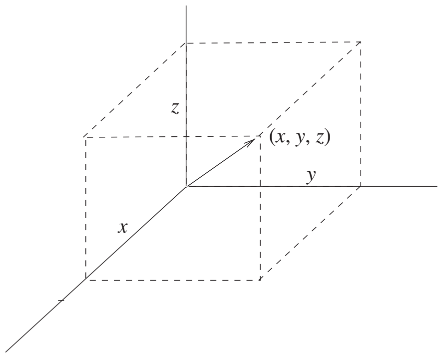



**FIGURE 1.4** Coordinates in $\mathbb{R}^3$.

matrix. Since this column represents the total profit for each branch, it is just the sum of the other columns in the profit matrix:

$$\begin{bmatrix} 11.4 \\ 9.1 \\ 14.3 \\ 10.0 \\ 44.8 \end{bmatrix} + \begin{bmatrix} 5.7 \\ 6.7 \\ 6.2 \\ 7.1 \\ 25.7 \end{bmatrix} + \begin{bmatrix} 6.3 \\ 5.5 \\ 5.0 \\ 5.7 \\ 22.5 \end{bmatrix} = \begin{bmatrix} 23.4 \\ 21.3 \\ 25.5 \\ 22.8 \\ 93.0 \end{bmatrix} \tag{1.2}$$

This last column does not tell us anything we did not already know in that we could have computed the sums ourselves. Thus, while it is useful to have the data explicitly displayed, it is not essential. We say that this data is "dependent on" the data in the other columns. Similarly, the last row of the profit matrix is dependent on the other rows in that it is just their sum.

For another example of dependence, consider the two profit matrices $Q$ and $P$ and their average

$$A = \tfrac{1}{2}(Q + P) = \tfrac{1}{2}Q + \tfrac{1}{2}P \tag{1.3}$$

The matrix $A$ depends on $P$ and $Q$—once we know $P$ and $Q$, we can compute $A$.

These examples exhibit an especially simple form of dependence. In each case, the matrix we chose to consider as dependent was produced by multiplying the other matrices by scalars and adding. This leads to the following concept.

**Definition 1.3** *Let $S = \{A_1, A_2, \dots, A_k\}$ be a set of elements of $M(m, n)$. An element $C$ of $M(m, n)$ is **linearly dependent on** $S$ if there are scalars $b_i$ such that*

$$C = b_1A_1 + b_2A_2 + \cdots + b_kA_k \tag{1.4}$$

*We also say that "$C$ is a **linear combination** of the $A_i$."*

*Remark.* In set theory, an object that belongs to a certain set is called an **element** of that set. The student must be careful not to confuse the terms "element" and "entry." The matrix below is *one element* of the set of $2 \times 2$ matrices. Every element of the set of $2 \times 2$ matrices has four *entries*.

$$\begin{bmatrix} 1 & 2 \\ 4 & -5 \end{bmatrix}$$

The expression "$a \in B$" means that $a$ is an element of the set $B$.

One particular element of $M(m, n)$ is linearly dependent on every other element of $M(m, n)$. This is the $m \times n$ matrix, which has all its entries equal to 0. We denote this matrix by 0. It is referred to as "the zero element of $M(m, n)$." Thus, the zero element of $M(2, 3)$ is

$$0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The $m \times n$ zero matrix depends on every other $m \times n$ matrix because, for any $m \times n$ matrix $A$,

$$0A = 0$$

We can also discuss linearly dependent sets of matrices.

**Definition 1.4** *Let $S = \{A_1, A_2, \dots, A_k\}$ be a set of elements of $M(m, n)$. Then $S$ is **linearly dependent** if at least one of the $A_j$ is a linear combination of the other elements of $S$—that is, $A_j$ is a linear combination of the set of elements $A_i$ with $i \neq j$. We also define the set $\{0\}$, where 0 is the zero element of $M(m, n)$, to be linearly dependent. $S$ is said to be **linearly independent** if it is not linearly dependent. Hence, $S$ is linearly independent if none of the $A_i$ are linear combinations of other elements of $S$.*

Thus, from formula (1.3), the set $S = \{P, Q, A\}$ is linearly dependent. In addition, from formula (1.2), the set of columns of $P$ is linearly dependent. The set of rows of $P$ is also linearly dependent since the last row is the sum of the other rows.

### EXAMPLE 1.2

Is $S = \{A_1, A_2, A_3, A_4\}$ a linearly independent set where the $A_i$ are the following $2 \times 2$ matrices?

$$A_1 = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

*Solution.* By inspection

$$A_2 = 2A_3 + A_4 = 0A_1 + 2A_3 + A_4$$

showing that $S$ is linearly dependent.

*Remark.* Note that $A_1$ is *not* a combination of the other $A_i$ since the (2, 1) entry of $A_1$ is nonzero, while all the other $A_i$ are zero in this position. This demonstrates that linear dependence does not require that each of the $A_i$ be a combination of the others.

### EXAMPLE 1.3

Let $B_1$, $B_2$, and $B_3$ be as shown. Is $S = \{B_1, B_2, B_3\}$ a linearly dependent set?

$$B_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -7 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1.3 \\ 0 \\ 2.2 \\ 0 \end{bmatrix}$$

*Solution.* We begin by asking ourselves whether $B_1$ is linearly dependent on $B_2$ and $B_3$—that is, are there scalars $a$ and $b$ such that

$$B_1 = aB_2 + bB_3$$

The answer is no since the last entries of both $B_2$ and $B_3$ are 0, while the last entry of $B_1$ is 1.

Similarly, we see that $B_2$ is not a linear combination of $B_1$ and $B_3$ (from the second entries) and $B_3$ is not a linear combination of $B_1$ and $B_2$ (from the third entries). Thus, the given three matrices form a linearly independent set.

Example 1.3 is an example of the following general principle that we use several times later in the text.

**Proposition 1.1** *Suppose that $S = \{A_1, A_2, \ldots, A_p\}$ is a set of $m \times n$ matrices such that each $A_k$ has a nonzero entry in a position where all the other $A_q$ are zero—that is, for each $k$ there is a pair of indices $(i, j)$ such that $(A_k)_{ij} \neq 0$ while $(A_q)_{ij} = 0$ for all $q \neq k$. Then $S$ is linearly independent.*

*Proof.* Suppose that $S$ is linearly dependent. Then there is a $k$ such that

$$A_k = c_1A_1 + c_2A_2 + \cdots + c_{k-1}A_{k-1} + c_{k+1}A_{k+1} + \cdots + c_qA_q$$

Let $(i, j)$ be as described in the statement of the proposition. Equating the $(i, j)$ entries on both sides of the above equation shows that

$$(A_k)_{ij} = c_1 0 + c_2 0 + \cdots + c_{k-1} 0 + c_{k+1} 0 + \cdots + c_q 0 = 0$$

contradicting the hypothesis that $(A_k)_{ij} \neq 0$. □

Linear independence also has geometric significance. Two vectors $X$ and $Y$ in $\mathbb{R}^2$ will be linearly independent if and only if neither is a scalar multiple of the other—that is, they are noncollinear (Figure 1.5, left). We will prove in Section 2.2 that any three vectors $X, Y$, and $Z$ in $\mathbb{R}^2$ are linearly dependent (Figure 1.5, right).

In $\mathbb{R}^3$, the set of linear combinations of a pair of linearly independent vectors lies in the plane they determine. Thus, three noncoplanar vectors will be linearly independent (Figure 1.6).

In general, the set of all matrices that depends on a given set of matrices is called the span of the set:

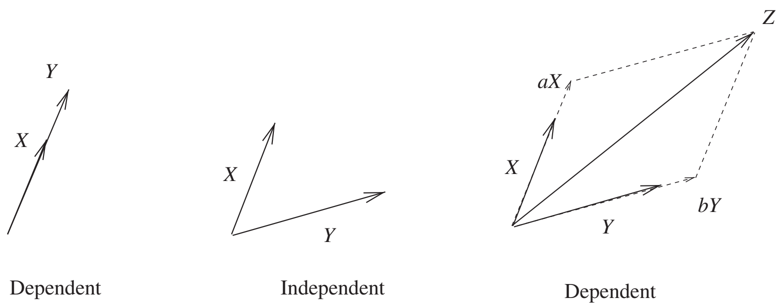



**FIGURE 1.5** Dependence in $\mathbb{R}^2$.

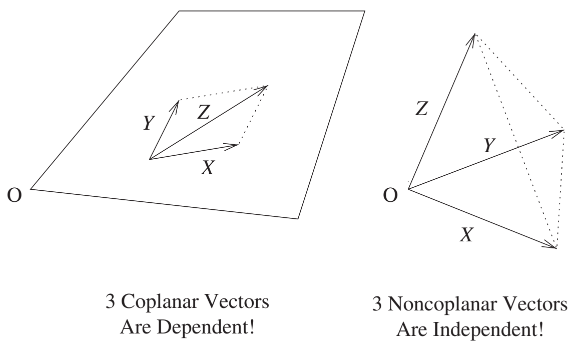



**FIGURE 1.6** Three vectors in $\mathbb{R}^3$.

**Definition 1.5** *Let* $S = \{A_1, A_2, \ldots, A_k\}$ *be a set of elements of* $M(m, n)$. *Then* span $S$ (*the* ***span*** *of* $S$) *is the set of all elements of the form*

$$B = c_1A_1 + c_2A_2 + \cdots + c_kA_k$$

*where the* $c_i$ *are scalars.*

The span of $S$, then, is just the set of all linear combinations of the elements of $S$. Thus, for example, if $B_1$, $B_2$, and $B_3$ are as in Example 1.3 on page 9, then

$$B_1 + B_2 + 10B_3 = \begin{bmatrix} 8 \\ 2 \\ 22 \\ 1 \end{bmatrix}$$

is one element of span $\{B_1, B_2, B_3\}$.

In $\mathbb{R}^2$ and $\mathbb{R}^3$, the span of a single vector is the line through the origin determined by it. From Figure 1.6, the span of a set of two linearly independent vectors will be the plane they determine.

### What Is a Vector Space?

One of the advantages of matrix notation is that it allows us to treat a matrix as if it were one single number. For example, we may solve for $Q$ in formula (1.3) on page 7:

$$\begin{aligned} A &= \tfrac{1}{2}(P + Q) \\ 2A &= Q + P \\ 2A - P &= Q \end{aligned}$$

The preceding calculations used a large number of properties of matrix arithmetic that we have not discussed. In greater detail, our argument was as follows:

$$\begin{aligned} A &= \tfrac{1}{2}(P + Q) \\ 2A &= 2[\tfrac{1}{2}(P + Q)] = \tfrac{2}{2}(Q + P) = Q + P \\ 2A + (-P) &= (Q + P) + (-P) \\ &= Q + (P - P) = Q + 0 = Q \end{aligned}$$

We certainly used the associative law $(A + B) + C = A + (B + C)$, the laws $A + (-A) = 0$ and $A + 0 = A$, as well as several other laws. In Theorem 1.1, we list the most important algebraic properties of matrix addition and scalar multiplication.

These properties are called the **vector space properties**. Experience has proved that these properties are all that one needs to effectively deal with any computations such as those just done with $A$, $P$, and $Q$. For the sake of this list, we let $\mathcal{V} = M(m, n)$ for some fixed $m$ and $n$.[1] Thus, for example, $\mathcal{V}$ might be the set of all $2 \times 3$ matrices.

**Theorem 1.1** (The Vector Space Properties). *Let $X, Y$, and $Z$ be elements of $\mathcal{V}$. Then:*

*(a)* $X + Y$ *is a well-defined element of* $\mathcal{V}$.
*(b)* $X + Y = Y + X$ *(commutativity).*
*(c)* $X + (Y + Z) = (X + Y) + Z$ *(associativity).*
*(d) There is an element denoted* $0$ *in* $\mathcal{V}$ *such that* $X + 0 = X$ *for all* $X \in \mathcal{V}$. *This element is referred to as the "zero element."*
*(e) For each* $X \in \mathcal{V}$, *there is an element* $-X \in \mathcal{V}$ *such that* $X + (-X) = 0$.

*Additionally, for all scalars k and l:*

*(f)* $kX$ *is a well-defined element of* $\mathcal{V}$.
*(g)* $k(lX) = (kl)X$.
*(h)* $k(X + Y) = kX + kY$.
*(i)* $(k + l)X = kX + lX$.
*(j)* $1X = X$.

The proofs that the properties from this list hold for $\mathcal{V} = M(m, n)$ are left as exercises for the reader. However, let us prove property (c) as an example of how such a proof should be written.

### EXAMPLE 1.4

Prove property (c) for $M(m, n)$.

*Solution.* Let $X = [x_{ij}]$, $Y = [y_{ij}]$, and $Z = [z_{ij}]$ be elements of $M(m, n)$. Then

$$\begin{aligned} X + (Y + Z) &= [x_{ij}] + ([y_{ij} + z_{ij}]) \\ &= [x_{ij} + (y_{ij} + z_{ij})] \\ &= [(x_{ij} + y_{ij}) + z_{ij}] \quad \text{(from the associative law for numbers)} \\ &= ([x_{ij} + y_{ij}]) + [z_{ij}] = (X + Y) + Z \end{aligned}$$

[1] We use $\mathcal{V}$ in order to avoid the necessity of re-listing these properties when we define the general notion of "vector space."

When we introduced linear independence, we mentioned that for any $m \times n$ matrix $A$

$$0A = 0$$

This is very simple to prove:

$$0A = 0[a_{ij}] = [0a_{ij}] = 0$$

This proof explicitly uses the fact that we are dealing with matrices. It is possible to give another proof that uses only the vector space properties. We first note from property (i) that

$$0A + 0A = (0 + 0)A = 0A$$

Next, we cancel $0A$ from both sides using the vector space properties:

$$\begin{aligned} -0A + (0A + 0A) &= -0A + 0A && \text{Property (e)} \\ (-0A + 0A) + 0A &= 0 && \text{Property (c)} \\ 0 + 0A &= 0 && \text{Properties (b) and (e)} \\ 0A &= 0 && \text{Properties (b) and (d)} \end{aligned} \tag{1.5}$$

Both proofs are valid for matrices. We, however, prefer the second. Since it used only the vector space properties, it will be valid in any context in which these properties hold. For example, let $\mathcal{F}(\mathbb{R})$ denote the set of all real-valued functions which are defined for all real numbers. Thus, the functions $y = e^x$ and $y = x^2$ are two elements of $\mathcal{F}(\mathbb{R})$. We define addition and scalar multiplication for functions by the formulas

$$\begin{aligned} (f + g)(x) &= f(x) + g(x) \\ (cf)(x) &= cf(x) \end{aligned} \tag{1.6}$$

Thus, for example,

$$y = 3e^x - 7x^2$$

defines an element of $\mathcal{F}(\mathbb{R})$. Since addition and scalar multiplication of functions are defined using the corresponding operations on numbers, it is easily proved that the vector space properties (a)–(j) hold if we interpret $A$, $B$, and $C$ as functions rather than matrices. (See Example 1.5.)

Thus, we can automatically state that $0f(x) = 0$, where $f(x)$ represents any function and 0 is the zero function. Admittedly, this is not an exciting result. (Neither, for that matter, is $0A = 0$ for matrices.) However, it demonstrates an extremely important principle: *Anything we prove about matrices using only the vector space properties will be true in any context for which these properties hold.*

As we progress in our study of linear algebra, it will be important to keep track of exactly which facts can be proved directly from the vector space properties and which require additional structure. We do this with the concept of "vector space."

**Definition 1.6** *A set $\mathcal{V}$ is a **vector space** if it has a rule of addition and a rule of scalar multiplication defined on it so that all the vector space properties* (a)–(j) *from Theorem 1.1 hold. By a **rule of addition** we mean a well-defined process for taking an arbitrary pair of elements $X$ and $Y$ from $\mathcal{V}$ and producing a third element $X + Y$ in $\mathcal{V}$. (Note that the sum must lie in $\mathcal{V}$.) By a **rule of scalar multiplication** we mean a well-defined process for taking an arbitrary scalar $c$ and an arbitrary element $X$ of $\mathcal{V}$ and producing a second element $cX$ of $\mathcal{V}$.*

The following theorem summarizes our discussion of functions. We leave most of the proof as an exercise.

**Theorem 1.2** *The set $\mathcal{F}(\mathbb{R})$ of real valued functions on $\mathbb{R}$ is a vector space under the operations defined by formula* (1.6).

### EXAMPLE 1.5

Prove vector space property (h) for $\mathcal{F}(\mathbb{R})$.

*Solution.* Let $f(x)$ and $g(x)$ be real-valued functions and let $k \in \mathbb{R}$. Then

$$\begin{aligned}(k(f+g))(x) &= k((f+g)(x)) \\ &= k(f(x)+g(x)) \\ &= kf(x)+kg(x) \\ &= (kf+kg)(x)\end{aligned}$$

showing that $k(f + g) = kf + kg$, as desired.

Any concept defined for $M(m, n)$ solely in terms of addition and scalar multiplication will be meaningful in any vector space $\mathcal{V}$. One simply replaces $M(m, n)$ by $\mathcal{V}$ where $\mathcal{V}$ is a general vector space. Specifically:

(a) *The concept an element $C$ in $\mathcal{V}$ depending on a set $S = \{A_1, A_2, \dots, A_k\}$ of elements of $\mathcal{V}$ is defined as in Definition* 1.3.
(b) *The concepts of linear independence/dependence for a set $S = \{A_1, A_2, \dots, A_k\}$ of elements of $\mathcal{V}$ are defined as in Definition* 1.4.
(c) *The concept of the span of a set $S = \{A_1, A_2, \dots, A_k\}$ of elements of $\mathcal{V}$ is defined as in Definition* 1.5.

### EXAMPLE 1.6

Show that the set of functions $\{\sin^2 x, \cos^2 x, 1\}$ is linearly dependent in $\mathcal{F}(\mathbb{R})$.

*Solution.* This is clear from the formula

$$\sin^2 x + \cos^2 x = 1$$

Theorem 1.1 states that for each $m$ and $n$, $M(m, n)$ is a vector space. The set of all possible matrices is not a vector space, at least under the usual rules of addition and scalar multiplication. This is because we cannot add matrices unless they are the same size: for example, we cannot add a $2 \times 2$ matrix to a $2 \times 3$ matrix. Thus, our "rule of addition" is not valid for all matrices.

At the moment, the $M(m, n)$ spaces, along with $\mathcal{F}(\mathbb{R})$, are the only vector spaces we know. This will change in Section 1.5 where we describe the concept of "subspace of a vector space." However, if we say that something is "true for all vector spaces," we are implicitly stating that it can be proved solely on the basis of the vector space properties. Thus, the property that $0A = 0$ is true for all vector spaces. Another important vector space property is the following. The proof (which *must* use only the vector space properties or their consequences) is left as an exercise.

**Proposition 1.2** *Let $X$ be an element of a vector space $\mathcal{V}$. Then* $(-1)X = -X$.

Before ending this section, we need to make a comment concerning notation. Writing column vectors takes considerable text space. There is a handy space-saving notation that we shall use often. Let $A$ be an $m \times n$ matrix. The "main diagonal" of $A$ refers to the entries of the form $a_{ii}$. (Note that all these entries lie on a diagonal line starting at the upper left-hand corner of $A$.) If we flip $A$ along its main diagonal, we obtain an $n \times m$ matrix, which is denoted $A^t$ and called the **transpose** of $A$. Mathematically, $A^t$ is the $n \times m$ matrix $[b_{ij}]$ defined by the formula

$$b_{ij} = a_{ji}$$

Thus if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \text{ then } A^t = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Notice that the columns of $A$ become rows in $A^t$. Thus, $[2, 3, -4, 10]^t$ is a space efficient way of writing the column vector

$$\begin{bmatrix} 2 \\ 3 \\ -4 \\ 10 \end{bmatrix}$$

*Remark.* The reader will discover in later sections that the transpose of a matrix has importance far beyond typographical convenience.

### Why Prove Anything?

There is a fundamental difference between mathematics and science. Science is founded on experimentation. If certain principles (such as Newton's laws of motion) seem to be valid every time experiments are done to verify them, they are accepted as a "law."

They will remain a law only as long as they agree with experimental evidence. Thus, Newton's laws were eventually replaced by the theory of relativity when they were found to conflict with the experiments of Michelson and Morley. Mathematics, on the other hand, is based on *proof*. No matter how many times some mathematical principle is observed to hold, we will not accept it as a "theorem" until we can produce a logical argument that shows the principle can *never* be violated.

One reason for this insistence on proof is the wide applicability of mathematics. Linear algebra, for example, is essential to a staggering array of disciplines including (to mention just a few) engineering (all types), biology, physics, chemistry, economics, social sciences, forestry, and environmental science. We must be certain that our "laws" hold, regardless of the context in which they are applied. Beyond this, however, proofs also serve as explanations of *why* our laws are true. We cannot say that we truly understand some mathematical principle until we can prove it.

Mastery of linear algebra, of course, requires that the student learn a body of computational techniques. Beyond this, however, the student should read and, most important, *understand* the proofs. The student will also be asked to create his or her own proofs. This is because it cannot be truly said that we understand something until we can explain it to someone else.

In writing a proof, the student should always bear in mind that *proofs are communication.* One should envision the "audience" as another student who wants to be convinced of the validity of what is being proved. This other student will question anything that is not a simple consequence of something that he or she already understands.

**True-False Questions:** Justify your answers.

**1.1** A subset of a linearly independent set is linearly independent.

**1.2** A subset of a linearly dependent set is linearly dependent.

**1.3** A set that contains a linearly independent set is linearly independent.

**1.4** A set that contains a linearly dependent set is linearly dependent.

**1.5** If a set of elements of a vector space is linearly dependent, then each element of the set is a linear combination of the other elements of the set.

**1.6** A set of vectors that contains the zero vector is linearly dependent.

**1.7** If $X$ is in the span of $A_1$, $A_2$, and $A_3$, then the set $\{X, A_1, A_2, A_3\}$ is linearly independent as long as the $A_i$ are independent.

**1.8** If $\{X, A_1, A_2, A_3\}$ is linearly dependent then $X$ is in the span of $A_1$, $A_2$, and $A_3$.

**1.9** The following set of vectors is linearly independent:

$$[1,0,1,1,0]^t, \quad [0,1,0,2,0]^t, \quad [2,0,0,3,4]^t$$

**1.10** The following matrices form a linearly independent set:

$$\begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 4 & 5 \\ 9 & -1 \\ 5 & 5 \end{bmatrix}$$

**1.11** If $\{A_1, A_2, A_3\}$ is a linearly dependent set of matrices, then $\{A_1^t, A_2^t, A_3^t\}$ is also a linearly dependent set.

**1.12** The set of functions $\{\tan^2 x, \sec^2 x, 1\}$ is a linearly independent set of elements of the vector space of all continuous functions on the interval $(-\pi/2, \pi/2)$.

## EXERCISES

*A check mark ✓ next to an exercise indicates that there is an answer/solution provided in the Student Resource Manual. A double check mark ✓✓ indicates that there is also an answer/hint provided in the Answers and Hints section at the back of the text.*

**1.1** In each case, explicitly write out the matrix $A$, where $A = [a_{ij}]$. Also, give the third row (written as a row vector) and the second column (written as a column vector).

**(a)** ✓✓ $a_{ij} = 2i - 3j$, where $1 \le i \le 3$ and $1 \le j \le 4$

**(b)** ✓ $a_{ij} = i^2 j^3$, where $1 \le i \le 3$ and $1 \le j \le 2$

**(c)** ✓✓ $a_{ij} = \cos(ij\pi/3)$, where $1 \le i \le 3$ and $1 \le j \le 2$

**1.2** For the matrices $A$, $B$, and $C$ below, compute (in the order indicated by the parentheses) $(A + B) + C$ and $A + (B + C)$ to illustrate that $(A + B) + C = A + (B + C)$. Also illustrate the distributive law by computing $3(A + B)$ and $3A + 3B$.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 2 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 1 & 0 \\ 2 & 3 & 2 \\ -1 & 2 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 1 & 3 \\ 4 & 2 & -2 \\ 4 & 3 & 3 \\ 2 & 4 & -1 \end{bmatrix}$$

**1.3** ✓✓The set of matrices $\{A, B, C\}$ from Exercise 1.2 is linearly dependent. Express one element of this set as a linear combination of the others. You should be able to solve this by inspection (guessing).

**1.4** Let $A$, $B$, and $C$ be as in Exercise 1.2. Give a fourth matrix $D$ (reader's choice) that belongs to the span of these matrices.

**1.5** Each of the following sets of matrices is linearly dependent. Demonstrate this by explicitly exhibiting one of the elements of the set as a linear combination of the others. You should be able to find the constants by inspection (guessing).

**(a)** ✓✓ $\{[1,1,2],[0,0,1],[1,1,4]\}$

**(b)** $\{[1,0,0],[0,1,0],[0,0,1],[1,2,3]\}$

**(c)** ✓✓ $\left\{\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right\}$

**(d)** ✓✓ $\left\{\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}, \begin{bmatrix}4\\ 5\\ 6\end{bmatrix}, \begin{bmatrix}3\\ 3\\ 3\end{bmatrix}, \begin{bmatrix}9\\ 12\\ 15\end{bmatrix}\right\}$

**(e)** $\left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}2 & 1\\ 3 & -4\end{bmatrix}\right\}$

**(f)** ✓ $\left\{\begin{bmatrix}3 & -1 & 2\\ 0 & 1 & 4\end{bmatrix}, \begin{bmatrix}-9 & 3 & -6\\ 0 & -3 & -12\end{bmatrix}\right\}$

**(g)** $\left\{\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix}, \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & -1\\ 0 & 1\end{bmatrix}\right\}$

**1.6** ✓✓Write the second row of the Acme profit matrix $P$ (Table (1.1) on page 2) as a linear combination of the other rows.

**1.7** Write the first column of the Acme profit matrix $P$ (Table (1.1) on page 2) as a linear combination of the other columns.

**1.8** Verify the Remark following Example 1.2 on page 8, that is, show that $A_1$ is not a linear combination of $A_2$, $A_3$, and $A_4$.

**1.9** ✓✓What general feature of the following matrices makes it clear that they are independent?

$$\begin{bmatrix}1\\ 0\\ 3\\ 0\end{bmatrix}, \quad \begin{bmatrix}0\\ 0\\ -5\\ 1\end{bmatrix}, \quad \begin{bmatrix}0\\ 1\\ 13\\ 0\end{bmatrix}$$

**1.10** ✓Prove that the rows of the following matrix are linearly independent. [*Hint:* Assume $A_3 = xA_1 + yA_2$, where $A_i$ is the $i$th row of $A$. Prove first that $x = 0$ and then show $y = 0$, which is impossible. Repeat for the other rows. [In Section 2.1 we discuss a more efficient way of solving such problems.]

$$A = \begin{bmatrix}1 & 2 & 3\\ 0 & 5 & 6\\ 0 & 0 & 8\end{bmatrix}$$

**1.11** Prove that the columns of the matrix in Exercise 1.10 are linearly independent. [*Hint:* See the hint for Exercise 1.10.]

**1.12** Let $X = [1, -1, 0]$ and $Y = [1, 0, 0]$. Give four row vectors (reader's choice) that belong to the span of $X$ and $Y$. Give an element of $M(1, 3)$ that does not belong to the span of $X$ and $Y$.

**1.13** ✓✓Let $X = [-1, 1, -1]$ and $Y = [-1, 3, 2]$.

**(a)** Find an element in the span of $X$ and $Y$ such that each of its entries is positive.

**(b)** Show that every element $[x, y, z]$ of the span of $X$ and $Y$ satisfies $5x + 3y - 2z = 0$.

**(c)** Give an element of $M(1, 3)$ that does not belong to the span of $X$ and $Y$.

**1.14** Find two elements of $\mathbb{R}^4$ which belong to the span of the following vectors. Find an element of $\mathbb{R}^4$ which does not belong to their span. [*Hint:* Compute the sum of the entries of each of the given vectors.]

$$X_1 = [1, 1, -1, -1]^t, \quad X_2 = [2, -1, -3, 2]^t, \quad X_3 = [1, 3, -2, -2]^t$$

**1.15** Find an element in the span of the vectors $X = [-1, 2, 1]^t$ and $Y = [2, 5, 1]^t$ which has its third entry equal to 0 and its other two entries positive.

**1.16** ✓Let $X = [1, -1, 0]^t$ and $Y = [1, 0, -1]^t$. Are there any elements in their span with all entries positive? Explain.

**1.17** Let $X = [1, -2, 4]^t$ and $Y = [-1, 2, 3]^t$. Are there any elements in their span with all entries positive? Explain.

**1.18** Let $X = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ and $Y = \begin{bmatrix} -2 & -3 \\ 4 & -1 \end{bmatrix}$. Are there any elements in their span in the vector space $M(2, 2)$ with all entries positive? Explain.

**1.19** For each of the following sets of functions either find a function $f(x)$ in their span such that $f(x) > 0$ for all $x$ or prove that no such function exists.

**(a)** $\{\sin x, 1\}$ **(b)** $\{\cos x, 1\}$

**(c)** ✓ $\{\sin x, \cos x\}$

**1.20** The $xy$ plane in $\mathbb{R}^3$ is the set of elements of the form $[x, y, 0]^t$. Find a nonzero element of the $xy$ plane that belongs to the span of the vectors $X$ and $Y$ from (a) Exercise 1.16 and (b) Exercise 1.17. (c)✓Find two nonzero vectors $X$ and $Y$ in $\mathbb{R}^3$, $X \neq Y$, for which there are NO *nonzero* vectors $Z$ in the $xy$ plane that also belong to the span of $X$ and $Y$.

**1.21** Let $X = [x_1, y_1, z_1]$ and $Y = [x_2, y_2, z_2]$ be elements of $M(1, 3)$. Suppose that $a$, $b$, and $c$ are such that $ax_i + by_i + cz_i = 0$ for $i = 1, 2$. Show that every element $[x, y, z]^t$ of the span of $X$ and $Y$ satisfies $ax + by + cz = 0$.

**1.22** In Exercise 1.16, find constants $a, b, c$, not all zero, such that every element $[x, y, z]$ of the span of $X$ and $Y$ satisfies the equation $ax + by + cz = 0$. Repeat for Exercise 1.17. Explain geometrically why such constants should exist. [*Hint:* The equation $ax + by + cz = 0$ describes a plane through 0, as long as at least one of $a$, $b$, and $c$ is nonzero.]

**1.23** ✓✓Let $X$, $Y$, and $Z$ be as shown. Give four matrices (reader's choice) that belong to their span. Give a matrix that does not belong to their span.

$$X = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \quad Y = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

**1.24** In the following questions we investigate geometric significance of spanning and linear independence.

**(a)** Sketch the span of $[1, 2]^t$ in $\mathbb{R}^2$.

**(b)** What do you guess that the span of $\{[1, 2]^t, [1, 1]^t\}$ in $\mathbb{R}^2$ is? Draw a diagram to support your guess.

**(c)** Do you think that it is possible to find three linearly independent matrices in $M(2, 1)$?

**(d)** On the basis of your answer to (c), do you think that it is possible to construct a $2 \times 3$ matrix with linearly independent columns? How about a $3 \times 2$ matrix with linearly independent rows?

**(e)** Sketch (as best you can) the span of $\{[1, 1, 0]^t, [0, 0, 1]^t\}$ in $\mathbb{R}^3$.

**(f)** How does the span of $\{[1, 1, 1]^t, [1, 1, 0]^t\}$ in $\mathbb{R}^3$ compare with that in part (e)?

**(g)** Sketch the span of $\{[1, 1, 1]^t, [2, 2, 2]^t\}$ in $\mathbb{R}^3$. Why is this picture so different from that in part (f)? Bring the phrase " linearly dependent" into your discussion.

**1.25** ✓✓Suppose that $V$ and $W$ both belong to the span of $X$ and $Y$ in some vector space. Show that all linear combinations of $V$ and $W$ also belong to this span.

**1.26** ✓✓The columns of the following matrix $A$ are linearly dependent. Exhibit one column as a linear combination of the other columns.

$$\begin{bmatrix} 6 & 6 & 4 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

**1.27** Let $A$ be as in Exercise 1.26. Exhibit one row of $A$ as a linear combination of the other rows.

**1.28** Is it possible to find a $2 \times 2$ matrix whose rows are linearly dependent but whose columns are linearly independent? Prove your answer.

**1.29** ✓✓Construct an example of your own choice of a $4 \times 4$ matrix with linearly dependent rows having all of its entries nonzero.

**1.30** Construct an example of your own choice of a $4 \times 4$ matrix with linearly dependent columns having all of its entries nonzero.

**1.31** ✓Let $S = \{A, B, C, D\}$ be some set of four elements of some vector space. Suppose that $D = 2A + B + 3C$ and $C = A - B$. (a) Is $\{A, B, D\}$ linearly dependent? Explain. (b) Is $\{A, C, D\}$ linearly dependent? (c) What can you conclude (if anything) about the linear dependence of $\{A, B\}$?

**1.32** Let $S = \{A, B, C, D\}$ be some set of four elements of some vector space. Suppose that $A = B - 3C + D$ and $C = A - B$. (a) Is $\{A, B, D\}$ linearly dependent? Explain. (b) Is $\{A, C, D\}$ linearly dependent? (c) What can you conclude (if anything) about the linear dependence of $\{A, C\}$?

**1.33** The following sets of functions are linearly dependent in $\mathcal{F}(\mathbb{R})$. Show this by expressing one of them as a linear combination of the others. (You may need to look up the definitions of the sinh and cosh functions as well as some trigonometric identities in a calculus book.)

**(a)** $\{3\sin^2 x, -5\cos^2 x, 119\}$

**(b)** ✓✓ $\{2e^x, 3e^{-x}, \sinh x\}$

**(c)** $\{\sinh x, \cosh x, e^{-x}\}$

**(d)** ✓✓ $\{\cos(2x), \sin^2 x, \cos^2 x\}$

**(e)** $\{\cos(2x), 1, \cos^2 x\}$

**(f)** ✓✓ $\{(x+3)^2, 1, x, x^2\}$

**(g)** $\{x^2 + 3x + 3, x + 1, 2x^2\}$

**(h)** ✓✓ $\{\sin x, \sin(x + \frac{\pi}{4}), \cos(x + \frac{\pi}{4})\}$

**(i)** ✓✓ $\{\ln[(x^2+1)^3/(x^4+7)], \ln\sqrt{x^2+1}, \ln(x^4+7)\}$

**1.34** ✓Give two examples of functions in the span of the functions $\{1, x, x^2\}$. Describe in words what the span of these three functions is. [Some useful terminology: the function $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is a polynomial of degree less than or equal to $n$. Its degree equals $n$ if $a_n \neq 0$.]

**1.35** Repeat Exercise 1.34 for the polynomials $\{1, x, x^2, x^3\}$.

**1.36** ✓✓Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

**(a)** Use the definition of matrix addition to prove that the only $2 \times 2$ matrix $B$ such that $A + B = A$ is the zero matrix. The point of this problem is that one should think of $A + 0 = A$ as the defining property of the zero matrix.

**(b)** Use the definition of matrix addition to prove that the only $2 \times 2$ matrix $B$ such that $A + B = 0$ is

$$B = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$$

The point of this problem is that one should think of $A + (-A) = 0$ as the defining property of $-A$.

**1.37** Prove vector space properties (b), (e), (g), (h), and (i)✓for $M(m, n)$.

**1.38** Let $B$, $C$, and $X$ be elements of some vector space. In the following discussion, we solved the equation $3X + B = C$ for $X$. At each step we used one of the vector space properties. Which property was used? [*Note:* We define $C - B = C + (-B)$.]

$$\begin{aligned}
3X + B &= C & & \\
(3X + B) + (-B) &= C + (-B) & &\text{Properties (a) and (e)}\\
3X + (B + (-B)) &= C - B & &\text{Definition of } C - B \text{ and property (?)}\\
3X + 0 &= C - B & &\text{Property (?)}\\
3X &= C - B & &\text{Property (?)}\\
\frac{1}{3}(3X) &= \frac{1}{3}(C - B) & &\text{Property (?)}\\
\left(\frac{1}{3}3\right)X &= \frac{1}{3}(C - B) & &\text{Property (?)}\\
1X &= \frac{1}{3}(C - B) & & \\
X &= \frac{1}{3}(C - B) & &\text{Property (?)}
\end{aligned}$$

**1.39** Let $X$ and $Y$ be elements of some vector space. Prove, putting in every step, that $-(2X + 3Y) = (-2)X + (-3)Y$. You may find Proposition 1.2 useful.

**1.40** ✓✓Let $X$, $Y$, and $Z$ be elements of some vector space. Suppose that there are scalars $a$, $b$, and $c$ such that $aX + bY + cZ = 0$. Show that if $a \neq 0$, then

$$X = \left(-\frac{b}{a}\right)Y + \left(-\frac{c}{a}\right)Z$$

Do your proof in a step-by-step manner to demonstrate the use of each vector space property needed. [*Note*: In a vector space, $X + Y + Z$ denotes $X + (Y + Z)$.]

**1.41** Prove that in any vector space, if $X + Y = 0$, then $Y = -X$. (Begin by adding $-X$ to both sides of the given equality.)

**1.42** Prove Proposition 1.2. [*Hint:* From Exercise 1.41, it suffices to prove that $X + (-1)X = 0$.]

### 1.1.1 Computer Projects

Our goal in this discussion is to plot some elements of the span of the vectors $A = [1, 1]$ and $B = [2, 3]$ using MATLAB. Before we begin, however, let us make a few general comments. When you start up MATLAB, you will see something like >> followed by a blank line. If the instructions ask you to enter $2 + 2$, then you should type $2 + 2$ on the screen behind the >> prompt and then press the enter key. Try it!

```
>> 2+2
ans =
     4
```

Entering matrices into MATLAB is not much more complicated. Matrices begin with "[" and end with "]". Entries in rows are separated with either commas or spaces. Thus, after starting MATLAB, our matrices $A$ and $B$ would be entered as shown. Note that MATLAB repeats our matrix, indicating that it has understood us.

```
>> A = [1  1]
A =
    1     1
>> B = [1  3]
B =
    1     3
```

Next we construct a few elements of the span of A and B. If we enter "2∗A+B", MATLAB responds

```
ans =
     3    5
```

(Note that ∗ is the symbol for "times." MATLAB will complain if you simply write 2A+B.)

If we enter (−5) ∗A +7 ∗B, MATLAB responds

```
ans =
     2    16
```

Thus, the vectors $[3, 5]$ and $[2, 16]$ both belong to the span.

We can get MATLAB to automatically generate elements of the span. Try entering the word "rand". This should cause MATLAB to produce a random number between 0 and 1. Enter "rand" again. You should get a different random number. It follows that entering the command C=rand∗A+rand∗B should produce random linear combinations of $A$ and $B$. Try it!

To see more random linear combinations of these vectors, push the up-arrow key. This should return you to the previous line. Now you can simply hit "enter" to produce a new random linear combination. By repeating this process, you can produce as many random elements of the span as you wish.

Next, we will plot our linear combinations. Begin by entering the following lines. Here "figure" creates a figure window, "hold on" tells MATLAB to plot all points on the same graph, and "axis([-5,5,-5,5])" tells MATLAB to show the range $-5 \le x \le 5$ and $-5 \le y \le 5$ on the axes. The command "hold on" will remain in effect until we enter "hold off":

```
>> figure
>> hold on
>> axis([-5,5,-5,5])
```

A window (the Figure window) showing a blank graph should pop up.

Points are plotted in MATLAB using the command "plot." For example, entering plot(3,4) will plot the point $(3, 4)$. Return to the MATLAB Command window and try plotting a few points of your own choosing. (To see your points, you will either need to return to the Figure window or move and resize the Command and Figure windows so that you can see them both at the same time. Moving between windows is accomplished by pulling down the Window menu.) When you are finished, clear the figure window by entering "cla" and then enter the following line:

```
C=rand*A+rand*B plot(C(1),C(2))
```

This will plot one point in the span. [ C(1) is the first entry of C and C(2) is the second.] You can plot as many points as you wish by using the up-arrow key as before.

## EXERCISES

**1.** Plot the points $[1, 1]$, $[1, -1]$, $[-1, 1]$, and $[-1, -1]$ all on the same figure. When finished, clear the figure window by entering the "cla" command.

**2.** Enter the vectors A and B from the discussion above.

- **(a)** Get MATLAB to compute several different linear combinations of them. (Reader's choice.)
- **(b)** Use C=rand*A+rand*B to create several "random" linear combinations of A and B.
- **(c)** Plot enough points in the span of A and B to obtain a discernible geometric figure. Be patient. This may require plotting over 100 points. What kind of geometric figure do they seem to form? What are the coordinates of the vertices?

  *Note*: If your patience runs thin, you might try entering the following three lines. The ";" keeps MATLAB from echoing the command every time it is being executed.

```
for i=1:200
  C=rand*A+rand*B; plot(C(1),C(2));
end
```

This causes MATLAB to execute any commands between the "for" and "end" statements 200 times.

**(d)** The plot in part (c) is only part of the span. To see more of the span, enter the commands

```
for i=1:200
  C=2*rand*A+rand*B; plot(C(1),C(2),'r');
end
```

The "r" in the plot command tells MATLAB to plot in red.

**3.** Describe in words the set of points $s*A + t*B$ for $-2 \leq s \leq 2$ and $-2 \leq t \leq 2$. Create a MATLAB plot that shows this set reasonably well. Use yet another color. (Enter "help plot" to see the choice of colors.) [*Hint:* "rand" produces random numbers between 0 and 1. What would "rand-0.5" produce?]

**4.** In Exercise 1.13 on page 19, it was stated that each element of the span of $X$ and $Y$ satisfies $5x + 3y - 2z = 0$.

**(a)** Check this by generating a random matrix $C$ in the span of $X$ and $Y$ and computing 5∗C(1)+3*C(2)-2*C(3). Repeat with another random element of the span.

**(b)** Plot a few hundred elements of this span in $\mathbb{R}^3$. Before doing so, close the Figure window by selecting Close from the File menu. Next, enter "figure", then "axis([-4,4,-4,4,-4,4])", and "hold on". A three-dimensional graph should pop up. The command plot3(C(1),C(2),C(3)) plots the three-dimensional vector C.

Describe the geometric figure so obtained. What are the coordinates of the vertices? Why is this to be expected?

### 1.1.2 Applications to Graph Theory I

Figure 1.7 represents the route map of an airline that serves four cities, A, B, C, and D. Each arrow represents a daily flight between the two cities.

The information from this diagram can be represented in tabular form, where the numbers represent the number of daily flights between the cities:

| from/to | A | B | C | D |
|---|---|---|---|---|
| A | 0 | 1 | 0 | 1 |
| B | 1 | 0 | 0 | 1 |
| C | 0 | 1 | 0 | 1 |
| D | 1 | 0 | 2 | 0 |

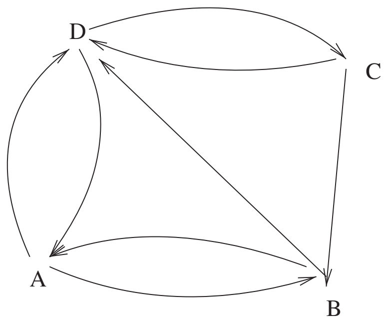



**FIGURE 1.7** Route map.

The $4 \times 4$ matrix obtained by deleting the labels from the preceding table is what we refer to as the route matrix. The route matrix could be stored in a computer. One could then, for example, use this information as the basis for a computer program to find the shortest connection for a customer.

Route maps are examples of what are called **directed graphs**. In general, a directed graph is a finite set of points (called **vertices**), together with arrows connecting some of the vertices. A directed graph may be described using a matrix just as was done for route maps. Specifically, if the vertices are $V_1, V_2, \ldots, V_n$, then the graph will be represented by the matrix $A$, where $a_{ij}$ is the number of arrows from $V_i$ to $V_j$.

Graph theory may also be applied to anthropology. Suppose an anthropologist is studying generational dominance in an extended family. The family members are M (mother), F (father), S1 (first born son), S2 (second born son), D1 (first born daughter), D2 (second born daughter), MGM (maternal grandmother), MGF (maternal grandfather), PGM (paternal grandmother), and PGF (paternal grandfather). The anthropologist represents the dominance relationships by a directed graph where an arrow is drawn from each individual to any individual he or she directly dominates. In the exercises you will study the dominance relationship given in Figure 1.8.

We will say more about the matrix of a graph in Section 3.2.

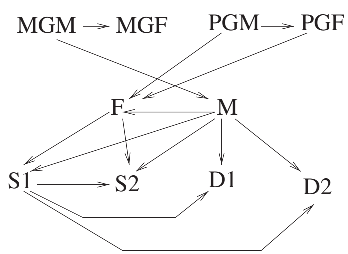



**FIGURE 1.8** Dominance in an extended family.

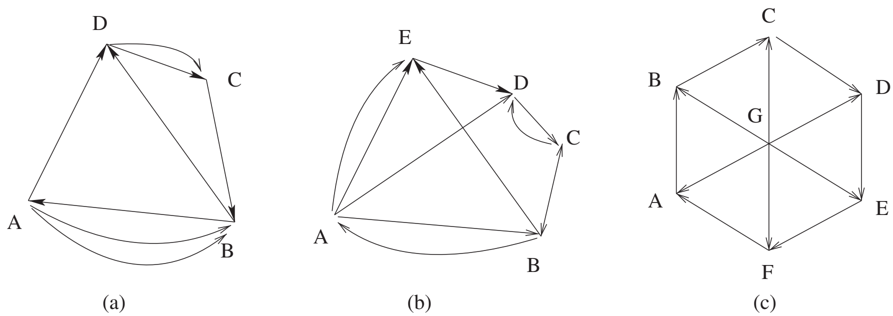



**FIGURE 1.9** Exercise 1.

## Self-Study Questions

**1.1** ✓Give the route matrix for each of the route maps in Figure 1.9 where the nodes are listed in alphabetical order.

**1.2** ✓For each of the following matrices, draw a route map that could correspond to the given matrix:

$$\text{(a)} \quad \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \text{(b)} \quad \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

**1.3** ✓Why are entries on the diagonal in a route matrix always zero?

## EXERCISES

**1.43** Under what circumstances does a route matrix $A$ satisfy $A = A^t$?

**1.44** What would be the significance for the route if all the entries in a given column were zero? What if a given row were zero? What if a row and a column were zero?

**1.45** ✓Give the dominance matrix (the matrix for the graph) for the dominance relationship described by Figure 1.8.

**1.46** Suppose that $A$ is the matrix of a dominance relationship. Explain why $a_{ij}a_{ji} = 0$.

**1.47** ✓We say that two points $A$ and $B$ of a directed graph are two-step connected if there is a point $C$ such that $A \to C \to B$. Thus, for example, in the route map in Figure 1.7, $A$ and $C$ are two-step connected, but $D$ and $C$ are not. Also $A$

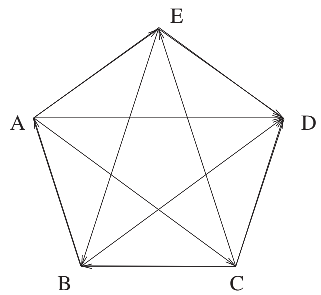



**FIGURE 1.10** Exercise 1.48.

is two-step connected with itself. Give the two-step route matrix for the route map in Figure 1.7.

**1.48** Figure 1.10 shows the end-of-season results from an athletic conference with teams A–D can be described using a graph. The arrows indicate which team beat which.

1. Find the matrix $A$ for the graph in Figure 1.10.
2. Compute the win–loss record of team $C$.

## 1.2 SYSTEMS

An equation in variables $x_1,\ x_2, \dots, x_n$ is a **linear equation** if and only if it is expressible in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1.7}$$

where $a_i$ and $b$ are all scalars. By a **solution** to equation (1.7) we mean a column vector $[x_1, x_2, \dots, x_n]^t$ of values for the variables that make the equation valid. Thus, $X = [1, 2, -1]^t$ is a solution of the equation

$$2x + 3y + z = 7$$

because

$$2(1) + 3(2) + (-1) = 7$$

More generally, a set of linear equations in a particular collection of variables is called a **linear system** of equations. Thus, the general system of linear equations in the variables $x_1, x_2, \dots, x_n$ may be written as follows:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{1.8}$$

A solution to the system is a column vector that is a solution to each equation in the system. The set of all column vectors that solve the system is the **solution set** for the system.

In particular,

$$\begin{aligned} x + 2y + z &= 1 \\ 3x + y + 4z &= 0 \\ 2x + 2y + 3z &= 2 \end{aligned} \tag{1.9}$$

is a linear system in the variables $x, y$, and $z$.

Finding all solutions to this system is not hard. We begin by subtracting three times the first equation from the second, producing

$$\begin{aligned} x + 2y + \ \ z &= 1 \\ -5y + \ \ z &= -3 \\ 2x + 2y + 3z &= 2 \end{aligned} \tag{1.10}$$

Any $x, y$, and $z$ that satisfy the original system also satisfy the system above.

Conversely, notice that we can transform the above system back into the original by *adding* three times the first equation onto the second equation. Thus, any variables that satisfy the second system must also satisfy the first. Hence, both systems have the same solution set. We say that these systems are equivalent:

**Definition 1.7** *Two systems of linear equations in the same variables are* ***equivalent*** *if they have the same solution set.*

To continue the solution process, we next subtract twice the first equation from the third, producing

$$\begin{aligned} x + 2y + z &= 1 \\ -5y + z &= -3 \\ -2y + z &= 0 \end{aligned}$$

Note that we have eliminated all occurrences of $x$ from the second and third equations. This system is equivalent with our second system for similar reasons that the second

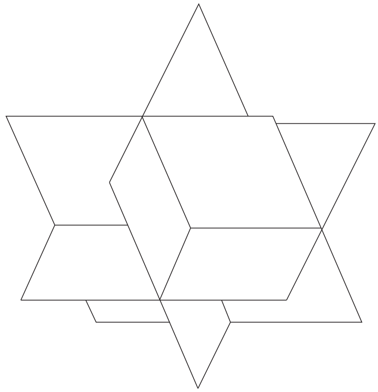


**FIGURE 1.11** Only one solution.

system was equivalent with the first. It follows that this system has the same solution set as the original system.

Next, we eliminate $y$ from the third equation by subtracting twice the second from five times the third, again producing an equivalent system:

$$\begin{aligned} x + 2y + \phantom{3}z &= 1 \\ -5y + \phantom{3}z &= -3 \\ 3z &= 6 \end{aligned} \tag{1.11}$$

Thus, $z = 2$. Then, from the second equation, $y = 1$, and finally, from the first equation, $x = -3$. Thus, our only solution is $[-3, 1, 2]^t$.

The fact that there was only one solution can be understood geometrically. Each of the equations in system (1.9) describes a plane in $\mathbb{R}^3$. A point that satisfies each equation in the system must lie on all three planes. Typically, three planes in $\mathbb{R}^3$ intersect at precisely one point, as shown in Figure 1.11.

*Remark.* The method we used to compute the solution from system (1.11) is referred to as **back substitution**. In general, in back substitution, we solve the last equation for one variable and then substitute the result into the preceding equations, obtaining a system with one fewer variable and one fewer equation, to which the same process may be repeated. In this way, we obtain all solutions to the system.

The process we used to reduce system (1.9) to system (1.11) is called **Gaussian elimination**. The general idea is to use the first equation to eliminate all occurrences of the first variable from the equations below it. One then attempts to use the second equation to eliminate the next variable from all equations below it, and so on. In the end, the last variable is determined first ($z$ in our example) and then the others are determined by substitution as in the example.

We will describe Gaussian elimination in detail in the next section, after considering several more examples. First, however, we introduce a "shorthand" notation for systems that can save considerable writing. We associate with a system of linear equations a matrix, called the **augmented matrix**, formed by both the coefficients of the variables and the constants on the right side of the equalities.[2] Thus, for example, the augmented matrix for system (1.9) (which we repeat on the left) is the matrix on the right:

$$\begin{aligned} x + 2y + \phantom{3}z &= 1 \\ 3x + \phantom{2}y + 4z &= 0 \\ 2x + 2y + 3z &= 2 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 1 & 4 & 0 \\ 2 & 2 & 3 & 2 \end{bmatrix}$$

We can do the whole solution process using the augmented matrix, avoiding the necessity of constantly writing both the names of the variables and the equality signs. Thus, for example, our first step in solving system (1.9) was to subtract three times the first equation from the second, producing system (1.10). This corresponds to subtracting three times the first row of the augmented matrix from the second, producing

$$\begin{aligned} x + 2y + \phantom{3}z &= 1 \\ -5y + \phantom{3}z &= -3 \\ 2x + 2y + 3z &= 2 \end{aligned} \qquad \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -5 & 1 & -3 \\ 2 & 2 & 3 & 2 \end{bmatrix}$$

which is the augmented matrix for the system (1.10).

Continuing as in the solution of (1.9), we eventually arrive at the augmented matrix

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -5 & 1 & -3 \\ 0 & 0 & 3 & 6 \end{bmatrix}$$

which we reinterpret as a system and solve just as before.

Back to examples!Consider the following system:

$$\begin{aligned} x + \phantom{3}y + \phantom{5}z &= 1 \\ 4x + 3y + 5z &= 7 \\ 2x + \phantom{3}y + 3z &= 5 \end{aligned} \tag{1.12}$$

Although this system appears similar to the other, close inspection reveals that the second equation equals the third plus twice the first. Thus, the second equation carries

[2]The matrix formed using just the coefficients of the variables, and not the constants on the right of the equalities, is called the coefficient matrix for the system. The augmented matrix is then the coefficient matrix *augmented* by the column of constants. We discuss the coefficient matrix in the next section.

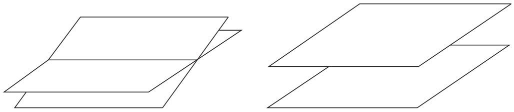


**FIGURE 1.12** Two planes.

no additional information: any solution to the first and third equations automatically satisfies the second. Hence, our system is equivalent to the system

$$\begin{aligned} x + y + \phantom{3}z &= 1 \\ 2x + y + 3z &= 5 \end{aligned}$$

In $\mathbb{R}^3$, two planes either have no intersection (if they happen to be parallel) or intersect in a line. (see Figure 1.12.) Thus, a system of two linear equations in three unknowns is never sufficient to uniquely determine the unknowns; either there are no solutions (the planes are parallel) or an infinite number of solutions (the planes intersect in a line). The same principle holds in general; *it takes at least n linear equations to uniquely determine n unknowns; a system of linear equations with more unknowns than equations either has no solutions or has an infinite number of solutions.* (We prove this in the next section.)

Let us apply the Gaussian elimination technique to system (1.12) and see what happens. The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 3 & 5 & 7 \\ 2 & 1 & 3 & 5 \end{bmatrix} \tag{1.13}$$

We begin by subtracting multiples of the first row from the other rows so as to eliminate $x$ from the last two equations. As an aid to the reader, we indicate beside each row the transformation used to produce it, where $R_i$ denotes the $i$th row. We obtain

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 3 \\ 0 & -1 & 1 & 3 \end{bmatrix} \qquad \begin{array}{l} \\ R_2 \rightarrow R_2 - 4R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{array}$$

Next, we subtract the second row from the third to eliminate $y$ from the third equation. Notice, however, that as we eliminate $y$ from the third equation, we also eliminate $z$. We obtain the augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} \\ \\ R_3 \rightarrow R_3 - R_2 \end{array}$$

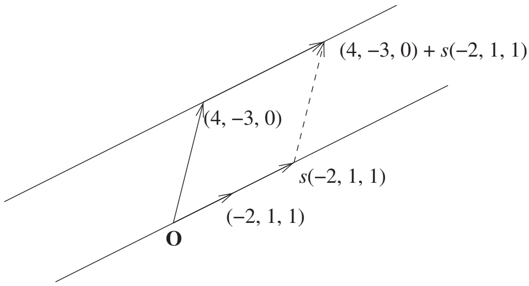



**FIGURE 1.13** Solution set is a line.

which corresponds to the system

$$\begin{aligned} x + y + z &= 1 \\ -y + z &= 3 \\ 0 &= 0 \end{aligned}$$

In this system, $z$ can be set equal to any arbitrary number, say, $z = s$. Then back substitution yields $y = z - 3 = s - 3$ and $x = 1 - y - z = 4 - 2s$. Hence, our solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 - 2s \\ s - 3 \\ s \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} -2s \\ s \\ s \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \tag{1.14}$$

If we think of $[x, y, z]^t$ as representing the point $(x, y, z)$ in $\mathbb{R}^3$, then the set of multiples of $[-2, 1, 1]^t$ is the line through 0 spanned by this vector. Adding $[4, -3, 0]^t$ translates this line away from 0 (Figure 1.13). Thus, our solution set is indeed a line.

*Remark.* Our original system (1.12), of course, had three equations. Since the solution set for this system is a line, the planes described by these equations must be configured as in Figure 1.14 on page 34.

Of course, in solving system (1.12), we could equally well have let $y = s$ and set $z = y + 3$. This yields $x = -2 - 2s$, so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 - 2s \\ s \\ s + 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} -2s \\ s \\ s \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \tag{1.15}$$

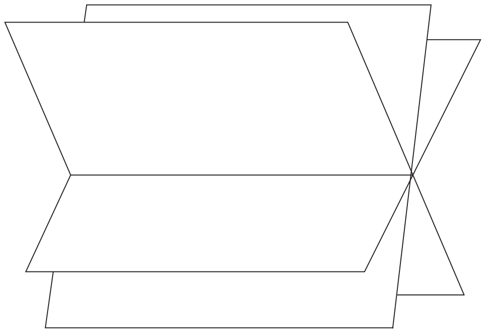


**FIGURE 1.14** Three planes intersecting in a line.

Although this solution looks different from the previous one, it is just a different parametric representation of the same line.

Formula (1.14) expresses our solution in what we refer to as "parametric form." In general, **parametric form** means that the solution, when expressed as a vector, is written as a constant vector (the **translation vector**) plus the span of some other vectors (the **spanning vectors**.)

Let us next produce an example of a system with no solutions. Consider the following system, which is identical to (1.12), except that the 5 in the last equation has been changed to 6:

$$\begin{aligned} x + \phantom{3}y + \phantom{5}z &= 1 \\ 4x + 3y + 5z &= 7 \\ 2x + \phantom{3}y + 3z &= 6 \end{aligned} \tag{1.16}$$

Now, however, the sum of twice the first equation with the third produces

$$4x + 3y + 5z = 8$$

which contradicts the second. Hence, there are no solutions. Geometrically, changing the 5 to 6 in the third equation of system (1.12) shifted one of the planes parallel to itself, transforming Figure 1.14 into a diagram similar to Figure 1.15, where there are no points of intersection.

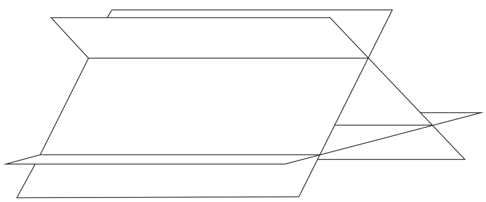


**FIGURE 1.15** No solution.

**Definition 1.8** *A system of linear equations that has no solution is **inconsistent**.*

Let us go through the elimination process to see how this problem manifests itself. The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 3 & 5 & 7 \\ 2 & 1 & 3 & 6 \end{bmatrix}$$

Again, we subtract multiples of the first row from the others. We obtain

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 3 \\ 0 & -1 & 1 & 4 \end{bmatrix} \qquad \begin{matrix} \\ R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

Finally, we subtract the second row from the third:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{matrix} \\ \\ R_3 \to R_3 - R_2 \end{matrix}$$

The last row corresponds to the obviously contradictory statement that $0 = 1$, showing that the system is indeed inconsistent.

## Rank: The Maximum Number of Linearly Independent Equations

As mentioned previously, the "problem" in our second example [system (1.12)] was that the second equation equaled twice the first plus the third. It follows that the second row of the augmented matrix [formula (1.13)] is also twice the first plus the third. Thus, the rows of the augmented matrix are linearly dependent. In general, if any row in the augmented matrix is linearly dependent on the other rows, the corresponding equation in the system may be eliminated without changing the solution set. This amounts to eliminating this row from the augmented matrix. If the rows of the new augmented matrix are still linearly dependent, we can eliminate another row without changing the solution set. We can continue dropping linearly dependent rows until a matrix with linearly independent rows is obtained. It is a remarkable theorem (which we prove in Section 2.3) that *the matrix so produced will always have the same number of rows, regardless of which specific rows were eliminated.*

**Definition 1.9** *The **rank** of a system of equations is the number of rows remaining in the augmented matrix after eliminating linearly dependent rows one at a time until a matrix with linearly independent rows is obtained.*

We give a better definition of rank in the next section. The significance of the rank is that a system of rank $r$ is equivalent to a system having only $r$ equations, regardless of the number of equations in the original system.

We close this section with another example of linear dependence in systems. Consider the system

$$\begin{aligned} 4x + 5y + 3z + 3w &= 1 \\ x + y + z + w &= 0 \\ 2x + 3y + z + w &= 1 \\ 5x + 7y + 3z + 3w &= 2 \end{aligned} \tag{1.17}$$

The augmented matrix is

$$A = \begin{bmatrix} 4 & 5 & 3 & 3 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 2 & 3 & 1 & 1 & 1 \\ 5 & 7 & 3 & 3 & 2 \end{bmatrix} \tag{1.18}$$

The rows of $A$ are linearly dependent. Specifically, if $A_i$ denotes the $i$th row, then

$$\begin{aligned} A_1 &= A_3 + 2A_2 \\ A_4 &= 2A_3 + A_2 \end{aligned} \tag{1.19}$$

Thus, in this system, the first and fourth equations may be eliminated without changing the solution set, producing the augmented matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 2 & 3 & 1 & 1 & 1 \end{bmatrix}$$

If we preferred, we could keep the first and fourth equations and eliminate the second and third since, from formula (1.19),

$$\begin{aligned} 2A_1 - A_4 &= 3A_2 \\ 2A_4 - A_1 &= 3A_3 \end{aligned}$$

This produces the system with augmented matrix

$$A = \begin{bmatrix} 4 & 5 & 3 & 3 & 1 \\ 5 & 7 & 3 & 3 & 2 \end{bmatrix}$$

Either way, our system is equivalent with one with only two equations: this is a rank 2 system. Without going further, we can say that either there are no solutions or there are an infinite number of solutions.

As in the second example, this system actually has an infinite number of solutions. Again, let us suppose that we did not notice the linear dependence of the system. We could begin by subtracting multiples of the first row of the augmented matrix from the others in order to eliminate $x$. It is easier to first interchange the first and second equations, however, since the coefficient of $x$ in the second is 1. This is equivalent to interchanging the first and second rows of the augmented matrix, producing

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 4 & 5 & 3 & 3 & 1 \\ 2 & 3 & 1 & 1 & 1 \\ 5 & 7 & 3 & 3 & 2 \end{bmatrix} \qquad \begin{matrix} R_2 & \to & R_1 \\ R_1 & \to & R_2 \\ & & \\ & & \end{matrix}$$

Subtracting multiples of the first equation from the others, we get

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 2 & -2 & -2 & 2 \end{bmatrix} \qquad \begin{matrix} & & \\ R_2 & \to & R_2 - 4R_1 \\ R_3 & \to & R_3 - 2R_1 \\ R_4 & \to & R_4 - 5R_1 \end{matrix}$$

Next we use the second row to eliminate $y$ from the last two equations, producing

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} & & \\ & & \\ R_3 & \to & R_3 - R_2 \\ R_4 & \to & R_4 - 2R_2 \end{matrix} \tag{1.20}$$

Our solution process stops since $z$ and $w$ were also eliminated from the last two equations. We are left with the system

$$\begin{aligned} x + y + z + w &= 0 \\ y - z - w &= 1 \end{aligned}$$

There are only two equations since this is a rank 2 system.

We solve the second equation by setting $z = r$ and $w = s$, where $r$ and $s$ are arbitrary real numbers. Then back substitution yields $y = 1 + r + s$ and $x = -2r - 2s - 1$. Thus,

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -2r - 2s - 1 \\ 1 + r + s \\ r \\ s \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2r \\ r \\ r \\ 0 \end{bmatrix} + \begin{bmatrix} -2s \\ s \\ 0 \\ s \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \tag{1.21}$$

Hence, the translation vector is $[-1, 1, 0, 0]^t$. The spanning vectors are $[-2, 1, 1, 0]^t$ and $[-2, 1, 0, 1]^t$.

We may describe our solution in geometric terms. We interpret $[x, y, z, w]^t$ as representing a point in $\mathbb{R}^4$. Our solution set is then the "plane" in $\mathbb{R}^4$ spanned by $[-2, 1, 1, 0]^t$ and $[-2, 1, 0, 1]^t$, translated by $[-1, 1, 0, 0]^t$.

**Definition 1.10** *A **line** in $\mathbb{R}^n$ is the set of points obtained by translating the span of a single, nonzero vector in $\mathbb{R}^n$. A **plane** in $\mathbb{R}^n$ is the set of points obtained by translating the span of a pair of linearly independent vectors in $\mathbb{R}^n$. A **hyperplane** in $\mathbb{R}^n$ is the set of points obtained by translating the span of $n - 1$ linearly independent vectors in $\mathbb{R}^n$.*

**True-False Questions:** Justify your answers.

**1.13** The solution set to a system of three equations in three unknowns cannot be a plane.

**1.14** A system of linear equations cannot have only two solutions.

**1.15** The solution set to a consistent rank 2 linear system in four unknowns would be a line in four-dimensional space.

**1.16** A system of four equations in four unknowns always has a solution.

**1.17** A system of four equations in four unknowns can have at most one solution.

**1.18** The rank of a system is always less than or equal to the number of equations in the system.

**1.19** Use geometric reasoning to answer the following questions concerning systems (i) and (ii) below:

**(a)** If (i) has exactly one solution, then the same is true for (ii).

**(b)** If the solution set of (i) is a line, then the same is true for (ii).

**(c)** If (i) has no solutions, then the same is true for (ii).

$$\text{(i)}\quad \begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \qquad \text{(ii)}\quad \begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 + 1 \end{aligned}$$

## EXERCISES

**1.49** ✓✓Let $X = [1, 1, 1, 1]^t$ and $Y = [1, 2, -1, 1]^t$. One of these vectors is a solution to the system below and one is not. Which is which?

$$\begin{aligned} 4x - 2y - z - w &= 0 \\ x + 3y - 2z - 2w &= 0 \end{aligned}$$

**1.50** ✓✓Let $X$ and $Y$ be as in Exercise 1.49. For which scalars $a$ and $b$ is it true that $aX + bY$ is a solution to the system in Exercise 1.49?

**1.51** Let $Z = [1, 1, 2, 0]^t$ and let $X$ be as in Exercise 1.49. For which scalars $a$ and $b$ is it true that $aX + bZ$ is a solution to the system in Exercise 1.49?

**1.52** ✓Let $U = [x, y, z, w]^t$ and $V = [x', y', z', w']^t$ be solutions to the system in Exercise 1.49. For which scalars $a$ and $b$ is $aU + bV$ also a solution?

**1.53** Let $U = [1, 1, 2, -1]^t$ and $V = [1, 1, 1, 0]^t$. For which scalars $a$ and $b$ is it true that $aU + bV$ is a solution to the following system?

$$\begin{aligned} 4x - 2y - z - w &= 1 \\ x + 3y - 2z - 2w &= 2 \end{aligned}$$

**1.54** ✓Let $U = [x, y, z, w]^t$ and $V = [x', y', z', w']^t$ be solutions to the system in Exercise 1.53. For which scalars $a$ and $b$ is $aU + bV$ also a solution?

**1.55** For each system: (i) Write the augmented matrix $A$. (ii) Find all solutions (if any exist). Express your answer in parametric form and give the translation vector and the spanning vectors. State whether the solution is a line or plane or neither. (iii) If one of the rows of the augmented matrix becomes zero during the solution process, explicitly exhibit one row of $A$ as a linear combination of the other rows.

**(a)**

$$\begin{aligned} x - 3y &= 2 \\ -2x + 6y &= -4 \end{aligned}$$

**(b)** ✓✓

$$\begin{aligned} x + 3y + z &= 1 \\ 2x + 4y + 7z &= 2 \\ 3x + 10y + 5z &= 7 \end{aligned}$$

**(c)** ✓✓

$$\begin{aligned} x + 3y + z &= 1 \\ 2x + 4y + 7z &= 2 \\ 4x + 10y + 9z &= 4 \end{aligned}$$

**(d)** ✓✓

$$\begin{aligned} x + 3y + z &= 1 \\ 2x + 4y + 7z &= 2 \\ 4x + 10y + 9z &= 7 \end{aligned}$$

**(e)**

$$\begin{aligned} x_1 + 2x_2 + x_3 \phantom{+3x_4} &= 1 \\ x_2 + 4x_3 + 3x_4 &= 2 \\ 2x_3 + 2x_4 &= 4 \end{aligned}$$

**(f)** ✓✓

$$\begin{aligned} x - y + 2z - 2w &= 1 \\ 2x + y \phantom{+2z} + 3w &= 4 \\ 2x + 3y + 2z \phantom{-2w} &= 6 \end{aligned}$$

**(g)**

$$\begin{aligned} 3x + 7y + 2z &= 1 \\ x - y + z &= 2 \\ 5x + 5y + 4z &= 5 \end{aligned}$$

**(h)**

$$\begin{aligned} 2x - 3y + 2z &= 1 \\ x - 6y + z &= 2 \\ -x - 3y - z &= 1 \end{aligned}$$

**(i)**

$$\begin{aligned} 2x + 3y - z &= -2 \\ x - y + z &= 2 \\ 2x + 3y + 4z &= 5 \end{aligned}$$

**(j)** ✓✓

$$\begin{aligned} x + y + z + w &= 1 \\ 2x - 2y + z + 2w &= 3 \\ 2x + 6y + 3z + 2w &= 1 \\ 5x - 3y + 3z + 5w &= 7 \end{aligned}$$

**(k)**

$$\begin{aligned} x + y + z + w &= 1 \\ 2x - 2y + z + 2w &= 3 \\ 2x + 6y + 3z + 2w &= 1 \\ 5x - 3y + 3z + 5w &= 8 \end{aligned}$$

**(l)** ✓

$$\begin{aligned} x + 2y - z - 2w &= 1 \\ -3x - 3y + z + 10w &= -6 \\ -5x - 4y + z + 18w &= -11 \\ -2x + 5y - 4z + 16w &= -11 \end{aligned}$$

**1.56** In Exercise 1.55.j, give two different solutions to the system (reader's choice). Call your solutions $X$ and $Y$. Show that $(X + Y)/2$ is a solution to the system but $X + Y$ is not. Repeat for part 1.55.j.

**1.57** Create an example of a system of five equations in five unknowns that has rank 2. How about one with rank 3? Rank 1?

**1.58** ✓✓The following system is obviously inconsistent. On a single sheet of graph paper, graph $y - 2x = 0$ and $y - 2x = 1$ as lines in $\mathbb{R}^2$. What geometric property of these lines do you notice? How is this relevant to the inconsistency of the system?

$$\begin{aligned} y - 2x &= 1 \\ y - 2x &= 0 \end{aligned}$$

**1.59** Let $a$ be a fixed number. On a single sheet of graph paper, graph $y - x = 1$, $y + x = 1$, and $y - 2x = a$ as lines in $\mathbb{R}^2$. Using only your graph, find all values of $a$ for which the following system is consistent:

$$\begin{aligned} y - x &= 1 \\ y + x &= 1 \\ y - 2x &= a \end{aligned}$$

### 1.2.1 Computer Projects

## EXERCISES

1. Solve (by hand) equation (1.22). You should obtain the general solution $X = [4, 0, 0]^t + s[-2, 1, 0]^t + t[-3, 0, 1]^t$ where $s$ and $t$ are arbitrary parameters. Then do (a)–(c).

$$x + 2y + 3z = 4 \tag{1.22}$$

   **(a)** Use the MATLAB command "rand" to create a "random" element $C$ of the span of the vectors $V = [-2, 1, 0]^t$ and $W = [-3, 0, 1]^t$. (see page 24.)
   
   *Note*: When entering matrices into MATLAB, one can use semicolons (";") to separate columns. Thus, we could enter the column vector $V$ into MATLAB as V=[-2;1;0].

   **(b)** Let $X = [4, 0, 0]^t + C$. Check by direct substitution that X solves equation (1.22). [You can ask MATLAB to compute $X(1) + 2 * X(2) + 3 * X(3)$.]
   
   This demonstrates that the general solution to our system is the span of the spanning vectors translated by the translation vector.

   **(c)** Substitute $C$ into equation (1.22) and describe what you get. Try substituting some other random linear combinations of $V$ and $W$ into this equation. What do you conjecture? Can you prove it?

2. Repeat 1a–1c for system (1.17) on page 36 whose solution appears on page 37. Specifically, (a) create a random element $C$ in the span of $V = [-2, 1, 1, 0]^t$ and $W = [-2, 1, 0, 1]^t$. Then (b) substitute $C + [-1, 1, 0, 0]^t$ into *each* equation in system (1.17) and (c) substitute $C$ into *each* equation in the system. Does the conjecture you made in Exercise 1 above still hold? Can you prove it?

### 1.2.2 Applications to Circuit Theory

In this section we study the flow of electricity through simple electrical circuits. In many respects, electricity flowing through a wire is similar to water flowing through a pipe. Water flow in a pipe is created by water molecules moving through the pipe. Electrical flow through a wire is created by electrons moving through the wire. The rate of flow of water at a point along the pipe is a measure of the number of water molecules that pass the given point during a stated amount of time. It might, for example, be measured in gallons/second (gal/sec). The number of electrons passing a given point along a wire in a given amount of time is measured by the **current** and is typically measured in **amperes** (amps, $i$). (One ampere represents a flow rate of $6.241 \times 10^{18}$ electrons/sec.) The force of the water in the pipe is called pressure which could, for example, be measured in pounds per square inch (psi). In electricity the force is called **voltage** and is measured in **volts** ($V$).

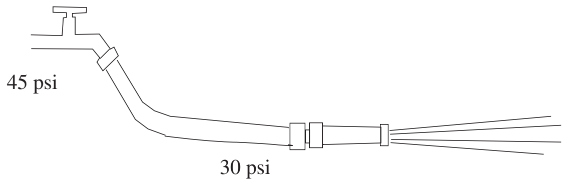



**FIGURE 1.16** Pressure drop.

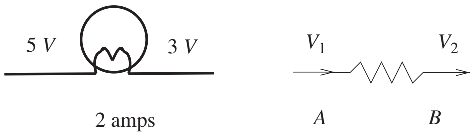



**FIGURE 1.17** Voltage drop.

When water passes through something that impedes its flow, such as a faucet or a nozzle, it's pressure drops. The amount by which it drops is called the pressure drop. Thus, in Figure 1.16, the pressure drop is $45 - 30 = 15$psi. The pressure will never increase in the direction of flow since water always flows from higher pressure to lower pressure.

Similarly, when electricity flows through something that impedes its flow, such as a light bulb, its voltage drops. The difference in voltage is called the **voltage drop** and is typically denoted "E". Thus, in Figure 1.17 the voltage drop across the light bulb as we go from the left side to the right side is

$$E = 5V - 3V = 2V$$

Like water, we think of electricity as flowing from higher voltage to lower voltage.[3]

In general, things that impede the flow of electricity are called **resistors** and are denoted by "-/\/\/\-" in circuit diagrams. The amount that that the resistor impedes the current flow is measured by the **resistance** which is measured in **ohms** and is described by **Ohm's law** which states that the current flow through the resistor is proportional to the voltage drop across the resistor.

**Ohm's Law:** *Let a resistor be connected to wires at points A and B as shown in Figure* 1.17. *Then there is a positive constant R such that*

$$E = iR \tag{1.23}$$

[3] Actually, since electrons are negatively charged and voltage is positive, the electrons move *toward* regions of higher voltage. However, it is common to *define* the direction of the current flow to be the opposite of the direction of the electron flow, allowing us to think of the current as flowing from higher voltage to lower.

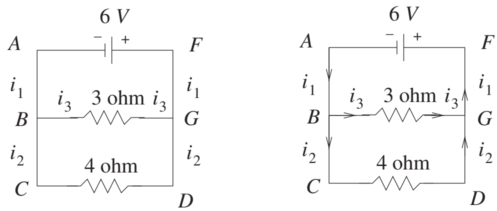



**FIGURE 1.18** Example 1.7.

*where i is the current flowing from A to B and $E = V_1 - V_2$ is the voltage drop across the resistor.*

Thus, the resistance of the light bulb in Figure 1.17 is

$$R = \frac{E}{i} = \frac{5-3}{2} = 1 \text{ ohm}.$$

Notice that if in Figure 1.17 $V_1 < V_2$ then $E = V_1 - V_2 < 0$ and, from (1.23), $i < 0$. Thus, the current flow from $A$ to $B$ is negative, indicating that the current flow is actually from $B$ to $A$, which is consistent with the principle that current flows from higher voltage to lower.

We study simple electrical circuits made up by connecting resistors and batteries as in Figure (1.18), left. We assume that the resistances of the wires are negligible. Batteries are in a sense, the opposite of resistors; they cause voltage to increase. The amount of increase from the minus side to the plus side of the battery is the voltage rating of the battery. Thus, for example, in Figure (1.18) the voltage at point F will be 6 volts higher than that at point A. The voltage "drop" from point $A$ to point $F$ is −6 volts since the voltage actually increases.

Kirchhoff's laws describe how the current and voltage change as electricity flows through a circuit.

**Kirchhoff's Current Law:** *The total signed current flow into any node in a circuit must equal the total signed current flow out of the node.*

**Kirchhoff's Voltage Law:** *The sum of the signed voltage drops at each resistor/battery along any path in a circuit must equal the difference in the voltages at the beginning of the path and at the end of the path. In particular the sum of the voltage drops along a closed path must be zero.*

### EXAMPLE 1.7

Find the directs of current flow and the currents $i_1$, $i_2$, and $i_3$ in Figure 1.18.

*Solution.* We do not initially know any of the directions of the current flows. We begin by arbitrarily choosing a direction of flow in each branch. If we happen to make a wrong choice, the corresponding current will turn out to be negative, allowing us to determine the actual direction of flow. Our initial choices are indicated on the right in Figure 1.18.

We first consider the loop ABGFA. From Ohm's law the voltage drop from G to B is $E = iR = 3i_3$ and the voltage drop from F to A is 6. (It is positive since we are going form the "+" terminal to the "-" terminal.) Thus, from the voltage law

$$0 = 6 + 3i_3$$

showing $i_3 = -2$ amps.

Next we consider the loop BGDCB. By similar reasoning we find

$$0 = 3i_3 - 4i_2$$

showing that $i_2 = (3/4)i_3 = -3/2$. amps. Finally, from the current law at the node at B

$$i_1 = i_3 + i_2 = -2 + \left(-\frac{3}{2}\right) = -\frac{7}{2}.$$

Since each current is negative we conclude that current flow is in the *opposite direction* of each of our arrows in Figure 1.18.

## EXAMPLE 1.8

Determine the currents $i_1$, $i_2$, and $i_3$ in the circuit indicated in Figure 1.19 where we have indicated our hypothetical directions of current flow.

*Solution.* We proceed as before. The loops ABGFA and BCDGB, together with the nodes at B and G, yield the following equations:

$$\begin{aligned} 2i_2 + 5i_1 - 18 &= 0 \\ 3 + 7i_3 - 2i_2 &= 0 \\ i_2 + i_3 &= i_1 \\ i_2 + i_3 &= i_1 \end{aligned}$$

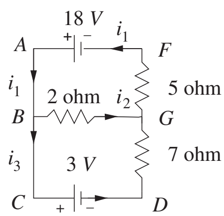



**FIGURE 1.19** Hypothetical flows.

We ignore the last equation as it is the same as the second to last. We obtain a system in the variables $i_1$, $i_2$, and $i_3$ with augmented matrix

$$\begin{bmatrix} 5 & 2 & 0 & 18 \\ 0 & -2 & 7 & -3 \\ -1 & 1 & 1 & 0 \end{bmatrix}$$

Next we exchange the first and last rows and add five times the new first row to the new third, producing

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & -2 & 7 & -3 \\ 0 & 7 & 5 & 18 \end{bmatrix}$$

We multiply the second row by $\frac{7}{2}$ and add it to the third:

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & -2 & 7 & -3 \\ 0 & 0 & \frac{59}{2} & \frac{15}{2} \end{bmatrix}$$

Interpreting as a system and solving yield $i_1 = \frac{156}{59}$, $i_2 = \frac{141}{59}$, and $i_3 = \frac{15}{59}$. In particular, the actual directions of current flow are the same as those indicated in Figure 1.19.

## Self-Study Questions

**1.4** ✓✓What is the voltage drop across the resistor in Figure 1.17 if the resistance is increased from 5 to 7 ohms while the current remains at 3 amperes?

**1.5** ✓✓What is the voltage drop across the resistor in Figure 1.17 if the current is decreased from 3 to 2 amperes while the resistance remains at 5 ohms?

**1.6** ✓✓Copy circuit (a) from Figure 1.20 onto a piece of paper. (You may download a copy of this figure and the others in this section from the companion website for the text.) Using arrows, indicate a hypothesized direction of current flow through each part of the circuit. Write (i) the equation that results from applying the current law to the node at the left side of the 6-ohm resistor. Write (ii) the equation that results from applying the voltage law to the loop containing the 6-ohm resistor and the 5-volt battery. Write (iii) the equation that results from applying the voltage law to the outer loop—that is, the loop containing both batteries and the 10-ohm resistor.

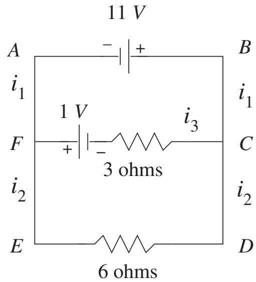



**FIGURE 1.20** Exercise 1.60(a)

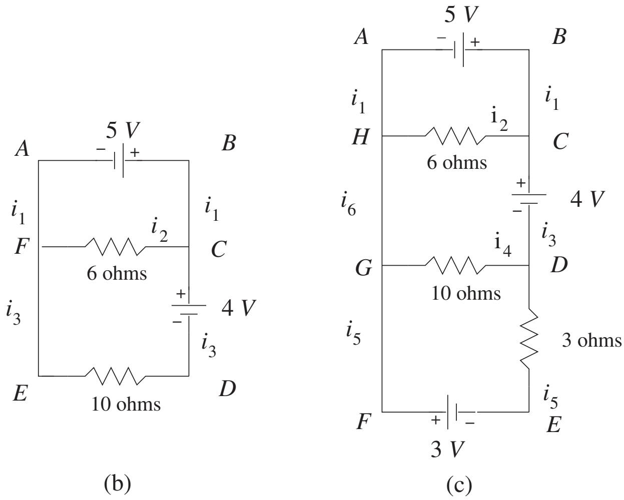



**FIGURE 1.21** Exercises 1.60.b and 1.60.c.

## EXERCISES

**1.60** Compute $i_1, i_2, \ldots, i_k$ (direction and magnitude) for

**(a)** ✓✓the circuit in Figure 1.20 **(b)** circuit (b) in Figure 1.21
**(c)** ✓✓circuit (c) in Figure 1.21.

**1.61** Find a system of equations in that could be solved to find the currents in Figure 1.22, left. Solve your system if you have appropriate technology available.

**1.62** ✓Compute the current $i_1$ in circuit (a) in Figure 1.23 as a function of the resistances $R_1$ and $R_2$. Show that we obtain the same current in circuit (b), provided $R_3 = R_1R_2/(R_1 + R_2)$.

*Remark.* This exercise demonstrates that parallel resistances of $R_1$ and $R_2$ ohms are equivalent to a single resistance of $R_1R_2/(R_1 + R_2)$ ohms.

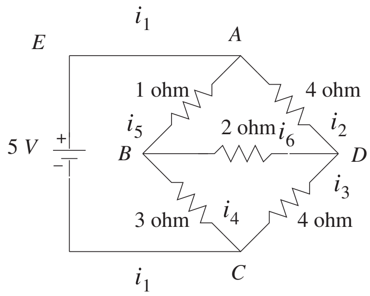



**FIGURE 1.22** Exercise 1.61.

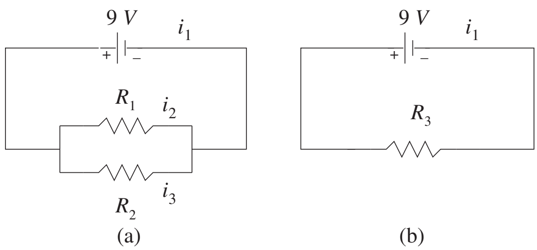



**FIGURE 1.23** Exercise 1.62.

## 1.3 GAUSSIAN ELIMINATION

In this section we would like to consider Gaussian elimination more systematically than in the last section. Consider, for example, the following system, which is the same as system (1.17):

$$\begin{aligned} 4x + 5y + 3z + 3w &= 1 \\ x + y + z + w &= 0 \\ 2x + 3y + z + w &= 1 \\ 5x + 7y + 3z + 3w &= 2 \end{aligned} \tag{1.24}$$

On page 37, we solved this system by transforming its augmented matrix $A$ into the matrix $R$ below.

$$A = \begin{bmatrix} 4 & 5 & 3 & 3 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 2 & 3 & 1 & 1 & 1 \\ 5 & 7 & 3 & 3 & 2 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.25}$$

At each step we performed an operation of one of the following types on the augmented matrix:[4]

(I) Interchange two rows.
(II) Add a multiple of one row onto another.
(III) Multiply one row by a nonzero constant.

These operations are called **elementary row operations**. Two matrices $A$ and $B$ are said to be **row equivalent** if there is a sequence of elementary row operations that transforms $A$ into $B$. Thus, the matrices $A$ and $R$ in formula (1.25) are row equivalent. The following theorem says that *elementary row operations do not change the solution sets for the corresponding systems.* In particular, matrix $R$ in formula (1.25) is the augmented matrix for a system that has exactly the same set of solutions as system (1.24).

**Theorem 1.3** *Suppose that A and B are row equivalent matrices. Then the system having A as an augmented matrix has the same solution set as the system having B as its augmented matrix.*

*Proof.* If we apply a single elementary row operation to $A$, we obtain a matrix $A_1$ that is the augmented matrix for a new system. We claim that this new system has the same solution set as the original system with augmented matrix $A$. To see this, note first that every row of $A_1$ is a linear combination of rows of the original matrix. Hence, every equation in the new system is a linear combination of equations from the original system, showing that every solution of the original system is also a solution of the new one.

Conversely, every solution of the new system is also a solution of the original. This is due to the reversibility of elementary row operations. We can undo the effect of adding a multiple of a given row onto another by subtracting the same multiple of the given row. Division by a nonzero constant undoes the effect of multiplication by a nonzero constant. Hence, we may transform $A_1$ back into $A$ using elementary row operations. The same argument as before now shows that every solution of the new system is also a solution of the original system, proving our claim.

Now, since $B$ is row equivalent with $A$, there is a sequence $A_0, A_1, \dots, A_n$ of matrices, where $A_0 = A$, $A_n = B$, and, for each $i$, $A_{i+1}$ was produced by applying a single elementary row operation to $A_i$. We refer to the system with augmented matrix $A_i$ as "system $i$" so that system 0 has augmented matrix $A$ and system $n$ has augmented matrix $B$. From the preceding argument, the solution set of system 0 equals the solution set of system 1, which equals the solution set of system 2, …, which equals the solution set of system $n$, proving our theorem. □

Matrix $R$ in formula (1.25) is in what is called **echelon form**, that is, recognizable by the "step like" arrangement of the zeros in the lower left corner.

[4]We did not actually use any steps of type III in this particular example.

**Definition 1.11** *A matrix A is in echelon form if:*

- *The first nonzero entry in any nonzero row occurs to the right of the first such entry in the row directly above it.*
- *All zero rows are grouped together at the bottom.*

*In this case, the first nonzero entry in each nonzero row is referred to as a* ***pivot entry****.*

Thus, any matrices of the following forms are in echelon form as long as all the entries in the positions marked "#" are nonzero. (The entries in the positions marked "$*$" can be arbitrary.) The "#" entries are the pivot entries:

$$\begin{bmatrix} \# & * & * & * & * \\ 0 & \# & * & * & * \\ 0 & 0 & 0 & \# & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \# & * & * & * & * & * \\ 0 & 0 & 0 & \# & * & * & * \\ 0 & 0 & 0 & 0 & \# & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.26}$$

### EXAMPLE 1.9

Which of the following matrices are in echelon form?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

*Solution.* The first three matrices are in echelon form; the last two matrices are not. In the fourth matrix, the first nonzero entry of row 2 is directly below the first nonzero entry of row 1. In the last matrix, rows 3 and 4 must be interchanged to get an echelon form.

A matrix that is not already in echelon form may be reduced further. Hence, we expect that if we keep reducing, we will eventually produce a matrix in echelon form. The following theorem proves this.

**Theorem 1.4** *Every matrix A is row equivalent to a matrix in echelon form.*

*Proof.* We may assume that the first column of $A$ is nonzero since null columns do not affect the reduction process. By exchanging rows (if necessary) to make $a_{11}$ nonzero and then dividing the first row by $a_{11}$, we may assume $a_{11} = 1$. Subtracting multiples of row 1 from the other rows produces a matrix with all the other entries in the first column equal to zero:

$$\begin{matrix} 1 & \% & \% & \% & \dots & \% \\ 0 & * & * & * & \dots & * \\ 0 & * & * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & * & * & \dots & * \end{matrix}$$

(The $\%$'s and $*$'s represent arbitrary numbers.)

We ignore the first row and column of the original matrix and apply the reduction process to the resulting smaller matrix, the matrix of $*$'s in our figure. Applying the reduction process to successively smaller matrices eventually results in an echelon form matrix, proving the theorem. □

Recall that the first nonzero entry in each nonzero row of an echelon form matrix is referred to as a pivot entry.

**Definition 1.12** *Let $R$ be an echelon form of a matrix $A$, where $A$ is the augmented matrix for a system in the variables $x_1, x_2, \dots, x_n$. The variable $x_i$ is said to be a* ***pivot variable*** *for the system, if the ith column of $R$ contains a pivot entry of $R$.*

Thus, from the echelon matrix $R$ in formula (1.25), we see that $x$ and $y$ are the pivot variables for system (1.24). It is somewhat remarkable that all echelon forms of a given matrix $A$ produce precisely the same set of pivot variables. (See Theorem 1.5 below.)

Let $A$ be the augmented matrix for a system in the variables $x_1$, $x_2$, …, $x_n$ and let $R$ be an echelon form for $A$. We refer to the system with augmented matrix $R$ as the "echelon system." The last equation in the echelon system expresses the last pivot variable in terms of the subsequent variables, all which may be set arbitrarily. The second to last equation in the echelon system expresses the second to last pivot variable in terms of the subsequent variables. Since the last pivot variable has already been expressed in terms of nonpivot variables, we see that, in fact, back substitution expresses the second to last pivot variable in terms of the subsequent nonpivot variables, all which may be set arbitrarily. In general, we see that:

1. All of the nonpivot variables may be set arbitrarily.
2. The values of the pivot variables are uniquely determined by the values of the nonpivot variables.

More specifically, if $X = [x_1, x_2, \dots, x_n]^t$ is a solution to a linear system, then, for all $1 \le i \le n$,

$$x_i = c_i + r_{i1}x_{j_1} + \cdots + r_{ik}x_{j_k}$$

where $x_{j_1}, x_{j_2}, \dots, x_{j_k}$ are the nonpivot variables and the $c_i$ and $r_{ij}$ are scalars that do not depend on $X$. Hence, the general solution may be expressed as

$$X = T + x_{j_1}X_1 + \cdots + x_{j_k}X_k \tag{1.27}$$

where $T$ and $X_j$ are fixed elements of $\mathbb{R}^n$ and the $x_{j_i}$ are the nonpivot variables. This, of course, is the familiar "parametric form" that we have already used extensively.

The nonpivot variables are examples of what are referred to as **free variables**. In a general linear system, a variable $x_i$ is said to be free if for all real numbers $t$ there exists a solution $X = [x_1, x_2, \dots, x_n]^t$ to the system with $x_i = t$. Pivot variables can also be free. For example, in the single equation

$$x - y - 7z = 0$$

$x$ is the pivot variable. It is also free since we may solve for, say, $y$, in terms of $x$ and $z$, allowing us to set $x$ and $z$ arbitrarily. We also refer to the nonpivot variables as "the" free variables. Exercises 1.87 and 1.88 investigate this issue further.

It should be noted that we typically refer to "an echelon form" rather than "the echelon form." This is because a given matrix $A$ can be row equivalent with many different echelon form matrices: the specific echelon form produced by applying Gaussian elimination to a given matrix $A$ will typically depend on the steps used to produce it. Suppose, for example, that in solving system (1.24), we do not first interchange the first and second rows. Then, the Gaussian elimination process proceeds as follows:

$$A = \begin{bmatrix} 4 & 5 & 3 & 3 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 2 & 3 & 1 & 1 & 1 \\ 5 & 7 & 3 & 3 & 2 \end{bmatrix}$$

We multiply rows 2–4 by constants and add (in this case, negative) multiples of the first row:

$$\begin{bmatrix} 4 & 5 & 3 & 3 & 1 \\ 0 & -1 & 1 & 1 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 3 & -3 & -3 & 3 \end{bmatrix} \qquad \begin{array}{lcl} \\ R_2 & \to & 4R_2 - R_1 \\ R_3 & \to & 2R_3 - R_1 \\ R_4 & \to & 4R_4 - 5R_1 \end{array}$$

Adding multiples of the second row to lower rows yields the echelon form:

$$\begin{bmatrix} 4 & 5 & 3 & 3 & 1 \\ 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{ccc} & & \\ & & \\ R_3 & \to & R_3 + R_1 \\ R_4 & \to & R_4 + 3R_2 \end{array} \tag{1.28}$$

Our answer is quite different from the echelon form found previously [$R$ in formula (1.25) on page 47]. Note, however, that just as before, $x$ and $y$ are pivot variables, while $z$ and $w$ are nonpivot variables. This demonstrates a general principle that we prove at the end of this section.

**Theorem 1.5** *Suppose $A$ is the augmented form of a consistent system of linear equations in the variables $x_1, x_2, \dots, x_n$. Then, the set of pivot variables does not depend on the particular echelon form of $A$ used to determine the pivots.*

Although the specific echelon form of a given matrix obtained from Gaussian elimination depends on the steps used in reducing the matrix, remarkably, the final form of the solution does not. For example, the matrix (1.28) is an echelon form for system (1.24) on page 47. This matrix is the augmented matrix for the system

$$\begin{aligned} 4x + 5y + 3z + w &= 1 \\ -y + z + w &= -1 \end{aligned}$$

We solve the second equation by setting $z = r$ and $w = s$, where $r$ and $s$ are arbitrary real numbers. Then back substitution yields $y = 1 + r + s$ and $x = -2r - 2s - 1$. Thus,

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \tag{1.29}$$

which is identical to equation (1.21) on page 37.

The fact that the expression for the solution does not depend on the reduced form used to derive it is a direct consequence of Theorem 1.5. In fact, from formula (1.27), the translation vector $T$ is the unique solution obtained by setting all nonpivot variables equal to 0. Since the set of nonpivot variables is uniquely determined, the translation vector is unique as well. Similarly, from formula (1.27), the spanning vector $X_j$ is $X - T$, where $X$ is the solution obtained by setting $x_{i_j} = 1$ and all the other nonpivot variables equal to 0, implying the uniqueness of $X_j$.

Some readers might be tempted to say that we get the same description of the solution because we are, after all, solving the same system. This, however, is not the correct explanation. Suppose, for example, that we were to write system (1.24) on page 47 in the equivalent formulation

$$\begin{aligned} 3z + 4x + 5y + 3w &= 1 \\ z + x + y + w &= 0 \\ z + 2x + 3y + w &= 1 \\ 3z + 5x + 7y + 3w &= 2 \end{aligned}$$

The augmented matrix is then

$$\begin{bmatrix} 3 & 4 & 5 & 3 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 1 & 1 \\ 3 & 5 & 7 & 3 & 2 \end{bmatrix}$$

An echelon form is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which describes the system

$$\begin{aligned} z + x + y + w &= 0 \\ x + 2y &= 1 \end{aligned}$$

Now $w$ and $y$ are free variables. Setting $y = t$ and $w = s$ yields

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \tag{1.30}$$

which is quite different from equation (1.29).

This example also demonstrates that Theorem 1.5, which states that the set of pivot variables does not depend on the echelon form, assumes a specific ordering of the variables. In system (1.24), if we order the variables $z$, $x$, $y$, $w$, then, as seen above, the pivots become $x$ and $z$.

It is sometimes useful to carry the elimination process beyond echelon form. In the matrix $R$ in formula (1.25) on page 47, we may make the entries above the pivot entry equal to zero by subtracting the second row from the row *above it*, getting

$$B = \begin{bmatrix} 1 & 0 & 2 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.31}$$

This matrix is in **row reduced echelon form** (or "reduced form" for short).

**Definition 1.13** *A matrix A is in* ***row reduced echelon form (RREF)*** *if all the following conditions hold:*

1. *A is in echelon form.*
2. *All the pivot entries of A are 1.*
3. *All the entries above the pivots are 0.*

The first two matrices in Example 1.9 are in reduced form, the third is not: it has nonzero entries above the pivot in the third row. Also, the pivots in the first and third rows are not 1.

It is clear from Theorem 1.4 that any matrix is row equivalent with a row reduced matrix: we can reduce each pivot entry to 1 by dividing each nonzero row by the value of its first nonzero entry. We can then reduce the entries above each pivot entry to zero by subtracting multiples of the row containing the pivot entry from the higher rows. The advantage of reduced form is that the answer may be obtained directly, without back substitution. Thus, the first row of the preceding matrix tells us that $x = -1 - 2z - 2w$ and the second tells us that $y = 1 + z + w$.

Example 1.10 demonstrates most of the "wrinkles" that can occur in reducing a matrix to row reduced echelon form.

## EXAMPLE 1.10

Bring the following matrix into echelon and reduced echelon form.

$$A = \begin{bmatrix} 1 & -1 & 1 & 3 & 0 & 3 & 6 \\ 2 & -2 & 2 & 6 & 0 & 1 & 7 \\ -1 & 1 & 1 & -1 & -2 & 0 & 1 \\ 4 & -4 & 1 & 9 & 3 & 0 & 6 \end{bmatrix} \tag{1.32}$$

*Solution.* We begin by eliminating the first variable from the equations below the first equation:

$$\begin{bmatrix} 1 & -1 & 1 & 3 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 & -5 & -5 \\ 0 & 0 & 2 & 2 & -2 & 3 & 7 \\ 0 & 0 & -3 & -3 & 3 & -12 & -18 \end{bmatrix} \quad \begin{array}{lcl} R_2 & \to & R_2 - 2R_1 \\ R_3 & \to & R_3 + R_1 \\ R_4 & \to & R_4 - 4R_1 \end{array}$$

$$\begin{bmatrix} 1 & -1 & 1 & 3 & 0 & 3 & 6 \\ 0 & 0 & 1 & 1 & -1 & 4 & 6 \\ 0 & 0 & 2 & 2 & -2 & 3 & 7 \\ 0 & 0 & 0 & 0 & 0 & -5 & -5 \end{bmatrix} \quad \begin{array}{lcl} R_4 & \to & R_4/(-3) \\ R_2 & \leftrightarrow & R_4 \end{array}$$

$$\begin{bmatrix} 1 & -1 & 1 & 3 & 0 & 3 & 6 \\ 0 & 0 & 1 & 1 & -1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & 0 & 0 & -5 & -5 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$$\begin{bmatrix} 1 & -1 & 1 & 3 & 0 & 3 & 6 \\ 0 & 0 & 1 & 1 & -1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{lcl} R_4 & \to & R_4 - R_3 \\ R_3 & \to & R_3/(-5) \end{array}$$

This is the echelon form. In producing the reduced echelon form, it is usually most efficient to begin with the last nonzero row and work from the bottom up. Thus, we subtract multiples of row 3 from the rows above it, yielding

$$\begin{bmatrix} 1 & -1 & 1 & 3 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{lcl} R_2 & \to & R_2 - 4R_3 \\ R_1 & \to & R_1 - 3R_3 \end{array}$$

Next, we subtract row 2 from row 1. This is the reduced echelon form.

$$\begin{bmatrix} 1 & -1 & 0 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \to R_1 - R_2$$

It is, of course, a simple matter to compute the spanning and translation vectors for a consistent linear system from the row reduced echelon form of its augmented matrix. Interestingly, we can go the other way; we can compute the entries of the row reduced form from the translation and spanning vectors. Since the translation and spanning vectors do not depend on the particular reduced form used in computing them, it follows that *the augmented matrix for a consistent linear system of equations is row equivalent to one, and only one, row reduced echelon form matrix.* This argument may be used to prove the following theorem. We do not present a formal proof as this result is not used elsewhere in the text.

**Theorem 1.6** *Each $m \times n$ matrix $A$ is row equivalent to one, and only one, row reduced echelon form matrix $R$.*

Theorem 1.4 has some very important theoretical consequences. One of the most important is the following:

**Theorem 1.7** (More Unknowns Theorem). *A system of linear equations with more unknowns than equations will either fail to have any solutions or will have an infinite number of solutions.*

*Proof.* Consider an echelon form of the augmented matrix. It might look something like the following matrix:

$$\begin{bmatrix} 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$

There is at most one pivot variable per equation. If there are fewer equations than variables, then there are nonpivot variables. If the system is consistent, the values of these variables may be set arbitrarily, proving the theorem. □

We know, of course, that not all systems have solutions. In the following pair of matrices, the matrix on the left is the augmented matrix for a system with no solutions, since the last nonzero row describes the equation $0 = 1$. The matrix on the right is the augmented matrix for a system that does have solutions. The last equation says $x_4 = 1$. A system is inconsistent if and only if its augmented matrix has an echelon form with a row describing the equation $0 = a$, where $a \neq 0$.

$$\begin{bmatrix} 1 & 2 & -1 & 4 & 5 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & -1 & 4 & 5 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

### EXAMPLE 1.11

Find conditions on $a$, $b$, $c$, and $d$ for the following system to be consistent:

$$\begin{aligned} x + y + 2z + w &= a \\ 3x - 4y + z + w &= b \\ 4x - 3y + 3z + 2w &= c \\ 5x - 2y + 5z + 3w &= d \end{aligned}$$

*Solution.* We reduce the augmented matrix:

$$\begin{bmatrix} 1 & 1 & 2 & 1 & a \\ 3 & -4 & 1 & 1 & b \\ 4 & -3 & 3 & 2 & c \\ 5 & -2 & 5 & 3 & d \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 & a \\ 0 & -7 & -5 & -2 & b-3a \\ 0 & -7 & -5 & -2 & c-4a \\ 0 & -7 & -5 & -2 & d-5a \end{bmatrix} \qquad \begin{aligned} R_2 &\to R_2 - 3R_1 \\ R_3 &\to R_3 - 4R_1 \\ R_4 &\to R_4 - 5R_1 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 & a \\ 0 & -7 & -5 & -2 & b-3a \\ 0 & 0 & 0 & 0 & c-a-b \\ 0 & 0 & 0 & 0 & d-2a-b \end{bmatrix} \qquad \begin{aligned} R_3 &\to R_3 - R_2 \\ R_4 &\to R_4 - R_2 \end{aligned}$$

To avoid inconsistencies, we require $c - a - b = 0$ and $d - 2a - b = 0$.

Now that we have an efficient way of solving systems, we can deal much more effectively with some of the issues discussed in Section 1.1, as the next example demonstrates.

### EXAMPLE 1.12

Decide whether or not the vector $B$ belongs to the span of the vectors $X_1, X_2$, and $X_3$ below.

$$B = [13, -16, 1]^t, \quad X_1 = [2, -3, 1]^t, \quad X_2 = [-1, 1, 1]^t, \quad X_3 = [-3, 4, 0]^t$$

*Solution.* The vector $B$ belongs to the span if there exist constants $x$, $y$, and $z$ such that

$$x\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + z\begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 13 \\ -16 \\ 1 \end{bmatrix}$$

We simplify the left side of this equality and equate coefficients, obtaining the system

$$\begin{aligned} 2x - y - 3z &= 13 \\ -3x + y + 4z &= -16 \\ x + y \qquad &= 1 \end{aligned}$$

The augmented matrix for this system is

$$\begin{bmatrix} 2 & -1 & -3 & 13 \\ -3 & 1 & 4 & -16 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

We now reduce this matrix, obtaining

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix represents an inconsistent system, showing that $B$ *is not* in the span of the $X_i$.

## Spanning in Polynomial Spaces

### EXAMPLE 1.13

Write the polynomial $q(x) = -3 + 14x^2$ as a linear combination of the polynomials

$$\begin{aligned} p_1(x) &= 2 - x + 3x^2 \\ p_2(x) &= 1 + x + x^2 \\ p_3(x) &= -5 + 4x + x^2 \end{aligned}$$

*Solution.* We seek constants $a$, $b$, and $c$ such that

$$\begin{aligned} a(2 - x + 3x^2) + b(1 + x + x^2) + c(-5 + 4x + x^2) &= -3 + 14x^2 \\ 2a + b - 5c + (-a + b + 4c)x + (3a + b + c)x^2 &= -3 + 14x^2 \end{aligned}$$

Equating the coefficients of like powers of $x$, we see that this is equivalent to the system of equations

$$\begin{aligned} 2a + b - 5c &= -3 \\ -a + b + 4c &= 0 \\ 3a + b + c &= 14 \end{aligned}$$

that has augmented matrix

$$\begin{bmatrix} 2 & 1 & -5 & -3 \\ -1 & 1 & 4 & 0 \\ 3 & 1 & 1 & 14 \end{bmatrix}$$

We reduce, obtaining

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Hence, $a = 5$, $b = -3$, and $c = 2$. As a check on our work, we compute

$$5(2 - x + 3x^2) - 3(1 + x + x^2) + 2(-5 + 4x + x^2) = -3 + 14x^2$$

as desired.

## EXAMPLE 1.14

Show that the span of the polynomials $p_1(x)$, $p_2(x)$, and $p_3(x)$ from Example 1.13 is the space $P_2$ of all polynomials of degree $\leq 2$.

*Solution.* We must show that every polynomial $q(x) = u + vx + wx^2$ is a linear combination of the $p_i(x)$—that is, there are scalars $a$, $b$, and $c$ such that

$$ap_1(x) + bp_2(x) + cp_3(x) = q(x) \tag{1.33}$$

Reasoning as in Example 1.13, we see that this equation is equivalent to the system

$$\begin{aligned} 2a + b - 5c &= u \\ -a + b + 4c &= v \\ 3a + b + c &= w \end{aligned}$$

which has augmented matrix

$$\begin{bmatrix} 2 & 1 & -5 & u \\ -1 & 1 & 4 & v \\ 3 & 1 & 1 & w \end{bmatrix}$$

We reduce, obtaining

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{9}u & -\frac{2}{3}v & +\frac{1}{3}w \\ 0 & 1 & 0 & \frac{17}{27}v & +\frac{13}{27}u & -\frac{1}{9}w \\ 0 & 0 & 1 & \frac{1}{9}w & -\frac{4}{27}u & +\frac{1}{27}v \end{bmatrix}$$

Hence, equation (1.33) holds with

$$\begin{aligned} a &= -\tfrac{1}{9}u - \tfrac{2}{3}v + \tfrac{1}{3}w \\ b &= \tfrac{17}{27}v + \tfrac{13}{27}u - \tfrac{1}{9}w \\ c &= \tfrac{1}{9}w - \tfrac{4}{27}u + \tfrac{1}{27}v \end{aligned}$$

We end this section with the promised proof of Theorem 1.5 on page 52, which asserts that all echelon forms of a given $m \times n$ matrix $A$ have the same set of pivot variables.

*Proof.* Let $R$ be an echelon form of $A$. We refer to the system with augmented matrix $A$ as the "original" system and the system with augmented matrix $R$ as the "echelon system."

While not essential for the proof, it helps our understanding to picture a specific $R$ along with the associated variables:

$$R = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{bmatrix} \# & * & * & * & * & * & * \\ 0 & 0 & 0 & \# & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \# & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{1.34}$$

Let $x_m$ be the last nonpivot variable in the echelon system [$x_5$ for matrix (1.34)]. The value of each pivot variable $x_i$ is determined by the values of those nonpivot variables $x_j$ with $j > i$. Since $x_m$ is the last nonpivot variable, it follows that the values of the $x_i$ for $i > m$ are all uniquely determined. Hence, *$x_m$ is also describable as the last free variable for the original system.* Thus, we have succeeded in describing the last nonpivot variable in the echelon system without referring to $R$. It follows that all reduced forms for $A$ have the same last nonpivot variable.

Now, suppose that we have succeeded in describing the last $k$ nonpivot variables for the original system without referring to $R$. Call these variables $x_{m_1}, x_{m_2}, \ldots, x_{m_k}$. Consider the new system obtained from the original system by setting all the $x_{m_j} = 0$ for $1 \le j \le k$. The augmented matrix for this new system is the matrix $A'$ obtained by deleting the corresponding columns of $A$. The matrix $R'$ obtained by deleting the same columns from $R$ is an echelon form of $A'$. [For the system corresponding to matrix (1.34), we would set $x_5 = 0$ and $R'$ would be as indicated below.]

$$R' = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_6 & \phantom{*} \end{matrix} \\ \begin{bmatrix} \# & * & * & * & * & * \\ 0 & 0 & 0 & \# & * & * \\ 0 & 0 & 0 & 0 & \# & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

From the preceding reasoning, we see that the last nonpivot variable in the system corresponding to $R'$ ($x_3$ in our example) is the last free variable for the system corresponding to $A'$. Furthermore, the last nonpivot variable for the system corresponding to $R'$ is the $(k+1)$st to last nonpivot variable in the echelon system. (In our example $x_3$ is the second to last nonpivot variable for $R$.)

Hence, each nonpivot variable in the echelon system is describable as the last free variable for the system obtained from the original system by setting all the subsequent nonpivot variables equal to zero. Since this description does not make use of a specific reduced form, it follows that all reduced forms of $A$ have the same set of nonpivots and, hence, the same set of pivots, proving the theorem. □

### Computational Issues: Pivoting

Division by very small numbers tends to create inaccuracies in numerical algorithms. Suppose, for example, that we wish to solve the system in the variables $x$ and $y$ with augmented matrix

$$\begin{bmatrix} 0.0001 & 1.0 & 1.0 \\ 1.0 & 1.0 & 2.0 \end{bmatrix}$$

We reduce

$$\begin{bmatrix} 1.0 & 10000.0 & 10,000 \\ 1.0 & 1.0 & 2.0 \end{bmatrix}$$

$$\begin{bmatrix} 1.0 & 10000.0 & 10000.0 \\ 0.0 & -9999.0 & -9998.0 \end{bmatrix}$$

yielding

$$y = 9998/9999 = 0.9998999900$$
$$x = 10000 - 10000y = 1.000100010001$$

Suppose that our computer only carries three places after the decimal. To three places, the answer is $y = 1.000, x = 1.000$. This, however, is not what our computer tells us! Our computer first rounds $y$ to 1.000 and then computes $x = 10000 - 10000y = 0$, which is not even close.

The problem was caused by the division by 0.0001 in our first step. This can be avoided by interchanging the rows before doing the reduction:

$$\begin{bmatrix} 1.0 & 1.0 & 2.0 \\ 0.0001 & 1.0 & 1.0 \end{bmatrix}$$

$$\begin{bmatrix} 1.0 & 1.0 & 2.0 \\ 0.0 & 0.9999 & 0.9998 \end{bmatrix}$$

yielding $y = 1.000$ and $x = 2 - 1 \cdot 1 = 1.000$.

To avoid such problems, many computational algorithms rearrange the rows before each reduction step so that the entry in the pivot position is the one of largest absolute value. This process is called **partial pivoting**. (There is also a process called **full pivoting**, which involves rearranging both rows and columns so as to obtain the largest pivot entry.) Unfortunately, even full pivoting will not eliminate all round-off difficulties. There are certain systems, called **ill-conditioned**, that are inherently sensitive to small inaccuracies in the values of the coefficients on the right sides of the equations in the system. The **condition number**, which is discussed in the computer exercises for Section 3.3 and in Section 8.1, measures this sensitivity.

**True-False Questions:** Justify your answers.

In these questions, assume that $R$ is the reduced echelon form of the augmented matrix for a system of equations.

**1.20** If the system has three unknowns and $R$ has three nonzero rows, then the system has at least one solution.

**1.21** If the system has three unknowns and $R$ has three nonzero rows, then the system can have an infinite number of solutions.

**1.22** The system below has an infinite number of solutions:

$$\begin{aligned} 2x + 3y + 5z + 6w - 7u - 8v &= 0 \\ 3x - 4y + 7z + 6w + 8u + 5v &= 0 \\ -7x + 9y - 2z - 4w - 5u + 2v &= 0 \\ -5x - 5y + 9z + 3w + 2u + 7v &= 0 \\ -9x + 3y - 9z + 5w - 3u - 4v &= 0 \end{aligned}$$

**1.23** The following matrix may be reduced to reduced echelon form *using only one elementary row operation*:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

**1.24** The matrix in question 1.23 is the coefficient matrix for a consistent system of equations.

## EXERCISES

*In these exercises, if you are asked for a general solution, the answer should be expressed in "parametric form" as in the text. Indicate the spanning and translation vectors.*

**1.63** ✓✓Which of the following matrices are in echelon form? Which are in reduced echelon form?

**(a)** $\begin{bmatrix} 1 & 0 & 2 & 4 & 0 & 1 \\ 1 & 1 & 0 & 4 & 3 & 2 \\ 0 & 1 & 0 & 4 & 3 & 2 \\ 0 & 0 & 0 & 2 & 1 & 3 \end{bmatrix}$ **(b)** $\begin{bmatrix} 0 & 1 & 2 & 2 & 4 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

**(c)** $\begin{bmatrix} 0 & 0 & 2 & 4 & 1 \\ 3 & 1 & 2 & 6 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 & 1 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & 0 & 2 & 4 & 0 & 1 \\ 0 & 1 & 0 & 4 & 3 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

**(e)** $\begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

**1.64** ✓Bring each of the matrices in Exercise 1.63 that are not already in echelon form to echelon form. Interpret each matrix as the augmented matrix for a system of equations. Give the system and general solution for each system.

**1.65** Find the reduced echelon form of each of the following matrices:

**(a)** ✓✓ $\begin{bmatrix} 2 & 7 & -5 & -3 & 13 \\ 1 & 0 & 1 & 4 & 3 \\ 1 & 3 & -2 & -2 & 6 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 & 1 \\ 1 & 0 & 2 & 3 & 2 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix}$

**(c)** ✓✓ $\begin{bmatrix} 3 & 9 & 13 \\ 2 & 7 & 9 \end{bmatrix}$

**(d)** $\begin{bmatrix} 2 & 1 & 3 & 4 & 0 & -1 \\ -2 & -1 & -3 & -4 & 5 & 6 \\ 4 & 2 & 7 & 6 & 1 & -1 \end{bmatrix}$

**(e)** ✓✓ $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

**(f)** $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $ad - bc \neq 0$

**(g)** ✓✓ $\begin{bmatrix} 2 & 4 & 3 & 0 & 6 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix}$

**(h)** $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 9 & 10 \\ 0 & 0 & 0 & 13 \end{bmatrix}$

**(i)** $\begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{bmatrix}$
where $a$, $e$, $h$, and $j$ are all nonzero.

**(j)** ✓ $\begin{bmatrix} 2 & 5 & 11 & 6 \\ 1 & 4 & 9 & 5 \\ -1 & 2 & 5 & 3 \\ 2 & -1 & -3 & -2 \end{bmatrix}$

**1.66** Suppose that the matrices in Exercise 1.65 are the augmented matrices for a system of equations. In each case, write the system down and find all solutions (if any) to the system. ✓[(a), (c), (e), (g)]

**1.67** In each of the following systems, find conditions on $a$, $b$, and $c$ for which the system has solutions:

**(a)** ✓✓
$$\begin{aligned} 3x + 2y - \phantom{2}z &= a \\ x + \phantom{2}y + 2z &= b \\ 5x + 4y + 3z &= c \end{aligned}$$

**(b)**
$$\begin{aligned} -3x + 2y + \phantom{2}4z &= a \\ -\phantom{3}x - 2y + \phantom{2}3z &= b \\ -\phantom{3}x - 6y + 23z &= c \end{aligned}$$

**(c)** ✓✓
$$\begin{aligned} 4x - 2y + 3z &= a \\ 2x - 3y - 2z &= b \\ 4x - 2y + 3z &= c \end{aligned}$$

**1.68** ✓The coefficient matrix (the augmented matrix without its last column) for the system in Exercise 1.67.a is the matrix $A$ below. Note that $A_3 = A_1 + 2A_2$, where $A_i$ are the rows of $A$. On the other hand, the consistency condition for this system is equivalent to $c = a + 2b$, which is essentially the same equation. Explain this "coincidence." [*Hint:* Compute the sum of the first and twice the second equations in the system in Exercise 1.67.a.] Check that there is a similar relationship between the rows of the coefficient matrices and the consistency conditions for the systems in Exercise 1.67.c.

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 2 \\ 5 & 4 & 3 \end{bmatrix}$$

**1.69** Find conditions on $a$, $b$, $c$, and $d$ for which the following system has solutions:

$$\begin{aligned} 2x + 4y + z + 3w &= a \\ -3x + y + 2z - 2w &= b \\ 13x + 5y - 4z + 12w &= c \\ 12x + 10y - z + 13w &= d \end{aligned}$$

**1.70** Show that the consistency conditions found in Exercise 1.69 are explainable in terms of linear combinations of the rows of the corresponding coefficient matrix.

**1.71** Prove that the solution to system (1.24) on page 47 found in equation (1.29) on page 52 is equivalent with the solution given in equation (1.30) on page 53.

[*Hint:* In deriving equation (1.30) on page 53, we set $t = y$ and $s = w$. According to equation (1.21) on page 37, $y = 1 + r + s$ and $w = s$. In equation (1.30), replace $t$ by $1 + r + s$ and simplify to obtain equation (1.21). What you have shown is that every vector $X$ expressible in the form given in equation (1.30) is also expressible in the form given in equation (1.21). You must also prove that every vector $X$ expressible in the form given in equation (1.21) is also expressible in the form given in equation (1.30).]

**1.72** ✓We proved that which variables are pivot variables and which are not do not depend on how we reduce the system. This is true only if we keep the variables in the same order.

**(a)** Find the free variables and the general solution for the following system:

$$\begin{aligned} 2x + 2y + 2z + 3w &= 4 \\ x + y + z + w &= 1 \\ 2x + 3y + 4z + 5w &= 2 \\ x + 3y + 5z + 11w &= 9 \end{aligned} \tag{1.35}$$

**(b)** Solve the equivalent system obtained by commuting the terms involving $z$ and $y$ in each equation. Express the general solution in the form $[x, y, z, w]^t = \ldots.$

**(c)** Prove the consistency of the answers to parts (a) and (b). [*Hint:* See the hint for Exercise 1.71.]

**1.73** Show that commuting the terms involving $x$ and $y$ in each equation of system (1.35) does not change the free variables. How can we know, without any further work, that the expression for the solution to system (1.35) obtained by solving this new system will be identical with that obtained in Exercise 1.72.a.

**1.74** ✓Create an example of each of the following. Construct your examples so that none of the coefficients in your equations are zero and explain why your examples work.

**(a)** A system of five equations in five unknowns that has a line as its solution set

**(b)** A system of five equations in five unknowns that has a plane as its solution set

**(c)** A system of five equations in three unknowns that has a line as its solution set

**(d)** A system of five equations in three unknowns that has a plane as its solution set

**1.75** ✓The following vectors are the translation vector $T$ and spanning vectors $X_1$ and $X_2$, obtained by using Gaussian elimination to solve a linear system of three equations in the variables $x$, $y$, $z$, and $w$.

$$T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad X_1 = \begin{bmatrix} -3 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix},$$

**(a)** Which are the free variables? How can you tell?

**(b)** ✓✓Find a row reduced echelon matrix that is the augmented matrix for a system having the stated translation and spanning vectors. Explain why there is only one such matrix.

**1.76** One of your engineers wrote a program to solve systems of equations using Gaussian elimination. In a "test" case, the program produced the following answers for the spanning vector $T$ and the translation vectors $X_1$ and $X_2$. You immediately knew that the program had an error in it. How? [*Hint:* Which are the free variables?]

$$T = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \quad X_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

**1.77** A linear system is homogeneous if the numbers $b_i$ in system (1.8) on page 29 are all 0. What can you conclude from the more unknowns theorem about a homogeneous system that has fewer equations than unknowns?

**1.78** ✓✓You are given a vector $B$ and vectors $X_i$. In each part, decide whether $B$ is in the span of the $X_i$ by attempting to solve the equation $B = x_1X_1 + x_2X_2 + x_3X_3$.

**(a)** $B = [3, 2, 1]^t, X_1 = [1, 0, -1]^t, X_2 = [1, 2, 1]^t, X_3 = [1, 1, 1]^t$

**(b)** $B = [a, b, c]^t, X_1 = [1, 0, -1]^t, X_2 = [1, 2, 1]^t, X_3 = [1, 1, 1]^t$

**(c)** $B = [3, 2, 1]^t, X_1 = [1, 0, -1]^t, X_2 = [1, 2, 1]^t, X_3 = [3, 4, 1]^t$

**1.79** In a vector space $\mathcal{V}$, a set of vectors "spans $\mathcal{V}$" if every vector in $\mathcal{V}$ is a linear combination of the given vectors. Show that the vectors $X = [1, 2]^t$ and $Y = [1, -2]^t$ span $\mathbb{R}^2$. Specifically, show that the equation $[a, b]^t = xX + yY$ is solvable regardless of $a$ and $b$.

**1.80** ✓✓Show that the vectors $X_1 = [3, 1, 5]^t, X_2 = [2, 1, 4]^t$, and $X_3 = [-1, 2, 3]^t$ *do not* span $\mathbb{R}^3$ by finding a vector that cannot be expressed as a linear combination of them.

**1.81** Prove that a consistent system of five equations in five unknowns will have an infinite number of solutions if and only if the row reduced echelon form of the augmented matrix has at least one row of zeros. [*Hint:* Think about the number of nonpivot variables in the echelon form.]

**1.82** ✓Prove that a consistent system of $n$ equations in $n$ unknowns will have an infinite number of solutions if and only if the row reduced echelon form of the augmented matrix has at least one row of zeros. [*Hint:* Think about the number of free variables in the echelon form.]

**1.83** ✓✓One possible reduced echelon form of a nonzero $2 \times 2$ matrix is shown below. What are the other possibilities?

$$\begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}$$

**1.84** What general feature makes it clear that in any matrix in row reduced echelon form the set of nonzero rows is linearly independent?

**1.85** ✓✓In any linear system, the last column of the augmented matrix is called the "vector of constants," while the matrix obtained from the augmented matrix by deleting the last column is called the "coefficient" matrix. For example, in Exercise 1.67.a, the vector of constants is $[a, b, c]^t$, which is the vector formed from the constants on the right sides of the equations in this system and the coefficient matrix is

$$\begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 2 \\ 5 & 4 & 3 \end{bmatrix}$$

In applications, it often happens that the coefficient matrix remains fixed while the vector of constants changes periodically. We shall say that the coefficient matrix is "nonsingular" if there is one and only one solution to the system, regardless of the value of the vector of constants.[5]

**(a)** Is the coefficient matrix for the system in Exercise 1.67.a nonsingular? Explain.

**(b)** Suppose a given system has three equations in three unknowns with a nonsingular coefficient matrix. Describe the row reduced form of the augmented matrix as explicitly as possible.

**(c)** Can a system with two equations and three unknowns have a nonsingular coefficient matrix? Explain in terms of the row reduced form of the system.

**1.86** Suppose that $ad - bc = 0$. Show that the rows of the matrix $A$ are linearly dependent.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

**1.87** ✓✓In the following system, show that for any choice of $x$ and $y$ there is one (and only one) choice of $z$ and $w$ that solves the system. Write the general solution in parametric form using $x$ and $y$ as free variables. Specifically, set $x = s$ and $y = t$, solve for $z$ and $w$ in terms of $s$ and $t$, and express the general solution as a linear combination of two elements in $\mathbb{R}^4$.

$$\begin{aligned} x \quad\;\; + \;\; z + w &= 0 \\ y + 2z + w &= 0 \end{aligned}$$

**1.88** Can $x$ and $y$ be set arbitrarily in this system? (If you are not sure, try a few values.)

$$\begin{aligned} x \quad\; + 2z + 2w &= 0 \\ y + \;\; z + \;\; w &= 0 \end{aligned}$$

## Computational Issues: Counting Flops

A rough measure of the amount of time a computer will take to do a certain task is the total number of algebraic operations ($+$, $-$, $\times$, and $/$) required. Each such operation is called a **flop** (floating point operation). The next exercise shows that it requires at most $2n^3/3 + 3n^2/2 - 7n/6$ flops to solve a system of $n$ equations in $n$ unknowns. Thus, for example, with 20 unknowns, the solution could require 5910 flops, while

[5]This exercise is intended as a "preview" of the concept of invertibility that is discussed in depth in Chapter 3.

with 100 unknowns, 681,550 might be required. Such large systems are not at all uncommon in "real-world" applications.

**1.89** Let $A$ be an $n \times (n+1)$ matrix.

**(a)** Explain why it takes at most $n$ flops to reduce $A$ to a matrix of the following form. Note that row exchanges do not count as flops.

$$\begin{bmatrix} 1 & * & \dots & * & * \\ * & * & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \dots & * & * \end{bmatrix}$$

**(b)** Explain why it takes at most $2n$ additional flops to create a 0 in the $(2,1)$ position of the matrix in part (a) using row reduction.

**(c)** Use parts (a) and (b) to explain why it takes at most $n + 2n(n-1) = 2n^2 - n$ flops to reduce $A$ to a matrix with a 1 in the $(1,1)$ position and 0's in the rest of column 1.

**(d)** Use part (c) to explain why it takes a total of at most

$$2(n^2 + (n-1)^2 + \cdots + 1) - (n + (n-1) + \cdots + 1) \tag{1.36}$$

flops to reduce $A$ to an echelon matrix.

**(e)** Explain why it only takes at most

$$2(n-1) + 2(n-2) + \cdots + 2$$

additional flops to bring the matrix from part (b) into row reduced echelon form.

**(f)** Use parts (d) and (e) and the following formulas to prove that it requires at most $2n^3/3 + 3n^2/2 - 7n/6$ flops to reduce $A$ to row reduced echelon form.

$$n + (n-1) + \cdots + 1 = \frac{n(n+1)}{2}$$
$$n^2 + (n-1)^2 + \cdots + 1 = \frac{n(n+1)(2n+1)}{6}$$

### 1.3.1 Computer Projects

MATLAB can row reduce many matrices almost instantaneously. One begins by entering the matrix into MATLAB. We mentioned in the last section that rows can be separated by semicolons. One can also separate rows by putting them on different lines. Thus, we can enter a matrix A by stating that

```
>> A = [  1   -1   1   3   0   6
          2   -2   2   6   0   7
         -1    1   1  -1  -2   1
          4   -4   1   9   3   6  ]
```

To row reduce A, we simply enter

```
>> rref(A)
```

We obtain the row reduced form almost instantaneously:

```
ans  =
      1  -1   0  2   1  0
      0   0   1  1  -1  0
      0   0   0  0   0  1
      0   0   0  0   0  0
```

## EXERCISES

1. Enter format long and then compute $(1/99) * 99 - 1$ in MATLAB. You should get zero. Next compute $(1/999) * 999 - 1$ and $(1/9999) * 9999 - 1$. Continue increasing the number of 9's until MATLAB does not produce 0. How many 9's does it require? Why do you not get 0?

   This calculation demonstrates a very important point: MATLAB can interpret some answers that should be zero as nonzero numbers. If this happens with a pivot in a matrix, MATLAB may produce totally incorrect reduced forms. MATLAB guards against such errors by automatically setting all sufficiently small numbers to zero. How small is "sufficiently small" is determined by a number called eps, which, by default, is set to around $10^{-17}$. MATLAB will also issue a warning if it suspects that its answer is unreliable.

   In some calculations (especially calculations using measured data), the default value of eps will be too small. For this reason, MATLAB allows you to include a tolerance in rref. If you wanted to set all numbers less than 0.0001 to 0 in the reduction process, you would enter rref(A,.0001).

2. Use rref to find all solutions to the system in Exercise 1.55.i on page 39.

3. We said that the rank of a system is the number of equations left after linearly dependent equations have been eliminated. We commented (but certainly did not prove) that this number does not depend on which equations were kept and which were eliminated. It seems natural to suppose that the rows in the row reduced form of a matrix represent linearly independent equations and hence, that the rank should also be computable as the number of nonzero rows in the row reduced form of the matrix. We can check this conjecture experimentally

using the MATLAB command rank(A), which computes the rank of a given matrix.

Use MATLAB to create four $4 \times 5$ matrices $A$, $B$, $C$, and $D$ having respective ranks of 1, 2, 3, and 4. Design your matrices so that none of their entries are zero. Compute their rank by (a) using the rref command and (b) using the rank command.

[*Hint:* Try executing the following sequence of commands and see what happens. This could save you some time!]

```
>> M = [ 1 2 1 5 3
         2 1 1 4 3 ]
>> M(3,:) = 2*M(2,:) + M(1,:)
```

This works because, in MATLAB, M(i,:) represents the *i*th row of M. [Similarly, M(:,j) represents the *j*th column.] The equation M(3,:)=2*M(2,:)+ M(1,:) both creates a third row for M and sets it equal to twice the second plus the first row.

(*Note:* If you choose "nasty" enough coefficients, then you may need to include a tolerance in the rank command in order to get the answer to agree with what you expect. This is done just as with the rref command.)

**4.** For each of the matrices in Exercise 1.65 on page 63, other than, (f) and (i), use rref to row reduce the *transpose*. The MATLAB symbol for the transpose of $A$ is A′. How does the rank of each matrix compare with that of its transpose?

**5.** Let

$$X = [1, 2, -5, 4, 3]^t, \quad Y = [6, 1, -8, 2, 10]^t, \quad Z = [-5, 12, -19, 24, 1]^t$$

**(a)** Determine which of the vectors $U$ and $V$ is in the span of $X$, $Y$, and $Z$ by using rref to solve a system of equations.

$$U = [-5, 23, -41, 46, 8]^t, \quad V = [22, 0, -22, 0, 34]^t$$

**(b)** Imagine as the head of an engineering group that you supervise a computer technician who knows *absolutely nothing* about linear algebra, other than how to enter matrices and commands into MATLAB. You need to get this technician to do problems similar to part (a). Specifically, you will be giving him or her an initial set of three vectors $X$, $Y$, and $Z$ from $\mathbb{R}^n$. You will then provide an additional vector $U$, and you want the technician to determine whether $U$ is in the span of $X$, $Y$, and $Z$.

Write a brief set of instructions that will tell your technician how to do this job. Be as explicit as possible. Remember that the technician cannot do linear algebra! You must provide instructions on constructing the necessary matrices, telling what to do with them and how to interpret the answers.

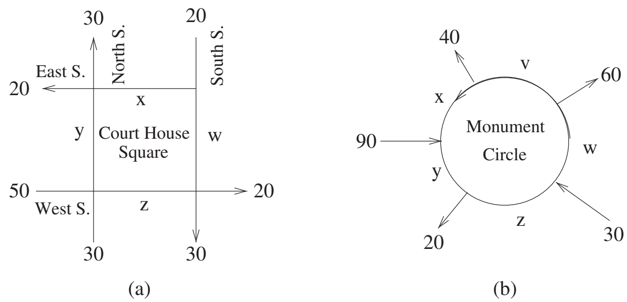



**FIGURE 1.24** Two traffic patterns.

The final output to you should be a simple Yes or No. You do not want to see matrices!

**(c)** One of your assistant engineers comments that it would be easier for the technician in part (b) to use MATLAB's rank command rather than rref. What does your assistant have in mind?

### 1.3.2 Applications to Traffic Flow

An interesting context in which linearly dependent systems of equations can arise is in the study of traffic flow. Figure 1.24a is a map of the downtown area of a city. Each street is one-way in its respective direction. The numbers represent the average number of cars per minute that enter or leave a given street at 3:30 p.m. The variables also represent average numbers of cars per minute. Of course, barring accidents, the total number of cars entering any intersection must equal the total number leaving. Thus, from the intersection of East and North, we see that $x + y = 50$. Continuing counterclockwise around the square, we get the system

$$\begin{aligned} x + y \qquad\quad &= 50 \\ y + z \qquad &= 80 \\ z + w &= 50 \\ x \qquad\quad + w &= 20 \end{aligned} \tag{1.37}$$

Notice also that in Figure 1.24, the total number of cars entering the street system per minute is $20 + 30 + 50 = 100$ while the total number leaving is $20 + 30 + 20 + 30 = 100$. All of our examples share the property that the total number of cars per minute that enter the street system per minute equals the total number that leave the street system per minute. *We assume that the total number of cars per minute on the street system at any given time equals the total number of cars entering the street system per minute which equals the total number of cars leaving the street system per minute.* Thus we augment system 1.37 with the additional equation

$$x + y + z + w = 100$$

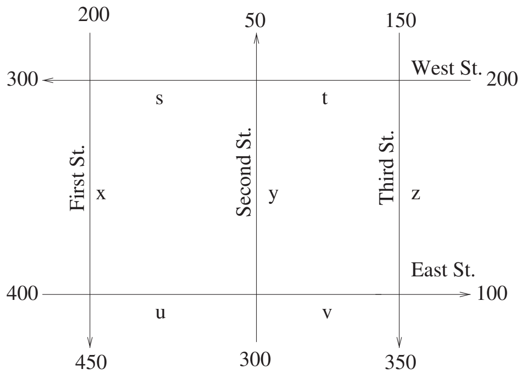



**FIGURE 1.25** Exercise 3.

In this case, this equation is dependent on the other equations; it equals half their sum. In other cases, however, it can add additional information.

In the exercises that follow, you will analyze this system as well as the system that describes figure (b). The type of analysis done in these exercises can be applied in any context where some quantity is flowing through prescribed channels. In circuit theory, the statement that the amount of current entering a node equals the amount leaving is called Kirchhoff's current law.

## Self-Study Questions

**1.7** ✓In Figure 1.25 write the linear equation that describes the traffic flow at the intersection of West and Third Streets.

**1.8** ✓In Figure 1.25, write the linear equation that describes the traffic flow at the intersection of East and Second Streets.

**1.9** ✓In Figure 1.24b, write the linear equation that describes the traffic flow at the intersection of Monument Circle with the street labeled 90.

## EXERCISES

**1.90** Find all solutions to the system that describes Figure 1.24a. Use $w$ as your free variable. Then answer the following:

**(a)** ✓Suppose that over the month of December the traffic on South Street in front of the courthouse at 3:30 p.m. ranged from six to eight cars per minute. Determine which of the streets on Courthouse Square had the greatest volume of traffic. What were the maximum and minimum levels of traffic flow on this street?

**(b)** Suppose that it is observed that in June the traffic flow past the courthouse is heaviest on West Street. Prove that $0 \le w < 10$.

**(c)** Sleezie's Construction wants to close down West Street for six months. The City Council refused to grant the permit. Why? Explain on the basis of the solution to the system of Figure 1.24a.

**1.91** Find all solutions to the system that describes Figure 1.24b. Use $v$ as your free variable.

**1.92** Figure 1.25 shows the traffic flow in a town at 3:30 p.m.

**(a)** Find the system of equations that describes the traffic flow?

**(b)** The section of Third Street between East and West is under construction. Assuming that as few cars as possible use this block, what are the possible ranges of traffic flow on each of the other blocks? [*Hint:* Note that none of the variables can take on negative values.]

## 1.4 COLUMN SPACE AND NULLSPACE

The two most fundamental questions concerning any system of equations are: (a) Is it solvable? (b) If it is solvable, what is the nature of the solution set? In this section we begin the discussion of both issues. This discussion will be completed in Section 2.3. Example 1.15 addresses the first question.

### EXAMPLE 1.15

Describe geometrically the set of vectors $B = [b_1, b_2, b_3]^t$ for which the system below is solvable.

$$\begin{aligned} x \quad\;\; + 2z &= b_1 \\ x + y + 3z &= b_2 \\ y + \;\; z &= b_3 \end{aligned}$$

*Solution.* The system may be written as a single matrix equation:

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} x \quad\;\; + 2z \\ x + y + 3z \\ y + \;\; z \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + z \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \tag{1.38}$$

We denote the three columns on the right by $A_1$, $A_2$, and $A_3$, respectively, so that our system becomes

$$B = xA_1 + yA_2 + zA_3$$

We conclude that our system is solvable if and only if $B$ is in the span of the $A_i$.

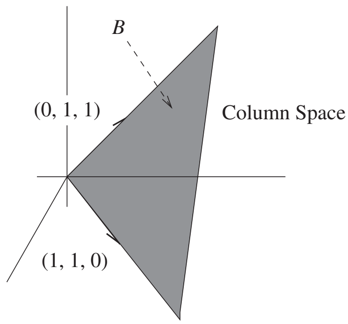



**FIGURE 1.26** The column space.

To describe this span geometrically, note that $A_3 = 2A_1 + A_2$. Hence, the foregoing equation for $B$ is equivalent to

$$B = xA_1 + yA_2 + z(2A_1 + A_2) = (x + 2z)A_1 + (y + z)A_2 = x'A_1 + y'A_2$$

where $x' = x + 2z$ and $y' = y + z$. This describes the plane spanned by the vectors $A_1$ and $A_2$. (See Figure 1.26.) We conclude that our system is solvable if and only if $B$ belongs to this plane.

The ideas we used to analyze Example 1.15 generalize readily to other systems. The columns on the right in equation (1.38) are the columns of the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

Note that $A$ is the augmented matrix for system (1.15), with its last column deleted. As commented in Section 1.2, this matrix is referred to as the coefficient matrix:

**Definition 1.14** *The* ***coefficient matrix*** *for a system of linear equations is the augmented matrix for the system with its last column deleted.*

The general system of equations

$$\begin{aligned} b_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ &\vdots \\ b_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{aligned}$$

is equivalent to the vector equation

$$\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Hence, just as in our example, the general system of equations may be written as

$$B = x_1 A_1 + x_2 A_2 + \cdots + x_n A_n \tag{1.39}$$

where $A_i$ are the columns of the coefficient matrix and $B$ is the matrix of constants. These considerations suggest the following definition:

**Definition 1.15** *The span of the columns of a matrix is called the* ***column space*** *of the matrix.*

The following theorem is a direct consequence of equation (1.39) and the definition of the span of a set of vectors.

**Theorem 1.8** *A linear system is solvable if and only if the vector of constants belongs to the column space of the coefficient matrix.*

### EXAMPLE 1.16

Find a system of two equations in three unknowns such that the system is solvable if and only if the vector of constants lies on the line spanned by the vector $[1, 4]^t$.

*Solution.* The coefficient matrix for our system is $2 \times 3$. From Theorem 1.8, the span of its columns must equal the line spanned by the vector $[1, 4]^t$. This will happen only if each column is a multiple of this vector. Thus, we could, for example, let the coefficient matrix be

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 8 & 12 \end{bmatrix}$$

The corresponding system would be

$$\begin{aligned} x + 2y + \ \ 3z &= a \\ 4x + 8y + 12z &= b \end{aligned}$$

Planes through the origin have a special algebraic property: if we take any two vectors $X$ and $Y$ lying in a given plane, then all linear combinations of $X$ and $Y$ also lie in this plane (see Figures 1.27 and 1.28). Sets with this property are said to be subspaces of the vector space containing them.

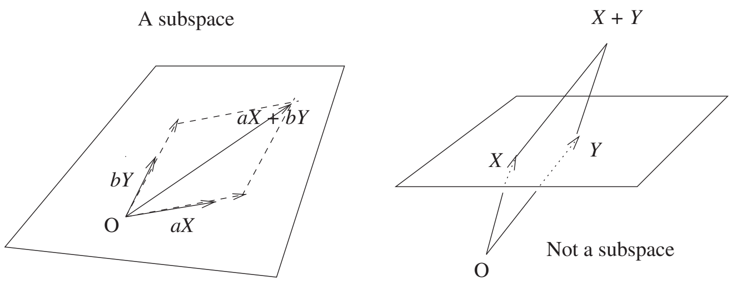



**FIGURE 1.27** Spaces in $\mathbb{R}^3$.

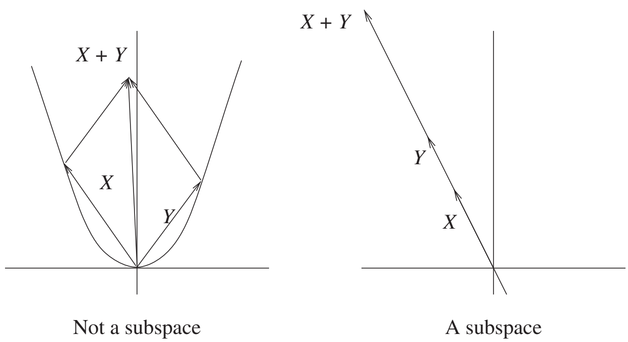



**FIGURE 1.28** Spaces in $\mathbb{R}^2$.

## Subspaces

**Definition 1.16** *A nonempty subset $\mathcal{W}$ of some vector space $\mathcal{V}$ is a subspace of $\mathcal{V}$ if it is "closed under linear combinations." This means that if $X$ and $Y$ belong to $\mathcal{W}$, then $sX + tY$ also belongs to $\mathcal{W}$ for all scalars $s$ and $t$.*

In $\mathbb{R}^3$, the only types of sets that are subspaces are (i) all $\mathbb{R}^3$, (ii) any plane containing the origin, (iii) any line containing the origin, and (iv) the set consisting of the origin by itself. Planes and lines not containing the origin, as well as any "curved" set, cannot be subspaces of $\mathbb{R}^3$ (Figures 1.27 and 1.28). In general, subspaces should be thought of as higher-dimensional analogues of planes and lines through the origin.

The following proposition asserts that the span of a set of vectors in a vector space $\mathcal{V}$ is a subspace of $\mathcal{V}$. This, together with Theorem 1.8, explains why, in Example 1.15, the set of vectors $B$ for which the system is solvable is a plane containing the origin; it had to be either all $\mathbb{R}^3$, a plane through the origin, or a line through the origin since it is a nonzero subspace.

**Proposition 1.3** *Let $\mathcal{W} = \text{span}\,\{A_1, \ldots, A_n\}$, where $A_1, A_2, \ldots, A_k$ are elements of some vector space $\mathcal{V}$. Then $\mathcal{W}$ is a subspace of $\mathcal{V}$.*

*Proof.* Let $X$ and $Y$ belong to $\mathcal{W}$. Then there are constants $x_i$ and $y_i$ such that

$$\begin{aligned} X &= x_1A_1 + x_2A_2 + \cdots + x_nA_n \\ Y &= y_1A_1 + y_2A_2 + \cdots + y_nA_n \end{aligned} \tag{1.40}$$

Then, for $s$ and $t$ any scalars,

$$\begin{aligned} sX &= sx_1A_1 + sx_2A_2 + \cdots + sx_nA_n \\ tY &= ty_1A_1 + ty_2A_2 + \cdots + ty_nA_n \\ sX + tY &= (sx_1 + ty_1)A_1 + (sx_2 + ty_2)A_2 + \cdots + (sx_n + ty_n)A_n \end{aligned} \tag{1.41}$$

Thus, $sX + tY$ is again in $\mathcal{W}$, proving that the column space is indeed closed under linear combinations. □

The process of forming linear combinations of the columns of the coefficient matrix is so important that it has its own notation.

**Definition 1.17** *Let $A = [A_1, A_2, \ldots, A_n]$ be an $m \times n$ matrix, where $A_i$ are the columns of $A$, and let $X \in \mathbb{R}^n$. We define the **product** of the matrix $A$ with the column vector $X$ by*

$$AX = x_1A_1 + x_2A_2 + \cdots + x_nA_n \tag{1.42}$$

*where $X = [x_1, x_2, \ldots, x_n]^t$.*

Hence, in the preceding notation,

$$(AX)_j = a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n \tag{1.43}$$

*Remark.* In Chapter 3 we state a generalization of this definition that allows $X$ to be any $n \times q$ matrix.

Thus, for example,

$$\begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3+4-3 \\ 1+2+6 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$$

Equation (1.43) says that the $j$th entry of $AX$ is the product of the $j$th row of $A$ with $X$. Thus,

$$\begin{bmatrix} 4 & -5 \\ 2 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \cdot 2 & + & -5 \cdot 3 \\ 2 \cdot 2 & + & 1 \cdot 3 \\ -3 \cdot 2 & + & 1 \cdot 3 \end{bmatrix} = \begin{bmatrix} -7 \\ 7 \\ -3 \end{bmatrix}$$

Equation (1.43) shows that any system of linear equations may be written in product form as

$$B = AX \tag{1.44}$$

where $A$ is the coefficient matrix, $X$ is the vector of unknowns, and $B$ is the matrix of constants. Thus, for example, the system from Example 1, written as a product, becomes

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Matrix multiplication has some important algebraic properties.

**Theorem 1.9** (Linearity Properties). *Let $A$ be an $m \times n$ matrix and let $X$ and $Y$ be $n \times 1$ matrices. Let $a$ be a scalar. Then*

$$\begin{aligned} A(X+Y) &= AX + AY \quad \textit{(distributive law)} \\ A(aX) &= aAX \quad \textit{(scalar law)} \end{aligned}$$

*Proof.* Let

$$\begin{aligned} X &= [x_1, x_2, \ldots, x_n]^t \\ Y &= [y_1, y_2, \ldots, y_n]^t \end{aligned}$$

so that

$$X + Y = [x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n]^t$$

Then

$$\begin{aligned} A(X+Y) &= (x_1 + y_1)A_1 + (x_2 + y_2)A_2 + \cdots + (x_n + y_n)A_n \\ &= (x_1A_1 + x_2A_2 + \cdots + x_nA_n) + (y_1A_1 + y_2A_2 + \cdots + y_nA_n) \\ &= AX + AY \end{aligned} \tag{1.45}$$

proving the distributive law. The scalar law is left as an exercise for the reader. □

Next, let us address the question of describing the solution set to a system. Consider, for example, the system

$$\begin{aligned} x + \ \ y + \ \ z &= 1 \\ 4x + 3y + 5z &= 7 \\ 2x + \ \ y + 3z &= 5 \end{aligned} \tag{1.46}$$

This is system (1.12) on page 31. From formula (1.14) on page 33, the general solution is

$$X = [4, -3, 0]^t + s[-2, 1, 1]^t \tag{1.47}$$

The translation vector $T_o = [4, -3, 0]^t$ is a solution to the system: it is the solution obtained by letting $s = 0$.

This is not the case for the spanning vector $[-2, 1, 1]^t$. In fact, the spanning vector satisfies the system

$$\begin{aligned} x + \ \ y + \ \ z &= 0 \\ 4x + 3y + 5z &= 0 \\ 2x + \ \ y + 3z &= 0 \end{aligned} \tag{1.48}$$

The system $AX = B$, where $A$, $X$, and $B$ are as in equation (1.44), is **homogeneous** if $B = 0$. We refer to the system $AX = 0$ as the **homogeneous system corresponding to** the system $AX = B$. Thus, system (1.48) is the homogeneous system corresponding to system (1.46).

**Definition 1.18** *The* ***nullspace*** *for a matrix $A$ is the solution space to the homogeneous system $AX = 0$.*

Just as in the preceding example, when we compute the solution to a general system of linear equations, the expression that is added onto the translation vector comes from the nullspace:

**Theorem 1.10** (Translation Theorem). *Let $A$ be an $m \times n$ matrix and let $T$ be any particular solution to*

$$AX = B$$

*Then the general solution is $X = T + Z$, where $Z$ belongs to the nullspace.*

Notice that in this theorem $T$ need not be the translation vector found by row reduction. In fact, $T$ can be *any arbitrary solution to the system.*

*Proof.* Suppose first that $X = T + Z$, where $Z$ belongs to the nullspace of $A$. Then

$$\begin{aligned} AX &= A(T + Z) \\ &= AT + AZ \\ &= B + 0 = B \end{aligned}$$

showing that $X = T + Z$ solves $AX = B$.

Conversely, suppose that $AX = B$. Then

$$\begin{aligned} AX - AT &= B - B = 0 \\ A(X - T) &= 0 \end{aligned}$$

Hence, $Z = X - T$ belongs to the nullspace of $A$, implying that $X = T + Z$, where $Z$ belongs to the nullspace. □

The zero vector always belongs to the nullspace since $A0 = 0$. The translation theorem tells us that the solution to $AX = B$ is unique if and only if the zero vector is the *only* element of the nullspace.

**Proposition 1.4** *Assume that the system $AX = B$ has a solution. Then the solution is unique if and only if the nullspace of $A$ is* $\{0\}$.

The conclusion of the translation theorem is often described by saying that the general solution to the system $AX = B$ is

$$T + \mathcal{W}$$

where $\mathcal{W}$ is the nullspace of $A$. The following theorem says that the nullspace is a subspace; hence, it is a higher-dimensional analogue of a line or a plane through the origin. The content of the translation theorem then is that the general solution to a nonhomogeneous linear system is obtained by translating an object that is something like a plane away from the origin (Figure 1.29).

**Theorem 1.11** *The nullspace of an $m \times n$ matrix $A$ is a subspace of* $\mathbb{R}^n$.

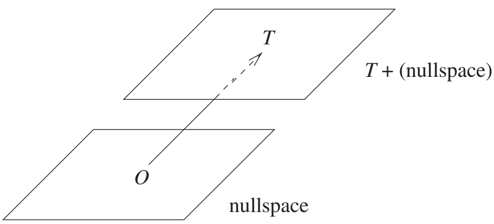



**FIGURE 1.29** The translation theorem.

*Proof.* Since 0 belongs to the nullspace of $A$, the nullspace is nonempty.

Next, suppose that $X$ and $Y$ belong to the nullspace; hence, $AX = AY = 0$. Then, for all scalars $s$ and $t$

$$A(sX + tY) = A(sX) + A(tY) = sAX + tAY = s0 + t0 = 0$$

showing that $sX + tY$ also belongs to the nullspace of $A$. Hence, the nullspace of $A$ is closed under linear combinations, proving that it is a subspace of $\mathbb{R}^n$. □

It turns out that the nullspace of a matrix is extremely important. Indeed, it is referred to as one of the three fundamental spaces associated with a matrix. (The other two are the column space, which was defined earlier, and the row space, which we study in Section 2.3.) Computing the nullspace is something that we already know how to do. All we need to remember is that the nullspace of a matrix $A$ is the solution set to $AX = 0$.

### EXAMPLE 1.17

Find the nullspace to the matrix $A$:

$$A = \begin{bmatrix} 1 & 2 & 4 & 1 \\ 1 & 1 & 3 & 2 \\ 2 & 3 & 7 & 3 \end{bmatrix}$$

*Solution.* The system $AX = 0$ has the augmented matrix

$$\begin{bmatrix} 1 & 2 & 4 & 1 & 0 \\ 1 & 1 & 3 & 2 & 0 \\ 2 & 3 & 7 & 3 & 0 \end{bmatrix}$$

The row reduced form of this matrix is

$$\begin{bmatrix} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The third and fourth variables are free. Hence, the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \tag{1.49}$$

Thus, the nullspace is the span of the two column vectors above.

*Remark.* The spanning vectors just computed are linearly independent since each has a 1 in a position where the other has a 0 (Proposition 1.1 on page 9). Specifically, the first has a 1 in the third position while the second is 0 in this position, and the second has a 1 in the fourth position while the first is 0 in this position. The reason for this is not difficult to understand. From formula (1.49), the first spanning vector is the solution obtained by setting $x_3 = 1$ and $x_4 = 0$. Hence, it *must* have a 1 in the third position and a 0 in the fourth position. Similarly, the second spanning vector is obtained by setting $x_3 = 0$ and $x_4 = 1$ and thus *must* have a 0 in the third position and a 1 in the fourth position. The following theorem asserts that the same is true for any system.

**Theorem 1.12** *Let $A$ be an $m \times n$ matrix. The spanning vectors for the solution set of the system $AX = 0$ span the nullspace of $A$ and are linearly independent.*

*Proof.* In the augmented matrix for the system $AX = 0$, all the entries in the last column are 0. Hence, the translation vector is 0. Thus, from formula (1.27) on page 51, the general solution to the system is expressible as

$$X = x_{i_1}X_1 + x_{i_2}X_2 + \cdots + x_{i_k}X_k$$

where $x_{i_j}$ is the $j$th nonpivot variable and $X_j$ the corresponding spanning vector. It follows that the spanning vectors span the nullspace of $A$.

From this equation, $X_j$ is the solution obtained by setting $x_{i_j} = 1$ and all the other nonpivot variables to 0. In particular, the vector $X_j$ has a 1 in the $i_j$th position while the others have a zero in this position. The linear independence now follows from Proposition 1.1 on page 9. □

We finish this section with an important comment concerning subspaces of vector spaces. The property of being a subspace can be broken down into three statements called the **subspace properties**. We leave the proof of the following theorem as an exercise (Exercise 1.113).

**Theorem 1.13** (Subspace Properties). *A subset $\mathcal{W}$ of a vector space is a subspace if and only if all of the following properties hold:*

1. *If $X$ and $Y$ belong to $\mathcal{W}$, then $X + Y$ is in $\mathcal{W}$.*
2. *For all $X$ in $\mathcal{W}$ and all scalars $a$, the element $aX$ belongs to $\mathcal{W}$.*
3. $0 \in \mathcal{W}$.

### EXAMPLE 1.18

Which of the following sets of functions is a subspace of the space $\mathcal{F}(\mathbb{R})$ of all real-valued functions on $\mathbb{R}$? Which of the subspace properties fail for the set which is not a subspace of $\mathcal{F}(\mathbb{R})$?

$$\mathcal{V} = \{f \mid f(2) = 0\}$$
$$\mathcal{W} = \{f \mid f(2) = 3\}$$

*Solution.* The set $\mathcal{V}$ is a subspace of $\mathcal{F}(\mathbb{R})$. It contains the zero function since $0(2) = 0$. Suppose that $f$ and $g$ belong to $\mathcal{V}$. Then

$$(f+g)(2) = f(2) + g(2) = 0 + 0 = 0$$

showing that $f + g \in \mathcal{V}$. Similarly, if $a$ is any scalar,

$$(af)(2) = af(2) = a0 = 0$$

showing that $af \in \mathcal{V}$. Hence, the subspace properties hold for $\mathcal{V}$, proving that it is a subspace of $\mathcal{F}(\mathbb{R})$.

The space $\mathcal{W}$ is not a subspace of $\mathcal{F}(\mathbb{R})$. In fact, *none* of the subspace properties hold: the zero function is not in $\mathcal{W}$ since $0(2) = 0$. To see that property 1 fails, let, for example, $f(x) = x + 1$ and $g(x) = x^2 - 1$. Then both $f$ and $g$ belong to $\mathcal{W}$ since, as the reader may check, $f(2) = 3 = g(2)$. However,

$$f(x) + g(x) = x^2 + x$$

which equals 6 at $x = 2$ and, thus, does not belong to $\mathcal{W}$. Similarly, $2f(x)$ equals 6 at $x = 2$ and thus does not belong to $\mathcal{W}$.

Recall that in Section 1.2 we said that a "rule of addition" for a set $\mathcal{W}$ is a way of adding pairs of elements of $\mathcal{W}$ that always produce elements of $\mathcal{W}$, and a "rule of scalar multiplication" is a way of multiplying elements of $\mathcal{W}$ by scalars so as to produce elements of $\mathcal{W}$. Subspace property 1 says that subspaces have a rule of addition, and subspace property 2 says that they have a rule of scalar multiplication. In fact, we have the following proposition:

**Proposition 1.5** *A subspace of a vector space $\mathcal{V}$ is itself a vector space under the addition and scalar multiplication of $\mathcal{V}$.*

*Proof.* Let $\mathcal{W}$ be a subspace of a vector space $\mathcal{V}$. We must show that the addition and scalar multiplication for $\mathcal{W}$ satisfy all the vector space properties [properties a–j on page 12]. Most of these properties, however, are automatic by virtue of being already valid for $\mathcal{V}$. For example, addition in $\mathcal{W}$ is certainly commutative because addition in $\mathcal{V}$ is commutative. In fact, a close inspection of the list of vector space properties shows that the only property which is not immediate is the statement that if $A$ belongs to $\mathcal{W}$, then $-A$ also belongs to $\mathcal{W}$. This, however, follows from part 2 of Theorem 1.13 with $a = -1$. □

**Proposition 1.6** *The space $C^\infty(\mathbb{R})$ of infinitely differentiable elements of $\mathcal{F}(\mathbb{R})$ is a vector space.*

*Proof.* It suffices to show that $C^\infty(\mathbb{R})$ is a subspace of $\mathcal{F}(\mathbb{R})$. However, this is clear since the sum of two infinitely differentiable functions is again infinitely differentiable, as is any scalar multiple of a differentiable function. □

**Summary** This section introduced a large number of new (and important) ideas. It is perhaps worth summarizing the main points:

1. The column space for an $m \times n$ matrix $A$ is the span of the columns of $A$. It is a subspace of $\mathbb{R}^n$. The column space can also be described as the set of vectors $B$ for which the system $AX = B$ is solvable.
2. The nullspace for an $m \times n$ matrix $A$ is the set of $X$ for which $AX = 0$. It is a subspace of $\mathbb{R}^m$. The nullspace describes the solutions to the system $AX = B$ in the sense that the general solution to this system is a translate of the nullspace.

**True-False Questions:** Justify your answers.

**1.25** The nullspace of a $3 \times 4$ matrix cannot consist of only the zero vector.

**1.26** The nullspace of a $4 \times 3$ matrix cannot consist of only the zero vector.

**1.27** Suppose that $A$ is a $2 \times 3$ matrix such that $A[1,1,1]^t = [2,3]^t = A[2,3,4]^t$. Then $[1,2,3]^t$ belongs to the nullspace of $A$.

**1.28** Suppose that $A$ is a $2 \times 3$ matrix such that $[1,2,3]^t$ belongs to the nullspace of $A$ and $A[1,1,1]^t = [2,3]^t$. Then $A[2,3,4]^t = [2,3]^t$.

**1.29** $X = [2,4,-2]^t + s[1,2,-1]^t$ cannot be the general solution of a nonhomogeneous system of linear equations.

**1.30** $X = [2,4,-2]^t + s[1,2,-1]^t$ cannot be the general solution of a homogeneous system of linear equations.

**1.31** The set of $3 \times 1$ column vectors $[b_1, b_2, b_3]^t$ for which the system below is solvable is a plane in $\mathbb{R}^3$.

$$\begin{aligned} x + 2y + 3z &= b_1 \\ 2x + 4y + 6z &= b_2 \\ 3x + 6y + 9z &= b_3 \end{aligned}$$

**1.32** Let $X_1$, $X_2$, and $X_3$ be elements of a vector space and let $Y_1 = X_1 + X_2$ and $Y_2 = X_3$. Then the span of $Y_1$ and $Y_2$ is contained in, but not equal to, the span of $X_1$, $X_2$, and $X_3$.

**1.33** The set of all vectors of the form $[1,x,y]^t$, where $x$ and $y$ range over all real numbers, is a subspace of $\mathbb{R}^3$.

**1.34** The set of all vectors of the form $[0,x,y]^t$, where $x$ and $y$ range over all real numbers, is a subspace of $\mathbb{R}^3$.

**1.35** The set of all vectors of the form $[0,x,y]^t$, where $x$ and $y$ range over all integers, is a subspace of $\mathbb{R}^3$.

**1.36** A subspace of a vector space $\mathcal{V}$ cannot contain only a finite number of elements.

## EXERCISES

**1.93** ✓✓For each matrix $A$ and each vector $X$, compute $AX$.

**(a)** $A = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 2 & -2 & 1 & 1 \\ 3 & 2 & -2 & -3 \end{bmatrix}, \quad X = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

**(b)** $A = \begin{bmatrix} -5 & 17 \\ 4 & 2 \\ 3 & 1 \\ 5 & -5 \end{bmatrix}, \quad X = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

**(c)** $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

**1.94** Write each of the systems in Exercise 1.55 on page 39 in the form $AX = B$.

**1.95** Give an example of a nonhomogeneous system of equations that has the matrix $A$ from Exercise 1.93.a as its coefficient matrix. Repeat for 1.90.b and 1.90.c✓✓.

**1.96** ✓Find the nullspace for each of the matrices $A$ in Exercise 1.93. Express each answer as a span.

**1.97** Find the nullspace for each of the matrices in Exercise 1.65 on page 63. Express each answer as a span.✓✓[ (a), (c), (e), (g)]

**1.98** Create a $2 \times 3$ matrix $A$ such that the equation $AX = B$ is solvable if and only if $B$ belongs to the line in $\mathbb{R}^2$ spanned by the vector $[1, 2]^t$. Choose your matrix so that no two of its entries are equal. As a check on your work, choose a specific $B$ not on this line and show (by row reduction) that the system $AX = B$ is inconsistent.

**1.99** ✓✓Create a $3 \times 4$ matrix $A$ such that the equation $AX = B$ is solvable if and only if $B$ belongs to the line in $\mathbb{R}^3$ spanned by the vector $[1, 2, 3]^t$. Choose your matrix so that no two of its entries are equal. As a check on your work, choose a specific $B$ on this line and show (by row reduction) that the system $AX = B$ is consistent.

**1.100** ✓Create a $3 \times 4$ matrix $A$ such that the equation $AX = B$ is solvable if and only if $B$ belongs to the plane in $\mathbb{R}^3$ spanned by the vectors $[1, 0, 1]^t$ and $[1, 1, 1]^t$. Choose your matrix so that none of its entries are 0. As a check on your work, choose a specific $B$ not on this plane and show (by row reduction) that the system $AX = B$ is inconsistent.

**1.101** The following exercises relate to Theorem 1.8 on page 76.

**(a)** Create a $4 \times 5$ matrix $A$ such that $AX = B$ is solvable if and only if $B$ belongs to the span of $[3, 0, 2, 0]^t$ and $[2, 0, 0, -3]^t$. Choose $A$ so that only five of its entries are 0.

**(b)** Explain why it is not possible to choose $A$ so that fewer than five entries are 0.

**1.102** Create a system of four equations in five unknowns (reader's choice) such that the solution space is a plane in $\mathbb{R}^5$ (Definition 1.10 on page 38). *Do not make any coefficients equal 0.* Explain why your example works.

**1.103** ✓✓Can a homogeneous system be inconsistent? Explain. What can you say about the number of solutions to a homogeneous system that has more unknowns than equations?

**1.104** ✓Find the nullspace for matrix $A$. Compare your answer with the general solution to system (1.17) on page 36. [Formula (1.21) on page 37.] What theorem does this exercise demonstrate?

$$A = \begin{bmatrix} 4 & 5 & 3 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 1 \\ 5 & 7 & 3 & 3 \end{bmatrix}$$

**1.105** Let $A$ be the matrix (1.32) in Example 1.10 on page 54.

**(a)** Find the general solution to the system of equations in the variables $x_1, \dots, x_6$ whose augmented matrix is $A$.

**(b)** Find the nullspace for the following matrix $B$. Compare your answer with part a). What theorem does this exercise demonstrate?

$$B = \begin{bmatrix} 1 & -1 & 1 & 3 & 0 & 3 \\ 2 & -2 & 2 & 6 & 0 & 1 \\ -1 & 1 & 1 & -1 & -2 & 0 \\ 4 & -4 & 1 & 9 & 3 & 0 \end{bmatrix}$$

**(c)** ✓✓I claim that the general solution to the system described in part (a) may be expressed as

$$\begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

where $s$, $t$, and $u$ are arbitrary parameters. Am I right? Explain your answer on the basis of the translation theorem.

**(d)** I claim that the general solution to the system described in part (a) may be expressed as

$$\begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

where $s$, $t$, and $u$ are arbitrary parameters. Am I right? Explain your answer on the basis of the translation theorem.

**1.106** Let $X = [1, 2, 1]^t$ and $Y = [1, 0, -3]^t$.

**(a)** ✓✓Find an equation in three unknowns whose solution space is the subspace of $\mathbb{R}^3$ spanned by $X$ and $Y$. [*Hint:* Is the equation homogeneous?]

**(b)** ✓✓Find a system of three equations in three unknowns whose solution space is the subspace of $\mathbb{R}^3$ spanned by $X$ and $Y$.

**(c)** ✓Find a system of three equations in three unknowns that has the set of vectors of the form $Z + aX + bY$ as its general solution, where $Z = [1, 1, 1]^t$.

**1.107** Let $X = [1, 0, 2, 0]^t$ and $Y = [1, -1, 0, 2]^t$.

**(a)** Find a system of two equations in four unknowns whose solution set is spanned by $X$ and $Y$.

**(b)** Find a system of three equations in four unknowns whose solution set is spanned by $X$ and $Y$.

**(c)** Find a system of four equations in four unknowns that has the set of vectors of the form $Z + aX + bY$ as its general solution where $Z = [1, 1, 1, 1]^t$.

**1.108** ✓✓The vectors $X_1 = [1, -1, 1]^t$ and $X_2 = [1, 0, 1]^t$ span the same plane as $Y_1 = [4, -2, 4]^t$ and $Y_2 = [0, -1, 0]^t$. True or false? Explain.

**1.109** "The vectors $X_1 = [1, 2, 1, 1]^t$, $X_2 = [1, 1, 1, 1]^t$, $X_3 = [1, 0, 1, 2]^t$ span the same subspace of $\mathbb{R}^4$ as $Y_1 = [2, 3, 2, 2]^t$, $Y_2 = [0, 1, 0, 0]^t$, and $Y_3 = [1, 1, 1, 1]^t$." True or false? Explain.

**1.110** ✓✓As CEO of an engineering firm, you have two groups of engineers working on solving the same linear system. Group I tells you that the solution is

$$[1, 0, 0]^t + s[-3, 1, 1]^t + t[-1, 0, 1]^t$$

where $s$ and $t$ are arbitrary parameters. Group II tells you that the solution is

$$[1,-1,1]^t + s[-4,1,2]^t + t[-3,1,1]^t$$

Are the answers consistent? Explain.

**1.111** Let $\{X, Y, Z, W\}$ be four nonzero elements in some vector space. Suppose that $Z = 3X$ and $W = -2Y$. Prove that span $\{X, Y, Z, W\}$ = span $\{X, Y\}$. Under what further conditions would span $\{X, Y, Z, W\}$ = span $\{Y, W\}$?

**1.112** ✓✓Let $\{X, Y, Z\}$ be three elements in some vector space. Suppose that $Z = 2X + 3Y$.

**(a)** Prove that span $\{X, Y, Z\}$ = span $\{X, Y\}$.

**(b)** Under the same hypothesis, prove that span $\{X, Y, Z\}$ = span $\{X, Z\}$.

**1.113** Prove Theorem 1.13 on page 83.

**1.114** In these exercises, you are given a subset $\mathcal{W}$ of $M(m, n)$ for some $m$ and $n$. You should (i) give a nonzero matrix that belongs to $\mathcal{W}$, (ii) give a matrix in $M(m, n)$ not in $\mathcal{W}$, (iii) use the subspace properties (Theorem 1.13 on page 83) to prove that $\mathcal{W}$ is a subspace of $M(m, n)$, and (iv) express $\mathcal{W}$ as a span. We present the solution to part (a) as an example of what is required.

Part (a): $\mathcal{W}$ is the set of all matrices of the form

$$\begin{bmatrix} a+b+3c & 2a-b \\ a & 2a+b+4c \end{bmatrix}$$

where $a$, $b$, and $c$ range over all real numbers.

*Solution.*

**(i)** Letting $a = 1, b = 0, c = -1$ yields the matrix

$$\begin{bmatrix} -2 & 2 \\ 1 & -2 \end{bmatrix}$$

which is one element of $\mathcal{W}$. Other assignments produce other elements.

**(ii)** To find a matrix that is not in $\mathcal{W}$, we choose some "random" matrix and check whether it is in $\mathcal{W}$. For example, let us try a matrix all whose entries are 1. This will be in $\mathcal{W}$ only if there are constants $a$, $b$, and $c$ such that

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a+b+3c & 2a-b \\ a & 2a+b+4c \end{bmatrix}$$

This yields the system

$$\begin{aligned}
a + b + 3c &= 1\\
2a - b \phantom{{}+ 3c} &= 1\\
a \phantom{{}- b + 3c} &= 1\\
2a + b + 4c &= 1
\end{aligned}$$

We solve this system by means of row reduction, finding it inconsistent; hence, our matrix does not belong to $\mathcal{W}$.

**(iii)** To prove subspace property 1, let

$$X = \begin{bmatrix} a+b+3c & 2a-b \\ a & 2a+b+4c \end{bmatrix}$$

and

$$Y = \begin{bmatrix} a'+b'+3c' & 2a'-b' \\ a' & 2a'+b'+4c' \end{bmatrix}$$

where $a$, $b$, $c$, $a'$, $b'$, and $c'$ are all scalars. Then

$$\begin{aligned}
X + Y &= \begin{bmatrix} (a+a') + (b+b') + 3(c+c') & 2(a+a') - (b+b') \\ a+a' & 2(a+a') + (b+b') + 4(c+c') \end{bmatrix}\\
&= \begin{bmatrix} a''+b''+3c'' & 2a''-b'' \\ a'' & 2a''+b''+4c'' \end{bmatrix}
\end{aligned}$$

where $a'' = a + a'$, $b'' = b + b'$, and $c'' = c + c'$. This is of the proper form to belong to $\mathcal{W}$.

To prove subspace property 2, suppose that $k$ is a scalar. Then

$$\begin{aligned}
kX &= \begin{bmatrix} ka+kb+3kc & 2ka-kb \\ ka & 2ka+kb+4kc \end{bmatrix}\\
&= \begin{bmatrix} a^*+b^*+3c^* & 2a^*-b^* \\ a^* & 2a^*+b^*+4c^* \end{bmatrix}
\end{aligned}$$

where $a^* = ka$, $b^* = kb$, and $c^* = kc$, proving that $kX$ also belongs to $\mathcal{W}$.

**(iv)** To find elements that span $\mathcal{W}$, we write the general element of $\mathcal{W}$ as

$$\begin{bmatrix} a+b+3c & 2a-b \\ a & 2a+b+4c \end{bmatrix} = \begin{bmatrix} a & 2a \\ a & 2a \end{bmatrix} + \begin{bmatrix} b & -b \\ 0 & b \end{bmatrix} + \begin{bmatrix} 3c & 0 \\ 0 & 4c \end{bmatrix}$$

$$= a\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} + b\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$$

This shows that $\mathcal{W}$ is the span of the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$$

*Remark.* Actually, we could have done the preceding exercise with much less work, had we not been required to "use the subspace properties." According to Proposition 1.3 on page 78, to show that $\mathcal{W}$ is a subspace, all we need to show is that it is the span of some set of vectors. This is what we did in part (iv). Knowing that $\mathcal{W}$ was a subspace of $M(m,n)$ helped us only in that it told us to look for spanning elements.

**(a)** $\mathcal{W}$ is the set of all matrices of the form

$$\begin{bmatrix} a+b+3c & 2a-b \\ a & 2a+b+4c \end{bmatrix}$$

where $a$, $b$, and $c$ range over all real numbers.

**(b)** $\mathcal{W}$ is the set of all elements of $\mathbb{R}^2$ of the form $[x, -x]^t$, where $x$ ranges over all real numbers.

**(c)** $\mathcal{W}$ is the set of all elements of $M(2,2)$ of the form

$$\begin{bmatrix} a & a+b+c \\ b & c \end{bmatrix}$$

where $a$, $b$, and $c$ range over all real numbers.

**(d)** $\mathcal{W}$ is the set of points in $\mathbb{R}^3$ of the form

$$[s-2t+u, -3s+t-u, -2s-t]^t$$

where $s$, $t$, and $u$ range over all real numbers.

**(e)** $\mathcal{W}$ is the set of points in $M(1,4)$ of the form

$$[a+2b+5c, 2a+6c, 3a-b+8c, a+4b+8c]$$

where $a$, $b$, and $c$ range over all real numbers.

**1.115** Let $\mathcal{W}$ denote the set of all elements $A$ of $M(2,2)$ such that the sum of all the entries of $A$ is zero.

**(a)** Give an example of two nonzero elements $A$ and $B$ in $\mathcal{W}$.

**(b)** For your $A$ and $B$, show that $3A + 4B \in \mathcal{W}$.

**(c)** ✓Show that $\mathcal{W}$ is a subspace of $M(2,2)$.

**1.116** ✓✓Let $\mathcal{W}$ be the set of all points $[x, y]^t$ in $\mathbb{R}^2$ such that both $x \geq 0$ and $y \geq 0$. Draw a graph representing $\mathcal{W}$. Is $\mathcal{W}$ a subspace of $\mathbb{R}^2$? Explain.

**1.117** Graph the set $\mathcal{W}$ of points $[m, n]^t$ in $\mathbb{R}^2$ such that both $m$ and $n$ are integers. Is $\mathcal{W}$ a subspace of $\mathbb{R}^2$? Explain.

**1.118** ✓Let $\mathcal{W}$ be the set of all points in $\mathbb{R}^3$ of the form $[x, y, z]^t$, where $x^2 = y + z$. Find two specific points (reader's choice) $X$ and $Y$ in $\mathcal{W}$ such that $X + Y$ is not in $\mathcal{W}$. Are there any nonzero points $X$ and $Y$ in $\mathcal{W}$ such that $X + Y$ also belongs to $\mathcal{W}$? Explain.

**1.119** ✓✓A square matrix $A$ is **upper triangular** if all the entries below the main diagonal are zero. Thus, for example, the following matrix $A$ is upper triangular. Prove that the set $\mathcal{T}$ of all $3 \times 3$ upper triangular matrices is a subspace of $M(3,3)$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

**1.120** An upper triangular matrix $B$ is unipotent if its diagonal entries are all equal to 1. Thus, for example, these matrices $A$ and $B$ are unipotent.

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

**(a)** Find nonzero scalars $a$ and $b$ such that $aA + bB$ is unipotent.

**(b)** Is the set of all unipotent matrices a subspace of $M(3,3)$? Explain.

**1.121** ✓✓Consider a system $AX = B$, where $B \neq 0$. Let $X$ and $Y$ satisfy this system. Find all constants $a$ and $b$ such that $aX + bY$ also satisfies this system.

**1.122** Consider the following differential equations. Prove that the set of solutions of each is a subspace of $C^\infty(\mathbb{R})$.

**(a)** $y'' + y = 0$

**(b)** ✓ $y'' + 3y' + 2y = 0$

**(c)** $y'' + 4y' + 13y = 0$

**(d)** $y''' + y'' - y' - y = 0$

**(e)** $y''' + 3y'' + 3y' + y = 0$

**(f)** $y''' = 0$

**1.123** Consider the following differential equations. Prove that the set of solutions of each is not a subspace of $C^\infty(\mathbb{R})$. In fact, prove that *none* of the subspace properties hold for these spaces.

**(a)** $y'' + y = 1$ **(b)** ✓✓ $y'' + 3y' + 2y = t$

**(c)** $y'' + 4y' + 13y = \sin(3t)$ **(d)** $y''' + y'' - y' - y = e^t$

**(e)** $y''' + 3y'' + 3y' + y = 3$ **(f)** $y''' = \ln(t^2 + 1)$

**1.124** A function of the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is said to be a polynomial of degree less than or equal to $n$. The space of all polynomials of degree less than or equal to $n$ is denoted by $\mathcal{P}_n$. Prove that $\mathcal{P}_n$ is a subspace of $C^\infty(\mathbb{R})$.

**1.125** Is the set of all polynomial functions (regardless of degree) a subspace of $C^\infty(\mathbb{R})$? What about the set of all polynomial functions with integral coefficients?

**1.126** ✓Let $\mathcal{W}$ denote the set of all functions $f \in C^\infty(\mathbb{R})$ such that $f(1) = 0$. Prove that $\mathcal{W}$ is a subspace of $C^\infty(\mathbb{R})$.

**1.127** Let $\mathcal{W}$ denote the set of all functions $f \in C^\infty(\mathbb{R})$ such that $f'(3) = 0$. Prove that $\mathcal{W}$ is a subspace of $C^\infty(\mathbb{R})$.

**1.128** Let $\mathcal{W}$ denote the set of all functions $f \in C^\infty(\mathbb{R})$ such that $f(1) = f(2) = 0$. Prove that $\mathcal{W}$ is a subspace of $C^\infty(\mathbb{R})$.

**1.129** Let $\mathcal{W}$ denote the set of all functions $f \in C^\infty(\mathbb{R})$ such that $f(1) = 2$.

**(a)** Give an example of a function that is in $\mathcal{W}$.

**(b)** Let $f$ and $g$ be in $\mathcal{W}$. For which scalars $a$ and $b$ is it true that $af + bg$ belongs to $\mathcal{W}$?

**(c)** Is $\mathcal{W}$ a subspace of $C^\infty(\mathbb{R})$?

**1.130** ✓Let $\mathcal{W}$ denote the set of all functions $f \in C^\infty(\mathbb{R})$ such that $\int_1^2 f(x)\,dx = 0$.

**(a)** Give an example of a nonzero element of $\mathcal{W}$.

**(b)** Prove that $\mathcal{W}$ is a subspace of $C^\infty(\mathbb{R})$.

**(c)** Prove that the set $\mathcal{V}$ of functions satisfying $\int_1^2 f(x)\,dx = -2$ is not a subspace of $C^\infty(\mathbb{R})$.

**1.131** Suppose that $\mathcal{S}$ and $\mathcal{T}$ are subspaces of some vector space $\mathcal{V}$. Is $\mathcal{S} \cap \mathcal{T}$ also a subspace of $\mathcal{V}$? Prove your answer. (Recall that $\mathcal{S} \cap \mathcal{T}$ denotes the set of all elements that belong to both $\mathcal{S}$ and $\mathcal{T}$.)

**1.132** ✓✓Suppose that $\mathcal{S}$ and $\mathcal{T}$ are subspaces of $\mathbb{R}^3$. Under what circumstances is $\mathcal{S} \cup \mathcal{T}$ a subspace of $\mathbb{R}^3$? What if $\mathcal{S}$ and $\mathcal{T}$ are subspaces of $\mathbb{R}^n$? Prove your answer. (Recall that $\mathcal{S} \cup \mathcal{T}$ denotes the set of all elements that belong to either $\mathcal{S}$ or $\mathcal{T}$ or both.)

**1.133** Let $\mathcal{S}$ and $\mathcal{T}$ be subspaces of some vector space $\mathcal{V}$. By $\mathcal{S} + \mathcal{T}$ we mean the set of all vectors of the form $X + Y$, where $X \in \mathcal{S}$ and $Y \in \mathcal{T}$. Prove that $\mathcal{S} + \mathcal{T}$ is a subspace of $\mathcal{V}$.

### 1.4.1 Computer Projects

MATLAB is very good at matrix multiplication. For example,

```
>> A = [   1  2  1  3
          -5  7  2  2
          13  4  4  3  ]

   A =
        1      2      1      3
       -5      7      2      2
       13      4      4      3

>> X y [1;3;-2;4]
   X =
        1
        3
       -2
        4

>> B = A * X



   B =
       17
       20
       29
```

1. Check the value of $AX$ computed above by asking MATLAB to compute $A_1 + 3A_2 - 2A_3 + 4A_4$, where $A_i$ is the $i$th column of $A$. [Recall that MATLAB for $A_i$ is A(:,i).]

2. For the matrices $A$ and $B$ above, find the general solution to the system $AX = B$ using rref. Express your solution in parametric form. For which value of the parameter do you obtain $X$?

   [*Note:* A neat way of producing the augmented matrix is to enter C=[A,B].]

3. Here is another way to find the general solution to the system in Exercise 2. Try this (with $X$ as before):

```
>> format short
>> Z=null(A)
>> A*(X+20*Z)
```

What the function null does is find a spanning vector for the nullspace. This example demonstrates the translation theorem in that it shows that the general solution is any one solution plus the nullspace. There is a catch though; null only gives *approximate* answers. To see this, try

```
>>format long
>> A*Z
```

The term 1.0e-15 means that the given vector should be multiplied by $10^{-15}$. Small, but not 0.

**4.** Let

$$A = \begin{bmatrix} 17 & -6 & 13 & 27 & 64 & 19 \\ 4 & -6 & -33 & 25 & 7 & 9 \\ 55 & -24 & 6 & 106 & 199 & 66 \\ 89 & -36 & 32 & 160 & 327 & 104 \end{bmatrix}, \quad B = \begin{bmatrix} 17 \\ 4 \\ 55 \\ 89 \end{bmatrix}$$

**(a)** Compute the rank of $A$. How many nonpivot variables does the system $AX = 0$ have?

**(b)** How many spanning vectors does the nullspace for $A$ have?

**(c)** Use the W=null(A) command to find a spanning set for the nullspace of $A$. (The columns of $W$ are the desired spanning set.)

**(d)** Find (by inspection) a vector $X_0$ such that $AX_0 = B$.

**(e)** Let the columns of $W$ be denoted by $W_i$. Check that $X_0 + 2W_1 - 3W_2 + 7W_3 - 9W_4$ is, at least approximately, a solution to $AX = B$.

**(f)** I claim that the general solution to $AX = B$ is $X_0 + WY$, where $Y \in \mathbb{R}^4$. Explain.

**(g)** Use rref to find a spanning for the nullspace of $A$ by solving $AX = 0$. Your answer will appear to be totally different from the answer that null gives. Demonstrate that the two answers are really equivalent by showing that the second column of $W$ is a linear combination of the spanning elements and that your third spanning element is a linear combination of the columns of $W$.

[*Note:* You may need to use rref([A,B],tol), where the tolerance tol is larger than the default value. Try to use the smallest possible tolerance.]

## CHAPTER SUMMARY

The central topic of this chapter is systems of linear equations, which we solved using ***Gaussian elimination*** (Section 1.3). Specifically, we used ***elementary row operations*** to reduce the ***augmented matrix*** for the system until either ***echelon form*** or ***reduced echelon form*** was obtained. The variables corresponding to the corners of the steps in the echelon form are the ***pivot variables*** and the nonpivot variables are ***free variables***. We obtain the ***general solution*** by expressing the pivot variables in terms of the free variables yielding the ***parametric form*** of the solution as well as the ***more unknowns*** theorem, which states that at least $n$ equations are required to uniquely determine $n$ unknowns.

In Section 1.4 we expressed linear systems in terms of ***matrix multiplication*** as $AX = B$, where $A$ is the ***coefficient matrix*** for the system and $B$ is the ***matrix of constants***. We proved that $AX = B$ is solvable if and only if $B$ belongs to the the span of the columns of $A$ (***column space*** of $A$). (In general the ***span*** of a set of vectors is the set of all their linear combinations.) The ***translation theorem*** says that the general solution to $AX = B$ is obtained by translating the ***nullspace*** of $A$, which is the solution space to the equation $AX = 0$, by any vector $T$ that satisfies $AT = B$.

It is important that both the column space and nullspace are ***subspaces*** of $\mathbb{R}^k$ for appropriate $k$. Algebraically, this means that they are ***closed under linear combinations***. Geometrically, this means that we may think of them as higher dimensional analogues of lines and planes through the origin. Since subspaces of vector spaces are themselves vector spaces, the nullspace and column space are vector spaces. This will allow us to use the theory that we develop in Chapter 2 to study these spaces.

Another fundamental concept was ***rank***. The rank of a system of linear equations is number of equations left after eliminating linearly dependent equations, one at a time, until a linearly independent system is obtained. It is a fundamental result (to be proved in Chapter 2) that the rank is independent of which equations were kept and which were eliminated. In fact, the rank is the number of nonzero rows of the reduced form of the augmented matrix.

The concept of rank is based on the concept of linear independence. A system of equations is linearly dependent if one row of the augmented matrix is a linear combination of the other rows. In general, a set of elements from some vector space is linearly dependent if one of the elements is a linear combination of other elements from the same set.

By a vector space, we mean any set of elements that can be added and multiplied by scalars so as to satisfy the ***vector space properties*** from Section 1.1. Examples of vector spaces include the space $M(m, n)$ of $m \times n$ matrices, function spaces such as $\mathcal{F}(\mathbb{R})$ and $C^\infty(\mathbb{R})$, nullspaces, and column spaces. The significance of the concept of vector space is that anything we prove using only the vector space properties will automatically be true for any vector space. In particular, it will apply to nullspaces, column spaces, function spaces, as well as $m \times n$ matrices. We will see more examples of vector spaces later.

# CHAPTER 2

# LINEAR INDEPENDENCE AND DIMENSION

## 2.1 THE TEST FOR LINEAR INDEPENDENCE

One important application of the Gaussian technique in linear algebra is deciding whether a given set of elements of $M(m,n)$ is linearly independent. In principle, this task could be quite tedious. For example, to show that $\{A_1, A_2, A_3\}$ is linearly independent, we must show that:

1. $A_1$ is not a linear combination of $A_2$ and $A_3$.
2. $A_2$ is not a linear combination of $A_1$ and $A_3$.
3. $A_3$ is not a linear combination of $A_1$ and $A_2$.

Fortunately, this can all be checked in a single step. Let $S = \{A_1, A_2, A_3\}$ be a set of three linearly dependent matrices. Suppose

$$A_1 = d_2A_2 + d_3A_3$$

We may write this as

$$\mathbf{0} = (-1)A_1 + d_2A_2 + d_3A_3$$

Thus, there are constants $x_1, x_2$, and $x_3$ with $x_1 \neq 0$ such that

$$\mathbf{0} = x_1A_1 + x_2A_2 + x_3A_3$$

*Linear Algebra: Ideas and Applications*, Fourth Edition. Richard C. Penney.

Companion website: www.wiley.com/go/penney/linearalgebra

On the other hand, suppose we did not know that $S$ is linearly dependent, but we did have constants $x_1, x_2$, and $x_3$ with $x_1 \neq 0$ such that

$$\mathbf{0} = x_1A_1 + x_2A_2 + x_3A_3$$

Then, we may write

$$-x_1A_1 = x_2A_2 + x_3A_3$$

Hence,

$$A_1 = \left(\frac{-x_2}{x_1}\right)A_2 + \left(\frac{-x_3}{x_1}\right)A_3$$

Thus, $A_1$ is a combination of $A_2$ and $A_3$. If $x_2 \neq 0$, then $A_2$ would be a combination of $A_1$ and $A_3$. If $x_3 \neq 0$, then $A_3$ would be a combination of $A_1$ and $A_2$. As long as one of the $x_i \neq 0$, then one of the $A_i$'s is a combination of the others, making $S$ linearly dependent.

The same reasoning, applied to $n$ elements instead of three, proves the following theorem, which is called the "test for linear independence." This property is taken as the definition of linear independence in many linear algebra texts. Notice that in the calculations above, we never explicitly used the fact that we were dealing with matrices; we only used the vector space properties [properties (a–j) on page 12]. Hence, we state our result as a vector space theorem.

**Theorem 2.1** (Test for Independence). *Let $S = \{A_1, A_2, \dots, A_n\}$ be a set of $n$ elements of a vector space $\mathcal{V}$. Consider the equation*

$$x_1A_1 + \cdots + x_nA_n = \mathbf{0} \tag{2.1}$$

*If the only solution to this equation is $x_1 = x_2 = \cdots = x_n = 0$, then $S$ is linearly independent.*

*On the other hand, if there is a solution to equation* (2.1) *with $x_k \neq 0$ for some $k$, then $S$ is linearly dependent and $A_k$ is a linear combination of the set of $A_j$ with $j \neq k$.*

*Proof.* Suppose first that we are given

$$x_1A_1 + \cdots + x_nA_n = \mathbf{0}$$

with $x_k \neq 0$. Then

$$x_1A_1 + \cdots + x_{k-1}A_{k-1} + x_{k+1}A_{k+1} + \cdots + x_nA_n = -x_kA_k$$
$$-\frac{x_1}{x_k}A_1 - \cdots - \frac{x_{k-1}}{x_k}A_{k-1} - \frac{x_{k+1}}{x_k}A_{k+1} + \cdots - \frac{x_n}{x_k}A_n = A_k$$

proving linear dependence.

Conversely, if $A_k$ is a linear combination of the other $A_j$, then there are constants $x_i$ such that

$$x_1A_1 + x_2A_2 + \cdots + x_{k-1}A_{k-1} + x_{k+1}A_{k+1} + \cdots + x_nA_n = A_k$$

Then

$$x_1A_1 + \cdots + x_{k-1}A_{k-1} + (-1)A_k + x_{k+1}A_{k+1} + \cdots + x_nA_n = \mathbf{0}$$

showing that there is a solution to equation (2.1) with $x_k \neq 0$. □

**Definition 2.1** *Equation* (2.1) *is referred to as the* ***dependency equation*** *for the sequence* $A_1, A_2, \ldots, A_n$ *of elements of* $\mathcal{V}$.

### EXAMPLE 2.1

Prove that the set $\{A_1, A_2, A_3\}$ is linearly independent where $A_1 = [1, 1, 2], A_2 = [1, 2, 1]$, and $A_3 = [0, 1, 1]$.

*Solution.* Consider the dependency equation

$$\begin{aligned} x_1A_1 + x_2A_2 + x_3A_3 &= \mathbf{0} \\ x_1[1, 1, 2] + x_2[1, 2, 1] + x_3[0, 1, 1] &= [0, 0, 0] \\ [x_1 + x_2, x_1 + 2x_2 + x_3, 2x_1 + x_2 + x_3] &= [0, 0, 0] \end{aligned}$$

Equating corresponding entries, we get the system

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 + x_2 + x_3 &= 0 \end{aligned}$$

The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 1 & 0 \end{bmatrix}$$

After row reduction, we obtain

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

From this, it follows that $x_3 = 0$, $x_2 = 0$, and $x_1 = 0$, proving linear independence.

### EXAMPLE 2.2

Check the set $\{A, B, C\}$ for linear independence using the test for independence, where

$$A = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

*Solution.* Before we begin, let us first note that the test will yield "dependent" as an answer, since

$$A = B + 3C$$

If we had not been required to use the test, we could stop here.

For the test, consider the dependency equation

$$x\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + z\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{0} \tag{2.2}$$

This is the same as

$$\begin{bmatrix} x + y \\ 3x + z \\ x + y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus,

$$\begin{aligned} x + y &= 0 \\ 3x + z &= 0 \\ x + y &= 0 \end{aligned}$$

The augmented matrix for this system is

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

We row reduce, obtaining

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that $z$ is arbitrary. In particular, $z$ may be chosen to be nonzero, showing that $C$ is linearly dependent on $A$ and $B$. In fact, if we choose some specific value for $z$, say, $z = 1$, we find $y = \frac{1}{3}$ and $x = -\frac{1}{3}$. Formula (2.2) then says that

$$-\frac{A}{3} + \frac{B}{3} + C = \mathbf{0}$$

Hence, $C = A/3 - B/3$. This is easily seen to be equivalent to the relation observed above.

## EXAMPLE 2.3

Test the set formed by the following matrices for linear dependence. If linear dependence is found, use your results to express one as a linear combination of the others.

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad \begin{bmatrix} -1 & -2 \\ -3 & -5 \end{bmatrix}, \quad \begin{bmatrix} -1 & -2 \\ 0 & -2 \end{bmatrix}$$

*Solution.* Let us denote these matrices by $A$, $B$, $C$, and $D$, respectively. Consider the equation

$$xA + yB + zC + wD = \mathbf{0} \tag{2.3}$$

Substituting for $A$, $B$, $C$, and $D$ and combining, we have

$$x\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} + y\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} + z\begin{bmatrix} -1 & -2 \\ -3 & -5 \end{bmatrix} + w\begin{bmatrix} -1 & -2 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x + y - z - w & 2x + 2y - 2z - 2w \\ x + 2y - 3z & 3x + 4y - 5z - 2w \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This yields the system

$$\begin{aligned} x + y - z - w &= 0 \\ 2x + 2y - 2z - 2w &= 0 \\ x + 2y - 3z \phantom{- 2w} &= 0 \\ 3x + 4y - 5z - 2w &= 0 \end{aligned}$$

The reduced augmented matrix for this system is

$$\begin{bmatrix} 1 & 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The variables $z$ and $w$ are free and $y = 2z - w$ and $x = -z + 2w$. Clearly, the test yields linear dependence, since we may, for example, choose any $z \neq 0$. Since $z$ is the coefficient of $C$ in formula (2.3), it follows that $C$ must be a combination of the other vectors. To explicitly realize $C$ as a combination, we must assign specific values to $z$ and $w$. We could use any values as long as $z \neq 0$. However, a convenient assignment is $z = -1$ and $w = 0$. This yields $x = 1$ and $y = -2$. From formula (2.3), we see that

$$(1)A + (-2)B + (-1)C + 0D = \mathbf{0}$$

Hence,

$$A - 2B = C$$

What if, instead, we had set $z = 0$ and $w = -1$? Then $x = -2$, $y = 1$, and thus

$$(-2)A + (1)B + 0C + (-1)D = \mathbf{0}$$

Therefore,

$$-2A + B = D$$

It follows that both $D$ and $C$ depend on $A$ and $B$. The elements $A$ and $B$ form a linearly independent set since neither is a multiple of the other.

Example 2.3 demonstrates a general technique for finding dependencies. To find dependencies among the elements of a sequence $A_1, A_2, \dots, A_k$ of $m \times n$ matrices:

1. Solve the homogeneous system of equations produced by equating the entries on both sides of the dependency equation (2.1). We refer to this system as the **dependency system**. We say that $A_i$ is a **pivot matrix** if $x_i$ is a pivot variable for the dependency system.
2. Each nonpivot matrix in $S$ will be linear combinations of the pivot matrices. To express the nonpivot matrix $A_j$ as a linear combination of the pivot matrices, compute the solution of the dependency system obtained by letting $x_j = -1$ and all the other nonpivot variables equal 0. Substitute these values into the dependency equation and solve for $A_j$.

The following theorem is a consequence of the preceding comments.

**Theorem 2.2** *Let $A_1, A_2, \dots, A_k$ be a sequence of $m \times n$ matrices. Then each nonpivot matrix is a linear combination of the set of pivot matrices. Furthermore, the set of pivot matrices is linearly independent.*

*Proof.* Statement 2 above Theorem 2.2 shows that the nonpivot matrices are linear combinations of the set pivot matrices. Hence, we need only prove the linear independence of the set pivot matrices. Consider the dependency equation

$$x_{i_1}A_{i_1} + x_{i_2}A_{i_2} + \cdots + x_{i_p}A_{i_p} = \mathbf{0}$$

where the $A_{i_j}$ are the pivot matrices. Then

$$x_1A_1 + x_2A_2 + \cdots + x_kA_k = \mathbf{0}$$

where $x_j = 0$ if $A_j$ is not a pivot matrix. But then

$$X = [x_1, x_2, \ldots, x_k]^t$$

is a solution to the dependency system for which all the nonpivot variables equal 0. The zero vector is also a solution to the dependency equation for which all the nonpivot variables equal 0. Since the solution is uniquely determined by the values of the nonpivot variables, it follows that $X = \mathbf{0}$, proving linear independence. □

Recall that the span of a set of elements in a vector space is the set of all their linear combinations. The following proposition says, roughly, that linearly dependent vectors do not contribute to the span.

**Proposition 2.1** *Let* $\{A_1, A_2, \ldots, A_k\}$ *be a set of elements of some vector space* $\mathcal{V}$ *such that* $A_k$ *is a linear combination of* $\{A_1, A_2, \ldots, A_{k-1}\}$. *Then*

$$\text{span}\{A_1, A_2, \ldots, A_{k-1}\} = \text{span}\{A_1, A_2, \ldots, A_k\} \tag{2.4}$$

*Proof.* Let

$$\begin{aligned}\mathcal{U} &= \text{span}\{A_1, A_2, \ldots, A_{k-1}\}\\ \mathcal{W} &= \text{span}\{A_1, A_2, \ldots, A_k\}\end{aligned}$$

We must show that every element of $\mathcal{U}$ also belongs to $\mathcal{W}$ and, conversely, every element of $\mathcal{W}$ also belongs to $\mathcal{U}$.[1]

If $X \in \mathcal{U}$, then there are scalars $c_1, c_2, \ldots, c_{k-1}$ such that

$$\begin{aligned}X &= c_1A_1 + c_2A_2 + \cdots + c_{k-1}A_{k-1}\\ &= c_1A_1 + c_2A_2 + \cdots + c_{k-1}A_{k-1} + 0A_k\end{aligned}$$

which is an element of $\mathcal{W}$. Hence every element of $\mathcal{U}$ also belongs to $\mathcal{W}$.

[1]If $S$ and $T$ are sets, then, by definition, $S \subset T$ means that every element of $S$ also belongs to $T$. We say that $S = T$ if $S \subset T$ and $T \subset S$ both hold.

Conversely, notice that, by hypothesis, $A_k$ belongs to $\mathcal{U}$ as do all the $A_j$ for $j = 1, \dots, k-1$. Since $\mathcal{U}$ is a subspace of $\mathcal{V}$, it follows that every linear combination of $\{A_1, \dots, A_k\}$ also belongs to $\mathcal{U}$, showing that every element of $\mathcal{W}$ belongs to $\mathcal{U}$. This finishes our proof. □

According to Proposition 2.1, if a subspace $\mathcal{W}$ of a vector space is spanned by a linearly dependent set $S$, then we may produce a smaller spanning set $S_1$ by deleting an element of $S$. If $S_1$ is still linearly dependent, we may delete one of its elements to obtain a yet smaller spanning set. We may continue deleting elements until a linearly independent set is obtained. Thus, *every spanning set for a subspace $\mathcal{W}$ contains a linearly independent spanning set.*

**Definition 2.2** *A basis for a vector space $\mathcal{V}$ is a linearly independent subset $\mathcal{B} = \{A_1, A_2, \dots, A_n\}$ of $\mathcal{V}$ that spans $\mathcal{V}$.*

### Bases for the Column Space

Theorem 2.2 can be stated using bases as follows.

**Theorem 2.3** *Let $A_1, A_2, \dots, A_k$ be a sequence of $m \times n$ matrices. Then the pivot elements form a basis for* $\text{span}\{A_1, A_2, \dots, A_k\}$.

If the $A_i$ are column vectors, then the dependency equation (2.1) is equivalent to the matrix equation $AX = \mathbf{0}$, where

$$A = [A_1, A_2, \dots, A_k]$$

Hence, the pivot and nonpivot vectors correspond to the pivot and nonpivot variables of the system whose augmented matrix is $[A_1, A_2, \dots, A_k, \mathbf{0}]$.

**Definition 2.3** *Let $A = [A_1, A_2, \dots, A_k]$ be a matrix where the $A_i$ are the columns of $A$. We say that $A_i$ is a **pivot column** for $A$ if $x_i$ is a pivot variable for the system $AX = \mathbf{0}$.*

The following theorem, which will be used in Section 2.4 to prove the crucial rank theorem, is an immediate consequence of Theorem 2.3.

**Theorem 2.4** *The pivot columns of a matrix $A$ form a basis for the column space of $A$.*

### EXAMPLE 2.4

Find a basis for the column space of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 2 & -4 & 4 \\ 1 & 3 & 0 & 4 \end{bmatrix}$$

*Solution.* From Theorem 2.4 we need to determine the pivot columns, which requires reducing the following matrix:

$$B = \begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ 2 & 2 & -4 & 4 & 0 \\ 1 & 3 & 0 & 4 & 0 \end{bmatrix}$$

We row reduce, finding the reduced form to be

$$R = \begin{bmatrix} 1 & 0 & -3 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot variables are $x_1$ and $x_2$. From Theorem 2.4, the corresponding columns of $A$ form a basis for the column space. Thus, our basis is formed by

$$A_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

*Remark.* It is important to note that our basis comes from the columns of $A$ and not from the echelon form of $A$. The echelon form is used only to determine which are the pivot columns. In fact, the pivot columns of $R$ are $[1, 0, 0]^t$ and $[0, 1, 0]^t$, which do not even belong to $A$'s column space. However, as we note below, relationships among the columns of $R$ do carry over to the columns of $A$.

Note that the columns $R_i$ of the matrix $R$ in Example 2.4 satisfy

$$R_3 = -3R_1 + R_2 \tag{2.5}$$

Precisely the same relation holds for the columns of $A$:

$$\begin{bmatrix} -1 \\ -4 \\ 0 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \tag{2.6}$$

Similarly,

$$R_4 = R_1 + R_2$$

and, as the reader may check,

$$A_4 = A_1 + A_2$$

The explanation of these "coincidences" is not hard to understand. Equation (2.5) is equivalent to

$$-3R_1 + R_2 - R_3 = \mathbf{0}$$

which implies that $X = [-3, 1, -1, 0]^t$ solves the system with augmented matrix $R$. Since $R$ is row equivalent to $B$, $X$ also solves the system with augmented matrix $B$, implying that

$$-3A_1 + A_2 - A_3 = \mathbf{0}$$

which is equivalent to equation (2.6). The explanation for the two formulas following equation (2.6) is similar. The general principle that this example illustrates is contained in the following theorem.

**Theorem 2.5** *Let $A$ and $B$ be row equivalent $m \times n$ matrices with columns $A_i$ and $B_i$, $1 \leq i \leq n$, respectively. Suppose that for some $j$*

$$B_j = c_1B_1 + \cdots + c_{j-1}B_{j-1} + c_{j+1}B_{j+1} + \cdots + c_nB_n$$

*for some scalars $c_i$, $1 \leq i \leq n$. Then*

$$A_j = c_1A_1 + \cdots + c_{j-1}A_{j-1} + c_{j+1}A_{j+1} + \cdots + c_nA_n$$

*Proof.* Our assumptions on the columns of $B$ imply that $BX = \mathbf{0}$, where

$$X = [c_1, \ldots, c_{j-1}, -1, c_{j+1}, \ldots, c_n]^t$$

Since $A$ is row equivalent to $B$, it follows that $AX = \mathbf{0}$, which is equivalent to our theorem. □

## Testing Functions for Independence

Linear independence is also important for functions. The identity

$$\sin^2 x + \cos^2 x = 1$$

can be thought of as expressing the function $y = 1$ as a linear combination of the functions $y = \cos^2 x$ and $y = \sin^2 x$. It is also important to know when certain functions *cannot* be expressed as a linear combination of other functions. For example, we might want to know if there is an identity of the form

$$k_1e^x + k_2e^{-x} = 1$$

that is true for all $x$, where $k_1$ and $k_2$ are fixed scalars.

It turns out that there is no such identity. If there were such an identity, then the set $\{e^x, e^{-x}, 1\}$ would be linearly dependent in the vector space $C^\infty(\mathbb{R})$ of all infinitely differentiable functions on $\mathbb{R}$. Hence, there would exist scalars $a$, $b$, and $c$, not all zero, such that

$$ae^x + be^{-x} + c\,1 = \mathbf{0} \tag{2.7}$$

where "0" is the constant function $y = 0$. Since this identity must hold for all $x$, we can set $x = 0$, concluding that

$$a + b + c = 0$$

One equation is not sufficient to determine three unknowns. However, we can differentiate both sides of equation (2.7) with respect to $x$, producing

$$ae^x - be^{-x} = \mathbf{0} \tag{2.8}$$

Setting $x = 0$ produces

$$a - b = 0$$

Next, we differentiate equation (2.8),

$$ae^x + be^{-x} = \mathbf{0}$$

and set $x = 0$,

$$a + b = 0$$

The equations $a + b = 0$ and $a - b = 0$ prove that $a = b = 0$. The equation $a + b + c = 0$ then shows that $c = 0$. Hence, the given functions are linearly independent, showing that no such identity is possible.

*Remark.* The preceding proof of the linear independence of $e^x$, $e^{-x}$, and 1 demonstrates a general technique. To prove the linear independence of a set of functions $\{f_1(x), f_2(x), \dots, f_n(x)\}$, we differentiate the dependency equation $n - 1$ times, producing the system of $n$ equations

$$\begin{array}{rcrcrl} c_1f_1(x) & + & c_2f_2(x) & + \cdots + & c_nf_n(x) & = \mathbf{0} \\ c_1f_1'(x) & + & c_2f_2'(x) & + \cdots + & c_nf_n'(x) & = \mathbf{0} \\ & & & & & \vdots \\ c_1f_1^{(n-1)}(x) & + & c_2f_2^{(n-1)}(x) & + \cdots + & c_nf_n^{(n-1)}(x) & = \mathbf{0} \end{array} \tag{2.9}$$

Substitution of some specific value of $x$ (e.g., $x = 0$) produces a linear system that can be solved for the $c_i$ to show (hopefully) that they are all 0.

**True-False Questions:** Justify your answers.

**2.1** A $5 \times 27$ matrix can have six linearly independent columns.

**2.2** I start with a certain $4 \times 6$ matrix $A$ and reduce, obtaining

$$M = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3 & -2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Which of the following statements about $A$ are guaranteed to be true, where $A_i$ denotes the $i$th column of $A$.

**(a)** The columns of $A$ are linearly independent.

**(b)** $A_4$ is a linear combination of $A_1$, $A_2$, and $A_5$.

**(c)** $A_3$ is a linear combination of $A_1$, $A_2$, and $A_5$.

**(d)** $A_3$ is a linear combination of $A_1$ and $A_2$.

**(e)** $A_4 = A_1 - 2A_2$.

**2.3** Suppose that $A$ and $B$ are $n \times n$ matrices that both have linearly independent columns. Then $A$ and $B$ have the same reduced echelon form.

**2.4** Suppose that $A$ is an $n \times n$ matrix and $B$ is an $n \times 1$ column vector such that the equation $AX = B$ has an infinite number of solutions. Then the columns of $A$ are linearly dependent.

**2.5** The functions $\cos^2 x$, $\sin^2 x$, $\sin 2x$, and $\cos 2x$ form a linearly independent set.

**2.6** Suppose that $f$ and $g$ in $C^\infty(\mathbb{R})$ are such that $f(17) = 1, f'(17) = 0$, $g(17) = 0$, and $g'(17) = 1$. It follows that $f$ and $g$ are linearly independent functions.

**2.7** Suppose that $f$ and $g$ in $C^\infty(\mathbb{R})$ are such that $f(17) = 1, f(19) = 0$, $g(17) = 0$, and $g(19) = 1$. It follows that $f$ and $g$ are linearly independent functions.

**2.8** Suppose that $f$ and $g$ in $C^\infty(\mathbb{R})$ are such that $f(0) = 1, f(1) = 1$, $g(0) = 0$, and $g(2) = 1$. It follows that $f$ and $g$ are linearly independent functions.

**2.9** Suppose that $f$ and $g$ in $C^\infty(\mathbb{R})$ are such that $f(1) = 1, f(-1) = 2$, $g(1) = 2$, and $g(-1) = 4$. It follows that $f$ and $g$ are linearly dependent functions.

## EXERCISES

**2.1** Test the given matrices for linear dependence using the test for linear independence. Then find a basis for their span and express the other vectors (if there are any) as linear combinations of the basis elements.

**(a)** ✓✓ $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

**(b)** $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

**(c)** $[1,2,1], \quad [3,-1,2], \quad [7,-7,4]$

**(d)** ✓✓ $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 17 & 0 \\ 9 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 5 \\ 0 & 6 \end{bmatrix}$

**(e)** $\begin{bmatrix} 2 & -2 \\ 3 & -1 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 4 & -1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 8 & -2 \\ 4 & 6 \\ 2 & 0 \end{bmatrix}$

**(f)** $\begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 6 & 5 \end{bmatrix}$

**(g)** ✓✓ $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, \quad \begin{bmatrix} 5 & -2 \\ 3 & 10 \end{bmatrix}$

**(h)** $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

**(i)** $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

**(j)** ✓✓ $[4,2,-1], \quad [3,3,2], \quad [1,0,1]$

**(k)** $\begin{bmatrix} 3 \\ 2 \\ 4 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} -4 \\ -3 \\ -5 \\ -6 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -3 \\ 7 \\ 12 \end{bmatrix}$

**2.2** In Exercise 2.1(b)✓, describe in words the span of the given vectors. Repeat for Exercise 2.1(h).

**2.3** For each of the following matrices $A$, determine the pivot columns and use the technique of Example 2.3 on page 101 to express each of the other columns as linear combinations of the pivot columns. *Your answer should be stated entirely in terms of the columns of A, not its reduced form.*

**(a)** ✓✓ $\begin{bmatrix} 1 & 2 & -3 & 3 \\ 2 & 1 & 3 & 4 \\ -8 & -1 & -21 & -14 \\ 4 & -1 & 15 & 6 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 2 & -8 & 4 \\ 2 & 1 & -1 & -1 \\ -3 & 3 & -21 & 15 \\ 3 & 4 & -14 & 6 \end{bmatrix}$

**(c)** ✓✓ $\begin{bmatrix} 2 & -1 & 2 \\ 3 & 0 & -1 \\ -1 & 2 & -5 \\ 19 & -8 & 15 \\ 2 & 5 & -14 \end{bmatrix}$ **(d)** $\begin{bmatrix} 2 & 3 & -1 & 19 & 2 \\ -1 & 0 & 2 & -8 & 5 \\ 2 & -1 & -5 & 15 & -14 \end{bmatrix}$

**(e)** ✓✓ $\begin{bmatrix} 3 & 2 & -1 & 2 \\ 1 & 4 & -2 & 2 \\ 2 & 0 & -1 & -3 \\ -7 & 8 & -2 & 8 \end{bmatrix}$ **(f)** $\begin{bmatrix} 3 & 1 & 2 & -7 \\ 2 & 4 & 0 & 8 \\ -1 & -2 & -1 & -2 \\ 2 & 2 & -3 & 8 \end{bmatrix}$

**2.4** Consider the accompanying matrix. Use the test for linear independence to find a basis for the space spanned by the rows of the matrix. Suppose that this matrix is the augmented matrix for a system of equations. What is the rank of this system? (Recall that the rank is the number of equations left after linearly dependent equations have been discarded, one at a time.)

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 0 \\ 3 & 3 & 6 & -3 \\ 4 & 1 & 5 & 2 \end{bmatrix}$$

**2.5** ✓✓Let $A$ be the matrix from Exercise 2.4. Find a basis for the column space and express the other columns as linear combinations of your basis.

**2.6** ✓✓Prove that the rows of the following $3 \times 6$ matrix are linearly independent:

$$A = \begin{bmatrix} 1 & a & 0 & b & 0 & c \\ 0 & 0 & 1 & d & 0 & e \\ 0 & 0 & 0 & 0 & 1 & f \end{bmatrix}$$

**2.7** ✓✓Use the test for linear independence to prove that the rows of the following $3 \times 6$ matrix are linearly independent:

$$A = \begin{bmatrix} 1 & a & b & c & d & e \\ 0 & 0 & 1 & f & g & h \\ 0 & 0 & 0 & 0 & 1 & k \end{bmatrix}$$

**2.8** Let $A = [A_1, A_2, A_3]$ be a $3 \times 3$ matrix with linearly independent columns $A_i$.

**(a)** Explain why the row reduced form of $A$ is the following matrix $R$.

[*Hint:* Think about the number of free variables in the system with augmented matrix $[A, \mathbf{0}]$.]

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**(b)** Let $A = [A_1, A_2, A_3, A_4, A_5]$ be a $3 \times 5$ matrix such that the first three columns are linearly independent. Explain why the pivot columns must be the first three. [*Hint:* If not, what would this say about the row reduced form of $[A_1, A_2, A_3]$?]

**2.9** ✓State and solve an analogue of Exercise 2.8 for $4 \times 3$ matrices $A$. For part (b), discuss $4 \times 7$ matrices whose first three columns are linearly independent.

**2.10** ✓Let $A$ be an $m \times n$ matrix. Is there a relationship between the number of pivot columns and the number of nonzero rows in an echelon form of $A$? Do you expect a relationship between the rank of the corresponding system of equations and the number of basis elements for the column space? Explain.

**2.11** ✓✓Suppose that $\{X_1, X_2\}$ is a linearly independent set of elements in some vector space $\mathcal{V}$. Let $Y_1 = 3X_1 - 2X_2$ and $Y_2 = X_1 + X_2$. Prove that $\{Y_1, Y_2\}$ is linearly independent in $\mathcal{V}$.

**2.12** ✓Suppose that $\{X_1, X_2\}$ is a linearly independent set of elements in some vector space $\mathcal{V}$. Let $Y_1 = aX_1 + bX_2$ and $Y_2 = cX_1 + dX_2$. Prove that $\{Y_1, Y_2\}$ is linearly independent in $\mathcal{V}$ if and only if the vectors $[a, b]^t$ and $[c, d]^t$ are linearly independent elements of $\mathbb{R}^2$.

**2.13** Suppose that $\{X_1, X_2, X_3\}$ is a linearly independent set of elements in some vector space $\mathcal{V}$. Let $Y_1 = X_1 + 2X_2 - X_3$, $Y_2 = X_1 + X_2$, and $Y_3 = 7X_2$. Prove that $\{Y_1, Y_2, Y_3\}$ is a linearly independent set in $\mathcal{V}$.

**2.14** State and prove a result similar to that stated in Exercise 2.12 which applies to problems such as Exercise 2.13.

**2.15** ✓Suppose that $X_1$, $X_2$, and $X_3$ are linearly independent elements of some vector space. Let $Y_1 = X_1 + 2X_2$, $Y_2 = X_1 + X_2$, and $Y_3 = X_1 + X_2 + X_3$. Use the result you stated in Exercise 2.14 to prove that $Y_1$, $Y_2$, and $Y_3$ are also independent.

**2.16** Suppose that $X_1$, $X_2$, and $X_3$ are elements in some vector space. Let $Y_1 = X_1 + X_2 + 2X_3$, $Y_2 = X_1 + X_2 - X_3$. Prove that if $\{Y_1, Y_2\}$ is linearly dependent then $\{X_1, X_2, X_3\}$ is also dependent.

**2.17** ✓Suppose that $\{X_1, X_2, X_3\}$ is a linearly independent set of elements in some vector space $\mathcal{V}$. Let $Y_1 = X_1 + 2X_2 - X_3$, $Y_2 = 2X_1 + 2X_2 - X_3$, and $Y_3 = 4X_1 + 2X_2 - X_3$. Prove that $\{Y_1, Y_2, Y_3\}$ is a linearly dependent set in $\mathcal{V}$. Does this exercise really require the linear independence of the $X_i$?

**2.18** Suppose that $\{X_1, X_2, X_3\}$ is a linearly independent set of elements in some vector space $\mathcal{V}$. Let $Y_1 = 2X_1 + 3X_2 + 5X_3$, $Y_2 = X_1 - X_2 + 3X_3$, and $Y_3 = 2X_1 + 13X_2 + 3X_3$. Prove that $\{Y_1, Y_2, Y_3\}$ is a linearly dependent set in $\mathcal{V}$. Does this exercise really require the linear independence of the $X_i$?

**2.19** Suppose that $X$ and $Y$ belong to some vector space $\mathcal{V}$. Let $Z_1 = a_1X + b_1Y$, $Z_2 = a_2X + b_2Y$, and $Z_3 = a_3X + b_3Y$, where the $a_i$ and $b_i$ are scalars. Prove that $\{Z_1, Z_2, Z_3\}$ is a linearly dependent set in $\mathcal{V}$.

**2.20** ✓✓Prove that the columns of a matrix $A$ are linearly independent if and only if the nullspace of $A$ consists of only the zero vector.

**2.21** ✓Prove that any set of three vectors in $\mathbb{R}^2$ is linearly dependent.

**2.22** Prove that any set of four vectors in $\mathbb{R}^3$ is linearly dependent.

**2.23** ✓✓Prove that any set of five matrices in $M(2, 2)$ is linearly dependent.

**2.24** Show that the following sets of functions are linearly dependent in $C^\infty(\mathbb{R})$ by expressing one of them as a linear combination of the others. Recall that $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$.

**(a)** $\{e^x, e^{-x}, \cosh x\}$
**(b)** ✓✓ $\{\cos(2x), \sin^2 x, \cos^2 x\}$
**(c)** $\{\sinh x, e^x, \cosh x\}$
**(d)** ✓✓ $\{\cos(2x), 1, \sin^2 x\}$
**(e)** ✓✓ $\{\ln(3x), \ln x, 1\}$
**(f)** $\{x(x-2)^2, 1, x, x^2, x^3\}$
**(g)** $\{1, (x-1), (x-1)^2, 2x^2 + 5x + 3\}$

**2.25** Show that the following sets of infinitely differentiable functions are linearly independent using the test for linear independence. [*Hint* In some cases you may need to differentiate the dependency equation additional times and/or evaluate at different values of $x$.]

**(a)** $\{e^x, e^{-x}, e^{3x}\}$
**(b)** ✓✓ $\{e^x, e^{2x}, e^{3x}\}$
**(c)** $\{1, x, x^2, x^3\}$
**(d)** $\{x^2, x^3, x^4\}$
**(e)** ✓ $\{e^x, \sin(2x), e^x \sin(2x)\}$
**(f)** $\{x^2e^x, 1, e^{-x}\}$
**(g)** ✓✓ $\{\ln x, x \ln x\}$
**(h)** $\{1, (x-1), (x-1)^2, (x-1)^3\}$

**2.26** Let $f$ and $g$ be elements of $C^\infty(\mathbb{R})$. Let $X = [f(0), f'(0)]^t$ and $Y = [g(0), g'(0)]^t$. (a) Prove that if $X$ and $Y$ are linearly independent in $\mathbb{R}^2$, then $f$ and $g$ are linearly independent in $C^\infty(\mathbb{R})$. (b) Give an example of two linearly independent functions $f$ and $g$ for which the corresponding vectors $X$ and $Y$ both equal zero. Prove that your $f$ and $g$ are linearly independent.

**2.27** We know that the functions $\{\cos^2 x, \sin^2 x, 1\}$ are linearly dependent. Attempt to prove that they are *linearly independent* by differentiating the dependency equation and substituting $x = 0$. How does the technique break down?

### 2.1.1 Computer Projects

## EXERCISES

**1.** Let

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 5 & -1 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Use MATLAB to find the row reduced form of $A$. How can you tell just from this reduced form that the columns of $A$ are linearly independent?

**2.** Let $A$ be a matrix with more rows than columns. State a general rule for using rref(A) to decide whether the columns of $A$ are linearly independent. Demonstrate your condition by (a) producing a $5 \times 4$ matrix $A$ with no nonzero coefficients that has linearly independent columns and computing rref(A) and (b) producing a $5 \times 4$ matrix $A$ that has no nonzero coefficients but does have linearly dependent columns and computing rref(A).

**3.** Let

$$A = \begin{bmatrix} -1 & 2 & 6 & -8 & -14 & 3 \\ 2 & 4 & 1 & -8 & 5 & -1 \\ -3 & 1 & 4 & -9 & -10 & 0 \\ 3 & -2 & -1 & 12 & -1 & 4 \\ 5 & 7 & 11 & -11 & -19 & 9 \end{bmatrix}$$

Use rref(A) to find the pivot columns of $A$. Write them out explicitly as columns. Then express the other columns of $A$ as linear combinations of the pivot columns. (Use Theorem 2.3.) You should discover that the first three columns of $A$ are the pivot columns.

**4.** In MATLAB, enter B=[A(:,4),A(:,1),A(:,2),A(:,3),A(:,5),A(:,6)], where $A$ is the matrix from Exercise 3. The columns of $B$ are just those of $A$, listed in a different order. Compute rref(B) to find the pivot columns. Do you obtain a different set of pivot columns? Use your answer to express the other columns as linear combinations of the pivot columns. Could you have derived these expressions from those in Exercise 3? If so, how?

**5.** Find a matrix $C$ whose columns are just those of $A$ listed in a different order, such that the column of $C$ that equals A(:,5) and the column that equals A(:,1) are both pivot columns. Is it possible to find such a $C$, where A(:,2) is a pivot column as well? If so, find an example. If not, explain why it is not possible.

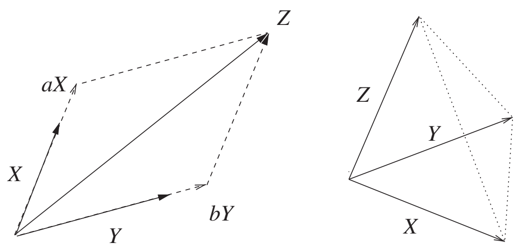



**FIGURE 2.1** Co-planar and non-co-planar vectors.

## 2.2 DIMENSION

The term "dimension" has slightly different meanings in mathematics and in ordinary speech. Roughly, a mathematician would say that the dimension of some set is the smallest number of variables necessary to describe the set. Thus, to a mathematician, space is three-dimensional because the position of any point may be described using exactly three coordinates. These coordinates may be freely chosen, and different choices yield different points. To a mathematician, a problem such as describing the flight of a cannon ball is at least six-dimensional; one must know the location of the cannon ($x$, $y$, and $z$ coordinates), the direction the barrel was pointed (elevation and compass bearing), and the initial speed of the cannon ball. Thus, from this point of view, the physical reality of six dimensions is clear.

This concept of dimension is consistent with the notion that a plane should be two dimensional. Recall that in Section 1.1 we said that in $\mathbb{R}^3$ two linearly independent elements $X$ and $Y$ span a plane through $\mathbf{0}$. This means that every element of the plane may be written as $xX + yY$ for real numbers $x$ and $y$.

Thus, it takes two numbers ($x$ and $y$) to describe the general element of the plane (Figure 2.1, left). (We consider $X$ and $Y$ as fixed.) No fewer than two vectors can span a plane; the span of a single vector is a line. These comments suggest the following definition:

**Definition 2.4** *The dimension of a vector space $\mathcal{V}$ is the smallest number of elements necessary to span $\mathcal{V}$—that is, $\mathcal{V}$ has dimension $n$ if there is a set $\{A_1, A_2, \ldots, A_n\}$ of $n$ elements in $\mathcal{V}$ whose span equals $\mathcal{V}$, while no set of $n - 1$ or fewer elements of $\mathcal{V}$ spans $\mathcal{V}$.*

The dimension of a space puts limits on the number of linearly independent elements that can exist in the space. For example, it appears that any three vectors $X$, $Y$, and $Z$ in $\mathbb{R}^2$ are always linearly dependent (Figure 2.1, left).

In $\mathbb{R}^3$, three non-coplanar vectors are linearly independent. (Figure 2.1, right.) Although the picture is difficult to draw, one feels that four vectors in $\mathbb{R}^3$ must always be linearly dependent. What we observed for $\mathbb{R}^2$ and $\mathbb{R}^3$ suggests the following theorem, which is one of the fundamental theorems of linear algebra.

**Theorem 2.6** *Suppose that the vector space $\mathcal{V}$ can be spanned by n elements. Then any set containing more than n elements must be linearly dependent, that is, if $S = \{Y_1, Y_2, \dots, Y_m\}$ is a subset of $\mathcal{V}$ where $m > n$, then S is linearly dependent.*

*Proof.* The proof is most easily understood in the case where $n = 2$ and $m = 3$. We first discuss the proof in this case and then describe how to modify the argument to cover the general case.

Thus, we assume initially that $\mathcal{V}$ is spanned by two elements $X_1$ and $X_2$. Let $Y_1, Y_2$, and $Y_3$ be any three elements of $\mathcal{V}$. We must prove that the $Y_i$ form a linearly dependent set. This is true if there are constants $x_1$, $x_2$, and $x_3$ such that

$$x_1Y_1 + x_2Y_2 + x_3Y_3 = \mathbf{0} \tag{2.10}$$

with at least one $x_i \neq 0$.

Since $\{X_1, X_2\}$ spans $\mathcal{V}$, each $Y_i$ may be written in terms of the $X_i$:

$$\begin{aligned} Y_1 &= a_{11}X_1 + a_{21}X_2 \\ Y_2 &= a_{12}X_1 + a_{22}X_2 \\ Y_3 &= a_{13}X_1 + a_{23}X_2 \end{aligned}$$

We substitute these expressions for $Y_i$ into formula (2.10) and factor out $X_1$ and $X_2$, obtaining an expression of the form

$$c_1X_1 + c_2X_2 = \mathbf{0}$$

where

$$\begin{aligned} c_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ c_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{aligned}$$

Equation (2.10) is satisfied if $c_1 = c_2 = 0$. Hence if

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= 0 \end{aligned}$$

Now, observe that this system consists of two homogeneous equations in three unknowns. A homogeneous system cannot be inconsistent. (Why?) Thus, from the more unknowns theorem, it has an infinite number of solutions and, in particular, it has one with at least one of the $x_i \neq 0$. This proves linear dependence.

In the general ($n$-dimensional) case, $\mathcal{V}$ is spanned by $n$ elements $X_1, X_2, \dots, X_n$ and we are given $m$ elements $Y_1$, $Y_2$, …, $Y_m$ where $m > n$. Now the dependency equation for the $Y_i$ is

$$x_1Y_1 + x_2Y_2 + \cdots + x_mY_m = \mathbf{0} \tag{2.11}$$

We wish to prove that there are nonzero $x_i$ that make this equation valid.

Just as before, we express each of the $Y_i$ as a linear combination of the $X_i$:

$$Y_i = a_{1i}X_1 + a_{2i}X_2 + \cdots + a_{ni}X_n$$

As before, we substitute each of these expressions into the dependency equation [equation (2.11)] and factor out the $X_i$. Now we obtain an expression of the form

$$c_1X_1 + c_2X_2 + \cdots + c_nX_n = \mathbf{0}$$

where now

$$c_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{im}x_m$$

Formula (2.11) is satisfied if each of the $c_i = 0$, $i = 1, \ldots, n$. However, setting the $c_i = 0$ produces a system of $n$-homogeneous, linear equations in the $m$ unknowns $x_1, \ldots, x_m$. As in the two-dimensional case, the crucial observations are that the system is homogeneous and that there are fewer equations than unknowns. (Recall that $n < m$.) Thus, the system has an infinite number of solutions and, in particular, it has one with at least one of the $x_i \neq 0$. This proves linear dependence. □

Since subspaces of vector spaces are themselves vector spaces, it is meaningful to discuss their dimension. *The reader should pay close attention to the following example.* Despite its simplicity, it demonstrates many key ideas.

### EXAMPLE 2.5

Show that the set of matrices of the form

$$\begin{bmatrix} a+b+3c & 2a-b \\ 0 & 2a+b+4c \end{bmatrix}$$

where $a$, $b$, and $c$ range over all real numbers, is a subspace of $M(2,2)$ and find its dimension.

*Solution.* We begin by finding a spanning set for $\mathcal{W}$. We write

$$\begin{aligned} \begin{bmatrix} a+b+3c & 2a-b \\ 0 & 2a+b+4c \end{bmatrix} &= \begin{bmatrix} a & 2a \\ 0 & 2a \end{bmatrix} + \begin{bmatrix} b & -b \\ 0 & b \end{bmatrix} + \begin{bmatrix} 3c & 0 \\ 0 & 4c \end{bmatrix} \\ &= a\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} + b\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \end{aligned}$$

This shows that $\mathcal{W}$ is the span of the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$$

In particular, since spans are subspaces, $\mathcal{W}$ is a subspace of $M(2,2)$.

The presence of three spanning elements suggests that $\mathcal{W}$ could be three-dimensional. However, our spanning set is linearly dependent since, as the reader may check,

$$C = A + 2B$$

Hence, from Proposition 2.1 on page 103, $A$ and $B$ span $\mathcal{W}$ by themselves.

Thus, the dimension is at most 2. In fact, the dimension must be 2, since $A$ and $B$ are linearly independent (neither is a multiple of the other) and (from Theorem 2.6) a set having two linearly independent elements cannot exist in a one-dimensional space.

The argument of Example 2.5 is used in the proof of the following theorem. (Recall that according to Definition 2.2 on page 104, a basis for a vector space $\mathcal{V}$ is a linearly independent subset $\mathcal{B}$ of $\mathcal{V}$ that spans $\mathcal{V}$.)

**Theorem 2.7** (Dimension Theorem). *If $\mathcal{V}$ is an n-dimensional vector space, then all bases of $\mathcal{V}$ have exactly n elements. In particular, the dimension of $\mathcal{V}$ is the number of elements in any basis of $\mathcal{V}$.*

*Proof.* Assume that $\mathcal{V}$ has a basis with $m$ elements. Then $n \leq m$ since the dimension is the smallest number of elements necessary to span $\mathcal{V}$. It is impossible that $m > n$ since, from Theorem 2.6, a set having $m$ linearly independent elements cannot exist in $\mathcal{V}$. Hence, $m = n$ as claimed. □

## EXAMPLE 2.6

Show that $\mathbb{R}^n$ is $n$-dimensional.

*Solution.* We begin by finding a spanning set for $\mathbb{R}^n$. Let $X = [x_1, x_2, \ldots, x_n]^t$ be in $\mathbb{R}^n$. Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{2.12}$$

The $j$th column vector on the right is usually denoted. Our formula says that

$$X = x_1 I_1 + x_2 I_2 + x_3 I_3 + \cdots + x_n I_n \tag{2.13}$$

Hence, the $I_j$ span $\mathbb{R}^n$.

These vectors are linearly independent since the $j$th entry of $I_j$ is 1 while all the other $I_k$ are 0 in this position (Proposition 1.1 on page 9). The $n$-dimensionality follows from Theorem 2.7.

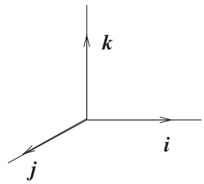



**FIGURE 2.2** The standard basis in $\mathbb{R}^3$.

The basis formed by the vectors $I_j$ in Example 2.6 is particularly important. It is referred to as the **standard basis** for $\mathbb{R}^n$. The reason for the notation $I_j$ is that $I_j$ is the $j$th column of the $n \times n$ **identity matrix** $I$. This is the matrix with 1's on its main diagonal and all its other entries **0**. Thus, for example, the $3 \times 3$ identity matrix is

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$I_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad I_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad I_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

In $\mathbb{R}^3$, the vectors $I_1$, $I_2$, and $I_3$ are usually denoted by $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$, respectively. Equation (2.13), in this case, is just the familiar expression (Figure 2.2).

$$[x, y, z]^t = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

### EXAMPLE 2.7

Show that $M(2, 2)$ is four-dimensional.

*Solution.* The general $2 \times 2$ matrix $X$ may be written as

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.14}$$

This equation says that the following matrices span $M(2, 2)$:

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Since each $E_{ij}$ has a 1 in a position where the others have a 0, they are linearly independent; hence they form a basis for $M(2, 2)$. Since there are four elements in this basis, the dimension is 4.

*Remark.* In general, in $M(m, n)$, the matrices $E_{ij}$ whose only nonzero element is a 1 in the $(i, j)$ position form a basis that is referred to as the **standard basis** for $M(m, n)$. The dimension of $M(m, n)$ is $mn$.

Recall that $\mathcal{P}_n$ is the space of polynomial functions of degree at most $n$.

### EXAMPLE 2.8

Prove that $\mathcal{P}_n$ has dimension $n + 1$.

*Solution.* The general polynomial of degree $n$ is

$$p(x) = a_o + a_1 x + \cdots + a_n x^n$$

This equation says exactly that the set

$$S = \{1, x, x^2, \ldots, x^n\} \tag{2.15}$$

spans $\mathcal{P}_n$.

Since $S$ has $n + 1$ elements, it suffices to prove that $S$ is linearly independent. The dependency equation is

$$a_o + a_1 x + \cdots + a_n x^n = \mathbf{0}$$

Substituting $x = 0$ shows that $a_o = 0$. Then, division by $x$ shows

$$a_1 + a_2 x + \cdots + a_n x^{n-1} = \mathbf{0}$$

Repetition of this argument shows that all the $a_i$ are 0, proving the linear independence.

*Remark.* The set $S$ in (2.15) is referred to as the **standard basis** for $\mathcal{P}_n$.

By definition, a basis for a vector space is a set of elements that is linearly independent and spans the space. However, Figure 2.1 suggests that in $\mathbb{R}^2$ any two linearly independent elements $X$ and $Y$ automatically span $\mathbb{R}^2$. Conversely, any two vectors that span $\mathbb{R}^2$ must be linearly independent; if one were a linear combination of the other, their span would be a line.

A similar principle holds in general:

**Theorem 2.8** *Let $\mathcal{V}$ be an n-dimensional vector space. Then:*

1. *Any set of n elements of $\mathcal{V}$ that spans $\mathcal{V}$ must be linearly independent and thus is a basis.*
2. *Any linearly independent set of n elements of $\mathcal{V}$ must span $\mathcal{V}$ and thus is a basis.*

*Proof.* To prove statement 1, suppose that $S = \{X_1, \dots, X_n\}$ spans $\mathcal{V}$. If $S$ is linearly dependent, then from Proposition 2.1 on page 103, some proper subset of $S$ still spans $\mathcal{V}$, making the dimension less than $n$, which is impossible. Statement (1) follows.

To prove statement 2, let $S = \{X_1, \dots, X_n\}$ be a set of $n$ linearly independent elements in $\mathcal{V}$. Let $W \in \mathcal{V}$. Then, from Theorem 2.6, $\{W, X_1, \dots, X_n\}$ is linearly dependent. Thus, there are scalars $c_i$, not all zero, such that

$$c_0 W + c_1 X_1 + \cdots + c_n X_n = \mathbf{0}$$

If $c_0 = 0$, then

$$c_1 X_1 + \cdots + c_n X_n = \mathbf{0}$$

which contradicts the linear independence of the $X_i$. Hence, $c_0 \neq 0$. We may therefore solve for $W$ in terms of the $X_i$:

$$W = -\frac{c_1}{c_0} X_1 - \cdots - \frac{c_n}{c_0} X_n$$

Thus, the $X_i$ span $\mathcal{V}$. □

Theorem 2.8 can, at times, save a considerable amount of work, as the following examples show.

### EXAMPLE 2.9

Show that the following vectors form a basis for $\mathbb{R}^4$:

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad X_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

*Solution.* $\mathbb{R}^4$ is four-dimensional and we are given four vectors. Hence, from Theorem 2.8, we need only show linear independence. For this, we proceed as usual, setting up the dependency equation and solving. We leave the details to the reader.

### EXAMPLE 2.10

Show that the following vectors do not span $\mathbb{R}^4$:

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad X_4 = \begin{bmatrix} 3 \\ 2 \\ 3 \\ -1 \end{bmatrix}$$

*Solution.* $\mathbb{R}^4$ is four-dimensional and we are given four vectors. Hence, if we show that the given vectors are linearly dependent, it follows from Theorem 2.8 that they do not span. We could set up the dependency equation and solve it, finding that there are indeed nontrivial solutions. However, we may also just note that $X_4 = X_1 + X_2$.

Theorem 2.8 also helps us describe certain spaces.

### EXAMPLE 2.11

Suppose that $\mathcal{W}$ is a two-dimensional subspace of $\mathbb{R}^3$ that contains the vectors $X_1 = [1, 2, -3]^t$ and $X_2 = [3, -7, 2]^t$. Does $\mathcal{W}$ contain $Y = [1, 1, 1]^t$?

*Solution.* Since $\mathcal{W}$ is two-dimensional, any set of two linearly independent vectors in $\mathcal{W}$ spans $\mathcal{W}$. Thus, $\{X_1, X_2\}$ spans $\mathcal{W}$. The vector $Y$ belongs to $\mathcal{W}$ if and only if there are scalars $x$ and $y$ such that $xX_1 + yX_2 = Y$. Substituting for $Y$, $X_1$, and $X_2$, we see that the vector equation is equivalent to the system

$$\begin{aligned} x + 3y &= 1 \\ 2x - 7y &= 1 \\ -3x + 2y &= 1 \end{aligned}$$

We row reduce the augmented matrix, obtaining

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the system is inconsistent, showing that $Y$ is not an element of $\mathcal{W}$.

*Remark.* We close this section with a rather sobering comment. All the vector spaces considered in this section were finite-dimensional in the sense that they could be spanned by a finite number of elements. There do, however, exist vector spaces so big that they cannot be spanned by any finite number of elements. Such spaces are called "infinite-dimensional." The simplest example is the space $\mathbb{R}^\infty$. This is by definition the set of all "vectors" of the form

$$[x_1, x_2, \ldots, x_n, \ldots$$

where $\{x_n\}_{n=1}^{\infty}$ is an infinite sequence of real numbers. Thus, for example, both

$$X = [1, 2, 3, 4, \ldots, n, \ldots$$

and

$$Y = \left[1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\right.$$

represent elements of $\mathbb{R}^\infty$.

We can add elements of $\mathbb{R}^\infty$ and multiply them by scalars just as we do for elements of $\mathbb{R}^n$. Thus, for $X$ and $Y$ as above

$$X + Y = \left[2, 2 + \frac{1}{2}, 3 + \frac{1}{3}, 4 + \frac{1}{4}, \dots, n + \frac{1}{n}, \dots\right.$$

Similarly,

$$2X = [2, 4, 6, 8, 10, \dots, 2n, \dots$$

It is easily seen that $\mathbb{R}^\infty$ satisfies all the vector space properties listed on page 12. Thus, $\mathbb{R}^\infty$ is a vector space.

Let $I_j$ be the element of $\mathbb{R}^\infty$ that has a 1 in the $j$th position and 0's in all other positions. Thus, for example,

$$I_3 = [0, 0, 1, 0, 0, \dots, 0, \dots$$

It is clear that each of the $I_j$ is linearly independent of the other $I_k$, since each $I_j$ has a 1 in a position where all the others have a 0. Thus, $\mathbb{R}^\infty$ has an infinite set of linearly independent vectors. This proves that $\mathbb{R}^\infty$ is infinite-dimensional because (by Theorem 2.6) in an $n$-dimensional space we can have at most $n$ linearly independent elements.

**True-False Questions:** Justify your answers.

**2.10** $\{[17, 6, -4]^t, [2, 3, 3]^t, [19, 9, -1]^t\}$ does not span $\mathbb{R}^3$.

**2.11** $\{[1, 1]^t, [1, 2]^t, [4, 7]^t\}$ spans $\mathbb{R}^2$.

**2.12** Let $\mathcal{W}$ be a two-dimensional subspace of $\mathbb{R}^3$. Then two of the following three vectors span $\mathcal{W}$: $X = [1, 0, 0]^t$, $Y = [0, 1, 0]^t$, $Z = [0, 0, 1]^t$.

**2.13** The following set of vectors is linearly independent:

$$\{[-36, 13, -11]^t, \quad [22, \pi, \sqrt{2}]^t, \quad [41, -37/17, -2]^t, \quad [\sqrt{3}, \sqrt{7}, -3]^t\}$$

**2.14** The nullspace of a nonzero $4 \times 4$ matrix cannot contain a set of four linearly independent vectors.

**2.15** Suppose that $\mathcal{W}$ is a four-dimensional subspace of $\mathbb{R}^7$ and $X_1$, $X_2$, $X_3$, and $X_4$ are vectors that belong to $\mathcal{W}$. Then $\{X_1, X_2, X_3, X_4\}$ spans $\mathcal{W}$.

**2.16** Suppose that $\{X_1, X_2, X_3, X_4, X_5\}$ spans a four-dimensional vector space $\mathcal{W}$ of $\mathbb{R}^7$. Then $\{X_1, X_2, X_3, X_4\}$ also spans $\mathcal{W}$.

**2.17** Suppose that $S = \{X_1, X_2, X_3, X_4, X_5\}$ spans a four-dimensional subspace $\mathcal{W}$ of $\mathbb{R}^7$. Then $S$ contains a basis for $\mathcal{W}$.

**2.18** Suppose that $\{X_1, X_2, X_3, X_4, X_5\}$ spans a four-dimensional subspace $\mathcal{W}$ of $\mathbb{R}^7$. Then one of the $X_i$ must be a linear combination of the others.

**2.19** Suppose that $\mathcal{W}$ is a four-dimensional subspace of $\mathbb{R}^7$ that is spanned by $\{X_1, X_2, X_3, X_4\}$. Then one of the $X_i$ must be a linear combination of the others.

## EXERCISES

**2.28** ✓✓The set $\mathcal{B} = \{A_1, A_2\}$, where $A_1 = [1,2]^t$ and $A_2 = [2,-3]^t$, is (obviously) linearly independent in $\mathbb{R}^2$. According to Theorem 2.8, $\mathcal{B}$ must then span $\mathbb{R}^2$. As an example of this, choose some random *nonzero* vector $B$ in $\mathbb{R}^2$ and find constants $x$ and $y$ such that $B = xA_1 + yA_2$. Do not choose either $A_1$ or $A_2$ as $B$. That is not "random"!

**2.29** Redo Exercise 2.28 with another choice of $B$. This is what spanning the space means: no matter which $B$ you choose, the constants $x$ and $y$ always exist.

**2.30** ✓✓Let $A_1 = [1,2]^t$ and $A_2 = [2,4]^t$ in $\mathbb{R}^2$. Find a vector $B$ in $\mathbb{R}^2$ for which there are no constants $x$ and $y$ such that $B = xA_1 + yA_2$. [*Hint:* Draw a picture.] Find another vector $C$ such that $x$ and $y$ do exist. What geometric condition must $C$ satisfy?

**2.31** ✓✓Does the subspace $\mathcal{W}$ in Example 2.11 on page 121 contain (a) $[9,-8,-5]^t$? (b) $[1,2,3]^t$?

**2.32** Prove that the given sets $\mathcal{W}$ are subspaces of $\mathbb{R}^n$ for the appropriate $n$. Find spanning sets for these spaces and find at least two different bases for each space. Give the dimension of each space. Reason as in Example 2.5 on page 116.

**(a)** $\mathcal{W} = \{[a+b+2c, 2a+b+3c, a+b+2c, a+2b+3c]^t \mid a,b,c \in \mathbb{R}\}$

**(b)** $\mathcal{W} = \{[a+2c, 2a+b+3c, a+b+c]^t \mid a,b,c \in \mathbb{R}\}$

**(c)** $\mathcal{W} = \{[a+b+2c, 2a+b+3c, a+b+c]^t \mid a,b,c \in \mathbb{R}\}$

**(d)** $\mathcal{W} = \{[a+2b-4c+5d, -2a-2b+2c-6d, 6a+4b+14d, 3a+b+3c+5d] \mid a,b,c,d \in \mathbb{R}\}$

**2.33** Show that each of the following sets $\mathcal{W}$ is a subspace of $M(m,n)$ for the appropriate $m$ and $n$, find a basis for $\mathcal{W}$, and give $\mathcal{W}$'s dimension.

**(a)** $\mathcal{W} = \left\{ \begin{bmatrix} a+3b+c & -2b+2c & 2a+8c \\ -3a+2b-14c & a-3b+7c & 5a+3b+17c \end{bmatrix} \mid a,b,c \in \mathbb{R} \right\}$

**(b)** ✓✓$\mathcal{W} = \left\{ \begin{bmatrix} a+b+2c+3d & a+b+2c+3d \\ 2a+2c+4d & a+b+2c+3d \end{bmatrix} \mid a,b,c,d \in \mathbb{R} \right\}$

**2.34** Show that

**(a)** $M(3,2)$ is six-dimensional.

**(b)** ✓✓ $M(2,3)$ is six-dimensional.

**(c)** $M(1,n)$ is $n$-dimensional.

**2.35** Find a basis and give the dimension for the following spaces of matrices $A$.

**(a)** $2 \times 2, A = A^t$.
**(b)** $3 \times 3, A = A^t$.
**(c)** ✓ $2 \times 2, A = -A^t$.
**(d)** ✓✓ $3 \times 3$ upper triangular.
**(e)** $3 \times 3$ lower triangular.

[Recall that $A$ is upper (respectively lower) triangular if its only nonzero entries lie on or above (respectively below) the main diagonal.]

**2.36** Find a basis for the subspace of $M(2,2)$ spanned by the following matrices. What is the dimension of this subspace? [*Hint:* These matrices were considered in Example 2.3 on page 101.]

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad \begin{bmatrix} -1 & -2 \\ -3 & -5 \end{bmatrix}, \quad \begin{bmatrix} -1 & -2 \\ 0 & -2 \end{bmatrix}$$

**2.37** The vectors in parts (a)–(d) form a spanning set for some subspace of $M(m, n)$ for some $m$ and $n$. Use Theorem 2.2 on page 102 in Section 2.1, to find a basis for this subspace. What is the dimension of the subspace?

**(a)** ✓✓$[5, -3, 2, 4]$, $[-2, 1, -1, -2]$, $[4, -3, 1, 2]$, $[-5, 1, -4, -8]$

**(b)** $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 6 \\ -15 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \\ -2 \\ 7 \end{bmatrix}$

**(c)** ✓✓ $\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

**(d)** $[1, 2, 0, 3]$, $[2, 1, 1, 1]$, $[1, 4, 3, -3]$, $[3, 15, 8, -4]$, $[3, -11, -9, 18]$

**2.38** Find a basis for the nullspace and the rank for each of the following matrices. What is the dimension of the nullspace? [*Hint:* Use Theorem 1.12 on page 83.] Do you see a relationship between the rank of the matrix and the dimension of the nullspace?[2]

**(a)** $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$

**(b)** ✓✓ $\begin{bmatrix} 1 & 3 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ 4 & 7 & 4 & 3 \end{bmatrix}$

**(c)** $\begin{bmatrix} 1 & -5 & 3 & 4 \\ 2 & -10 & 6 & 8 \\ 3 & -15 & 9 & 12 \end{bmatrix}$

**2.39** Prove that $\mathcal{B} = \{X_1, X_2, X_3\}$ is a basis for $\mathbb{R}^3$, where

$$X_1 = [2, -3, 1]^t, \quad X_2 = [1, 2, 3]^t, \quad X_3 = [7, -2, 4]^t$$

[2]*Note to instructor:* This exercise serves as a preview of Section 2.3, where the relationship between the rank and the dimension of the nullspace is discussed in depth.

**2.40** The most efficient way to do Exercise 2.39 is to prove that $\mathcal{B}$ is linearly independent and then use Theorem 2.8. To see how much work this saves, prove directly that $\mathcal{B}$ spans $\mathbb{R}^3$ without using dimension theory.

**2.41** The following sequence of Exercises studies forming bases that contain a given set of vectors.

**(a)** ✓✓Let $X_1 = [-3, 4, 1]^t$ and $X_2 = [1, 1, 1]^t$. Find a third vector $X_3$ (reader's choice) such that $\{X_1, X_2, X_3\}$ is a basis for $\mathbb{R}^3$.

**(b)** ✓✓Given linearly independent vectors $X_1$ and $X_2$ in $\mathbb{R}^3$, is it always possible to find $X_3$ such that $\{X_1, X_2, X_3\}$ is a basis for $\mathbb{R}^3$? Explain.

**(c)** Let $X_1 = [1, 0, 1, 0]^t$ and $X_2 = [1, 2, 3, 4]^t$. Find a vector $X_3$ (reader's choice) in $\mathbb{R}^4$ such that $\{X_1, X_2, X_3\}$ is a linearly independent subset of $\mathbb{R}^4$. Find a fourth vector $X_4$ such that $\{X_1, X_2, X_3, X_4\}$ forms a basis for $\mathbb{R}^4$.

**(d)** Given a linearly independent subset $\{X_1, X_2\}$ of $\mathbb{R}^4$, is it always possible to find $X_3$ and $X_4$ such that $\{X_1, X_2, X_3, X_4\}$ forms a basis for $\mathbb{R}^4$? Explain.

**(e)** Given a linearly independent subset $\{X_1, X_2, \dots, X_k\}$ of an $n$-dimensional vector space $\mathcal{V}$, is it always possible to find $X_{k+1}, X_{k+2}, \dots, X_n$ such that $\{X_1, X_2, \dots, X_n\}$ is a basis for $\mathcal{V}$? Explain.

**2.42** Let $X$, $Y$, $Z$, and $W$ be elements of some vector space. Suppose that $W = 2X + 3Y - Z$ and $Z = X - Y$. What are the possible values for the dimension of span $\{X, Y, Z, W\}$? Suppose that this dimension is 2. Must $Z$ and $W$ be independent? Explain.

**2.43** A certain $5 \times 4$ matrix $A$ is known to have a three-dimensional nullspace. It is also known that the three elements $X_1$, $X_2$, and $X_3$ below satisfy $AX = 0$.

$$X_1 = [3, -2, 1, -1]^t, \quad X_2 = [2, 2, 1, 2]^t, \quad X_3 = [-1, 4, 2, 3]^t$$

**(a)** ✓Does $\{X_1, X_2, X_3\}$ span the nullspace of $A$? Explain.

**(b)** ✓Is the vector $[1, 1, 1, 1]^t$ in the nullspace of $A$? Explain.

**(c)** Do the vectors below satisfy $AX = 0$? Does $\{Y_1, Y_2, Y_3\}$ span the nullspace?

$$Y_1 = [3, -2, 1, -1]^t, \quad Y_2 = [2, 2, 1, 2]^t, \quad Y_3 = [1, -4, 0, -3]^t$$

**2.44** Suppose that $\mathcal{W}$ is a two dimensional subspace of $R^3$ that contains the vectors $X_1 = [1, 2, -3]^t$ and $X_2 = [-2, -4, 6]^t$. Is it possible to determine if $\mathcal{W}$ contains $Y = [1, 1, 1]^t$?

**2.45** ✓✓Suppose that $\mathcal{W}$ is a vector space spanned by three elements $\{A, B, C\}$. Suppose that $\{A, B\}$ forms a basis for $\mathcal{W}$. Does it follow that $\{A, B, C\}$ is linearly dependent? Explain. Suppose that $\{A, C\}$ is also a basis of $\mathcal{W}$. Does it follow that $\{B, C\}$ is a basis of $\mathcal{W}$? Explain. (Think about a plane in $\mathbb{R}^3$.)

**2.46** Suppose that $\{A, B\}$ is a basis for a vector space $\mathcal{W}$. Prove that $\{X, Y\}$ is also a basis for $\mathcal{W}$, where $X = 2A + 3B$ and $Y = 3A - 5B$.

**2.47** In our proof of Theorem 2.6 on page 115, we began with the case of a set of three elements in a two-dimensional space. Give a similar proof of Theorem 2.6 for the case of a set of four elements $\{Y_1, Y_2, Y_3, Y_4\}$ in a three-dimensional space. You are not allowed to use ellipses.

**2.48** We have seen that in $\mathbb{R}^3$, every line through 0 and every plane through 0 is a subspace of $\mathbb{R}^3$. Additionally $\mathbb{R}^3$ and $\{0\}$ are both subspaces. Prove that every subspace of $\mathbb{R}^3$ is one of these four types. For your proof, let $\mathcal{W}$ be a nonzero subspace of $\mathbb{R}^3$. Any linearly independent subset of $\mathcal{W}$ can contain at most three elements. (Why?) If $\mathcal{W}$ contains a linearly independent set with three independent elements, what is $\mathcal{W}$? Suppose that the maximal number of elements in any linearly independent subset of $\mathcal{W}$ is two. Prove that then any two linearly independent elements in $\mathcal{W}$ span $\mathcal{W}$. What is $\mathcal{W}$ in this case? What if the maximal number is one?

**2.49** ✓✓Let $\mathcal{W}$ be a subspace of a finite-dimensional vector space $\mathcal{V}$. Prove that $\mathcal{W}$ has a finite spanning set. For the proof, it suffices to assume that $\mathcal{W} \neq \{0\}$. (Why?) Let $S = \{X_1, X_2, \dots, X_k\}$ be a linearly independent subset of $\mathcal{W}$, where $k$ is as large as possible. (Why is there a limit on the number of elements $S$ can contain?) Prove that $S$ spans $\mathcal{W}$.

*Remark.* It follows from this exercise that any subspace of a finite-dimensional vector space is itself finite-dimensional.

**2.50** Let $\mathcal{W}$ be a subspace of an $n$-dimensional vector space $\mathcal{V}$. It follows from Exercise 2.49 that $\mathcal{W}$ is finite-dimensional. Prove that $\dim \mathcal{W} \leq n$. Prove that if $\dim \mathcal{W} = n$, then $\mathcal{W} = \mathcal{V}$.

**2.51** Let $\{X_1, X_2, \dots, X_n\}$ be a linearly independent subset of some vector space $\mathcal{V}$. Suppose that there are scalars $c_i$ and $d_i$ such that $c_1X_1 + c_2X_2 + \cdots + c_nX_n = d_1X_1 + d_2X_2 + \cdots + d_nX_n$. Prove that $c_i = d_i$ for $i =, \dots, n$. Thus, elements of $\mathcal{V}$ are representable in at most one way as a linear combination of the $X_i$. This statement is called the "uniqueness theorem." [*Hint:* For the proof, note that $X = Y$ can also be expressed as $X - Y = 0$.]

**2.52** Prove that the uniqueness theorem from Exercise 2.51 is false for spanning sets that are not bases. That is, suppose that $\{X_1, X_2, \dots, X_n\}$ spans a vector space $\mathcal{V}$ and is linearly dependent. Prove that every element $X \in \mathcal{V}$ has at least two different expressions as a linear combination of the spanning vectors. [*Hint:* First do the $X = 0$ case. Then note that $X = X + 0$.]

**2.53** ✓In $\mathbb{R}^\infty$, let $\mathcal{W}$ be the set of vectors $[x_1, x_2, \dots, x_n, \dots$, where

$$\lim_{n\to\infty} x_n = 0$$

**(a)** Give an example of an element $X \in \mathbb{R}^\infty$ that belongs to $\mathcal{W}$ and an element $Y \in \mathbb{R}^\infty$ that does not.

**(b)** Prove that $\mathcal{W}$ is a subspace of $\mathbb{R}^\infty$.

**(c)** Prove that $\mathcal{W}$ is infinite-dimensional.

**(d)** Exhibit a three-dimensional subspace of $\mathbb{R}^\infty$.

**2.54** ✓Show that the space $\mathcal{P}$ of all polynomial functions (no restriction on degree) is an infinite-dimensional vector space. [*Hint:* It suffices to exhibit an infinite set of linearly independent elements of $\mathcal{P}$.]

**2.55** Show that the space $\mathcal{P}_e$ of all polynomial functions containing no odd powers of $x$ (no restriction on degree) is an infinite dimensional vector space. [*Hint:* It suffices to exhibit an infinite set of linearly independent elements of $\mathcal{P}$.]

**2.56** Show that the space $C^\infty(\mathbb{R})$ of infinitely differentiable functions is an infinite dimensional vector space. [*Hint:* It suffices to exhibit an infinite set of linearly independent elements of $C^\infty(\mathbb{R})$.]

**2.57** An $m \times \infty$ matrix is a matrix

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix}$$

where $A_i \in \mathbb{R}^\infty$. The space of all such matrices is denoted by $M(n, \infty)$.

**(a)** Exhibit three elements of $M(2, \infty)$.

**(b)** Prove that $M(2, \infty)$ is infinite-dimensional.

**(c)** Prove that $M(m, \infty)$ is infinite-dimensional.

### 2.2.1 Computer Projects

Let

$$A = \begin{bmatrix} -1 & 2 & 6 & -8 & -14 & 3 \\ 2 & 4 & 1 & -8 & 5 & -1 \\ -3 & 1 & 4 & -9 & -10 & 0 \\ 3 & -2 & -1 & 12 & -1 & 4 \\ 5 & 7 & 11 & -11 & -19 & 9 \end{bmatrix}$$

Please use $A$ for the following exercises.

## EXERCISES

**1.** Use the "rank" command in MATLAB to find the rank of $A$. *Using only the value of the rank*, explain why statements (a) and (b) are true.

**(a)** The reduced form for the augmented matrix for the system $AX = 0$ has three nonpivot variables. (Recall that in Section 1.3.1 we commented that the rank is the number of nonzero rows in the reduced form.)

**(b)** The nullspace of $A$ is at most three-dimensional. [*Hint:* How many spanning vectors are there in the general solution to $AX = \mathbf{0}$?]

**2.** Show that each of the following vectors satisfies $AX = \mathbf{0}$:

$$\begin{aligned} X_1 &= [-5, 13, -10, 2, -3, 1]^t \\ X_2 &= [3, -6, 11, 1, 2, -5]^t \\ X_3 &= [-4, 7, 9, 5, 1, -6]^t \end{aligned}$$

**3.** Prove that $X_1, X_2$, and $X_3$ are linearly independent by computing the rank of the matrix $X = [X_1, X_2, X_3]$. (Recall that the maximal number of linearly independent columns equals the rank.)

**4.** How does it follow that the dimension of the nullspace of $A$ is 3? How does it follow that the $X_i$ constitute a basis for the nullspace?

**5.** Use Y=null(A) to find a basis for the nullspace of $A$. Express each column of Y as a linear combination of the $X_i$.

*Remark.* An efficient way of doing this is as follows. In MATLAB, set B=[X,Y], where $X$ is as in Exercise 3. Then $B = [X_1, X_2, X_3, Y_1, Y_2, Y_3]$, where the $Y_i$ are the columns of $Y$. Let C=rref(B) and write the last three columns of $C$ as linear combinations of the pivot columns, which are the first three columns. According to Theorem 2.5 on page 106, the same coefficients express the $Y_i$ as linear combinations of the $X_i$. You should check that this really works.

[*Note*: The matrix $B$ should have rank 3. (Why?) You may need to increase the tolerance in rref to get $B$ to have rank 3.]

**6.** In Exercise 5, what made us so sure that the first three columns would be the pivot columns? Why, for example, could not the pivot columns be columns 1, 3, and 4? [*Hint:* Think about what this would say regarding the reduced form of $[X_1, X_2, X_3]$.]

**7.** Express each of the $X_i$ as a linear combination of the $Y_i$. This shows that the $Y_i$ also span the nullspace of $A$.

**8.** Find (by inspection) a vector $T$ such that $AT = [6, 1, 4, -1, 11]^t$.

**9.** Generate a $1 \times 3$ random matrix $[r, s, t]$ and let $Z = T + r * X_1 + s * X_2 + t * X_3$, where $T$ is as in Exercise 8. Compute $AZ$. Explain why you get what you do. Find constants $u$, $v$, and $w$ such that $Z = T + u * Y_1 + v * Y_2 + w * Y_3$. What theorem does this demonstrate?

### 2.2.2 Applications to Differential Equations

Informally, a differential equation may be defined as an equation relating a number of the derivatives of a function with each other. For example the functions $y_1(t) = \cos t$ and $y_2(t) = \sin t$ both satisfy the equation

$$y'' + y = \mathbf{0} \tag{2.16}$$

In fact, the solution space is a subspace of $C^\infty(\mathbb{R})$ (Exercise 1.119, page 92). Hence, any function of the form

$$y(t) = A\cos t + B\sin t$$

where $A$ and $B$ are scalars, is also a solution of equation (2.16). It turns out that all solutions of equation (2.16) are of the preceding form—that is, the functions $y_1$ and $y_2$ span the solution space. Theorem 2.9 below proves our comment. Specifically, Theorem 2.9 shows that the solution space is two-dimensional and Theorem 2.8 says that any two linearly independent elements of the solution space (such as $y_1$ and $y_2$) span the solution space.

**Theorem 2.9** *Consider the differential equation*

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = \mathbf{0} \tag{2.17}$$

*where $y^{(k)}$ denotes the kth derivative of y and the $a_i$ are scalars with $a_n \neq 0$. Then the set of solutions to this equation is an n-dimensional subspace of $C^\infty(\mathbb{R})$.*

Using Theorem 2.9 to solve an equation of the form (2.17) typically involves the following steps:

1. Find all solutions of the form $y = e^{rt}$.
2. If there are $n$ linearly independent solutions of this form, then your work is finished: the general solution is their span.
3. If not, you must look for more solutions. See Example 2.13 below.

The following theorem, which we do not prove, saves proving the linear independence of the functions found in step (2).

**Theorem 2.10** *Let $r_1, r_2, \ldots, r_n$ be n distinct real numbers and let $y_i(t) = e^{r_i t}$. Then $\{y_1, \ldots, y_n\}$ is a linearly independent subset of $C^\infty(\mathbb{R})$.*

### ■ EXAMPLE 2.12

Find all solutions to the following equation:

$$y''' + 3y'' + 2y' = \mathbf{0}$$

*Solution.* Substituting $y = e^{rt}$ into this equation produces

$$\begin{aligned} r^3 e^{rt} + 3r^2 e^{rt} + 2re^{rt} &= \mathbf{0} \\ r^3 + 3r^2 + 2r &= 0 \\ r(r+1)(r+2) &= 0 \end{aligned}$$

The roots are $r = 0$, $r = -1$, and $r = -2$ corresponding to the solutions

$$1, \quad e^{-t}, \quad e^{-2t}$$

These functions are linearly independent by Theorem 2.10. From Theorem 2.9, the solution space is three-dimensional. Hence, the general solution to the differential equation is

$$y(x) = c_1 + c_2 e^{-t} + c_3 e^{-2t}$$

where the $c_i$ are arbitrary constants.

As in Example 2.12, $y = e^{rt}$ solves equation (2.17) if and only if $p(r) = 0$, where

$$p(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0 = 0 \tag{2.18}$$

The polynomial $p(r)$ is called the **characteristic polynomial** for the equation.

### EXAMPLE 2.13

Find the general solution to the following differential equation:

$$y'' - 2y' + y = \mathbf{0} \tag{2.19}$$

*Solution.* Setting the characteristic polynomial equal to 0 yields

$$0 = r^2 - 2r + 1 = (r - 1)^2$$

We obtain only one solution $y = e^t$. A single function cannot, however, span the solution space since, according to Theorem 2.9, the solution space is two-dimensional. There must be another linearly independent solution.

However, Theorem 2.11 below states that since $(r - 1)^2$ is a factor of the characteristic polynomial, both $e^t$ and $te^t$ are solutions. To prove the linear independence of these solutions, we write the dependency equation together with its derivative:

$$\begin{aligned} c_1 e^t + c_2 te^t &= \mathbf{0} \\ (c_1 + c_2)e^t + c_2 te^t &= \mathbf{0} \end{aligned}$$

Setting $t = 0$ yields

$$\begin{aligned} c_1 \qquad &= 0 \\ c_1 + c_2 &= 0 \end{aligned}$$

Hence, $c_1 = c_2 = 0$, proving linear independence.

Since the solution space is two-dimensional, the general solution is

$$y(t) = c_1 e^t + c_2 t e^t \tag{2.20}$$

**Theorem 2.11** *If $(r - a)^n$ is a factor of the characteristic polynomial for equation (2.17), then $t^k e^{rt}$ is a solution to this equation for all integers $k$, $0 \le k < n$. If the characteristic polynomial factors as*

$$p(r) = a(r - r_1)^{n_1} \dots (r - r_k)^{n_k}$$

*then the functions*

$$y_{m,j}(t) = t^j e^{r_m t}, 0 \le j < n_m,\ 1 \le m \le k$$

*form a basis for the solution set to equation (2.17).*

### Self-Study Questions

**2.1** ✓The functions $y_1(t) = e^{2t}$ and $y_2(t) = e^{-3t}$ satisfy the differential equation

$$y'' + y' - 6y = 0$$

**(a)** Write three more solutions to this equation.

**(b)** What is the dimension of the solution space of this system?

**(c)** What theorems from this subsection prove that $y_1$ and $y_2$ span the solution space?

**(d)** What differential equation has characteristic polynomial $p(r) = r^2(r - 1)^2(r + 2)$?

**(e)** What is the dimension of the solution space of the differential equation in (d)?

**(f)** Write the general solution to the differential equation in (a).

**2.2** ✓True or False: The general solution to the following differential equation is $y = Ae^t - Be^{2t} + Ce^{-3t}$:

$$y^{(4)} - y^{(3)} - 7y^{(2)} + 13y' - 6y = 0$$

## EXERCISES

**2.58** Prove that the solution space to equation (2.17) on page 129 is a subspace of $C^\infty(\mathbb{R})$.

**2.59** For each equation find all solutions of the form $y(t) = e^{rt}$ and, if possible, find a basis for the solution space.

**(a)** ✓✓ $y'' + 3y' + 2y = \mathbf{0}$ **(b)** ✓✓ $y'' + 4y' + 13y = \mathbf{0}$
**(c)** ✓✓ $y''' + y'' - y' - y = \mathbf{0}$ **(d)** $y''' + 3y'' + 3y' + y = \mathbf{0}$
**(e)** $y''' = \mathbf{0}$

**2.60** In part (a) of Exercise 2.59, find all solutions $y$ that satisfy $y(0) = 1, y'(0) = 2$. Repeat for part (e).

**2.61** Show that the functions $e^{-2x}\cos 3x$ and $e^{-2x}\sin 3x$ both satisfy the equation in part (b) of Exercise 2.59. Prove that these functions are linearly independent. What is the general solution to the differential equation in part (b)?

**2.62** ✓✓Use the answer to Exercise 2.61 to find all solutions $y$ to the differential equation in part (b) of Exercise 2.59 that satisfy $y(0) = 1, y'(0) = 2$.

**2.63** Let $\mathcal{W}$ denote the set of all solutions $y$ to the equation in part (c) of Exercise 2.59 satisfying $y(1) = 0$. Prove that $\mathcal{W}$ is a subspace of $C^\infty(\mathbb{R})$. What is its dimension?

## 2.3 ROW SPACE AND THE RANK-NULLITY THEOREM

Let us take a moment to review an idea discussed in Section 1.2. Consider the following system of equations:

$$\begin{aligned} x + 2y + \phantom{2}z &= 1 \\ 2x + 4y + \phantom{2}z &= 3 \\ x + 2y + 2z &= 0 \end{aligned} \tag{2.21}$$

The augmented matrix for this system is

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 1 & 3 \\ 1 & 2 & 2 & 0 \end{bmatrix}$$

The set of rows of $A$ is linearly dependent because $3A_1 - A_2 = A_3$. Thus, the third equation in system (2.21) contributes no information; our system is equivalent to the system formed by the first two equations. If we prefer, we can write $A_2 = 3A_1 - A_3$. Thus, we can ignore the second equation in system (2.21), considering only the first and third equations. In either case, our original system is equivalent with one having only two equations.

In Section 1.3, we stated a remarkable fact: if one begins with a matrix $A$ having dependent rows and eliminates dependent rows, one at a time, until a matrix with independent rows is obtained, then the final number of rows will be the same, regardless of which rows were eliminated and which were kept. In this section, we present a proof of this fact using the concept of dimension. In the process, we obtain an efficient method for finding bases for vector spaces.

Forming linear combinations of the equations in some system corresponds to forming linear combinations of the rows of the augmented matrix of the system. This suggests the following definition.

**Definition 2.5** *The row space of an $m \times n$ matrix $A$ is the subspace of $M(1, n)$ spanned by the rows of $A$. It is denoted by "row space $(A)$".*

### EXAMPLE 2.14

Show that the row vector $[1, 2, 3, 5]$ belongs to the row space of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 1 & 3 & 1 & 7 \end{bmatrix}$$

*Solution.* In this case, the problem is easily solved by inspection. Direct computation shows the following:

$$[1, 2, 3, 5] = [1, 1, 1, 1] + [0, 1, 2, 4] + 0[1, 3, 1, 7]$$

Recall that matrices $A$ and $B$ are said to be row equivalent if $B$ is obtainable from $A$ by a sequence of elementary row operations.

**Proposition 2.2** *Let $A$ and $B$ be row equivalent matrices. Then $A$ and $B$ have the same row space.*

*Proof.* Recall that the elementary row operations are:

1. Exchange two rows.
2. Replace a row by a scalar multiple of itself.
3. Replace a row by the sum of itself and a scalar multiple of another row.

All these operations, when applied to a matrix $A$, replace rows of $A$ with elements of the row space of $A$. Hence, if we apply a single elementary row operation to $A$, we obtain a matrix $A_1$, each of whose rows belong to the row space of $A$. Since the row space of $A$ is a vector space, it follows that all linear combinations of the rows of $A_1$ also belong to the row space of $A$—that is,

$$\text{row space}(A_1) \subset \text{row space}(A)$$

Since row operations are reversible, $A_1$ is also row equivalent to $A$. Repeating the preceding argument shows that

$$\text{row space}(A) \subset \text{row space}(A_1)$$

Hence,

$$\text{row space}(A) = \text{row space}(A_1)$$

(See the comments on proving set equalities in the footnote on page 103.)

Now, by hypothesis, there is a sequence $A_0, A_1, \ldots, A_n$ of matrices, where $A_0 = A$, $A_n = B$ and, for $1 \le i \le n$, each $A_i$ is obtained by applying a single elementary row operation to $A_{i-1}$. By the preceding reasoning,

$$\text{row space}(A) = \text{row space}(A_1) = \cdots = \text{row space}(A_n) = \text{row space}(B)$$

□

proving our theorem.

### Bases for the Row Space

Finding a basis for the row space is simple:

**Theorem 2.12** (Nonzero Rows Theorem). *The nonzero rows of any echelon form of a matrix A form a basis for the row space of A. In particular, the dimension of the row space of A is the number r of nonzero rows in any echelon form of A.*

*Proof.* We will first prove the theorem in the case of the row reduced echelon form $R$ of $A$. Since (from Proposition 2.2) the rows of $R$ span the row space of $A$, we need only show that the set of nonzero rows of $R$ is linearly independent. It helps to visualize the typical row echelon matrix:

$$R = \begin{bmatrix} 1 & * & 0 & * & 0 & * \\ 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The linear independence follows from Proposition 1.1 on page 9 since in each nonzero row there is a 1 in a position where every other row has a 0. Note that it follows that the dimension $r$ of the row space of $A$ is the number of nonzero rows in $R$.

It now follows from Theorem 2.8 on page 119 that the set of nonzero rows of any (not necessarily reduced) echelon form of $A$ also constitutes a basis for the row space of $A$ since there are $r$ of them and they span the row space. This finishes the proof. □

The nonzero rows theorem provides one of the most important ways of constructing bases.

#### EXAMPLE 2.15

Find a basis for the subspace $\mathcal{W}$ of $\mathbb{R}^5$ spanned by the following vectors. What is the dimension of this subspace?

$$\begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -6 \\ 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -6 \\ 1 \\ 10 \\ 13 \end{bmatrix} \tag{2.22}$$

*Solution.* We interpret these vectors as the *rows* of the matrix $A$, where

$$A = \begin{bmatrix} 1 & -2 & 0 & 2 & 3 \\ 1 & -6 & 1 & 4 & 7 \\ 2 & 0 & -1 & 2 & 2 \\ 1 & 2 & 1 & 2 & 1 \\ 5 & -6 & 1 & 10 & 13 \end{bmatrix} \tag{2.23}$$

We reduce $A$, obtaining

$$R_1 = \begin{bmatrix} 1 & -2 & 0 & 2 & 3 \\ 0 & -4 & 1 & 2 & 4 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The three nonzero rows of $R$ form a basis for the row space. To obtain a basis for $\mathcal{W}$, we reinterpret these vectors as columns, yielding the basis $\{X_1, X_2, X_3\}$ where

$$X_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 \\ -4 \\ 1 \\ 2 \\ 4 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 2 \\ 2 \end{bmatrix} \tag{2.24}$$

The dimension of $\mathcal{W}$ is 3.

We may obtain a different basis for $\mathcal{W}$ by using a different echelon form for $A$. For example, the reduced echelon form of $A$ is

$$R_2 = \begin{bmatrix} 1 & 0 & 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Writing the three nonzero rows of $R_2$ as columns yields another basis $\{Y_1, Y_2, Y_3\}$ for $\mathcal{W}$ where

$$Y_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -\frac{1}{4} \\ -\frac{3}{4} \end{bmatrix}, \quad Y_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{2.25}$$

This latter basis has the important property that each basis element has a 1 in a position where the others have a 0. This property makes expressing other vectors in terms of the basis particularly easy, as Example 2.16 illustrates.

*Remark.* We could also have solved the preceding problem by interpreting the given vectors as the *columns* of a matrix $B$ and reducing $B$. According to Theorem 2.2 on page 102, the pivot columns form a basis for the span. The difference between these two techniques is that the basis produced using Theorem 2.2 in Section 2.1 comes from the given set of vectors while bases produced using the nonzero rows theorem often have a particularly simple form, particularly if we reduce all the way to reduced echelon form.

## EXAMPLE 2.16

Express the first vector formula (2.22) of Example 2.15 as a linear combination of the vectors $Y_1$, $Y_2$, and $Y_3$ from formula (2.25).

*Solution.* We seek scalars $a$, $b$, and $c$ such that

$$\begin{bmatrix} 2 \\ 0 \\ -1 \\ 2 \\ 2 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \\ \frac{3}{2} \\ \frac{3}{2} \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ -\frac{1}{4} \\ -\frac{3}{4} \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Comparison of the first, second, and third entries on both sides of this equation shows $a = 2$, $b = 0$, and $c = -1$. The reader may check that for the stated values of $a$, $b$, and $c$, the other entries are also equal.

Now, let us consider the process of eliminating linearly dependent equations one at a time. Eliminating linearly dependent equations corresponds to eliminating linearly dependent rows of the augmented matrix. This process eventually produces a basis for the row space. Thus, the final number of equations is just the dimension of the row space, which is the same as the number of nonzero rows in any echelon form of the augmented matrix. In Section 1.3, we defined the rank of a system of equations to be the number of equations left after linearly dependent equations have been eliminated. Here is a much better way of saying this:

**Definition 2.6** *The rank of a matrix $A$ is the dimension of the row space. It is computable as the number of nonzero rows in any echelon form of the matrix.*

Theorem 2.2 on page 102 proves that the dimension of the column space of $A$ also equals the number of nonzero rows in an echelon form of $A$. Thus, we have proved the following result, which is one of the most important in linear algebra.

**Theorem 2.13** (Rank). *For any matrix A, the row space and column space have the same dimension. This dimension is the rank of the matrix.*

In some linear algebra texts, the dimension of the row space for a matrix $A$ is called the "row rank of $A$" and the dimension of the column space is called the "column rank of $A$." For this reason, Theorem 2.13 is often stated as: "The row rank of $A$ equals the column rank of $A$." Another version of the same theorem is the following result, which follows from the observation that the rows of $A^t$ are the columns of $A$, written "sideways."

**Theorem 2.14** *For any matrix A,* $\operatorname{rank} A = \operatorname{rank} A^t$.

*Proof.* The rows of $A^t$ are the columns of $A$. A basis for the row space of $A^t$ may be produced by transposing each element in a basis for the column space of $A$. Hence, the dimension of the row space of $A^t$ equals the rank of $A$, proving our theorem. □

The conclusion of Theorem 2.14 is quite remarkable. It implies, for example, that the echelon forms of the following two matrices have the same number of nonzero rows. This is surprising, granted the complexity of the reduction process.

$$\begin{bmatrix} 1 & 3 & 0 & -2 \\ 2 & 1 & \pi & -5 \\ 3 & 4 & -2 & 17 \\ 4 & 7 & -2 & 15 \\ -7 & 6 & e & \sqrt{31} \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 & 4 & -7 \\ 3 & 1 & 4 & 7 & 6 \\ 0 & \pi & -2 & -2 & e \\ -2 & -5 & 17 & 15 & \sqrt{31} \end{bmatrix}$$

The rank theorem has a simple, but important, consequence: *The rank of an $m \times n$ matrix $A$ cannot exceed either $m$ or $n$* since $m$ is the number of rows and $n$ is the number of columns. Thus, for example, without doing any work at all, we know that the preceding two matrices both have a rank of at most 4.

Another consequence of the rank theorem is the following useful result:

**Theorem 2.15** *Let A be an $m \times n$ matrix. Then the set of columns of A is linearly independent if and only if* $\operatorname{rank} A = n$. *The set of rows of A is linearly independent if and only if* $\operatorname{rank} A = m$.

*Proof.* The statement about the columns follows from the observation that a set of $n$ elements in a vector space $\mathcal{V}$ is linearly independent if and only if its span is $n$-dimensional. Thus, the $n$ columns of $A$ will form a linearly independent set if and only if the column space is $n$-dimensional, which is equivalent to $\operatorname{rank} A = n$. The statement about the rows follows similarly. □

## EXAMPLE 2.17

Determine whether or not the following vectors are linearly independent:

$$\begin{bmatrix} 3 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 5 \\ 2 \end{bmatrix}$$

*Solution.* These vectors are the columns of

$$A = \begin{bmatrix} 3 & -2 & 4 \\ 1 & 2 & 3 \\ -2 & 0 & 5 \\ 4 & 3 & 2 \end{bmatrix}$$

The row reduced echelon form of $A$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

showing that $A$ has rank 3. Hence, the vectors are linearly independent.

Let $A$ be an $m \times n$ matrix. In Section 1.4 we posed the question, "When is the system $AX = B$ solvable for all $B \in \mathbb{R}^m$?" We can now answer this question. From Theorem 1.8 on page 76, the set of $B$ for which this equation is solvable is the column space. Since the column space is contained in $\mathbb{R}^m$, it can equal all $\mathbb{R}^m$ only if it is $m$-dimensional. Thus, we arrive at the following theorem.

**Theorem 2.16** *Let $A$ be an $m \times n$ matrix. The equation $AX = B$ is solvable for all $B \in \mathbb{R}^m$ if and only if rank $A = m$.*

*Proof.* As explained preceding the statement of the theorem, if $AX = B$ is solvable for all $B \in \mathbb{R}^m$, then the column space of $A$ is $\mathbb{R}^m$ and hence, $m$-dimensional. Thus, rank $A = m$.

Conversely, if rank $A = m$, its column space contains $m$ linearly independent vectors. These vectors span $\mathbb{R}^m$ due to Theorem 2.8 on page 119. Hence every $B$ in $\mathbb{R}^m$ belongs to the column space. Thus, $AX = B$ is solvable for all $B$ in $\mathbb{R}^n$. □

From the translation theorem (Theorem 1.10 on page 80), the general solution to $AX = B$ is a translate of the nullspace. Hence, the dimension of the nullspace measures the size of the solution set. We refer to this dimension as the nullity of $A$:

**Definition 2.7** *The dimension of the nullspace of a matrix $A$ is called the **nullity** of $A$ and is denoted by null $A$.*

The following theorem says that there is a complementary relationship between the rank and the nullity: for matrices of a given size, the larger the rank, the smaller the nullity and vice versa. Thus, among consistent systems, the smaller the rank, the larger the solution set. This makes sense. The rank measures the number of linearly independent equations. We certainly expect that fewer equations put fewer constraints on the solutions, allowing for the possibility of more solutions.

**Theorem 2.17** (Rank-Nullity Theorem). *For an $m \times n$ matrix $A$,*

$$\text{rank}\, A + \text{null}\, A = n$$

*Proof.* The set of spanning vectors for the system $AX = \mathbf{0}$ forms a basis for the nullspace (Theorem 1.12 on page 83). Hence, the dimension of null $A$ is the number of nonpivot variables for the system $AX = \mathbf{0}$.

On the other hand, the rank is the number of nonzero rows in an echelon form of $A$ (Definition 2.6). Since there is exactly one pivot variable in each such row, the rank is the number of pivot variables. Our theorem follows from the observation that there are $n$ variables in the system $AX = \mathbf{0}$ and each variable is either a pivot or nonpivot variable. □

It follows from the rank-nullity theorem that the nullspace of an $m \times n$ matrix $A$ is 0 if and only if rank $A = n$. Hence, we have the following theorem:

**Theorem 2.18** *Let $A$ be an $m \times n$ matrix. There is at most one solution to $AX = B$ for all $B \in \mathbb{R}^m$ if and only if rank $A = n$.*

### EXAMPLE 2.18

Discuss the existence and uniqueness for the solutions to the system $AX = B$ for each of the following matrices. You may use any observed linear relationships among the rows or columns but not do any formal row reduction.

(a) $A = \begin{bmatrix} 2 & 4 & -3 & 5 \\ 1 & 1 & 1 & 1 \\ 4 & 6 & -1 & 7 \end{bmatrix}$ (b) $A = \begin{bmatrix} 2 & 1 & 4 \\ 4 & 1 & 6 \\ -3 & 1 & -1 \\ 5 & 1 & 7 \end{bmatrix}$

(c) $A = \begin{bmatrix} 2 & 1 & 8 \\ 1 & 0 & 0 \\ 0 & 0 & 7 \\ 0 & 1 & 0 \end{bmatrix}$ (d) $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 8 & 0 & 7 & 0 \end{bmatrix}$

*Solution.*
(a): The third row of $A$ is the first plus twice the second. Since the first two rows are linearly independent, the rank of $A$ is 2. Hence, the dimension of the column space is 2, implying that there are $B \in \mathbb{R}^3$ such that the system $AX = B$ is not solvable. From the rank-nullity theorem, the dimension of the nullspace is $4 - 2 = 2$. Hence, when the solution exists, it is not unique.

(b): The matrix in (b) is the transpose of the one in (a). Hence, its rank is also 2. Since the column space is only two-dimensional, there are many vectors $B \in \mathbb{R}^4$ such that $AX = B$ is not solvable. From the rank-nullity theorem, the dimension of the nullspace is $3 - 2 = 1$. Hence, when the solution exists, it is not unique.

(c): The columns of $A$ are clearly linearly independent since each has a nonzero entry in a place where the other two have a zero. Hence, the rank of $A$ is 3. Since the column space is only three-dimensional, there are many vectors $B \in \mathbb{R}^4$ such that $AX = B$ is not solvable. From the rank-nullity theorem, the dimension of the nullspace is $3 - 3 = 0$. Hence, when the solution exists, it is unique.

(d): The matrix in (d) is the transpose of the one in (c). Hence, its rank is also 3. Since the column space is three-dimensional, the equation $AX = B$ is solvable for all $B \in \mathbb{R}^3$. From the rank-nullity theorem, the dimension of the nullspace is $4 - 3 = 1$. Hence, when the solution exists, it is not unique.

The ideal situation is, of course, when the solution to $AX = B$ both exists and is unique for all $B \in \mathbb{R}^n$:

**Definition 2.8** *An $m \times n$ matrix* A *is said to be* ***nonsingular*** *if the solution to $AX = B$ both exists and is unique for all $B \in \mathbb{R}^n$.*

According to Theorems 2.16 and 2.18, for an $m \times n$ matrix, this happens if and only if $m = \operatorname{rank} A = n$—that is, the system $AX = B$ must consist of $n$ independent equations in $n$ unknowns. Thus, we have proved the following theorem:

**Theorem 2.19** *The matrix $A$ is nonsingular if and only if $A$ is an $n \times n$ matrix with rank $n$.*

We said that an $m \times n$ matrix $A$ is nonsingular if the solution to $AX = B$ both exists and is unique for all $B$ in $R^n$. Actually, for an $n \times n$ matrix, we do not need both statements to conclude nonsingularity; if the solution exists for all $B$ in $\mathbb{R}^n$, then it will be unique. Similarly, if the solution is unique for a single $B$ in $R^n$, it will exist for all $B$ in $\mathbb{R}^n$. This is a consequence of the following theorem:

**Theorem 2.20** *Let $A$ be an $n \times n$ matrix. Then the following statements are equivalent to each other*[3]*:*

*(a) The nullspace of $A$ is $\{\mathbf{0}\}$.*

[3]In mathematics, saying that two statements are equivalent means that each implies the other.

*(b) For each $B$ in $\mathbb{R}^n$, the system $AX = B$ has at most one solution.*

*(c) The system $AX = B$ has a solution for all $B \in \mathbb{R}^n$.*

*(d) $A$ is nonsingular.*

*Proof.* From the translation theorem in Section 1.5, if the system $AX = B$ has a solution, then the general solution is

$$T + \mathcal{W}$$

where $\mathcal{W}$ is the nullspace of $A$ and $T$ is any single solution. It follows that the solution to $AX = B$ is unique if and only if the nullspace is $\mathbf{0}$, which proves the equivalence of (a) and (b).

Theorem 2.18 proves the equivalence of statements (b) and (d), whereas Theorem 2.16 proves the equivalence of statements (c) and (d). □

Nonsingular matrices have an especially simple reduced form. Since the system $AX = \mathbf{0}$ has only one solution, the reduced form of the matrix $[A, \mathbf{0}]$ has no free variables; every column of $A$ is a pivot column of $[A, \mathbf{0}]$. Hence, the reduced form of $A$ must take the form

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & 1 \end{bmatrix}$$

which is just the $n \times n$ identity matrix mentioned in Section 2.1. We state this as a proposition:

**Proposition 2.3** *An $n \times n$ matrix $A$ is nonsingular if and only if it is row equivalent to the identity matrix.*

We will discuss nonsingular matrices further in Section 3.3.

## Summary

This section introduced a number of important ideas. It is worthwhile to summarize the main points:

1. The row space of a matrix is the span of its rows. A basis for the row space may be found by reducing the matrix to echelon form and using the nonzero rows. The dimension of the row space is the rank of the matrix.
2. The dimension of the column space of a matrix is also equal to the rank of the matrix.
3. $\operatorname{rank} A = \operatorname{rank} A^t$.

4. The dimension of the nullspace, $\operatorname{null} A$, is the number of nonpivot variables in the system $AX = \mathbf{0}$. The set of spanning vectors for the solution set to this system forms a basis for the nullspace.
5. For an $m \times n$ matrix $A$, $\operatorname{rank} A + \operatorname{null} A = n$.
6. For an $m \times n$ matrix $A$, if $\operatorname{rank} A = m$, the system $AX = B$ is always solvable. If $\operatorname{rank} A = n$, the solution is unique.
7. For an $n \times n$ matrix $A$, if $\operatorname{rank} A = n$, then $AX = B$ always has a unique solution.

## Computational Issues: Computing Rank

One has to be very careful in computing the rank by counting the nonzero rows in the reduced form. Suppose, for example, we attempt to compute the rank of the following matrix using row reduction on a computer:

$$M = \begin{bmatrix} 1 & 7 \\ \frac{1}{13} & \frac{7}{13} \end{bmatrix}$$

$M$ is rank 1 because the second column is seven times the first. However, to nine digits, $M = M'$, where

$$M' = \begin{bmatrix} 1.0000000000 & 7.000000000 \\ 0.0769230769 & 0.538461539 \end{bmatrix}$$

This matrix however has rank 2 since

$$7 \times 0.0769230769 = 0.5384615383 \neq 0.538461539$$

Hence, row reducing $M'$ would not produce a row of zeros!The problem is that we have replaced the actual values of the matrix with approximations so we can only expect that the second column is close to, but not equal to, seven times the first. Hence, small errors, such as measurement error, and round-off error can convert low rank matrices into maximal rank matrices.

A good computational package, such as MATLAB or Maple, is not so easily fooled. When asked to reduce $M$ (entered as written), MATLAB produces

$$\begin{bmatrix} 1 & 7 \\ 0 & 0 \end{bmatrix}$$

showing that $M$ has rank 1, and when asked to reduce $M'$, MATLAB produces the identity matrix, which is correct. The key is that in the reduction process MATLAB sets sufficiently small numbers equal to zero. However, for large matrices, even the best of algorithms can be fooled. Fortunately, there are other ways of computing the rank, such as using the singular value decomposition discussed in Section 6.6, which are not so susceptible to this kind of error.

**True-False Questions:** Justify your answers.

**2.20** Suppose that $A$ is a $3 \times 5$ matrix such that the vectors $X = [1, 1, 1, 1, 1]^t$, $Y = [0, 1, 1, 1, 1]^t$, and $Z = [0, 0, 1, 1, 1]^t$ belong to the nullspace of $A$. Classify the following statements as true or false.

**(a)** The rows of $A$ are dependent.

**(b)** $AX = B$ has a solution for all $B \in \mathbb{R}^3$.

**(c)** The solution to $AX = B$, when it exists, is unique.

**2.21** Suppose that $A$ is an $m \times n$ matrix such that $AX = B$ has a solution for all $B \in \mathbb{R}^m$. Then the solution to $A^tX = B'$, when it exists, is unique.

**2.22** Suppose that $A$ is an $m \times n$ matrix such that the solution to $A^tX = B'$, when it exists, is unique. Then $AX = B$ has a solution for all $B \in \mathbb{R}^m$.

**2.23** Suppose that $A$ is an $m \times n$ matrix such that $AX = B$ has a solution for all $B \in \mathbb{R}^m$. Then the equation $A^tX = B'$ has a solution for all $B' \in \mathbb{R}^n$.

**2.24** The following matrix has a set of three linearly independent columns:

$$\begin{bmatrix} 1 & 2 & -4 & 5 & -7 & 6 \\ -3 & 2 & -17 & 33 & 11 & 7 \\ -2 & 4 & -21 & 38 & 4 & 13 \end{bmatrix}$$

**2.25** Suppose that $A$ is a $3 \times 7$ matrix such that the equation $AX = B$ is solvable for all $B$ in $\mathbb{R}^3$. Then $A$ has rank 3.

**2.26** Suppose that $A$ is a $4 \times 9$ matrix such that the column space of $A$ is a line in $\mathbb{R}^4$. Then every row of $A$ is a multiple of the first row of $A$ provided that the first row is nonzero.

**2.27** Suppose that $A$ is a matrix for which the column space and nullspace are both two-dimensional. Then $A$ *must* be a $4 \times 4$ matrix.

## EXERCISES

**2.64** ✓✓For the matrix $A$, decide whether the given row vectors $B$ and $C$ belong to the row space.

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 1 \\ 3 & 3 & 8 & 2 \end{bmatrix}, \quad B = [4, 1, 2, 5], \quad C = [1, 2, 3, 4]$$

**2.65** For the matrix $A$ in Exercise 2.64:

**(a)** Compute the *reduced* echelon form $R$ of $A$.

**(b)** Use $R$ to find a basis for the row space of $A$.

**(c)** Express each row of $A$ as a linear combination of these basis elements.

**(d)** Do the columns of $R$ span the column space of $A$? Explain.

**(e)** Check (by row reduction) that rank $A$ = rank $A^t$.

**2.66** For each matrix (a)–(d), find its rank and bases for its column and row spaces. (Use Theorem 2.3 on page 104 to find a basis for the column space.)

**(a)** ✓✓ $\begin{bmatrix} 1 & 2 & 0 & -2 \\ 2 & 3 & 2 & 3 \\ 2 & 10 & 4 & 10 \end{bmatrix}$ **(b)** $\begin{bmatrix} -1 & 4 & -2 \\ 4 & 4 & 2 \\ 3 & 0 & -3 \end{bmatrix}$

**(c)** ✓✓ $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 5 & 0 & 1 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 3 & 6 & 6 \\ -2 & -4 & -4 \end{bmatrix}$

**2.67** Repeat Exercise 2.66 for the transposes of the stated matrices. Then use the results to give different bases for the row and column spaces for the matrices in each part of Exercise 2.66. ✓[(a), (c)]

**2.68** It is claimed that the vectors $[1, 0, 0]^t$, $[0, 1, 0]^t$, and $[0, 0, 1]^t$ form a basis for the column space Exercise 2.66(b). Explain.

**2.69** Let $\mathcal{W}$ be the span of the vectors in Example 2.15 on page 134.

**(a)** Use Theorem 2.3 on page 104 to find a basis for $\mathcal{W}$.

**(b)** Write each of the basis elements from part (a) as a linear combination of the basis elements $Y_1$, $Y_2$, and $Y_3$ found in Example 2.15.

**2.70** Let $\mathcal{W}$ be the span of the following vectors.

$$[2, 3, 1, 2]^t, \quad [5, 2, 1, 2]^t, \quad [1, -4, -1, -2]^t, \quad [11, 0, 1, 2]^t$$

**(a)** ✓✓Use Theorem 2.12 on page 134 to find a basis for $\mathcal{W}$.

**(b)** ✓Express each the given vectors as a linear combination of the basis elements.

**(c)** ✓Use Theorem 2.3 on page 104 to find a basis for $\mathcal{W}$.

**2.71** ✓✓Use Theorem 2.12 on page 134 to find a basis for the span of the following matrices.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 3 \\ 7 \end{bmatrix}$$

**2.72** $A$ is a $4 \times 5$, rank 2, matrix for which the vectors $X_1 = [17, 6, -13, 1, 2]^t$, $X_2 = [1, 2, 3, 4, 5]^t$, and $X_3 = [5, 4, 3, 2, 1]^t$ all satisfy $AX = \mathbf{0}$.

**(a)** ✓Prove that $X_1, X_2$, and $X_3$ span the solution space to the system $AX = \mathbf{0}$.

**(b)** Is it possible to find a $4 \times 5$, rank 3 matrix $B$ such that the given $X_i$ all satisfy $BX_i = \mathbf{0}$?

**2.73** Suppose that $A$ is a rank 2, $4 \times 5$, matrix for which $X_1 = [1, 2, 3, 2, 1]^t$, $X_2 = [1, -1, 0, 1, 2]^t$, and $X_3 = [2, 1, 4, 4, 4]^t$ all satisfy $AX = \mathbf{0}$. Do these vectors span the nullspace of $A$?

**2.74** Each row of the table indicated in Figure 2.3 summarizes data collected for some matrix $A$. The "always solvable" column refers to whether the system $AX = B$ is solvable for all $B$ and the "unique solution" column refers to whether or not there is at most one solution. Your assignment is to fill in all the missing data from the table. Where the data is inconsistent, write "impossible."

| *Size* | *Always Solvable* | *Unique Solution* | *Dimension of Column Space* | *Dimension of Nullspace* | *Rank* |
|---|---|---|---|---|---|
| $4 \times 3$ | yes | | | | |
| $3 \times 4$ | yes | | | | |
| $5 \times 5$ | | | | 1 | |
| $5 \times 5$ | | | 5 | | |
| $3 \times 2$ | | yes | | | |
| $4 \times 4$ | yes | | | | |
| $5 \times 4$ | | | | | 3 |
| $5 \times 4$ | | | | | 5 |

**FIGURE 2.3** Exercise 2.74.

**2.75** Which of the rows in Figure 2.3 describe nonsingular matrices?

**2.76** Let

$$A = \begin{bmatrix} 1 & 1 & 3 & 2 \\ 2 & 2 & 6 & 4 \\ 10 & 2 & 14 & 20 \\ 2\sqrt{2} & -\sqrt{2} & 0 & 4\sqrt{2} \\ \pi & e & \pi + 2e & 2\pi \\ \sqrt{2} & \sqrt{3} & \sqrt{2} + 2\sqrt{3} & 2\sqrt{2} \\ \ln 5 & 6 & \ln 5 + 12 & 2\ln 5 \\ -7 & 4 & 1 & -14 \\ 17 & -24 & -31 & 34 \\ 2 & 2 & 6 & 4 \end{bmatrix}$$

**(a)** What is the rank of $A$? Explain.

**(b)** What is the dimension of the nullspace of $A$?

**(c)** Is the equation $AX = B$ solvable for all $B$?

**(d)** Does the equation $AX = B$ have at most one solution? Explain.

**(e)** Find *two* different bases for the row space of $A$. Use some theorems from linear algebra to justify your answer.

**(f)** I claim that the nullspace of $A$ is the solution set of the following system of equations. Explain.

$$\begin{aligned} x + \phantom{4}y + 3z + \phantom{1}2w &= 0 \\ -7x + 4y + \phantom{3}z - 14w &= 0 \end{aligned}$$

**2.77** Answer questions (a)–(d) from Exercise 2.76 for $A^t$, where $A$ is as in Exercise 2.76.

**2.78** ✓✓Consider the accompanying matrix $A$. Without doing any row reduction at all, answer the following. Be sure to justify your answers.

**(a)** What is the rank of $A$?

**(b)** Find two different bases for the column space of $A$.

**(c)** Show that the vectors $X_1$ and $X_2$ belong to the nullspace of $A$ where $X_1 = [-\pi, 0, 0, 0, 0, 1]^t$ and $X_2 = [-2, 1, 0, 0, 0, 0]^t$. Do they span the nullspace?

**(d)** Find a vector $T$ such that $AT = [1, 1, 2, 3]^t$.

**(e)** Find an infinite number of vectors $X$ such that $AX = [1, 1, 2, 3]^t$. Remember, no row reduction allowed!

$$A = \begin{bmatrix} 1 & 2 & e & \ln 4 & 7 & \pi \\ 1 & 2 & \pi & \sqrt{2} & 1 & \pi \\ 2 & 4 & e+\pi & \sqrt{2}+\ln 4 & 8 & 2\pi \\ 3 & 6 & 2e+\pi & 2\ln 4+\sqrt{2} & 15 & 3\pi \end{bmatrix}$$

**2.79** Prove that an $n \times n$ matrix $A$ is nonsingular if and only if $A^t$ is.

**2.80** ✓✓An $m \times n$ matrix $A$ has a $d$-dimensional nullspace. What is the dimension of the nullspace of $A^t$?

**2.81** Suppose that $A$ is $n \times n$ and has rank $n$. What is its row space? What is its nullspace? What is its column space?

**2.82** Let $A$ be an $m \times n$ matrix of rank $m$. Prove that the rows of $A$ form a basis of the row space.

**2.83** Let $A$ be an $m \times n$ matrix of rank $n$. Prove that the columns of $A$ form a basis of the column space.

### 2.3.1 Computer Projects

### EXERCISES

**1.** In MATLAB, use the command M=rand(3,5) to construct a random $3 \times 5$ matrix $M$. What do you expect the rank of $M$ to be? Check your guess using MATLAB. Is it conceivable that the rank could have turned out otherwise? Why is this unlikely?

**2.** Let $M$ be as in the preceding exercise. Let $s$ and $t$ be random $1 \times 3$ matrices (constructed using the rand command). Set

```
>> M(4,:)=s(1)*M(1,:)+s(2)*M(2,:)+s(3)*M(3,:)
```

Similarly, let

```
>> M(5,:)=t(1)*M(1,:)+t(2)*M(2,:)+t(3)*M(3,:)
```

Note that M is now $5 \times 5$. What is the rank of $M$? Check this using rref. What is the maximal number of linearly independent *columns* in $M$?

**3.** For the preceding $5 \times 5$ matrix $M$, use Theorem 2.3 on page 104 to find a set of columns of $M$ that forms a basis for the column space. Express the other columns of $M$ as linear combinations of these columns. Check your answer using MATLAB.

**4.** Are the coefficients you computed in the previous exercise exact? To check this, issue the command "format long" and then form the linear combinations of the basis columns using the coefficients you computed. Do you reproduce the columns of $M$? You can return to the regular format with the command "format short".

**5.** For $M$ as above, find a basis for the column space of $M$ by reducing the transpose of $M$ (this is denoted by M′ in MATLAB) and then using the nonzero rows theorem. Try to express each of the basis columns you found in Exercise 3 as linear combinations of these columns. Again, you may need to include a tolerance in rref.

**6.** According to the comments in the text, there is an inverse relationship between the dimension of the nullspace of a matrix and its rank. Demonstrate this by creating (as in Exercise 1) four random $4 \times 4$ matrices with rank 1, 2, 3, and 4, respectively. For each of your matrices, use the "null" command to find a basis for the nullspace. (For a discussion of the "null" command, see Exercise 3 on page 94.)

## CHAPTER SUMMARY

The key concept from this chapter is ***dimension*** (Section 2.2). The dimension of a vector space $\mathcal{W}$ is the smallest number of elements that it takes to span $\mathcal{W}$. Given a spanning set for a space, we can produce a linearly independent spanning set by deleting linearly dependent elements. The ***test for linear independence*** provides a systematic way of doing this (Theorem 2.3 in Section 2.1). A linearly independent spanning set is called a ***basis***. Bases are important because they span the space as efficiently as possible. Bases also allow us to find the dimension of a vector space:

the number of elements in any basis is the dimension of the space (Theorem 2.7, the ***dimension theorem***, Section 2.2).

The dimension of a space is one of the most fundamental parameters in determining the properties of the space. In an $n$-dimensional space, there can exist no more than $n$ linearly independent elements (Theorem 2.6, Section 2.2); in an $n$-dimensional vector space, $n$ linearly independent elements always span the whole space and $n$ elements that span the space must be linearly independent (Theorem 2.8, Section 2.2).

Dimension is also a powerful tool for studying matrices. The ***row space*** of a matrix is the span of its rows and the ***column space*** is the span of its columns. We may find bases for both of these spaces by row reducing the given matrix. The nonzero rows of the echelon form of the matrix form a basis for the row space of the matrix (Theorem 2.12, the ***nonzero rows theorem***, in Section 2.3), and the pivot columns of the original matrix form a basis for the column space (Theorem 2.4, Section 2.1). Since the number of such nonzero rows is the same as the number of pivot variables, we see that the dimension of the row space equals both the dimension of the column space and the rank of the matrix (Theorem 2.13, the ***rank theorem***, Section 2.3).

Another important space that we associate with a matrix A is its ***nullspace***, which is the set of vectors $X$ such that $AX = 0$. The nullspace measures the size of the solution set to the equation $AX = B$ in that the general solution to this equation is a translate of the nullspace. The spanning vectors for the system $AX = 0$ form a basis for the nullspace, and the dimension of the nullspace is the number of nonpivot variables found in solving the system $AX = 0$. The dimension of the nullspace ($\operatorname{null} A$) for an $m \times n$ matrix satisfies $\operatorname{null} A + \operatorname{rank} A = n$ (Theorem 2.17, the ***rank-nullity theorem***, Section 2.3). Hence, for matrices of the same size, the larger the rank, the smaller the nullity.

# CHAPTER 3

# LINEAR TRANSFORMATIONS

## 3.1 THE LINEARITY PROPERTIES

Suppose that the triangle pictured in Figure 3.1 appears on a computer screen and we wish to rotate it about the origin by $\pi/6$ radians counterclockwise. How should we instruct the computer to plot the new image of the triangle?

To answer this question, suppose that $X = [x, y]^t$ is some point on the triangle and $X' = [x', y']^t$ is 00$X$ rotated by $\pi/6$ radians. Let $r$ and $\theta$ be the polar coordinates of $X$ so that

$$x = r\cos\theta, \quad y = r\sin\theta$$

Then, the polar coordinates of $X'$ will be $r$ and $\theta + \frac{\pi}{6}$. Hence

$$\begin{aligned} x' &= r\cos(\theta + \pi/6) = r\cos\theta\cos(\pi/6) - r\sin\theta\sin(\pi/6) \\ &= (\sqrt{3}/2)x - (1/2)y \\ y' &= r\sin(\theta + \pi/6) = r\cos\theta\sin(\pi/6) + r\sin\theta\cos(\pi/6) \\ &= (1/2)x + (\sqrt{3}/2)y \end{aligned}$$

(We used the angle addition formulas for the sine and cosine functions.)

*Linear Algebra: Ideas and Applications*, Fourth Edition. Richard C. Penney.

Companion website: www.wiley.com/go/penney/linearalgebra

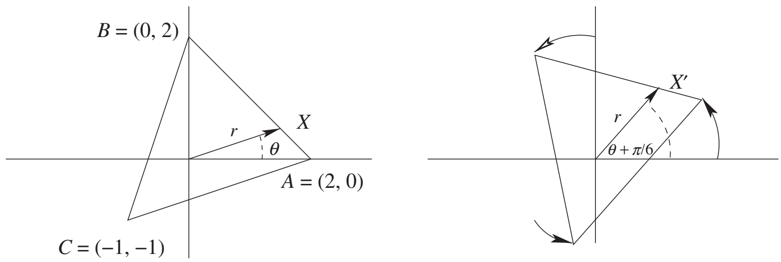



**FIGURE 3.1** Rotating a triangle.

Now comes the main point. The preceding formulas may be written as a matrix equality as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = x\begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix} + y\begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Let

$$A = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

Multiplication of points by $A$ rotates them by $\pi/6$ radians. Thus, we solve our problem by multiplying each vertex of the triangle by $A$. We obtain

$$A' = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} \approx \begin{bmatrix} 1.73 \\ 1.00 \end{bmatrix}$$
$$B' = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} \approx \begin{bmatrix} -1.00 \\ 1.73 \end{bmatrix}$$
$$C' = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\sqrt{3}/2 + 1/2 \\ -1/2 - \sqrt{3}/2 \end{bmatrix} \approx \begin{bmatrix} -0.37 \\ -1.37 \end{bmatrix}$$

We then instruct the computer to plot the triangle with these vertices.

In general, multiplication by the matrix

$$R_\psi = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} \tag{3.1}$$

rotates points by $\psi$ radians counterclockwise about the origin.

Rotation is an example of a transformation. In general, a **transformation** $T$ is a well-defined process for taking elements $X$ from one set $\mathcal{V}$ (the **domain of the**

**transformation**) and using them to produce elements $T(X)$ of another set $\mathcal{V}$ (the **target space**). In this case, we write $T : \mathcal{U} \to \mathcal{V}$. "Well-defined" means that each element of the domain is transformed onto only one element of the target space. Two transformations $S$ and $T$ are equal if they have the same domain and target spaces and $S(X) = T(X)$ for all $X$ in their domain. For the rotation above, the domain and the target space are both $\mathbb{R}^2$.

The functions studied in elementary mathematics are all examples of transformations. For example, the function

$$f(x) = x^2$$

transforms each real number into its square. Hence, $f$ is a transformation with domain and target space both equal to $\mathbb{R}$. In fact, the terms "function" and "transformation" mean exactly the same thing, although in this text we use function only for transformations whose domain and target space are both subsets of $\mathbb{R}$.

Since rotation is describable in terms of matrix multiplication, we say that it is a matrix transformation:

**Definition 3.1** *Let A be a $m \times n$ matrix. The transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by*

$$T_A(X) = AX \tag{3.2}$$

*is the **matrix transformation** defined by A. In general, a **matrix transformation** is a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ for which there is an $m \times n$ matrix A such that $T = T_A$.*

Notice that to rotate our triangle it was not necessary to rotate each point on the triangle—only the vertices need to be rotated. This works because rotation transforms line segments onto line segments. The same is true for any matrix transformation:

**Proposition 3.1** *Multiplication by an $m \times n$ matrix A transforms a given line segment in $\mathbb{R}^n$ onto either a line segment in $\mathbb{R}^m$ or a single point in $\mathbb{R}^m$.*

*Proof.* Let $X$ and $Y$ be points in $\mathbb{R}^n$. The equation

$$Z = X + t(Y - X)$$

is the parametric description of a line that passes through $X$ ($t = 0$) and $Y$ ($t = 1$). The segment between $X$ and $Y$ is the portion of the line corresponding to $0 \le t \le 1$.

From the linearity properties of matrix multiplication,

$$AZ = A(X + t(Y - X)) = AX + t(AY - AX)$$

If $AX = AY$, this describes the single point $AX$. Otherwise, for $0 \le t \le 1$, this equation describes the line segment between $AX$ and $AY$, proving the proposition. □

**Definition 3.2** *If $S$ is a subset of the domain of a transformation $T$, then the* ***image*** *of $S$ under $T$ is the set $T(S)$ consisting of all points $T(X)$, where $X \in S$.*

Proposition 3.1 is useful in computing images.

## EXAMPLE 3.1

Let

$$M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Let $S$ be the square in $\mathbb{R}^2$ with vertices $A = [0, 0]^t$, $B = [1, 0]^t$, $C = [1, 1]^t$, and $D = [0, 1]^t$. What is the image of $S$ under multiplication by $M$?

*Solution.* It is easily computed that $T_M$ transforms $A$, $B$, $C$, and $D$ onto $A' = [0, 0]^t$, $B' = [1, 1]^t$, $C' = [1, 2]^t$, and $D' = [0, 1]^t$, respectively. These points form the vertices of a parallelogram, as shown in Figure 3.2. Multiplication by $M$ transforms the square onto this parallelogram, because segments transform onto segments. We picture this transformation as "tilting" the square. Such a transformation is called a "shear."

The proof of Proposition 3.1 demonstrates the importance of the linearity properties of matrix multiplication. It is both interesting and important that among transformations from $\mathbb{R}^n$ into $\mathbb{R}^m$ matrix transformations are the *only* transformations that exhibit these properties. First, however, we make a general definition:

**Definition 3.3 (Linear Transformation).** *Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces. Let $T : \mathcal{V} \to \mathcal{W}$ be a transformation. We say that $T$ is a linear transformation if it satisfies two properties:*

1. *For all $X$ and $Y$ in $\mathcal{V}$,*

$$T(X + Y) = T(X) + T(Y) \qquad \textit{(additivity)}$$

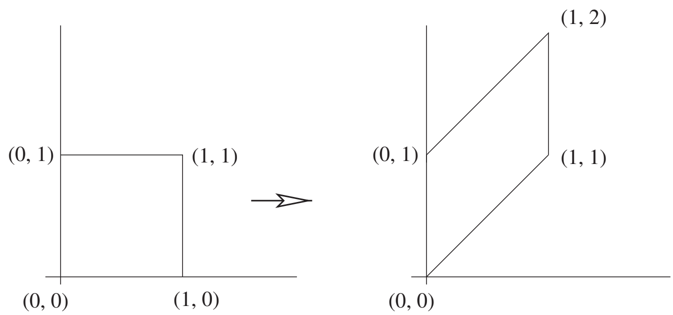



**FIGURE 3.2** A shear.

2. *For all $X$ in $\mathcal{V}$ and all scalars $c$,*

$$T(cX) = cT(X) \qquad\qquad \textit{(scalar)}$$

Matrix transformations are linear transformations since the properties just defined are simply restatements of the linearity properties for matrix multiplication. The following theorem states that matrix transformations are the *only* transformations from $\mathbb{R}^n$ into $\mathbb{R}^m$ with these properties. For an example of a nonlinear transformation, see Exercise 3.14.

**Theorem 3.1** (Matrix Representation Theorem). *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be any linear transformation. Then there is a unique matrix $A$ such that $T(X) = AX$ for all $X \in \mathbb{R}^n$.*

*Proof.* If $X = [x_1, x_2, \cdots, x_n]^t$, then we may write

$$X = x_1 I_1 + x_2 I_2 + \cdots + x_n I_n$$

where $I_j$ is the $j$th standard basis element for $\mathbb{R}^n$.

Applying the additivity and scalar properties of $T$, we obtain

$$T(X) = x_1 T(I_1) + x_2 T(I_2) + \cdots + x_n T(I_n)$$

The elements $T(I_j)$ are $m \times 1$ column vectors. From the definition of matrix multiplication [formula (1.42) on page 78], the term on the right is exactly $AX$, where $A = [T(I_1), T(I_2), \ldots, T(I_n)]$. This proves the existence of $A$.

To prove uniqueness, suppose that $B$ is some other $m \times n$ matrix that defines the same transformation as $A$. Then, for all $j$

$$BI_j = AI_j$$

However, if $C$ is an $m \times n$ matrix, $CI_j$ is just the $j$th column of $C$. Thus, each column of $A$ equals the corresponding column of $B$, proving that $A = B$. □

## EXAMPLE 3.2

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T([x, y]^t) = [2x + 3y, x - y, x]^t$. Use the definition of linearity to show that $T$ is linear, and find a matrix $A$ that represents $T$.

*Solution.* To prove linearity, we must prove the additivity and scalar properties. For additivity let $X = [x_1, y_1]^t$ and $Y = [x_2, y_2]^t$ be two points in $\mathbb{R}^2$. Then

$$T(X) = [2x_1 + 3y_1, x_1 - y_1, x_1]^t$$
$$T(Y) = [2x_2 + 3y_2, x_2 - y_2, x_2]^t$$

Hence,

$$\begin{aligned} T(X) + T(Y) &= [2x_1 + 3y_1, x_1 - y_1, x_1]^t + [2x_2 + 3y_2, x_2 - y_2, x_2]^t \\ &= [2(x_1 + x_2) + 3(y_1 + y_2), (x_1 + x_2) - (y_1 + y_2), (x_1 + x_2)]^t \end{aligned}$$

On the other hand,

$$\begin{aligned} T(X + Y) &= T([x_1 + x_2, y_1 + y_2]^t) \\ &= [2(x_1 + x_2) + 3(y_1 + y_2), (x_1 + x_2) - (y_1 + y_2), (x_1 + x_2)]^t \\ &= T(X) + T(Y) \end{aligned}$$

For the scalar property, let $X = [x, y]^t$. We compute

$$\begin{aligned} T(c[x, y]^t) &= T([cx, cy]^t) = [2cx + 3cy, cx - cy, cx]^t \\ &= c[2x + 3y, x - y, x]^t = cT(X) \end{aligned}$$

To find the matrix that describes $T$, we write

$$T([x, y]^t) = \begin{bmatrix} 2x + 3y \\ x - y \\ x \end{bmatrix} = x \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The $3 \times 2$ matrix on the right is $A$.

A surprisingly important role is played by the "identity transformation," which could be called the "do nothing" transformation since it "transforms" each $X$ into itself:

**Definition 3.4** *Let $\mathcal{V}$ be a set. The **identity transformation** on $\mathcal{V}$ is the transformation $i_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ defined by $i_{\mathcal{V}}(X) = X$ for all $X \in \mathcal{V}$.*

It is clear that if $\mathcal{V}$ is a vector space, then $i_{\mathcal{V}}$ is a linear transformation. Formula (2.12) on page 117 says that for all $X \in \mathbb{R}^n$

$$IX = X$$

where $I$ is the $n \times n$ matrix that has all the entries on the main diagonal equal to 1 and all other entries equal to 0 (the $n \times n$ identity matrix). Hence, $I$ is the matrix that describes the identity transformation on $\mathbb{R}^n$.

Linearity can at times be used to compute values for a linear transformation without having a formula for the transformation, as the next example illustrates.

### EXAMPLE 3.3

Let $A$ be a $2 \times 3$ matrix such that multiplication by $A$ transforms $X_1 = [4, 7, 3]^t$ onto $[1, 3]^t$, $X_2 = [1, 1, 0]^t$ onto $[1, 4]^t$, and $X_3 = [1, 0, 0]^t$ onto $[1, 1]^t$. Determine what multiplication by $A$ transforms $[1, 2, 1]^t$ to.

*Solution.* If we can write $[1, 2, 1]^t$ as a linear combination of $X_1$, $X_2$, and $X_3$, then we may compute its image using the linearity properties. Hence, we solve the system of equations defined by the following vector equality:

$$[1, 2, 1]^t = x[4, 7, 3]^t + y[1, 1, 0]^t + z[1, 0, 0]^t$$

finding that $x = \frac{1}{3}$, $y = -\frac{1}{3}$, and $z = 0$. Hence,

$$[1, 2, 1]^t = \tfrac{1}{3}[4, 7, 3]^t - \tfrac{1}{3}[1, 1, 0]^t$$

Next, we multiply both sides of this equality on the left by $A$:

$$\begin{aligned} A[1, 2, 1]^t &= A\left(\tfrac{1}{3}[4, 7, 3]^t - \tfrac{1}{3}[1, 1, 0]^t\right) = \tfrac{1}{3}A[4, 7, 3]^t - \tfrac{1}{3}A[1, 1, 0]^t \\ &= \tfrac{1}{3}[1, 3]^t - \tfrac{1}{3}[1, 4]^t = \left[0, -\tfrac{1}{3}\right]^t \end{aligned}$$

The last vector on the right is the image.

The reader who has studied calculus has already seen the linearity properties. Consider, for example, how one differentiates

$$f(x) = 3 \sin x + 7x^3$$

We write

$$\begin{aligned} f'(x) &= (3 \sin x + 7x^3)' \\ &= 3(\sin x)' + 7(x^3)' \\ &= 3 \cos x + 21x^2 \end{aligned}$$

The properties of the derivative we used were

$$\begin{aligned} (f(x) + g(x))' &= f'(x) + g'(x) \\ (cf(x))' &= cf'(x) \end{aligned}$$

*These are just the linearity properties.* Differentiation is a linear transformation from the vector space $C^\infty(\mathbb{R})$ of infinitely differentiable functions on $\mathbb{R}$ into itself.

The definite integral also has linearity properties:

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

Hence, the integral from $a$ to $b$ defines a linear transformation, in this case from the vector space $C([a, b])$ of continuous functions on the closed interval $[a, b]$ into $\mathbb{R}$.

The exercises discuss other linear transformations on function spaces.

**True-False Questions**: Justify your answers.

**3.1** A linear transformation of $\mathbb{R}^2$ into $\mathbb{R}^2$ that transforms $[1, 2]^t$ to $[7, 3]^t$ and $[3, 4]^t$ to $[-1, 1]^t$ will also transform $[5, 8]^t$ to $[13, 7]^t$.

**3.2** It is impossible for a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$ to transform a parallelogram onto a pentagon.

**3.3** It is impossible for a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$ to transform a parallelogram onto a square.

**3.4** It is impossible for a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$ to transform a parallelogram onto a line segment.

**3.5** All transformations of $\mathbb{R}^2$ into $\mathbb{R}^2$ transform line segments onto line segments.

**3.6** Suppose that $T$ is a linear transformation of $\mathbb{R}^2$ into itself and I know what $T$ transforms $[1, 1]^t$ and $[2, 3]^t$ to. Then, I can compute the effect of $T$ on any vector.

**3.7** The transformation $T$ on the set of all continuous functions that is defined by

$$T(f) = 3\int_0^4 f(t)\cos t\,dt$$

is a linear transformation.

**3.8** The transformation $T$ on the set of all continuous functions that is defined by

$$T(f) = 3\int_0^4 (f(t))^2 \cos t\,dt$$

is a linear transformation.

**3.9** The transformation $T$ on the set of all continuous functions that is defined by

$$T(f) = f(1)$$

is a linear transformation.

**3.10** The transformation $T$ on the set of all continuous functions that is defined by

$$T(f) = 1$$

is a linear transformation.

**3.11** The transformation $T$ on the set of all differentiable functions that is defined by

$$T(f) = \frac{df}{dx} + 1$$

is a linear transformation.

## EXERCISES

**3.1** ✓✓Indicate on a graph the image of the square in Example 3.1 on page 152 under multiplication by each of the following matrices.

**(a)** $M = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$ **(b)** $M = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

**(c)** $M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ **(d)** $M = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

**3.2** Repeat Exercise 3.1 for the parallelogram with verticies $[1, 1]^t$, $[1, 2]^t$, $[2, 2]^t$, and $[2, 3]^t$. Sketch both the parallelogram and its image.

**3.3** ✓✓Let

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Show that the image of the circle $x^2 + y^2 = 1$ under multiplication by $A$ is the ellipse pictured in Figure 3.3. [*Hint:* It is easily computed that $A([x, y]^t) = [2x, 3y]^t$. Let $u = 2x$ and $v = 3y$ . If $x$ and $y$ lie on the circle, what equation must $u$ and $v$ satisfy?]

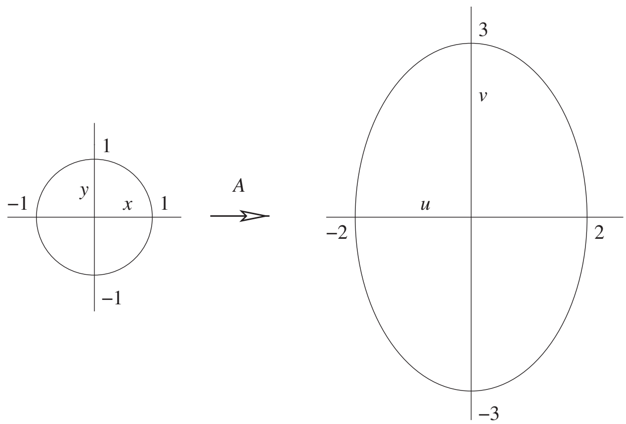



**FIGURE 3.3** Exercise 3.3.

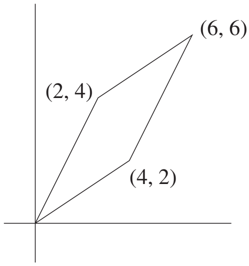



**FIGURE 3.4** Exercises 3.5 and 3.6.

**3.4** Indicate on a graph the image of all $\mathbb{R}^2$ under multiplication by the matrix from Exercise 3.1.d. How, geometrically, would you describe the image of the general point $[x, y]^t$?

On the same graph, draw the circle $x^2 + y^2 = 1$ and indicate what you would expect its image to be. A transformation is said to be "two-to-one" if every point in the image is the image of exactly two points in the domain. Is this transformation two-to-one on the circle? Explain.

**3.5** ✓✓Find a matrix $A$ such that multiplication by $A$ transforms the square in Figure 3.2 on page 152 onto the parallelogram indicated in Figure 3.4.

**3.6** ✓Find a matrix $B$ such that multiplication by $B$ transforms the parallelogram in Figure 3.4 onto the square in Figure 3.2 on page 152.

**3.7** Find a $2 \times 2$ matrix $A$ such that multiplication by $A$ transforms the parallelogram on the left in Figure 3.5 onto the parallelogram on the right. How many possible answers does this question admit?

**3.8** ✓✓Show (using a diagram) that the transformation which reflects points in $\mathbb{R}^2$ on the $x$ axis is linear. Find a matrix that describes this transformation.

**3.9** Describe geometrically the effect of the transformation of $\mathbb{R}^3$ into $\mathbb{R}^3$ defined by multiplication by the following matrices. [*Hint:* For a) compare the effect of multiplying points of the form $[0, y, z]^t$ by $R^x_\psi$ with that of multiplying $[y, z]^t$ by the matrix $R_\psi$ from formula (3.1) on page 150. Next compute how points

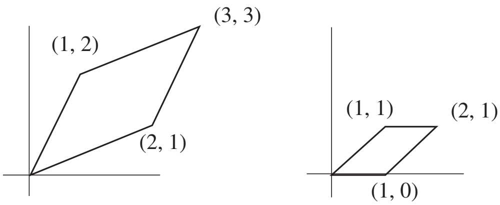



**FIGURE 3.5** Exercise 3.7.

of the form $[x, 0, 0]^t$ transform under multiplication by $R_\psi^x$. Reason similarly for (b) and (c).]

**(a)** $R_\psi^x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}$ **(b)** $R_\psi^y = \begin{bmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{bmatrix}$

**(c)** ✓✓ $R_\psi^z = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**3.10** ✓✓Consider the points

$$X_1 = [1, 1]^t, \quad X_2 = [2, 2]^t$$
$$Y_1 = [4, 5]^t, \quad Y_2 = [5, 6]^t$$

Is it possible to find a $2 \times 2$ matrix $A$ for which multiplication by $A$ transforms $X_1$ into $Y_1$ and $X_2$ into $Y_2$? [*Hint:* How are $X_1$ and $X_2$ related?]

**3.11** Consider the points

$$X_1 = [1, 1]^t, \quad X_2 = [2, 3]^t, \quad X_3 = [3, 4]^t$$
$$Y_1 = [1, 0]^t, \quad Y_2 = [0, 1]^t, \quad Y_3 = [2, 2]^t$$

Is it possible to find a $2 \times 2$ matrix $A$ for which multiplication by $A$ transforms $X_1 \to Y_1, X_2 \to Y_2$, and $X_3 \to Y_3$?

**3.12** Let $A$ be a $2 \times 3$ matrix for which multiplication by $A$ transforms $[2, 1, 1]^t$ onto $[1, 1]^t$ and $[1, 1, 1]^t$ onto $[1, -2]^t$.

**(a)** ✓✓Determine, if possible, what multiplication by $A$ transforms $[0, 1, 1]^t$ to. [*Hint:* How does $[0, 1, 1]^t$ relate to $[1, 1, 1]^t$ and $[2, 1, 1]^t$?])

**(b)** Suppose that it is also true that multiplication by $A$ transforms $[0, 0, 1]^t$ onto $[3, -5]^t$. Determine what $A$ transforms $[2, 2, 3]^t$ to.

**(c)** ✓✓Find $A$. [*Hint*: The columns of $A$ are $AI_1$, $AI_2$, and $AI_3$.]

**3.13** Although it seems silly, it is possible to do elementary row operations on $n \times 1$ matrices. Every such operation defines a transformation of $\mathbb{R}^n$ into $\mathbb{R}^n$. For example, if we define a transformation of $\mathbb{R}^3$ into $\mathbb{R}^3$ by "add twice row 1 to row 3," this transformation transforms

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \begin{bmatrix} x_1 \\ x_2 \\ x_3 + 2x_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Since this transformation is described by a matrix, we see that our elementary row operation defines a *linear* transformation. A transformation defined by a single elementary row operation is called an **elementary transformation** and the matrix that describes such a transformation is called an **elementary matrix**.

Find matrices that describe the following elementary row operations on $\mathbb{R}^n$ for the given value of $n$.

**(a)** Add twice row 3 to row 2 in $\mathbb{R}^4$. **(b)** Multiply row 2 by 17 in $\mathbb{R}^3$.

**(c)** Interchange rows 1 and 2 in $\mathbb{R}^4$.

**3.14** ✓Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T([x, y]^t) = [x, y(1 + x^2)]^t$$

**(a)** Show that $T$ is not a linear transformation by finding two specific points (reader's choice) $X$ and $Y$ such that $T(X + Y) \neq T(X) + T(Y)$.

**(b)** On a graph, draw the lines $x = n$ and $y = m$ for $n$ and $m$ having the values 0, 1, 2. (There will be six lines.) On a separate graph, draw the image of these lines under $T$. The purpose here is to understand how $T$ transforms squares. You will notice that the squares become increasingly "warped" as you move away from the origin.

**3.15** Prove that each of the transformations in (a)–(f) is linear, and find a matrix $A$ such that $T(X) = AX$.

**(a)** ✓✓ $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T([x, y, z]^t) = [2x + 3y - 7z, 0]^t$

**(b)** $T : \mathbb{R}^2 \to \mathbb{R}^3$, $T([x, y]^t) = [x + y, x - y, x + 2y]^t$

**(c)** ✓ $T : \mathbb{R}^2 \to \mathbb{R}^3$, $T([x, y]^t) = [2x + 3y, 2y + x, -3x - y]^t$

**(d)** $T : \mathbb{R}^5 \to \mathbb{R}^3$, $T([x_1, x_2, x_3, x_4, x_5]^t) = [x_1, x_2, x_3]^t$

**(e)** $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T([x, y, z]^t) = [2x + 3y - 7z, y - z]^t$

**(f)** $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T([x, y, z]^t) = [z, y, x]^t$

**3.16** ✓What is the image of all $\mathbb{R}^3$ under the transformation $T$ in Exercise 3.15.a? The equation $2x + 3y - 7z = 0$ describes a plane in $\mathbb{R}^3$. What is the image of this plane under $T$? How, geometrically, would you describe the set of $[x, y, z]^t$ such that $T([x, y, z]^t) = [-3, 0]^t$?

**3.17** What matrix describes rotation in $\mathbb{R}^2$ *clockwise* by $\theta$ radians?

**3.18** In the text, it was shown that a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ transforms line segments onto line segments. Prove that such a transformation will in fact transform midpoints onto midpoints. [The midpoint of the segment from $X$ to $Y$ is $(X + Y)/2$.]

**3.19** Let $C([-1, 1])$ be the vector space of continuous functions on the interval $[-1, 1]$ and define $T : C([-1, 1]) \to \mathbb{R}$ by

$$T(f) = \int_{-1}^{1} f(x)\, dx$$

Thus, for example,

$$T(x^3) = \int_{-1}^{1} x^3 dx = \frac{x^4}{4} \Big|_{-1}^{1} = 0$$

**(a)** Let $f(x) = e^x$ and $g(x) = x^2$ for $-1 \le x \le 1$. Show by direct computation that

$$T(2f + 3g) = 2T(f) + 3T(g)$$

**(b)** Prove that $T$ is a linear transformation.

**(c)** Let $S$ be the transformation defined by

$$S(f) = \int_{-1}^{1} (f(x))^2 \, dx$$

Show by explicit computation that $S(2e^x) \neq 2S(e^x)$. Is $S$ linear?

**(d)** For the transformation $S$ in part (c), find two functions $f$ and $g$, $f \neq g$, such that $S(f + g) \neq S(f) + S(g)$.

**(e)** Let $U$ be defined by

$$U(f) = \int_{-1}^{1} f(x)x \, dx$$

Is $U$ linear? Prove your answer.

**3.20** Let $D : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ be defined by

$$D(f) = \frac{df}{dx}$$

**(a)** Let $f(x) = e^x$ and $g(x) = x^2$. Show by direct computation that

$$D(2f + 3g) = 2D(f) + 3D(g)$$

**(b)** Prove that $T$ is a linear transformation.

**(c)** Let $U : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ be defined by

$$U(f) = D(xf)$$

Is $U$ linear? If so, prove it. If not, show by an explicit example that one of the linearity properties fails.

**3.21** Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces. Let $T : \mathcal{V} \to \mathcal{W}$ be a linear transformation. Prove that $T(0) = 0$. (*Note*: You should not assume that $T$ is a matrix transformation. Instead, think about the property that in any vector space $0, X = 0$.)

**3.22** ✓✓Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces. Let $T : \mathcal{V} \to \mathcal{W}$ be a linear transformation. Let $\{X_1, X_2, X_3\}$ be a set of three linearly dependent elements in $\mathcal{V}$. Prove that the set $\{T(X_1), T(X_2), T(X_3)\}$ is also linearly dependent.

**3.23** ✓✓Let $T : \mathcal{V} \to \mathcal{W}$ be a linear transformation between two vector spaces. We define the nullspace of $T$ to be the set of $X$ such that $T(X) = 0$. Show that the nullspace of $T$ is a subspace of $\mathcal{V}$.

**3.24** Let $T : \mathcal{V} \to \mathcal{W}$ be a linear transformation between two vector spaces. We define the **image** $T(\mathcal{V})$ by $T(\mathcal{V}) = \{T(X) \mid X \in \mathcal{V}\}$. Thus, the image is the set of $Y \in \mathcal{W}$ such that the equation $T(X) = Y$ is solvable for $X \in \mathcal{V}$. Show that the image of $T$ is a subspace of $\mathcal{V}$.

**3.25** Let $T : \mathcal{V} \to \mathcal{W}$ be a linear transformation between two vector spaces. Let $Y_o = T(X_o)$ for some $X_o \in \mathcal{W}$. Prove that $T(X) = Y_o$ if and only if $X = X_o + Z$, where $Z \in \text{null } T$. [*Hint:* See the proof of the translation theorem (Theorem 1.10 on page 80.)]

### 3.1.1 Computer Projects

Here we present a very simple-minded program for drawing stick figures in MATLAB. To enter this into MATLAB, select the File box and pull down the menu, selecting New M-File. A new window will open. Type the commands below into this window. When finished, select File and then Save As. Save it as stick.m. Then select Close from the File menu. [*Note:* You might want to save your file to a disk as this program will be used again in Section 3.2.]

To run stick, simply type stick in the MATLAB command window. You should see a blank graph with axes $0 \le x \le 1$ and $0 \le y \le 1$. You place points on the graph by pointing to where you want each one and clicking with the (left) mouse button. Each subsequent point will be connected to the previous one by a line. At the last point of your figure, use the right mouse button instead of the left. This will cause the program to end. (On a Mac, hold down the shift button while clicking the mouse.)

The program saves each of your points, in the order that you placed them, as the columns of a large matrix called FIG.

```
cla; hold on; axis([0 1 0 1]); grid on;
FIG=[0;0]; i=1; z=1;
while z==1,
   [x,y,z]=ginput(1);
   FIG(:,i)=[x,y]';
   plot(FIG(1,:),FIG(2,:));
   i=i+1;
end
```

## EXERCISES

1. Use stick to draw the first letter of your initials. It is necessary to go over some lines twice to produce some letters (such as B). After exiting stick, save your work in an appropriately named matrix. For example, my initials are RCP. I use stick to draw an R. After exiting stick, I save my work in a matrix called R by entering R=FIG at the MATLAB prompt.

2. To see the fruits of your labor from Exercise 1, you can use the MATLAB plot command. For example, to see my (presumably beautiful) R, I write

```
cla;
plot(R(1,:),R(2,:))
```

   This tells MATLAB to first clear the figure and then plot the first row of R as $x$ values and the second as $y$ values. Plot your first initial. [*Note*: If you have exited the Figure window, you will need to first enter the line hold on;axis equal in order to get the right proportions for your initial.]

3. The transformation defined by the matrix in Example 1 above is a shear along the $y$ axis. To see what a shear would do to my R, I enter the matrix M from Example 1 into MATLAB and set SR=M∗R. This has the effect of multiplying each column of R by M and storing in a matrix SR. I then plot SR just as in Exercise 2. Plot the image of your first initial under the shear. [*Note:* You will need to enlarge the viewing window to see your output. The command axis([0,2,0,2]) should make it large enough.]

4. Use a rotation matrix to plot the image of your first initial under rotation by 20 degrees counterclockwise. [*Notes*: MATLAB works in radians, so you will need to convert from degrees to radians. You might want to read the MATLAB help entry for cos. See the note in Exercise 3 about enlarging the viewing window.]

5. Use stick to create appropriately named matrices to represent your other initials. For example, I would draw a C and save it in a matrix called C, and draw a P and save it in matrix P. Then get MATLAB to plot each of the letters you created on the same graph: quite a mess.

   Try to find a way of shifting the letters over so that your initials appear in order. [*Note*: If M is a matrix in MATLAB, then the command M=M+1 will add 1 onto every entry of M. (You will need to change the size of the viewing window. See the note in Exercise 3.)]

6. Let $S$ be the transformation of $\mathbb{R}^2$ into itself defined by stipulating that $S(X)$ is the result of shifting $X$ one unit to the right. Show graphically that $S$ is not linear. Specifically, use the first letter of your initial to show that $S(2X) \neq 2S(X)$.

7. For items (a)–(d), find a matrix $M$ for which multiplication by $M$:

   **(a)** Flips your initials upside down

   **(b)** Flips your initials left to right

**(c)** Rotates your initials by 20 degrees

**(d)** Shears your initials along the $x$-axis

For each of your matrices, plot the effect of the matrix on your *first* initial.

**8.** Plot the effects of the following transformations on your *first* initial:

**(a)** A shear along the $x$ axis followed by a rotation of 20 degrees

**(b)** A rotation of 20 degrees followed by a shear along the $x$ axis

**(c)** A shear along the $x$ axis followed by a shear along the $y$ axis

**(d)** A shear along the $y$ axis followed by a shear along the $x$ axis

## 3.2 MATRIX MULTIPLICATION (COMPOSITION)

In applications, one sometimes needs to construct transformations that transform one specific set onto another.

### EXAMPLE 3.4

Find a matrix $C$ such that multiplication by $C$ transforms the circle $x^2 + y^2 = 1$ onto the ellipse indicated in Figure 3.6.

*Solution.* In Exercise 3.3 on page 157 we found that multiplication by the following matrix $A$ "stretches" the circle $x^2 + y^2 = 1$ onto the first ellipse in Figure 3.7.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

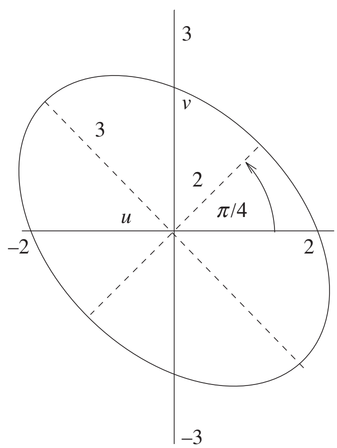



**FIGURE 3.6** A rotated ellipse.

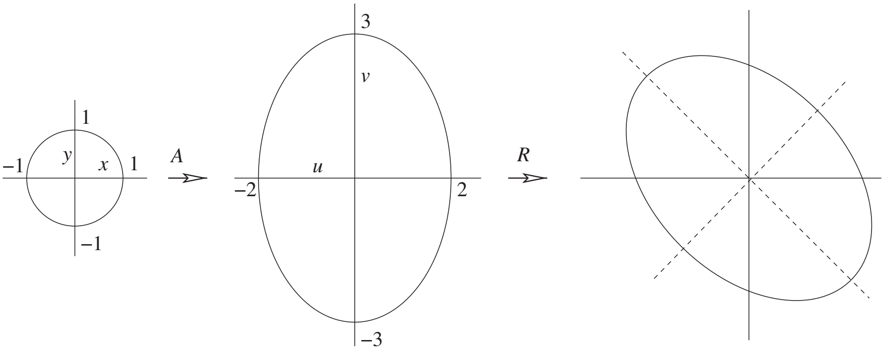



**FIGURE 3.7** A composite transformation.

From formula (3.1) on page 150, multiplication by

$$R_{\pi/4} = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

rotates this ellipse onto the desired ellipse. Thus, the transformation that transforms $X$ into

$$Y = R_{\pi/4}(AX)$$

will transform the circle onto the rotated ellipse (Figure 3.7).

To express this as a matrix transformation, let the columns of $A$ be $A_1$ and $A_2$ and let $X = [x_1, x_2]^t$. Then,

$$Y = R_{\pi/4}(AX) = R_{\pi/4}(x_1 A_1 + x_2 A_2) = x_1 R_{\pi/4} A_1 + x_2 R_{\pi/4} A_2$$

This formula describes multiplication by the matrix $C$ whose columns are $R_{\pi/4}A_1$ and $R_{\pi/4}A_2$. Explicitly, then, we have

$$C = \left[ \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right] = \begin{bmatrix} \sqrt{2} & -3\sqrt{2}/2 \\ \sqrt{2} & 3\sqrt{2}/2 \end{bmatrix}$$

In words, we would say that $C$ is "$R_{\pi/4}$ times the columns of $A$."

In Example 1, the desired transformation was constructed by successively applying two transformations: first we stretched the circle; then we rotated the result. The process of successively applying transformations is called **composition**, and the resulting transformation is called the **composite transformation**.

$$\begin{tikzcd} \mathbb{R}^p \arrow[r, "T_B"'] \arrow[rr, bend left, "T_A \circ T_B"] & \mathbb{R}^n \arrow[r, "T_A"'] & \mathbb{R}^m \end{tikzcd}$$

**FIGURE 3.8** Composite transformations.

**Definition 3.5** *Let $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ be sets and let $T : \mathcal{U} \to \mathcal{V}$ and $S : \mathcal{V} \to \mathcal{W}$ be transformations. Then the **composite** of $S$ with $T$ is the transformation $S \circ T$ of $\mathcal{U}$ into $\mathcal{W}$ defined by*

$$S \circ T(X) = S(T(X))$$

It is crucially important that the composite of two matrix transformations (if defined) is itself a matrix transformation. The proof is virtually identical to the computations done in Example 1. Specifically, let $A$ and $B$ be matrices of size $m \times n$ and $n \times p$, respectively. Then $T_B : \mathbb{R}^p \to \mathbb{R}^n$ and $T_A : \mathbb{R}^n \to \mathbb{R}^m$ so $T_A \circ T_B$ is defined and transforms $\mathbb{R}^p$ into $\mathbb{R}^n$ as in Figure 3.8.

If we successively multiply $X = [x_1, x_2, \dots, x_n]^t$ by $B$ and then $A$, we obtain

$$A(BX) = A(x_1B_1 + x_2B_2 + \cdots + x_nB_n) = x_1AB_1 + x_2AB_2 + \cdots + x_nAB_n$$

where the $B_i$ are the columns of $B$. The expression on the right is the product of $X$ with the matrix whose $i$th column is $AB_i$. These comments inspire the following definition and prove the following proposition.

**Definition 3.6** *Let $A$ and $B$ be $m \times n$ and $n \times p$ matrices, respectively. Write $B = [B_1, B_2, \dots, B_p]$, where the $B_i$ are the columns of $B$. Then the **matrix product** of $A$ and $B$ is the $m \times p$ matrix*

$$AB = [AB_1, AB_2, \cdots, AB_p] \tag{3.3}$$

*In words, $AB$ is "$A$ times the columns of $B$."*

**Proposition 3.2** *Let $A$ and $B$ be, respectively, $m \times n$ and $n \times p$ matrices. Then for all $X \in \mathbb{R}^q$*

$$A(BX) = (AB)X$$

It is important to note that the product of an $m \times n$ and a $q \times p$ matrix is only defined if $n = q$. The product of an $m \times n$ with an $n \times p$ matrix has size $m \times p$:

$$\begin{array}{cccc} A & B & = & AB \\ m \times n & n \times p & & m \times p \end{array}$$

It follows from formula (1.43) on page 78 that

$$\begin{aligned}(AB)_{ij} &= (AB_j)_i \\ &= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} \\ &= A_iB_j\end{aligned} \tag{3.4}$$

where $A_i$ is the $i$th row of $A$ and $B_j$ is the $j$th column of $B$.

### EXAMPLE 3.5

Compute $AB$, where $A$ and $B$ are as follows:

(a) $$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

(b) $$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -3 & 3 \\ 6 & -5 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 3 & -1 \\ 7 & 1 \end{bmatrix}$$

*Solution.* From formula (3.3), the product in part (a) is

$$AB = \left[ A\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad A\begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad A\begin{bmatrix} 3 \\ 6 \end{bmatrix} \right] = \begin{bmatrix} 9 & 12 & 15 \\ 14 & 19 & 24 \end{bmatrix}$$

Similarly, for part (b),

$$AB = \left[ A\begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}, \quad A\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} 14 & -1 \\ 14 & 6 \\ 19 & 9 \end{bmatrix}$$

Proposition 3.2 has an interesting consequence. Recall that the identity transformation $i_{\mathcal{V}}$ on a set $\mathcal{V}$ transforms each $X$ into itself. It follows that if $T : \mathcal{V} \to \mathcal{W}$ is any transformation, then

$$T \circ i_{\mathcal{V}} = T = i_{\mathcal{W}} \circ T$$

The $n \times n$ identity matrix $I$ is the matrix for the identity transformation on $\mathbb{R}^n$. It follows from Proposition 3.2 that for any $m \times n$ matrix $A$

$$AI = A = IA$$

where the first $I$ is the $m \times m$ identity matrix and the second is the $n \times n$ identity matrix. This equation says that the identity matrix behaves like the number 1 in arithmetic. Thus, for example, without doing any arithmetic, we can immediately conclude that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

We have described $AB$ as $A$ times the columns of $B$. There is another description of matrix multiplication that can be demonstrated using the matrix product in part (a) of Example 3.5. The product of the first row of $A$ with $B$ is

$$[1,2] \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \left[ [1,2] \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad [1,2] \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad [1,2] \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right] = [9, 12, 15]$$

which is the first row of $AB$.

Similarly, the product of the second row of $A$ by $B$ is the second row of $AB$:

$$[2,3] \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = [14, 19, 24]$$

This example demonstrates the general principle that *the rows of AB are the rows of A times B*.

**Theorem 3.2** *Let A and B be $m \times n$ and $n \times q$ matrices, respectively, and let $A_i$ be the rows of A. Then*

$$AB = \begin{bmatrix} A_1 B \\ A_2 B \\ \vdots \\ A_m B \end{bmatrix}$$

*Proof.* Let $B = [B_1, \dots, B_q]$, where the $B_j$ are columns. From formula (3.4),

$$\begin{aligned} A_i[B_1, \cdots, B_q] &= [A_i B_1, \cdots, A_i B_q] \\ &= [(AB)_{i1}, \cdots, (AB)_{iq}] \end{aligned}$$

which is the $i$th row of $AB$, proving our theorem. □

The rank of $AB$ is the dimension of both its column space and its row space. The following proposition follows from the observation that the columns of $AB$ are $A$ times the columns of $B$ and the rows of $AB$ are $B$ times the rows of $A$.

**Theorem 3.3** (Rank of Products). *Let $A$ and $B$ be matrices such that $AB$ is defined. Then*

$$\begin{aligned} \operatorname{rank}(AB) &\leq \operatorname{rank} B \\ \operatorname{rank}(AB) &\leq \operatorname{rank} A \end{aligned}$$

*Proof.* Let $B = [B_1, B_2, \ldots, B_n]$, where the $B_i$ are the columns of $B$. There are indices $i_1, i_2, \ldots, i_r$, where $r = \text{rank } B$, such that $B_{i_1}, B_{i_2}, \ldots, B_{i_r}$ form a basis for the column space of $B$. Then, for each $j$, there are scalars $c_i$ (which also depend on $j$) such that

$$B_j = c_1 B_{i_1} + c_2 B_{i_2} + \cdots + c_r B_{i_r}$$

Hence,

$$AB_j = c_1 AB_{i_1} + c_2 AB_{i_2} + \cdots + c_r AB_{i_r}$$

implying that the $AB_{i_j}$, $1 \leq j \leq r$, span the column space of $AB$. The first inequality follows since the dimension is the smallest number of elements it takes to span the space.

The second inequality is proved similarly using the observation that the rank of $AB$ is the dimension of its row space. The proof is left as an exercise. □

It is, of course, possible to multiply elements of $M(1, n)$ with elements of $M(n, 1)$. Thus, for example,

$$[a_1, a_2, \ldots, a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \tag{3.5}$$

The final result is a $1 \times 1$ matrix, which we interpret as a number. Note that this is the same as

$$[b_1, b_2, \ldots, b_n] \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = b_1 a_1 + b_2 a_2 + \cdots + b_n a_n$$

The property demonstrated here is not exactly commutativity. Instead, what we are saying is that if $A$ is a row and $B$ is a column of the same length as $A$, then

$$(AB)^t = B^t A^t$$

This observation implies the following more general result:

**Theorem 3.4** *If A and B are matrices such that AB is defined, then*

$$(AB)^t = B^t A^t \tag{3.6}$$

*Proof.* For any matrix $C$, $(C^t)_{ij} = C_{ji}$. Hence, from formula (3.4) and the preceding remarks,

$$\begin{aligned}((AB)^t)_{ij} &= (AB)_{ji} \\ &= A_j B_i = B_i^t A_j^t = (B^t A^t)_{ij}\end{aligned}$$

where $A_j$ is the $j$th row of $A$ and $B_i$ is the $i$th column of $B$. This proves our theorem. □

One of the most important algebraic properties of matrix multiplication is the following.

**Theorem 3.5** (Associative Law). *Let A, B, and C be matrices such that the product A(BC) is defined. Then A(BC) = (AB)C.*

*Proof.* Let $C = [C_1, C_2, \dots, C_n]$, where $C_i$ are the columns of $C$. Then, from Proposition 3.2,

$$\begin{aligned}A(BC) &= A[BC_1, BC_2, \dots, BC_n] \\ &= [A(BC_1), A(BC_2), \dots, A(BC_n)] \\ &= [(AB)C_1, (AB)C_2, \dots, (AB)C_n] \\ &= (AB)C\end{aligned}$$

□

Matrix multiplication also has some important distributive laws. The left distributive law follows directly from the linearity properties of matrix multiplication and the right distributive law is a consequence of the left, together with Theorem 3.4. We leave the details as an exercise. It is necessary to distinguish between the left and right distributive laws because, as the reader will see in the exercises, *matrix multiplication is not commutative*.

**Proposition 3.3** *Let A and B be $m \times n$ matrices and let C and D be $n \times q$ matrices. Let a be a scalar. Then*

$$\begin{aligned}A(C + D) &= AC + AD && \textit{(left distributive law)} \\ (A + B)C &= AC + BC && \textit{(right distributive law)} \\ A(aC) &= aAC && \textit{(scalar law)}\end{aligned}$$

## Partitioned Matrices

Theorem 3.2 is an instance of a more general result.

### EXAMPLE 3.6

Compute $CD$, where

$$C = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 \\ 4 & 5 & 6 & 0 & 1 \\ 0 & 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 9 & 10 \end{bmatrix}$$

*Solution.* At first glance, this appears to be a long, messy computation. However, $C$ and $D$ are both made up of four simpler matrices, which we indicate by drawing vertical and horizontal "partition" lines:

$$C = \left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \\ \hdashline 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{array}\right], \quad D = \left[\begin{array}{ccc:cc} 1 & 2 & 3 & 1 & 0 \\ 4 & 5 & 6 & 0 & 1 \\ \hdashline 0 & 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 9 & 10 \end{array}\right]$$

Note that the matrices in the upper left-hand corners of $C$ and $D$ are, respectively, the matrices $A$ and $B$ from Example 3.5, part (a), whose product we already know. In fact, schematically,

$$C = \begin{bmatrix} A & I \\ 0 & -3I \end{bmatrix}, \quad D = \begin{bmatrix} B & I \\ 0 & E \end{bmatrix} \tag{3.7}$$

where $I$ is the $2 \times 2$ identity matrix and

$$E = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}$$

The entries of the matrices in formula (3.7) are themselves matrices. If they were numbers, then we could write

$$\begin{aligned} CD &= \begin{bmatrix} A & I \\ 0 & -3I \end{bmatrix} \begin{bmatrix} B & I \\ 0 & E \end{bmatrix} \\ &= \begin{bmatrix} AB + I0 & AI + IE \\ 0B + (-3I)0 & 0I + (-3I)E \end{bmatrix} \end{aligned} \tag{3.8}$$

Remarkably, *if we interpret these products and sums as matrix operations*, we get the correct answer for $CD$. Hence, copying $AB$ from Example 3.5 and computing $A+E$ and $-3E$, we find

$$
\begin{aligned}
CD &= \begin{bmatrix} AB & A+E \\ 0 & -3E \end{bmatrix} \\
&= \left[\begin{array}{ccc:cc} 9 & 12 & 15 & 8 & 10 \\ 14 & 19 & 24 & 11 & 13 \\ \hdashline 0 & 0 & 0 & -21 & -24 \\ 0 & 0 & 0 & -27 & -30 \end{array}\right]
\end{aligned}
$$

where it is understood that we would erase the partition lines before reporting our answer.

Similar comments hold in general. Drawing vertical and horizontal lines through a matrix is referred to as **partitioning** the matrix. Thus, for example, the matrix (3.9) represents a partitioning of the following matrix:

$$
\begin{bmatrix} 1 & 3 & 5 & -3 & 4 & 0 & 5 \\ 0 & 2 & -1 & 1 & 6 & 5 & -2 \\ 11 & 9 & 8 & -2 & -7 & 0 & -2 \\ 6 & 5 & 4 & 1 & 1 & 7 & 2 \end{bmatrix}
$$

$$
\left[\begin{array}{cc:ccc:cc} 1 & 3 & 5 & -3 & 4 & 0 & 5 \\ \hdashline 0 & 2 & -1 & 1 & 6 & 5 & -2 \\ \hdashline 11 & 9 & 8 & -2 & -7 & 0 & -2 \\ 6 & 5 & 4 & 1 & 1 & 7 & 2 \end{array}\right] \tag{3.9}
$$

As in Example 3.6, partitioning a matrix allows us to represent it as a matrix of matrices. The following theorem (which we do not prove) states that, just as in Example 3.6, we may compute matrix products by multiplying the partitioned matrices.

**Theorem 3.6** *Let $A$ and $B$ be matrices for which $C = AB$ is defined. Let $[A_{ij}]$ and $[B_{ij}]$ be, respectively, partitionings of $A$ and $B$ into $m \times n$ and $n \times p$ matrices of matrices. Assume that the size of the partitions is such that $A_{ij}B_{jk}$ is defined for all $i, j, k$. Then $[C_{ij}]$ is a partitioning of $C$, where*

$$
C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{in}B_{nj}
$$

## Computational Issues: Parallel Computing

Partitioning large matrices allows us to use "team work" to multiply large matrices. For example, we could have used a separate team to compute each of the 4 partitions in equation (3.8). For a large matrix, this could reduce the time to compute the product by a factor of four. Humans rarely divide into teams to compute products — but computers do. Using "parallel processing," computers routinely multiply large matrices by partitioning them and using different parts of the computer simultaneously to serve as the teams for computing each partition.

**True-False Questions**: Justify your answers.

**3.12** If $A$ and $B$ are $2 \times 2$ matrices, $(AB)^2 = A^2B^2$.

**3.13** Let $A$, $B$, and $C$ be, respectively, $3 \times 2$, $2 \times 3$, and $3 \times 3$ matrices. Then $A(BC)$, $B(CA)$, and $B(AC)$ are all defined.

**3.14** Let $A = R_{\pi/2}$ be the matrix that describes rotation by $\pi/2$ radians [formula (3.1) on page 150]. Then $A^4 = I$, where $I$ is the $2 \times 2$ identity matrix.

**3.15** Assume that $A$ and $B$ are matrices such that $AB$ is defined and $B$ has a column that has all its entries equal to zero. Then one of the columns of $AB$ also has all its entries equal to zero.

**3.16** Assume that $A$ and $B$ are matrices such that $AB$ is defined and $A$ has a row that has all its entries equal to zero. Then one of the rows of $AB$ also has all its entries equal to zero.

**3.17** Assume that $A$ and $B$ are matrices such that $AB$ is defined and $AB$ has a column that has all its entries equal to zero. Then one of the columns of $B$ also has all its entries equal to zero.

**3.18** Assume that $A$ and $B$ are matrices such that $AB$ is defined and the columns of $B$ are linearly dependent. Then the columns of $AB$ are also linearly dependent.

**3.19** Assume that $A$ and $B$ are matrices such that $AB$ is defined and the rows of $A$ are linearly dependent. Then the rows of $AB$ are also linearly dependent.

**3.20** Suppose that matrices $A$ and $B$ satisfy $AB = 0$. Then either $A = 0$ or $B = 0$.

## EXERCISES

*The "unit square" mentioned in some of the exercises below is the square S defined in Example 3.1, page 152.*

**3.26** For the matrices $A$, $B$, and $C$, demonstrate the associative law $A(BC) = (AB)C$ by directly computing the given products in the given orders.

**(a)** ✓✓ $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \end{bmatrix}$

**(b)** $A = \begin{bmatrix} -1 & 0 & 3 \\ 4 & -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ -4 & 1 \\ 0 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

**3.27** For the matrices in part (a) of Exercise 3.26:

**(a)** ✓✓Check that $BA \neq AB$.

**(b)** Is $(CA)B$ defined? $B(AC)$? $A(CB)$?

**(c)** ✓✓Show by direct calculation that $(AB)^t = B^tA^t \neq A^tB^t$.

**3.28** ✓✓What is the analogue of formula (3.6) for a product of three matrices? Prove your formula. [*Hint:* You may use formula (3.6) in your proof.]

**3.29** What is the analogue of formula (3.6) for a product of $n$ matrices?

**3.30** Let $A$ and $B$ be as in Exercise 3.26.a.

**(a)** ✓✓Explicitly write the following matrices, where $U$ is $4 \times 4$ and $V$ is $4 \times 5$:

$$U = \begin{bmatrix} I & A \\ B & 0 \end{bmatrix}, \quad V = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$$

**(b)** Use the answers to Exercises 3.26 and 3.27 to compute $UV$.

**3.31** Is the following formula valid for partitioned matrices? If so, prove it. If not, give a counterexample and state the correct formula.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^t = \begin{bmatrix} A & C \\ B & D \end{bmatrix}$$

**3.32** Prove the left distributive law of Proposition 3.3 on page 170. [*Hint:* Use formula (3.3) on page 166, along with the observation that if $C = [C_1, C_2, \ldots, C_q]$ and $D = [D_1, D_2, \ldots, D_q]$, then $C + D = [C_1 + D_1, C_2 + D_2, \ldots, C_q + D_q]$.]

**3.33** Use formula (3.6) on page 170 along with the left distributive law to prove the right distributive law of Proposition 3.3 on page 170.

**3.34** Define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by the following rule: $T(X)$ is the result of first rotating $X$ counterclockwise by $\pi/6$ radians and then multiplying by

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

**(a)** What is the image of the circle $x^2 + y^2 = 1$ under $T$?

**(b)** What is the image of the unit square under $T$?

**(c)** Find a matrix $B$ such that $T(X) = BX$ for all $X \in \mathbb{R}^2$.

**3.35** ✓✓Define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by the following rule: $T(X)$ is the result of first rotating $X$ counterclockwise by $\pi/4$ radians and then multiplying by

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

**(a)** What is the image of the unit square under $T$?

**(b)** Find a matrix $B$ such that $T(X) = BX$ for all $X \in \mathbb{R}^2$.

**3.36** Suppose that in Exercise 3.35 we multiply first by $A$ and then rotate. Are your answers to parts (a) and (b) different? How?

**3.37** ✓Define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(X) = A(AX)$, where $A$ is the matrix from Exercise 3.35. What is the image of the unit square under $T$? Find a matrix that represents this transformation. What if we multiply $X$ by $A$, $n$ times? Find a matrix that represents this transformation.

**3.38** Define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by the following rule: $T(X)$ is the result of first rotating $X$ counterclockwise about the origin by $\frac{\pi}{4}$ radian, then multiplying by

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

and, finally, rotating the result by $\frac{\pi}{4}$ radian clockwise.

**(a)** What is the image of the circle $x^2 + y^2 = 4$ under $T$?

**(b)** What is the image of the unit square under $T$?

**(c)** Find a matrix $B$ such that $T(X) = BX$ for all $X \in \mathbb{R}^2$.

**3.39** ✓✓Define a transformation $U : \mathbb{R}^2 \to \mathbb{R}^2$ by $U(X) = A(AX)$, where $A$ is the matrix from Exercise 3.34.

**(a)** What is the image of the circle $x^2 + y^2 = 1$ under $U$?

**(b)** Find a matrix $C$ such that $U(X) = CX$ for all $X \in \mathbb{R}^2$.

**3.40** ✓✓Let $S$ be the unit square. I rotate $S$ by $\pi/4$ radians counterclockwise and then reflect the result in the $x$ axis. You first reflect $S$ in the $x$-axis and then rotate.

**(a)** Indicate on separate graphs my result and your result.

**(b)** Compute matrices $M$ and $Y$ that describe my composite transformation and yours.

**3.41** ✓For the given matrix $A$, find a $3 \times 2$ nonzero matrix $B$ such that $AB = 0$. Prove that any such matrix $B$ must have rank 1. [*Hint:* The columns of $B$ belong to the nullspace of $A$.]

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

**3.42** For the given matrix $C$ find a $3 \times 3$, nonzero matrix $B$ such that $CB = 0$. Prove that any such matrix $B$ must have rank 1. *Hint:* See Exercise 3.41.

$$C = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

**3.43** Find a pair of $2 \times 2$ matrices $A$ and $B$ of your own choice such that $AB \neq BA$. *Do not use either of the matrices from Exercise 3.26 or their transposes.* [*Hint:* This is not hard. It is a theorem that with probability 1 any two randomly selected matrices will not commute.]

**3.44** Find all $2 \times 2$ matrices $B$ such that $AB = BA$, where

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

**3.45** ✓✓Find a pair of $2 \times 2$ matrices $A$ and $B$ of your own choice with $A \neq B$ such that $AB = BA$.

**3.46** Find a pair of $2 \times 2$ matrices $A$ and $B$ such that $(A + B)(A + B) \neq A^2 + 2AB + B^2$. Under what conditions would equality hold?

**3.47** ✓✓Find a pair of $2 \times 2$ matrices $A$ and $B$ such that $(A + B)(A - B) \neq A^2 - B^2$. Under what conditions would equality hold?

**3.48** ✓Find a $2 \times 2$ nonzero matrix $A$ such that $A^2 = 0$. [*Hint:* Try making most of the entries equal to zero.]

**3.49** Find a $3 \times 3$ matrix $A$ such that $A^3 = 0$ but $A^2 \neq 0$. [*Hint:* Try making most of the entries equal to zero.]

**3.50** Find an $n \times n$ matrix $A$ such that $A^n = 0$ but $A^{n-1} \neq 0$.

**3.51** ✓✓In an $n \times n$ matrix $A$, the entries of the form $a_{ii}$ form the **main** or **first diagonal**. The entries of the form $a_{i,i+k-1}$ form the $k$**th diagonal**. An $n \times n$ matrix is said to be **diagonal** if its only nonzero entries lie on the main diagonal. Thus, an $n \times n$ diagonal matrix would have the form

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Find eight different $3 \times 3$ diagonal matrices $A$ such that $A^2 = I$, where $I$ is the $3 \times 3$ identity matrix.

**3.52** An $n \times n$ matrix $C$ is said to be **nilpotent** if there is a $k$ such that $C^k = 0$. Prove that $A$ is nilpotent where

$$A = \begin{bmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

**3.53** Let $C$ be an $n \times n$ upper triangular matrix for which all of its entries on the main diagonal equal 0. Thus $C$ has the form

$$C = \begin{bmatrix} 0 & c_{12} & c_{13} & \cdots & c_{1,n} \\ 0 & 0 & c_{23} & \cdots & c_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & c_{n-1,n} \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

**(a)** Suppose that $X_k = [x_1, x_2, \ldots, x_k, 0, \ldots, 0]^t \in \mathbb{R}^n$. Prove that $CX_k$ has has its last $n-k+1$ entries equal to 0—i.e., $CX_k = [y_1, y_2, \ldots, y_{k-1}, 0, 0, \ldots, 0]^t$ for some $y_j \in \mathbb{R}$.

**(b)** Suppose that $C$ is as above and $D$ is an $n \times n$ matrix such that all of the entries above the $k$th diagonal are 0. (See Exercise 3.51.) Prove that all of the entries of $CD$ below the $(k+1)$st diagonal are 0. How does it follow that $C^n = 0$? Thus, in the terminology of Exercise 3.52, $C$ is nilpotent. [*Hint: CD* is $C$ times the columns of $D$.]

**3.54** Let $P = [1, 1, 1]^t$.

**(a)** $P^tP$ is a $1 \times 1$ matrix, which we interpret as a number. Find this number.

**(b)** $PP^t$ is a $3 \times 3$ matrix. Compute $M_P$ where $I$ is the $3 \times 3$ identity matrix and

$$M_P = I - \frac{2}{P^tP} PP^t \tag{3.10}$$

**(c)** Prove that $M_P = M_P^t$ and $M_P^2 = I$.

**3.55** Let $P \in \mathbb{R}^n$ and let $M_P$ be as in formula (3.10) where now $I$ is the $n \times n$ identity matrix. Repeat Exercise 3.54.c.

**3.56** For $A$ and $B$ $n \times n$ matrices, we define their **Lie bracket** by

$$[A, B] = AB - BA$$

**(a)** Compute $[A, B]$ for $A$ and $B$ as in Exercise 3.26a.

**(b)** Under what conditions is it true that $[A, B] = 0$?

**(c)** Prove the following identities for $n \times n$ matrices $A$, $B$, and $C$ and scalars $\alpha$:

$$\begin{aligned} [A, B] &= -[B, A] \\ [A, \alpha B] &= \alpha[A, B] \\ [A, B + C] &= [A, B] + [A, C] \\ [A, [B, C]] &= [[A, B], C] + [B, [A, C]] \end{aligned}$$

*Remark.* A vector space $\mathcal{V}$ [such as the vector space $M(n, n)$] together with a "multiplication" $[\cdot, \cdot]$ satisfying the preceding properties is called a **Lie algebra**. (Here "Lie" is pronounced as in the name"Lee.") Lie algebras are important in many areas of study (e.g., structural engineering) but especially in physics.

**3.57** Prove that $[A, B]^t = -[A^t, B^t]$, where the notation is as in Exercise 3.56.

**3.58** For $f \in C^\infty(\mathbb{R})$ define

$$D(f) = \frac{df}{dx},$$
$$S(f) = xf$$

(By "$x$" we mean the function $y = x$.) Thus, for example,

$$S(x^3) = x^4, \qquad S(e^x) = xe^x$$

**(a)** Compute $S \circ D(f)$ and $D \circ S(f)$ for (i) $f(x) = \sin x$, (ii) $f(x) = e^{3x}$, and (iii) $f(x) = \ln(x^2 + 1)$. Does $D \circ S = S \circ D$? [Two transformations $U$ and $V$ are defined to be equal if they (i) have the same domain, (ii) have the same target space, and (iii) satisfy $U(X) = V(X)$ for all $X$ in the domain.]

**(b)** Let $I$ be the transformation of $\mathcal{W}$ into $\mathcal{W}$ defined by $I(f) = f$. Prove that

$$D \circ S - S \circ D = I \tag{3.11}$$

*Remark.* In the mathematical theory of quantum mechanics, the operator $D$ represents velocity and the operator $S$ position. The observation that $D \circ S \neq S \circ D$ indicates that we cannot simultaneously measure both the position and velocity; any measurement of one of these quantities must necessarily change the other. The Heisenberg uncertainty principle is a consequence of formula (3.11).

**3.59** ✓✓Suppose that $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ are vector spaces and $T : \mathcal{U} \to \mathcal{V}$ and $S : \mathcal{V} \to \mathcal{W}$ are linear transformations. Prove that $S \circ T$ is a linear transformation.

### 3.2.1 Computer Projects

### EXERCISES

1. Use the program "stick" from page 163 to draw a stick silhouette of a car similar to that shown in Figure 3.9 and store it in a matrix called $C$. Then transform $C$ into $\mathbb{R}^3$ by setting D=[C(1,:); 0∗C(1,:);C(2,:)]. Plot your car in three dimensions

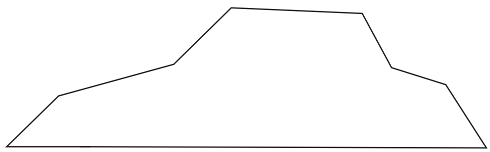


**FIGURE 3.9** My car picture.

with plot3(D(1,:),D(2,:),D(3,:)). You should first close the Figure window and then enter "hold" on and "axis([0 1 0 1 0 1])".

**2.** To add some dimension to your car, set E=D and then E(2,:)=E(2,:)+.25 and F=[D,E]. Plot $F$. It still is not much like a car.

**3.** To add some substance to the car, enter the following commands:

```
>>d=size(C,2);
>>for i=1:d,
   F(:,2*d+2*i-1)=D(:,i);
   F(:,2*d+2*i)=E(:,i);
>>end
```

Then plot $F$. You should get a reasonably good stick picture of a car.

**4.** You are about ready to start playing with your car. First, however, you should move it so that the origin is at the center of the car. This will ensure that as you rotate it, it will not move out of the viewing window. For this, you simply need to subtract a suitable constant from each row of $F$. (You should determine what suitable means.) You will need to clear the screen (with cla) and center the axes. Try: axis([−0.6 0.6 −0.6 0.6 −0.6 0.6])

Rotate F by 30 degrees about the $x$-axis and then rotate this image by 20 degrees about the $z$-axis. Plot both images. Be careful to save your original $F$! (The three-dimensional rotation matrices are in Exercise 3.9 on page 158.)

**5.** Find a single matrix that transforms your car into the final image from Exercise 4. Plot the image of $F$ under this transformation. (Be careful to save $F$.)

**6.** How do you suppose the image would appear if you were to transform $F$ by a rank 2 matrix transformation? Create a random rank 2 matrix and test your guess. Clear the figure window first. (See Exercise 2 on page 147, for information on creating random matrices with specific ranks.) Plot $F$ on a separate graph and compare this plot with your plot of the image. Get both plots printed. Attempt to label on the plot of $F$ several points, where the transformation is many-to-one. (This will only be approximate.)

**7.** How do you suppose the image would appear if you were to transform $F$ by a rank 1 matrix transformation? Create a random rank 1 matrix and test your guess.

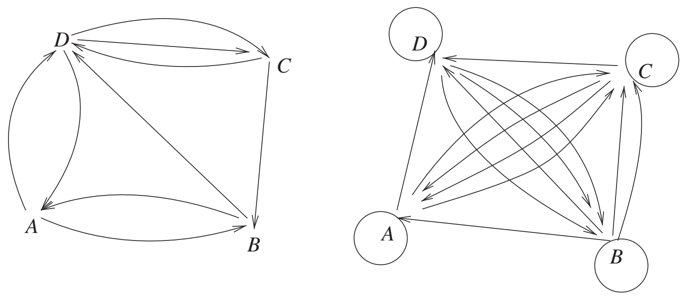



**FIGURE 3.10** Two-step connections.

### 3.2.2 Applications to Graph Theory II

In Section 1.1.2 on page 25 we showed that a graph such as the one shown on the left in Figure 3.10 could be represented by a matrix. Specifically, if the vertices are $V_1, V_2, \dots, V_n$, then the graph is represented by the matrix $M$, where $m_{ij}$ is the number of arrows from $V_i$ to $V_j$.

Hence, from the table on page 25, the graph on the left in Figure 3.10 is represented by the matrix

$$M = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

We say that two points $A$ and $B$ of a directed graph are 2-step connected if there is a point $C$ such that $A \to C \to B$.

In Section 1.1, we interpreted the graph on the left in Figure 3.10 as representing daily flights between cities, in which case the graph on the right shows the 2-step connections between the cities. (We allow overnight stays so that, e.g., one could go out on one flight and return on an earlier flight.) Circles represent a trip starting and ending at the same city. In Exercise 1.47 on page 27, you discovered that the matrix of this graph is

$$N = \begin{bmatrix} 2 & 0 & 2 & 1 \\ 1 & 1 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 0 & 3 & 0 & 3 \end{bmatrix}$$

There is a remarkably easy way to produce $N$; it is just $M^2$!Similarly, $M^3$ represents the graph of 3-step connections in the graph. The reason this works is explained in the proof of the following theorem.

**Theorem 3.7** *Let $M$ be the matrix of a directed graph $G$. Then for all $k$ the matrix $M^k$ is the matrix of the graph of $k$-step connections in $G$.*

*Proof.* Assume that $M$ is $n \times n$ and let $B = M^2$. Then

$$b_{ij} = m_{i1}m_{1j} + m_{i2}m_{2j} + \cdots + m_{in}m_{nj}$$

The entry $m_{il}m_{lj}$ is nonzero if and only if there is both an arrow from $V_i$ to $V_l$ *and* an arrow from $V_l$ to $V_j$—that is, there is a 2-step connection between $V_i$ and $V_j$ through $V_l$. In this case, $m_{il}m_{lj} = 1$. Hence, summing over $l$ yields the total number of 2-step connections between $V_i$ and $V_j$. This proves our theorem for $k = 2$.

More generally, suppose we have shown our theorem for all $m < k$, where $k$ is some integer. Let $B = M^k = MC$ where $C = M^{k-1}$. Then

$$b_{ij} = m_{i1}c_{1j} + m_{i2}c_{2j} + \cdots + m_{in}c_{nj}$$

Note that $C$ is the matrix of (($k-1$)-step connections of our graph. Hence, $m_{il}c_{lj}$ is nonzero if and only if there is both an arrow from $V_i$ to $V_l$ *and* a ($k-1$)-step connection from $V_l$ to $V_j$—that is, there is a $k$-step connection between $V_i$ and $V_j$ that begins by connecting $V_i$ with $V_l$. As before, $m_{il}c_{lj}$ is the number of such connections and summing over $l$ yields the total number of $k$-step connections between $V_i$ and $V_j$, proving the theorem. □

## Self-Study Questions

**3.1** ✓A certain airline has two round-trip flights daily between Indianapolis and Chicago. How many 2-step connections are there between Indianapolis and itself, allowing overnight stays?

**3.2** ✓The same airline has a daily round-trip flight between Chicago and Denver. How many 2-step connections are there between Indianapolis and Denver, allowing overnight stays?

**3.3** ✓What is the 2-step connection matrix for the graph described by the following matrix?

$$\begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 0 \end{bmatrix}$$

## EXERCISES

**3.60** Compute the matrix for the route maps in Figure 1.9 on page 27. Use these matrices to find the 2-step connectivity matrices. Finally, use these matrices to draw the 2-step connectivity graphs. (*Note:* To indicate many arrows between two vertices, draw one arrow and write next to it the number of arrows it represents. You can download a copy of this figure and the others in this section from the companion website for the text.)

**3.61** Find the 3-step connectivity matrices for the route maps in Figure 1.9 on page 25.

**3.62** Find the 2-step connectivity matrix for the graph in Figure 1.8 on page 26. Explain, in terms of dominance, the meaning of the non-zero entries of the 2-step connectivity matrix.

## 3.3 INVERSES

In Section 2.3, we said that an $n \times n$ matrix $A$ is nonsingular if for all $Y$ in $\mathbb{R}^n$ there is one, and only one, $X$ such that $AX = Y$. In principle, then, if $A$ is nonsingular, there is a transformation that computes $X$ in terms of $Y$. This transformation is called the "inverse transformation."

**Definition 3.7** *Let $T : \mathcal{V} \to \mathcal{W}$ be a transformation between two sets $\mathcal{V}$ and $\mathcal{W}$. We say that $T$ is* ***invertible*** *if for all $Y \in \mathcal{W}$ there is a unique solution $X$ to the equation $T(X) = Y$. In this case, the inverse transformation $T^{-1} : \mathcal{W} \to \mathcal{V}$ is defined by $T^{-1}(Y) = X$ if and only if $T(X) = Y$.*

Finding inverse transformations (when they exist) is one very important technique for solving systems.

### EXAMPLE 3.7

Find a formula for $T_A^{-1}$ where $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ is multiplication by

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 4 & 1 \end{bmatrix} \tag{3.12}$$

*Solution.* The inverse transformation computes $X$ in terms of $Y$ in the system $AX = Y$. If $Y = [y_1, y_2, y_3]^t$, then this system has augmented matrix

$$\begin{bmatrix} 1 & 2 & 1 & y_1 \\ 1 & 3 & 4 & y_2 \\ 2 & 4 & 1 & y_3 \end{bmatrix}$$

We solve this system by computing the row reduced echelon form of the preceding matrix, obtaining (after some work)

$$\begin{bmatrix} 1 & 0 & 0 & 13y_1 - 2y_2 - 5y_3 \\ 0 & 1 & 0 & -7y_1 + y_2 + 3y_3 \\ 0 & 0 & 1 & 2y_1 - y_3 \end{bmatrix}$$

This yields the answer

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 13y_1 - 2y_2 - 5y_3 \\ -7y_1 + y_2 + 3y_3 \\ 2y_1 - y_3 \end{bmatrix} = y_1 \begin{bmatrix} 13 \\ -7 \\ 2 \end{bmatrix} + y_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + y_3 \begin{bmatrix} -5 \\ 3 \\ -1 \end{bmatrix}$$

This may be written in the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 13 & -2 & -5 \\ -7 & 1 & 3 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Hence, the inverse transformation is defined by multiplication by the matrix

$$B = \begin{bmatrix} 13 & -2 & -5 \\ -7 & 1 & 3 \\ 2 & 0 & -1 \end{bmatrix}$$

Knowing $B$ allows us to solve any system with coefficient matrix $A$. For example, the unique solution to $AX = [1, 2, 3]^t$ is

$$X = \begin{bmatrix} 13 & -2 & -5 \\ -7 & 1 & 3 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \\ -1 \end{bmatrix} \tag{3.13}$$

The reader may check that this solution really works.

In Example 1, the inverse transformation turned out to be a matrix transformation. This is an example of the following general proposition.

**Proposition 3.4** *Let $T : \mathcal{V} \to \mathcal{W}$ be an invertible linear transformation, where $\mathcal{V}$ and $\mathcal{W}$ are vector spaces. Then $T^{-1} : \mathcal{W} \to \mathcal{V}$ is a linear transformation.*

*Proof.* We need to show that for $Y_1, Y_2 \in \mathcal{W}$

$$T^{-1}(Y_1 + Y_2) = T^{-1}(Y_1) + T^{-1}(Y_2) \tag{3.14}$$

Let $X_i = T^{-1}(Y_i)$. Since $T$ is linear,

$$\begin{aligned} T(X_1 + X_2) &= T(X_1) + T(X_2) \\ &= Y_1 + Y_2 \end{aligned}$$

Hence,

$$T^{-1}(Y_1 + Y_2) = X_1 + X_2 = T^{-1}(Y_1) + T^{-1}(Y_2)$$

proving formula (3.14).

We also need to show that for all $Y \in \mathcal{W}$ and all scalars $c$

$$T^{-1}(cY) = cT^{-1}(Y) \tag{3.15}$$

We leave the proof to the reader as an exercise. □

Let $A$ be a nonsingular matrix. The matrix that defines the inverse transformation is referred to as the **inverse matrix** and denoted by $A^{-1}$. Thus, in Example 1, $B = A^{-1}$.

We noted in Chapter 1 that in solving a system, it is not necessary to keep repeating the names of the variables. We can represent the whole solution process in terms of the augmented matrix. The same is true for computing inverse matrices. The only difference is that now we need to keep track of the coefficients of both the $x_i$ and the $y_i$. To explain this, consider the first few steps in solving Example 3.7. We are solving the system

$$\begin{aligned} x_1 + 2x_2 + \phantom{4}x_3 &= \phantom{0}y_1 + 0y_2 + 0y_3 \\ x_1 + 3x_2 + 4x_3 &= 0y_1 + \phantom{0}y_2 + 0y_3 \\ 2x_1 + 4x_2 + \phantom{4}x_3 &= 0y_1 + 0y_2 + \phantom{0}y_3 \end{aligned}$$

We represent this system by a "double" matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

We subtract the first equation in our system from the second, producing

$$\begin{aligned} x_1 + 2x_2 + \phantom{4}x_3 &= \phantom{-}y_1 + 0y_2 + 0y_3 \\ x_2 + 3x_3 \phantom{{}+x_3} &= -y_1 + \phantom{0}y_2 + 0y_3 \\ 2x_1 + 4x_2 + \phantom{4}x_3 &= \phantom{-}0y_1 + 0y_2 + \phantom{0}y_3 \end{aligned}$$

The double matrix that represents this system is

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

Notice that the effect was to subtract the first row from the second in the whole double matrix.

Next we might subtract twice the first equation from the third. In terms of the double matrix, this would be described by subtracting twice the whole first row from the third, producing

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array}\right]$$

We keep reducing our double matrix until the $3 \times 3$ matrix on its left side is reduced to the identity matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 13 & -2 & -5 \\ 0 & 1 & 0 & -7 & 1 & 3 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right]$$

This tells us that

$$\begin{aligned} x_1 &= 13y_1 - 2y_2 - 5y_3 \\ x_2 &= -7y_1 + \phantom{2}y_2 + 3y_3 \\ x_3 &= 2y_1 \phantom{{}-2y_2} - \phantom{5}y_3 \end{aligned}$$

which, of course, is the same answer as before. Notice that the $3 \times 3$ matrix on the right of the final double matrix is just $A^{-1}$.

In general, to invert an $n \times n$ matrix $A$, we first form the double matrix $[A \mid I]$, where $I$ is the $n \times n$ identity matrix. If we can row reduce this double matrix until $A$ is reduced to the identity matrix, then the matrix on the right side is $A^{-1}$. This process cannot fail unless during the reduction $A$ some row reduces to 0, in which case $A$ does not have rank $n$, and hence, it is not nonsingular. Here is another example of this process.

## EXAMPLE 3.8

Find $A^{-1}$, where $A$ is as follows:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 3 & 3 & 1 \end{bmatrix}$$

*Solution.*

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & -3 & -2 & -3 & 0 & 1 \end{array}\right] \quad \begin{array}{lcl} R_2 & \to & R_2 - R_1 \\ R_3 & \to & R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -3 & -2 & -3 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right] \quad \begin{array}{lcl} R_2 & \to & R_3 \\ R_3 & \to & R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 2 & -1 & 0 \\ 0 & -3 & 0 & -5 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right] \quad \begin{array}{lcl} R_1 & \to & R_1 - R_3 \\ R_2 & \to & R_2 + 2R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & \frac{5}{3} & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right] \quad R_2 \to -\frac{1}{3} R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{4}{3} & \frac{1}{3} & \frac{2}{3} \\ 0 & 1 & 0 & \frac{5}{3} & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right] \quad R_1 \to R_1 - 2R_2$$

Thus, the inverse of $A$ is

$$A^{-1} = \begin{bmatrix} -\frac{4}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{5}{3} & -\frac{2}{3} & -\frac{1}{3} \\ -1 & 1 & 0 \end{bmatrix}$$

There is an important relation between an $n \times n$ matrix $A$ and its inverse $B$. Saying that multiplication by $B$ solves the equation $AX = Y$ means that for all $X$ and $Y$ in $\mathbb{R}^n$

$$AX = Y \text{ if and only if } BY = X \tag{3.16}$$

Substituting the second equation in (3.16) into the first shows

$$A(BY) = Y$$

which is equivalent to

$$(AB)Y = Y$$

The only $n \times n$ matrix $C$ with the property that $CY = Y$ for all $Y \in \mathbb{R}^n$ is the $n \times n$ identity matrix $I$. (The matrix that describes a transformation is unique!) Hence, $AB = I$.

We may also substitute the first equation in (3.16) into the second, showing that' for all $X \in \mathbb{R}^n$, $(BA)X = X$, and hence, $BA = I$. Thus, we have proved the following theorem:

**Theorem 3.8** *Let A be a nonsingular matrix. Then* $AA^{-1} = A^{-1}A = I$.

Theorem 3.8 suggests the following definition:

**Definition 3.8** *An $n \times n$ matrix A is **invertible** if there is a matrix B such that $AB = I = BA$, where I is the $n \times n$ identity matrix. In this case we write $B = A^{-1}$ and refer to B as the inverse of A.*

Logically, Definition 3.8 should refer to $B$ as "an inverse" of $A$ rather than "the inverse" of $A$ as, conceivably, there might be more than one such matrix. However, suppose that $C$ is another matrix such that $AC = CA = I$. Then,

$$B = BI = B(AC) = (BA)C = IC = C$$

showing that there can be at most one inverse for $A$.

The inverse of a matrix is, in some respects, similar to the inverse of a number. Suppose, for example, we wish to solve the equation

$$2x = y$$

We simply multiply both sides by $\frac{1}{2}$, producing $x = \frac{1}{2}y$. Similarly, suppose we want to solve

$$AX = Y$$

where $A$ is an invertible $n \times n$ matrix and $Y \in \mathbb{R}^n$. We multiply both sides *on the left* by $A^{-1}$, producing

$$A^{-1}Y = A^{-1}(AX) = (A^{-1}A)X = IX = X \tag{3.17}$$

Thus, $X = A^{-1}Y$, as before.

Conversely, if we let $X = A^{-1}Y$, then

$$AX = A(A^{-1}Y) = (AA^{-1})Y = IY = Y$$

Hence, *if A is invertible, then the system $AX = Y$ has one and only one solution,* $A^{-1}Y$. From Definition 2.8 on page 140, $A$ must be nonsingular. Since we have already seen that all nonsingular matrices have inverses, we have proved the following:

**Theorem 3.9** *Let A be an $n \times n$ matrix. Then A is invertible if and only if A is nonsingular—that is, if and only if A has rank n.*

Notice that in equation (3.17) we used only $A^{-1}A = I$, not $AA^{-1} = I$. In general, a matrix $B$ is said to be a **left inverse** for $A$ if $BA = I$. Our calculation suggests that perhaps all we really need to solve systems are left inverses. It turns out, however, if an $n \times n$ matrix has a left inverse, the matrix is invertible and the left inverse is, in fact, just the usual inverse. Similar comments apply for right inverses.

**Theorem 3.10** *Let $A$ be an $n \times n$ matrix. If either of the following statements hold, then $A$ is invertible and $B = A^{-1}$.*

*(a) There is a matrix $B$ such that*

$$BA = I \tag{3.18}$$

*(b) There is a matrix $B$ such that*

$$AB = I \tag{3.19}$$

*Proof.* We begin with part (a). To prove the invertibility of $A$, recall that rank $(BA) \leq$ rank $A$. Hence, from equation (3.18),

$$n = \text{rank } I \leq \text{rank } A$$

Since $A$ is $n \times n$, this shows that rank $A = n$, proving the invertibility.

Then $B = A^{-1}$ follows from multiplying both sides of equation (3.18) on the right by $A^{-1}$:

$$\begin{aligned} (BA)A^{-1} &= IA^{-1} \\ B(AA^{-1}) &= A^{-1} \\ BI &= A^{-1} \\ B &= A^{-1} \end{aligned}$$

Part (b) is similar and left as an exercise for the reader. □

### Computational Issues: Reduction versus Inverses

One rough measure of the amount of time a computer takes to do a certain task is the total number of algebraic operations (additions, subtractions, multiplications, and divisions) required.

Each such operation is called a **flop** (floating point operation). Other activities, such as recollection and storage of data and changing the contents of registers, take considerably less time. In MATLAB the "flops" command yields the total number of flops done so far.

It can be shown that computing the inverse of a typical $n \times n$ matrix $A$ can require $2n^3 - 3n^2 + n$ flops and multiplying an $n \times 1$ vector by $A^{-1}$ can require an additional $2n^2$ flops. Hence, solving the system $AX = B$ using inverses can require $2n^3 - n^2 + n$ flops. Note that

$$2n^3 - n^2 + n = 2n^3 \left(1 - \frac{1}{n} + \frac{1}{n^2}\right)$$

For large $n$, the quantity in the parentheses on the right is very close to 1. Hence, for large $n$, solving a system using inverses takes roughly $2n^3$ flops.

On the other hand, it was seen in Exercise 1.89 on page 69 that solving $AX = B$ using row reduction can require $2n^3/3 + 3n^2/2 - 7n/6$ flops, that, from similar reasoning, is roughly $2n^3/3$ for large $n$. *Hence, solving a large system using inverses typically takes roughly three times as long as solving the same system using row reduction.*

This does not mean that inverses are useless for large systems. It often happens that we want to solve the system $AX = B$ for a fixed matrix $A$ but several different $B$. If we will need to use four or more different $B$s, then computing $A^{-1}$ might be more efficient since doing four row reductions would require around $\frac{4}{3}(2n^3)$ flops, whereas using inverses would require

$$(2n^3 - 3n^2 + n) + 4(2n^2) = 2n^3 + 5n^2 + n$$

flops, which is still around $2n^3$ for large $n$. However, in the next section we will describe a technique ("*LU* factorization") for dealing with multiple $B$'s that is often faster and more accurate than computing $A^{-1}$.

Incidentally, it is quite remarkable that the number of flops required to compute $A^{-1}$ is only around three times the number required to solve $AX = B$, since computing $A^{-1}$ requires solving the $n$ systems $AX_j = I_j$, where $I_j$ is the $j$th column of the identity matrix. What saves us is that each of these $n$ systems has the same coefficient matrix so the algebraic operations on the left sides of the equations only get done once. We are also aided by the fact that the identity matrix is a **sparse** matrix, implying that most of its entries are 0.

**True-False Questions**: Justify your answers.

**3.21** The following matrix is invertible:

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 1 & 0 \\ 4 & 6 & 2 & 6 \end{bmatrix}$$

**3.22** Let $A$ be a $4 \times 3$ matrix and $B$ a $3 \times 4$ matrix. Then $AB$ cannot be invertible. [*Hint:* Think about rank.]

**3.23** Suppose that $A$ is an invertible matrix and $B$ is any matrix for which $BA$ is defined. Then the matrices $BA$ and $B$ need not have the same rank.

**3.24** Suppose that $A$ is an invertible matrix and $B$ is any matrix for which $AB$ is defined. Then the matrices $AB$ and $B$ need not have the same rank.

**3.25** Suppose that $A$ and $B$ are $n \times n$ matrices such that $AB$ is invertible. Then both $A$ and $B$ are invertible. [*Hint:* Compare the rank of $AB$ with that of $A$ and of $B$.]

**3.26** Suppose that $A$ is an $n \times n$ matrix such that $AA^t = I$. Then $A^tA = I$ as well.

**3.27** Suppose that $A$ is an $n \times n$ matrix that satisfies $A^2 + 7A - I = 0$. Then $A$ is invertible.

**3.28** Suppose that $A$ is invertible and $B$ is row equivalent to $A$. Then $B$ is invertible.

**3.29** Suppose that $A$ is invertible and $B$ is any matrix such that $AB$ is defined. Then $AB$ and $B$ have the same nullspace.

**3.30** Suppose that $A$ is an $n \times n$ invertible matrix and $B$ is any $n \times n$ matrix. Then $ABA^{-1} = B$.

## EXERCISES

**3.63** Use the method of Example 3.7 on page 182 to find the inverse of

**(a)** ✓✓ $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

**3.64** Use the method of Example 3.8 on page 185 to invert the following matrices (if possible).

**(a)** ✓✓ $\begin{bmatrix} 1 & 0 & 3 \\ 4 & 4 & 2 \\ 2 & 5 & -4 \end{bmatrix}$ **(b)** ✓✓ $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

**(c)** ✓✓ $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ **(d)** ✓✓ $\begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

**(e)** ✓✓ $\begin{bmatrix} 3 & 10 & 3 & 8 \\ 3 & -2 & 8 & 7 \\ 2 & 1 & 4 & -5 \\ 5 & 11 & 7 & 3 \end{bmatrix}$ **(f)** $\begin{bmatrix} 2 & -1 & 2 & 0 \\ 4 & -1 & 4 & -2 \\ 8 & -3 & 10 & 0 \\ 6 & -3 & 8 & 9 \end{bmatrix}$

**(g)** ✓✓ $\begin{bmatrix} 2 & 1 & 0 & 3 \\ 0 & 3 & 1 & 1 \\ 1 & -1 & 1 & 2 \\ 2 & 7 & 0 & 4 \end{bmatrix}$

**(h)** ✓✓ $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

**(i)** ✓✓ $\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$

**(j)** $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 3 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

**(k)** $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

**(l)** $\begin{bmatrix} 5 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

**(m)** $\begin{bmatrix} 5 & 1 & 0 \\ -1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

**(n)** $\begin{bmatrix} 5 & -1 & 2 \\ 1 & 0 & 1 \\ 6 & 0 & 3 \end{bmatrix}$

**(o)** $\begin{bmatrix} 5 & 1 & 6 \\ -1 & 0 & 0 \\ 2 & 1 & 3 \end{bmatrix}$

**(p)** $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & -1 & 2 \end{bmatrix}$

**3.65** For each invertible $3 \times 3$ matrix $A$ from Exercise 3.64, use the inverse to solve the equation $AX = Y$, where $Y = [1, 2, 3]^t$. For each invertible $4 \times 4$ matrix, solve the equation $AX = Y$, where $Y = [1, 2, 3, 4]^t$. ✓✓[(a)-(e), (g)– (i)]

**3.66** In Exercise 3.64, for each noninvertible matrix, express one row as a linear combination of the others. Then do the same for the columns.✓[(c)] [*Hint:* Theorem 2.5 on page 106 might be useful for part (c).]

**3.67** For each invertible matrix $A$ in Exercise 3.64, check directly that $AA^{-1} = A^{-1}A = I$.

**3.68** ✓Give an example of a noninvertible $5 \times 5$ matrix in which none of the entries are equal to zero and none are equal to each other. How can you be sure that this matrix is not invertible without doing any further computations?

**3.69** ✓Solve the system in Exercise 1.55.b, on page 39, using inverses. Do you think this was less work than simply reducing the augmented matrix? Explain.

**3.70** Attempt to solve the system in Exercise 1.55.c, on page 39, using inverses. Why does your attempt fail?

**3.71** ✓✓Assume that $ad - bc \neq 0$. Find the inverse of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

**3.72** ✓✓Compute the inverse of the matrix $A$:

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

*Remark.* An upper (or lower) triangular matrix is called unipotent if it has only 1's on the diagonal. This exercise proves that the inverse of a $3 \times 3$, upper triangular, unipotent matrix is unipotent.

**3.73** Explain why the inverse of an $n \times n$, upper triangular, unipotent matrix $A$ is unipotent. [*Hint:* Show that (a) a unipotent matrix may be reduced to $I$ by adding multiples of lower rows onto higher rows and (b) applying such operations to a unipotent matrix produces a unipotent matrix.]

**3.74** A matrix $N$ is said to be nilpotent if there is a $k$ such that $N^k = 0$. The smallest such $k$ is called the **degree of nilpotency** of $N$.

**(a)** Prove that the following matrix is nilpotent of degree at most 3:

$$N = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}$$

**(b)** ✓✓Suppose that $N$ is nilpotent of degree 3. Prove that $(I - N)^{-1} = I + N + N^2$. Use this to explicitly compute the inverse of the matrix $I - N$, where $N$ is as in part (a).

**(c)** Suppose that $N$ is nilpotent of degree 4. What would be the corresponding formula for $(I - N)^{-1}$? Prove your answer.

**3.75** A certain matrix $A$ has inverse

$$B = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

Find a matrix $X$ such that $XA = C$, where

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

**3.76** ✓✓In Exercise 3.75, find a matrix $Y$ such that $AY = C$.

**3.77** ✓✓Suppose that $A$ is an $n \times n$ matrix such that $A^2 + 3A + I = 0$. Show that $A$ is invertible and $A^{-1} = -A - 3I$.

**3.78** Suppose that $A$ is an $n \times n$ matrix such that $A^3 + 3A^2 + 2A + 5I = 0$. Show that $A$ is invertible.

**3.79** Suppose that $A$ is invertible. Prove that $A^{-1}$ is invertible and $(A^{-1})^{-1} = A$.

**3.80** ✓✓Let $A$ and $B$ be invertible $n \times n$ matrices. Prove that $AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. For your proof, recall that $(AB)^{-1}$ is the matrix that expresses $X$ in terms of $Y$ in the equation $ABX = Y$. If you know $A^{-1}$, what would be your first step in solving this equation?

**3.81** Using properties of inverses and matrix multiplication, prove that if $A$ and $B$ are invertible $n \times n$ matrices, then

$$(B^{-1}A^{-1})(AB) = I$$

How does this prove that $(AB)^{-1} = B^{-1}A^{-1}$?

**3.82** Using properties of inverses and matrix multiplication, prove that if $A$ and $B$ are invertible $n \times n$ matrices, then

$$(AB)(B^{-1}A^{-1}) = I$$

How does this prove that $(AB)^{-1} = B^{-1}A^{-1}$?

**3.83** Prove that if $A$ is invertible, then so are $A^2$, $A^3$, and $A^4$. What are the inverses of these matrices? (Assume that you know $A^{-1}$.)

**3.84** ✓✓Prove that, for an invertible matrix $A$, $(A^t)^{-1} = (A^{-1})^t$. [*Hint:* Simplify $(A^{-1})^t A^t$.]

**3.85** Suppose that $A$ and $B$ are invertible $n \times n$ matrices. Prove that $(AB)^2 = A^2B^2$ if and only if $AB = BA$.

**3.86** Let $Q$ and $D$ be $n \times n$ matrices with $Q$ invertible.

**(a)** ✓✓Prove that $(Q^{-1}DQ)^2 = Q^{-1}D^2Q$.

*Warning: Matrix multiplication is not necessarily commutative! It is **not** usually true that $Q^{-1}DQ = D$. It is also not in general true that for matrices* $(AB)^2 = A^2B^2$. (See Exercise 3.85.)

**(b)** What is the corresponding formula for $(Q^{-1}DQ)^3$? $(Q^{-1}DQ)^n$? Prove your answer.

**3.87** Compute a formula for $A^n$, where

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

*Suggestion:* Show that $A = QDQ^{-1}$ where

$$Q = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

Then use the result from Exercise 3.86.

**3.88** We know that only square matrices can be invertible. We also know that if a square matrix has a right inverse, the right inverse is also a left inverse. It is possible, however, for a non square matrix to have either a right inverse or a left inverse (but not both). Parts (a)–(d) explore these possibilities.

**(a)** ✓✓For the given matrix $A$ find a $3 \times 2$ matrix $B$ such that $AB = I$, where $I$ is the $2 \times 2$ identity matrix. [*Hint:* If $B_1$ and $B_2$ are the columns of $B$, then $AB_j = I_j$.]

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

**(b)** Suppose that $A$ is any $2 \times 3$ matrix with rank 2. Prove that $A$ has a right inverse. [*Hint:* Is the equation $AX = B$ solvable for all $B \in \mathbb{R}^2$]?

**(c)** ✓✓Show conversely that if $A$ is a $2 \times 3$ matrix that has a right inverse, then $A$ has rank 2.

**(d)** Under what circumstances does an $m \times n$ matrix have a right inverse? State your condition in terms of rank and prove your answer.

**3.89** For $A'$ as shown, find a matrix $B'$ with $B'A' = I$, where $I$ is the $2 \times 2$ identity matrix. (The answer to Exercise 3.88.a might help.) Under what circumstances does an $m \times n$ matrix have a left inverse? Prove your answer.

$$A' = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$$

**3.90** Without using the result from Exercise 3.88, prove that if $A$ is an $m \times n$ matrix that has a right inverse, then the equation $AX = B$ has a solution for all $B \in \mathbb{R}^m$.

**3.91** ✓Without using the result of Exercise 3.89, prove that if $A$ is an $m \times n$ matrix that has a left inverse, the nullspace of $A$ is zero.

**3.92** Let $C$ be the $3 \times 3$ matrix shown. Let $A$ and $B$ be, respectively, the indicated $3 \times 2$ and $2 \times 3$ matrices such that $AB = C$. In this exercise we explore a few of the numerous ways of seeing that $C$ cannot be invertible.

$$AB = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 2 \\ 6 & -3 & 0 \end{bmatrix} = C$$

**(a)** Since the columns of $B$ are three vectors in $\mathbb{R}^2$, they cannot be linearly independent. Express one of them as a linear combination of the other two. Show that the columns of $C$ satisfy the same dependency relation.

Explain this fact in terms of the definition of $C$ as $AB$. (Recall that the columns of $AB$ are $A$ times the columns of $B$.) How does this show that $C$ is not invertible?

**(b)** ✓What general theorem guarantees that the rows of $A$ are linearly dependent? Find an explicit dependency relation. Show that the rows of $C$ satisfy the same dependency relation. Explain this fact in terms of the definition of $C$ as $AB$. How does this prove that $C$ is not invertible?

**(c)** Can the dimension of the nullspace of $B$ be zero? Suppose that $BX = 0$. Prove that $CX = 0$. What can you say about the nullspace of $C$? How does this prove that $C$ is not invertible?

**3.93** Let $A$ be an $m \times n$ matrix and let $B$ be $n \times m$, where $n < m$. Give three different proofs that $AB$ is not invertible. Your proofs should parallel the three parts of the previous exercise.

**3.94** Use the rank of products theorem (Theorem 3.3 on page 169) to prove the result in Exercise 3.93.

### 3.3.1 Computer Projects

**Ill-Conditioned Systems**

MATLAB provides an excellent facility for solving systems of the form $AX = B$, where $A$ is invertible. Once $A$ and $B$ have been entered, $X$ is computed by the MATLAB command X = A\B. The reader should think of this as shorthand for $A^{-1}B$, although MATLAB actually uses row reduction rather than inverses in solving this equation.

## EXERCISES

1. Let $A$ be the matrix from Exercise 3.64.a above. Use the MATLAB command X = A\B to solve the system $AX = B$ where $B = [2.1, 3.2, -4.4]^t$.

2. In most applications of linear algebra, we get our numerical data from measurements that are susceptible to error. Suppose that the vector $B$ in Exercise 1 was obtained by measuring a vector $B'$ whose actual value is $B' = [2.1, 3.21, -4.4]^t$. Compute the solution to the equation $AX' = B'$. Which component of the $X$ you just computed has the largest error? (We measure error as the absolute value of the difference between the computed value and the actual value.) Explain why this is to be expected in terms of the magnitude of the components of $A^{-1}$. [In MATLAB, $A^{-1}$ is computed with the command inv(A). ] Which component of $B$ would you change to produce the greatest change in $X$? Why? Back up your answer with a numerical example. How much error could you tolerate in the measured values of the components of $B$ if each entry of $X$ is to have an error of at most $\pm 0.001$?

**3.** Let

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

and let $B = [83, 46, 32]^t$. Use X = A\B to find a solution to $AX = B$. As before, suppose that the vector $B$ was obtained by measuring a vector $B'$ whose actual value is $B' = [82.9, 46.07, 31.3]^t$. Compute the solution to the equation $AX' = B'$. What is the percentage of error in the least accurate entry of $X$? What is there about the components of $A^{-1}$ that accounts for the large error? How much error could you tolerate in the measured values of the components of $B$ if each entry of $X$ is to have an error of at most $\pm 0.001$?

**4.** Exercise 2 demonstrates that the process of solving a system of equations can magnify errors in disastrous ways. One quantitative measure of the inaccuracy of a calculation is *the ratio of the percentage of error of the final answer to the percentage of error of the input data.* But what do we mean by the percentage of error in a vector in which every entry might have an error of a different magnitude? In this discussion we define the total error to be the sum of the errors in each component. More precisely, we say that the error in the approximation $X \approx X'$ where $X$ and $X'$ belong to $\mathbb{R}^n$ is $|X - X'|_1$ where

$$|X|_1 = |x_1| + |x_2| + \cdots + |x_n| \tag{3.20}$$

In MATLAB, this can be computed as norm(X,1). If $X'$ is the computed answer and $X$ is the actual answer, we define the percentage of error to be

```
P=100*norm(X-X',1)/norm(X,1)
```

**(a)** Let $B$, $B'$, $X$, and $X'$ be as in Exercise 2. Compute (i) the percentage of error in $B$, (ii) the percentage of error in $X$, and (iii) the ratio of the percentage of error in $X$ to that in $B$. This is the inaccuracy of the calculation of $X$ from $B$.

**Question** Assuming that accuracy is desired, do we want this number to be large or small? Explain.

**(b)** Compute the inaccuracy of the computation of $X$ from $B$ in Exercise 3.

**(c)** For each $n \times n$ invertible matrix $A$, there is a number cond$(A, 1)$ (the **one norm condition number** of $A$) such that the inaccuracy in solving the system $AX = B$, measured using $|\cdot|_1$, is at most cond$(A, 1)$, regardless of $B$ and regardless of the amount of error in $B$. This means that if, say, cond$(A, 1)$ and the error in $B$ is 0.001%, then the computed value of $X$ has at most $20 \times 0.001 = 0.02\%$ error. In general, cond$(A, 1) \geq 1$. (This says that we cannot expect the answer to be more accurate than the input data.) MATLAB recognizes the command cond($A$, 1). Compute the condition numbers for the matrices $A$ in Exercises 2 and 3 above.

*Remark.* Matrices with large condition numbers are called "ill-conditioned." If the coefficient matrix of a system is ill-conditioned, any slight error in the input data can make the solution very inaccurate, and we must be extremely suspicious of answers obtained by solving the system. Notice that these inaccuracies are not related to round-off error. Ill-conditioning is inherent to the matrix and not to the method of solution. See Section 8.1 for a more complete discussion of conditioning.

### 3.3.2 Applications to Economics

Linear algebra is used extensively in the study of economics. Here, we describe an economic model (the Leontief open model), which was developed by Wassily Leontief, who won the Nobel Prize in economics in 1973. We begin with an example.

**EXAMPLE 3.9**

In many areas of the world, the main source of energy is coal. Coal is used to both heat homes and produce electricity. One of the most basic jobs of any manager is to determine the levels of production necessary to meet demand.

In this example, we consider a small village in which there is a coal mine and an electric plant. All the residents get their coal from the mine and their electricity from the electric company. Let us assume that the currency used by the people is the slug (S).

It is known that to produce 1 S worth of coal, the mine must purchase 0.20 S worth of electricity. To produce 1 S worth of electricity, the power plant must purchase 0.60 S worth of coal and 0.07 S worth of electricity (from itself). Assume that the village demands 2500 S worth of electricity and 10,000 S worth of coal. What level of production is necessary to meet this demand?

*Solution.* Let $c$ and $e$ denote the respective values of the coal and electricity produced to meet the demand of the village. The consumers demand 10,000 S worth of coal and the electric company demands $0.60e$ S worth of coal. Thus, the total demand for coal is

$$10{,}000 + 0.60e$$

Similarly, the demand for electricity is

$$2500 + 0.20c + 0.07e$$

Setting production equal to demand, we obtain the following system

$$\begin{aligned} c &= 10{,}000 + 0.60e \\ e &= 2500 + 0.20c + 0.07e \end{aligned} \tag{3.21}$$

which is equivalent to

$$\begin{aligned} c - 0.60e &= 10{,}000 \\ -0.20c + 0.93e &= 2500 \end{aligned}$$

We solve this system by inverting the coefficient matrix. We find that

$$\begin{aligned} \begin{bmatrix} c \\ e \end{bmatrix} &= \begin{bmatrix} 1 & -0.60 \\ -0.20 & 0.93 \end{bmatrix}^{-1} \begin{bmatrix} 10000 \\ 2500 \end{bmatrix} \\ &\approx \begin{bmatrix} 1.1481 & 0.7407 \\ 0.2469 & 10.2346 \end{bmatrix} \begin{bmatrix} 10000 \\ 2500 \end{bmatrix} \\ &\approx \begin{bmatrix} 13300 \\ 5500 \end{bmatrix} \end{aligned}$$

Thus, we must produce 13,300 S worth of coal and 5500 S worth of electricity.

Notice that in the example the computed values of both $c$ and $e$ turned out to be positive. This is due to the (somewhat surprising) property that the inverse of the coefficient matrix in the above system has only positive entries. Had this not been the case, there would be levels of demand resulting in a negative value for either $c$ or $e$. If, say, $c$ were negative, the villagers and the electric company would together be giving coal to the coal company which we certainly do not expect. Thus, a negative value of either $c$ or $e$ would suggest a serious flaw in our model. Hence, the positivity of the coefficients of the inverse matrix is one reality check on our model.

In the general case, we will have $n$ industries numbered $1, \dots, n$. The production of our economy is described by an $n \times 1$ column vector $X$ where $x_i$ is the value produced by industry $i$. In Example 3.9, this was $[c, e]^t$.

Each industry must purchase raw materials from all the local industries (including itself). Let $c_{ij}$ represent the cost (in dollars) of the goods or services that industry $j$ must purchase from industry $i$ in order to produce \$1's worth of output. If the output vector is $X$, industry $i$ will sell $c_{ij}x_j$ units to industry $j$. In total, then, industry $i$ sells a total of

$$c_{i1}x_1 + c_{i2}x_2 + \cdots + c_{in}x_n$$

to itself and other industries. This is the industrial demand for industry $i$. Notice that this is just the $i$th row of the matrix product $CX$, where $C = [c_{ij}]$. $C$ is called the **consumption** matrix because $CX$ gives the units of output consumed by industry. The vector $CX$ is called the **intermediate demand vector**. In Example 3.9,

$$C = \begin{bmatrix} 0.00 & 0.60 \\ 0.20 & 0.07 \end{bmatrix}$$

**TABLE 3.1 Demand Table**

| *Purchases from* | Ag | Mn | Sv |
|---|---|---|---|
| Agriculture | 0.2 | 0.1 | 0.3 |
| Manufacturing | 0.1 | 0.3 | 0.2 |
| Service | 0.2 | 0.2 | 0.2 |

Now, suppose that the consumers demand $d_i$ units from industry $i$. Then the total demand for the $i$th industry is

$$d_i + c_{i1}x_1 + c_{i2}x_2 + \cdots + c_{in}x_n$$

We write this in matrix format as

$$D + CX$$

The vector $D$ is referred to as the **final demand vector**. In the above example, $D = [10{,}000,\ 2500]^t$.

As in the example, we set total demand equal to output, obtaining the equation

$$X = D + CX$$

Usually, $D$ and $C$ are given and our goal is to find $X$. The above equation is equivalent to $(I - C)X = D$, which we solve as $X = (I - C)^{-1}D$.

### EXAMPLE 3.10

Suppose that we have an economy consisting of three sectors: agriculture (Ag), manufacturing (Mn), and service (Sv). To produce \$1's worth of output, each sector purchases units as shown in Table 3.1.

Answer the following questions:

1. If the production vector is $[\text{Ag}, \text{Mn}, \text{Sv}]^t = [10, 20, 30]^t$, what is the industrial demand for manufacturing?
2. What is the consumption matrix for this problem?
3. What level of production would be necessary to sustain a demand of $D = [70, 80, 90]^t$?

*Solution.* From the table, we see that at the levels of production given in 1, manufacturing will sell a total of

$$(0.1 \cdot 10) + (0.3 \cdot 20) + (0.2 \cdot 30) = 13$$

units to industry. This is the demand for manufacturing.

For question 2, recall that the product of the consumption matrix with the output vector should equal the amount sold to industry. From part 1, it is clear that the rows of our table should be the rows of $C$. Thus, our consumption matrix is

$$C = \begin{bmatrix} 0.2 & 0.1 & 0.3 \\ 0.1 & 0.3 & 0.2 \\ 0.2 & 0.2 & 0.2 \end{bmatrix}$$

For part 3, we write $X = (I - C)^{-1}D$ as above. Using MATLAB, we see

$$(I - C)^{-1} = \begin{bmatrix} 0.8 & -0.1 & -0.3 \\ -0.1 & 0.7 & -0.2 \\ -0.2 & -0.2 & 0.8 \end{bmatrix}^{-1} = \begin{bmatrix} 1.4607 & 0.3933 & 0.6461 \\ 0.3371 & 1.6292 & 0.5337 \\ 0.4494 & 0.5056 & 1.5449 \end{bmatrix}$$

It is of course striking that all the entries of $(I - C)^{-1}$ are positive. Multiplying by $D$ and rounding yield the production vector $X = [192, 202, 211]^t$.

Recall that $D = [70, 80, 90]^t$. The need to produce at such high levels to meet a relatively small demand means that our industries are inefficient. Notice, for example, that the sum of the first column of $C$ is 0.5. This means that 50% of our agricultural output is consumed by other industries! This would not be typical of a real economy.

The general method of finding the production vector works only if $I - C$ is invertible. It turns out, however, that the following theorem holds. It should be noted that the assumption on the row sums is very natural. Consider, for example, the economy in Example 3.10. To produce a dollar's worth of goods, agriculture must purchase goods worth a total of $0.2 + 0.1 + 0.3 = 0.6$ dollars from itself and other industries. This is just the sum of the first row of $C$. Similarly, the second row sum is the amount manufacturing must purchase to produce one unit of output, and the third row sum is the amount that service must purchase. If any of these had been greater than 1, that sector would have been operating at a loss. If the sum were equal to 1, then that sector would have been just breaking even. Thus, the condition in this theorem is described by saying that all our industries are profitable.

**Theorem 3.11** *Suppose that $C$ is a matrix with nonnegative entries such that the sum of each of its rows is strictly less than 1. Then $(I - C)^{-1}$ exists and has nonnegative entries.*

The proof of this theorem is based on the well-known fact that for $x$ a real number, $|x| < 1$,

$$(1 - x)^{-1} = 1 + x + x^2 + \cdots + x^n + \cdots$$

This formula is simple to prove, provided that we grant the convergence of the series. Specifically, note that

$$\begin{aligned}(1-x)(1+x+x^2+\cdots+x^n+\cdots) &= 1(1+x+x^2+\cdots+x^n+\cdots)\\ &\quad -x(1+x+x^2+\cdots+x^n+\cdots)\\ &= 1+x+x^2+\cdots+x^n+\cdots\\ &\quad -(x+x^2+\cdots+x^n+\cdots)=1\end{aligned}$$

proving the formula.

Remarkably, the same formula works if we replace $x$ with an $n\times n$ matrix $C$. We say that a sequence $X_k$ of $m\times n$ matrices converges to an $m\times n$ matrix $X$ if

$$\lim_{k\to\infty}(X_k)_{ij}=X_{ij} \tag{3.22}$$

for all $1\le i\le m$ and $1\le j\le n$. The foregoing argument shows that if $C$ is an $n\times n$ matrix for which the series

$$I+C+C^2+\cdots+C^n+\cdots \tag{3.23}$$

converges to a matrix $B$, then $(I-C)B=I$. Hence, $I-C$ is invertible and $B$ is its inverse.

If we can prove that the series (3.23) converges for matrices of the form stated in Theorem 3.11, then we have proved Theorem 3.11, since if $C$ is positive, so is any power of $C$, and thus, so is $B$.

The idea behind the convergence proof is most easily demonstrated with a specific matrix. Let

$$C=\begin{bmatrix}0.00 & 0.60\\ 0.20 & 0.07\end{bmatrix}$$

(This is the consumption matrix from Example 3.9.)

Let $S=[1,1]^t$. Note that for any $2\times 2$ matrix $B$

$$BS=\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}b_{11}+b_{12}\\ b_{21}+b_{22}\end{bmatrix}$$

Thus, the components of $BS$ are the row sums of $B$. In particular, for $C$ as above,

$$CS=[0.6,0.27]\le 0.27[1,1]^t=0.27S$$

where we define $\le$ for matrices by saying that each entry of the first matrix is less than or equal to the corresponding entry for the second.

Multiplying both sides of this inequality by $C$ on the left, we get

$$C^2S \leq 0.27CS \leq (0.27)^2S$$

(You should think about why multiplying both sides of a matrix inequality by a matrix with nonnegative coefficients is allowed.)

Continuing, we see that

$$C^nS \leq (0.27)^nS = [(0.27)^n, (0.27)^n]$$

This means that each of the row sums in $C^n$ is $\leq (0.27)^n$, and hence, each entry of $C^n$ is $\leq (0.27)^n$. The convergence follows from the comparison test, since $\sum_0^\infty (0.27)^n$ converges.

The argument for general $C$ is similar. In the $n \times n$ case, one uses $S = [1, 1, \ldots, 1]^t$. Then, for any $n \times n$ matrix $B$, the matrix $BS$ is, once again, the vector of row sums of $B$. In particular, $CS \leq mS$, where $m$ is the largest row sum in $C$. Repeated multiplication by $C$ shows $C^nS \leq m^nS$. The convergence of the series then follows exactly as before, since, by assumption, $m$ is less than 1.

## Self-Study Questions

**3.4** ✓Suppose that due to seasonal variations in the economy described in Example 3.9 on page 197, the demand for coal drops to 4000 units, while the demand for electricity rises to 3000 units. What production levels of coal and electricity are necessary to meet this demand? What is the corresponding intermediate demand vector?

**3.5** ✓For the economy described in Example 3.10 on page 199:

**(a)** If the production vector is $[\text{Ag}, \text{Mn}, \text{Sv}]^t = [20, 15, 10]^t$, what is the industrial demand for manufacturing?

**(b)** What level of production would be necessary to sustain a demand of $D = [50, 35, 20]^t$?

**(c)** What property of $(I - C)^{-1}$ guarantees that if the demand of any quantity rises, then the production levels must also rise?

## EXERCISES

**3.95** Suppose that we have an economy governed by three sectors: lumber (Lu), paper (Pa), and labor (Lb). Make up a table using your own numbers similar to that from Example 3.10 to describe the units each industry would need to purchase from the other industries to produce one unit of output. Try to keep your numbers reasonable. For example, how would the amount the lumber company spends on paper to produce one \$1's worth of lumber compare with the amount the paper company would spend on lumber? Would the labor sector typically spend more on paper or lumber? Note also the comments preceding Theorem 3.11.

**3.96** For the economy you invented in Exercise 3.95, what quantity of lumber does industry demand if the production vector is $[Lu, Pa, Lb] = [30, 20, 10]^t$?

**3.97** For the economy you invented in Exercise 3.95, give the consumption matrix. At what level does the economy need to produce if the demand is $[Lu, Pa, Lb]^t = [70, 80, 100]^t$? (*Note*: This is best done with software that can invert matrices. If none is available, express your answer in terms of the inverse of a specific matrix but do not compute the inverse.)

**3.98** Suppose that in Example 3.9, besides coal and electricity, there is a train company whose sole business is transporting coal from the mine to the electric company. Assume also that the trains run on coal, and hence, the train company uses large amounts of coal and only a little electricity. Invent a consumption matrix that might correspond to this information. You may assume that the coal company does not use the train. Note also the comments preceding Theorem 3.11.

Find the output level for the three companies necessary to satisfy the demand described in Example 3.9. (*Note*: This is best done with software that can invert matrices. If none is available, express your answer in terms of the inverse of a specific matrix but do not compute the inverse.)

**3.99** Suppose that in Exercise 3.98, the trains run on electricity. Give a potential consumption matrix to describe this situation.

## 3.4 THE *LU* FACTORIZATION

You are assigned to run some manufacturing process that involves solving a system of $n$ equations in $n$ unknowns to determine how to set your machines. Let us suppose that the system is given in the form

$$AX = Y$$

where $A$ is an $m \times n$ matrix. Let us also assume that the value of $A$ is reset each hour while the value of $Y$ changes each minute.

One approach to finding $X$ would be to row reduce the coefficient matrix for the system $AX = Y$ to, say, echelon form once a minute. (Let us hope you have a computer!) Note, however, that the steps in the row reduction depend only on the coefficient matrix $A$. By keeping track of the steps in the reduction process, we need only redo the reduction every hour when $A$ changes. We use the double matrix formulation from Section 3.3 to keep track of the steps in the reduction process.

To explain this, suppose that

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 7 & 8 \end{bmatrix} \tag{3.24}$$

The general system we wish to solve is, then,

$$\begin{aligned} x_1 + 2x_2 + \phantom{4}x_3 &= y_1 \\ x_1 + 3x_2 + 4x_3 &= y_2 \\ 2x_1 + 7x_2 + 8x_3 &= y_3 \end{aligned}$$

As in Section 3.3, we describe this system with the double matrix

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 2 & 7 & 8 & 0 & 0 & 1 \end{array}\right] \tag{3.25}$$

where the $3 \times 3$ matrix on the left is the matrix of coefficients for the $x_i$ and the one on the right is the matrix of coefficients for the $y_i$. We reduce this double matrix until the $3 \times 3$ matrix on the left is in echelon form:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 3 & 6 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array} \tag{3.26}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & -3 & 1 & -3 & 1 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 - 3R_2 \end{array} \tag{3.27}$$

Let $U$ be the $3 \times 3$ matrix on the left of this double matrix and let $B$ be the one on the right. As explained in Section 3.3, the system $AX = Y$ is equivalent to the system $UX = BY$. Thus, for example, to solve

$$\begin{aligned} x_1 + 2x_2 + \phantom{4}x_3 &= 1 \\ x_1 + 3x_2 + 4x_3 &= 2 \\ 2x_1 + 7x_2 + 8x_3 &= 3 \end{aligned}$$

we compute $B[1, 2, 3]^t = [1, 1, -2]^t$ and solve

$$\begin{aligned} x_1 + 2x_2 + \phantom{3}x_3 &= \phantom{-}1 \\ x_2 + 3x_3 &= \phantom{-}1 \\ -\,3x_3 &= -2 \end{aligned}$$

by back substitution, finding $x_3 = \frac{2}{3}$, $x_2 = -1$, and $x_1 = \frac{7}{3}$.

The idea just demonstrated, in principle, solves the problem with which we began this section. Every time the coefficient matrix $A$ changes, we form the double matrix $[A|I]$ and reduce until we have obtained a matrix of the form $[U|B]$ where $U$ is in echelon form. Then, $X$ is found by solving the system

$$UX = BY \tag{3.28}$$

There is, however, an additional improvement we can make to this procedure. Before describing it, however, we first note a few general features of the reduction process.

**Proposition 3.5** *Let $A$ be an $m \times n$ matrix and let $I$ be the $m \times m$ identity matrix. Suppose that the double matrix $[A|I]$ is row equivalent to the double matrix $[U|B]$, where $U$ is $m \times n$ and $B$ is $m \times m$. Then $B$ is invertible and $A = B^{-1}U$.*

*Proof.* Let $X \in \mathbb{R}^n$ and let $Y = AX$. From the preceding discussion, $AX = Y$ implies $UX = BY$—that is, $UX = BAX$. Since this holds for all $X \in \mathbb{R}^n$, it follows (from the uniqueness of the matrix of a linear transformation) that

$$U = BA$$

Furthermore, $B$ is row equivalent to $I$and, hence, invertible. Our theorem follows by multiplying both sides of the above equation by $B^{-1}$. □

In Theorem 1.4 on page 49, we proved that an $m \times n$ matrix $A$ may be reduced to echelon form using a sequence of elementary row operations of the following form:

1. Interchange two rows.
2. Add a multiple of one row to a *lower* row—that is, transform the row $R_i$ into $R_i + cR_j$, where $j < i$.
3. Multiply one row by a nonzero constant.

We say that an $m \times n$ matrix $A$ is **type-2** if $A$ may be brought into echelon using a sequence of operations of only type-2 above. The number $c$ in 2 is referred to as the **multiplier**. A matrix will be type-2 if, in the reduction process, we never are forced to exchange two rows. The only situation in the Gaussian reduction process where we are *forced* to exchange rows is if we need to produce a nonzero entry in a pivot position. Thus, for example, the following matrix is not type-2:

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 4 \end{bmatrix}$$

Notice that applying a type-2 row operation to the identity matrix produces a lower triangular matrix with 1's on the main diagonal—that is, a **unipotent matrix**. More generally, applying operations of type-2 to lower triangular unipotent matrices produces lower triangular unipotent matrices. These comments lead to the following important theorem. The decomposition in Theorem 3.12 is referred to as the ***LU* decomposition** of $A$.

**Theorem 3.12** (LU Decomposition Theorem). *Let $A$ be an $m \times n$ type-2 matrix. Then there is an $m \times m$ lower triangular unipotent matrix $L$ and an $m \times n$ echelon matrix $U$ such that*

$$A = LU$$

*Proof.* Let $U$ be an echelon form for $A$ obtained by applying a sequence of type-2 row operations to $A$. Then, from the preceding comments, the matrix $B$ in Proposition 3.5 is unipotent and invertible. Our theorem follows from Proposition 3.5 together with the observation that the inverse of a unipotent matrix is unipotent. (See Exercise 3.73 on page 192.) □

What makes the $LU$ decomposition so useful is that *there is a way of computing $L$ as we reduce $A$ that does not require any more computations than just computing $U$ itself.* To explain this, notice that from the proof of Theorem 3.12, $L = B^{-1}$, where $[U|B]$ is obtained by applying type-2 row operations to $[A|I]$. Elementary row operations are reversible. We could, if we wish, transform $[U|B]$ back into $[A|I]$ by applying the inverse row operations in reverse order. If we apply the same operations to the *triple* matrix $[U|B|I]$, we obtain $[A|I|L]$, where $L = B^{-1}$.

Specifically, for the matrix (3.24), $[U|B|I]$ is [from (3.27)]

$$\left[\begin{array}{ccc|ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 & -3 & 1 & 0 & 0 & 1 \end{array}\right] \tag{3.29}$$

which we transform to $[A|I|L]$ by doing the inverse operations to those indicated in (3.27) followed by those in (3.26). Note that the inverse of an operation of the form $R_i \to R_i + cR_j$ is $R_i \to R_i - cR_j$.

$$\left[\begin{array}{ccc|ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 6 & -2 & 0 & 1 & 0 & \mathbf{3} & 1 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 + \mathbf{3}R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 & \mathbf{1} & 1 & 0 \\ 2 & 7 & 8 & 0 & 0 & 1 & \mathbf{2} & 3 & 1 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 + \mathbf{1}R_1 \\ R_3 \to R_3 + \mathbf{2}R_1 \end{array}$$

$L$ is the matrix on the right—that is,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

Notice that each entry of $L$ is one of the multipliers used in transforming $[U|B|I]$ into $[A|I|L]$, as indicated in bold in (3.4) and (3.4). Furthermore, the multiplier for the transformation $R_i \to R_i + cR_j$ becomes the $(i,j)$ entry of $L$. Thus, for example, the transformation $R_3 \to R_3 + 3R_2$ in (3.4) tells us that $L_{32} = 3$. Similarly, the transformations $R_2 \to R_2 + 1R_1$ and $R_3 \to R_3 + 2R_1$ used in (3.4) tell us that $L_{21} = 1$ and $L_{31} = 2$, respectively. Finally, notice that the multipliers in (3.5) and (3.4) are just the *negatives* of those in (3.27) and (3.26). Hence, *the nonzero, nondiagonal, entries of L are just the negatives of the multipliers used in reducing A to echelon form*, provided that the echelon form was obtained using only type-2 row operations. Specifically, if the transformation $R_i \to R_i + c_{ij}R_j$ is one of the transformations used in reducing $A$, then $L_{ij} = -c_{ij}$.

We will prove that these comments are valid in general shortly. However, we first consider another example.

### EXAMPLE 3.11

Compute the $LU$ factorization for the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

*Solution.* We reduce $A$ using only type-2 row operations, recording the negatives of the multipliers in the appropriate positions in $L$ at each step:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{array}{lcl} \\ R_2 & \to & R_2 - \mathbf{2}R_1\,, \\ R_3 & \to & R_3 - \mathbf{1}R_1 \end{array} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{2} & 1 & 0 \\ \mathbf{1} & & 1 \end{bmatrix}$$

Now we are tempted to exchange rows 2 and 3. Row exchanges, however, are not type-2 operations. For similar reasons, we cannot divide row 2 by $-3$. Instead, we continue as follows:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & \frac{2}{3} \end{bmatrix} \begin{array}{lcl} \\ \\ R_3 & \to & R_3 + (\mathbf{1/3})R_2 \end{array} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & \mathbf{-1/3} & 1 \end{bmatrix} \tag{3.30}$$

The matrix on the left is $U$.

It is, of course, very nice that $L$ is so easy to compute. What is even nicer is that when we know $L$ and $U$, our system is, for all intents and purposes, solved. To explain this, note that if $A = LU$ is the $LU$ decomposition of $A$, then the system $AX = Y$ is equivalent to

$$L(UX) = Y$$

We solve this in two steps. First we find $W$ such that $LW = Y$. Then we solve the system $UX = W$. Both these systems can be solved *without any row reduction*, since $L$ is lower triangular and $U$ is in echelon form. (This is why these matrices are called "$L$" and "$U$.")

### EXAMPLE 3.12

Let $A$ be as in Example 3.11 on page 207. Use the $LU$ factorization to find all solutions to $AX = Y$, where $Y = [1, 2, 3]^t$.

*Solution.* We first solve the system $LW = Y$, where $L$ is as in (3.30). Thus, we solve

$$\begin{aligned} w_1 \qquad\qquad\quad &= 1 \\ 2w_1 + w_2 \qquad &= 2 \\ w_1 - \tfrac{1}{3}w_2 + w_3 &= 3 \end{aligned}$$

by back substitution, beginning with the first equation, finding $W = [1, 0, 2]^t$.

Next, we solve $UX = W$. This is equivalent to

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ -3x_2 - x_3 &= 0 \\ \tfrac{2}{3}x_3 &= 2 \end{aligned}$$

We find $x_3 = 3$, which yields (by back substitution) $x_2 = -1$ and, finally, $x_1 = 0$.

*Remark.* At this point, the reader might be questioning exactly how easy it is to solve equations using the $LU$ factorization. Certainly, in most of the problems that follow, directly solving the system is less work. In fact, not being allowed to switch equations or divide out scalars can make hand calculations tedious. The reader should bear in mind that the $LU$ factorization is designed for the convenience of computers, not humans. "Ugly" numbers are of no concern. The only reason we have exercises in computing the $LU$ factorization is to demonstrate to the reader what the computer is doing. Bear in mind, as well, that using the $LU$ factorization is efficient only if in the system $AX = Y$ we expect to use many different $Y$'s with the same $A$. If $A$ changes every time $Y$ changes, we should simply reduce each system.

As commented previously, not all matrices are type-2. Specifically, if we must exchange rows during the reduction process, then the matrix will not be type-2. Interestingly, it is possible to do all the row interchanges first, as the next proposition shows.

**Proposition 3.6** *Suppose that $A$ is an $m \times n$ non-type-2 matrix. Then there is an $m \times n$ type-2 matrix $A'$ whose rows are the same as those of $A$, only listed in a different order.*

*Proof.* Suppose first that $A$ is $2 \times m$. We may ignore all null columns of $A$ since they do not play a role in the reduction process. Thus, we may assume that at least one of $A_{11}$ and $A_{21}$ is nonzero. In fact, since $A$ is by hypothesis non type-2, $A_{11} = 0$. Switching the first two rows of $A$ produces a type-2 matrix, proving our result.

Now suppose by induction that we have proved the result for all matrices of size $k \times n$, where $k < m$. Let $A$ have size $m \times n$. Once again, we may assume that the first column of $A$ is nonzero. By exchanging rows of $A$, we may also assume that $A_{11} \neq 0$. Subtracting multiples of the first row of $A$ from lower rows produces a partitioned matrix of the form

$$R_1 = \begin{bmatrix} A_{11} & W \\ 0 & A^o \end{bmatrix}$$

where 0 is the $(m-1) \times 1$ zero matrix, $W$ has size $1 \times (n-1)$, and $A^o$ has size $(m-1) \times (n-1)$. From the induction hypothesis, there is a type-2 matrix $B$ obtained by rearranging the rows of $A^o$.

The matrix

$$R_1' = \begin{bmatrix} A_{11} & W \\ 0 & B \end{bmatrix}$$

is then a type-2 matrix obtained by rearranging the last $n-1$ rows of $R_1$. Let $A'$ be the matrix obtained from $A$ by rearranging its last $n-1$ in the same order as the rows of $R_1$ were rearranged to produce $R_1'$. Then $A'$ can be transformed into the type-2 matrix $R_1'$ using only type-2 operations, proving that $A'$ is type-2. Our theorem follows. □

There is an important way of stating Proposition 3.5 using matrices. A **permutation** of the sequence $1, 2, \ldots, n$ is a listing of these elements in a different order. Thus, a permutation of this sequence is a sequence $\sigma = \{i_j\}_{j=1}^n$, where $\{i_1, i_2, \ldots, i_n\} = \{1, 2, \ldots, n\}$. For each such $\sigma$ we define a transformation $T_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ by

$$T_\sigma([x_1, x_2, \ldots, x_n]^t) = [x_{i_1}, x_{i_2}, \ldots, x_{i_n}]^t$$

Thus, $T_\sigma$ permutes the rows of $X$ as dictated by the permutation $\sigma$. For example, if $\sigma$ is the sequence $2, 1, 3, \ldots, n$, then $T_\sigma(X)$ is $X$ with its first two rows interchanged.

It is easily seen that for all permutations $\sigma$ the transformation $T_\sigma$ is a *linear* transformation. Hence, there is a matrix $P_\sigma$ such that for all $X \in \mathbb{R}^n$

$$T_\sigma(X) = P_\sigma X$$

Finding $P_\sigma$ is simple. Let $I = [I_1, I_2, \ldots, I_m]$ be the $n \times n$ identity matrix. Then

$$\begin{aligned} P_\sigma &= P_\sigma I \\ &= [P_\sigma I_1, P_\sigma I_2, \ldots, P_\sigma I_n] \end{aligned}$$

$P_\sigma I_j$ is just $I_j$ with its rows permuted according to $\sigma$. Thus, $P_\sigma = P_\sigma I$ is just $I$ with its rows permuted according to $\sigma$. This leads to the following definition:

**Definition 3.9** *An $n \times n$ permutation matrix $P$ is a matrix obtained by permuting the rows of the $n \times n$ identity matrix.*

## EXAMPLE 3.13

Find a $3 \times 3$ permutation matrix $P$ such that for all $3 \times m$ matrices $A$, the matrix $PA$ is $A$ with its first two rows exchanged.

*Solution.* $P$ is the matrix obtained by exchanging the first two rows of the $3 \times 3$ identity matrix $I$. Hence,

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

As a check on our work, we compute

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \\ x_3 \end{bmatrix}$$

The existence of the $LU$ decomposition for the matrix $A'$ in Proposition 3.6 implies the following generalization of Theorem 3.12:

**Theorem 3.13** (*PLU* Decomposition Theorem). *Let $A$ be an $m \times n$ matrix. Then there is an $m \times m$ lower triangular unipotent matrix $L$, an $m \times n$ echelon form matrix $U$, and an $m \times m$ permutation matrix $P$ such that*

$$A = PLU \tag{3.31}$$

We promised to prove the comments we made concerning how to compute $L$. According to the discussion in this section, these comments follow directly from the following proposition:

**Proposition 3.7** *Suppose that a sequence of row transformations of the form*

$$R_i \to R_i + c_{ij}R_j \tag{3.32}$$

*where* $1 \le j < i \le n$ *is applied to the* $n \times n$ *identity matrix I, producing a matrix L. Suppose also that the transformations corresponding to higher values of j precede those corresponding to lower values of j—that is, if* $j_1 > j_2$*, then* $R_i \to R_i + c_{ij_1}R_{j_1}$ *is applied prior to* $R_k \to R_k + c_{kj_2}R_{j_2}$*, regardless of the values of i and k. Then L is an* $n \times n$ *unipotent matrix such that* $L_{ij} = c_{ij}$ *for all* $1 \le j < i \le n$.

*Proof.* The largest allowed value for $j$ in (3.32) is $j = n - 1$, in which case $i = n$. Thus, the first transformation applied to $I$ is $R_n \to R_n + c_{n,n-1}R_{n-1}$, which produces the matrix

$$L^1 = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & c_{n,n-1} & 1 \end{bmatrix}$$

The next transformations to be applied correspond to $j = n - 2$ in (3.32). Specifically, they are $R_n \to R_n + c_{n,n-2}R_{n-2}$ and $R_{n-1} \to R_{n-1} + c_{n-1,n-2}R_{n-2}$. These transformations can only change the $(n - 2)$nd column of $L^1$ since the only nonzero entry in either the $(n - 2)$nd column and the $(n - 2)$nd row is $L^1_{n-2,n-2} = 1$. For the same reason, the resulting matrix $L^2$ satisfies $L^2_{n,n-2} = c_{n,n-2}$ and $L^2_{n-1,n-2} = c_{n-1,n-2}$. Thus,

$$L^2 = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & c_{n-1,n-2} & 1 & 0 \\ 0 & 0 & \cdots & c_{n,n-2} & c_{n,n-1} & 1 \end{bmatrix}$$

The next group of transformations to be applied will be those corresponding to $j = n - 3$ in (3.32). For reasons similar to those explained in the preceding paragraph, the result is a matrix $L^3$ that differs from $L^2$ only in that $L^3_{j,n-3} = c_{j,n-3}$ for $n - 3 < j \le n$.

We continue, applying the transformations corresponding to $j = n - 4$, $j = n - 5, \ldots, j = n - k$, producing a matrix $L^k$. In each case, $L^k$ differs from $L^{k-1}$ only in that

$L^k_{j,n-k} = c_{j,n-k}$ for $n - k < j \leq n$. This is due to the fact that the only nonzero entry in both the $(n-k)$th column and the $(n-k)$th row of $L^{k-1}$ is $L^{k-1}_{n-k,n-k} = 1$. The final result is as claimed in the proposition. □

## EXERCISES

**3.100** For the following matrices $L$, $U$, and $Y$, solve the system $AX = Y$ for $X$ where $A = LU$ by first finding a $Z$ such that $LZ = Y$ and then finding an $X$ such that $UX = Z$. (*Note*: It is not necessary to compute $A$.)

**(a)** ✓✓ $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 3 & 2 & 0 & 4 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $Y = \begin{bmatrix} 6 \\ 5 \\ 4 \\ 3 \end{bmatrix}$

**(b)** ✓✓ $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, $Y = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

**(c)** $L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $Y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

**3.101** Compute the $LU$ factorization for the following matrices.

**(a)** ✓✓ $\begin{bmatrix} 1 & 4 & -1 \\ 2 & 10 & 2 \\ 1 & 5 & 2 \end{bmatrix}$

**(b)** $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 3 & 4 & 1 \\ 1 & 0 & 0 & 4 \\ 3 & 7 & 3 & 4 \end{bmatrix}$

**(c)** $\begin{bmatrix} 2 & 4 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

**(d)** ✓✓ $\begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix}$

**3.102** ✓✓Let $[U|B]$ be the second double matrix in Example 3.8 on page 185. Compute $BA$. You should get $U$. Why? (See the proof of Theorem 3.12 on page 206.)

**3.103** ✓✓What would be the result if you were to repeat Exercise 3.102 for the fifth double matrix? Why? Can you state a theorem about the double matrices one gets in computing inverses?

**3.104** ✓✓A lower (or upper) triangular matrix is called unipotent if it has only 1's on the main diagonal. In Example 3.11 on page 207 in the text, find a diagonal matrix $D$, a lower triangular, unipotent matrix $L$, and an upper triangular, unipotent matrix $U$ such that $A = LDU$. This is called the ***LDU* factorization** of $A$.

**3.105** Prove that the $LDU$ factorization of an $n \times n$ invertible matrix $A$ is unique when it exists—that is, suppose that $A = LDU = L'D'U'$, where $L$ and $L'$ are lower triangular unipotent matrices, $U$ and $U'$ are upper triangular unipotent matrices, and $D$ and $D'$ are diagonal matrices. Prove that then $L = L'$, $U = U'$, and $D = D'$. [*Hint:* Note that $(L')^{-1}L = D'U'U^{-1}D^{-1}$. Explain why the matrix on the left is lower triangular unipotent and the matrix on the right is upper triangular. How can this be?]

**3.106** ✓✓Find a $3 \times 3$ permutation matrix $P$ such that for all $3 \times m$ matrices $A$ the matrix $PA$ is $A$ with its last two rows exchanged. Verify your answer by computing $PX$, where $X = [x_1, x_2, x_3]^t$.

**3.107** Find a $3 \times 3$ permutation matrix $P$ such that for all $3 \times m$ matrices $A$ the matrix $PA$ is $A$ with its rows permuted according to the rule $R_1 \to R_2$, $R_2 \to R_3$, $R_3 \to R_1$. Verify your answer by computing $PX$, where $X = [x_1, x_2, x_3]^t$.

**3.108** Redo Exercise 3.106 in the $4 \times 4$ case, suitably adapting the instructions.

**3.109** Find a $4 \times 4$ permutation matrix $P$ such that for all $4 \times m$ matrices $A$ the matrix $PA$ is $A$ with $R_1$ and $R_3$ exchanged and $R_2$ and $R_4$ exchanged. Verify your answer by computing $PX$, where $X = [x_1, x_2, x_3, x_4]^t$.

**3.110** ✓✓Attempt to compute the $LU$ factorization for the matrix $A$:

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 5 \\ 2 & 1 & 4 \end{bmatrix}$$

Compute the $LU$ factorization for $A'$, where $A'$ is $A$ with its last two rows exchanged. Finally, compute the $PLU$ factorization of $A$.

**3.111** Write all $2 \times 2$ permutation matrices. (*Note:* $I$ is considered to be a permutation matrix.)

**3.112** Write all six $3 \times 3$ permutation matrices. (*Note:* $I$ is considered to be a permutation matrix.)

**3.113** These exercises refer to the permutation transformation $T : \mathbb{R}^5 \to \mathbb{R}^5$ defined by

$$T([x_1, x_2, x_3, x_4, x_5]^t) = [x_5, x_1, x_4, x_2, x_3]^t$$

**(a)** Find a $5 \times 5$ permutation matrix $P$ such that, for all $X \in \mathbb{R}^5$, $T(X) = PX$.

**(b)** The inverse transformation $T^{-1}$ satisfies

$$T^{-1}([x_5, x_1, x_4, x_2, x_3]^t) = [x_1, x_2, x_3, x_4, x_5]^t$$

Give a formula for

$$T^{-1}([y_1, y_2, y_3, y_4, y_5]^t)$$

Then find the $5 \times 5$ matrix $Q$ such that, for all $Y \in \mathbb{R}^5$, $QY = T^{-1}(Y)$. Check that $QP = I$.

*Remark.* The preceding exercise demonstrates the general fact that the inverse of a permutation matrix is a permutation matrix. Roughly, this is true because the operation of rearranging the rows of a matrix can always be undone by putting the rows back into their original order. The permutation that puts them back into the original order is the **inverse permutation**. We will not describe this concept in detail.

### 3.4.1 Computer Projects

## EXERCISES

**1.** Would the designers of MATLAB have omitted the $LU$ decomposition? Certainly not! Enter the matrix $A$ from Example 1 in the text and give the command [L,U]=lu(A). MATLAB should respond almost immediately with the $LU$ factorization.

**2.** We commented earlier that the $LU$ factorization exists only if the matrix can be reduced without row interchanges. What would the lu command in MATLAB do if it came across a matrix that required row interchanges? To investigate this, enter a $3 \times 3$ upper triangular matrix into MATLAB. (Recall that a matrix is upper triangular if all the entries below the main diagonal are zero.) In order to avoid trivialities, make all the entries on or above the main diagonal nonzero. Call your matrix $A$. Let

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In MATLAB, compute B=E∗A. How are $B$ and $A$ related? The matrix $E$ is called a permutation matrix. (See Exercise 3.113 on page 213.)

**3.** The matrix $B$ computed in Exercise 2 cannot be row reduced without interchanging rows. Ask MATLAB to compute [L,U]=B. Note that the $L$ you get is just the permutation matrix $E$. Use this to explain why $LU = B$.

**4.** Enter a $3 \times 3$ lower triangular, unipotent matrix into MATLAB. (Recall that a triangular matrix is unipotent if all the entries on the main diagonal are 1.) To avoid trivialities, make all the entries below the main diagonal nonzero. Call your matrix $V$. Let C=V∗B, where $B$ is as in Exercise 2 and then ask MATLAB to compute [L,U]=lu(C). Show that $EL$ is lower triangular, where $E$ is as above.

*Remark.* In general, in the MATLAB command [L,U]=lu(A), the $L$ at least has the property that it becomes unipotent after interchanging a few rows. The $U$ will always be upper triangular. It is a theorem that such a decomposition always exists.

**5.** Enter a new upper triangular matrix $A$ and again let $B = EA$, where $E$ is as in Exercise 2. Then, using the same $V$ as in Exercise 4, compute [L,U]=lu(V∗B). You should get the same $L$ as before. Why? [*Hint:* Compute the $LU$ decomposition for $VE$.]

## 3.5 THE MATRIX OF A LINEAR TRANSFORMATION

### Coordinates

When introducing Cartesian coordinates in the plane, we begin by choosing a point 0 to serve as the origin. We then put a horizontal and a vertical axis through 0 and choose a unit of measure on each axis. We plot points as indicated in Figure 3.11, left. In mathematics, one generally uses the same unit of measure on both axes, although in science it is often necessary to use different units.

As Figure 3.11 shows, we can also define coordinates using non perpendicular axes. Let $X_1$ be the vector of unit length (measured in the units of measure on the $A$ axis) pointing in the positive direction of the $A$ axis and let $X_2$ be similarly defined with respect to the $B$ axis. According to Figure 3.11, if $X$ has coordinate vector $[x', y']^t$ with respect to the $A$-$B$ coordinate system, then

$$X = x'X_1 + y'X_2 \tag{3.33}$$

Conversely, since $X_1$ and $X_2$ are noncollinear, they are linearly independent, and hence, they form a basis of $\mathbb{R}^2$. Thus, every $X$ in $\mathbb{R}^2$ can be written as in equation (3.33). We can therefore *define* the coordinate vector of $X$ to be that vector $X' = [x', y']^t$ for which equation (3.33) holds.

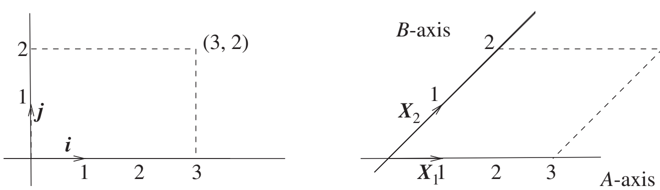



**FIGURE 3.11** Natural and Skewed Coordinates in $\mathbb{R}^2$.

Notice that in equation (3.33) the coefficient of the first basis vector ($X_1$) defines the first entry of $X'$ and the coefficient of the second basis vector ($X_2$) defines the second entry. It follows that the coordinates depend on both the basis vectors and the order in which they are listed. These comments suggest the following definition:

**Definition 3.10** *An **ordered basis** $X_1, X_2, \dots, X_n$ for a vector space $\mathcal{V}$ is a sequence of elements of $\mathcal{V}$ such that the set $\mathcal{B} = \{X_1, X_2, \dots, X_n\}$ is a basis for $\mathcal{V}$. If $X \in \mathcal{V}$, then the **coordinate vector** for $X$ with respect to the given basis is the vector $X' = [x_1', x_2', \dots, x_n']^t \in \mathbb{R}^n$ such that*

$$X = x_1'X_1 + x_2'X_2 + \cdots + x_n'X_n \tag{3.34}$$

*The vector $X'$ is also referred to as the "$\mathcal{B}$ coordinate vector of $X$."*

*Remark.* We will at times refer to "the ordered basis $\mathcal{B} = \{X_1, X_2, \dots, X_n\}$." This is an abuse of notation since an ordered basis is a *sequence* of vectors and the braces "{" and "}" denote a *set*. By definition, such a reference means that $\mathcal{B}$ is the sequence of vectors formed by listing the elements of the stated set in the stated order.

When we refer to *the* coordinate vector for $X$ with respect to the given basis, we are implying that $X$ may be written as a linear combination of the $X_i$ in only one way. This follows from the proposition below.

**Proposition 3.8** *Let $\mathcal{V}$ be a vector space and let $\{X_1, X_2, \dots, X_n\}$ be an ordered basis for $\mathcal{V}$. Suppose that there are scalars $x_i$ and $y_i$, $1 \le i \le n$, such that*

$$X = x_1X_1 + x_2X_2 + \cdots + x_nX_n$$
$$X = y_1X_1 + y_2X_2 + \cdots + y_nX_n$$

*Then $x_i = y_i$ for all $i$.*

*Proof.* Subtracting the preceding equalities yields

$$0 = (x_1 - y_1)X_1 + (x_2 - y_2)X_2 + \cdots + (x_n - y_n)X_n$$

Since the $X_i$ are linearly independent, $x_i - y_i = 0$ for all $i$, proving our proposition. □

If $\mathcal{B} = \{X_1, X_2, \dots, X_n\}$ is an ordered basis of $\mathbb{R}^n$, then formula (3.34) is equivalent to the matrix equality

$$X = P_{\mathcal{B}}X'$$

where $P_{\mathcal{B}} = [X_1, X_2, \ldots, X_n]$ is the $n \times n$ matrix whose columns are the basis vectors. [See formula (1.42) on page 78.] $P_{\mathcal{B}}$ is called the **point matrix** for the ordered basis $\mathcal{B}$ because multiplying a coordinate vector $X'$ by $P_{\mathcal{B}}$ produces the point $X$ with $\mathcal{B}$ coordinates $X'$. Since the columns of $P_{\mathcal{B}}$ are linearly independent, $P_{\mathcal{B}}$ has rank $n$, and hence, is invertible. Let $C_{\mathcal{B}} = P_{\mathcal{B}}^{-1}$. The preceding equality implies

$$C_{\mathcal{B}} X = X'$$

$C_{\mathcal{B}}$ is called the **coordinate matrix** for the basis because multiplying a point by it produces the coordinates of the point.

### EXAMPLE 3.14

In $\mathbb{R}^2$, consider the ordered basis $\mathcal{B}$ of $\mathbb{R}^2$ formed by $X_1 = [1, 0]^t$ and $X_2 = [1, 1]^t$.

(a) What point $X$ has $\mathcal{B}$ coordinate vector $X' = [-3, 2]^t$?
(b) What is the $\mathcal{B}$ coordinate vector $X'$ for the point $X = [3, 1]^t$?

*Solution.* The point matrix is

$$P_{\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Hence, the answer to (a) is

$$X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

This answer is illustrated on the left in Figure 3.12.

The inverse of $P_{\mathcal{B}}$ is computed to be

$$C_{\mathcal{B}} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

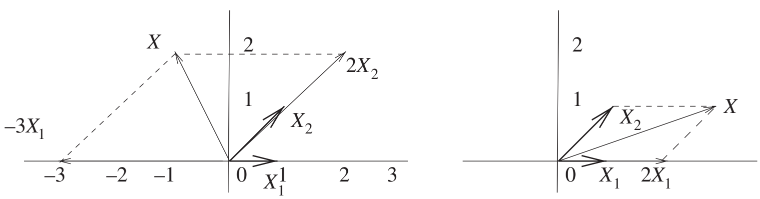



**FIGURE 3.12** Example 1.

Hence, the answer to (b) is

$$X' = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

This answer is illustrated on the right in Figure 3.12.

Here is another example of the same concept:

### EXAMPLE 3.15

Consider the ordered basis $\mathcal{B}$ of $\mathbb{R}^3$ formed by the vectors

$$X_1 = [1, 1, 0]^t, \quad X_2 = [1, -1, 1]^t, \quad X_3 = [1, -1, -2]^t$$

Calculate the $\mathcal{B}$ coordinate vector of the point $X = [1, 1, 1]^t$. Show that the coordinates do express $X$ as a linear combination of the elements of $\mathcal{B}$.

*Solution.* The point matrix is

$$P_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & -2 \end{bmatrix}$$

The coordinate matrix is $C_{\mathcal{B}} = P_{\mathcal{B}}^{-1}$, which we compute to be

$$C_{\mathcal{B}} = \frac{1}{6} \begin{bmatrix} 3 & 3 & 0 \\ 2 & -2 & 2 \\ 1 & -1 & -2 \end{bmatrix}$$

The $\mathcal{B}$ coordinate vector of $X$ is $X' = C_{\mathcal{B}}X = [1, \frac{1}{3}, -\frac{1}{3}]^t$. As a check of our computations, we see that

$$X_1 + \tfrac{1}{3}X_2 - \tfrac{1}{3}X_3 = [1, 1, 0]^t + \tfrac{1}{3}[1, -1, 1]^t - \tfrac{1}{3}[1, -1, -2]^t = [1, 1, 1]^t = X$$

The reader might be wondering why one would want to use non perpendicular axes in $\mathbb{R}^n$. The answer is that many seemingly complicated problems become simpler when viewed in the correct coordinates, as the next example shows.

### EXAMPLE 3.16

Describe geometrically the effect of the transformation of $\mathbb{R}^2$ into $\mathbb{R}^2$ defined by multiplication by the matrix

$$A = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$$

*Solution.* Let $X_1 = [2, 1]^t$ and $X_2 = [1, 1]^t$. (We choose these particular vectors using techniques that will be explained in Chapter 5.) Then, as the reader may check,

$$AX_1 = [2, 1]^t = X_1$$
$$AX_2 = [2, 2]^t = 2X_2$$

For each $X$ in $\mathbb{R}^2$, let $X' = [x', y']$ denote its coordinates with respect to the ordered basis $\mathcal{B} = \{X_1, X_2\}$ so that

$$X = x'X_1 + y'X_2$$

Hence,

$$AX = x'AX_1 + y'AX_2 = x'X_1 + 2y'X_2$$

These equations say that multiplication by $A$ transforms the vector with $\mathcal{B}$ coordinates $[x', y']^t$ into the vector with $\mathcal{B}$ coordinates $[x', 2y']^t$. Thus, multiplication of $X$ by $A$ doubles the $y'$ coordinate of $X$ while leaving the $x'$ coordinate unchanged. (See Figure 3.13.)

*Remark.* The basis used in Example 3.16 had the special property that $AX_1 = X_1$ and $AX_2 = 2X_2$. In general, an **eigenvector** for an $n \times n$ matrix $A$ is a column vector $X$ such that $AX = \lambda X$ for some scalar $\lambda$. Typically, matrix transformations are most readily described using bases in which every basis element is an eigenvector (an **eigenbasis**), provided such a basis exists. We reintroduce and study this subject in depth in Chapter 5.

The conclusion of Example 3.16 can be stated algebraically as

$$(AX)' = \begin{bmatrix} x' \\ 2y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

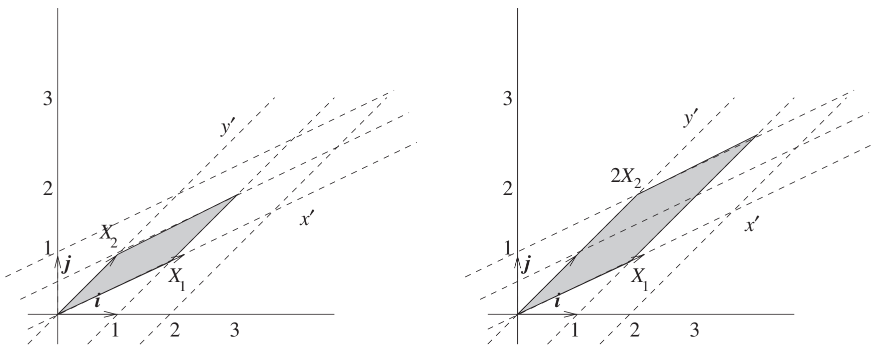



**FIGURE 3.13** The $y'$ coordinate is doubled.

which is equivalent to

$$(AX)' = M(X')$$

where $M$ is the $2 \times 2$ matrix in the last equality. In words, this equality says, "The coordinate vector of $AX$ is $M$ times the coordinate vector of $X$."

It turns out that any linear transformation between two finite-dimensional vector spaces can be described in similar terms. Just as before, we would paraphrase the following theorem as, "The coordinate vector of $L(X)$ is $M$ times the coordinate vector of $X$."

**Theorem 3.14** *Let $\mathcal{V}$ and $\mathcal{W}$ be finite-dimensional vector spaces where* $\dim \mathcal{V} = n$ *and* $\dim \mathcal{W} = m$, *and let $L : \mathcal{V} \to \mathcal{W}$ be a linear transformation. Let $\mathcal{B}$ and $\overline{\mathcal{B}}$ be, respectively, ordered bases for $\mathcal{V}$ and $\mathcal{W}$. Then there exists a unique $m \times n$ matrix $M$ such that for all $X \in \mathcal{V}$*

$$(L(X))' = M(X') \tag{3.35}$$

*where $(L(X))'$ is the $\overline{\mathcal{B}}$ coordinate vector of $L(X)$ and $X'$ is the $\mathcal{B}$ coordinate vector of $X$.*

The matrix $M$ is referred to as **the matrix of $L$ with respect to the bases $\mathcal{B}$ and $\overline{\mathcal{B}}$**. We will prove Theorem 3.14 for general vector spaces shortly. However, the proof for matrix transformations is simple. Let $A$ be an $m \times n$ matrix. For $L = T_A$ equality (3.35) is equivalent to

$$C_{\overline{\mathcal{B}}}AX = MC_{\mathcal{B}}X$$

Since the matrix of a linear transformation is unique, this equality is equivalent to

$$C_{\overline{\mathcal{B}}}A = MC_{\mathcal{B}}$$

Multiplying both sides of this equality by $C_{\mathcal{B}}^{-1} = P_{\mathcal{B}}$ shows both that $M$ exists and

$$M = C_{\overline{\mathcal{B}}}AP_{\mathcal{B}}$$

These calculations prove the following theorem.

**Theorem 3.15** *Let $A$ be an $m \times n$ matrix and let $\mathcal{B}$ and $\overline{\mathcal{B}}$ be, respectively, ordered bases for $\mathbb{R}^n$ and $\mathbb{R}^m$. Then the matrix of $T_A : \mathbb{R}^n \to \mathbb{R}^m$ with respect to the bases $\mathcal{B}$ and $\overline{\mathcal{B}}$ is*

$$M = C_{\overline{\mathcal{B}}}AP_{\mathcal{B}} \tag{3.36}$$

In Example 3.16, we used the same bases for the domain and the range, although this is not necessary, as the next example shows.

### EXAMPLE 3.17

Find the matrix $M$ of $T_A$ relative to the ordered bases $\mathcal{B}$ of $\mathbb{R}^3$ and $\overline{\mathcal{B}}$ of $\mathbb{R}^2$, where

$$\mathcal{B} = \{[1,0,0]^t, [0,1,0]^t, [2,7,-11]^t\}$$
$$\overline{\mathcal{B}} = \{[2,-3]^t, [1,4]^t\}$$

and

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -3 & 4 & 2 \end{bmatrix}$$

*Solution.* The point matrices are

$$P_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 7 \\ 0 & 0 & -11 \end{bmatrix}$$
$$P_{\overline{\mathcal{B}}} = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}$$

We compute that

$$C_{\overline{\mathcal{B}}} = P_{\overline{\mathcal{B}}}^{-1} = \frac{1}{11}\begin{bmatrix} 4 & -1 \\ 3 & 2 \end{bmatrix}$$

Hence,

$$\begin{aligned} M &= C_{\overline{\mathcal{B}}} A P_{\mathcal{B}} \\ &= \frac{1}{11}\begin{bmatrix} 4 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ -3 & 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 7 \\ 0 & 0 & -11 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{aligned} \tag{3.37}$$

finishing the example.

*Remark.* It is a theorem that *the transformation defined by any* $2 \times 3$, *rank 2 matrix* $A$ *can be described by the same matrix* $M$ *of formula (3.37).* Explicitly, there are bases $\mathcal{B} = \{X_1, X_2, X_3\}$ of $\mathbb{R}^3$ and $\overline{\mathcal{B}} = \{X_1', X_2'\}$ of $\mathbb{R}^2$ such that the matrix of $A$ with respect to these bases is $M$.

Coordinates are also important in studying vector spaces other than $\mathbb{R}^n$. Recall that $\mathcal{P}_n$ is the space of all polynomial functions

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

where the $a_i$ are scalars. From the results of Section 2.2, the polynomials $\{1, x, x^2, \ldots, x^n\}$ form an ordered basis for $\mathcal{P}_n$ (the standard basis for $\mathcal{P}_n$). According to Definition 3.10, the coordinate vector for the polynomial $p(x)$ in the standard basis is, then,

$$[a_0, a_1, \ldots, a_n]^t$$

Thus, for example, the coordinate vector for $p(x) = 2 - 3x + x^3$ relative to the standard basis for $\mathcal{P}_3$ is $[2, -3, 0, 1]^t$. We can, of course, use other bases for this space as well, as the next example shows.

### EXAMPLE 3.18

Let $\mathcal{B} = \{1, (x-1), (x-1)^2\}$ form an ordered basis for $\mathcal{P}_2$. (a) What polynomial has $\mathcal{B}$ coordinate vector $[1, 2, 3]^t$? (b) What is the $\mathcal{B}$ coordinate vector of $p(x) = 3 - 2x + 5x^2$?

*Solution.* The answer to (a) is simple:

$$\begin{aligned} p(x) &= 1 + 2(x-1) + 3(x-1)^2 \\ &= 2 - 4x + 3x^2 \end{aligned}$$

For (b) we need to write $p(x)$ as a linear combination of the basis elements. Since $x = (x-1) + 1$,

$$\begin{aligned} 3 - 2x + 5x^2 &= 3 - 2[(x-1) + 1] + 5[(x-1) + 1]^2 \\ &= 3 - 2(x-1) - 2 + 5[(x-1)^2 + 2(x-1) + 1] \\ &= 6 + 8(x-1) + 5(x-1)^2 \end{aligned}$$

Hence, the $\mathcal{B}$ coordinate vector of $p(x)$ is $[6, 8, 5]^t$.

In studying vector spaces other than $\mathbb{R}^n$, we use the point and coordinate *transformations* rather than the point and coordinate *matrices*. For an ordered basis $\mathcal{B}$ of $\mathbb{R}^n$, the point transformation is, by definition, the transformation $T_{P_\mathcal{B}}$ defined by multiplication by the point matrix $P_\mathcal{B}$, and the coordinate transformation is, by definition, the transformation $T_{C_\mathcal{B}}$ obtained by multiplication by the coordinate matrix $C_\mathcal{B}$. In order to avoid having subscripts on subscripts, we will usually denote these operators by $T_\mathcal{B}^P$ and $T_\mathcal{B}^C$, respectively. We define generalizations of these operators as follows:

**Definition 3.11** *Let $\mathcal{B} = \{X_1, X_2, \dots, X_n\}$ be an ordered basis for a vector space $\mathcal{V}$. The* ***point transformation*** *for $\mathcal{V}$ with respect to $\mathcal{B}$ is the transformation $T_{\mathcal{B}}^P : \mathbb{R}^n \to \mathcal{V}$ defined by*

$$T_{\mathcal{B}}^P([x_1, \dots, x_n]^t) = x_1X_1 + x_2X_2 + \cdots + x_nX_n$$

*The* ***coordinate transformation*** *is the mapping $T_{\mathcal{B}}^C = (T_{\mathcal{B}}^P)^{-1} : \mathcal{V} \to \mathbb{R}^n$ that transforms the point $X \in \mathcal{V}$ into its coordinate vector $X' \in \mathbb{R}^n$ with respect to the given basis.*

Thus, for example, relative to the standard basis $\mathcal{B}$ of $\mathcal{P}_n$,

$$T_{\mathcal{B}}^P : [a_0, a_1, \dots, a_n]^t \to a_0 + a_1x + \cdots + a_nx^n$$

and

$$T_{\mathcal{B}}^C : a_0 + a_1x + \cdots + a_nx^n \to [a_0, a_1, \dots, a_n]^t$$

Coordinates allow us to "convert" elements of general finite-dimensional vector spaces into vectors. This works because *the point and coordinate transformations are both linear.* The proof of the linearity of the point transformation is virtually identical to the proof of the linearity properties for matrix multiplication (Theorem 1.9 on page 79) and the linearity of the coordinate transformation is then automatic since it is the inverse of the point transformation. (See Proposition 3.8 on page 216.)

### EXAMPLE 3.19

Determine if the polynomials $p(x) = 3 + x + x^2$, $q(x) = 1 - 4x + 5x^2$, and $r(x) = 7 + 11x - 7x^2$ form a linearly dependent set in $\mathcal{P}_2$.

*Solution.* We use the standard basis $\mathcal{B} = \{1, x, x^2\}$ to define coordinates on $\mathcal{P}_2$. We note that the $\mathcal{B}$ coordinate vectors for $p(x)$, $q(x)$, and $r(x)$ are linearly dependent since

$$[7, 11, -7]^t = 3[3, 1, 1]^t - 2[1, -4, 5]^t$$

Applying $T_{\mathcal{B}}^P$ to both sides of the equality and using linearity, we find

$$\begin{aligned} T_{\mathcal{B}}^P([7, 11, -7]^t) &= 3T_{\mathcal{B}}^P([3, 1, 1]^t) - 2T_{\mathcal{B}}^P([1, -4, 5]^t) \\ 7 + 11x - 7x^2 &= 3(3 + x + x^2) - 2(1 - 4x + 5x^2) \\ r(x) &= 3p(x) - 2q(x) \end{aligned}$$

Hence, the polynomials are linearly dependent.

We may use coordinate transformations to prove Theorem 3.14 for general finite dimensional vector spaces. Explicitly, equality (3.35) states that

$$T_{\overline{\mathcal{B}}}^{C}(L(X)) = M\left(T_{\mathcal{B}}^{C}(X)\right)$$

which is equivalent to

$$\begin{aligned} T_{\overline{\mathcal{B}}}^{C} \circ L &= T_M \circ T_{\mathcal{B}}^{C} \\ T_{\overline{\mathcal{B}}}^{C} \circ L \circ \left(T_{\mathcal{B}}^{C}\right)^{-1} &= T_M \end{aligned}$$

The matrix $M$ exists due to the matrix representation theorem (Theorem 3.1 on page 153) and the observation that the transformation of the left side of this equality is a linear transformation of $\mathbb{R}^n \to \mathbb{R}^m$.

## Application to Differential Equations

### EXAMPLE 3.20

Let $L : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear transformation defined by

$$L(y) = x^2y'' - y' + y$$

Find the matrix $M$ of $L$ with respect to the standard basis $\mathcal{B}$ of $\mathcal{P}_2$. Use your answer to find all solutions $y$ in $\mathcal{P}_2$ to the following differential equation:

$$x^2y'' - y' + y = 3 + 3x + x^2 \tag{3.38}$$

*Solution.* We compute that

$$\begin{aligned} L(a + bx + cx^2) &= x^2(a + bx + cx^2)'' - (a + bx + cx^2)' + (a + bx + cx^2) \\ &= a - b + (b - 2c)x + 3cx^2 \end{aligned}$$

Thus, $L$ transforms the polynomial with coordinates $[a, b, c]^t$ into the one with coordinates

$$\begin{bmatrix} a - b \\ b - 2c \\ 3c \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

The $3 \times 3$ matrix on the right of the preceding equality is $M$.

The differential equation (3.38), in our coordinates, is equivalent to the matrix equation

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$

which is easily solved to yield

$$[a, b, c]^t = \tfrac{1}{3}[20, 11, 1]^t$$

which corresponds to the polynomial $y(x) = \frac{1}{3}(20 + 11x + x^2)$. The reader may check that $y$ does solve equation (3.38).

Just as there is a standard basis for $\mathcal{P}_n$, a standard basis also exists for the space $M(m, n)$ of $m \times n$ matrices. For example, the general $2 \times 2$ matrix $X$ may be written as

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.39}$$

This equation says that the set $\mathcal{B}$ consisting of following matrices spans $M(2, 2)$:

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Since $\mathcal{B}$ is clearly linearly independent, it is a basis (the "standard basis") for $M(2, 2)$, which we order as listed above. Note that from formula (3.39) the coordinate vector for $X$ is then $X' = [a, b, c, d]^t$, which is just the rows of $X$ stretched out into one long vector and then transposed.

Recall that in $M(m, n)$, the matrices $E_{ij}$ whose only nonzero element is a 1 in the $(i, j)$ position form a basis that is referred to as the standard basis for $M(m, n)$. We order this basis by saying that $E_{ij}$ precedes $E_{pq}$ if either (a) $i < p$ or (b) $i = p$ and $j < q$. Thus, the **standard ordered basis for** $M(m, n)$ is the ordered basis

$$\mathcal{B} = \{E_{11}, E_{12}, \ldots, E_{1n}, E_{21}, E_{22}, \ldots, E_{2n}, \ldots, E_{m1}, E_{m2}, \ldots, E_{mn}\}$$

Just as in the $2 \times 2$ case, the $\mathcal{B}$ coordinate vector of an $m \times n$ matrix $X$ is obtained by laying the rows of $X$ side by side and then transposing so as to obtain an element of $\mathbb{R}^{mn}$.

### EXAMPLE 3.21

Let $L : M(2,2) \to M(3,2)$ be the linear transformation defined by $L(X) = AX$, where

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Compute the matrix $M$ of $L$ with respect to the standard bases for $M(2,2)$ and $M(3,2)$.

*Solution.* We compute that

$$\begin{bmatrix} 3 & 2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3a+2c & 3b+2d \\ -a & -b \\ c & d \end{bmatrix}$$

Hence, $L$ transforms the matrix with coordinates $[a,b,c,d]^t$ into one with coordinates

$$\begin{bmatrix} 3a+2c \\ 3b+2d \\ -a \\ -b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 0 & 3 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

The $6 \times 4$ matrix on the right is $M$.

Examples 3.20 and 3.21 worked out fairly easily because we were using standard bases. While it is possible to use the same technique for other bases, the computations can become involved. With some thought, however, we can break the process down into a series of more manageable calculations. Specifically, let $L : \mathcal{V} \to \mathcal{W}$ be a linear transformation. Let coordinates on $\mathcal{V}$ and $\mathcal{W}$ be defined respectively by ordered bases $\mathcal{B} = \{X_1, X_2, \dots, X_n\}$ and $\overline{\mathcal{B}} = \{Y_1, Y_2, \dots, Y_m\}$.

Let $X \in \mathcal{V}$ have $\mathcal{B}$ coordinate vector $X' = [x_1, x_2, \dots, x_n]^t$ so that

$$X = x_1X_1 + x_2X_2 + \cdots + x_nX_n$$

Then

$$L(X) = x_1L(X_1) + x_2L(X_2) + \cdots + x_nL(X_n)$$

Hence,

$$L(X)' = x_1L(X_1)' + x_2L(X_2)' + \cdots + x_nL(X_n)'$$

where the prime indicates $\overline{\mathcal{B}}$ coordinates. It follows that

$$L(X)' = M(X')$$

where

$$M = [L(X_1)', L(X_2)', \ldots, L(X_n)'] \tag{3.40}$$

Thus, *the ith column of M is the $\overline{\mathcal{B}}$ coordinate vector of $L(X_i)$.*

### EXAMPLE 3.22

Compute the matrix $M$ that represents the linear transformation $L$ in Example 3.20 using the ordered basis $\mathcal{B} = \{1, (x-1), (x-1)^2\}$ for the domain and $\overline{\mathcal{B}} = \{1, (x-2), (x-2)^2\}$ for the target space.

*Solution.* The columns of $M$ are the $\overline{\mathcal{B}}$ coordinates for the images of the elements of $\mathcal{B}$. We compute

$$\begin{aligned}
L(1) &= 1 \\
&= 1 + 0(x-2) + 0(x-2)^2 \\
L(x-1) &= x^2(x-1)'' - (x-1)' + (x-1) = x-2 \\
&= 0 + (x-2) + 0(x-2)^2 \\
L((x-1)^2) &= x^2((x-1)^2)'' - ((x-1)^2)' + (x-1)^2 = 3x^2 - 4x - 3 \\
&= 3((x-2)+2)^2 - 4((x-2)+2) - 3 \\
&= 7 + 8(x-2) + 3(x-2)^2
\end{aligned}$$

Hence,

$$\begin{aligned}
L(1)' &= [1, 0, 0]^t \\
L(x-1)' &= [0, 1, 0]^t \\
L((x-2)^2)' &= [7, 8, 3]^t
\end{aligned}$$

Thus,

$$M = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 3 \end{bmatrix}$$

The coordinate transformation on a vector space $\mathcal{V}$ provides what is called an "isomorphism" of $\mathcal{V}$ with $\mathbb{R}^n$, which simply means a way of converting elements of $\mathcal{V}$ into elements of $\mathbb{R}^n$ and vice versa.

### Isomorphism

**Definition 3.12** *Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces. Then an isomorphism of $\mathcal{V}$ onto $\mathcal{W}$ is an invertible linear transformation $L : \mathcal{V} \to \mathcal{W}$. In this case we say that $\mathcal{V}$ is isomorphic with $\mathcal{W}$.*

If $L : \mathcal{V} \to \mathcal{W}$ is an isomorphism of $\mathcal{V}$ onto $\mathcal{W}$, then $L^{-1} : \mathcal{W} \to \mathcal{V}$ is an isomorphism of $\mathcal{W}$ onto $\mathcal{V}$. Hence, if $\mathcal{V}$ is isomorphic with $\mathcal{W}$, then $\mathcal{W}$ is also isomorphic with $\mathcal{V}$.

The existence of the coordinate transformation proves that *every n-dimensional vector space is isomorphic with* $\mathbb{R}^n$. Isomorphic vector spaces behave identically with respect to all the basic linear algebraic concepts. For example, the following simple proposition holds:

**Proposition 3.9** *Suppose $\mathcal{W}$ is isomorphic with an n-dimensional vector space $\mathcal{V}$. Then $\mathcal{W}$ is n-dimensional.*

*Proof.* Let $\mathcal{B} = \{X_1, X_2, \dots, X_n\}$ be a basis of $\mathcal{V}$ and let $L : \mathcal{V} \to \mathcal{W}$ be an isomorphism. We claim that $\overline{\mathcal{B}} = \{L(X_1), L(X_2), \dots, L(X_n)\}$ is a basis for $\mathcal{W}$, and hence, $\mathcal{W}$ is $n$-dimensional.

**$\overline{\mathcal{B}}$ spans $\mathcal{W}$:**

Let $Y \in \mathcal{W}$ and let $X \in \mathcal{V}$ satisfy $L(X) = Y$. Then there are scalars $x_i$ such that

$$X = x_1X_1 + x_2X_2 + \cdots + x_nX_n$$

Then

$$Y = L(X) = x_1L(X_1) + x_2L(X_2) + \cdots + x_nL(X_n)$$

showing that $\overline{\mathcal{B}}$ spans $\mathcal{W}$.

**$\overline{\mathcal{B}}$ is linearly independent:**

Suppose that $x_i$ are scalars such that

$$x_1L(X_1) + x_2L(X_2) + \cdots + x_nL(X_n) = 0$$

Then

$$L(x_1X_1 + x_2X_2 + \cdots + x_nX_n) = 0$$

It follows that

$$x_1X_1 + x_2X_2 + \cdots + x_nX_n = 0$$

since, from the invertibility of $L$, there is only one $X$ such that $L(X) = 0$. The linear independence of the $X_i$ now shows that the $x_i$ are zero, proving the linear independence of $\overline{\mathcal{B}}$. Our proposition follows. □

### Invertible Linear Transformations

Being able to represent linear transformations by matrices allows us to extend theorems about matrices to theorems about linear transformations between finite dimensional vector spaces. For example, in Section 2.3, we said that an $n \times n$ matrix $A$ is nonsingular if for all $Y$ in $\mathbb{R}^n$, there is one, and only one, $X$ such that $AX = Y$. In Section 3.3, we extended this concept to linear transformations, stating that a linear transformation $L$ between two vector spaces $\mathcal{V}$ and $\mathcal{W}$ is invertible if for all $Y \in \mathcal{W}$, there is one, and only one, $X \in \mathcal{V}$ such that $L(X) = Y$.

One of the the most important results concerning nonsingular matrices is Theorem 2.20 on page 140 which characterizes nonsingular $n \times n$ matrices in terms of solving equations. The following result is the linear transformation version of this result. It is a direct consequence of Theorem 2.20 on page 140 and the ability to represent linear transformations by matrices. The proof is left as an exercise (Exercise 3.129).[1]

**Theorem 3.16** *Let $\mathcal{V}$ and $\mathcal{W}$ be finite-dimensional vector spaces with* $\dim \mathcal{V} = \dim \mathcal{W}$. *Let $L : \mathcal{V} \to \mathcal{W}$ be a linear transformation. Then the following statements are equivalent:*

*(a) The nullspace of $L$ is* $\{0\}$.

*(b)* The equation $L(X) = Y$ *has at least one solution for all $Y \in \mathcal{W}$.*

*(c) $L$ is invertible.*

**True-False Questions**: Justify your answers.

**3.31** Let $A$ be an $m \times n$ matrix and let $M$ be the matrix of $T_A$ with respect to bases $\mathcal{B}$ of $\mathbb{R}^m$ and $\overline{\mathcal{B}}$ of $\mathbb{R}^n$. Then rank $A$ = rank $M$. [*Hint:* Consider formula (3.36).]

**3.32** Let $\mathcal{B}$ and $\overline{\mathcal{B}}$ be ordered bases for $\mathbb{R}^n$. Then the matrix of the identity transformation of $\mathbb{R}^n$ into itself with respect to $\mathcal{B}$ and $\overline{\mathcal{B}}$ is the $n \times n$ identity matrix $I$.

**3.33** Let $\mathcal{B}$ and $\overline{\mathcal{B}}$ be ordered bases for $\mathbb{R}^n$ where $\mathcal{B} = \overline{\mathcal{B}}$. Then the matrix of the identity transformation of $\mathbb{R}^n$ into itself with respect to $\mathcal{B}$ and $\overline{\mathcal{B}}$ is the $n \times n$ identity matrix $I$.

**3.34** Let $A$ be an $m \times n$ matrix and let $M$ be the matrix of $T_A$ with respect to bases $\mathcal{B}$ and $\overline{\mathcal{B}}$. Then the dimensions of the column spaces of $A$ and $M$ are equal.

**3.35** Let $A$ be an $m \times n$ matrix and let $M$ be the matrix of $T_A$ with respect to bases $\mathcal{B}$ and $\overline{\mathcal{B}}$. Then the the column spaces of $A$ and $M$ are equal.

[1]The nullspace of a linear transformation was defined in Exercise 3.23 on page 162.

**3.36** Let $A$ be an $m \times n$ matrix and let $M$ be the matrix of $T_A$ with respect to bases $\mathcal{B}$ and $\overline{\mathcal{B}}$. Then the dimensions of the row spaces of $A$ and $M$ are equal.

**3.37** Let $A$ be an $m \times n$ matrix and let $M$ be the matrix of $T_A$ with respect to bases $\mathcal{B}$ and $\overline{\mathcal{B}}$. Then the row spaces of $A$ and $M$ are equal.

**3.38** Let $A$ be an $m \times n$ matrix and let $M$ be the matrix of $T_A$ with respect to bases $\mathcal{B}$ and $\overline{\mathcal{B}}$. Then the dimensions of the nullspaces of $A$ and $M$ are equal.

**3.39** Let $A$ be an $m \times n$ matrix and let $M$ be the matrix of $T_A$ with respect to bases $\mathcal{B}$ and $\overline{\mathcal{B}}$. Then $A$ and $M$ have the same nullspace.

## EXERCISES

*The first three exercises demonstrate that an appropriate choice of coordinates can simplify the form of some equations. Techniques for choosing coordinates are discussed in Section 6.6.*

**3.114** Sketch the graph of the curve $xy = 1$. Use the ordered basis $X_1 = [1, 1]^t$ and $X_2 = [1, -1]^t$ to define new coordinates for $\mathbb{R}^2$.

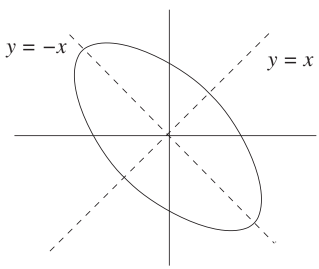
**(a)** Sketch the new coordinate axes on your graph.

**(b)** ✓✓Show that in these coordinates the curve is described by the equation

$$(x')^2 - (y')^2 = 1$$

This proves that $y = 1/x$ represents a hyperbola. [*Hint:* Show that if $[x', y']^t$ is the coordinate vector for $[x, y]^t$, then $x = x' + y'$ and $y = x' - y'$.]

**3.115** Figure 3.14 is a rough sketch of the graph of the equation $x^2 + y^2 + xy = 1$. It appears to represent an ellipse with its major axis along the line $y = -x$




$x^2 + y^2 + xy = 1$

**FIGURE 3.14** An ellipse.

and its minor axis along the line $y = x$. Show that in the coordinates defined in Exercise 3.114 this curve is described by the equation

$$3(x')^2 + (y')^2 = 1$$

and thus is indeed an ellipse.

**3.116** A curve is described by the equation $13x^2 - 8xy + 7y^2 = 5$. Use the basis $X = [1, 2]^t$ and $Y = [-2, 1]^t$ to define new coordinates. What is the formula for this curve in these coordinates? Draw a graph of this curve. [*Hint* Draw the graph in skewed coordinates and then transfer it to the old coordinates.]

**3.117** For each of the following ordered bases $\mathcal{B}$ for $\mathbb{R}^3$, (i) find $P_{\mathcal{B}}$ and $C_{\mathcal{B}}$, (ii) find the $\mathcal{B}$ coordinate vector for $X = [1, 2, 3]^t$, and (iii) check your answer to (ii) by showing that your coordinates do express $X$ as a linear combination of the basis elements.

**(a)** $\mathcal{B} = \{[1, 1, 1]^t, [0, 1, 1]^t, [0, 0, 1]\}$

**(b)** ✓✓ $\mathcal{B} = \{[1, -2, 1]^t, [2, 3, 2]^t, [1, 1, 0]^t\}$

**(c)** $\mathcal{B} = \{[1, 3, 2]^t, [-1, 1, -1]^t, [5, 1, -4]^t\}$

**(d)** $\mathcal{B} = \{[1, 0, 1]^t, [1, 2, 3]^t, [1, 1, 1]^t\}$

**3.118** Let $\mathcal{B}$ be the standard basis for $\mathcal{P}_3$. In each part of this exercise, (i) give $\mathcal{B}$ coordinate vectors for each polynomial, (ii) express one of the coordinate vectors as a linear combination of the others, (iii) express one of the given polynomials as a linear combination of the others, and (iv) find a basis for the span of the set of polynomials.

**(a)** $\{10x + 14x^2, 1 + 2x + 3x^2, 2 + 14x + 20x^2\}$

**(b)** ✓✓ $\{1 - 4x + 4x^2 + 4x^3, 2 - x + 2x^2 + x^3, -17 - 2x - 8x^2 + 2x^3\}$

**(c)** $\{1 + 2x + 3x^3, 2 + x + x^2 + x^3, 1 + 4x + 3x^2 - 3x^3, 3 + 15x + 8x^2 - 4x^3, 3 - 11x - 9x^2 + 18x^3\}$

**3.119** In each part of the following exercise, you are given a matrix $A$ and an eigenbasis $\mathcal{B}$. Find the matrix of $T_A$ with respect to the given basis. [*Hint:* Use the comments immediately preceding Example 3.22 on page 227.]

**(a)** ✓✓

$$\begin{bmatrix} 13 & -4 \\ -4 & 7 \end{bmatrix}, \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$$

**(b)**

$$\begin{bmatrix} 7 & 1 \\ -20 & -2 \end{bmatrix}, \quad \mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \end{bmatrix} \right\}$$

**(c)** ✓✓

$$\begin{bmatrix} 1 & -2 & 2 \\ 0 & 2 & -5 \\ 0 & 0 & -3 \end{bmatrix}, \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

**(d)**
$$\begin{bmatrix} 7 & 4 & 0 \\ 4 & 13 & 10 \\ 0 & 0 & -5 \end{bmatrix}, \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -5 \end{bmatrix} \right\}$$

**3.120** In (a)–(c) of this exercise, you are given a $2 \times 3$, rank 2 matrix $A$. In each part find a nonzero vector $X$ in the nullspace of $A$. Use $\mathcal{B} = \{[1,0,0]^t, [0,1,0]^t, X\}$ as an ordered basis for $\mathbb{R}^3$ and $\overline{\mathcal{B}} = \{A_1, A_2\}$ as an ordered basis for $\mathbb{R}^2$, where $A_1$ and $A_2$ are the first two columns of $A$. Compute the matrix $M$ of $T_A : \mathbb{R}^3 \to \mathbb{R}^2$ with respect to these bases. You should find that in each case $M$ is the matrix from formula (3.37).

**(a)** $A = \begin{bmatrix} 1 & -3 & 6 \\ 2 & -1 & 5 \end{bmatrix}$ **(b)** ✓ $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

**(c)** $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

**3.121** Compute the matrix $M$ of $T_A : \mathbb{R}^3 \to \mathbb{R}^2$ for each of the $2 \times 3$ matrices in Exercise using the standard ordered basis for the target space and the ordered basis $\mathcal{B} = \{[1,1,1]^t, [1,2,-1]^t, [1,0,-1]^t\}$ for the domain.✓✓[(a)]

**3.122** For the following matrix $A$, first find a nonzero vector $X$ in the nullspace of $A$. Then compute the matrix of $T_A$ using the ordered basis $\mathcal{B} = \{[1,0]^t, X\}$ for the domain and the ordered basis $\overline{\mathcal{B}} = \{A_1, [0,1,0]^t, [0,0,1]^t\}$ for the target space where $A_1$ is the first column of $A$. [*Hint:* Use the comments immediately preceding Example 3.22 on page 227.]

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$$

**3.123** Redo Exercise 3.122 using the standard ordered basis for the target space and the ordered basis $\{[1,7]^t, [1,2]^t\}$ for the domain.

**3.124** Assume that the first two columns $A_1$ and $A_2$ of the following matrix $A$ are linearly independent so that $\overline{\mathcal{B}} = \{A_1, A_2\}$ forms an ordered basis for $\mathbb{R}^2$. Let $\{X_o\}$ be a basis of the nullspace of $A$. Then $\mathcal{B} = \{[1,0,0]^t, [0,1,0]^t, X_o\}$ is a linearly independent set. (Why?) Prove that the matrix of $T_A : \mathbb{R}^3 \to \mathbb{R}^2$ with respect to the ordered bases $\mathcal{B}$ and $\overline{\mathcal{B}}$ is the matrix $M$ from formula (3.37). [*Hint:* Use the comments immediately preceding Example 3.22.]

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

**3.125** The first two columns of the following matrix $A$ are linearly dependent. If $A$ has rank 2, the set $\overline{\mathcal{B}} = \{A_2, A_3\}$ formed by the last two columns of $A$ must be independent. Let $X_o = [-1, 2, 0]^t$. Then $\mathcal{B} = \{[0, 1, 0]^t, [0, 0, 1]^t, X_o\}$ is a linearly independent set. Prove that the matrix of $T_A$ with respect to the ordered bases $\mathcal{B}$ and $\overline{\mathcal{B}}$ is the matrix $M$ from formula (3.37). [*Hint:* Use the comments immediately preceding Example 3.22.]

$$A = \begin{bmatrix} 2a & a & c \\ 2d & d & f \end{bmatrix}$$

**3.126** Compute the matrix $M$ with respect to the standard ordered basis of $M(2,2)$ for the linear transformation $L : M(2,2) \to M(2,2)$, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

**(a)** ✓✓ $L(X) = AX$.

**(b)** $L(X) = XA$.

**(c)** ✓✓ $L(X) = AXA^t$.

**(d)** $L(X) = X^t$.

**(e)** ✓✓ $L(X) = X + X^t$.

**(f)** $L(X)$ is $X$ with its first and second rows interchanged.

**(g)** $L(X)$ is $X$ with its second row replaced by the sum of the second plus twice the first.

**3.127** Compute the matrix $M$ with respect to the standard ordered bases for the linear transformation $L : M(3,2) \to M(2,3)$, where $L(X) = X^t$.

**3.128** Let $\mathcal{V}$ and $\mathcal{W}$ be finite-dimensional vector spaces. The nullspace of a linear transformation $L : \mathcal{V} \to \mathcal{W}$ is the set of $X \in \mathcal{V}$ such that $L(X) = 0$. It was shown in Exercise 3.23 on page 162 that the nullspace is a subspace of $\mathcal{V}$.

**(a)** Let ordered bases $\mathcal{B}$ and $\overline{\mathcal{B}}$ for $\mathcal{V}$ and $\mathcal{W}$ be given and let $M$ be the matrix of $L$ with respect to these bases. Show that $L(X) = 0$ if and only if $MT_{\mathcal{B}}^{\mathcal{C}}X = 0$.

**(b)** Show that $T_{\mathcal{B}}^{\mathcal{C}}$ restricts to an isomorphism of the nullspace of $L$ with the nullspace of $M$.

**(c)** Explain why the nullspace of the transformation in Exercise 3.126.d is $\{0\}$. It follows from part (b) that the nullspace of the corresponding matrix $M$ is also $\{0\}$. What then is the rank of $M$? Check this by row reducing $M$. Repeat for Exercise 3.126.f

**(d)** Describe the nullspace of the transformation in Exercise 3.126.d and prove that its dimension is 1. What is the rank of $M$? Check this by row reducing $M$.

**3.129** In this exercise you will prove Theorem 3.16 on page 229. Let our assumptions be as in the statement of this theorem and let the notation be as in Exercise 3.128.a where $\dim \mathcal{V} = \dim \mathcal{W} = n$.

**(a)** Assume that the nullspace of $L = \{0\}$. Explain how it follows from Exercise 3.128.a that the nullspace of $M$ is $\{0\}$? How does it follow that the equation $L(X) = Y$ has at least one solution for all $Y \in \mathcal{V}$? *Hint:* First explain why $M$ is an $n \times n$ matrix. Then apply Theorem 2.20 on page 140.

**(b)** Assume that the equation $L(X) = Y$ has at least one solution for all $Y \in \mathcal{V}$. Prove that then this equation has only one solution. *Hint:* First prove that for each $B$ in $\mathbb{R}^n$ the system $MX = B$ has at least one solution. Then apply Theorem 2.20 on page 140.

**3.130** Compute the matrix $M$ with respect to the standard ordered basis of $\mathcal{P}_2$ for the linear transformation $L : \mathcal{P}_2 \to \mathcal{P}_2$, where:

**(a)** ✓✓ $L(y) = y'$

**(b)** $L(y) = y''$

**(c)** ✓✓ $L(y) = y' - y$

**(d)** $L(y) = y' + 3y$

**(e)** ✓✓ $L(y) = x^2y'' + 3xy' - 7y$

**(f)** $L(y) = 3x^2y'' - xy' + 2y$

**3.131** The formulas from Exercise 3.130 also define linear transformations $L : \mathcal{P}_3 \to \mathcal{P}_3$. Compute the matrices of each of these linear transformations with respect to the standard ordered bases. Use your answer to find all polynomials $y$ in $\mathcal{P}_3$ such that $L(y) = x^2$.

**3.132** The formulas from Exercise 3.130 also define linear transformations $L : \mathcal{P}_4 \to \mathcal{P}_4$. Compute the matrices of each of these linear transformations with respect to the standard ordered bases.

**3.133** Compute the matrix $M$ for the linear transformation $L : \mathcal{P}_2 \to \mathcal{P}_2$ from each part of Exercise 3.130 where we use the ordered basis $\{1, (x-1), (x-1)^2\}$ for the domain and the standard ordered basis for the target space. [(a)✓✓, (c)✓✓, (e)✓✓]

**3.134** Compute the matrix $M$ for the linear transformation $L : \mathcal{P}_2 \to \mathcal{P}_2$ from each part of Exercise 3.130 where we use the ordered bases $\{1, (x-1), (x-1)^2\}$ for the domain and $\{1, (x+1), (x+1)^2\}$ for the target space.

**3.135** Let $\mathcal{B} = \{X_1, X_2, \ldots, X_k\}$ be a subset of a vector space $\mathcal{V}$ and let $L : \mathcal{V} \to \mathcal{W}$ be an isomorphism of vector spaces. Show that $\{L(X_1), L(X_2), \ldots, L(X_k)\}$ is linearly independent in $\mathcal{W}$ if and only if $\mathcal{B}$ is linearly independent in $\mathcal{V}$.

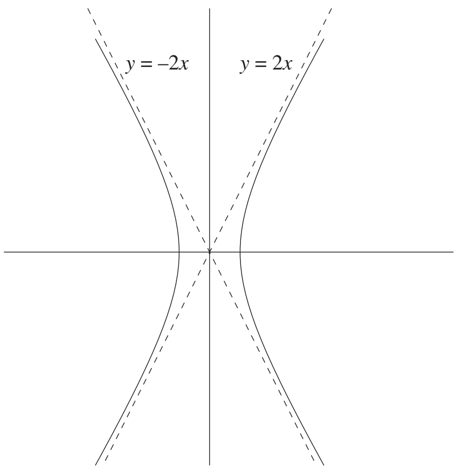
### 3.5.1 Computer Projects

The following MATLAB commands generate a piece of graph paper for the coordinates in $\mathbb{R}^2$ defined by the ordered basis $\mathcal{B} = \{Y_1, Y_2\}$, where $Y_1 = [1, 2]^t$ and $Y_2 = [-1, 1]^t$. The "grid on" command causes it to be overlaid on top of a rectangular grid. Begin by initializing the coordinate axes with the commands:

```
cla; hold on; axis equal; grid on;
Q1=[1;2]; Q2=[-1; 1];
A=[5*Q1,-5*Q1] ; B=[5*Q2,-5*Q2]
plot(A(1,:),A(2,:),'r'); plot(B(1,:),B(2,:),'r');
for n=-5:5,
  plot(A(1,:)+n*Q2(1),A(2,:)+n*Q2(2),'r:');
  plot(B(1,:)+n*Q1(1),B(2,:)+n*Q1(2),'r:');
end
```

## EXERCISES

1. Modify the above sequence of code to produce a sheet of graph paper for the coordinates defined by the ordered basis $\overline{\mathcal{B}} = \{Y_1, Y_2\}$, where $Y_1 = [1, 2]^t$ and $Y_2 = [1, -2]^t$.
2. The curve defined by the equation $x^2 - y^2/4 = 1$ represents the hyperbola of Figure 3.15. Show that this hyperbola is given by the equation $x'y' = 4$ in coordinates relative to the basis from Exercise 1.




$x^2 - y^2/4 = 1$

**FIGURE 3.15** A hyperbola.

**3.** The following sequence of commands plots the right side of the hyperbola of Figure 3.15. (Note the period before the caret.)

```
x=.8:.1:5; y=4*(x.^(-1)); C=Q1*x+Q2*y;
plot(C(1,:),C(2,:));
```

Execute the above sequence of commands to produce a plot of the hyperbola. Explain what each command does. To help you in understanding what these commands do, get MATLAB to print out both $x$ and $y$. How would you describe $x$? How does $y$ relate to $x$? What is the size of the matrix Q1$*$ $x$? How does each column of this matrix relate to $x$ and Q1? The "plot" command plots the first row of Q1$*$ $x$+Q2$*$ $y$ against the second. Why does this produce the right side of Figure 3.15?

**4.** Pick a point (reader's choice) on the graph of the hyperbola from Exercise 3 and estimate both its natural coordinates $[x, y]^t$ and its $\overline{\mathcal{B}}$ coordinates $[x', y']^t$. Show that (approximately) $x'y' = 4$ and $x^2 - y^2/4 = 1$. Repeat for another point on the graph.

## CHAPTER SUMMARY

In Section 3.1, we noted that multiplication by an $m \times n$ matrix $A$ defines a ***transformation*** of $\mathbb{R}^n$ into $\mathbb{R}^m$, which is called a ***matrix transformation***. Matrix transformations are ***linear transformations*** due to the linearity properties of matrix multiplication. The ***matrix representation theorem*** says that every linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$ is a matrix transformation.

In Section 3.2, we studied composition of transformations, which is the process of following one transformation with another. Composition of matrix transformations is given by ***matrix multiplication***: "the columns of $AB$ are $A$ times the columns of $B$" and, equivalently, "the rows of $AB$ are the rows of $A$ times $B$.". Important properties include the***left and right distributive laws***, the ***associative law***, and the ***rank of products theorem***: rank $AB \leq \min\{\text{rank}\, A, \text{rank}\, B\}$.

In Section 3.3 we studied invertible matrices i.e., $n \times n$ matrices $A$ for which the equation$AX = Y$ has one and only one solution $X$ for every $Y$ in $\mathbb{R}^n$. Then the inverse matrix (the ***inverse matrix***) $A^{-1}$ is the unique matrix such that $AX = Y$ if and only if $X = A^{-1}Y$. This matrix is also describable as the unique matrix, such that $AA^{-1} = A^{-1}A = I$. Not all $n \times n$ matrices $A$ have an inverse: An $n \times n$ matrix $A$ has an inverse if and only if its rank is $n$. A technique for computing matrix inverses, when they exist, was given in Section 3.3.

In Section 3.4, we studied the $LU$ factorization, which is a very efficient method of solving the system $AX = Y$ based on writing $A$ as a product $A = LU$, where $L$ is lower triangular and $U$ is upper triangular with 1's on its diagonal. In this form, the $LU$ factorization only exists if $A$ can be reduced to echelon form without switching rows and without multiplying any row by a scalar. However, there are more general versions of this factorization that apply to more general matrices.

Finally, in Section 3.5 we discussed how coordinates can be used to describe any $n$-dimensional vector space in terms of $\mathbb{R}^n$. This then allows us to describe linear transformations between finite-dimensional vector spaces using matrices. This is even useful when the vector space is $\mathbb{R}^n$ since the proper choice of coordinates can simplify many problems.

# CHAPTER 4

# DETERMINANTS

## 4.1 DEFINITION OF THE DETERMINANT

You may have wondered why we need to row reduce whenever we compute the inverse of a matrix. Should we not be able to write down a general formula for the inverse and simply plug in our specific numbers? The answer is, yes, formulas do exist. Using them for large matrices, however, usually requires more work than simply doing the row reduction! The formula for a $2 \times 2$ matrix is, nevertheless, very simple:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The factor $\delta = ad - cb$, which occurs in the formula for $A^{-1}$, is called the determinant of $A$. We denote it by $\det A$. Thus,

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - cb$$

Hence,

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = (1 \cdot 4) - (3 \cdot 2) = -2$$

*Linear Algebra: Ideas and Applications*, Fourth Edition. Richard C. Penney.

Companion website: www.wiley.com/go/penney/linearalgebra

It is also common to denote the determinant of a matrix by replacing the square brackets with straight lines. Thus, we define

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb \tag{4.1}$$

In Section 4.3, we give a formula for the inverse of the general $n \times n$ matrix $A$ that has the form

$$A^{-1} = \frac{1}{\delta} B$$

where $\delta$ is a number that, once again, is called the determinant of $A$ and denoted by $\det A$. Just as in the $2 \times 2$ case, it turns out that $A$ is invertible if and only if $\delta \neq 0$. (We prove this in Section 4.2.) As commented above, using formulas to compute matrix inverses is usually very inefficient. However, the fact that a matrix is invertible if and only if the determinant is nonzero is one of the most important facts in linear algebra. In particular, it becomes important when we study eigenvalues in Chapter 5.

Our first goal is to describe how to compute determinants for matrices larger than $2 \times 2$. The determinant of a $3 \times 3$ matrix is computable in terms of determinants of $2 \times 2$ matrices. Specifically, we define

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \tag{4.2}$$

(Note the use of the vertical lines to denote determinants.)

Notice that in this formula the first $2 \times 2$ matrix on the right is what we get if we eliminate the row and column containing $a_{11}$ from the original $3 \times 3$ matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

In formula (4.2), the determinant of this matrix is multiplied by $a_{11}$. Similarly, the term that is multiplied by $a_{12}$ is the determinant of the $2 \times 2$ matrix obtained by eliminating the row and column containing $a_{12}$, and the term that is multiplied by $a_{13}$ is the determinant of the $2 \times 2$ matrix obtained by eliminating the row and column containing $a_{13}$. We add the first product, subtract the second, and add the third.

## EXAMPLE 4.1

Compute $\det A$, where

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 6 & 1 & 7 \\ 4 & 5 & 9 \end{bmatrix}$$

*Solution.* From formula (4.2)

$$\begin{vmatrix} 2 & 0 & 1 \\ 6 & 1 & 7 \\ 4 & 5 & 9 \end{vmatrix} = 2\det\begin{bmatrix} \cancel{2} & \cancel{0} & \cancel{1} \\ \cancel{6} & 1 & 7 \\ \cancel{4} & 5 & 9 \end{bmatrix} - 0\det\begin{bmatrix} \cancel{2} & \cancel{0} & \cancel{1} \\ 6 & \cancel{1} & 7 \\ 4 & \cancel{5} & 9 \end{bmatrix} + 1\det\begin{bmatrix} \cancel{2} & \cancel{0} & \cancel{1} \\ 6 & 1 & \cancel{7} \\ 4 & 5 & \cancel{9} \end{bmatrix}$$

$$= 2\begin{vmatrix} 1 & 7 \\ 5 & 9 \end{vmatrix} - 0\begin{vmatrix} 6 & 7 \\ 4 & 9 \end{vmatrix} + 1\begin{vmatrix} 6 & 1 \\ 4 & 5 \end{vmatrix}$$

$$= 2[(1 \cdot 9) - (5 \cdot 7)] - 0 + [(6 \cdot 5) - (4 \cdot 1)] = -26$$

In general, the determinant of an $n \times n$ matrix is defined in terms of determinants of $(n-1) \times (n-1)$ matrices. This type of definition is referred to as an *inductive definition.*

**Definition 4.1** *If $A$ is any $n \times n$ matrix, then*

$$\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + \cdots + (-1)^{n-1}a_{1n}\det A_{1n} \tag{4.3}$$

*where $A_{ij}$ denotes the matrix obtained from $A$ by deleting the ith row and jth column.*

Thus, as in the $3 \times 3$ case, we multiply each entry in the first row by the determinant of the matrix obtained by eliminating the row and column containing the entry. Also, as in the $3 \times 3$ case, we alternately add and subtract the terms. This method of computing determinants is called either the **Laplace expansion** or the **cofactor expansion**.

### EXAMPLE 4.2

Compute

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix}$$

*Solution.* We compute

$$1\begin{vmatrix} 6 & 7 & 8 \\ 10 & 11 & 12 \\ 14 & 15 & 16 \end{vmatrix} - 2\begin{vmatrix} 5 & 7 & 8 \\ 9 & 11 & 12 \\ 13 & 15 & 16 \end{vmatrix} + 3\begin{vmatrix} 5 & 6 & 8 \\ 9 & 10 & 12 \\ 13 & 14 & 16 \end{vmatrix} - 4\begin{vmatrix} 5 & 6 & 7 \\ 9 & 10 & 11 \\ 13 & 14 & 15 \end{vmatrix}$$

Each of these $3 \times 3$ determinants is zero. For example,

$$\begin{aligned}
\begin{vmatrix} 6 & 7 & 8 \\ 10 & 11 & 12 \\ 14 & 15 & 16 \end{vmatrix} &= 6\begin{vmatrix} 11 & 12 \\ 15 & 16 \end{vmatrix} - 7\begin{vmatrix} 10 & 12 \\ 14 & 16 \end{vmatrix} + 8\begin{vmatrix} 10 & 11 \\ 14 & 15 \end{vmatrix} \\
&= 6[(11 \cdot 16) - (15 \cdot 12)] - 7[(10 \cdot 16) - (14 \cdot 12)] \\
&\quad + 8[(10 \cdot 15) - (11 \cdot 14)] \\
&= 0
\end{aligned}$$

Thus, the answer is zero.

Formula (4.3) makes it appear that the first row plays a special role in computing determinants. This is not the case due to the following **row interchange property** that is proved at the end of this section.

**Theorem 4.1** (Row Interchange Property). *Let $A$ be an $n \times n$ matrix and suppose $B$ is obtained by interchanging two rows of $A$. Then* $\det B = -\det A$.

Thus, for example,

$$\begin{aligned}
\begin{vmatrix} 6 & 7 & 8 \\ 9 & 10 & 11 \\ 12 & 13 & 14 \end{vmatrix} &= -\begin{vmatrix} 9 & 10 & 11 \\ 6 & 7 & 8 \\ 12 & 13 & 14 \end{vmatrix} \\
&= -\left(9\begin{vmatrix} 7 & 8 \\ 13 & 14 \end{vmatrix} - 10\begin{vmatrix} 6 & 8 \\ 12 & 14 \end{vmatrix} + 11\begin{vmatrix} 6 & 7 \\ 12 & 13 \end{vmatrix}\right) \qquad (4.4)
\end{aligned}$$

which works out to be 0 as before.

The last line of equation (4.4) is the *negative* of the result of applying the expansion process for the determinant to the second row:

$$9\det\begin{bmatrix} \cancel{6} & 7 & 8 \\ \cancel{9} & \cancel{10} & \cancel{11} \\ \cancel{12} & 13 & 14 \end{bmatrix} - 10\det\begin{bmatrix} 6 & \cancel{7} & 8 \\ \cancel{9} & \cancel{10} & \cancel{11} \\ 12 & \cancel{13} & 14 \end{bmatrix} + 11\det\begin{bmatrix} 6 & 7 & \cancel{8} \\ \cancel{9} & \cancel{10} & \cancel{11} \\ 12 & 13 & \cancel{14} \end{bmatrix}$$

The row interchange property can be used to prove that in the general case, if we expand along any *even-numbered* row, we get the negative of the determinant.

The row interchange property also implies that if we expand along an odd-numbered row, we get the determinant:

$$\begin{vmatrix} 6 & 7 & 8 \\ 9 & 10 & 11 \\ 12 & 13 & 14 \end{vmatrix} = 12\begin{vmatrix} 7 & 8 \\ 10 & 11 \end{vmatrix} - 13\begin{vmatrix} 6 & 8 \\ 9 & 11 \end{vmatrix} + 14\begin{vmatrix} 6 & 7 \\ 9 & 10 \end{vmatrix}$$

The following theorem is a general statement of the preceding discussion. The proof appears at the end of this section.

**Theorem 4.2** *Let $A$ be an $n \times n$ matrix. Then, for each $i$,*

$$\det A = (-1)^{i+1}(a_{i1} \det A_{i1} - a_{i2} \det A_{i2} + \cdots + (-1)^{n-1} a_{in} \det A_{in}) \tag{4.5}$$

*where the $A_{ik}$ are as defined in Definition 4.1.*

### EXAMPLE 4.3

Redo Example 4.1 by expanding along the second row.

*Solution.* Since we are using the second row, we begin by subtracting the first product and alternate the signs thereafter. Thus,

$$\begin{bmatrix} 2 & 0 & 1 \\ 6 & 1 & 7 \\ 4 & 5 & 9 \end{bmatrix} = -6 \det \begin{bmatrix} 2 & 0 & 1 \\ 6 & 1 & 7 \\ 4 & 5 & 9 \end{bmatrix} + 1 \det \begin{bmatrix} 2 & 0 & 1 \\ 6 & 1 & 7 \\ 4 & 5 & 9 \end{bmatrix} - 7 \det \begin{bmatrix} 2 & 0 & 1 \\ 6 & 1 & 7 \\ 4 & 5 & 9 \end{bmatrix}$$

$$\begin{aligned} &= -6 \begin{vmatrix} 0 & 1 \\ 5 & 9 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ 4 & 9 \end{vmatrix} - 7 \begin{vmatrix} 2 & 0 \\ 4 & 5 \end{vmatrix} \\ &= 30 + 14 - 70 = -26 \end{aligned}$$

From the preceding calculation, if we were to double each entry in the second row, the value of the determinant would also double. Hence, without any additional work, we can say

$$\begin{vmatrix} 2 & 0 & 1 \\ 12 & 2 & 14 \\ 4 & 5 & 9 \end{vmatrix} = -52$$

Since we may expand along any row, multiplying any row of a matrix by a constant multiplies the determinant by the same constant:

**Theorem 4.3** (Row Scalar Property). *Suppose the $n \times n$ matrix $B$ is obtained from $A$ by multiplying each element in the $i$th row by some scalar $c$. Then* $\det B = c \det A$.

*Proof.* As before, we let $B_{ik}$ and $A_{ik}$ be the matrices obtained, respectively, by deleting the $i$th row and $k$th column from $A$ and $B$. Since $B$ differs from $A$ only in the $i$th row, $B_{ik} = A_{ik}$ for all $k$. Expanding $\det B$ along the $i$th row shows

$$\begin{aligned} \det B &= (-1)^{i+1}(ca_{i1} \det A_{i1} - ca_{i2} \det A_{i2} + \cdots + (-1)^{n-1} ca_{in} \det A_{in}) \\ &= c\left((-1)^{i+1}(a_{i1} \det A_{i1} - a_{i2} \det A_{i2} + \cdots + (-1)^{n-1} a_{in} \det A_{in})\right) \\ &= c \det A \end{aligned}$$

□

## EXAMPLE 4.4

Without computing determinants, how are the determinants of $A$ and $B$ related?

$$A = \begin{bmatrix} 1 & 7 & 6 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 3 & 3 \\ 1 & 7 & 6 \end{bmatrix}$$

*Solution.* We use the scalar and interchange properties on $B$ to conclude that

$$\det B = 3 \begin{vmatrix} 2 & 3 & 4 \\ 1 & 1 & 1 \\ 1 & 7 & 6 \end{vmatrix} = -3 \begin{vmatrix} 1 & 7 & 6 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{vmatrix} = 3 \begin{vmatrix} 1 & 7 & 6 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{vmatrix} = 3 \det A$$

We can also compute determinants by expanding along columns. Again, expansion along even-numbered columns yields the negative of the determinant. This theorem too is proved at the end of this section.

**Theorem 4.4** *Let $A$ be an $n \times n$ matrix. Then, for each $j$,*

$$\det A = (-1)^{j+1}(a_{1j} \det A_{1j} - a_{2j} \det A_{2j} + \cdots + (-1)^{n-1} a_{nj} \det A_{nj}) \qquad (4.6)$$

*where the $A_{ik}$ are as defined in Definition 4.1.*

## EXAMPLE 4.5

Compute $\det A$ by expanding along any row or column:

$$A = \begin{bmatrix} 1 & 0 & 2 & 0.700 \\ 1 & 0 & 3 & 0.000 \\ 17 & 5 & -37 & 0.002 \\ 1 & 0 & 6 & -9.000 \end{bmatrix}$$

*Solution.* Since the second column has only one nonzero term, we use it. Since we are expanding along an even-numbered column, the first term is subtracted:

$$\begin{vmatrix} 1 & 0 & 2 & 0.700 \\ 1 & 0 & 3 & 0.000 \\ 17 & 5 & -37 & 0.002 \\ 1 & 0 & 6 & -9.000 \end{vmatrix} = -0 \det A_{12} + 0 \det A_{22} - 5 \det A_{32} + 0 \det A_{42}$$

$$= -5 \begin{vmatrix} 1 & 2 & 0.700 \\ 1 & 3 & 0.000 \\ 1 & 6 & -9.000 \end{vmatrix}$$

We expand this $3 \times 3$ determinant along the second row:

$$-5\left(-1\begin{vmatrix} 2 & 0.700 \\ 6 & -9.000 \end{vmatrix} + 3\begin{vmatrix} 1 & 0.700 \\ 1 & -9.000 \end{vmatrix}\right) = 34.5$$

Recall that a matrix is said to be upper triangular if all the entries below the main diagonal are zero. Computing the determinant of an upper triangular matrix is simple.

### EXAMPLE 4.6

Find the determinant of

$$A = \begin{bmatrix} 2 & 7 & -1 & 16 \\ 0 & 3 & -2 & 9 \\ 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

*Solution.* Expanding determinants along the first column, we see that

$$\begin{aligned} \begin{vmatrix} 2 & 7 & -1 & 16 \\ 0 & 3 & -2 & 9 \\ 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 6 \end{vmatrix} &= 2\begin{vmatrix} 3 & -2 & 9 \\ 0 & 4 & 8 \\ 0 & 0 & 6 \end{vmatrix} \\ &= 2 \cdot 3 \begin{vmatrix} 4 & 8 \\ 0 & 6 \end{vmatrix} \\ &= 2 \cdot 3 \cdot 4 \cdot 6 = 144 \end{aligned}$$

Thus, the determinant is just the product of the entries on the main diagonal.

If we use the same reasoning as in the above example, it is not difficult to prove the following result:

**Theorem 4.5** *The determinant of an upper triangular matrix is the product of the entries on the main diagonal.*

In Section 4.2 we describe more efficient methods for computing determinants than presented so far. These techniques are based on some general properties of the determinant that we describe below.

**Theorem 4.6** (Row Additive Property). *Let $U$, $V$, and $A_i$ be $1 \times n$ row matrices, where $i = 2, 3, \ldots, n$. Then*

$$\det \begin{bmatrix} U+V \\ A_2 \\ A_3 \\ \vdots \\ A_n \end{bmatrix} = \det \begin{bmatrix} U \\ A_2 \\ A_3 \\ \vdots \\ A_n \end{bmatrix} + \det \begin{bmatrix} V \\ A_2 \\ A_3 \\ \vdots \\ A_n \end{bmatrix}$$

*Similar equalities hold for other rows.*

*Proof.* Let $A$ be the matrix on the left in the preceding equality. Let $A_{ik}$ be as defined in Definition 4.1. Expansion along the first row shows

$$\begin{aligned} \det A &= (u_1 + v_1) \det A_{11} - (u_2 + v_2) \det A_{12} + (u_3 + v_3) \det A_{13} + \cdots \\ &\quad + (-1)^{n-1}(u_n + v_n) \det A_{1n} \\ &= u_1 \det A_{11} - u_2 \det A_{12} + u_3 \det A_{13} + \cdots + (-1)^{n-1} u_n \det A_{1n} \\ &\quad + v_1 \det A_{11} - v_2 \det A_{12} + v_3 \det A_{13} + \cdots + (-1)^{n-1} v_n \det A_{1n} \\ &= \det B + \det C \end{aligned}$$

where $B$ and $C$ are, respectively, the first and second matrices on the right in the statement of the additive property.

This proves the result for the first row. The validity for other rows follows similarly since we may expand determinants along any row (Theorem 4.2). □

## EXAMPLE 4.7

Given that

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & 0 & 0 \end{vmatrix} = -2$$

compute

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & 0 & 5 \end{vmatrix}$$

*Solution.* We shall apply the additive property to the third row. We note that

$$[1, 0, 5] = [1, 0, 0] + [0, 0, 5]$$

Hence, from additivity,

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 5 \end{vmatrix} = -2 + 1 \cdot 2 \cdot 5 = 8$$

since the matrix on the right is triangular.

### 4.1.1 The Rest of the Proofs

In this section, we prove the properties of the determinant that were stated without proof. *Throughout this section, if $A$ is any matrix, then $A_i$ is $A$ with its ith row deleted and $A^j$ is $A$ with its jth column deleted. As before, $A_{ij}$ is $A$ with both the ith row and jth column deleted.* We begin by proving that we may expand down the first column:

**Proposition 4.1** *Formula (4.6) holds for $j = 1$—that is, we can compute determinants by expanding down the first column.*

*Proof.* It is easily checked that expansion along the first column yields the correct answer for $2 \times 2$ determinants. Assume that we have succeeded in proving the proposition for all $k \times k$ matrices, where $k < n$. We will show that it then follows for $n \times n$ matrices, proving the theorem by mathematical induction.

Thus, let $A$ be an $n \times n$ matrix. Write $A$ as a partitioned matrix

$$A = \begin{bmatrix} a & B \\ C & D \end{bmatrix}$$

where $D$ is an $(n-1) \times (n-1)$ matrix, $B = [b_1, b_2, \dots, b_{n-1}]$ is a row vector, $C = [c_1, c_2, \dots, c_{n-1}]^t$ is a column vector, and $a$ is a scalar. We compute $\det A$ by expansion along the first row:

$$\begin{aligned} \det A = {} & a \det D - b_1 \det[C, D^1] \\ & + b_2 \det[C, D^2] + \cdots + (-1)^{n-1} b_{n-1} \det[C, D^{n-1}] \end{aligned}$$

where, as mentioned previously, $D^i$ is $D$ with its $i$th column deleted.

Since we have (by assumption) already proved our theorem for $(n-1) \times (n-1)$ matrices, we may compute $\det[C, D^i]$ by expanding down the first column, yielding

$$\det A = a \det D - \sum_{i,j} (-1)^{i+j-1} b_i c_j \det D_{ij} \tag{4.7}$$

On the other hand, if we attempt to expand det $A$ down the first column, we get

$$a \det D - c_1 \det \begin{bmatrix} B \\ D_1 \end{bmatrix} + c_2 \det \begin{bmatrix} B \\ D_2 \end{bmatrix} + \cdots + (-1)^{n-1} c_{n-1} \det \begin{bmatrix} B \\ D_{n-1} \end{bmatrix}$$

Expanding all but det $D$ along the first row results in the expression in formula (4.7), proving the proposition. □

Next we prove the row interchange property.

**Row Interchange Property**. Again, the result is clear for $2 \times 2$ matrices. Assume that we have succeeded in proving the row interchange property for all $k \times k$ matrices, where $k < n$. We will show that it then follows for $n \times n$ matrices, proving the theorem by mathematical induction.

As long as neither of the rows being interchanged is the first, our theorem follows directly from formula (4.3) since interchanging the rows in $A$ interchanges the corresponding rows in $A_{ij}$ and thus negates the determinant.

Hence, we may assume that one of the rows in question is the first. Assume, for the moment, that the other row is the second. We write $A$ as a partitioned matrix

$$A = \begin{bmatrix} a & B \\ c & D \\ E & F \end{bmatrix}$$

where $a$ and $c$ are scalars, $B$ and $D$ are row vectors, and $E = [e_1, \ldots, e_{n-2}]^t$ is a column vector. Expanding down the first column shows

$$\begin{aligned} \det A = a \det \begin{bmatrix} c & D \\ E & F \end{bmatrix} - c \det \begin{bmatrix} a & B \\ E & F \end{bmatrix} + e_1 \det \begin{bmatrix} a & B \\ c & D \\ E_1 & F_1 \end{bmatrix} + \cdots \\ + (-1)^{n-1} e_{n-2} \det \begin{bmatrix} a & B \\ c & D \\ E_{n-2} & F_{n-2} \end{bmatrix} \end{aligned} \tag{4.8}$$

Similarly

$$\begin{aligned} \det \begin{bmatrix} c & D \\ a & B \\ E & F \end{bmatrix} = c \det \begin{bmatrix} a & B \\ E & F \end{bmatrix} - a \det \begin{bmatrix} c & D \\ E & F \end{bmatrix} + e_1 \det \begin{bmatrix} c & D \\ a & B \\ E_1 & F_1 \end{bmatrix} + \cdots \\ + (-1)^{n-1} e_{n-2} \det \begin{bmatrix} c & D \\ a & B \\ E_{n-2} & F_{n-2} \end{bmatrix} \end{aligned}$$

which is the negative of the value in formula (4.8) since the row interchange property holds for $(n-1) \times (n-1)$ determinants.

Finally, if the first and $i$th rows are being interchanged, where $i \neq 2$, we (1) interchange the $i$th and second rows, (2) interchange the first and second rows, and (3) interchange the second and $i$th rows. A little thought convinces one that the net effect is to interchange the first and $i$th rows. Since there are three interchanges, the determinant is once again negated, proving our theorem. □

We commented that the row interchange property immediately implies the ability to expand determinants along any row. We leave the argument as an exercise for the reader.

*Remark.* Our proofs of the row interchange, row additive, and row scalar properties used only the observation that we may expand determinants using either the first row or the first column. The same arguments prove that there are corresponding column interchange, column additive, and column scalar properties. It follows in particular that we may expand down any column.

**True-False Questions**: Justify your answers. *It is given that none of the matrices have 0 determinant.*

**4.1** The following matrices have the same determinant. [*Hint:* Factor some scalars out from the determinants.]

$$\begin{bmatrix} 2 & 4 & 2 & 6 \\ 3 & 3 & 27 & 33 \\ 2 & 1 & 5 & 2 \\ 6 & 1 & -3 & 3 \end{bmatrix} \qquad \begin{bmatrix} 3 & 6 & 3 & 9 \\ 2 & 1 & 5 & 2 \\ 6 & 1 & -3 & 3 \\ 2 & 2 & 18 & 22 \end{bmatrix}$$

**4.2** Let $A$ be a $3 \times 3$ matrix. Then $\det(5A) = 5\det(A)$.

**4.3** Let $A$ and $B$ be $3 \times 3$ matrices. Then $\det(A + B) = \det(A) + \det(B)$.

**4.4** The following statement is true:

$$\begin{vmatrix} 2 & 4 & 2 & 6 \\ 3 & 3 & 27 & 33 \\ 2 & 1 & 5 & 2 \\ 6 & 1 & -3 & 3 \end{vmatrix} = 2\begin{vmatrix} 2 & 4 & 2 & 6 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 5 & 2 \\ 6 & 1 & -3 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 4 & 2 & 6 \\ 1 & 1 & 23 & 33 \\ 2 & 1 & 5 & 2 \\ 6 & 1 & -3 & 3 \end{vmatrix}$$

**4.5** The following statement is true:

$$\begin{vmatrix} 2 & 4 & 2 & 6 \\ 3 & 3 & 7 & 8 \\ 2 & 1 & 5 & 2 \\ 6 & 1 & -3 & 3 \end{vmatrix} = 4\begin{vmatrix} 3 & 7 & 8 \\ 2 & 5 & 2 \\ 6 & -3 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 & 6 \\ 2 & 5 & 2 \\ 6 & -3 & 3 \end{vmatrix}$$
$$+\begin{vmatrix} 2 & 2 & 6 \\ 3 & 7 & 8 \\ 6 & -3 & 3 \end{vmatrix} - \begin{vmatrix} 2 & 2 & 6 \\ 3 & 7 & 8 \\ 2 & 5 & 2 \end{vmatrix}$$

**4.6** The following matrices have the same determinant:

$$\begin{bmatrix} 1753 & 0 & 0 & 0 \\ 27 & 33 & 0 & 0 \\ 13 & 911 & 1411 & 0 \\ -15 & 44 & 32 & 1001 \end{bmatrix}, \quad \begin{bmatrix} 1753 & 27 & 13 & -15 \\ 0 & 33 & 911 & 44 \\ 0 & 0 & 1411 & 32 \\ 0 & 0 & 0 & 1001 \end{bmatrix}$$

**4.7** The following matrices have the same determinant:

$$\begin{bmatrix} 1753 & 0 & 0 & 0 \\ 27 & 33 & 0 & 0 \\ 13 & 911 & 1411 & 0 \\ -15 & 44 & 32 & 1001 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 1753 \\ 0 & 0 & 33 & 27 \\ 0 & 1411 & 911 & 13 \\ 1001 & 32 & 44 & -15 \end{bmatrix}$$

## EXERCISES

**4.1** Compute the following determinants:

**(a)** ✓✓ $\begin{vmatrix} 1 & 4 \\ -3 & 2 \end{vmatrix}$ **(b)** $\begin{vmatrix} 1 & 4 \\ 2 & 8 \end{vmatrix}$

**(c)** ✓✓ $\begin{vmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ -2 & 2 & 2 \end{vmatrix}$ **(d)** $\begin{vmatrix} 7 & 1 & 1 \\ 0 & a & b \\ 0 & d & c \end{vmatrix}$

**(e)** ✓✓ $\begin{vmatrix} 0 & 5 & 1 \\ -1 & 1 & 3 \\ -2 & -2 & 2 \end{vmatrix}$ **(f)** $\begin{vmatrix} 2 & 1 & 1 \\ 5 & 4 & 3 \\ 7 & 5 & 4 \end{vmatrix}$

**(g)** ✓✓ $\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix}$ **(h)** $\begin{vmatrix} -3 & 2 & 2 \\ 1 & 4 & 1 \\ 7 & 6 & -2 \end{vmatrix}$

**(i)** ✓✓ $\begin{vmatrix} 2 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 2 \\ 1 & 0 & 0 & 5 \end{vmatrix}$ **(j)** $\begin{vmatrix} 3 & 1 & 3 & 0 \\ 3 & 1 & 3 & 1 \\ 0 & 0 & 2 & 1 \\ 6 & 3 & 4 & 5 \end{vmatrix}$

**4.2** In Section 4.2 we prove that the determinant of a matrix is zero if and only if the rows are linearly dependent. The analogous statement holds for the columns. Check this by expressing one row as a combination of the others in each of the parts of Exercise 4.1 where the determinant is zero. Repeat for the columns. ✓[(a), (c), (e), (g), (i)]

**4.3** ✓✓Write a formula similar to formula (4.2) on page 239 that describes the expansion of the determinant of a $3 \times 3$ matrix along the second row. Prove that your formula agrees with formula (4.2).

**4.4** Use expansion along the third row to express each of the following determinants as a sum of determinants of $4 \times 4$ matrices. Without expanding further, explain why $3\alpha = \beta$. What theorem from the text does this exercise demonstrate?

$$\alpha = \begin{vmatrix} 24 & -13 & 7 & 9 & 5 \\ 11 & 16 & -37 & 99 & 64 \\ 1 & 4 & 2 & 2 & -3 \\ 31 & -42 & 78 & 55 & -3 \\ 62 & 47 & 29 & -14 & -8 \end{vmatrix}, \quad \beta = \begin{vmatrix} 24 & -13 & 7 & 9 & 5 \\ 11 & 16 & -37 & 99 & 64 \\ 3 & 12 & 6 & 6 & -9 \\ 31 & -42 & 78 & 55 & -3 \\ 62 & 47 & 29 & -14 & -8 \end{vmatrix}$$

**4.5** ✓✓Use expansion along the third row to express $\beta$, $\delta$, and $\gamma$ as a sum of determinants of $4 \times 4$ matrices, where $\beta$ is as in Exercise 4.4 and $\delta$ and $\gamma$ are as follows. Without expanding further, explain why $\beta = \delta + \gamma$. What theorem from the text does this exercise demonstrate?

$$\delta = \begin{vmatrix} 24 & -13 & 7 & 9 & 5 \\ 11 & 16 & -37 & 99 & 64 \\ 1 & 7 & 3 & 3 & -13 \\ 31 & -42 & 78 & 55 & -3 \\ 62 & 47 & 29 & -14 & -8 \end{vmatrix}, \quad \gamma = \begin{vmatrix} 24 & -13 & 7 & 9 & 5 \\ 11 & 16 & -37 & 99 & 64 \\ 2 & 5 & 3 & 3 & 4 \\ 31 & -42 & 78 & 55 & -3 \\ 62 & 47 & 29 & -14 & -8 \end{vmatrix}$$

**4.6** This exercise discusses the proof of the statement that a matrix with integral entries has integral determinant.

**(a)** Prove that if all the entries of a $2 \times 2$ matrix are integers, then its determinant must be an integer.

**(b)** Use part (a) and formula (4.2) on page 239 to prove the statement in part (a) for $3 \times 3$ matrices.

**(c)** Use part (b) to prove the statement in part (a) for $4 \times 4$ matrices. If you are familiar with mathematical induction, prove the statement in part (a) for all $n \times n$ matrices.

**4.7** This exercise discusses the proof of the statement that a matrix with two equal rows has zero determinant.

**(a)** Use formula (4.1) on page 239 to prove that a $2 \times 2$ matrix with two equal rows has a zero determinant.

**(b)** ✓✓Use part (a) to prove that a $3 \times 3$ matrix with two equal rows has a zero determinant. [*Hint:* Expand along the row that is not one of those which are assumed equal.]

**(c)** ✓✓Use part (b) to prove the statement in part (a) for $4 \times 4$ matrices. If you are familiar with mathematical induction, prove the statement in part (a) for all $n \times n$ matrices.

**4.8** Use the row interchange property to prove that an $n \times n$ matrix with two equal rows has a zero determinant.

**4.9** Let $X = [a_1, a_2, a_3]^t$, $Y = [b_1, b_2, b_3]^t$, and $Z = [c_1, c_2, c_3]^t$ be vectors in $\mathbb{R}^3$. Let $X \times Y = [d_1, d_2, d_3]^t$, where

$$d_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \quad d_2 = -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \quad d_3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Prove that

$$Z^t(X \times Y) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

**4.10** Use Exercise 4.9 and properties of the determinant to prove the following identities:

**(a)** $X^t(X \times Y) = 0$

**(b)** $Z^t(X \times Y) = Y^t(Z \times X)$

**(c)** $X \times (Y + Z) = X \times Y + X \times Z$

**4.11** ✓Let $A$ and $B$ be $n \times n$ matrices.

**(a)** Prove that

$$p(\lambda) = \det(A - \lambda B)$$

is a polynomial of degree at most $n$. [*Hint:* Do a mathematical induction argument that is, first prove it for $2 \times 2$ matrices. Then use formula (4.3) to show that if it is true for all $k \times k$ matrices with $k < n$, then it is true for $n \times n$ matrices.]

**(b)** Prove that the coefficient of $\lambda^n$ in $p(\lambda)$ is $(-1)^n \det B$. [*Hint:* If $p(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$, then $a_n = \lim_{n\to\infty} \lambda^{-n}p(\lambda)$.]

### 4.1.2 Computer Projects

Determinants are extremely useful in many contexts. You will, for example, use them constantly when you study eigenvalues and eigenvectors later in the text. In addition, you will see them used to write formulas for the solutions to many applied problems. In particular, determinants are used extensively in the study of differential equations and in the study of advanced calculus. Determinants are also used extensively in studying the mathematical foundations of linear algebra. Computers, however, do not generally use determinants for computations. Much faster and more efficient numerical techniques have been found. Thus, we do not provide any computer exercises for this chapter.

The reader should be aware, however, that MATLAB will compute determinants. The appropriate command is det(A). Incidentally, MATLAB uses the methods of the next section to compute determinants rather than the methods already described.

## 4.2 REDUCTION AND DETERMINANTS

Usually, computers (both human and electronic) do not use the formulas described in Section 4.1 for computing determinants. There are vastly more efficient methods based on

(a) the row interchange property,
(b) the row additive property, and
(c) the row scalar property.

Before describing these techniques, however, we need to note one consequence of these properties. Consider the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 4 & 2 & 1 & 5 \\ -1 & 5 & 2 & 6 \\ 4 & 2 & 1 & 5 \end{bmatrix}$$

If we interchange the second and fourth rows, the determinant of $A$ does not change because $A$ does not change. And yet, from the row interchange property, interchanging two rows must negate the value of the determinant. Zero is the only real number that equals its own negative. Thus, we see that $\det A = 0$. In general, *any* $n \times n$ *matrix with two equal rows has a zero determinant.*

This observation leads us to a very important principle. Consider the following matrix $A$. If we add twice row 1 onto row 3, we get the matrix $B$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & 7 \\ -2 & -4 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -1 & 7 \\ 0 & 0 & 4 \end{bmatrix} \tag{4.9}$$

It turns out that, remarkably, $A$ and $B$ have the same determinant. To explain why, let $A_1$, $A_2$, and $A_3$ denote the rows of $A$. Then, from the additive and scalar properties

$$\det B = \det \begin{bmatrix} A_1 \\ A_2 \\ A_3 + 2A_1 \end{bmatrix} = \det \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} + 2 \det \begin{bmatrix} A_1 \\ A_2 \\ A_1 \end{bmatrix} = \det \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \det A$$

(Note that a matrix with two equal rows has a zero determinant.)

Hence, expanding $\det B$ from (4.9) along the third row, we see that

$$\det A = \det B = 4\begin{vmatrix} 1 & 2 \\ 4 & -1 \end{vmatrix} = -36$$

The same argument proves the following theorem.

**Theorem 4.7** *In an $n \times n$ matrix $A$, adding a multiple of one row of $A$ onto a different row does not change the determinant of $A$.*

*Proof.* From the row interchange property, we may reduce to the case where we are adding a multiple of the first row onto the second. Then, letting $A_i$ be the rows of $A$, we get

$$\det B = \det\begin{bmatrix} A_1 \\ A_2 + cA_1 \\ \vdots \\ A_n \end{bmatrix} = \det\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} + c\det\begin{bmatrix} A_1 \\ A_1 \\ \vdots \\ A_n \end{bmatrix} = \det\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} = \det A$$

since, once again, a matrix with two equal rows has determinant 0. □

The above theorem tells us that we can use row reduction to compute determinants.

### EXAMPLE 4.8

Use row reduction methods to compute the determinant of the matrix $A$:

$$A = \begin{bmatrix} 3 & 6 & 9 & 12 \\ 1 & 2 & 2 & 1 \\ 3 & 5 & 2 & 1 \\ 0 & 2 & 4 & 2 \end{bmatrix}$$

*Solution.* We begin by using the scalar property on row 1.

$$\det A = 3\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 1 \\ 3 & 5 & 2 & 1 \\ 0 & 2 & 4 & 2 \end{vmatrix}$$

$$= 3\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & -3 \\ 0 & -1 & -7 & -11 \\ 0 & 2 & 4 & 2 \end{vmatrix} \qquad \begin{matrix} R_2 & \to & R_2 - R_1 \\ R_3 & \to & R_3 - 3R_1 \end{matrix}$$

$$= -3\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -7 & -11 \\ 0 & 0 & -1 & -3 \\ 0 & 2 & 4 & 2 \end{vmatrix} \qquad \begin{matrix} R_2 & \to & R_3 \\ R_3 & \to & R_2 \end{matrix}$$

$$= -3\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -7 & -11 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & -10 & -20 \end{vmatrix} \qquad R_4 \to R_4 + 2R_2$$

$$= 30\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{vmatrix} \qquad \text{(constants factored from rows 2, 3, 4)}$$

$$= 30\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{vmatrix} = -30 \qquad R_4 \to R_4 - R_3$$

### EXAMPLE 4.9

Compute det $A$, where

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 7 & 5 & 2 \end{bmatrix}$$

*Solution.*

$$\det A = 2\begin{vmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 3 & 2 & 1 \\ 7 & 5 & 2 \end{vmatrix} \qquad \text{(We factored 2 out of row 1.)}$$

$$= 2\begin{vmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & \frac{1}{2} & -\frac{7}{2} \\ 0 & \frac{3}{2} & -\frac{17}{2} \end{vmatrix} \qquad \begin{matrix} R_2 & \to & R_2 - 3R_1 \\ R_3 & \to & R_3 - 7R_1 \end{matrix}$$

$$= \frac{2}{4}\begin{vmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & -7 \\ 0 & 3 & -17 \end{vmatrix} \qquad \text{(We factored } \tfrac{1}{2} \text{ out of rows 2 and 3.)}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & -7 \\ 0 & 0 & 4 \end{vmatrix} = 2 \qquad R_3 \to R_3 - 3R_2$$

Since we may also expand determinants along columns, all the row properties also apply to columns. In particular, there are column interchange, scalar, and additive properties. Thus, we may, if we wish, do "elementary column" operations instead of elementary row operations. For example, by subtracting twice column 1 from column 3, we see that

$$\begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 4 \\ 3 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & -3 \end{vmatrix} = -3$$

We may even mix the two types of operations within the same problem.

Recall that an $n \times n$ matrix $A$ fails to have an inverse if and only if some echelon form $R$ of $A$ has a row of zeros.

$$\begin{array}{cc} \text{Invertible} & \text{Not invertible} \\ \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\ \det = 6 & \det = 0 \end{array}$$

Since echelon forms are triangular matrices, this happens if and only if $\det R = 0$. These ideas give rise to the following crucial theorem.

**Theorem 4.8** *An $n \times n$ matrix $A$ is invertible if and only if* $\det A \neq 0$.

*Proof.* The only elementary row operations that change the determinant of a matrix $A$ are the row interchange property, which multiplies the determinant by $-1$, and the multiplication of a given row by a nonzero scalar, which multiplies the determinant by the same nonzero scalar. Hence, the determinant of any matrix is a nonzero multiple of the determinant of its row reduced echelon form. If $A$ is invertible, then its row reduced form is $I$, and hence, the determinant of $A$ is nonzero. On the other hand, if $A$ is not invertible, its row reduced form has a row of zeros. A matrix with a row of zeros has a zero determinant. (This follows by expansion along the row in question.) Thus, $\det A = 0$. □

The fact that we may use row operations to compute determinants has another, subtler consequence. The theorem below is significant in that it says that the properties of the determinant function uniquely determine its form. We will give a formal proof of the uniqueness principle at the end of this section. However, the basic idea of the proof is that the value of any such function on a given matrix can be computed using row reduction and this computation is, step by step, identical to the computation of the determinant. Hence, this function must equal the determinant.

## Uniqueness of the Determinant

**Theorem 4.9** (Uniqueness Theorem). *Suppose that $D$ is a function that transforms $n \times n$ matrices into numbers such that*

*(a) $D(I) = 1$ and*

*(b) $D$ satisfies the row interchange, the row scalar, and the row additivity properties found on pages 241, 242, and 245, respectively.*

*Then $D(A) = \det(A)$ for all $n \times n$ matrices $A$.*

Uniqueness principles are very important in mathematics. They often help us to derive many deep properties. The following theorem (and its proof) is a beautiful example of this. There is a more computational proof based on factoring $A$ into a product of simpler matrices for which the theorem can be proved directly. We have chosen to present the more "abstract" proof since it stresses the importance of the properties of the determinant. To gain an appreciation for the depth of this result, the reader should attempt to give a direct proof of it in the $2 \times 2$ and $3 \times 3$ cases.

**Theorem 4.10** (Product Theorem). *For all $n \times n$ matrices $A$ and $B$,* $\det(AB) = (\det A)(\det B)$.

*Proof.* If $\det B = 0$, then $B$ has rank less than $n$. It follows that $AB$ also has rank less than $n$ so $\det(AB) = 0$, proving the theorem in this case. Thus, we may assume that $\det B \neq 0$.

For each $n \times n$ matrix $A$, let

$$D(A) = \frac{\det(AB)}{\det B}$$

(We consider $B$ as fixed and $A$ as varying.) Our theorem is equivalent to showing that $D(A) = \det A$ for all $A$. This is true if $D$ satisfies the hypotheses of parts (a) and (b) of the uniqueness principle.

For (a), note that

$$D(I) = \frac{\det(IB)}{\det B} = \frac{\det B}{\det B} = 1$$

To prove the row interchange property, recall that in Section 3.2 we proved $AB$ may be computed as the rows of $A$ times $B$. In symbols,

$$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} B = \begin{bmatrix} A_1 B \\ A_2 B \\ \vdots \\ A_n B \end{bmatrix} \tag{4.10}$$

It follows that if two rows of $A$ are interchanged, the corresponding rows of $AB$ become interchanged. Thus, $D(A)$ is negated.

Since the row interchange property is true, it suffices to prove the row additivity property for the first row. Thus, suppose that the rows of $A$ are denoted by $A_i$ and that $A_1 = U + V$, where $U$ and $V$ are row vectors. Then

$$AB = \begin{bmatrix} (U+V)B \\ A_2B \\ \vdots \\ A_nB \end{bmatrix} = \begin{bmatrix} UB + VB \\ A_2B \\ \vdots \\ A_nB \end{bmatrix}$$

Hence, from the row additivity of the determinant

$$\det AB = \det \begin{bmatrix} UB \\ A_2B \\ \vdots \\ A_nB \end{bmatrix} + \det \begin{bmatrix} VB \\ A_2B \\ \vdots \\ A_nB \end{bmatrix}$$

Dividing both sides of this equality by $\det B$ shows that

$$D(A) = D\left(\begin{bmatrix} U \\ A_2 \\ \vdots \\ A_n \end{bmatrix}\right) + D\left(\begin{bmatrix} V \\ A_2 \\ \vdots \\ A_n \end{bmatrix}\right)$$

which proves additivity for $D$. We leave the row scalar property as an exercise for the reader. □

Another consequence of the uniqueness principle is the following result that we leave as an exercise (Exercise 4.23).

**Theorem 4.11** *For any $n \times n$ matrix $A$,* $\det A = \det A^t$.

*Proof of the uniqueness principle.* Suppose that $D$ is some function which transforms $n \times n$ matrices into numbers and satisfies properties (a) and (b) from the statement of the uniqueness principle.

Let us first assume that $A$ is in reduced echelon form. If $A$ has rank $n$, then $A = I$ and [from property (a)] $D(A) = 1 = \det A$. If $A$ has rank less than $n$, then the last row of $A$ is 0 and $\det A = 0$. From the row scalar property

$$D(A) = D\left(\begin{bmatrix} A_1 \\ \vdots \\ A_{n-1} \\ \mathbf{0} \end{bmatrix}\right) = D\left(\begin{bmatrix} A_1 \\ \vdots \\ A_{n-1} \\ 0 \cdot \mathbf{0} \end{bmatrix}\right) = 0\,D\left(\begin{bmatrix} A_1 \\ \vdots \\ A_{n-1} \\ \mathbf{0} \end{bmatrix}\right) = 0 = \det A$$

Next, suppose that $A$ can be brought into reduced echelon form using only one elementary row operation. Let $A'$ be the reduced echelon form of $A$. If two rows of $A$ were interchanged to produce $A'$, then $D(A) = -D(A')$ and $\det A = -\det A'$, showing that $D(A) = \det A$. If some row of $A$ was multiplied by a nonzero scalar $c$ to produce $A'$, then $D(A) = c^{-1}D(A')$ and $\det A = c^{-1} \det A'$. Again the desired equality follows. Finally, suppose that a multiple of some row of $A$ was added to another row of $A$ to produce $A'$. The argument used in the proof of Theorem 4.7 shows that $D(A) = D(A')$. Also, from Theorem 4.7, $\det A = \det A'$. Again the desired equality follows. Thus, regardless of how $A'$ was produced, $D(A) = \det A$.

Now, suppose that we have proved $D(A) = \det A$ for any matrix that can be brought into reduced echelon form using $n$ elementary row operations. Let $A$ be a matrix that is reducible using $n + 1$ elementary row operations and let $A'$ be the matrix produced by applying the first of these operation to $A$. Since $A'$ can be reduced using only $n$ elementary row operations, we know that $D(A') = \det A'$. The argument from the previous paragraph shows that then $D(A) = \det A$. Our theorem follows by mathematical induction. □

**True-False Questions**: Justify your answers.

**4.8** Given that none of the determinants are 0, the following matrices have the same determinant:

**(c)** $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \\ -1 & 2 & -1 \end{bmatrix}$, $\quad 4\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ -1 & 2 & -1 \end{bmatrix}$

**(d)** $\begin{vmatrix} 7 & -3 & 4 & -2 \\ 13 & 6 & 5 & 11 \\ 1 & 2 & 1 & 0 \\ 17 & 16 & 5 & 6 \end{vmatrix}$, $\quad \begin{vmatrix} 7 & -3 & 4 & -2 \\ 1 & 2 & 1 & 0 \\ 10 & 0 & 2 & 11 \\ 17 & 16 & 5 & 6 \end{vmatrix}$

**(e)** $\begin{bmatrix} 2 & 3 & -5 & 3 \\ 7 & 1 & 2 & 3 \\ 1 & 4 & 4 & 1 \\ 3 & 2 & 4 & 5 \end{bmatrix}$, $\quad \begin{bmatrix} 2 & 7 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ -5 & 2 & 4 & 4 \\ 3 & 3 & 1 & 5 \end{bmatrix}$

**4.9** For all $n \times n$ matrices $A$ and $B$, $\det(AB) = \det(BA)$.

**4.10** Suppose that $\det(A + I) = 3$, and $\det(A - I) = 5$. Then

$$\det(A^2 - I) = 20$$

**4.11** Suppose that $\det A = 2$, $\det(A + I) = 3$, and $\det(A + 2I) = 5$. Then

$$\det(A^4 + 3A^3 + 2A^2) = 48$$

## EXERCISES

**4.12** Use row reduction to compute the following determinants:

**(a)** ✓✓ $\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix}$ **(b)** $\begin{vmatrix} -3 & 2 & 2 \\ 1 & 4 & 1 \\ 7 & 6 & -2 \end{vmatrix}$

**(c)** ✓✓ $\begin{vmatrix} 2 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 2 \\ 1 & 0 & 0 & 5 \end{vmatrix}$ **(d)** $\begin{vmatrix} 3 & 1 & 3 & 0 \\ 3 & 1 & 3 & 1 \\ 0 & 0 & 2 & 1 \\ 6 & 3 & 4 & 5 \end{vmatrix}$

**(e)** ✓✓ $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix}$ **(f)** $\begin{vmatrix} 2 & 3 & 2 & 0 \\ 9 & 0 & 1 & 1 \\ 1 & 0 & 1 & 4 \\ 13 & 10 & 0 & 9 \end{vmatrix}$

**4.13** Compute the determinants in Exercise 4.12, using column reduction instead of row reduction.

**4.14** Compute the determinants in Exercise 4.1 on page 249, using row reduction.

**4.15** ✓✓Suppose that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5$$

Compute the following determinants:

**(a)** $\begin{vmatrix} 2a & 2b & 2c \\ 3d-a & 3e-b & 3f-c \\ 4g+3a & 4h+3b & 4i+3c \end{vmatrix}$ **(b)** $\begin{vmatrix} a+2d & b+2e & c+2f \\ g & h & i \\ d & e & f \end{vmatrix}$

**4.16** Use row reduction to prove that for all numbers $x$, $y$, and $z$

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (y-x)(z-x)(z-y)$$

reduction.

**4.17** Let

$$A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

represent a $3 \times 3$ matrix and suppose that $A_3 = 2A_1 + A_2$. Use the row scalar and additive properties to prove that $\det A = 0$.

**4.18** ✓✓Use the row interchange, scalar, and additive properties to prove that any $3 \times 3$ matrix with linearly dependent rows has a zero determinant.

**4.19** Let

$$A = [A_1, A_2, A_3]$$

represent a $3 \times 3$ matrix and suppose that $A_3 = 2A_1 + A_2$. Use the column scalar and additive properties to prove that $\det A = 0$.

**4.20** ✓✓Use the column interchange, scalar, and additive properties to prove that any $3 \times 3$ matrix with linearly dependent columns has a zero determinant.

**4.21** Prove the row scalar property for the function $D$ used in the proof of the product theorem.

**4.22** Suppose $D$ is a function that transforms $n \times n$ matrices into numbers and satisfies the row scalar property. Let $A$ be an $n \times n$ matrix whose first row is $\mathbf{0}$. Prove that $D(A) = 0$. Does the same proof apply to other rows?

**4.23** Prove Theorem 4.11 on page 257. *Hint:* Let $D(A) = \det(A^t)$ and make use of the Remark on page 248.

**4.24** Let $A$ be an $n \times n$ invertible matrix. Prove that $\det A^{-1} = 1/\det A$. (Think about the fact that $AA^{-1} = I$.)

**4.25** ✓✓Two $n \times n$ matrices $A$ and $B$ are said to be similar if there is an invertible matrix $Q$ such that $A = QBQ^{-1}$. Prove that similar matrices have the same determinant.

**4.26** Later we shall study $n \times n$ matrices with the property that $AA^t = I$. What are the possible values of the determinant of such a matrix?

**4.27** Let $A$ be an $n \times n$ matrix. By

$$B = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & A \end{bmatrix}$$

we mean the $(n+1) \times (n+1)$ matrix $B$, which has $B_{11} = 1$, all other entries in the first row and column equal to 0, and $A$ is an $n \times n$ submatrix in the lower right-hand corner. Thus, for example, if

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

then

$$\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & A \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 3 & 4 \end{bmatrix}$$

For any $n \times n$ matrix $A$, let

$$D(A) = \det \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & A \end{bmatrix}$$

**(a)** Compute $D(A)$, where $A$ is as in Exercise 4.1(a) on page 249.

**(b)** Use the uniqueness principle to prove that $D(A) = \det A$ for all $n \times n$ matrices $A$.

**(c)** Prove that $D(A) = \det A$ for all $n \times n$ matrices $A$ by expanding along the first row of $B$.

### 4.2.1 Volume

For a $2 \times 2$ matrix $A = [A_1, A_2]$, let area $A$ be the area of the parallelogram with sides $A_1$ and $A_2$. (If $A_1$ and $A_2$ are linearly dependent, we define area $A = 0$.) Thus, for example, if

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

then area $A$ is the area of the parallelogram indicated in Figure 4.1. Since the area of a parallelogram is the length of the base times the height, in this case,

$$\text{area } A = 2 \cdot 2 = 4$$

which equals $\det A$.

This illustrates a general principle: for any $2 \times 2$ matrix $A$, area $A = |\det A|$. The reason is that the area function satisfies properties similar to those of the determinant with respect to columns:

**Proposition 4.2** *The area function satisfies the following properties:*

*(a) area* $[A_1, A_2] =$ *area* $[A_2, A_1]$ (*column interchange property*)

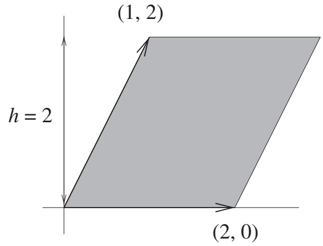



**FIGURE 4.1** The area of a parallelogram.

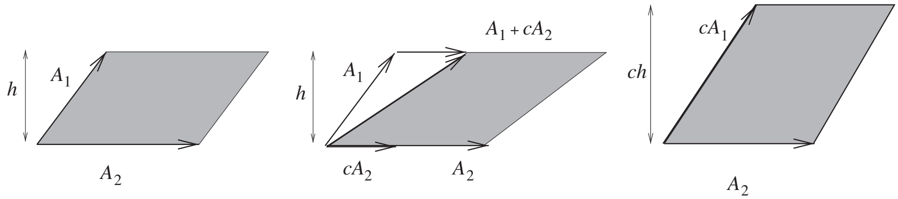



**FIGURE 4.2** Proof of Proposition 4.2.

*(b) area* $[A_1, A_2] =$ *area* $[A_1 + cA_2, A_2]$ (*column additive property*)
*(c) area* $[cA_1, A_2] = |c|$*area* $[A_1, A_2]$ (*column scalar property*)
*(d) area* $I = 1$

*Proof.* Property (a) is clear. Property (b) follows from the observation that the two parallelograms in question have the same base and height (Figure 4.2, middle). Property (c) follows from the observation that the height of a parallelogram is proportional to the length of each side (Figure 4.2, right). □

Although we stated (b) and (c) for the first column, they actually hold for the second column as well, due to property (a). It now follows from Proposition 4.2 that we may use *column reduction* to compute area $A$. For example,

$$\begin{aligned}\text{area}\begin{bmatrix}1 & 2\\ 3 & -1\end{bmatrix} &= \text{area}\begin{bmatrix}1 & 0\\ 3 & -7\end{bmatrix} && (\text{add } -2 \text{ col. 1 to col. 2})\\ &= |-7|\,\text{area}\begin{bmatrix}1 & 0\\ -\frac{3}{7} & 1\end{bmatrix} && (\text{factor } -7 \text{ from col. 2})\\ &= 7\,\text{area}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = 7 && (\text{add } \tfrac{3}{7} \text{ col. 2 to col. 1})\end{aligned}$$

But we may also use column reduction to compute $|\det A|$. The steps in either calculation are the same, proving that $|\det A| = \text{area } A$.

What we said for area applies equally to volume. For a $3 \times 3$ matrix $A = [A_1, A_2, A_3]$, let vol $A$ be the volume of the parallelepiped with sides $A_1$, $A_2$, and $A_3$. (If the $A_i$ are linearly dependent, we define vol $A = 0$.) (See Figure 4.3.) The

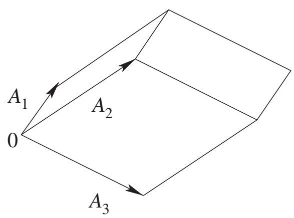



**FIGURE 4.3** Volume of a parallelepiped.

volume of a parallelepiped is the area of the base times the height. The following proposition follows from arguments similar to those in the two-dimensional case:

**Proposition 4.3** *The vol function satisfies the following properties, where $A$ is a* $3 \times 3$ *matrix:*

*(a) Interchanging any two columns does not change vol $A$ (column interchange property)*
*(b) vol $[A_1, A_2, A_3] =$ vol $[A_1 + cA_2, A_2, A_3]$ (column additive property)*
*(c) vol $[cA_1, A_2, A_3] = |c|$vol $[A_1, A_2, A_3]$ (column scalar property)*
*(d) vol $I = 1$*

Again, (a) implies that (b) and (c) hold for any column. Reasoning as before, we find that vol $A = |\det A|$. We summarize this discussion in the following theorem.

**Theorem 4.12** *Let $A = [A_1, A_2]$ be a $2 \times 2$ matrix. Then $|\det A|$ is the area of the parallelogram with sides $A_1$ and $A_2$. If $A = [A_1, A_2, A_3]$ is a $3 \times 3$ matrix, then $|\det A|$ is the volume of the parallelepiped with sides $A_1$, $A_2$, and $A_3$.*

Theorem 4.12 has an important consequence. Let $A = [A_1, A_2]$ be a $2 \times 2$ matrix and let $B$ be another $2 \times 2$ matrix. Then multiplication by $B$ transforms the parallelogram with sides $A_1, A_2$ into one with sides $BA_1$, $BA_2$. But these are just the columns of $BA$. Hence, the area of this new parallelogram is

$$|\det(BA)| = |\det B|\,|\det A|$$

Thus, multiplication of a parallelogram by $B$ multiplies its area by $|\det B|$. Similar comments hold for volume:

**Theorem 4.13** *Let $B$ be a $2 \times 2$ matrix. The image of a parallelogram of area $A$ under multiplication by $B$ has area $|\det B|A$. Similarly, the image of a parallelepiped of volume $V$ under multiplication by a $3 \times 3$ matrix $B$ has volume $|\det B|V$.*

Actually, the same theorem holds for any shape that has area or volume: Multiplication by $B$ multiplies the area (volume) by $|\det B|$.

What about higher dimensions? It is possible to define a concept of $n$-dimensional volume in such a way that the analogue of Theorem 4.12 holds for any $n \times n$ matrix $A = [A_1, \dots, A_n]$, in which case Theorem 4.13 follows just as before.

## EXERCISES

**4.28** ✓Sketch the parallelogram with the stated vectors as sides and compute its area.

**(a)** $[2, 0]^t$, $[0, 1]^t$ **(b)** $[2, 0]^t$, $[1, 1]^t$
**(c)** $[1, 1]^t$, $A_2 = [1, -1]^t$

**4.29** ✓Compute the volume of the parallelepiped with the stated vectors as sides.

**(a)** $[1, 1, 1]^t$, $[2, -1, 1]^t$, $[1, 0, 1]^t$ **(b)** $[2, -1, 0]^t$, $[1, 2, 0]^t$, $[0, 0, 1]^t$
**(c)** $[2, -5, 1]^t$, $[0, 4, 1]^t$, $[2, 0, 1]^t$.

**4.30** Use area to explain why it is expected that the determinant of a $2 \times 2$ matrix with linearly dependent columns has determinant zero.

**4.31** Use volume to explain why it is expected that the determinant of a $3 \times 3$ matrix with linearly dependent columns has determinant zero.

**4.32** Use area to explain why it is expected that the determinant of the matrix $R_\psi$ form formula (3.1) on page 150 is $\pm 1$. [*Hint:* Use Theorem 4.13.]

**4.33** Use volume to explain why it is expected that for $3 \times 3$ matrices $B$ and $C$ $|\det(BC)| = |\det B|\,|\det C|$. [*Hint:* Use Theorem 4.13.]

## 4.3 A FORMULA FOR INVERSES

In the first section of this chapter, we promised a formula for inverses. In this section, we state such a formula. To derive the formula, first let $A$ be a $3 \times 3$ invertible matrix. We compute $A^{-1}$ by solving $AX = Y$. This equation may be written

$$x_1 A_1 + x_2 A_2 + x_3 A_3 = Y$$

where $A_i$ are the columns of $A$ and $X = [x_1, x_2, x_3]^t$.

Since $A$ is invertible, we know that a unique solution exists. To find it, consider the following computation:

$$\begin{aligned} \det[Y, A_2, A_3] &= \det[x_1 A_1 + x_2 A_2 + x_3 A_3, A_2, A_3] \\ &= x_1 \det[A_1, A_2, A_3] + x_2 \det[A_2, A_2, A_3] + x_3 \det[A_3, A_2, A_3] \\ &= x_1 \det A \end{aligned}$$

(Note that a matrix with two equal columns has a determinant of zero.) We conclude that

$$x_1 = \det[Y, A_2, A_3](\det A)^{-1}$$

In words, to compute $x_1$, we replace $A_1$ by $Y$, take the determinant, and divide by the determinant of $A$. A similar rule applies to computing the other variables:

$$\begin{aligned} x_2 &= \det[A_1, Y, A_3](\det A)^{-1} \\ x_3 &= \det[A_1, A_2, Y](\det A)^{-1} \end{aligned}$$

These formulas are known collectively as Cramer's rule.

### EXAMPLE 4.10

Find $x_1$, $x_2$, and $x_3$ in terms of the $y_i$ in the system below:

$$\begin{aligned}
5x_1 + 2x_2 + \phantom{2}x_3 &= y_1\\
2x_1 + 2x_2 + 2x_3 &= y_2\\
2x_1 + \phantom{2}x_2 + \phantom{2}x_3 &= y_3
\end{aligned}$$

Use your answer to find $A^{-1}$, where $A$ is the coefficient matrix for the system.

*Solution.* According to Cramer's rule,

$$x_1 = \begin{vmatrix} y_1 & 2 & 1\\ y_2 & 2 & 2\\ y_3 & 1 & 1 \end{vmatrix} \left( \begin{vmatrix} 5 & 2 & 1\\ 2 & 2 & 2\\ 2 & 1 & 1 \end{vmatrix} \right)^{-1} = \frac{-y_2 + 2y_3}{2}$$

$$x_2 = \begin{vmatrix} 5 & y_1 & 1\\ 2 & y_2 & 2\\ 2 & y_3 & 1 \end{vmatrix} \left( \begin{vmatrix} 5 & 2 & 1\\ 2 & 2 & 2\\ 2 & 1 & 1 \end{vmatrix} \right)^{-1} = \frac{2y_1 + 3y_2 - 8y_3}{2}$$

and

$$x_3 = \begin{vmatrix} 5 & 2 & y_1\\ 2 & 2 & y_2\\ 2 & 1 & y_3 \end{vmatrix} \left( \begin{vmatrix} 5 & 2 & 1\\ 2 & 2 & 2\\ 2 & 1 & 1 \end{vmatrix} \right)^{-1} = \frac{-2y_1 - y_2 + 6y_3}{2}$$

We write

$$\begin{bmatrix} x_1\\ x_2\\ x_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -y_2 + 2y_3\\ 2y_1 + 3y_2 - 8y_3\\ -2y_1 - y_2 + 6y_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -1 & 2\\ 2 & 3 & -8\\ -2 & -1 & 6 \end{bmatrix}\begin{bmatrix} y_1\\ y_3\\ y_3 \end{bmatrix}$$

The $3 \times 3$ matrix on the right is $A^{-1}$.

The general case of Cramer's rule is almost identical, as is the derivation:

**Theorem 4.14** (Cramer's Rule). *Let $A$ be an $n \times n$ invertible matrix. Suppose that $AX = Y$, where $X$ and $Y$ belong to $\mathbb{R}^n$. Then $X = [x_1, x_2, \ldots, x_n]^t$ where*

$$x_i = \det[A_1, \ldots, A_{i-1}, Y, A_{i+1}, \ldots, A_n](\det A)^{-1}$$

*Proof.* Since $A$ is invertible, there is an $X$ such that $AX = Y$. Thus, there are scalars $x_i$ such that

$$x_1A_1 + x_2A_2 + \cdots + x_nA_n = Y$$

where the $A_i$ are the columns of $A$. Then, from the column additivity property of the determinant,

$$\begin{aligned}\det[Y, A_2, A_3, \ldots, A_n] &= x_1 \det[A_1, A_2, \ldots, A_n] + x_2 \det[A_2, A_2, \ldots, A_n] \\ &\quad + \cdots + x_n \det[A_n, A_2, \ldots, A_n] \\ &= x_1 \det A\end{aligned}$$

The last equality, which is true because the determinant of a matrix with two equal columns is zero, proves Cramer's rule for $x_1$. The proof for the other $x_i$ is similar. □

### EXAMPLE 4.11

Use Cramer's rule to solve for $z$ in the following system:

$$\begin{aligned}4x + 5y + 3z + 3w &= 1\\ 2x + y + z + w &= 0\\ 2x + 3y + z + w &= 1\\ 5x + 7y + 3z + 4w &= 2\end{aligned}$$

*Solution.* According to Cramer's rule, we replace the third column of the coefficient matrix by the constants on the right side of the equation, take the determinant, and divide by the determinant of the coefficient matrix. Thus, $z = a/b = -\frac{1}{2}$ where

$$a = \begin{vmatrix} 4 & 5 & 1 & 3 \\ 2 & 1 & 0 & 1 \\ 2 & 3 & 1 & 1 \\ 5 & 7 & 2 & 4 \end{vmatrix}, \quad b = \begin{vmatrix} 4 & 5 & 3 & 3 \\ 2 & 1 & 1 & 1 \\ 2 & 3 & 1 & 1 \\ 5 & 7 & 3 & 4 \end{vmatrix}$$

Example 4.10 demonstrates that we may use Cramer's rule to find inverses. To derive a general formula for inverses, let $A = [a_{ij}]$ be an $n \times n$ invertible matrix and let $B = A^{-1}$. The equation $AB = I$ is equivalent to the $n$ systems equations

$$AB_j = I_j, \qquad 1 \le j \le n$$

where $I_j$ is the $j$th column of the $n \times n$ identity matrix and $B_j$ is the $j$th column of $B$.

From Cramer's rule, for all $1 \le i \le n$,

$$b_{ij} = \frac{\det[A_1, \ldots, A_{i-1}, I_j, A_{i+1}, \ldots, A_n]}{\det A}$$

where the $A_k$ are the columns of $A$. We expand the determinant in the numerator along the $i$th column using formula (4.6). The only nonzero entry in this column is a 1 in the $j$th position. Hence, we find that

$$b_{ij} = \frac{(-1)^{i+j}\det(A_{ji})}{\det A}$$

where $A_{ji}$ is the matrix obtained from $A$ by deleting the $j$th row and $i$th columns. Hence, we have proved the following theorem:

**Theorem 4.15** *Let $A$ be $n \times n$ and invertible. Then*

$$(A^{-1})_{ij} = \frac{(-1)^{i+j}\det(A_{ji})}{\det A} \tag{4.11}$$

*where $A_{ji}$ is the matrix obtained from $A$ by deleting the $j$th row and $i$th columns.*

### EXAMPLE 4.12

Let $A$ be the matrix below. Find $(A^{-1})_{23}$ and $(A^{-1})_{41}$.

$$A = \begin{bmatrix} 1 & 7 & 6 & -5 \\ 2 & 4 & 3 & 1 \\ 3 & 2 & 1 & 0 \\ 6 & -1 & 2 & 4 \end{bmatrix}$$

*Solution.* It is easily seen that $\det A = -264$. Hence, from Theorem 4.14,

$$\begin{aligned}(A^{-1})_{23} &= \frac{(-1)^{2+3}\det A_{32}}{\det A} \\ &= \frac{-\begin{vmatrix} 1 & 6 & -5 \\ 2 & 3 & 1 \\ 6 & 2 & 4 \end{vmatrix}}{-264} = \frac{68}{264}\end{aligned}$$

Also

$$(-1)^{1+4}\det A_{14} = -\begin{vmatrix} 2 & 4 & 3 \\ 3 & 2 & 1 \\ 6 & -1 & 2 \end{vmatrix} = 35$$

yielding

$$(A^{-1})_{41} = -\frac{35}{264}$$

**True-False Questions**: Justify your answers.

**4.12** In the following system, you may assume that the determinant of the coefficient matrix is not zero. Then, $z = 0$. [*Hint:* Compute the sum of the first two columns of the coefficient matrix.]

$$\begin{aligned} 2x + 3y - 5z + 3w &= 5 \\ 7x + y + 2z + 3w &= 8 \\ x + 4y + 4z + w &= 5 \\ 3x + 2y + 4z + 5w &= 5 \end{aligned}$$

**4.13** If the following matrix is invertible, then the $(1,4)$ entry of $A^{-1}$ is zero. [*Hint:* Compute the sum of the first two rows of $A$.]

$$A = \begin{bmatrix} 1 & 2 & 1 & 13 \\ 2 & 3 & -5 & -1 \\ 0 & 5 & -4 & 12 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

## EXERCISES

**4.34** ✓✓Use Cramer's rule to solve the following system:

$$\begin{aligned} x + 2y + z + w &= 1 \\ x \phantom{+2y} + 3z - w &= 0 \\ 2x + y + z + w &= 0 \\ x - y + z \phantom{+w} &= 1 \end{aligned}$$

**4.35** Use Cramer's rule to express the value of $y$ in system (a) and the value of $z$ in system (b) as a ratio of two determinants. Do not evaluate the determinants.

**(a)** $$\begin{aligned} 2x + y + 3z + w &= 4 \\ x + 4y + 2z - 3w &= -1 \\ -x + y + z + w &= 0 \\ 4x - y + z + 2w &= 0 \end{aligned}$$

**(b)** $$\begin{aligned} x + 3y + z &= 1 \\ 3x + 4y + 5z &= 7 \\ 2x + 5y + 7z &= 2 \end{aligned}$$

**4.36** Use Cramer's rule to solve the following system for $z$ in terms of $p_1$, $p_2$, and $p_3$. (Note that we have not asked for $x$ or $y$.)

$$\begin{aligned} x + 2y - 3z &= p_1 \\ 3x + y - z &= p_2 \\ 2x + 3y + 5z &= p_3 \end{aligned}$$

**4.37** ✓✓Use Cramer's rule to find functions $c_1(x)$ and $c_2(x)$ that satisfy the following equations for all $x$:

$$(e^x \cos x)c_1(x) + (\sin x)c_2(x) = 1 + x^2$$
$$(-e^x \sin x)c_1(x) + (\cos x)c_2(x) = x$$

**4.38** Use Theorem 4.15 to find the $(1, 2)$ entry for the inverse of each invertible matrix in Exercise 4.12 on page 258. ✓[(a), (c), (e)]

**4.39** ✓✓Use Theorem 4.15 to find the entry in the $(1, 3)$ position of the inverse of the coefficient matrix for the system in Exercise 4.34 on page 268.

**4.40** ✓✓In Example 4.10 on page 265, we computed $A^{-1}$, where $A$ is the coefficient matrix of the given system. Use Theorem 4.15 to compute $(A^{-1})^{-1}_{32}$. What should you get?

**4.41** Find the inverse of the following matrix for all values of $x$ for which the inverse exists. [*Hint*: $\det A = -x^{14} + 2x^9 - x^4$.]

$$A = \begin{bmatrix} x^2 & x^3 & x^7 \\ x & x^2 & x \\ x^5 & x & x^2 \end{bmatrix}$$

**4.42** As you may have noted, computing inverses often produces some nasty fractions. However, some matrices have inverses that involve no fractions. For example, from the formula for the inverse of a $2 \times 2$ matrix given at the beginning of Section 4.1,

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

Find a $3 \times 3$ matrix $A$ with all nonzero entries such that both $A$ and $A^{-1}$ have only integral entries. Note that we have not asked for $A^{-1}$. [*Hint:* Compute the determinant of the above matrix. How is this relevant?]

**4.43** Let $A$ be an $n \times n$ matrix that has only integers as entries. State a necessary and sufficient condition on the determinant of such a matrix that guarantees that the inverse has only integers as entries. Prove your condition. [*Hint:* Consider the property $AA^{-1} = I$.]

## CHAPTER SUMMARY

This chapter was devoted to the ***determinant***, which is a number that we can compute from a given square matrix. One reason for the importance of the determinant is that it occurs as the denominator in the general formula for the inverse of an $n \times n$

matrix [formula (4.11) on page 267]. This suggests the important result that *an $n \times n$ matrix is invertible if and only if its determinant is nonzero* (Theorem 4.8, page 255, Section 4.2).

Our definition of the determinant was inductive, expressing the determinant of an $n \times n$ matrix in terms of determinants of $(n-1) \times (n-1)$ matrices. Explicitly, we defined the determinant in terms of a process called ***expansion along the first row*** (formula (4.3), page 240, Section 4.1). It turned out, however, that we can expand along any row we wish using the ***cofactor (Laplace)*** expansion (Theorem 4.2 on page 242). In fact, we can even expand along columns (Theorem 4.4 on page 243).

The determinant has several important properties, the most notable being the ***row additivity***, ***row scalar***, and ***row interchange*** properties from Section 4.1. In Section 4.2, these properties allowed us to understand exactly how the determinant of a matrix changes as we row reduce the matrix, which in turn allowed us to use row reduction to compute determinants.

The ability to use reduction to compute determinants leads to some deep insights into properties of the determinant. For example, reduction was the basis for our proof that an $n \times n$ matrix is invertible if and only if its determinant is nonzero. Also, the ability to compute determinants via reduction told us that the determinant is the only transformation that assigns numbers to matrices and that satisfies the row additivity, row scalar, and row interchange properties, together with the fact that the determinant of the identity matrix is 1 (the ***uniqueness theorem*** in Section 4.2). The facts that $\det AB = (\det A)(\det B)$ (the ***product theorem*** in Section 4.2) and $\det A = \det A^t$ are consequences of the Uniqueness Theorem.

In Section 4.3, we showed that the determinant yields an important method for solving systems of equations (***Cramer's rule***). Typically, in solving a system where the coefficients are numbers, Cramer's rule involves many more computations than row reduction. However, if the coefficients contain variables, or if they are, say, functions, then Cramer's rule can be useful. Cramer's rule also yielded a general formula for the inverse of a matrix (Theorem 4.15 on page 267).

# CHAPTER 5

# EIGENVECTORS AND EIGENVALUES

## 5.1 EIGENVECTORS

Consider the following example:

**EXAMPLE 5.1**

Metropolis is served by two newspapers, the Planet and the Jupiter. The Jupiter, however, seems to be in trouble. Currently, the Jupiter has only a 38% market share. Furthermore, every year, 10% of its readership switches to the Planet, whereas only 7% of the Planet's readers switch to the Jupiter. Assume that no one subscribes to both papers and that total newspaper readership remains constant. What is the long-term outlook for the Jupiter?

*Solution.* Currently, out of every 100 readers, 38 read the Jupiter and 62 read the Planet. Next year, the figures for the Jupiter and Planet will be, respectively,

$$0.9(38) + 0.07(62) = 38.54$$
$$0.1(38) + 0.93(62) = 61.46$$

This may be expressed as the matrix product

$$\begin{bmatrix} 0.9 & 0.07 \\ 0.1 & 0.93 \end{bmatrix} \begin{bmatrix} 38 \\ 62 \end{bmatrix} = \begin{bmatrix} 38.54 \\ 61.46 \end{bmatrix} \tag{5.1}$$

*Linear Algebra: Ideas and Applications*, Fourth Edition. Richard C. Penney.

Companion website: www.wiley.com/go/penney/linearalgebra

The vectors $V_1 = [38, 62]^t$ and $V_2 = [38.54, 61.46]^t$ are referred to as the (percentage) **state vectors** for the first and second years, respectively.

The $2 \times 2$ matrix $P$ in equation (5.1) is called the **transition matrix** because multiplication of a state vector by it produces the state vector for the next year. Thus, for the third year (rounding to two decimal places), we have

$$V_3 = PV_2 = \begin{bmatrix} 0.9 & 0.07 \\ 0.1 & 0.93 \end{bmatrix} \begin{bmatrix} 38.54 \\ 61.46 \end{bmatrix} = \begin{bmatrix} 38.99 \\ 61.01 \end{bmatrix}$$

Repeatedly multiplying by $P$, we obtain the state vectors through year 6:

$$V_4 = PV_3 = \begin{bmatrix} 39.36 \\ 60.64 \end{bmatrix}, \quad V_5 = PV_4 = \begin{bmatrix} 39.67 \\ 60.33 \end{bmatrix}, \quad V_6 = PV_5 = \begin{bmatrix} 39.93 \\ 60.07 \end{bmatrix}$$

Not only is the Jupiter not in trouble; it is actually thriving. The market share grows year after year! What is happening is that even though the Jupiter is less popular than the Planet, there are enough disgruntled Planet subscribers to keep Jupiter growing.

It does seem, though, that the *rate* of growth is slowing; in the first year the circulation grows by 1.4%. In the fifth year it only grows by 0.65%. We expect that eventually we will reach a state vector $X$ for which there is no noticeable growth—that is, $PX = X$.

If $X = [x, y]^t$, the system $PX = X$ is equivalent to

$$\begin{aligned} 0.9x + 0.07y &= x \\ 0.1x + 0.93y &= y \end{aligned}$$

which is the same as

$$\begin{aligned} -0.1x + 0.07y &= 0 \\ 0.1x - 0.07y &= 0 \end{aligned}$$

It is striking that this system is rank 1. This means that there really is a nonzero vector $X$ such that $PX = X$. Explicitly, we find that $x = 0.7y$. Since $x$ and $y$ represent percentages of readers, we also know that $x + y = 100$. This yields $y = 100/1.7 \approx 58.82$ and $x = 70/1.7 \approx 41.18$. Thus, we guess that Jupiter will eventually wind up with 41.18% of the market and the Planet will get 58.82%. That is, $X = [41.18, 58.82]^t$.

We can, in fact, prove that our guess is correct. For this, note that

$$V_1 - X = \begin{bmatrix} 38 \\ 62 \end{bmatrix} - \begin{bmatrix} 41.18 \\ 58.82 \end{bmatrix} = \begin{bmatrix} -3.18 \\ 3.18 \end{bmatrix} = 3.18 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Hence,

$$V_1 = X + (3.18)Y$$

where $Y = [-1, 1]^t$. Note also that

$$PY = \begin{bmatrix} 0.9 & 0.07 \\ 0.1 & 0.93 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.83 \\ 0.83 \end{bmatrix} = (0.83) \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (0.83)Y$$

Now, let us start allowing time to pass. Since $V_1 = X + (3.18)Y$, we see that after one year we are in the state

$$V_2 = PV_1 = P(X + (3.18)Y) = PX + P(3.18)Y = X + 0.83(3.18)Y$$

In the next year, we are in the state

$$V_3 = P(X + 0.83(3.18)Y) = PX + 0.83(3.18)PY = X + (0.83)^2(3.18)Y$$

After $n$ years, our state is

$$V_n = X + (0.83)^n(3.18)Y \tag{5.2}$$

As $n \to \infty$, $(0.83)^n \to 0$. Thus, our state vectors do, in fact, converge to $X = [41.18, 58.82]^t$.

The analysis done in our newspaper example was quite remarkable. Initially, the only way we had of predicting the readership after a given number of years was to repeatedly multiply the state vector by the transition matrix. Once we had written our state $V_1$ as a linear combination of the vectors $X$ and $Y$, however, we were able to predict the readership after any number of years [equation (5.2)]. This worked because each of the vectors $X$ and $Y$ has the property that multiplication by $P$ transforms it into a multiple of itself. Specifically, $PX = 1X$ and $PY = (0.83)Y$. This allowed us to compute the effect of repeated multiplication by $P$ *without doing any matrix multiplication*.

The same sort of analysis can be applied to a surprisingly large number of $n \times n$ matrices $A$. All one needs is sufficiently many nonzero column vectors $X$ such that $AX = \lambda X$ for some scalar $\lambda$. Such vectors are called "eigenvectors."

**Definition 5.1** *Let $A$ be an $n \times n$ matrix. A nonzero vector $X$ such that*

$$AX = \lambda X$$

*for some scalar $\lambda$ is called a **$\lambda$-eigenvector** for $A$. The scalar $\lambda$ is called the **eigenvalue**. The set of vectors $X$ that solve the preceding equation is the **$\lambda$-eigenspace** for $A$.*

Example 5.2 demonstrates how to find eigenvectors and eigenvalues.

## EXAMPLE 5.2

Compute $A^{10}B$ where

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

*Solution.* We begin by looking for vectors $X$ and scalars $\lambda$ such that $AX = \lambda X$. If $X = [x, y]^t$, this equality is equivalent to

$$\begin{aligned} 3x + y &= \lambda x \\ x + 3y &= \lambda y \end{aligned}$$

which is the same as

$$\begin{aligned} (3-\lambda)x + y &= 0 \\ x + (3-\lambda)y &= 0 \end{aligned} \tag{5.3}$$

The coefficient matrix for this system is

$$C = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$$

If $C$ is invertible, then $X = C^{-1}\mathbf{0} = \mathbf{0}$. This would be a disappointing conclusion: we want nonzero $X$.

Thus, nonzero $X$ can exist only when $C$ is *not* invertible. This is true if and only if the determinant of $C$ is zero—that is,

$$0 = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4)$$

Hence, $\lambda = 2$ or $\lambda = 4$.

For $\lambda = 2$, the system in equation (5.3) becomes

$$\begin{aligned} x + y &= 0 \\ x + y &= 0 \end{aligned}$$

The general solution to this system is $y[1, -1]^t$. In particular, $X = [1, -1]^t$ is an eigenvector corresponding to $\lambda = 2$. The reader should check that, indeed, $AX = 2X$.

For $\lambda = 4$, the system (5.3) becomes

$$\begin{aligned} -x + y &= 0 \\ -x + y &= 0 \end{aligned}$$

The general solution to this system is $y[1, 1]^t$. In particular, $Y = [1, 1]^t$ is an eigenvector corresponding to $\lambda = 4$. Again, the reader should check that indeed $AY = 4Y$.

Finally, to finish our example, we note that

$$B = [1, 2]^t = -\tfrac{1}{2}[1, -1]^t + \tfrac{3}{2}[1, 1]^t = -\tfrac{1}{2}X + \tfrac{3}{2}Y \tag{5.4}$$

Since $AX = 2X$ and $AY = 4Y$, we see that

$$AB = -\frac{1}{2}AX + \frac{3}{2}AY = -\frac{1}{2}(2X) + \frac{3}{2}(4Y)$$

Multiplying both sides of this equation by $A$ again produces

$$A^2B = -\frac{1}{2}(2AX) + \frac{3}{2}(4AY) = -\frac{1}{2}(2^2X) + \frac{3}{2}(4^2Y)$$

Repeating the same argument 10 times, we see that

$$A^{10}B = -\frac{1}{2}(2^{10}X) + \frac{3}{2}(4^{10}Y) = [1572352, 1573376]^t \tag{5.5}$$

Thus, we have multiplied $B$ by $A$ 10 times without computing a single matrix product! We could just as easily multiply $B$ by $A$ 100 times.

A few general comments about the procedure just illustrated are in order. Notice that the matrix $C$ from Example 5.2 may be written as

$$\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = A - \lambda I$$

For a general $n \times n$ matrix $A$, the equation $AX = \lambda X$ may be written as $AX = \lambda IX$, which is equivalent to

$$(A - \lambda I)X = \mathbf{0} \tag{5.6}$$

Thus, *the $\lambda$-eigenspace is the nullspace of $A - \lambda I$*. In particular, eigenspaces of $n \times n$ matrices are subspaces of $\mathbb{R}^n$.

Just as in Example 5.2, we use determinants to determine the eigenvalues.

**Theorem 5.1** *Let $A$ be an $n \times n$ matrix. Then $\lambda$ is an eigenvalue for $A$ if and only if $p(\lambda) = 0$, where*

$$p(\lambda) = \det(A - \lambda I)$$

*The function $p(\lambda)$ is a polynomial of degree $n$ with lead coefficient $(-1)^n$ that is called the* ***characteristic polynomial*** *for $A$.*

*Proof.* Equation (5.6) has a nonzero solution $X$ if and only if the nullspace of $A - \lambda I$ is nonzero which, according to Theorem 2.20 on page 140, is equivalent to the noninvertibility of $A - \lambda I$ which, in turn, is equivalent to $\det(A - \lambda I) = 0$ (Theorem 4.8, page 255). That $p(\lambda)$ is a polynomial of degree at most $n$ with lead coefficient $(-1)^n$ follows from Exercise 4.11 on page 251. Our theorem follows. □

Let us demonstrate these comments with another example.

### EXAMPLE 5.3

Find all eigenvalues and a basis for the corresponding eigenspace for the matrix

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$$

Use your answer to compute $A^{100}B$, where $B = [2, 2, 8]^t$.

*Solution.* We compute

$$A - \lambda I = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 4 & 2 & 1-\lambda \end{bmatrix}$$

Then

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 4 & 2 & 1-\lambda \end{vmatrix} = -\lambda^3 + 5\lambda^2 - 7\lambda + 3 = -(\lambda - 1)^2(\lambda - 3)$$

(One could, for example, do a Laplace expansion along the first row.)

The roots are $\lambda = 1$ and $\lambda = 3$. These are the eigenvalues.

To find the eigenvectors, let us first consider the $\lambda = 1$ case. The equation $(A - I)X = 0$ says

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 4 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By inspection, this is a rank 1 system. Hence, we have a two-dimensional solution set. The reduced form of the augmented matrix is

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot variable is $x_1$, while $x_2$ and $x_3$ are free. Also, $x_1 = -x_2/2$. We set $x_2 = 2s$ (to avoid fractions) and $x_3 = t$, finding

$$X = \left[ -\frac{2s}{2}, 2s, t \right]^t = s[-1, 2, 0]^t + t[0, 0, 1]^t$$

A basis for the $\lambda = 1$ eigenspace is $X_1 = [-1, 2, 0]^t$ and $X_2 = [0, 0, 1]^t$.

Next consider $\lambda = 3$. Solving the equation $(A - 3I)X = \mathbf{0}$ yields the general solution $X = sY$, where $Y = [1, 0, 2]^t$.

Finally, to compute $A^{100}B$, we note that

$$B = [2, 2, 8]^t = X_1 + 2X_2 + 3Y$$

Hence,

$$\begin{aligned} A^{100}B &= A^{100}X_1 + 2A^{100}X_2 + 3A^{100}Y \\ &= X_1 + 2X_2 + 3^{100}3Y \\ &= [-1 + 3^{101}, 2, 2(3^{101}) + 2]^t \end{aligned}$$

In Example 5.3, we could have computed $A^{100}B$ for any $B$ in $\mathbb{R}^3$. All that is needed is to be able to express $B$ as a linear combination of $X_1$, $X_2$, and $Y$. This is always possible since the set formed by these three vectors is easily seen to be linearly independent. (Note that three linearly independent vectors in $\mathbb{R}^3$ span $\mathbb{R}^3$.)

Unfortunately, not all matrices are so nice:

### EXAMPLE 5.4

Attempt to compute $A^{100}B$ using the techniques just given, where $B = [1, 1, 1]^t$ and

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

*Solution.* Clearly (since $A$ is triangular),

$$\det(A - \lambda I) = (3 - \lambda)^2(5 - \lambda)$$

Thus, we have two eigenvalues: $\lambda = 3$ and $\lambda = 5$. The equation $(A - 3I)X = \mathbf{0}$ says

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

An echelon form of the augmented matrix for this system is

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solving yields $x_2 = x_3 = 0$, while $x_1$ is free. Setting $x_1 = s$ shows that the general solution is $X = s[1, 0, 0]^t$. This yields only one basis element: $X = [1, 0, 0]^t$.

Similarly, $(A - 5I)X = \mathbf{0}$ yields the general solution $X = tY$, where $Y = [3, 2, 4]^t$.

So far, so good. When we attempt to express $B$ as a linear combination of the eigenvectors, however, we run into difficulty. Clearly, there are no scalars $a$ and $b$ such that

$$[1, 1, 1]^t = a[1, 0, 0]^t + b[3, 2, 4]^t$$

Thus, the technique breaks down: we cannot compute $A^{100}B$ in this manner.

The difference between Examples 5.3 and 5.4 is a matter of dimension. In Example 5.3, we found a set containing three linearly independent eigenvectors. This set then formed a basis for $\mathbb{R}^3$, allowing us to express any given vector $B$ in $\mathbb{R}^3$ as a linear combination of eigenvectors. In Example 5.4, we found only two linearly independent eigenvectors, which were not sufficient to span $\mathbb{R}^3$. We use the terms "diagonalizable" and "deficient" to distinguish between these two types of behavior.

**Definition 5.2** *An $n \times n$ matrix $A$ is* ***diagonalizable over*** $\mathbb{R}$ *if there is a basis for* $\mathbb{R}^n$ *consisting of eigenvectors for $A$. Such a basis is called an* **eigenbasis**. *Otherwise, $A$ is said to be* ***deficient*** *(over $\mathbb{R}$).*

Notice that in Example 5.3 the characteristic polynomial factors as $-(\lambda - 1)^2(\lambda - 3)$. In general, the **multiplicity** of an eigenvalue $\lambda_o$ of an $n \times n$ matrix $A$ is the number of times $\lambda - \lambda_o$ is a factor of the characteristic polynomial $p(\lambda)$ of $A$. Thus, in Example 5.3, $\lambda = 1$ is an eigenvalue of multiplicity 2 and $\lambda = 3$ is an eigenvalue of multiplicity 1. The dimensions of the corresponding eigenspaces were 2 and 1 as well. Theorem 5.2 below implies that the dimension of the eigenspace of an $n \times n$ matrix $A$ is at most equal to the multiplicity of the corresponding eigenvalue.

In Example 5.4, the polynomial factors as $-(\lambda - 3)^2(\lambda - 5)$. As the preceding comments indicate, the root $\lambda = 5$ can produce only one linearly independent eigenvector. To get three linearly independent eigenvectors, we would need to get two from the $\lambda = 3$ root. Thus, the deficiency of this matrix is attributable to the fact that $\lambda = 3$ is an eigenvalue of multiplicity 2, which produces only one linearly independent eigenvector. In general, *an $n \times n$ matrix is deficient if any one of its eigenvalues produces fewer than $m$ linearly independent eigenvectors*, where $m$ is the multiplicity of the eigenvalue. The following theorem, which follows from Theorem 7.1 on page 431, summarizes these comments.

**Theorem 5.2** *Let $p(\lambda)$ be the characteristic polynomial for an $n \times n$ matrix $A$ and let $\lambda_1, \lambda_2, \dots, \lambda_k$ be the roots of $p(\lambda)$. Then the dimension $d_i$ of the $\lambda_i$-eigenspace of $A$ is at most the multiplicity $m_i$ of $\lambda_i$ as a root of $p(\lambda)$. Thus, $A$ is deficient if $d_i < m_i$ for any $i$.*

The techniques described in this section also require modification if the characteristic polynomial does not have enough roots. Consider, for example, the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Then (as the reader may check)

$$\det(A - \lambda I) = \lambda^2 + 1$$

There are no real roots to $\lambda^2 + 1 = 0$—hence, no eigenvectors—at least if we use only real numbers.

This is not a serious problem. Its solution, however, requires using complex matrices. This explains the expression "over $\mathbb{R}$" in Definition 5.2; matrices diagonalizable using complex numbers are said to be "diagonalizable over $\mathbb{C}$." The matrix $A$ in the preceding paragraph is not diagonalizable over $\mathbb{R}$ but is diagonalizable over $\mathbb{C}$. Diagonalization over $\mathbb{C}$ is discussed in Section 5.3.

**True-False Questions:** Justify your answers.

**5.1** If $A$ is an $n \times n$ matrix that has zero for an eigenvalue, then $A$ cannot be invertible.

**5.2** There is no $3 \times 3$ matrix $A$ with $p_A(\lambda) = (\lambda - 2)^2(\lambda - 3)$.

**5.3** If $A$ is a $3 \times 3$ matrix with $p_A(\lambda) = (2 - \lambda)^2(3 - \lambda)$ then:

**(a)** There is a nonzero vector $X$ such that $AX = 2X$.

**(b)** Then there is at most one nonzero vector $X$ such that $AX = 3X$.

**(c)** There are two linearly independent vectors $X_1$ and $X_2$ such that $AX_i = 2X_i$.

**5.4** The sum of two eigenvectors is an eigenvector.

**5.5** If $X$ is an eigenvector for $A$ with eigenvalue 3, then $2X$ is an eigenvector for $A$ with eigenvalue 6.

**5.6** If $X$ is an eigenvector for an $n \times n$ matrix $A$, then $X$ is also an eigenvector for $2A$.

**5.7** If 3 is an eigenvalue for $A$, then 9 is an eigenvalue for $A^2$.

**5.8** Suppose that $A$ is a $3 \times 3$ matrix with 2 and 3 as its only eigenvalues. Then $A$ is deficient.

**5.9** The only nondeficient $3 \times 3$ matrix that has 1 as its only eigenvalue is the identity matrix. [*Hint:* What is the dimension of the $\lambda = 1$ eigenspace in this context.]

**5.10** Suppose that $p_A(\lambda) = -\lambda^3(\lambda - 2)(\lambda + 3)^2$. Then, the nullspace of $A$ is at most two-dimensional.

**5.11** There is a $3 \times 3$ matrix with eigenvalues 1, 2, 3, and 4.

## EXERCISES

**5.1** ✓✓For the following matrices $A$, determine which of the given vectors are eigenvectors and which are not. For those that are, give the eigenvalue.

**(a)** $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

**(b)** $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $X = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $Y = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

**(c)** $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ -2 & 1 & -2 & 0 \\ 0 & 0 & 3 & 2 \end{bmatrix}$, $X = \begin{bmatrix} 1 \\ 3 \\ 1 \\ -1 \end{bmatrix}$, $Y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

**5.2** For the matrices in Exercises 5.1(a)✓✓ and 5.1(b), find the characteristic polynomial and use this to find all eigenvalues. Finally, find a formula for $A^n B$, where $B = [3, 1, 1]^t$.

**5.3** In each part let $A$ be the given $3 \times 3$ matrix. Verify that the given vectors are eigenvectors and find a formula for $A^n B$, where $B = [1, 1, 2]^t$.

**(a)** $\begin{bmatrix} 0 & 3 & -3 \\ 2 & 2 & -2 \\ -4 & -1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

**(b)** ✓ $\begin{bmatrix} 4 & -1 & 1 \\ 2 & 1 & 1 \\ 2 & -1 & 3 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

**(c)** $\begin{bmatrix} 2 & 2 & -2 \\ -1 & 5 & -1 \\ -1 & 1 & 3 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

**5.4** For the matrix $A$ part (b)✓✓of Exercise 5.3 find an eigenvector $Y$ with only *positive* entries and eigenvalue 2. Repeat for part (c) and eigenvalue 4. In each case, check that $Y$ is an eigenvector. [*Note*: 0 is not positive!]

**5.5** For the following matrices, find all eigenvalues and a basis for each eigenspace. State whether or not the given matrix is diagonalizable over $\mathbb{R}$.

**(a)** ✓✓ $\begin{bmatrix} -18 & 30 \\ -10 & 17 \end{bmatrix}$

**(b)** $\begin{bmatrix} 10 & -17 \\ 6 & -10 \end{bmatrix}$

**(c)** ✓✓ $\begin{bmatrix} -12 & 21 \\ -6 & 11 \end{bmatrix}$

**(d)** $\begin{bmatrix} 2 & 12 & -8 \\ 0 & -8 & 6 \\ 0 & -9 & 7 \end{bmatrix}$

**(e)** ✓✓ $\begin{bmatrix} 2 & 0 & 0 \\ -2 & -2 & 2 \\ -5 & -10 & 7 \end{bmatrix}$

**(f)** $\begin{bmatrix} 0 & -5 & 2 \\ -2 & -2 & 2 \\ -7 & -15 & 9 \end{bmatrix}$

**(g)** ✓✓ $\begin{bmatrix} 10 & -24 & 7 \\ 6 & -14 & 4 \\ 6 & -15 & 5 \end{bmatrix}$

**(h)** $\begin{bmatrix} -2 & -1 & 1 \\ -6 & -2 & 0 \\ 13 & 7 & -4 \end{bmatrix}$

**(i)** ✓✓ $\begin{bmatrix} 1 & 2 & 0 \\ -3 & 2 & 3 \\ -1 & 2 & 2 \end{bmatrix}$

**(j)** $\begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & -4 \end{bmatrix}$

**5.6** For each matrix $A$ in Exercise 5.3 use the technique of Example 5.3 on page 276 to find a basis for each eigenspace of $A$. Prove that your basis is consistent with the basis from Exercise 5.3. [*Hint:* You do not need to compute determinants to find the eigenvalues.] ✓[(b)]

**5.7** ✓✓For the following matrix, show that $\lambda = 2$ and $\lambda = 3$ are the only eigenvalues. Describe geometrically the corresponding eigenspaces.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

**5.8** ✓✓The $\lambda = 0$ eigenspace was studied in Section 1.4. What name did it go by in that section?

**5.9** In the context of Example 5.1 on page 271, choose a state vector (reader's choice) that initially has more Jupiter readers than Planet readers. (Make sure that the entries total 100%.) Use an argument similar to that of Example 5.1 to prove that readership levels eventually will approach the same vector $X$ as in Example 5.1. Will the readership always approach $X$, regardless of the initial readership? Explain.

**5.10** The Fibonacci sequence $F_n$ is defined by $F_1 = 1$, $F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$. Thus,

$$F_3 = F_2 + F_1 = 1 + 1 = 2$$
$$F_4 = F_3 + F_2 = 2 + 1 = 3$$

**(a)** Compute $F_n$ for $n = 5, 6, 7$.

**(b)** Let $A$ and $X$ be as shown. Compute $A^n X$ for $n = 1, 2, 3, 4, 5, 6$. What do you observe? Can you explain why this happens?

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

**(c)** Find the eigenvalues and a basis for the corresponding eigenspaces for $A$ and express $X$ as a linear combination of the basis elements.

**(d)** Use the expression in part (c) to compute $A^8 X$. (Give an explicit decimal answer.) Use this answer to find $F_9$.

**(e)** Give a general formula for $F_n$.

**5.11** ✓Suppose that $A$ is an $n \times n$ matrix such that $A^2 = I$ and that $\lambda$ is an eigenvalue for $A$. Prove that $\lambda = \pm 1$.

**5.12** Let $A$ be an $n \times n$ matrix and let $\lambda$ be an eigenvalue for $A$. Prove that $\lambda^2$ is an eigenvalue for $A^2$.

**5.13** ✓✓Suppose that $A$ in Exercise 5.12 is invertible. Prove that $\lambda^{-1}$ is an eigenvalue for $A^{-1}$.

**5.14** ✓✓Suppose that $A$ is a square matrix with characteristic polynomial $p(\lambda) = \lambda^2(\lambda+5)^3(\lambda-7)^5$.

**(a)** What is the size of $A$?

**(b)** Can $A$ be invertible?

**(c)** What are the possible dimensions for the nullspace of $A$?

**(d)** What can you say about the dimension of the $\lambda = 7$ eigenspace?

**5.15** For the following matrix $A$ prove that $p_A(\lambda) = \lambda^2 - (a+d)\lambda + \det A$. Find two different matrices $A$ with $p_A(\lambda) = (\lambda-2)(\lambda-3)$.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

**5.16** Using Exercise 5.15, state a condition on the entries of $A$ that guarantees that $A$ has no real eigenvalues.

### 5.1.1 Computer Projects

In the computer projects Section 4.1.1, we commented that for virtually any computation for which you might use determinants, a computer would do otherwise. This includes finding eigenvalues. MATLAB uses a sophisticated computational technique which does not involve finding the roots of the characteristic polynomial. (See Section 8.2.) In fact, the algorithms for finding eigenvalues are often used to find roots of polynomials! The exercises here explore this idea.

## EXERCISES

**1.** Show that for the following matrix $A$, $p_A(\lambda) = \lambda^2 + a\lambda + b$. Then use the MATLAB command eig(A) with $a = 7$ and $b = 1$ to compute the roots of $p(\lambda) = \lambda^2 + 7\lambda + 1$. Check your calculation using the quadratic formula.

$$A = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}$$

**2.** Compute $p_A(\lambda)$ for the following matrix $A$. Then use MATLAB eig command to approximate the roots of $p(\lambda) = \lambda^3 + 8\lambda^2 + 17\lambda + 10$. Test your answer by substituting your roots into $p(\lambda)$.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & -b & -a \end{bmatrix}$$

**3.** Let $A$ be the matrix that you used to approximate the roots of $p(\lambda)$ in Exercise 2. The command [X,D]=eig(A) produces a matrix $X$ whose columns form a basis of eigenvectors for $A$. The corresponding eigenvalues appear as the diagonal entries of $D$. In MATLAB, let $V$ be the column of $X$ corresponding to $\lambda = -5$. Normalize $V$ by entering V=V/V(1,1). What do you notice about the entries of $V$? Do the other eigenvectors of $A$ exhibit a similar pattern? Use this observation to describe the eigenvectors for the matrix in the formula given in Exercise 2 in terms of the roots of the characteristic polynomial. Prove your answer.

**4.** Find a $4 \times 4$ matrix $A$ whose characteristic polynomial is $p(\lambda) = \lambda^4 + 3\lambda^2 - 5\lambda + 7$. Then use the MATLAB eig command to approximate the roots.

### 5.1.2 Application to Markov Processes

Example 5.1 on page 271 can be described in terms of a concept from probability called a "Markov process." A probabilist would say that a news reader in Metropolis can be in one of two states: he or she is either a "Jupiter reader" or a "Planet reader." Let us refer to Jupiter reading as state 1 and Planet reading as state 2. Saying that 10% of the Jupiter readers change to the Planet every year means that the probability of a given reader changing from state 1 to state 2 in a given year is 0.1, which is exactly the (2, 1) entry of the transition matrix. In general, the $(i, j)$ entry is the probability of changing from state $j$ to $i$. What makes this a Markov process is the independence of the transition probabilities of either the current or past readership.

Here is another example of a Markov process:

**EXAMPLE 5.5**

The Gauss Auto Rental Agency in Chicago has offices at two airports, O'Hare and Midway, as well as one downtown in the Loop. Table 5.1 shows the probabilities that a car rented at one particular location will be returned to some other location. Assuming that all cars are returned by the end of the day and that the agency does not move cars itself from office to office, what fraction of its total fleet, on the average, is available at each of its offices?

*Solution.* In this example we define the three states by saying that a given car can be at O'Hare, at Midway, or in the Loop. Let $H$, $M$, and $L$ denote, respectively, the

**TABLE 5.1 Gauss Auto Rental**

| | Returns | | |
|---|---|---|---|
| Rentals | O'Hare | Midway | Loop |
| O'Hare | 0.7 | 0.1 | 0.2 |
| Midway | 0.2 | 0.6 | 0.2 |
| Loop | 0.5 | 0.3 | 0.2 |

fraction of the total fleet initially in each place: O'Hare, Midway, and the Loop. Then at the end of a day we expect the fractions for each of the respective offices to be

$$\begin{aligned} H_1 &= 0.7H + 0.2M + 0.5L \\ M_1 &= 0.1H + 0.6M + 0.3L \\ L_1 &= 0.2H + 0.2M + 0.2L \end{aligned}$$

Thus, the transition matrix is

$$P = \begin{bmatrix} 0.7 & 0.2 & 0.5 \\ 0.1 & 0.6 & 0.3 \\ 0.2 & 0.2 & 0.2 \end{bmatrix} \tag{5.7}$$

*Remark.* It should be noted that, in matrix (5.7), $P_{ij}$ is the probability of changing *from* the $j$th state *to* the $i$th, contrary to what one might expect. Thus, $P_{23} = 0.3$, which is the probability of changing from the third state $L$ to the second state $M$.

In a general (finite) Markov process, we are studying a system that is describable in terms of a finite number of states numbered 1 through $n$. We assume that each individual (a probabilist would say "sample") occupies one and only one of the states. It is also assumed that the system changes periodically and the probability $p_{ij}$ of changing from state $j$ to state $i$ is known and independent of how many individuals occupied any of the states at any given time. The matrix $P = [p_{ij}]$ is referred to as the **transition matrix**.

The $p_{ij}$ are of course nonnegative and less than or equal to 1, since they are probabilities. Notice that the entries in each column of the transition matrix from Example 5.5 total to 1. The same statement is true for all transition matrices because an individual who was in state $j$ before the transition occurred must wind up in one of the $n$ possible states after the transition. These comments lead to the following definition:

**Definition 5.3** *An $n \times n$ matrix $P$ is a probability matrix if the entries of $P$ are all nonnegative and the sum of the entries in each column is 1.*

For a general Markov process, we usually use the fraction of the total population rather than percentages to measure the occupancy of any given state. Thus, our state vectors are $n \times 1$ column vectors having nonnegative entries that total 1. Such vectors are called **probability vectors**.

If we begin in a state defined by a probability vector $V_0$, then after the first transition we will be in state $V_1 = PV_0$, where $P$ is the transition matrix. After the next transition we will be in state $V_2 = PV_1 = P^2V_0$. In general, the $n$th state will be $V_n = P^nV_0$. The sequence of vectors $V_0, V_1, \dots, V_n, \dots$ is a "Markov chain."

The following theorem, whose proof is beyond the scope of this text, is fundamental to the study of Markov processes. Some of the exercises indicate the proof of Theorem 5.3 in specific cases.

**Theorem 5.3** *Let P be a probability matrix with no entries equal to* 0. *Then there is a unique probability vector X such that* $PX = X$. *If* $V_0$ *is any probability vector, then* $\lim_{n\to\infty} P^n V_0 = X$.

The vector $X$ in Theorem 5.3 is called the **equilibrium vector**.

## EXERCISES

**5.17** Find the equilibrium vector for the following matrices:

**(a)** ✓✓ $\begin{bmatrix} 0.3 & 0.5 \\ 0.7 & 0.5 \end{bmatrix}$

**(b)** ✓✓ $\begin{bmatrix} 0.1 & 0.3 & 0.2 \\ 0.4 & 0.7 & 0.1 \\ 0.5 & 0 & 0.7 \end{bmatrix}$

**(c)** $\begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}$

**(d)** $\begin{bmatrix} 0.2 & 0.8 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.4 & 0.1 & 0.6 \end{bmatrix}$

**5.18** ✓✓Denote the matrix in Exercise 5.17(a) by $P$. Let $V_0 = [1, 0]^t$. Compute values of $P^n V_0$ until the answer agrees with the computed equilibrium vector to within two decimal places. Repeat for the matrix in Exercise 5.17(c).

**5.19** In Exercises 5.17(a) and 5.17(c), prove that $Y = [1, -1]^t$ is an eigenvector. What is the corresponding eigenvalue? Then reason as in Example 5.1, page 271, to show that Theorem 5.3 on page 285 is true for these matrices.

**5.20** I tend to be rather moody at times. If I am in a good mood today, there is an 80% chance I will still be in a good mood tomorrow. But if I am grumpy today, there is only a 60% chance that my mood will be good tomorrow.

**(a)** Describe the transition matrix for this situation.

**(b)** If I am in a good mood today, what is the probability that I will be in a good mood three days from now?

**(c)** Over the long term, what percentage of the time am I in a good mood?

**5.21** ✓✓The following exercises refer to Example 5.5 on page 283:

**(a)** Suppose that on Monday morning we have equal numbers of cars at each location—that is, $V_0 = \frac{1}{3}[1, 1, 1]^t$. What will the distribution be on Tuesday? On Wednesday? Do not compute eigenvectors to answer this part.

**(b)** Find the equilibrium vector $X$.

**(c)** Given that the characteristic polynomial for $P$ is $p(\lambda) = -\lambda(\lambda - 1)(\lambda - 0.5)$, find eigenvectors $Y_1$ and $Y_2$ corresponding, respectively, to the eigenvalues $\lambda = 0$ and $\lambda = 0.5$.

**(d)** Find constants $a$, $c_1$, and $c_2$ such that

$$V_0 = aX + c_1 Y_1 + c_2 Y_2 \tag{5.8}$$

where $V_0$ is as in part (a). Use your answer to find the distribution in five weeks.

**(e)** Show (without using Theorem 5.3) that for all probability vectors $V_0$ [not just the $V_0$ in part (e)] $P^n V_0$ approaches $X$ as $n \to \infty$, where $X$ is as in part (b). [*Hint:* Express $V_0$ as in equation (5.8) and show that $a = 1$ by summing the components of the vectors on each side of this equation.]

**5.22** In a certain market, there are three competing brands of cereal: brands $X$, $Y$, and $Z$. In any given month 40% of Brand $X$ users switch to either Brand $Y$ or Brand $Z$, with equal numbers going to each. Similarly, 30% of Brand $Y$ users and 20% of Brand $Z$ users switch, with the switchers divided equally between the other two brands.

**(a)** ✓✓What is the transition matrix for this situation?

**(b)** What is the expected market share for each brand?

**5.23** Let $P$ be an $n \times n$ probability matrix and let $Z$ be an $n \times 1$ matrix. Prove that the sum of the entries in $PZ$ is the same as that of the entries in $Z$.

**5.24** Prove that the product of two probability matrices is a probability matrix. [*Hint:* Use Exercise 5.23.]

**5.25** Let

$$P = \begin{bmatrix} 0.1 & 0.5 & 0.2 \\ 0.3 & 0 & 0.4 \\ 0.6 & 0.5 & 0.4 \end{bmatrix}$$

**(a)** Which of the hypotheses of Theorem 5.3 does $P$ not satisfy?

**(b)** Show that $P^2$ satisfies all the hypotheses of Theorem 5.3.

**(c)** Use the result from part (b), together with Theorem 5.3, to prove that, for all probability vectors $V$, $\lim_{n\to\infty} P^{2n} V = X$, where $X$ is the unique probability vector such that $P^2 X = X$.

**(d)** For $X$ as in part (c), prove that $PX = X$. [*Hint:* $PX = P \lim_{n\to\infty} P^{2n} V = \lim_{n\to\infty} P^{2n}(PV)$.]

**(e)** Use the result of part (d) to prove that for all probability vectors $V$, $\lim_{n\to\infty} P^{2n+1} V = X$, where $X$ is as in part (c). This result, together with part (c), shows that $\lim_{n\to\infty} P^n V = X$, proving that Theorem 5.3 holds for $P$.

**5.26** Let $P$ be an $n \times n$ probability matrix and let $X = [1, 1, \ldots, 1]^t$ be the $n \times 1$ matrix, all whose entries are 1. What is $P^t X$? (Try some examples if you are not sure.) How does it follow that $\det(P^t - I) = 0$? How does it follow that $\det(P - I) = 0$? How does this relate to proving Theorem 5.3?

## 5.2 DIAGONALIZATION

In the last section we saw that we could use the eigenvectors of an $n \times n$ diagonalizable matrix $A$ to compute values of $A^k X$ for any $X$ and $k$. If we can compute $A^k X$ for all $X$, we should be able to compute $A^k$ itself, since a matrix is uniquely determined by the linear transformation it defines.

To consider this idea further, let $\{Q_1, Q_2, \dots, Q_n\}$ be a set of $n$ linearly independent eigenvectors for $A$ and let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the corresponding eigenvalues. Let

$$Q = [Q_1, Q_2, \dots, Q_n]$$

be the $n \times n$ matrix with the $Q_i$ as columns. Then

$$\begin{aligned} AQ &= [AQ_1, AQ_2, \dots, AQ_n] \\ &= [\lambda_1 Q_1, \lambda_2 Q_2, \dots, \lambda_n Q_n] \\ &= [Q_1, Q_2, \dots, Q_n] \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \end{aligned}$$

Hence,

$$AQ = QD$$

where

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \tag{5.9}$$

This argument proves the following theorem.

**Theorem 5.4** *Let $A$ be an $n \times n$ diagonalizable matrix and let $\mathcal{B} = \{Q_1, Q_2, \dots, Q_n\}$ be a set of $n$ linearly independent eigenvectors for $A$, where $Q_i$ corresponds to the eigenvalues $\lambda_i$. Then*

$$A = QDQ^{-1}$$

*where $Q = [Q_1, Q_2, \dots, Q_n]$ is the $n \times n$ matrix with the $Q_i$ as columns and $D$ is as in equation* (5.9).

*Remark.* The reader should review Example 3.16 on page 218 which provides a geometric interpretation of Theorem 5.4. Explicitly, the set $\mathcal{B}$ from Theorem 5.4 above forms an ordered basis for $\mathbb{R}^n$. It is easily seen that the matrix of $T_A$ with respect to this basis is the matrix $D$. In fact, Theorem 5.4 can be proved from Theorem 3.15 on page 220. The corresponding point matrix is $P_{\mathcal{B}} = Q$ and the corresponding coordinate matrix is $C_{\mathcal{B}} = Q^{-1}$. Then Theorem 3.15 on page 220 implies that $D = Q^{-1}AQ$, which is equivalent to Theorem 5.4 above.

We can use Theorem 5.4 to compute $A^k$:

$$\begin{aligned}
A^2 &= (QDQ^{-1})(QDQ^{-1}) = QD^2Q^{-1}\\
A^3 &= A \cdot A^2 = (QDQ^{-1})(QD^2Q^{-1}) = QD^3Q^{-1}\\
&\vdots\\
A^k &= A \cdot A^{k-1} = (QDQ^{-1})(QD^{k-1}Q^{-1}) = QD^kQ^{-1}
\end{aligned}$$

Any $n \times n$ matrix such as $D$ whose only nonzero entries lie on the main diagonal is said to be **diagonal**. For this reason, $Q$ and $D$ are said to provide a **diagonalization** of $A$. It is easily seen that diagonal matrices multiply by multiplying their diagonal entries—that is,

$$\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} \beta_1 & 0 & \dots & 0 \\ 0 & \beta_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \beta_n \end{bmatrix} = \begin{bmatrix} \lambda_1\beta_1 & 0 & \dots & 0 \\ 0 & \lambda_2\beta_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n\beta_n \end{bmatrix}$$

Thus,

$$D^k = \begin{bmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n^k \end{bmatrix} \tag{5.10}$$

## Powers of Matrices

### EXAMPLE 5.6

Find a formula for $A^k$, where

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$$

*Solution.* In Example 5.3 on page 276, we found that $Q_1 = [-1, 2, 0]^t$, $Q_2 = [0, 0, 1]^t$, and $Q_3 = [1, 0, 2]^t$ are three linearly independent eigenvectors for $A$, where $Q_1$ and $Q_2$ correspond to the eigenvalues $\lambda_1 = 1$ and $Q_3$ corresponds to $\lambda_2 = 3$. Hence, from Theorem 5.4 and the argument above equation (5.10),

$$A^k = QD^kQ^{-1} \tag{5.11}$$

where

$$Q = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \qquad D^k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3^k \end{bmatrix}$$

We compute

$$Q^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ -4 & -2 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

Hence,

$$A^k = QD^kQ^{-1} = \frac{1}{2}\begin{bmatrix} 2(3^k) & -1+3^k & 0 \\ 0 & 2 & 0 \\ -4+4(3^k) & -2+2(3^k) & 2 \end{bmatrix}$$

Before ending this section, we should make one final comment concerning bases of eigenvectors. Notice that in all the examples we never bothered to show that our sets of eigenvectors were linearly independent. This is because sets formed by eigenvectors corresponding to different eigenvalues are *always* linearly independent due to the following theorem.

**Theorem 5.5** *Let $A$ be an $n \times n$ matrix and let $Q_1, Q_2, \ldots, Q_k$ be eigenvectors for $A$ corresponding to the eigenvalues $\lambda_i$. Suppose that the $\lambda_i$ are all different. Then $\{Q_1, Q_2, \ldots, Q_k\}$ is a linearly independent subset of $\mathbb{R}^n$.*

*Proof.* Suppose that $\{Q_1, Q_2, \ldots, Q_k\}$ is a linearly dependent set. Then there is some smallest number $j$ such that $S = \{Q_1, Q_2, \ldots, Q_j\}$ is linearly dependent. Of course, $j > 1$ since $Q_1 \neq \mathbf{0}$.

Consider the dependency equation

$$c_1Q_1 + c_2Q_2 + \cdots + c_jQ_j = \mathbf{0} \tag{5.12}$$

We multiply both sides of this equation by $A$, producing

$$\begin{aligned} \mathbf{0} &= c_1AQ_1 + c_2AQ_2 + \cdots + c_jAQ_j \\ &= \lambda_1c_1Q_1 + \lambda_2c_2Q_2 + \cdots + \lambda_jc_jQ_j \end{aligned} \tag{5.13}$$

We now multiply both sides of formula (5.12) above by $\lambda_j$ and subtract it from formula (5.13). Notice that $Q_j$ drops out. We get

$$\mathbf{0} = (\lambda_1 - \lambda_j)c_1Q_1 + (\lambda_2 - \lambda_j)c_2Q_2 + \cdots + (\lambda_{j-1} - \lambda_j)c_{j-1}Q_{j-1}$$

From the choice of $j$, $\{Q_1, \ldots, Q_{j-1}\}$ is linearly independent. Furthermore $\lambda_i - \lambda_j \neq 0$ for $i < j$. It follows that $c_i = 0$ for $1 \leq i < j$. Then formula (5.12) shows that $c_j = 0$ as well. This contradicts the fact that $S$ is linearly dependent, proving our theorem. □

**True-False Questions:** Justify your answers.

**5.12** Suppose that $A$ is a $3 \times 3$ matrix with eigenvalues 2, 3, and 4. Then $\det A = 9$.

**5.13** Suppose that $A$ is a $3 \times 3$ diagonalizable matrix such that $A^2$ has eigenvalues 1, 4, and 9. Then $A$ has eigenvalues 1, 2, and 3.

**5.14** There are at least two $2 \times 2$ matrices, with eigenvectors $Q_1 = [1, 2]^t$ and $Q_2 = [-3, 2]^t$ corresponding to the respective eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 0$.

**5.15** There is no matrix $A$ that has the vectors $X_1 = [1, 1, 1]^t$, $X_2 = [1, 0, 1]^t$, and $X_3 = [2, 1, 2]^t$ as eigenvectors corresponding, respectively, to the eigenvalues 1, 2, and 3.

**5.16** There is no matrix $A$ that has the vectors $X_1 = [1, 1, 1]^t$, $X_2 = [1, 0, 1]^t$, and $X_3 = [2, 1, 3]^t$ as eigenvectors corresponding, respectively, to the eigenvalues 1, 2, and 3.

## EXERCISES

**5.27** For the matrices in Example 5.3, page 276, find a diagonal matrix $D$ and an invertible matrix $Q$ such that $A = QDQ^{-1}$. Use this information to find a formula for $A^n$. (You may leave $Q^{-1}$ unevaluated.)✓✓[(a), (b)]

**5.28** For each of the following matrices we give vectors $X_1, X_2, \ldots$ such that all but one of the $X_i$ is an eigenvector. In each case (i) find the eigenvalues of $A$ and (ii) find matrices $B$ and $C$ such that $A^{10} = BCB^{-1}$. *All that is asked for is B and C. You need compute neither $B^{-1}$ nor $BCB^{-1}$.*

**(a)** $X_1 = [1, 1, 1]^t, X_2 = [2, 3, 3]^t,$
$X_3 = [2, 1, 2]^t, X_4 = [1, 3, 4]^t.$

$$A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$$

**(b)** $X_1 = [1, -1, 1]^t, X_2 = [1, 2, 3]^t,$
$X_3 = [2, 3, 6]^t, X_4 = [1, 3, 4]^t.$

$$A = \begin{bmatrix} 11 & 6 & -7 \\ 18 & 11 & -12 \\ 30 & 18 & -20 \end{bmatrix}$$

**5.29** For the following matrices, find (if possible) an invertible matrix $Q$ and a diagonal matrix $D$ such that $A = QDQ^{-1}$. (*Note*: You are not asked to find $Q^{-1}$.) You may take as given that in (a), $p_A(\lambda) = (\lambda + 8)(\lambda^2 + 1)$, in (b) $p_A(\lambda) = (\lambda - 1)(\lambda - 2)^2$, in (g) $p_A(\lambda) = -(\lambda + 3)^2(\lambda - 6)$, and in (h) $p_A(\lambda) = -(\lambda - 2)^2(\lambda - 3)$.

**(a)** ✓✓ $\begin{bmatrix} -2 & -1 & 1 \\ -6 & -2 & 0 \\ 13 & 7 & -4 \end{bmatrix}$

**(b)** ✓✓ $\begin{bmatrix} 1 & 2 & 0 \\ -3 & 2 & 3 \\ -1 & 2 & 2 \end{bmatrix}$

**(c)** ✓✓ $\begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & -4 \end{bmatrix}$

**(d)** ✓✓ $\begin{bmatrix} -7 & 3 \\ -18 & 8 \end{bmatrix}$

**(e)** $\begin{bmatrix} 7 & -1 \\ 9 & 1 \end{bmatrix}$

**(f)** ✓✓ $\begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

**(g)** $\begin{bmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{bmatrix}$

**(h)** $\begin{bmatrix} 14 & -8 & -4 \\ 18 & -10 & -6 \\ -3 & 2 & 3 \end{bmatrix}$

**5.30** Let $A$, $D$, and $Q$ be as in Example 5.6 on page 288. Prove (by direct computation) that $AQ = QD$.

**5.31** Suppose that $A$ is an $n \times n$ matrix with $n$ distinct eigenvalues $\lambda_i$. Prove that $A$ is diagonalizable and $\det A = \lambda_1 \lambda_2 \dots \lambda_n$.

**5.32** In Example 5.6, on page 288, find a matrix $B$ such that $B^2 = A$. Check your answer by direct computation.

**5.33** ✓✓Suppose that $A$ is diagonalizable over $\mathbb{R}$ and $A$ has only $\pm 1$ as eigenvalues. Show that $A^2 = I$.

**5.34** Suppose that $A$ is diagonalizable over $\mathbb{R}$ and $A$ has only 0 and 1 as eigenvalues. Show that $A^2 = A$.

**5.35** ✓✓Suppose that $A$ is diagonalizable over $\mathbb{R}$ and $A$ has only 2 and 4 as eigenvalues. Show that $A^2 - 6A + 8I = \mathbf{0}$.

**5.36** Suppose that $A$ is diagonalizable over $\mathbb{R}$ with eigenvalues $\lambda_i$. Suppose also that

$$q(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_0$$

is some polynomial such that $q(\lambda_i) = 0$ for all $i$. Prove that

$$a_n A^n + a_{n-1} A^{n-1} + \cdots + a_0 I = \mathbf{0}$$

*Remark.* The characteristic polynomial $p_A(\lambda) = \det(A - \lambda I)$ is zero at all of the $\lambda_i$. Hence, from Exercise 5.36, $p_A(A) = \mathbf{0}$. This result, known as the Cayley-Hamilton theorem, remains true in the non-diagonalizable case. See Exercise 7.22 on page 445.

**5.37** ✓✓Find values of $a$, $b$, and $c$, all nonzero, such that the matrix $A$ below *is* diagonalizable over $\mathbb{R}$:

$$A = \begin{bmatrix} 2 & a & b \\ 0 & -5 & c \\ 0 & 0 & 2 \end{bmatrix}$$

**5.38** In each part, find specific values of $a$, $b$, $c$, and $d$, all nonzero, for which the given matrix is diagonalizable over $\mathbb{R}$:

**(a)** $A = \begin{bmatrix} 2 & 1 & a & b \\ 0 & 3 & -1 & c \\ 0 & 0 & 2 & d \\ 0 & 0 & 0 & 3 \end{bmatrix}$ **(b)** $A = \begin{bmatrix} 2 & -2 & a & b \\ 0 & 1 & c & d \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

**5.39** Suppose that $A$ and $B$ are $n \times n$ matrices such that $A = QBQ^{-1}$ for some invertible matrix $Q$. Prove that $A$ and $B$ have the same characteristic polynomials. Suppose that $X$ is an eigenvector for $B$. Show that $QX$ is an eigenvector for $A$.

### 5.2.1 Computer Projects

In this exercise, you create your own eigenvalue problem. First declare "format short" in MATLAB. Then, enter a $3 \times 3$ rank 3 matrix $P$ of your own choice into MATLAB. (You can use the MATLAB rank command to check the rank of your matrix.) To avoid trivialities, make each of the entries of $P$ nonzero. Next, enter a $3 \times 3$ *diagonal* matrix $D$. Choose $D$ so that its diagonal entries are 2, 2, and 3 (in that order). Finally, let

```
A=P*D*inv(P)
```

The general theory makes two predictions:

(a) Each column of $P$ is an eigenvector for $A$.

(b) The eigenvalues of $A$ are 2 and 3.

## EXERCISES

**1.** Verify prediction (a) above by multiplying each column of $P$ by $A$ and checking to see that they are indeed eigenvectors corresponding to the stated eigenvalues.

**2.** Verify prediction (b) above with the MATLAB command eig(A).

**3.** MATLAB can also compute eigenvectors. Enter the command [Q,E]=eig(A). This produces a matrix whose first three columns are eigenvectors for $A$ and whose last three columns form the corresponding diagonal form of $A$. As in Exercise 1, check that the columns of $Q$ really are eigenvectors of $A$. The column of $Q$ that corresponds to the eigenvalue 3 should be a multiple of the third column of $P$. (Why?) Check that this really is the case. How should the other two columns of $Q$ relate to the columns of $P$? Check that this really is true.

**4.** In the text, we defined the characteristic polynomial of $A$ to be $p(x) = \det(A - xI)$. MATLAB uses $q(x) = \det(xI - A)$. The relationship between these two is simple: If $A$ is $n \times n$, then $q(x) = (-1)^n p(x)$. (Explain!) The advantage of MATLAB's version is that the program's characteristic polynomials always have $x^n$ as their highest-order term. There is only one degree 3 polynomial with roots 2, 2, and 3 that has $x^3$ as its highest degree term. What is this polynomial? [*Hint:* Write it as a product of linear factors and then expand.] You can check your work with the MATLAB command poly(A), which produces a vector whose components are the coefficients of the characteristic polynomial.

### 5.2.2 Application to Systems of Differential Equations

As usual, we begin with an example.

**EXAMPLE 5.7**

Find a pair of functions $x$ and $y$ in $C^\infty(\mathbb{R})$ such that

$$\begin{aligned} x' &= 3x + y \\ y' &= x + 3y \end{aligned} \tag{5.14}$$

*Solution.* Equation (5.14) is equivalent to

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

which we write as

$$X' = AX$$

where $A$ is the preceding $2 \times 2$ matrix, $X = [x, y]^t$, and $X' = [x', y']^t$.

From Example 5.2 on page 273, $A$ has eigenvectors $Q_1 = [-1, 1]^t$ and $Q_2 = [1, 1]^t$ with 2 and 4 as the corresponding eigenvectors. Hence,

$$A = QDQ^{-1}$$

where

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \qquad Q = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

Equation (5.14) is equivalent to

$$\begin{aligned} X' &= QDQ^{-1}X \\ Q^{-1}X' &= DQ^{-1}X \\ Z' &= DZ \end{aligned}$$

where $Z = Q^{-1}X$. If we let $Z = [z_1, z_2]^t$, the last equation is equivalent to

$$\begin{aligned} z_1' &= 2z_1 \\ z_2' &= 4z_2 \end{aligned}$$

The general solution to $y' = ay$ is $y = Ce^{at}$, where $C$ is an arbitrary constant. Hence,

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} C_1 e^{2t} \\ C_2 e^{4t} \end{bmatrix}$$

where $C_1$ and $C_2$ are arbitrary constants. Since $Z = Q^{-1}X$, $QZ = X$, implying

$$X = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 e^{2t} \\ C_2 e^{4t} \end{bmatrix}$$

Hence,

$$\begin{aligned} x &= -C_1 e^{2t} + C_2 e^{4t} \\ y &= \phantom{-}C_1 e^{2t} + C_2 e^{4t} \end{aligned} \tag{5.15}$$

In general, if $X = [x_1, x_2, \dots, x_n]^t$ is a vector of functions, we define $X' = [x_1', x_2', \dots, x_n']^t$. If $A$ is an $n \times n$ matrix, then the equation

$$X' = AX \tag{5.16}$$

is said to be a first-order **system of linear differential equations**. Solving the system means finding functions $x_1, x_2, \dots, x_n$ that make it valid. If $A$ is diagonalizable, then the system may always be solved just as in Example 5.7. Specifically, we write $A = QDQ^{-1}$, where $D$ is diagonal. As in Example 5.7, we set $Z = Q^{-1}X$, in which case our system becomes

$$Z' = DZ$$

Then $z_i' = \lambda_i z_i$, where $\lambda_i$ is the $i$th diagonal entry of $D$, implying that

$$Z = \left[C_1 e^{\lambda_1 t}, C_2 e^{\lambda_2 t}, \dots, C_n e^{\lambda_n t}\right] \tag{5.17}$$

where $C_i$ are arbitrary constants. Our solution is then computed from $X = QZ$.

## EXAMPLE 5.8

Find functions $x_1$, $x_2$, and $x_3$ in $C^\infty(\mathbb{R})$ such that

$$\begin{aligned} x_1' &= 3x_1 + x_2 \\ x_2' &= x_2 \\ x_3' &= 4x_1 + 2x_2 + x_3 \end{aligned} \tag{5.18}$$

*Solution.* Equation (5.18) is equivalent to

$$X' = AX$$

where

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$$

This matrix was analyzed in Example 5.6 on page 288, where it was shown that $A = QDQ^{-1}$ with

$$Q = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

According to the discussion immediately preceding the current example,

$$X = QZ$$

where $Z = [C_1 e^t, C_2 e^t, C_3 e^{3t}]^t$. Hence,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -C_1\, e^t + C_3\, e^{3\,t} \\ 2\, C_1\, e^t \\ C_2\, e^t + 2\, C_3\, e^{3\,t} \end{bmatrix} \tag{5.19}$$

### Self-Study Questions

**5.1** ✓Show by substitution into equation (5.14) on page 293 that the functions $x$ and $y$ from equation (5.15) on page 294 are solutions.

**5.2** ✓If we express the following system as $X' = AX$, what is $A$?

$$\begin{aligned} x' &= 3x - y + z \\ y' &= x + 2y \\ z' &= 2x - y - z \end{aligned}$$

**5.3** ✓Write the general solution to the systems $X' = AX$, where $A$ is the $3 \times 3$ matrix Exercise 5.3(a) on page 280. Repeat for the matrix in Exercise 5.3(b) on page 280.

## EXERCISES

**5.40** Suppose that $X_o$ is an eigenvector for the $n \times n$ matrix $A$ corresponding to the eigenvalue $\lambda$. Show that $X(t) = e^{\lambda t} X_o$ satisfies

$$X' = AX$$

**5.41** Find the solution to the system from Example 5.7 on page 293 that satisfies $x(0) = 5$ and $y(0) = 3$.

**5.42** Find the solution to the system from Example 5.8 on page 294 that satisfies $x_1(0) = -3$, $x_2(0) = 1$, and $x_3(0) = -1$.

**5.43** Let $A = QDQ^{-1}$, where $D$ is diagonal. Prove that the unique solution to the system $X' = AX$ satisfying $X(0) = X_o$ is $X = QZ$, where $Z$ is as in formula (5.17) on page 294 and $C = Q^{-1}X_o$.

## 5.3 COMPLEX EIGENVECTORS

Before beginning our discussion of complex eigenvectors, let us take a moment to review the complex numbers. The symbol $i$ represents $\sqrt{-1}$. Thus, $i^2 = -1$. A complex number is an expression of the form $a + bi$, where $a$ and $b$ are real numbers. Thus, $2 + 3i$ is a complex number. We say that two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ and $b = d$.

We add complex numbers according to the formula

$$(a + bi) + (c + di) = a + c + (b + d)i \tag{5.20}$$

The formula for multiplication is based on the fact that $i^2 = -1$. Hence,

$$\begin{aligned}(a + bi)(c + di) &= a(c + di) + bi(c + di)\\ &= ac + adi + bci + bdi^2\\ &= ac - bd + (ad + bc)i\end{aligned} \tag{5.21}$$

Thus, for example,

$$\begin{aligned}(2 + 3i) + (7 - 6i) &= 9 - 3i\\ (2 + 3i)(7 - 6i) &= 2 \cdot 7 - 3 \cdot (-6) + (3 \cdot 7 + 2 \cdot (-6))i = 32 + 9i\end{aligned}$$

Students often feel uneasy about using complex numbers because they have a belief that they do not really exist—they are not "real." This is wrong. We think of $2 + 3i$ as just another way of representing the point $[2, 3]^t$ in $\mathbb{R}^2$. Thus, the set of complex numbers is just $\mathbb{R}^2$. Formula (5.20) says that addition of complex numbers is simply vector addition (Figure 5.1).

The set of real numbers is the set of numbers of the form $x + 0i$. This is the same as the set of points $[x, 0]^t$ that is just the $x$-axis. From this point of view, formula (5.21) is just a special way of multiplying points in $\mathbb{R}^2$. Note that if we accept formula (5.21) as a definition, then we can prove that $i^2 = -1$ since

$$i^2 = (0 + 1i)(0 + 1i) = 0 \cdot 0 - 1 \cdot 1 + (0 \cdot 1 + 1 \cdot 0)i = -1$$

Thus, $\sqrt{-1}$ does exist; it is the point $[0, 1]^t$ in $\mathbb{R}^2$.

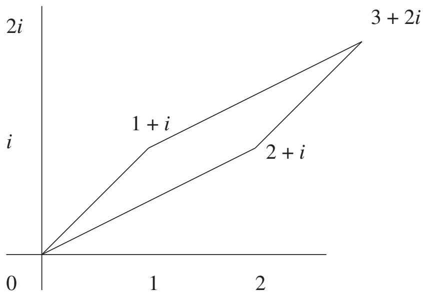



**FIGURE 5.1** Addition of complex numbers.

Actually, the multiplication of complex numbers has an interesting geometric interpretation. Any point $[x, y]^t$ in $\mathbb{R}^2$ may be expressed in polar coordinates. This amounts to writing

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{5.22}$$

where $r$ and $\theta$ are as in Figure 5.2. From Figure 5.2, in the first quadrant,

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\frac{y}{x}$$

Notice that $r$ is just the length of the vector $[x, y]^t$. From formula (5.22)

$$x + iy = r\cos\theta + (r\sin\theta)i = r(\cos\theta + i\sin\theta)$$

The angle $\theta$ is called the **argument** of the complex number.

Now, let $u + vi$ be another complex number written in polar form as

$$u + vi = s(\cos\psi + i\sin\psi)$$

Then, from formula (5.21),

$$\begin{aligned}(x + yi)(u + vi) &= rs(\cos\theta\cos\psi - \sin\theta\sin\psi + (\sin\theta\cos\psi + \cos\theta\sin\psi)i)\\ &= rs(\cos(\theta + \psi) + i\sin(\theta + \psi))\end{aligned}$$

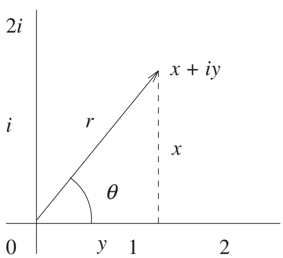



**FIGURE 5.2** Polar form.

The last equality came from the angle addition formulas for the sine and cosine functions. The term on the right side defines a point in polar form with argument $\theta + \psi$ and length $rs$. Thus, *we multiply two complex numbers by adding their arguments and multiplying their length.*

In particular, we square a complex number by squaring its length and doubling its argument. In fact, if we apply the above formula repeatedly, we obtain the following formula, which is known as De Moivre's theorem:

$$(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta) \tag{5.23}$$

### EXAMPLE 5.9

Compute $(1+\sqrt{3}\,i)^{20}$.

*Solution.* As Figure 5.3 indicates, the argument of $1+\sqrt{3}\,i$ is

$$\theta = \tan^{-1}\sqrt{3} = \pi/3$$

and the magnitude is

$$r = \sqrt{1^2 + (\sqrt{3})^2} = 2$$

Hence,

$$1+\sqrt{3}i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

Thus, from formula (5.23),

$$(1+\sqrt{3}\,i)^{20} = 2^{20}\left(\cos\frac{20\pi}{3} + i\sin\frac{20\pi}{3}\right) = -2^{19} + 2^{19}\sqrt{3}\,i$$

Now, let us turn to the main topic of this section. As usual, we begin with an example.

### EXAMPLE 5.10

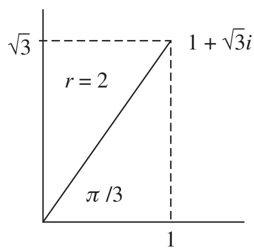



**FIGURE 5.3** Example 5.9.

Compute $A^{20}$ where

$$A = \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}$$

*Solution.* This problem looks very much like some problems that we solved in Section 5.2. The technique was to diagonalize $A$. Specifically, we found a diagonal matrix $D$ and an invertible matrix $Q$ such that $A = QDQ^{-1}$. Then Theorem 5.4 on page 287 in Section 5.2 shows that $A^{20} = QD^{20}Q^{-1}$.

To find $Q$, we need to find the eigenvalues of $A$. We compute

$$\begin{vmatrix} 1-\lambda & -3 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 4$$

Using the quadratic formula, we find

$$\lambda = \frac{2 \pm \sqrt{4-16}}{2} = 1 \pm \sqrt{3}\, i$$

Now we are concerned—the roots are complex.

Let us, however, proceed with our computation and see what happens. To find the eigenvectors, we must solve the equation $(A - \lambda I)X = \mathbf{0}$. Assume first that $\lambda = 1 + \sqrt{3}\, i$. Then our equation is

$$\begin{bmatrix} -\sqrt{3}\, i & -3 \\ 1 & -\sqrt{3}\, i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This corresponds to the system

$$\begin{aligned} (-\sqrt{3}\, i)x - 3y &= 0 \\ x - (\sqrt{3}\, i)y &= 0 \end{aligned}$$

Notice that if we multiply the second equation by $-\sqrt{3}\, i$ we obtain

$$0 = -\sqrt{3}\, i(x - (\sqrt{3}\, i)y) = (-\sqrt{3}\, i)x + 3i^2 y = (-\sqrt{3}\, i)x - 3y$$

This is the first equation. Thus our system is equivalent with the system consisting of only the second equation. The dependency of this system should not surprise us since a system of two equations in two unknowns cannot have any nonzero solutions unless it is linearly dependent.

We solve this system by letting $y$ be arbitrary and setting $x = (\sqrt{3}\, i)y$. In vector form, this becomes

$$[x, y]^t = [\sqrt{3}\, i, 1]^t y$$

A basis for the solution set is the single complex vector

$$Q_1 = [\sqrt{3}\,i, 1]^t$$

As a check on our work, we compute

$$\begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3}\,i \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}\,i - 3 \\ \sqrt{3}\,i + 1 \end{bmatrix} = (1 + \sqrt{3}\,i) \begin{bmatrix} \sqrt{3}\,i \\ 1 \end{bmatrix}$$

Thus, $Q_1$ really is an eigenvector with eigenvalue $1 + \sqrt{3}\,i$!

Similarly, if we use $\lambda = 1 - \sqrt{3}\,i$, we find the eigenvector

$$Q_2 = [-\sqrt{3}\,i, 1]^t$$

Hence, our diagonalizing matrix is

$$Q = \begin{bmatrix} \sqrt{3}\,i & -\sqrt{3}\,i \\ 1 & 1 \end{bmatrix}$$

Does this matrix really diagonalize $A$? To check it, we recall that the inverse of a general $2 \times 2$ matrix is given by the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Applying this to $Q$ (and hoping that it works for complex numbers), we find

$$Q^{-1} = \frac{1}{2\sqrt{3}i} \begin{bmatrix} 1 & \sqrt{3}\,i \\ -1 & \sqrt{3}\,i \end{bmatrix} = \frac{-\sqrt{3}i}{6} \begin{bmatrix} 1 & \sqrt{3}\,i \\ -1 & \sqrt{3}\,i \end{bmatrix}$$

Now, it is easily computed that, indeed, $Q^{-1}Q = I$ and

$$Q^{-1}AQ = \begin{bmatrix} 1 + \sqrt{3}\,i & 0 \\ 0 & 1 - \sqrt{3}\,i \end{bmatrix}$$

Let us call this latter matrix $D$. From Example 5.1, and a similar computation for $1 - \sqrt{3}i$, we see that

$$D^{20} = \begin{bmatrix} (1 + \sqrt{3}\,i)^{20} & 0 \\ 0 & (1 - \sqrt{3}\,i)^{20} \end{bmatrix} = 2^{19} \begin{bmatrix} -1 + \sqrt{3}\,i & 0 \\ 0 & -1 - \sqrt{3}\,i \end{bmatrix}$$

Hence, doing the required matrix multiplication, we find that

$$A^{20} = QD^{20}Q^{-1} = 2^{19}\begin{bmatrix} -1 & 3 \\ -1 & 1 \end{bmatrix}$$

Remarkably, the final answer only involves real numbers.

Having done this work and having arrived at an answer, let us step back to ask whether we can be confident that our answer is correct. Since $A = QDQ^{-1}$, $A^2$ should equal

$$(QDQ^{-1})(QDQ^{-1}) = QD(Q^{-1}Q)DQ^{-1} = QDIDQ^{-1} = QD^2Q^{-1}$$

We can use similar arguments to prove that $A^k = QD^kQ^{-1}$, justifying our work. But the preceding proof used many properties of matrix multiplication such as the associative law and $DI = D$. Can we be sure that such laws are valid for matrices with complex entries?

The answer is "Yes!" The reason is that, for the most part, real and complex numbers have the same algebraic properties. Specifically, all the following properties hold for both real and complex numbers:

(a) $a + b = b + a$.
(b) $a + (b + c) = (a + b) + c$.
(c) $a + 0 = 0 + a = a$.
(d) For every element $a$ there is an element $-a$ such that $a + (-a) = 0$.
(e) $a(bc) = (ab)c$.
(f) $a1 = 1a = a$.
(g) For every element $a \neq 0$ there is an element $a^{-1}$ such that $aa^{-1} = 1$.
(h) $ab = ba$.
(i) $a(b + c) = ab + ac$.

The above properties are referred to as the **field properties** and any system satisfying these properties is said to be a **field**. The set of all $m \times n$ complex matrices (matrices with complex entries) is denoted $M_c(m, n)$. Addition, multiplication by scalars, and matrix multiplication are defined just as in the real case. All of the basic properties of matrix algebra (such as the associative law for matrix multiplication) may be proved directly from the field properties. Hence, they are all true for complex matrices. Thus, our calculation really is correct.

Let us ask a more philosophical question: why were we forced to use the complex numbers to solve a problem that involved only real numbers? The answer is, of course, that the characteristic polynomial for our matrix had complex roots. This demonstrates why the complex numbers were invented in the first place. To find the roots of a second-degree polynomial, one often needs to be able to take the square root of negative numbers. It is one of the most surprising and useful facts of mathematics that once we are able to form square roots of negative numbers, we can (in principle) factor any polynomial. This result is called the fundamental theorem of algebra.

**Theorem 5.6** (Fundamental Theorem of Algebra). *Every polynomial $p(x)$ of degree $n$ with complex coefficients has a unique factorization of the form*

$$p(x) = a(x - r_1)^{n_1}(x - r_2)^{n_2} \dots (x - r_k)^{n_k}$$

*where the $n_i$ are natural numbers, $a$ and $r_i$ are complex numbers, and*

$$n = n_1 + n_2 + \cdots + n_k$$

A few comments are necessary. First, any real number, such as 2, is also a complex number: $2 = 2 + 0i$. Hence, the theorem does not deny the possibility that all the roots are real. Similarly, the theorem is valid for polynomials with real coefficients as well.

The concepts of complex eigenvectors and complex eigenvalues are defined just as in the case of real numbers. We define $\mathbb{C}^n = M_c(n, 1)$.

**Definition 5.4** *Let $A$ be a complex $n \times n$ matrix. A nonzero element $X$ in $\mathbb{C}^n$ such that*

$$AX = \lambda X$$

*for some scalar $\lambda$ is called a* ***complex eigenvector*** *for $A$. The scalar $\lambda$ is called the corresponding* ***complex eigenvalue****.*

The complex eigenvalues are the complex roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$. One consequence of the fundamental theorem of algebra is that the characteristic polynomial of any $n \times n$ matrix $A$ always has at least one complex root. It follows that $A$ must therefore have at least one eigenvalue. Thus, we conclude:

**Theorem 5.7** *Every $n \times n$ matrix $A$ has at least one (possibly complex) eigenvalue.*

Notice that the eigenvalues for the matrix $A$ from Example 5.10 were $1 + \sqrt{3}\,i$ and $1 - \sqrt{3}\,i$. Recall that for a complex number $z = a + bi$ the **complex conjugate** is

$$\bar{z} = a - b\,i$$

Thus, the two eigenvalues are conjugate to each other. The corresponding eigenvectors are also conjugate:

$$\overline{\begin{bmatrix} \sqrt{3}\,i \\ 1 \end{bmatrix}} = \begin{bmatrix} -\sqrt{3}\,i \\ 1 \end{bmatrix}$$

(If $A = [a_{ij}]$ is a matrix with complex entries, then we define $\overline{A} = [\overline{a}_{ij}]$.)

*The eigenvalues and eigenvectors for real matrices always occur in conjugate pairs*. To prove this, let $A$ be a real matrix and let $\lambda$ be an eigenvalue for $A$. Then there is a complex vector $X$ such that

$$\lambda X = AX$$

For all complex numbers $z$ and $w$, $\overline{z}\,\overline{w} = \overline{zw}$. This extends to matrices as

$$\overline{A}\,\overline{B} = \overline{AB} \tag{5.24}$$

Hence,

$$\overline{\lambda}\,\overline{X} = \overline{\lambda X} = \overline{AX} = \overline{A}\,\overline{X} = A\overline{X}$$

The last equality follows because $A$ is real. This shows that $\overline{X}$ is an eigenvector for $A$ corresponding to the eigenvalue $\overline{\lambda}$, as claimed.

*Remark.* Theorem 5.5 on page 289 holds for complex eigenvectors and eigenvalues with $\mathbb{R}^n$ replaced with $\mathbb{C}^n$.

### Complex Vector Spaces

We commented that the laws of matrix algebra are the same for real and complex matrices. It follows that all of the vector space properties (Theorem 1.1 on page 12) hold for $\mathbb{C}^n$, where now "scalar" means "complex number." Hence $\mathbb{C}^n$ is a **complex vector space**, meaning that it satisfies the requirements of Definition 1.6 on page 14 where now "scalar" means "complex number." $\mathbb{C}^n$ is also a real vector space since we can in particular multiply elements of $\mathbb{C}^n$ by real numbers.

In a complex vector space, the concepts of linear combination, linear independence, span, and basis are defined exactly as before (Definition 1.3 on page 8, Definition 1.4 on page 8, and Definition 2.2 on page 104), except that now the scalars can be complex. Thus, for example, in $\mathbb{C}^2$, the set $\{[1+i,2]^t, [2,2-2i]^t\}$ is complex linearly dependent because

$$[2, 2-2i]^t = (1-i)[1+i, 2]^t$$

This set is linearly independent in the sense of real vector spaces; there are no real numbers $r$ such that

$$[2, 2-2i]^t = r[1+i, 2]^t$$

The set $\mathcal{B} = \{[1,0]^t, [0,1]^t\}$ spans $\mathbb{C}^2$ as a complex vector space since for all $z, w \in \mathbb{C}$

$$[z, w]^t = z[1,0]^t + w[0,1]^t$$

In fact $\mathcal{B}$ is a complex basis for $\mathbb{C}^2$ since it is clearly independent; hence the complex dimension of $\mathbb{C}^2$ is 2. $\mathcal{B}$ *does not* span $\mathbb{C}^2$ as a real vector space. For example, there are no *real* numbers $x$ and $y$ such that

$$[i, 0]^t = x[1, 0]^t + y[0, 1]^t$$

The set

$$\mathcal{B}' = \{[1, 0]^t, [0, 1]^t, [i, 0]^t, [0, i]^t\} \tag{5.25}$$

spans $\mathbb{C}^2$ as a real vector space. This set is in fact a real basis for $\mathbb{C}^2$. Hence, the dimension of $\mathbb{C}^2$ as a real vector space is 4. This illustrates a general theorem; *the real dimension of a complex vector space is always twice its complex dimension.*

A **complex linear transformation** $T$ between complex vector spaces $\mathcal{V}$ and $\mathcal{W}$ is a mapping $T : \mathcal{V} \to \mathcal{W}$ satisfying the requirements of Definition 3.3 on page 152 where "scalar" now means "complex number." Any $m \times n$ complex matrix $A$ gives rise to a complex linear transformation $T_A : \mathbb{C}^n \to \mathbb{C}^m$ defined by formula (3.2) on page 151.

## EXERCISES

**5.44** ✓✓Compute $AB$ and $BA$ for the matrices $A$ and $B$.

$$A = \begin{bmatrix} 1+i & 2i \\ 2 & 3i \end{bmatrix}, \qquad B = \begin{bmatrix} -i & 3 \\ 2+i & 4i \end{bmatrix}$$

**5.45** For the matrix $A$ below, find complex matrices $Q$ and $D$ where $D$ is diagonal such that $A = QDQ{-}1$. Use your answer to find $A^{20}$. (Use a calculator to approximate the answer.)

$$A = \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}$$

**5.46** ✓✓Find all eigenvalues for the following matrix $A$:

$$A = \begin{bmatrix} 2 & 3 & 6 \\ 6 & 2 & -3 \\ 3 & -6 & 2 \end{bmatrix}$$

It might help to know that $-7$ is one eigenvalue.

**5.47** Consider the transformation $T : \mathbb{C} \to \mathbb{C}$ given by $T(z) = (2 + 3i)z$. We commented that we may interpret the complex numbers as being $\mathbb{R}^2$. Thus, we may think of $T$ as transforming $\mathbb{R}^2$ into $\mathbb{R}^2$. Explicitly, for example, $T$ transforms the point $[1, 2]^t$ into $[-4, 7]^t$ since $(2 + 3i)(1 + 2i) = -4 + 7i$.

Show that as a transformation of $\mathbb{R}^2$ into $\mathbb{R}^2$, $T$ is linear. Find a real $2 \times 2$ matrix $A$ such that $T = T_A$. Find all eigenvalues of this matrix.

**5.48** ✓✓Repeat Exercise 5.47 with $a + bi$ in place of $2 + 3i$.

**5.49** Compute the eigenvalues and eigenvectors for the rotation matrix

$$R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

**5.50** ✓✓Prove that, for all complex numbers $z$ and $w$, $\overline{zw} = \overline{z}\,\overline{w}$. Use this to prove formula (5.24) on page 303 in the text.

**5.51** ✓✓Prove that the set $\mathcal{B}'$ in formula (5.25) on page 304 is a basis for $\mathbb{C}^2$ considered as a real vector space.

**5.52** ✓✓Prove that the dimension of $\mathbb{C}^n$ considered as a real vector space is $2n$.

**5.53** ✓✓Let $\mathcal{V}$ be a complex vector space which is $n$-dimensional as a complex vector space. Prove that the dimension of $\mathcal{V}$ considered as a real vector space is $2n$.

**5.54** Let $A$ be a real $n \times n$ matrix and let $X \in \mathbb{C}^n$. Write $X = U + iV$, where $U$ and $V$ are elements of $\mathbb{R}^n$. Show that $X$ is an eigenvector for $A$ corresponding to the eigenvalue $\lambda = c + id$ if and only if both of the following equations hold:

$$AU = cU - dV$$
$$AV = dU + cV$$

**5.55** ✓✓Let $A$ be a complex $n \times n$ matrix and let $X \in \mathbb{C}^n$ be a complex eigenvector for $A$ with eigenvalue $\lambda$. Show that $\overline{X}$ is an eigenvector for $\overline{A}$. What is the corresponding eigenvalue?

**5.56** Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_o$, where the $a_i$ are real numbers. Show that if $\alpha$ is a complex root of $p(x)$, then so is $\overline{\alpha}$. Use this fact to prove that if $\lambda$ is an eigenvalue for a real $n \times n$ matrix $A$, then so is $\overline{\lambda}$.

**5.57** Prove that a real $3 \times 3$ matrix must have at least one real eigenvalue.

**5.58** Prove that an $n \times n$ matrix can have at most $n$ different complex eigenvalues. (There is a proof based on the linear independence of the eigenvectors. There is also a proof based on the characteristic polynomial.)

### 5.3.1 Computer Projects

## EXERCISES

1. Try the MATLAB command [B,D]=eig(A), where $A$ is as in Example 5.10 on page 298. This demonstrates that MATLAB knows about complex eigenvalues.
2. A complex matrix $A$ is Hermitian symmetric if $A^* = A$. Give an example of a $3 \times 3$ Hermitian matrix. Keep all entries nonzero and use as few real entries as possible. Enter your matrix into MATLAB. (Complex numbers such as $2 + 3i$

are entered into MATLAB as 2+3*i.) Then get MATLAB to find the eigenvalues. They are all real!

**3.** Change one of the entries of the matrix $A$ from Exercise 2 to make it non-Hermitian. Are the eigenvalues still real?

**4.** You may be under the impression that $A'$ is MATLAB for the transpose of $A$. Try it on a few complex matrices and see what you get. Read the MATLAB help entry for transpose for an explanation.

**5.** Enter into MATLAB a $3 \times 3$, real, symmetric matrix $A$. Make as many of the entries of $A$ as possible distinct. In MATLAB, let B=eye(3)+i*A. [eye(3) is MATLAB's notation for the $3 \times 3$ identity matrix.] Let C=eye(3)+inv(B)/2 and compute $C * C'$ and $C' * C$. Can you prove that what you observe is always true? [*Hint:* Begin with the equality $B + B^* = 2I$ and multiply by $B^{-1}$ on the left and $(B^*)^{-1}$ on the right.]

## CHAPTER SUMMARY

In this chapter we studied ***eigenvalues*** and ***eigenvectors***. An eigenvector for an $n \times n$ matrix $A$ is a nonzero column vector $X$ such that $AX = \lambda X$ for some scalar $\lambda$ (the eigenvalue). The computation of eigenvalues is done in two steps. First, we find the roots of the polynomial $p(\lambda) = \det(A - \lambda I)$ (the ***characteristic polynomial***). These are the eigenvalues. Next, for each eigenvalue $\lambda$, we find all solutions to $(A - \lambda I)X = \mathbf{0}$. These are the eigenvectors. Usually, we are most interested in finding a basis for this solution space.

Our first application of these concepts was computing formulas for $A^m B$, where $B$ was a given column vector and $A$ is an $n \times n$ matrix. The idea was that if we could find scalars $c_i$ such that

$$B = c_1 X_1 + \cdots + c_k X_k$$

where $X_i$ are eigenvectors, then

$$A^m B = c_1 \lambda_1^m X_1 + \cdots + c_k \lambda_k^m X_k$$

where the $\lambda_i$ are the eigenvalues corresponding to the $X_i$. This can be done for all $B$ in $\mathbb{R}^n$ if the $X_i$ form a basis for $\mathbb{R}^n$, in which case we say the $A$ is ***diagonalizable*** (over $\mathbb{R}$). Matrices that are not diagonalizable over $\mathbb{R}$ are ***deficient*** over $\mathbb{R}$.

Theorem 5.4 in Section 5.2 says that if $A$ is diagonalizable and $Q_1, Q_2, \ldots Q_n$ is a basis of $\mathbb{R}^n$ consisting of eigenvectors for $A$, then $A = QDQ^{-1}$ where $D$ is the diagonal matrix having the eigenvalues of $A$ corresponding to the $Q_i$ as its diagonal entries and $Q = [Q_1, Q_2, \ldots Q_n]$. (A matrix is diagonal if and only if all the entries off of the main diagonal are zero.) This formula allowed us to prove that $A^n = QD^nQ^{-1}$ [formula (5.2)], which then allowed us to compute powers of matrices.

Initially, we worked only with real numbers. However, in Section 5.3, we were forced to work with complex numbers because the characteristic polynomial could have complex roots. The ***fundamental theorem of algebra*** (which says that every polynomial factors completely over the complex numbers) allowed us to prove that every matrix has at least one (possibly complex) eigenvector. Moreover, the use of complex eigenvalues allowed us to compute powers of some matrices that did not diagonalize over the real numbers.

# CHAPTER 6

# ORTHOGONALITY

## 6.1 THE SCALAR PRODUCT IN $\mathbb{R}^N$

In geometry, distance and angle are obviously fundamental concepts. From Figure 6.1 on page 309, left, the length of the vector $X = [x, y]^t$ is

$$|X| = \sqrt{x^2 + y^2}$$

From Figure 6.1 on page 309, right, the distance between two points $X$ and $Y$ is the length of the vector $X - Y$. Thus, the distance between $X = [x_1, x_2]^t$ and $Y = [y_1, y_2]^t$ is

$$|X - Y| = |[x_1 - y_1, x_2 - y_2]^t| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \tag{6.1}$$

This, of course, is just the usual distance formula in $\mathbb{R}^2$.

**EXAMPLE 6.1**

Compute the distance between the points $X = [2, 4]^t$ and $Y = [4, 3]^t$.

*Solution.* The distance is

$$\begin{aligned} |X - Y| &= |[2, 4]^t - [4, 3]^t| \\ &= |[-2, 1]^t| = \sqrt{(-2)^2 + 1^2} = \sqrt{5} \end{aligned}$$

*Linear Algebra: Ideas and Applications*, Fourth Edition. Richard C. Penney.

Companion website: www.wiley.com/go/penney/linearalgebra

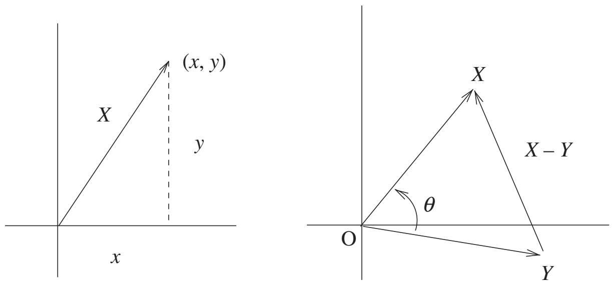



**FIGURE 6.1** Distance in $\mathbb{R}^2$.

Similar formulas hold in $\mathbb{R}^3$. The length of the vector $X = [x, y, z]^t$ is given by

$$|X| = \sqrt{r^2 + z^2} = \sqrt{x^2 + y^2 + z^2}$$

In fact, there is no reason to stop with three dimensions. We *define* the length of the vector $X = [x_1, x_2, \ldots, x_n]^t$ in $\mathbb{R}^n$ by

$$|X| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \tag{6.2}$$

and the distance between $X = [x_1, x_2, \ldots, x_n]^t$ and $Y = [y_1, y_2, \ldots, y_n]^t$ to be

$$\begin{aligned}|X - Y| &= |[x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n]^t| \\ &= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}\end{aligned} \tag{6.3}$$

## EXAMPLE 6.2

Find the distance in $\mathbb{R}^4$ between the vectors $X = [2, 4, 1, -1]^t$ and $Y = [5, -3, 7, 2]^t$.

*Solution.* The distance between $X$ and $Y$ is $|X - Y|$. We compute

$$\begin{aligned}X - Y &= [2, 4, 1, -1]^t - [5, -3, 7, 2]^t = [-3, 7, -6, -3]^t \\ |X - Y| &= |[-3, 7, -6, -3]^t| = \sqrt{(-3)^2 + 7^2 + (-6)^2 + (-3)^2} = \sqrt{103}\end{aligned}$$

If we square both sides of formula (6.3), we see that

$$\begin{aligned}|X - Y|^2 &= (x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2 \\ &= x_1^2 + y_1^2 + x_2^2 + y_2^2 + \cdots + x_n^2 + y_n^2 \\ &\quad - 2(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n) \\ &= |X|^2 + |Y|^2 - 2(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)\end{aligned} \tag{6.4}$$

The expression in parentheses on the right is called the **dot product** of $X$ and $Y$:

**Definition 6.1** *Let* $X = [x_1, x_2, \ldots, x_n]^t$ *and* $Y = [y_1, y_2, \ldots, y_n]^t$. *Then the* ***dot product*** *of* $X$ *and* $Y$ *is*

$$X \cdot Y = x_1y_1 + x_2y_2 + \cdots + x_ny_n \tag{6.5}$$

Then, formula (6.4) may be stated as

$$|X - Y|^2 = |X|^2 + |Y|^2 - 2X \cdot Y \tag{6.6}$$

We refer to this identity as the "law of cosines." To understand why, consider Figure 6.1 again. The law of cosines from geometry states that

$$|X - Y|^2 = |X|^2 + |Y|^2 - 2|X|\,|Y| \cos\theta$$

Comparing this with formula (6.6), we see that

$$X \cdot Y = |X|\,|Y| \cos\theta \tag{6.7}$$

Hence, if $X$ and $Y$ are both nonzero,

$$\cos\theta = \frac{X \cdot Y}{|X|\,|Y|} \tag{6.8}$$

In $\mathbb{R}^2$ and $\mathbb{R}^3$, our discussion proved formula (6.8). In $\mathbb{R}^n$, for $n > 3$, we *cannot* prove this formula since angles have no apparent meaning. Instead, we use formula (6.8) to *define* the angle between two vectors:

**Definition 6.2** *Let* $X$ *and* $Y$ *be nonzero vectors in* $\mathbb{R}^n$. *Then the angle* $\theta$ *between* $X$ *and* $Y$ *is defined by*

$$\theta = \cos^{-1}\left(\frac{X \cdot Y}{|X|\,|Y|}\right)$$

We must, however, be careful. Only numbers between $-1$ and 1 can be the cosine of an angle. In order to make such a definition, we need to know that quantity on the right in formula (6.8) lies in this interval. In $\mathbb{R}^2$ and $\mathbb{R}^3$, this is true since this quantity is, after all, the cosine of the angle between $X$ and $Y$. In $\mathbb{R}^n$ this follows from the following important result whose proof is developed in Exercise 6.14 on page 318.

**Theorem 6.1** (Cauchy-Schwartz). *Let* $X$ *and* $Y$ *be vectors in* $\mathbb{R}^n$. *Then*

$$|X \cdot Y| \le |X|\,|Y|$$

## EXAMPLE 6.3

Compute the cosine of the angle $\theta$ between $X = [2, 4, 1, -1]^t$ and $Y = [5, -3, 7, 2]^t$.

*Solution.* We compute

$$\begin{aligned} X \cdot Y &= 2 \cdot 5 + 4 \cdot (-3) + 1 \cdot 7 - 1 \cdot 2 = 3 \\ |X| &= \sqrt{2^2 + 4^2 + 1^2 + (-1)^2} = \sqrt{22} \\ |Y| &= \sqrt{5^2 + (-3)^2 + 7^2 + 2^2} = \sqrt{87} \\ \cos\theta &= \frac{3}{\sqrt{22}\sqrt{87}} = 0.0686 \end{aligned}$$

Note that

$$x_1y_1 + x_2y_2 + \cdots + x_ny_n = [x_1, x_2, \ldots, x_n]^t \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Hence, if $X$ and $Y$ belong to $\mathbb{R}^n$,

$$X \cdot Y = X^tY \tag{6.9}$$

The following theorem follows from formula (6.9) together with the linearity properties of matrix multiplication (Proposition 3.3 on page 170).

**Theorem 6.2** (Dot Product Theorem). *Let X, Y, and Z be vectors in $\mathbb{R}^n$. Let c be a scalar. Then:*

*(a)* $X \cdot Y = Y \cdot X$ *(commutative law)*
*(b)* $Z \cdot (X + Y) = Z \cdot X + Z \cdot Y$ *(additive law)*
*(c)* $(cX) \cdot Y = c(X \cdot Y) = X \cdot (cY)$ *(scalar law)*
*(d)* $|X|^2 = X \cdot X$

The next result is a consequence of Theorems 6.1 and 6.2. (See Exercise 6.16.)

**Theorem 6.3** (Triangle Inequality). *Let $X, Y \in \mathbb{R}^n$. Then*

$$|X + Y| \leq |X| + |Y|$$

When we discussed coordinates in Section 3.5, we emphasized the case of non-perpendicular axes. Certainly, however, the case of perpendicular axes is both the most common and the most useful. Since $\cos \pi/2 = 0$, two vectors $X$ and $Y$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ are perpendicular if and only if $X \cdot Y = 0$. Thus, the following definition seems very natural:

**Definition 6.3** *Two vectors X and Y in $\mathbb{R}^n$ are* ***orthogonal*** *(****perpendicular****) if $X \cdot Y = 0$.*

Thus, for example, $X = [1, 2, 3]^t$ and $Y = [7, 1, -3]^t$ are orthogonal because

$$X \cdot Y = 1 \cdot 7 + 2 \cdot 1 + 3 \cdot (-3) = 0$$

while $Z = [1, 2, 3, 4, 5]^t$ and $W = [1, 1, 1, 1]^t$ are not orthogonal because

$$Z \cdot W = 1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1 + 4 \cdot 1 + 5 \cdot 1 = 15 \neq 0$$

Recall that coordinates in $\mathbb{R}^n$ are defined by ordered bases. Specifically, if the coordinates are defined by the ordered basis $\mathcal{B} = \{P_1, P_2, \ldots, P_n\}$, then $X$ has $\mathcal{B}$ coordinate vector $X' = [x'_1, x'_2, \ldots, x'_n]^t$ if and only if

$$X = x'_1 P_1 + x'_2 P_2 + \cdots + x'_n P_n$$

In $\mathbb{R}^2$, the axes are perpendicular if and only if the basis vectors are orthogonal, that is, $P_1 \cdot P_2 = 0$. In $\mathbb{R}^3$, the axes are perpendicular if and only if each basis vector is orthogonal to both of the other two. In $\mathbb{R}^n$ we make the following definition:

### Orthogonal/Orthonormal Bases and Coordinates

**Definition 6.4** *A set $\mathcal{B} = \{P_1, P_2, \ldots, P_k\}$ of vectors in $\mathbb{R}^n$ is* ***orthogonal*** *if none of the $P_j = \mathbf{0}$ and $P_i \cdot P_j = 0$ for $i \neq j$. An orthogonal basis for a subspace $\mathcal{W}$ of $\mathbb{R}^n$ is an orthogonal set of vectors in $\mathcal{W}$ that forms a basis for $\mathcal{W}$.*

The second part of the next example is intended as a review of some of the discussion in Section 3.5.

**EXAMPLE 6.4**

The set $\mathcal{B} = \{P_1, P_2, P_3\}$, where the $P_i$ are as stated below, forms an ordered basis for $\mathbb{R}^3$. Show that this basis is orthogonal and find the $\mathcal{B}$ coordinate vector $X'$ for $X = [1, 1, 1]^t$.

$$P_1 = [1, -1, 1]^t, \quad P_2 = [1, -1, -2]^t, \quad P_3 = [1, 1, 0]^t$$

*Solution.* To prove orthogonality we must show that each basis vector is perpendicular to every other basis vector. Thus, we compute

$$\begin{aligned}
P_1 \cdot P_2 &= [1, -1, 1]^t \cdot [1, -1, -2]^t = 1 \cdot 1 + (-1) \cdot (-1) + 1 \cdot (-2) = 0 \\
P_1 \cdot P_3 &= [1, -1, 1]^t \cdot [1, 1, 0]^t = 1 \cdot 1 + (-1) \cdot 1 + 1 \cdot 0 = 0 \\
P_2 \cdot P_3 &= [1, -1, -2]^t \cdot [1, 1, 0]^t = 1 \cdot 1 + (-1) \cdot 1 + (-2) \cdot 0 = 0
\end{aligned}$$

showing orthogonality.

The coordinates are defined by the equality

$$X = x'_1 P_1 + x'_2 P_2 + x'_3 P_3 \tag{6.10}$$

which is equivalent to $X = PX'$, where $P = [P_1, P_2, P_3]$. Thus,

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & -2 & 0 \end{bmatrix}$$

Then $X' = CX$, where $C = P^{-1}$. We compute

$$P^{-1} = \frac{1}{6}\begin{bmatrix} 2 & -2 & 2 \\ 1 & -1 & -2 \\ 3 & 3 & 0 \end{bmatrix}$$

and $X' = [\frac{1}{3}, -\frac{1}{3}, 1]^t$.

It turns out that, due to the orthogonality of the basis, there is a way of solving Example 6.4 *without computing* $P^{-1}$. We begin by taking the dot product of both sides of equation (6.10) with $P_1$:

$$\begin{aligned} X \cdot P_1 &= (x'_1 P_1 + x'_2 P_2 + x'_3 P_3) \cdot P_1 \\ &= x'_1 P_1 \cdot P_1 + x'_2 P_2 \cdot P_1 + x'_3 P_3 \cdot P_1 \\ &= x'_1 P_1 \cdot P_1 + 0 + 0 = x'_1 P_1 \cdot P_1 \end{aligned}$$

Hence,

$$x'_1 = \frac{X \cdot P_1}{P_1 \cdot P_1} = \frac{1 - 1 + 1}{1^2 + (-1)^2 + 1^2} = \frac{1}{3}$$

Similarly,

$$x'_2 = \frac{X \cdot P_2}{P_2 \cdot P_2} = \frac{1 - 1 - 2}{1^2 + (-1)^2 + (-2)^2} = -\frac{1}{3}$$

and

$$x'_3 = \frac{X \cdot P_3}{P_3 \cdot P_3} = \frac{1 + 1 + 0}{1^2 + 1^2 + 0^2} = 1$$

This is, of course, in agreement with what we found in Example 6.4.

This technique generalizes to higher dimensions:

**Theorem 6.4** *Let $\mathcal{B} = \{P_1, P_2, \ldots, P_n\}$ be an ordered orthogonal basis for $\mathbb{R}^n$ and let $X \in \mathbb{R}^n$. Then*

$$X = x'_1 P_1 + x'_2 P_2 + \cdots + x'_n P_n \tag{6.11}$$

*where*

$$x'_i = \frac{X \cdot P_i}{P_i \cdot P_i} \tag{6.12}$$

*In particular, the $\mathcal{B}$ coordinate vector for $X$ is $X' = [x'_1, x'_2, \ldots, x'_n]^t$, where the $x'_i$ are given by formula* (6.12).

*Proof.* Since $\mathcal{B}$ is a basis, we may expand $X$ as in equation (6.11). We take the dot product of both sides of this equation with $P_j$ and use $P_i \cdot P_j = 0$, $i \neq j$, finding

$$X \cdot P_i = 0 + 0 + \cdots + x'_i P_i \cdot P_i + 0 + \cdots + 0 = x'_i P_i \cdot P_i \tag{6.13}$$

which is equivalent to our theorem. □

Notice that in Example 6.4 it was given that $\mathcal{B}$ forms a basis of $\mathbb{R}^3$. We, of course, could have checked this by showing that $\mathcal{B}$ is linearly independent. (We need to show only linear independence, since $\mathbb{R}^3$ is three-dimensional.) However, since the $P_i$ are mutually perpendicular, it seems apparent that they are linearly independent—how could, say, $P_3$ be a linear combination of $P_1$ and $P_2$ if it is perpendicular to the plane they span? (See Figure 6.2.) The following theorem is proved using much the same argument as in the proof of Theorem 6.4. The proof is indicated in Exercise 6.17.

**Theorem 6.5** *Any orthogonal set of vectors in $\mathbb{R}^n$ is linearly independent.*

The set $\mathcal{B} = \{P_1, P_2\}$, where $P_1 = [2, 0]^t$ and $P_2 = [0, 1]^t$, is an ordered, orthogonal basis for $\mathbb{R}^2$. The point with $\mathcal{B}$ coordinate vector $X' = [x', 0]^t$ is

$$X = x'[2, 0]^t = [2x', 0]^t$$

Hence, the scale on the $x'$ axis is such that every unit of measure of $x'$ corresponds to two units of measure in $\mathbb{R}^2$. This is due to the fact that $|P_1| = 2$ since then

$$|X| = |x'P_1| = 2|x'|$$

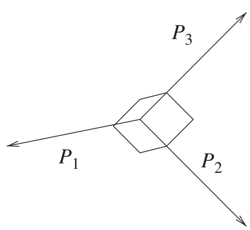



**FIGURE 6.2** Three orthogonal vectors.

Similarly, since $|P_2| = 1$, one unit of measure along the $P_2$ axis corresponds to one unit of distance in $\mathbb{R}^2$. Indeed, the point with coordinate vector $[0, y']^t$ is $y'[0,1]^t = [0, y']^t$. In general, the measure on each axis corresponds to actual distance in $\mathbb{R}^n$ if and only if each basis vector has a length of 1. These comments lead to the following definition:

**Definition 6.5** *A vector $Q \in \mathbb{R}^n$ is* ***normal*** *if $|Q| = 1$. An orthogonal subset $\{Q_1, Q_2, \dots, Q_k\}$ of $\mathbb{R}^n$ is* ***orthonormal*** *if $|Q_i| = 1$ for all i. An* ***orthonormal basis*** *for a subspace $\mathcal{W}$ of $\mathbb{R}^n$ is an orthonormal set of vectors in $\mathcal{W}$ which forms a basis for $\mathcal{W}$.*

Given a nonzero vector $X \in \mathbb{R}^n$, we may produce a normal vector $Q$ that points in the same direction as $X$ by dividing $X$ by $|X|$:

$$Q = \frac{X}{|X|}$$

This process is referred to as **normalization**. Any orthogonal set $\{P_1, P_2, \dots, P_k\}$ in $\mathbb{R}^n$ may be converted into an orthonormal set by normalizing each $P_i$. This is referred to as **normalizing the basis**.

*Remark.* We typically represent elements of orthogonal bases by "$P_i$" unless the basis is orthonormal, in which case we use "$Q_i$."

## EXAMPLE 6.5

Normalize the basis $\mathcal{B}$ from Example 6.4.

*Solution.* We compute that $|P_1| = \sqrt{3}$, $|P_2| = \sqrt{6}$, and $|P_3| = \sqrt{2}$. Hence, the normalized basis is $\mathcal{B}' = \{Q_1, Q_2, Q_3\}$, where

$$\begin{aligned} Q_1 &= \frac{P_1}{|P_1|} = \frac{1}{\sqrt{3}}[1, -1, 1]^t \\ Q_2 &= \frac{P_2}{|P_2|} = \frac{1}{\sqrt{6}}[1, -1, -2]^t \\ Q_3 &= \frac{P_3}{|P_3|} = \frac{1}{\sqrt{2}}[1, 1, 0]^t \end{aligned}$$

Coordinates taken with respect to orthonormal bases have the remarkable property that the formulas for the dot product and the length remain unchanged:

**Theorem 6.6** *Let $\mathcal{B} = \{Q_1, Q_2, \dots, Q_k\}$ be an orthonormal set of vectors in $\mathbb{R}^n$ and let*

$$X = x_1'Q_1 + \cdots + x_k'Q_k, \qquad Y = y_1'Q_1 + \cdots + y_k'Q_k$$

*Then*

$$\begin{aligned} X \cdot Y &= x_1' y_1' + \cdots + x_k' y_k' \\ |X| &= \sqrt{(x_1')^2 + \cdots + (x_k')^2} \end{aligned} \tag{6.14}$$

*Proof.* From Theorem 6.4

$$y_i' = \frac{Y \cdot Q_i}{Q_i \cdot Q_i} = \frac{Y \cdot Q_i}{|Q_i|^2} = Y \cdot Q_i$$

Hence,

$$\begin{aligned} X \cdot Y &= (x_1' Q_1 + x_2' Q_2 + \cdots + x_k' Q_k) \cdot Y \\ &= x_1'(Q_1 \cdot Y) + x_2'(Q_2 \cdot Y) + \cdots + x_k'(Q_k \cdot Y) \\ &= x_1' y_1' + x_2' y_2' + \cdots + x_k' y_k' \end{aligned}$$

This proves the first formula in equation (6.14). The second follows by letting $Y = X$ in the first formula since $|X|^2 = X \cdot X$. □

**True-False Questions**: Justify your answers.

**6.1** Let $\{P_1, P_2, P_3\}$ be an orthogonal subset of $\mathbb{R}^3$. Suppose that $X = P_1 - 2P_2 + 3P_3$ satisfies $X \cdot P_3 = 6$. Then $|P_3| = 2$.

**6.2** There exist five nonzero, mutually perpendicular vectors $X_1$, $X_2$, $X_3$, $X_4$, $X_5$ in $\mathbb{R}^4$.

**6.3** It is possible to find a two-dimensional subspace of $\mathbb{R}^5$ that contains three mutually perpendicular, nonzero vectors.

**6.4** The vectors $P_1 = [1, 1]^t$ and $P_2 = [1, 3]^t$ are perpendicular.

**6.5** If $X \cdot Y = 0$, then either $X = \mathbf{0}$ or $Y = \mathbf{0}$.

**6.6** $X \cdot (Y \cdot Z) = (X \cdot Y) \cdot Z$.

**6.7** $(X - Y) \cdot (X + Y) = |X|^2 - |Y|^2$.

**6.8** If all the entries of $X$, $Y$, and $Z$ are positive, then $(X + Y + Z)/(X \cdot Y \cdot Z)$ is defined.

**6.9** Suppose that $\{P_1, P_2, P_3\}$ is an orthogonal subset of $\mathbb{R}^3$. Then the matrix $[P_1, P_2, P_3]$ is invertible.

**6.10** Suppose that $\{P_1, P_2, P_3\}$ is an orthogonal subset of $\mathbb{R}^5$. Then the rank $r$ of the matrix $[P_1, P_2, P_3]$ can be any integer satisfying $1 \leq r \leq 3$.

## EXERCISES

**6.1** For each pair of vectors $X$ and $Y$ below, find (i) the distance between $X$ and $Y$, (ii) $|X|$ and $|Y|$, (iii) $X \cdot Y$, and (iv) the angle between $X$ and $Y$.

**(a)** ✓✓ $[3,4]^t$, $[-1,2]^t$ **(b)** $[-3,2]^t$, $[-4,7]^t$

**(c)** ✓✓ $[1,2,3]^t$, $[-1,1,2]^t$ **(d)** $[1,1,0,2]^t$, $[1,1,1,-3]^t$

**6.2** Which of the following pairs of vectors are perpendicular to each other?

**(a)** ✓✓ $X = [7,-3]^t$, $Y = [2,1]^t$ **(b)** $X = [3,-2,1]^t$ $Y = [1,1,-1]^t$

**(c)** ✓✓ $X = [a,b]^t$, $Y = [-b,a]^t$ **(d)** $X = [3,4,3]^t$, $Y = [1,-1,-2/3]^t$

**(e)** ✓✓ $X = [1,2,3,6]^t$, $Y = [1,1,1,-1]^t$

**6.3** Find $c, d, e$, and $f$ such that $[c,d,e,f]^t$ is perpendicular to $[a,b,a,b]^t$. [*Hint:* Look at Exercise 6.2.c.]

**6.4** Let $X = [1,1]^t$. Describe geometrically the set of all vectors $Y$ in $\mathbb{R}^2$ such that $Y \cdot X \leq |Y|$. [*Hint:* Look at formula (6.8) on page 310.]

**6.5** ✓✓Let $X = [1,1,1]^t$. Describe geometrically the set of all vectors $Y$ in $\mathbb{R}^3$ such that $Y \cdot X \geq \frac{3}{2}|Y|$. [*Hint:* Look at formula (6.8) on page 310.]

**6.6** ✓✓These questions refer to the equation

$$2x + 3y - 6z = 1$$

**(a)** Find three different points (reader's choice) that satisfy the equation. Call your points $A$, $B$, and $C$.

**(b)** Show that the vectors $A - B$ and $C - B$ are both perpendicular to $[2,3,-6]^t$.

**(c)** Explain: "A point $[x,y,z]^t$ satisfies the equation if and only if the vector $[x,y,z]^t - B$ is perpendicular to $[2,3,-6]^t$. Therefore, the set of points that satisfy the equation is a plane." (You may want to draw a diagram.)

**6.7** ✓✓One (and only one) of the ordered bases $\mathcal{B}$ in Exercise 3.117 on page 231 is orthogonal.

**(a)** Which one is it?

**(b)** Use Theorem 6.4 on page 313 to compute the $\mathcal{B}$ coordinate vector of $X = [1,2,3]^t$.

**(c)** Use Theorem 6.4 to compute the $\mathcal{B}$ coordinate vector $X'$ of $X = [x,y,z]^t$.

**(d)** *Use your answer* to part (c) to write down the coordinate matrix $C_{\mathcal{B}}$ for this basis. (Remember that $X' = C_{\mathcal{B}}X$.)

**6.8** Repeat Exercise 6.7.b–6.7.d for the basis $\mathcal{B}'$ in Example 6.5 on page 315.

**6.9** ✓✓Show that the following set $\mathcal{B}$ is an orthogonal basis for $\mathbb{R}^4$. Find the $\mathcal{B}$ coordinate vector for $X = [1,2,-1,-3]^t$.

$$\mathcal{B} = \{[2,-1,-1,-1]^t,\ [1,3,3,-4]^t,\ [1,1,0,1]^t,\ [1,-2,3,1]^t\}$$

**6.10** ✓For $P_i$ as defined below, show that $\{P_1, P_2, P_3\}$ is an orthogonal subset of $\mathbb{R}^4$. Find a fourth vector $P_4$ such that $\{P_1, P_2, P_3, P_4\}$ forms an orthogonal basis in $\mathbb{R}^4$. To what extent is $P_4$ unique?

$$P_1 = [1, 1, 1, 1]^t, \quad P_2 = [1, -2, 1, 0]^t, \quad P_3 = [1, 1, 1, -3]^t$$

**6.11** ✓Let $\{P_1, P_2, P_3\}$ be a set of three orthogonal vectors in $\mathbb{R}^4$. Prove that there is a fourth vector $P_4$ such that $\{P_1, P_2, P_3, P_4\}$ forms an orthogonal basis for $\mathbb{R}^4$. To what extent is $P_4$ unique? Prove your answer. What if the vectors were in $\mathbb{R}^5$? Does $P_4$ still exist? To what extent is it unique? [*Hint:* What is the rank of the coefficient matrix for the system of equations defined by $P_i \cdot X = 0$, $1 \le i \le 3$?]

**6.12** Let $X$ and $Y$ be vectors in $\mathbb{R}^n$. Prove that

$$|X + Y|^2 + |X - Y|^2 = 2|X|^2 + 2|Y|^2$$

Interpret this equality geometrically.

**6.13** Let $X$ and $Y$ be vectors in $\mathbb{R}^n$. Prove that $|X|^2 + |Y|^2 = |X + Y|^2$ holds if and only if $X \cdot Y = 0$. Using a diagram, explain why this identity is referred to as the Pythagorean theorem.

**6.14** Prove Theorem 6.1 on page 310—that is, prove that if $X$ and $Y$ are vectors in $\mathbb{R}^n$, then

$$-|X|\,|Y| \le X \cdot Y \le |X|\,|Y|$$

[*Hint:* If either $X$ or $Y$ is zero, the result is clear, so assume that both are nonzero. Let $U = X/|X| + Y/|Y|$ and note that $U \cdot U \ge 0$. This should yield one of the desired inequalities. For the other inequality, make a slightly different choice of $U$.]

**6.15** Prove that if $X \cdot Y = \pm|X|\,|Y|$ where $X$ and $Y$ are nonzero vectors in $\mathbb{R}^n$, then $X/|X| = \pm Y/|Y|$; hence, $X$ and $Y$ are scalar multiples of each other. [*Hint:* Let $U = X/|X| \mp Y/|Y|$ and compute $U \cdot U$.]

**6.16** Explain why Theorem 6.3 on page 311 is called the "triangle inequality." Then use the result of Exercise 6.14 to prove Theorem 6.3. [*Hint:* Replace $Y$ with $-Y$ in formula (6.6) on page 310.]

**6.17** Prove that any orthogonal set of three vectors in $\mathbb{R}^n$ is linearly independent. [*Hint:* Apply the reasoning used in the proof of Theorem 6.4 on page 313 to the dependency equation.]

**6.18** Prove Theorem 6.5 on page 314.

## 6.2 PROJECTIONS: THE GRAM-SCHMIDT PROCESS

Let us imagine that we are studying a particle that is moving in a direction defined by a vector $P$ and that there is a force acting on the particle defined by a vector $Y$.

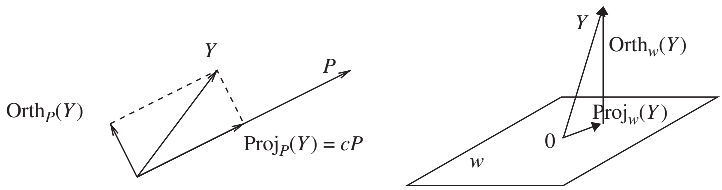



**FIGURE 6.3** Projections to line and planes.

Often in such problems it is useful to think of the force $Y$ as the sum of two forces $Y_1$ and $Y_0$, where $Y_0$ acts in the direction of the motion and $Y_1$ is perpendicular to the direction of motion, as in Figure 6.3, left. The vector $Y_0$ is called the **projection of** $Y$ **onto** $P$ and is denoted by $\mathrm{Proj}_P(Y)$. $Y_1$ is referred to as the **complementary vector** and is denoted by $\mathrm{Orth}_P(Y)$.

Computing $\mathrm{Proj}_P(Y)$ and $\mathrm{Orth}_P(Y)$ is not hard. Since $\mathrm{Proj}_P(Y)$ points along $P$, there is a scalar $c$ such that

$$\mathrm{Proj}_P(Y) = cP$$

We must find $c$ so that $\mathrm{Orth}_P(Y) = Y - cP$ is perpendicular to $P$, that is,

$$0 = (Y - cP) \cdot P = Y \cdot P - c(P \cdot P)$$

Solving for $c$ yields

$$c = \frac{Y \cdot P}{P \cdot P}$$

and thus,

$$\begin{aligned} \mathrm{Proj}_P(Y) &= \frac{Y \cdot P}{P \cdot P} P \\ \mathrm{Orth}_P(Y) &= Y - \frac{Y \cdot P}{P \cdot P} P \end{aligned} \tag{6.15}$$

## EXAMPLE 6.6

Find $\mathrm{Orth}_P(Y)$ and $\mathrm{Proj}_P(Y)$, where $P = [1, -1, 1]^t$ and $Y = [1, 2, 3]^t$.

*Solution.* According to formula (6.15),

$$\begin{aligned} \mathrm{Proj}_P(Y) &= \frac{Y \cdot P}{P \cdot P} P = \frac{1 - 2 + 3}{1 + 1 + 1}[1, -1, 1]^t = \left[\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}\right]^t \\ \mathrm{Orth}_P(Y) &= Y - \mathrm{Proj}_P(Y) = \left[\frac{1}{3}, \frac{8}{3}, \frac{7}{3}\right]^t \end{aligned}$$

It is easy to check that $\mathrm{Orth}_P(Y) \cdot P = 0$.

In Example 6.6, we projected a vector onto a line determined by another vector. We can also project vectors onto planes, as indicated in Figure 6.3, right. In this case, we wish to resolve our vector into a sum of two vectors, one lying in the plane and one perpendicular to the plane, that is, perpendicular to every vector in the plane.

More generally, we make the following definition:

**Definition 6.6** *Let $S$ be a subset of $\mathbb{R}^n$. We define a subset (**$S$-perp**) of $\mathbb{R}^n$ by*

$$S^{\perp} = \{X \in \mathbb{R}^n \mid X \cdot W = 0 \textit{ for all } W \in S\}$$

*If $\mathcal{W}$ is a subspace of $\mathbb{R}^n$, then $\mathcal{W}^{\perp}$ is also referred to as the **orthogonal complement** of $\mathcal{W}$.*

It is easily seen (Exercise 6.28) that for any subset $S$ of $\mathbb{R}^n$, $S^{\perp}$ is a subspace of $\mathbb{R}^n$. If $\mathcal{W}$ is a line through the origin in $\mathbb{R}^3$, then $\mathcal{W}^{\perp}$ is a plane through the origin. Similarly, if $\mathcal{W}$ is a plane through the origin in $\mathbb{R}^3$, then $\mathcal{W}^{\perp}$ is a line through the origin (Figure 6.4).

Given a subspace $\mathcal{W}$ of $\mathbb{R}^n$ and a vector $Y \in \mathbb{R}^n$, we can hope to find a point $\text{Proj}_{\mathcal{W}}(Y) \in \mathcal{W}$ such that

$$\text{Orth}_{\mathcal{W}}(Y) = Y - \text{Proj}_{\mathcal{W}}(Y) \tag{6.16}$$

belongs to $\mathcal{W}^{\perp}$.

We shall show that $\text{Proj}_{\mathcal{W}}(Y)$ exists and is unique. Our starting point is an orthogonal basis $\mathcal{P} = \{P_1, P_2, \dots, P_m\}$ for $\mathcal{W}$. (We shall show shortly that every subspace of $\mathbb{R}^n$ has such a basis.) We first prove uniqueness.

Since $\text{Proj}_{\mathcal{W}}(Y)$ (if it exists) lies in $\mathcal{W}$, there are constants $c_i$ such that

$$\text{Proj}_{\mathcal{W}}(Y) = c_1P_1 + c_2P_2 + \cdots + c_mP_m \tag{6.17}$$

As in the case of projection onto the line through a single vector, the $c_i$ are determined by the requirement that

$$Y_1 = Y - \text{Proj}_{\mathcal{W}}(Y)$$

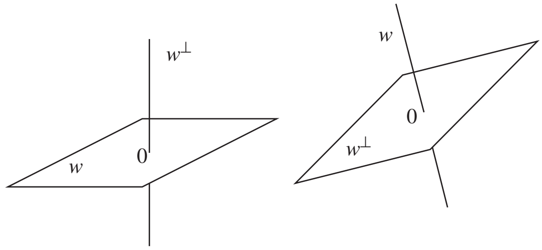



**FIGURE 6.4** Orthogonal complement.

belongs to $\mathcal{W}^{\perp}$. Specifically, note first that since $P_i \cdot P_j = 0$ for $i \neq j$, it follows from equation (6.17) that

$$\mathrm{Proj}_{\mathcal{W}}(Y) \cdot P_i = c_i P_i \cdot P_i \tag{6.18}$$

If $Y_1 \in \mathcal{W}^{\perp}$, then $Y_1 \cdot P_i = 0$ for all $1 \leq i \leq m$. This equality is true if and only if

$$\begin{aligned} 0 &= (Y - \mathrm{Proj}_{\mathcal{W}}(Y)) \cdot P_i \\ &= Y \cdot P_i - \mathrm{Proj}_{\mathcal{W}}(Y) \cdot P_i \\ &= Y \cdot P_i - c_i P_i \cdot P_i \end{aligned}$$

Hence, if $\mathrm{Proj}_{\mathcal{W}}(Y)$ exists,

$$c_i = \frac{Y \cdot P_i}{P_i \cdot P_i} \tag{6.19}$$

proving uniqueness.

To prove existence, let $\mathrm{Proj}_{\mathcal{W}}(Y)$ be defined by formula (6.17), where the $c_i$ are defined by formula (6.19), and let $Y_1 = Y - \mathrm{Proj}_{\mathcal{W}}(Y)$. From the above reasoning, $Y_1 \cdot P_i = 0$ for $1 \leq i \leq m$. It follows easily that $Y_1$ is orthogonal to every element of the span of $\mathcal{P}$—that is, $Y_1 \in \mathcal{W}^{\perp}$. (See Exercise 6.34.) Thus, we have proved the following important theorem:

**Theorem 6.7** (Fourier Theorem). *Let $\{P_1, P_2, \ldots, P_m\}$ be an orthogonal basis for a subspace $\mathcal{W}$ of $\mathbb{R}^n$ and let $Y \in \mathbb{R}^n$. Then the vector $\mathrm{Proj}_{\mathcal{W}}(Y)$ in formula* (6.20) *below is the unique vector in $\mathcal{W}$ such that $\mathrm{Orth}_{\mathcal{W}}(Y) = Y - \mathrm{Proj}_{\mathcal{W}}(Y)$ belongs to $\mathcal{W}^{\perp}$.*

$$Proj_{\mathcal{W}}(Y) = \frac{Y \cdot P_1}{P_1 \cdot P_1} P_1 + \frac{Y \cdot P_2}{P_2 \cdot P_2} P_2 + \cdots + \frac{Y \cdot P_m}{P_m \cdot P_m} P_m \tag{6.20}$$

The formula for $\mathrm{Proj}(Y)$ in equation (6.20) should look familiar. If the $P_i$ were an orthogonal basis for all $\mathbb{R}^n$, this formula would be exactly the expression of $Y$ in terms of the $P_i$ from Theorem 6.4 on page 313. The computations involved in doing specific examples are almost identical to those done in Section 6.1.

## EXAMPLE 6.7

Let $\mathcal{W}$ be the subspace of $\mathbb{R}^3$ spanned by $\mathcal{P} = \{P_1, P_2\}$ where

$$P_1 = [1, -1, 1]^t, \qquad P_2 = [1, -1, -2]^t$$

Show that $\mathcal{P}$ is an orthogonal basis for $\mathcal{W}$ and compute $Y_0 = \mathrm{Proj}_{\mathcal{W}}([1, 1, 1]^t)$ and $Y_1 = \mathrm{Orth}_{\mathcal{W}}([1, 1, 1]^t)$. Check that $Y_1$ belongs to $\mathcal{P}^{\perp}$.

*Solution.* It is easily seen that $P_1 \cdot P_2 = 0$; hence, $\mathcal{P}$ is an orthogonal set. From Theorem 6.5 on page 314, we see that $\mathcal{P}$ is linearly independent and hence, a basis of $\mathcal{W}$. From formula (6.20), we can write

$$Y_0 = \frac{Y \cdot P_1}{P_1 \cdot P_1} P_1 + \frac{Y \cdot P_2}{P_2 \cdot P_2} P_2 = \frac{1}{3} P_1 + \frac{-2}{6} P_2 = [0, 0, 1]^t$$

Then

$$Y_1 = [1, 1, 1]^t - [0, 0, 1]^t = [1, 1, 0]^t$$

It is easily checked that $Y_1 \cdot P_1 = Y_1 \cdot P_2 = 0$, showing that $Y_1 \in \mathcal{P}^{\perp}$.

*Remark.* The reader should compare the work done in this example with the work following Example 6.4 on page 312.

To compute projections using formula (6.20), we need an orthogonal basis for $\mathcal{W}$. Unfortunately, the algorithms we have developed for finding bases for row spaces, column spaces, and null spaces all tend to produce nonorthogonal bases. Fortunately, there is a process for converting nonorthogonal bases into orthogonal bases, the "Gram-Schmidt process."

To describe this process, suppose that we are given a basis $\mathcal{Y} = \{X_1, X_2, \ldots, X_m\}$ for a subspace $\mathcal{W}$ of $\mathbb{R}^n$. We hope to define an orthogonal basis $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$ for $\mathcal{W}$. The basis we produce will actually have the additional property that for each $k$ the set $\{P_1, P_2, \ldots, P_k\}$ is an orthogonal basis for the span of $\{X_1, X_2, \ldots, X_k\}$.

We begin by setting

$$P_1 = X_1 \tag{6.21}$$

Now assume that for some $k < m$ we have produced an orthogonal basis $\{P_1, P_2, \ldots, P_{k-1}\}$ for

$$\mathcal{V} = \text{span}\ \{X_1, X_2, \ldots, X_{k-1}\}$$

Then we define

$$\begin{aligned} Y_k &= \text{Proj}_{\mathcal{V}}(X_k) \\ P_k &= \text{Orth}_{\mathcal{V}}(X_k) = X_k - Y_k \end{aligned} \tag{6.22}$$

From formula (6.20),

$$Y_k = \frac{X_k \cdot P_1}{P_1 \cdot P_1} P_1 + \frac{X_k \cdot P_2}{P_2 \cdot P_2} P_2 + \cdots + \frac{X_k \cdot P_{k-1}}{P_{k-1} \cdot P_{k-1}} P_{k-1} \tag{6.23}$$

Thus, for example, $P_2$ is the component of $X_2$ orthogonal to $X_1$ (Figure 6.5, left) and $P_3$ is the component of $X_3$ orthogonal to the plane containing $X_1$ and $X_2$ (Figure 6.5, right).

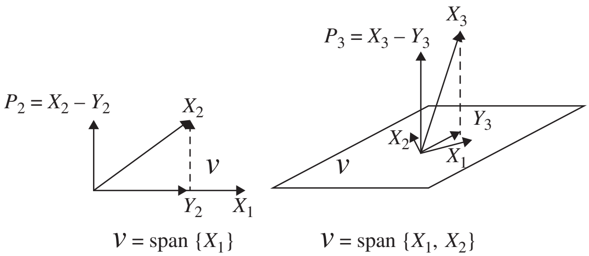



**FIGURE 6.5** The Gram-Schmidt process.

We make the following observations:

1. From formula (6.22 )

$$P_k \in X_k + \text{span}\ \{X_1, \dots, X_{k-1}\} \subset \text{span}\ \{X_1, \dots, X_k\}$$

2. From formula (6.22), $P_k$ is orthogonal to span $\{X_1, \dots, X_j\}$ for all $1 \le j \le k-1$. In particular, $P_k$ is orthogonal to $P_j$ for all $1 \le j \le k-1$.
3. From formulas (6.22) and (6.23),

$$\begin{aligned} X_k &= Y_k + P_k \\ &= u_{1,k}P_1 + \cdots + u_{k-1,k}P_{k-1} + P_k \end{aligned} \tag{6.24}$$

where

$$u_{j,k} = \frac{X_k \cdot P_j}{P_j \cdot P_j}$$

It follows that $X_k \in \text{span}\ \{P_1, \dots, P_k\}$. This, together with our first observation, shows that

$$\text{span}\ \{P_1, \dots, P_k\} = \text{span}\ \{X_1, \dots, X_k\}$$

This comment, together with our second observation, shows that $\{P_1, \dots, P_k\}$ *is an orthogonal basis for* span $\{X_1, \dots, X_k\}$. Continuing in this fashion, we produce an orthogonal basis for $\mathcal{W}$.

These comments prove the following important theorem. The basis $\mathcal{P}$ is referred to as the **Gram-Schmidt basis** for the ordered basis $\mathcal{B}$, and the process that produced $\mathcal{P}$ is called the **Gram-Schmidt process**.

**Theorem 6.8** *Let $\mathcal{B} = \{X_1, X_2, \dots, X_m\}$ be an ordered basis for an m-dimensional subspace $\mathcal{W}$ of $\mathbb{R}^n$ and let $\mathcal{P} = \{P_1, P_2, \dots, P_m\}$, where the $P_j$ are defined inductively* *by formula* (6.22). *Then the $P_i$ form an orthogonal basis for $\mathcal{W}$. In particular, every subspace of $\mathbb{R}^n$ has an orthogonal basis.*

### EXAMPLE 6.8

Find an orthogonal basis for the solution set to

$$2x + y + 3z - w = 0$$

*Solution.* The general solution to this system is

$$\left[-\tfrac{1}{2}, 1, 0, 0\right]^t s + \left[-\tfrac{3}{2}, 0, 1, 0\right]^t t + \left[\tfrac{1}{2}, 0, 0, 1\right]^t u$$

where $s$, $t$, and $u$ are arbitrary.

To avoid fractions, we multiply each basis element by 2, producing the basis

$$X_1 = [-1, 2, 0, 0]^t, \quad X_2 = [-3, 0, 2, 0]^t, \quad X_3 = [1, 0, 0, 2]^t$$

Then, from formulas (6.22) and (6.23),

$$\begin{aligned}
P_1 &= X_1 = [-1, 2, 0, 0]^t \\
Y_2 &= \frac{X_2 \cdot P_1}{P_1 \cdot P_1} P_1 = \tfrac{3}{5}\,[-1, 2, 0, 0]^t \\
P_2 &= X_2 - Y_2 = [-3, 0, 2, 0]^t - \tfrac{3}{5}\,[-1, 2, 0, 0]^t = \tfrac{2}{5}\,[-6, -3, 5, 0]^t
\end{aligned} \tag{6.25}$$

Now

$$\begin{aligned}
Y_3 &= \frac{X_3 \cdot P_1}{P_1 \cdot P_1} P_1 + \frac{X_3 \cdot P_2}{P_2 \cdot P_2} P_2 \\
&= \left(-\tfrac{1}{5}\right) P_1 + \left(-\tfrac{3}{14}\right) P_2 = \tfrac{1}{7}\,[5, -1, -3, 0]^t \\
P_3 &= X_3 - Y_3 = \tfrac{1}{7}\,[2, 1, 3, 14]^t
\end{aligned} \tag{6.26}$$

Formula (6.16) is, of course, equivalent to

$$Y = \mathrm{Proj}_{\mathcal{W}}(Y) + \mathrm{Orth}_{\mathcal{W}}(Y)$$

Since every subspace of $\mathbb{R}^n$ has an orthogonal basis (Theorem 6.8), we have proved the following theorem:

**Theorem 6.9** *Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$. Then every $Y \in \mathbb{R}^n$ may be written uniquely as*

$$Y = Y_0 + Y_1 \tag{6.27}$$

*where* $Y_0 \in \mathcal{W}$ *and* $Y_1 \in \mathcal{W}^\perp$. *In this decomposition,* $Y_0 = Proj_{\mathcal{W}}(Y)$ *and* $Y_1 = Orth_{\mathcal{W}}(Y)$.

### The *QR* Decomposition

Equation (6.24) has a very important consequence. Let $A = [A_1, A_2, \dots, A_n]$ be an $m \times n$, rank $n$ matrix, where the $A_i$ are the columns of $A$. Then $\mathcal{B} = \{A_1, A_2, \dots, A_n\}$ is a set of vectors in $\mathbb{R}^m$ whose span $\mathcal{W}$ is the column space of $A$. From the assumption on the rank of $A$, the set $\mathcal{B}$ is a linearly independent set and, hence, a basis for $\mathcal{W}$. Let $\mathcal{P} = \{P_1, P_2, \dots, P_n\}$ be the Gram-Schmidt basis of $\mathcal{W}$ produced by applying the Gram-Schmidt process to $\mathcal{B}$. Then, from equation (6.24),

$$\begin{aligned}
A_1 &= P_1 \\
A_2 &= u_{12}P_1 + P_2 \\
A_3 &= u_{13}P_1 + u_{23}P_2 + P_3 \\
&\vdots \\
A_n &= u_{1n}P_1 + u_{2n}P_2 + \cdots + u_{n-1,n}P_{n-1} + P_n
\end{aligned}$$

The preceding equations are equivalent to the single matrix equation

$$[A_1, A_2, \dots, A_n] = [P_1, P_2, \dots, P_n]\begin{bmatrix} 1 & u_{12} & u_{13} & \dots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ 0 & 0 & 1 & \cdots & u_{3n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{6.28}$$

which we write as

$$A = PU$$

where $U$ is the $n \times n$ matrix on the right in equation (6.28) and $P = [P_1, P_2, \dots, P_n]$ is the $m \times n$ matrix having the $P_i$ as columns. The matrix $U$ is an **upper triangular unipotent** matrix, meaning that it is upper triangular and all the entries on the main diagonal are equal to 1.

We may further refine this decomposition by normalizing the $P_i$, that is, letting $Q_i = P_i/|P_i|$ for $1 \leq i \leq n$. Then $P_i = Q_i|P_i|$ and, hence,

$$\begin{aligned}
P &= [P_1, P_2, \dots, P_n] = [|P_1|Q_1, |P_2|Q_2, \dots, |P_n|Q_n] \\
&= [Q_1, Q_2, \dots, Q_n]\begin{bmatrix} |P_1| & 0 & \dots & 0 \\ 0 & |P_2| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |P_n| \end{bmatrix} \\
&= QD
\end{aligned} \tag{6.29}$$

where $D$ is the $n \times n$ matrix on the right in equation (6.29) and $Q = [Q_1, Q_2, \ldots, Q_n]$ is the $m \times n$ matrix having the $Q_i$ as columns. The matrix $D$ is a **diagonal** matrix in that its only nonzero entries lie on the main diagonal. An $m \times n$ matrix, such as $Q$, whose columns form an orthonormal subset of $\mathbb{R}^m$, is said to be **partially orthogonal**. We study such matrices in Section 6.7 on page 396.

Equation (6.28) may be written as

$$A = QDU$$

The matrix $DU$, which is easily seen to be upper triangular, is often denoted by "$R$." The resulting factorization $A = QR$ is called the "$QR$ decomposition." We summarize these comments in the following theorem:

**Theorem 6.10** (QR Factorization Theorem). *Let $A$ be an $m \times n$ matrix with linearly independent columns. Then we may write $A = QDU$ where $Q$ is an $m \times n$ partially orthogonal matrix, $D$ is an $n \times n$ diagonal matrix with strictly positive entries, and $U$ is an $n \times n$ upper triangular, unipotent matrix. The matrix $R = DU$ is an upper triangular matrix.*

### EXAMPLE 6.9

Find the $QDU$ factorization for

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

*Solution.* The columns of $A$ are $A_1 = [1, 0, 1]^t$, $A_2 = [1, -2, 0]^t$, and $A_3 = [1, 1, 3]^t$. The Gram-Schmidt process produces

$$P_1 = A_1 = [1, 0, 1]^t \tag{6.30}$$

and

$$\begin{aligned} P_2 &= A_2 - \frac{A_2 \cdot P_1}{P_1 \cdot P_1} P_1 \\ &= A_2 - \tfrac{1}{2} P_1 \\ &= \tfrac{1}{2}[1, -4, -1]^t \end{aligned} \tag{6.31}$$

and

$$\begin{aligned} P_3 &= A_3 - \frac{A_3 \cdot P_1}{P_1 \cdot P_1} P_1 - \frac{A_3 \cdot P_2}{P_2 \cdot P_2} P_2 \\ &= A_3 - 2P_1 + \tfrac{2}{3} P_2 \\ &= \tfrac{1}{3}[-5, -4, -7] \end{aligned} \tag{6.32}$$

We solve the middle equation in each of formulas (6.30), (6.31), and (6.32) for $A_i$, finding

$$\begin{aligned} A_1 &= P_1 + 0P_2 + 0P_3 \\ A_2 &= \tfrac{1}{2}P_1 + P_2 + 0P_3 \\ A_3 &= 2P_1 - \tfrac{2}{3}P_2 + P_3 \end{aligned}$$

which implies

$$[A_1, A_2, A_3] = [P_1, P_2, P_3]\begin{bmatrix} 1 & \frac{1}{2} & 2 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

The $3 \times 3$ matrix on the right is $U$.

$$\begin{aligned} D &= \begin{bmatrix} |P_1| & 0 & 0 \\ 0 & |P_2| & 0 \\ 0 & 0 & |P_3| \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 3/2\sqrt{2} & 0 \\ 0 & 0 & \sqrt{10} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} Q &= \left[\frac{P_1}{|P_1|}, \frac{P_2}{|P_2|}, \frac{P_3}{|P_3|}\right] \\ &= \frac{\sqrt{2}}{30}\begin{bmatrix} 15 & 5 & -5\sqrt{5} \\ 0 & -20 & -4\sqrt{5} \\ 15 & -5 & -7\sqrt{5} \end{bmatrix} \end{aligned}$$

### Uniqueness of the *QR* Factorization

A natural and important question is to what extent the matrices $Q$, $D$, and $U$ in Theorem 6.10 on page 326 are unique. The following result, which is needed in Section 8.3, answers this question for square matrices.

**Theorem 6.11** *Suppose that $A$ is an $n \times n$ nonsingular matrix. Then the matrices $Q$, $D$, and $U$ in Theorem 6.10 on page 326 are unique.*

*Proof.* Suppose that

$$A = Q_1 D_1 U_1$$

is a second $QDU$ factorization of $A$. Let $R = DU$ and $R_1 = D_1U_1$. Note that $R$ and $R_1$ are upper triangular.

$Q_1$ is an $n \times n$ orthogonal matrix and hence invertible. Furthermore, since $Q_1R_1 = A = QR$,

$$\begin{aligned} Q_1R_1 &= QR \\ Q^{-1}Q_1 &= RR_1^{-1} \end{aligned}$$

It follows that

$$P = RR_1^{-1} \tag{6.33}$$

is orthogonal. On the other hand both the inverse and the product of an upper triangular matrix is upper triangular. Hence (1) $P$ is upper triangular and (2) $P^t = P^{-1}$ is upper triangular. But if $P$ and $P^t$ are both upper triangular, $P$ must be diagonal. In fact, since $P$ is also orthogonal, the diagonal entries of $P$ are $\pm 1$.

From (6.33), $R = PR_1$. Since both $R$ and $R_1$ have positive diagonal entries, $P = I$, proving that $R = R_1$, from which $Q = Q_1$ follows. $D = D_1$ also follows since the diagonal entries of $D$ and $D_1$ are, respectively, the diagonal entries of $R$ and $R_1$. Finally, $U = D^{-1}R = D_1^{-1}R_1 = U_1$, finishing the proof. □

**True-False Questions**: Justify your answers.

**6.11** $[1,1,1]^t$ is perpendicular to the span of $[1,-1,0]^t$ and $[2,2,-4]^t$ in $\mathbb{R}^3$.

**6.12** Let $\mathcal{W}$ be a two-dimensional subspace of $\mathbb{R}^5$. Suppose that $\{Q_1, Q_2\}$ and $\{P_1, P_2\}$ are two orthogonal bases for $\mathcal{W}$. Then for all $X \in \mathbb{R}^5$,

$$\frac{X \cdot Q_1}{Q_1 \cdot Q_1}Q_1 + \frac{X \cdot Q_2}{Q_2 \cdot Q_2}Q_2 = \frac{X \cdot P_1}{P_1 \cdot P_1}P_1 + \frac{X \cdot P_2}{P_2 \cdot P_2}P_2$$

**6.13** Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$ and let $X \in \mathcal{W}$. Then $\text{Proj}_{\mathcal{W}}(X) = \mathbf{0}$.

**6.14** Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$ and let $X \in \mathbb{R}^n$. Then $\text{Proj}_{\mathcal{W}}(X - \text{Proj}_{\mathcal{W}}(X)) = \mathbf{0}$.

## EXERCISES

**6.19** In each part let $\mathcal{W}$ be the subspace of $\mathbb{R}^4$ spanned by the set $\mathcal{B}$. Show that $\mathcal{B}$ is an orthogonal basis for $\mathcal{W}$ and find $\text{Proj}_{\mathcal{W}}([1,2,-1,-3]^t)$.

**(a)** ✓✓ $\mathcal{B} = \{[2,-1,-1,-1]^t,\ [1,3,3,-4]^t,\ [1,1,0,1]^t\}$

**(b)** $\mathcal{B} = \{[1,1,1,1]^t, [1,-2,1,0]^t, [1,1,1,-3]^t\}$

**6.20** Below, you are given two sets of vectors $\mathcal{B}_1$ and $\mathcal{B}_2$ in $\mathbb{R}^2$.

$$\begin{aligned} \mathcal{B}_1 &= \{[-1,1,-1]^t,\ [1,3,2]^t\} \\ \mathcal{B}_2 &= \{[3,1,4]^t,\ [-4,16,-1]^t\} \end{aligned}$$

**(a)** Show that both $\mathcal{B}_1$ and $\mathcal{B}_2$ are orthogonal sets.

**(b)** ✓Show that $\mathcal{B}_1$ and $\mathcal{B}_2$ both span the *same* subspace $\mathcal{W}$ of $\mathbb{R}^3$.

**(c)** ✓✓ Find $\text{Proj}_{\mathcal{W}}([1,2,2]^t)$ using formula (6.20) on page 321 and $\mathcal{B}_1$.

**(d)** Find $\text{Proj}_{\mathcal{W}}([1,2,2]^t)$ using formula (6.20) on page 321 and $\mathcal{B}_2$. You should get the same answer as in part (c). Why?

**(e)** Let $X = [x, y, z]^t$. Use formula (6.20) on page 321 to find $\text{Proj}_{\mathcal{W}}(X)$ using the basis $\mathcal{B}_1$. Then find a matrix $R$ such that $\text{Proj}_{\mathcal{W}}(X) = RX$.

**(f)** Repeat part (e) using basis $\mathcal{B}_2$. You should get the same matrix $R$.

**(g)** Show that the matrix $R$ from part (e) satisfies $R^2 = R$. Explain the geometric meaning of this equality.

**6.21** Repeat Exercises 6.20.a–6.20.g for the following sets with $\mathbb{R}^4$ in place of $\mathbb{R}^3$ and $[1,1,1,1]^t$ in place of $[1,2,2]^t$. ✓✓[(e)]

$$\mathcal{B}_1 = \{[1,2,1,-1]^t,\ [0,-1,1,-1]^t\}$$
$$\mathcal{B}_2 = \{[1,1,2,-2]^t,\ [3,13,-4,4]^t\}$$

**6.22** Use the Gram-Schmidt process to find an orthogonal basis for the subspaces of $\mathbb{R}^n$ spanned by the following ordered sets of vectors for the appropriate $n$:

**(a)** ✓✓ $\{[0,1,1]^t,\ [1,1,1]^t\}$

**(b)** $\{[-2,1,3]^t,\ [2,1,2]^t\}$

**(c)** ✓✓ $\{[1,2,1,1]^t,\ [-2,1,1,-1]^t,\ [1,2,1,3]^t\}$

**(d)** $\{[1,1,-1,1]^t,\ [1,1,0,0]^t,\ [1,1,1,-1]^t\}$

**6.23** Compute $\text{Proj}_{\mathcal{W}}([1,2,3]^t)$, where $\mathcal{W}$ is the subspace of $\mathbb{R}^3$ spanned by the vectors in Exercise 6.22.a. Repeat for the subspace spanned by the vectors in Exercise 6.22.b.

**6.24** Compute $\text{Proj}_{\mathcal{W}}([1,2,3,4]^t)$, where $\mathcal{W}$ is the subspace of $\mathbb{R}^4$ spanned by the vectors in Exercise 6.22.c. Repeat for the subspace spanned by the vectors in Exercise 6.22.d.

**6.25** For each of the following matrices $A$, (i) find an orthogonal basis for the column space of $A$, (ii) write the columns of $A$ as linear combinations of the basis elements, and (iii) use your results to find $Q$, $D$, and $U$ as in Theorem 6.10 such that $A = QDU$. [*Hint:* For (ii) use Theorem 2.5 on page 106.]

**(a)** ✓✓ $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ **(b)** $A = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 3 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

**6.26** In Exercises 6.22.a–6.22.c, let $A$ be the matrix whose $i$th column is the $i$th element of $\mathcal{B}$. Find $Q$, $D$, and $U$ as in Theorem 6.10 such that $A = QDU$.

**6.27** Let $A$ be an $m \times n$ matrix and let $\mathcal{W} \subset \mathbb{R}^n$ be the nullspace of $A$. Prove that for all $X \in \mathbb{R}^n$, $AX = A(\mathrm{Orth}_{\mathcal{W}}X)$. [*Hint:* $\mathrm{Orth}_{\mathcal{W}}(X) = X - \mathrm{Proj}_{\mathcal{W}}(X)$.]

**6.28** Prove that if $S$ is a subset of $\mathbb{R}^n$, then $S^{\perp}$ is a subspace of $\mathbb{R}^n$.

**6.29** Let $\mathcal{W}$ be the nullspace of

$$A = \begin{bmatrix} 1 & 3 & 1 & -1 \\ 2 & 6 & 0 & 1 \\ 4 & 12 & 2 & -1 \end{bmatrix}$$

**(a)** ✓✓Find an orthogonal basis for $\mathcal{W}$.

**(b)** ✓✓Let $X = [1, 1, 1, 1]^t$. Compute $X_1 = \mathrm{Orth}_{\mathcal{W}}([1, 1, 1, 1]^t)$.

**(c)** Verify that $AX = AX_1$. Use equation (6.16) to explain why this result is expected.

**6.30** Find an orthogonal basis for $S^{\perp}$ for the following sets of vectors. [*Hint:* Find the basis for the solution set to the homogeneous system of equations defined by $A_i \cdot X = 0$ where $S = \{A_1, \dots, A_k\}$.]

**(a)** ✓✓ $S = \{[1, 3, 1, -1]^t,\ [2, 6, 0, 1]^t,\ [4, 12, 2, -1]^t\}$

**(b)** $S = \{[1, 3, 1, -1]^t,\ [2, 6, 0, 1]^t\}$

**(c)** $S = \{[1, 1, 1, 1, 1]^t,\ [1, 1, 0, 1, -1]^t,\ [1, -1, 0, 0, 1]^t\}$

**6.31** ✓✓For the sets $S$ in Exercise 6.30, find an orthogonal basis for the subspace $\mathcal{W}$ of $\mathbb{R}^n$ spanned by $S$, where $S \subset \mathbb{R}^n$. Show that in each case the basis found in this exercise, together with the basis found in Exercise 6.30, constitutes an orthogonal basis for all $\mathbb{R}^n$.

**6.32** Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$. Let $\{P_1, P_2, \dots, P_k\}$ be an orthogonal basis for $\mathcal{W}$ and let $\{X_1, X_2, \dots, X_m\}$ be an orthogonal basis for $\mathcal{W}^{\perp}$. Show that the set

$$\mathcal{B} = \{P_1, P_2, \dots, P_k, X_1, X_2, \dots, X_m\}$$

is an orthogonal set. Prove that this set is a basis for $\mathbb{R}^n$ and, hence, $\dim(\mathcal{W}) + \dim(\mathcal{W}^{\perp}) = n$. [*Hint:* Use the Fourier theorem to show that $\mathcal{B}$ spans $\mathbb{R}^n$.]

**6.33** In Exercises 6.30.a–6.30.c, how does the coefficient matrix of the system you solved to find a basis for $S^{\perp}$ relate to the vectors in $S$? Use the rank-nullity theorem to prove that if $S = \{X_1, \dots, X_k\}$ is a linearly independent set of elements in $\mathbb{R}^n$, then $\dim S^{\perp} = n - k$. [(a)✓]

**6.34** Let $S = \{X_1, \dots, X_k\}$ be a set of elements in $\mathbb{R}^n$. Prove that $X \in \mathbb{R}^n$ is orthogonal to each element of $S$ if and only if $X$ is orthogonal to each element in the span of $S$—that is, if $\mathcal{W} = \operatorname{span} S$, then $\mathcal{W}^{\perp} = S^{\perp}$.

**6.35** Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$. Prove that $\dim \mathcal{W} + \dim \mathcal{W}^{\perp} = n$. [*Hint:* Choose a basis $\{X_1, \dots, X_k\}$ for $\mathcal{W}$ and apply the results of Exercises 6.33 and 6.34.]

**6.36** Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$ and let $\mathcal{B} = \{P_1, P_2, \dots, P_k\}$ be an orthogonal basis for $\mathcal{W}$. Let $X \in \mathcal{W}$. Without using any of the theorems in this section, prove that

$$X = x_1 P_1 + x_2 P_2 + \cdots + x_k P_k$$

where

$$x_i = \frac{X \cdot P_i}{P_i \cdot P_i}$$

For your proof, reason as in the proof of Theorem 6.4 on page 313. The difference between what you just proved and Theorem 6.4 is that in the current context $\mathcal{B}$ is not assumed to be a basis for $\mathbb{R}^n$.

**6.37** Let $\{P_1, P_2\}$ be an ordered orthogonal (but not orthonormal) basis for some subspace $\mathcal{W}$ of $\mathbb{R}^n$. Let $X$ and $Y$ be elements of $\mathcal{W}$ whose coordinate vectors with respect to these bases are $X' = [x_1, x_2]^t$ and $Y' = [y_1, y_2]^t$. Prove that

$$X \cdot Y = x_1 y_1 |P_1|^2 + x_2 y_2 |P_2|^2 \tag{6.34}$$

What is the corresponding formula for $|X|^2$?

**6.38** State a generalization of formula (6.34) that is valid an orthogonal basis $\mathcal{B} = \{P_1, P_2, \dots, P_m\}$ of a subspace $\mathcal{W}$ of $\mathbb{R}^n$ and prove your formula.

**6.39** Let $B \in \mathbb{R}^n$ and let $B_o = \mathrm{Proj}_{\mathcal{W}}(B)$, where $\mathcal{W}$ is a subspace of $\mathbb{R}^n$. Prove that, for all $Y \in \mathcal{W}$, $|B - (B_o + Y)|^2 = |B - B_o|^2 + |Y|^2$. It follows that $B_o$ is the closest point in $\mathcal{W}$ to $B$. [*Hint:* Use the result of Exercise 6.13 on page 318.]

### 6.2.1 Computer Projects

You are working for an engineering firm and your boss insists that you find one single solution to the following system:

$$\begin{aligned}
2x + 3y + 4z + 3w &= 12.9 \\
4x + 7y - 6z - 8w &= -7.1 \\
6x + 10y - 2z - 5w &= 5.9
\end{aligned}$$

You object, noting that:

(a) The system is clearly inconsistent: The sum of the first two equations contradicts the third.

(b) You need at least four equations to uniquely determine four unknowns. Even if the system were solvable, you could not produce just one solution.

The boss will not take no for an answer. Concerning objection (a), you are told that the system was obtained from measured data and any inconsistencies must be due to experimental error. Indeed, if any one of the constants on the right sides of the equations were modified by 0.1 unit in the appropriate direction, the system would be consistent.

Concerning objection (b), the boss says, "Do the best you can. We will pass this data on to our customers and they would not know what to do with multiple answers."

After some thought, you realize that projections can help with the inconsistency problem. The above system can be written in vector format as

$$x\begin{bmatrix}2\\4\\6\end{bmatrix}+y\begin{bmatrix}3\\7\\10\end{bmatrix}+z\begin{bmatrix}4\\-6\\-2\end{bmatrix}+w\begin{bmatrix}3\\-8\\-5\end{bmatrix}=\begin{bmatrix}12.9\\-7.1\\5.9\end{bmatrix}$$

You realize that this system would be solvable if the vector on the right of the equality were in the space spanned by the four vectors on the left. Using MATLAB's "rank" command, you quickly compute that the rank of the following matrix $A$ is 2, showing that these four vectors, in fact, span a plane (call it $\mathcal{W}$):

$$A=\begin{bmatrix}2&3&4&3\\4&7&-6&-8\\6&10&-2&-5\end{bmatrix}$$

Your idea is to let $B_o = \mathrm{Proj}_{\mathcal{W}}(B)$, where $B = [12.9, -7.1, 5.9]^t$. Since the system is so nearly consistent, $B_o$ should be very close to $B$. Furthermore, the system $AX = B_o$ should certainly be solvable, and one of the solutions should be what the boss is looking for.

Point (b) requires some further thought. However, you eventually come up with an idea, which is developed in Exercises 1–5.

## EXERCISES

**1.** Begin by computing an orthogonal basis for the span of the columns of $A$. You could use the Gram-Schmidt process. Instead, however, try typing help orth in MATLAB. Once you have found your orthogonal basis, use the Fourier theorem from the text to compute the projection $BO$. Finally, use rref( [A,BO]) to find all solutions to $AX = BO$. Express your solution in parametric form. (*Hint:* Let $W = \mathrm{Orth}(A)$. Then BO$'$ $*$W(:,i) computes the dot product of $BO$ with the $i$th basis element.)

**2.** Concerning objection (b), your first thought is that maybe you could just report the translation vector as the solution. But there is nothing special about the translation vector. The translation theorem says that the general solution can be expressed using any particular solution instead of the translation vector.

Your next idea, however, is a good one. Let $T$ be the translation vector and let $TO$ be its projection to the nullspace of $A$. Let $X = T - TO$. Then $X$ is what you report to your boss. Why is $X$ a solution?

Find $X$. For some help on computing $X$, enter help null at the MATLAB prompt.

**3.** Try computing $X$, starting with some solution other than $T$. You should get the same $X$. Why does it work out this way?

*Remark.* It can be shown that your $X$ is the solution of minimal length.

**4.** Although your solution was ingenious, MATLAB is way ahead of you. In MATLAB, let $B = [12.9, -7.1, 5.9]^t$ and ask MATLAB to compute $A\backslash B$. This solution is called the "pseudoinverse" solution, which we discuss in Section 6.7.

**5.** After giving the boss your answer, you delete all your data except for $A$ and $X$. A month later the customer calls, saying, we know that there must be other solutions. Will you please provide us with the general solution?

You are able to provide the desired information immediately by executing a single MATLAB command. How?

## 6.3 FOURIER SERIES: SCALAR PRODUCT SPACES

Imagine that you have been hired by a company that makes music synthesizers. You are assigned to design a model that produces a rasping noise. Being expert in sound design, you know that sound is produced by vibrating air. For simplicity, let us assume that your synthesizer produces only longitudinal waves—that is, the particles of air vibrate back and forth in the direction of the motion of the wave. Such a wave may be described by giving the displacement (forward or back) of the typical particle as a function of time. You also know that the wave graphed in Figure 6.6 produces an excellent rasp. (The time units are not in seconds. To be audible, your rasp would need to oscillate hundreds or even thousands of times a second. We assume that our time units are such that one oscillation of the rasp takes two units of time to complete.)

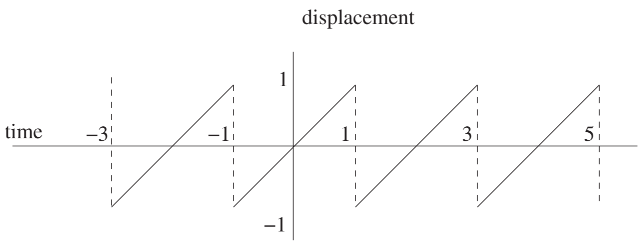



**FIGURE 6.6** A rasp.

For $-1 < x \leq 1$, the rasp is defined by the function

$$f(x) = x$$

Since the rasp is periodic of period 2, it also satisfies

$$f(x + 2) = f(x)$$

These two conditions uniquely determine the rasp for all $x$.

You also know where to buy computer chips that produce "pure tones". A pure tone is a sound that consists of a single note. Mathematically, pure tones are described by sine (or cosine) functions. For example, the function $q(x) = 2\sin 3\pi x$ produces the wave graphed in Figure 6.7. The maximum height (amplitude) of this wave is 2, which is the coefficient multiplying the sine function. The wave has three complete cycles between $-1$ and 1. In general, a function of the form

$$q_n(x) = \sin n\pi x \tag{6.35}$$

describes a wave with amplitude 1, having $n$ cycles between $-1$ and 1.

To build your synthesizer, you buy a chip that produces the pure tones described by the functions $q_i(x)$ in equation (6.35) for $1 \leq i \leq 4$. Your idea is to use this chip to produce four pure tones, each with a different amplitude. You then mix these tones to produce a sound that, hopefully, at least approximates the sound of a rasp. In physical terms, your problem is determining what amplitudes would best imitate the rasp.

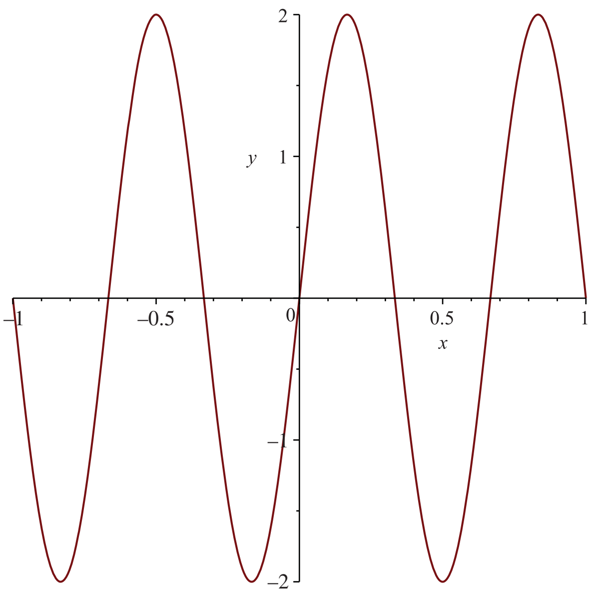



**FIGURE 6.7** A "pure" tone.

Mathematically, mixing simply means adding the graphs together. Thus, in mathematical terms, what you want is constants $c_1, c_2, c_3$, and $c_4$ (the amplitudes) such that the function

$$f_o = c_1 q_1 + c_2 q_2 + c_3 q_3 + c_4 q_4 \tag{6.36}$$

approximates the rasp function $f$ as closely as possible.

Notice that both the rasp function $f$ and the pure tone functions $q_k$ are periodic of period 2. This means that if $f_o$ approximates $f$ over the interval $-1 < x \leq 1$, the approximation will hold for all $x$. Thus, we shall think of our functions as being defined over the interval $(-1, 1]$.

To solve our approximation problem we shall use a quite remarkable line of reasoning. Let us imagine, for the moment, that instead of being functions, $f, q_1, q_2, q_3$, and $q_4$ are vectors in $\mathbb{R}^n$. Let $\mathcal{W}$ be the subspace of $\mathbb{R}^n$ spanned by $q_1$, $q_2$, $q_3$, and $q_4$. The "vector" $f_o = c_1 q_1 + c_2 q_2 + c_3 q_3 + c_4 q_4$ is then an element of $\mathcal{W}$. The constants $c_i$ should be chosen so that $f_o$ is as close to $f$ as possible while still lying in $\mathcal{W}$. Thus, $f_o$ should be the projection of $f$ onto $\mathcal{W}$.

This, of course, is nonsense, since $f$ is not a vector in $\mathbb{R}^n$; $f$ is a function. There are, however, ways in which functions are like vectors. A vector $[x_1, x_2, x_3]^t$ is uniquely defined by giving each of its entries $x_1$, $x_2$, and $x_3$. A function $f$ would be uniquely defined by giving each of its values $f(x)$. We add vectors by adding their entries; the third entry of $X + Y$ is $x_3 + y_3$. We add functions by adding their values; the value of $f + g$ at $x = 3$ is $f(3) + g(3)$. To multiply a vector by a scalar, we multiply each entry by the scalar. To multiply a function by a scalar, we multiply each value by the scalar. Just as all the entries of the zero vector equal zero, there is a zero function (denoted $\mathbf{0}$) having all its values equal to zero. In fact, the set of bounded, piecewise continuous functions defined over the interval $(-1, 1]$ is a vector space that contains both the rasp function $f$ and pure tone functions $q_n$. We shall call this space $\mathcal{F}((-1, 1])$. We define $\mathcal{F}(I)$ where $I$ is any interval in $\mathbb{R}$ similarly.

There is also a "dot product" for functions. For vectors, the dot product of two vectors is the sum of the products of their entries. For functions, the corresponding concept is the integral of the products of the functional values. For functions, we prefer not to denote the dot product by means of the symbol $f \cdot g$, which is too easily confused with the usual notion of product of functions. Instead, we use the symbol $(f, g)$. We call this the **scalar product** of the functions. Thus, for $f$ and $g$ in $\mathcal{F}((-1, 1])$, we define

$$(f, g) = \int_{-1}^{1} f(x)g(x)\,dx \tag{6.37}$$

We define the scalar product in $\mathcal{F}(I)$ for any interval $I$ similarly.

### EXAMPLE 6.10

Compute $(q_1, q_2)$, where $q_1$ and $q_2$ are as in formula (6.35).

*Solution.* From formula (6.37)

$$(q_1, q_2) = \int_{-1}^{1} (\sin \pi x)(\sin 2\pi x)\, dx$$

This integral is easily evaluated using the identity

$$\sin A \sin B = \frac{\cos(A - B) - \cos(A + B)}{2}$$

We get

$$\left( \frac{-\sin 3\pi x}{6\pi} + \frac{\sin \pi x}{2\pi} \right) \Big|_{-1}^{1} = 0$$

In $\mathbb{R}^n$, two vectors whose dot products are zero are called orthogonal. We use the same terminology for functions: two functions $f$ and $g$ are orthogonal if $(f, g) = 0$. Thus, according to Example 1, the functions $q_1$ and $q_2$ are orthogonal. In fact, it is not hard to prove the following proposition, where the $q_n$ are defined in formula (6.35).

**Proposition 6.1** *The functions $q_n(x) = \sin n\pi x$ form an orthogonal set of elements of $\mathcal{F}((-1, 1])$—that is, $(q_i, q_j) = 0$ for all $i \neq j$. Furthermore, $(q_n, q_n) = 1$ for all $n$.*

The scalar product in formula (6.37) has properties very similar to those of the dot product. (See Theorem 6.2 on page 311.) Specifically, it is easily seen that the following is true:

**Theorem 6.12** (Scalar Product Theorem). *Let $f$, $g$, and $h$ be functions in $\mathcal{F}((-1, 1])$. Let $c$ be a scalar. Then:*

*(a)* $(f, g) = (g, f)$ (*commutative law*)
*(b)* $(f + g, h) = (f, h) + (g, h)$ (*additive law*)
*(c)* $(cf, g) = c(f, g) = (f, cg)$ (*scalar law*)
*(d)* $(f, f) > 0$ *for* $f \neq \mathbf{0}$

Now, let us recall that our goal is to find constants $c_i$ such that the function $f_o$ of formula (6.36) approximates $f$ as closely as possible. As we have seen, if $f$ and $q_i$ were vectors, the answer would be found by letting $f_o$ be the projection of $f$ onto the subspace of $\mathbb{R}^n$ spanned by the $q_i$. The $q_i$, of course, are not vectors in $\mathbb{R}^n$. What if we were to ignore this fact and use Theorem 6.7 on page 321, replacing dot products with scalar products? This seems, in a way, reasonable in that the $q_i$ do form an orthogonal set of functions. What we find is

$$f_o = \frac{(f, q_1)}{(q_1, q_1)} q_1 + \frac{(f, q_2)}{(q_2, q_2)} q_2 + \frac{(f, q_3)}{(q_3, q_3)} q_3 + \frac{(f, q_4)}{(q_4, q_4)} q_4$$

We noted earlier that $(q_k, q_k) = 1$. The terms $(f, q_k)$ are computed by integration as before. Since $f(x) = x$ for $-1 < x \leq 1$,

$$\begin{aligned} c_k = (f, q_k) &= \int_{-1}^{1} x \sin(k\pi x)\, dx \\ &= \frac{\sin(k\pi x) - k\pi x \cos(k\pi x)}{k^2\pi^2}\Big|_{-1}^{1} \\ &= \frac{2(-1)^{k+1}}{k\pi} \end{aligned} \tag{6.38}$$

(We used integration by parts.) Hence, our candidate for $f_o$ is

$$f_o = \frac{2}{\pi}q_1 - \frac{2}{2\pi}q_2 + \frac{2}{3\pi}q_3 - \frac{2}{4\pi}q_4$$

Figure 6.8 shows the graph of $f_o$. The graph makes it clear that $f_o$ does a reasonable (but far from perfect) job of approximating a rasp. If we want a better approximation, we must use more pure tones. For example, if we used 10 pure tones, then [from formula (6.38)], we would have

$$f_o(x) = \sum_{1}^{10} \frac{2(-1)^{k+1}}{k\pi} \sin k\pi x \tag{6.39}$$

Our new approximation has the graph shown in Figure 6.9.

Is there any sense in which our original $f_o$ is the closest approximation to $f$ possible using the four pure tones? Answering this question is, once again, a matter of thinking about $f$ as a vector. If $X$ is a vector, we define the magnitude $|X|$ of $X$ to be $\sqrt{X \cdot X}$. We use the analogous definition for functions, except we usually denote the magnitude

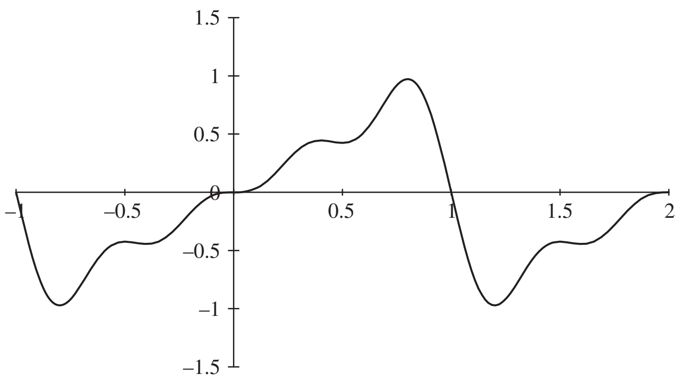



**FIGURE 6.8** Four-tone approximation.

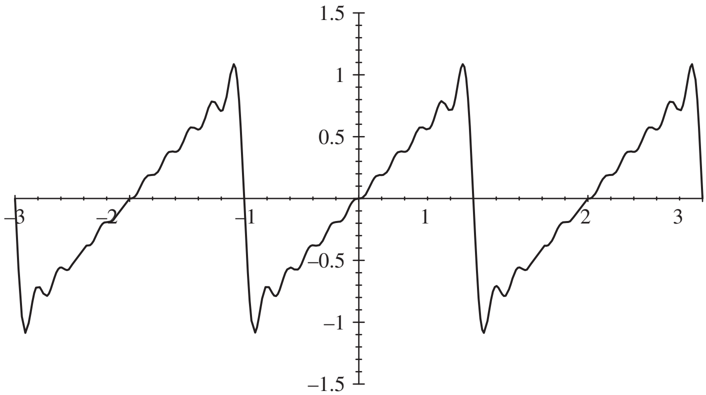



**FIGURE 6.9** Ten-tone approximation.

of a function by the symbol $||f||$ to prevent confusion of the magnitude of $f$ with the absolute value of $f$. Thus, for $f \in \mathcal{F}((-1, 1])$, we define

$$||f|| = \sqrt{(f, f)}$$

In formulas

$$||f|| = \sqrt{\int_{-1}^{1} (f(x))^2 \, dx}$$

This is very similar to vectors; the magnitude of a vector is the square root of the sum of the squares of its entries. The magnitude of a function is the square root of the integral of the squares of its values.

In what sense does $||f||$ represent the "size" of the function $f$? In Exercise 6.60 you will prove the following proposition:

**Proposition 6.2** *For all $f \in \mathcal{F}((-1, 1])$,*

$$\int_{-1}^{1} |f(x)| \, dx \leq \sqrt{2}||f||$$

The integral on the left side of this inequality represents the total area under the graph of $f$. The proposition says that if $||f||$ is small, then $f$ has very little area under its graph. If $f$ and $g$ belong to $\mathcal{F}((-1, 1])$, then we define the "distance" between $f$ and $g$ to be $||f - g||$. If the distance between $f$ and $g$ is small, then the graphs of $f$ and $g$ are close in the sense that there is very little area between the graphs (Figure 6.10).

This discussion is meant to lead to the following theorem. This result is called the Fourier sine theorem after Jean Fourier, the discoverer of this type of analysis,

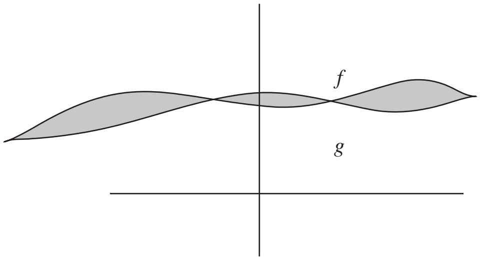



**FIGURE 6.10** "Close" functions.

and the coefficients $c_i$ are called Fourier coefficients. The proof is discussed in the exercises in the context of general scalar product spaces.

**Theorem 6.13** (Fourier Sine Theorem). *Let $f$ belong to $\mathcal{F}((-1, 1])$ and let $c_i$ be scalars for $i = 1, 2, \ldots, n$. Let*

$$f_o = c_1 q_1 + c_2 q_2 + \cdots + c_n q_n$$

*where*

$$q_k(x) = \sin k\pi x$$

*Then, setting $c_k = (f, q_k)$ yields the best approximation to $f$ by functions of the form $f_o$ in the sense that this choice of constants minimizes $||f - f_o||$.*

*Remark.* The approximations to waves by pure tones that we have been discussing have a remarkable amount of physical reality. We showed that it is possible to use appropriate combinations of pure tones to find better and better approximations to our rasp. It turns out that in real life waves, such as our rasp, behave as if they somehow were produced by combining an infinite number of pure tones precisely as we have described. For example, suppose that we fed our rasp (the real one, not an approximation) into an audio system. All audio systems, no matter how good, can only reproduce sounds whose pitch lies within a certain range—a tone whose pitch is either too high or too low will not be reproduced. This means that some of the pure tones used in producing the rasp will not be heard. If the amplifier reproduces, say, the first four pure tones perfectly, while completely attenuating all others, the sound we hear would be precisely that described by the graph in Figure 6.8. In essence, our amplifier projects us down to the subspace of $\mathcal{F}((-1, 1])$ spanned by the first four pure tones.

The method we used to solve our synthesizer problem occurs in many different contexts.

**Definition 6.7** *Let $\mathcal{V}$ be a vector space. A* ***scalar product*** *on $\mathcal{V}$ is a function $(\quad,\quad)$ that takes pairs of vectors from $\mathcal{V}$ and produces real numbers such that the properties (a)–(d) from Theorem 6.12 on page 336 hold, where $f$, $g$, and $h$ now represent elements* *of $\mathcal{V}$. A vector space on which a scalar product is given is called a* ***scalar product space****.*

Thus, for example, $\mathbb{R}^n$ is a scalar product space with the dot product as a scalar product. Our space $\mathcal{F}((-1, 1])$ is a scalar product space with the scalar product defined by formula (6.37).

Almost all the concepts already discussed above are meaningful for any scalar product space. For example, if $V$ is an element of $\mathcal{V}$, then we define

$$||V|| = \sqrt{(V, V)}$$

A set $P_1, P_2, \dots, P_n$ of elements of $\mathcal{V}$ is an orthogonal set if each of the $P_i$ is nonzero and $(P_i, P_j) = 0$ for all $i \neq j$. The projection of $V$ to the span of the $P_i$ is (from Theorem 6.7 in Section 6.2)

$$V_o = \frac{(V, P_1)}{(P_1, P_1)} P_1 + \frac{(V, P_2)}{(P_2, P_2)} P_2 + \frac{(V, P_3)}{(P_3, P_3)} P_3 + \cdots + \frac{(V, P_n)}{(P_n, P_n)} P_n$$

The coefficients

$$c_i = \frac{(V, P_i)}{(P_i, P_i)} \tag{6.40}$$

are called the Fourier coefficients of $V$. (Exercise 6.55 outlines a proof that $V_o$ is the closest point to $V$ in the span of the $P_i$.)

If $\mathcal{V}$ is $n$-dimensional, then the $P_i$ form a basis of $\mathcal{V}$ and

$$V = c_1 P_1 + \cdots + c_n P_n$$

where the $c_i$ are still given by (6.40). In the infinite dimensional case, the same formula can hold for certain orthogonal sets, except that the sum must be an infinite sum.

**Definition 6.8** *Let $\mathcal{B} = \{P_1, P_2, \dots, P_k, \dots\}$ be an infinite orthogonal subset of an infinite dimensional scalar product space $\mathcal{V}$. Then $\mathcal{B}$ is an* ***orthonormal basis*** *for $\mathcal{V}$ if for all $V \in \mathcal{V}$*

$$\lim_{n\to\infty} ||V - \sum_{i=1}^{n} c_i P_i|| = 0$$

*where the $c_i$ are given by (6.40). We then say that*

$$V = \sum_{i=1}^{\infty} c_i P_i.$$

Exercise 6.45 on page 342 illustrates the following theorem, which is beyond the scope of this text to prove:

**Theorem 6.14** *The set*

$$\mathcal{B} = \{\sin k\pi x \mid k \in \mathbb{Z}\} \cup \{\cos j\pi x \mid j \in \mathbb{Z}\}$$

*forms an orthogonal basis for* $\mathcal{F}((-1,1])$.

## EXERCISES

**6.40** A continuous function $f$ defined on $(-1,1]$ is said to be **even** if $f(-x) = f(x)$ for all $x \in (-1,1)$ and **odd** if $f(-x) = -f(x)$ for all $x \in (-1,1)$. Prove that if $f$ is even and $g$ is odd, $(f,g) = 0$.

**6.41** ✓✓The synthesizer company wants you to build a synthesizer to produce the wave shown on the left in Figure 6.11 which is defined by $f(x) = x^3$ on $(-1,1]$. Find the best approximation to $f$ using Theorem 6.13 on page 339 with $n = 5$.

**6.42** Now the synthesizer company wants you to build a synthesizer to produce the wave shown on the right in Figure 6.11 which is defined by $f(x) = x^2$ on $(-1,1]$. Find the best approximation to $f$ using Theorem 6.13 on page 339 with $n = 20$. *Hint:* Use Exercise 6.40.

**6.43** ✓✓Exercise 6.42 illustrates that it is not possible to approximate an even function over $(-1,1]$ by sine functions. This is because the sine functions are odd and any linear combination of odd functions is odd. Hence we try cosine functions.

**(a)** Show that $p_n(x) = \cos n\pi x$ for $n = 0, 1, 2, \ldots$ defines an orthogonal set of functions in $\mathcal{F}((-1,1])$.

**(b)** Compute $(p_n, p_n)$ for all $n$ (including $n = 0$).

**(c)** Compute the Fourier coefficients for $f(x) = x^2$, $x \in (-1,1]$, relative to the $p_n$.

**6.44** Next the synthesizer company wants a model that produces the saw-tooth wave shown in Figure 6.12, left. This wave is defined by the function $g(x) = |x|$ for

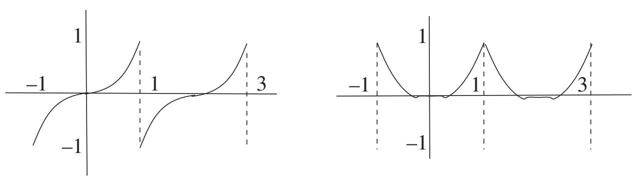



**FIGURE 6.11** Exercises 6.41 and 6.42.

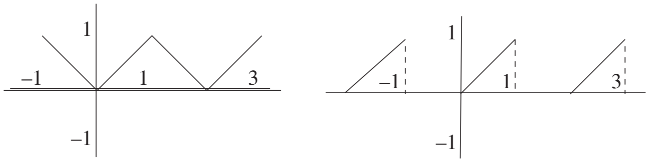



**FIGURE 6.12** Exercises 6.44 and 6.45.

$-1 \leq x \leq 1$. Compute the Fourier coefficients for $g$ relative to the $p_n$ from Exercise 6.43.

**6.45** ✓✓For a function that is neither odd nor even, we must use both sine and cosine functions to obtain a good approximation. Consider the wave graphed in Figure 6.12, right. This wave is described by the function

$$h(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 0 & -1 < x \leq 0 \end{cases}$$

**(a)** Show that the functions $q_i$, together with the $p_i$, form an orthogonal set of functions in $\mathcal{F}((-1, 1])$.

**(b)** Compute the Fourier coefficients of $h$ with respect to this orthogonal set. (*Note*: If you completed Exercise 6.44, this part can be done with essentially no work. Specifically, let $f$ be the rasp function from the text and let $g$ be the saw-tooth function from Exercise 6.44. How does the function $f + g$ relate to the function in the present problem?)

**6.46** Use the Gram-Schmidt process to find an orthogonal basis $\{L_0(x), L_1(x), L_2(x)\}$ in $\mathcal{F}((-1, 1])$ for $\mathcal{W} = \text{span}\,\{1, x, x^2\}$. Then compute the orthogonal projection of $x^3$ onto $\mathcal{W}$.

*Remark.* The functions $L_i(x)$ are the first three Legendre polynomials. In general, the **normalized Legendre polynomials** $\{L_0(x), L_1(x), \ldots, L_n(x)\}$ are the Gram-Schmidt basis in $\mathcal{F}((-1, 1])$ for the span $\mathcal{W}$ of the ordered basis $\{1, x, x^2, \ldots, x^n\}$.

**6.47** ✓✓Show that the functions $q_n(x) = \sin(2\pi nx/T)$ form an orthogonal subset of $\mathcal{F}((-T/2, T/2])$ with the scalar product defined following formula (6.37) on page 335. Suppose that $T = 0.002$ and that $f(x) = x$ for $-0.001 < x \leq 0.001$. Compute the corresponding Fourier coefficients. [*Warning*: $(q_n, q_n) \neq 1$.]

**6.48** We define a product on $\mathcal{F}(\mathbb{R})$ by setting

$$(f, g) = f(0)g(0) + f(1)g(1) + f(2)g(2) \tag{6.41}$$

**(a)** ✓✓Compute $(x^2 + x - 2, x^2 + 1)$.

**(b)** ✓✓Prove that ( , ) is a scalar product on $\mathcal{P}_2$ (Exercise 1.121 on page 93).

**(c)** Let $P_1(x) = (x-1)(x-2)$, $P_2(x) = x(x-2)$ and $P_3(x) = x(x-1)$. Show that $\{P_1, P_2, P_3\}$ is an orthogonal set in $\mathcal{P}_2$.

**(d)** ✓✓Compute the Fourier coefficients $c_i$ for $f(x) = x^2$ with respect to the orthogonal set $\mathcal{B}$ in part (c). Show by direct computation that $x^2 = c_1P_1(x) + c_2P_2(x) + c_3P_3(x)$ for all $x$.

**(e)** Let $f \in \mathcal{F}(\mathbb{R})$ and let $f_o = c_1P_1 + c_2P_2 + c_3P_3$ with the $c_i$ are given by (6.40) on page 340 where $(\cdot, \cdot)$ is as in (6.41), $V = f$, and $n = 3$. Show that $f_o(0) = f(0)$, $f_o(1) = f(1)$, and $f_o(2) = f(2)$.

*Remark.* This exercise illustrates the technique of **Legrange Interpolation** that produces an $n$th degree polynomial having specified $y$-values at $n$ given values of $x$.

**6.49** Prove that $(\cdot, \cdot)$ is a scalar product on $\mathbb{R}^2$ where

$$([x_1, y_1]^t, [x_2, y_2]^t) = 2x_1x_2 + 3y_1y_2$$

**6.50** Let $B$ be an invertible $n \times n$ matrix and let $A = B^tB$. Prove that

$$(X, Y) = X^tAY$$

a scalar product on $\mathbb{R}^n$. [*Hint:* Use formula (6.9) on page 311.]

**6.51** Prove Proposition 6.1 on page 336.

**6.52** Prove Theorem 6.12 on page 336. [(c)✓]

**6.53** Let $\mathcal{B} = \{V_1, V_2, \ldots, V_n\}$ be an orthogonal set of vectors in a scalar product space. Prove that $\mathcal{B}$ is linearly independent. For your proof, see Exercise 6.17 on page 318.

**6.54** Prove the following results where $V$ and $W$ are elements of a scalar product space. Interpret (a) and (b) geometrically.

**(a)** $||V + W||^2 = ||V||^2 + ||W||^2$ if and only if $(V, W) = 0$.

**(b)** ✓✓ $||V - W||^2 + ||V + W||^2 = 2(||V||^2 + ||W||^2)$.

**6.55** Let the notation be as in formula (6.40) on page 340 and the immediately preceding paragraph.

**(a)** Let $B = V - V_o$. Prove that $B$ is perpendicular to each of the elements $V_1, V_2, \ldots, V_n$. [*Hint:* See the proof of the Fourier theorem in Section 6.2.] How does it follow that $B$ is perpendicular to the span $\mathcal{W}$ of the $V_i$?

**(b)** Prove that $V_o$ is the closest element of $\mathcal{W}$ to $V$ in the sense that $||V - (V_o + Y)|| > ||V - V_o||$ for all nonzero $Y$ in $\mathcal{W}$. [*Hint:* Use part (a) along with Exercise 6.54 to show that $||V - (V_o + Y)||^2 = ||V - V_o||^2 + || - Y||^2$.]

**6.56** ✓✓Let $\mathcal{B} = \{V_1, V_2, \dots, V_n\}$ be an orthogonal set of vectors in a scalar product space. Let $W = c_1V_1 + c_2V_2 + \cdots + c_nV_n$ where the $c_i$ are scalars. Prove that

$$||W||^2 = |c_1|^2||V_1||^2 + |c_2|^2||V_2||^2 + \cdots + |c_n|^2||V_n||^2$$

**6.57** Let $\mathcal{V}$ be a finite-dimensional vector space that is also a scalar product space. Prove that $\mathcal{V}$ has an orthogonal basis. [*Hint:* Look in Section 6.2.]

**6.58** State and prove a version of the Cauchy-Schwarz theorem (Theorem 6.1 on page 310) that is valid in any scalar product space. (*For the proof, see Exercise 6.14 on page 318.*)

**6.59** Let $f$ and $g$ belong to $\mathcal{F}((-1,1])$. Use the result of Exercise 6.58 to prove that

$$\left(\int_{-1}^{1} f(x)g(x)\,dx\right)^2 \le \left(\int_{-1}^{1} (f(x))^2\,dx\right)\left(\int_{-1}^{1} (g(x))^2\,dx\right)$$

**6.60** Use Exercise 6.59 to prove Proposition 6.2 on page 338. For your proof, use $|f(x)|$ in place of $f(x)$ and make an appropriate choice for $g(x)$.

### 6.3.1 Application to Data Compression: Wavelets[1]

Our goal in this section is to discuss a new and rapidly developing area of mathematics, the theory of orthogonal wavelets. In Section 6.3 and its exercises, we focused on the orthogonal system of functions defined by the sine and cosine functions. We begin by introducing a different orthogonal system, the non-normalized Haar functions. They are real valued functions defined for all real numbers.

The "mother" Haar function (Figure 6.13, left) is

$$\tilde{H}_{0,0}(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2}, \\ -1, & \frac{1}{2} \le x < 1 \\ 0, & \text{all other x} \end{cases} \tag{6.42}$$

In general we define

$$\tilde{H}_{j,k}(x) = H_{0,0}(2^k(x - 2^{-k}j)),\ j,k \in \mathbb{Z} \tag{6.43}$$

Then $\tilde{H}_{j,k}$ is $\tilde{H}_{0,k}$ translated by a factor of $2^{-k}j$, either to the left or right, depending on the sign of $j$ (Figure 6.13, middle) and the graph of $\tilde{H}_{0,k}$ is the graph of $\tilde{H}_{0,0}$ either squeezed or expanded by a factor of $2^{-k}$ parallel to the $x$-axis (Figure 6.13, right), depending on the sign of $k$.

[1] In Section 6.4 we present an independent discussion of the discrete wavelet transform.

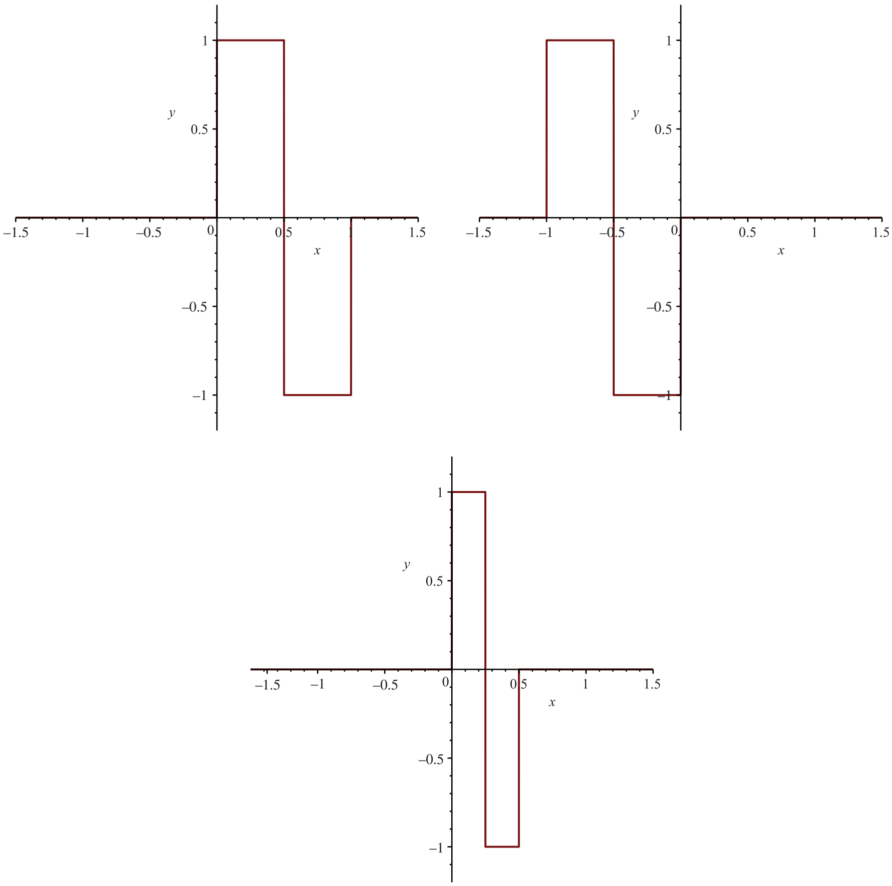


**FIGURE 6.13** $\tilde{H}_{0,0}$, $\tilde{H}_{-1,0}$ and $\tilde{H}_{0,1}$.

Then (Exercise 6.63)

$$\tilde{H}_{j,k}(x) = \begin{cases} 1, & 2^{-k}j \le x < 2^{-k}(j+0.5), \\ -1, & 2^{-k}(j+0.5) \le x < 2^{-k}(j+1) \\ 0, & \text{all other x} \end{cases} \tag{6.44}$$

We define the scalar product of two real valued functions $f$ and $g$ defined on all of $\mathbb{R}$ by

$$(f, g) = \int_{-\infty}^{\infty} f(x)g(x)\, dx \tag{6.45}$$

provided the integral exists. The following theorem implies that the set of Haar functions forms an orthogonal set of functions relative to this scalar product.

**Theorem 6.15** *For $j, k \in \mathbb{Z}$*

$$(\tilde{H}_{j,k}, \tilde{H}_{j',k'}) = 2^{-k}\delta$$

*where $\delta = 0$ if $(i,j) \neq (i',j')$ and equals 1 if $(i,j) = (i',j')$.*

*Proof.* Let

$$I_{j,k} = [j2^{-k}, (j+1)2^{-k}) \tag{6.46}$$

By definition, $\tilde{H}_{j,k}(x) \neq 0$ if and only if $x \in I_{j,k}$. For $j \neq j'$, $I_{j,k}$ and $I_{j',k}$ are disjoint. Hence

$$\tilde{H}_{k,j}(x)\tilde{H}_{k,j'}(x) = 0$$

showing first statement in Theorem 6.15 for $k = k'$. Thus we assume without loss of generality that $k' > k$.

Since

$$I_{j,k} = I_{2j,k+1} \cup I_{2j+1,k+1} \tag{6.47}$$

we have three cases:

1. $I_{j',k'} \subset I_{2j,k+1}$,
2. $I_{j',k'} \subset I_{2j+1,k+1}$, or
3. $I_{j',k'}$ and $I_{j,k,}$ are disjoint.

Hence

$$\tilde{H}_{j,k}(x)\tilde{H}_{j',k'}(x) = \epsilon\tilde{H}_{j',k'}(x)$$

where $\epsilon = 1$ in case 1, $\epsilon = -1$ in case 2, and $\epsilon = 0$ in case 3. The orthogonality follows since

$$\int_{-\infty}^{\infty} \epsilon H_{j',k'}(x) = 0 \tag{6.48}$$

The second statement in our theorem follows from the observation that $\tilde{H}^2_{j,k}$ equals 1 if and only if $x \in I_{j,k}$ so its integral is the length of $I_{j,k}$ which is $2^{-k}$. □

It follows from Theorem 6.15 that $\|\tilde{H}_{j,k}\| = 2^{\frac{k}{2}}$. Hence the set of functions

$$H_{j,k} = 2^{-\frac{k}{2}}\tilde{H}_{j,k}$$

is an orthonormal sequence of functions. These functions are the **Haar functions**.

For a given function $f$ the **Haar series** of $f$ is the series

$$\sum_{j,k} h(f)_{j,k} H_{j,k}(x) \tag{6.49}$$

where $h(f)_{j,k} = (f, H_{j,k})$. By analogy with the sine and cosine series, (Theorem 6.14 on page 341) we hope that the partial sum of this series for $|j| \leq a$ and $|k| \leq b$, where $a$ and $b$ are large numbers, should be a good approximation to $f(x)$.

Note that from (6.44) on page 345

$$h(f)_{j,k} = 2^{\frac{k}{2}} \left( \int_{2^{-k}j}^{2^{-k}(j+0.5)} f(x)\,dx - \int_{2^{-k}(j+0.5)}^{2^{-k}(j+1)} f(x)\,dx \right) \tag{6.50}$$

## EXAMPLE 6.11

Find $h(f)_{j,k}$ where $f(x) = x$.

*Solution.* From (6.50)

$$\begin{aligned} h(f)_{j,k} &= 2^{\frac{k}{2}} \left( \int_{2^{-k}j}^{2^{-k}(j+0.5)} x\,dx - \int_{2^{-k}(j+0.5)}^{2^{-k}(j+1)} x\,dx \right) \\ &= \frac{2^{\frac{k}{2}}}{2} \left( x^2\Big|_{2^{-k}j}^{2^{-k}(j+0.5)} - x^2\Big|_{2^{-k}(j+0.5)}^{2^{-k}(j+1)} \right) \\ &= 2^{\frac{k}{2}-1} \left( 2^{-2k}((j+.5)^2 - j^2) - 2^{-2k}((j+1)^2 - (j+.5)^2) \right) \\ &= -2^{-\frac{3k+4}{2}} \end{aligned} \tag{6.51}$$

In applications we are often interested in functions defined only on some interval $[0, 2^{-k_o}) = I_{0,k_o}$ where $k_o \in \mathbb{Z}$. More specifically, we consider the space $\mathcal{F}([0, 2^{-k_o}))$ defined on page 335. For $k \geq k_o$, $I_{j,k} \subset I_{0,k_o}$ if and only is $0 \leq j < 2^{k-k_o}$. Hence the Haar coefficient $h(f)_{j,k} = (f, H_{j,k})$ is only defined for such $i, j$, in which case it can still be computed by (6.50) on page 347. Note, however, that from (6.48) on page 346, the function 1 is orthogonal to all of the Haar functions. Let $1_{k_o}$ be the element of $\mathcal{F}(I_{0,k_o})$ which equals 1 for all $x \in I_{0,k_o}$. The following theorem, which we do not prove, implies that we can use Haar series to approximate elements of $\mathcal{F}([0, 2^{k_o}))$. (See Theorem 6.14 on page 341.)

**Theorem 6.16** *The set*

$$\mathcal{H}_{k_o} = \{H_{j,k} \mid k \geq k_o, 0 \leq j < 2^{k-k_o}\} \cup \{2^{\frac{k_o}{2}} 1_{k_o}\}$$

*forms an orthonormal basis for* $\mathcal{F}([0, 2^{k_o}))$.

We define successive approximations to $f \in \mathcal{F}([0, 2^{k_o}))$ by:

$$G_m(x) = h_0(f)1_{k_o} + \sum_{k=k_o}^{k_o+m} \sum_{j=0}^{2^k-1} h(f)_{j,k} H_{j,k}(x) \tag{6.52}$$

where $h(f)_{j,k}$ are as in (6.50) and

$$h_0(f) = 2^{\frac{k_o}{2}} (f, 1_{k_o}) = 2^{\frac{k_o}{2}} \int_0^{2^{k_o}} f(x)\, dx \tag{6.53}$$

### EXAMPLE 6.12

Find $h(f)_{j,k}$ and $h_0(f)$ for the function $f \in \mathcal{F}([0, 1))$ indicated below and graphed in Figure 6.14 where the vertical lines indicate discontinuities.

$$f(x) = \begin{cases} x, & 0 \le x < 0.5 \\ 1, & 0.5 \le x < 1 \end{cases}$$

*Solution.* Since $[0, 1) = [0, 2^0)$, $k_o = 0$. Thus we need to compute $h_0(f)$ and $h_{j,k}(f)$ for $k \ge 0$ and $0 \le j < 2^k$. Finding $h_0(f)$ is simple:

$$\begin{aligned} h_0(f) &= \int_0^1 f(x)\, dx \\ &= \int_0^{0.5} x\, dx + \int_{0.5}^1 1\, dx = \frac{5}{8} \end{aligned}$$

**Case 1:** $k = 0$ Since $0 \le j < 2^0$, we need only compute $h_{0,0}$. From (6.50)

$$h_{0,0}(f) = \left( \int_0^{.5} x\, dx - \int_{.5}^1 1\, dx \right) = -\frac{3}{8}$$

**Case 2:** $k > 0$ In this case the value of $h(f)_{j,k}$ depends on whether $I_{j,k} \subset [0, 0.5)$ or $I_{j,k} \subset [.5, 1)$. In the latter case, from (6.48) on page 346, $h(f)_{j,k} = 0$, and in the former case, from (6.51) on page 347, $h(f)_{j,k} = -2^{-\frac{3k+4}{2}}$. This occurs for $0 \le j < 2^{k-1}$.

Figure 6.14 shows graphs of $G_3$ and $G_7$. (Discontinuities are indicated with vertical lines.) They look very similar to the graph of $f$ shown on the left in Figure 6.14.

Now suppose that we want to transmit the graph of the function $f(x)$ in Example 6.12 over the internet so that it can be displayed on a computer screen of size $2^{10} \times 2^{10}$ pixels. The bottom edge of the screen has $2^{10}$ pixels corresponding tothe values $x_j = 2^{-10}j$, $0 \le j < 2^{10}$. To send an exact copy if the graph we need to send all of the values $y_j = f(x_j)$; hence we must send $2^{10}$ numbers.

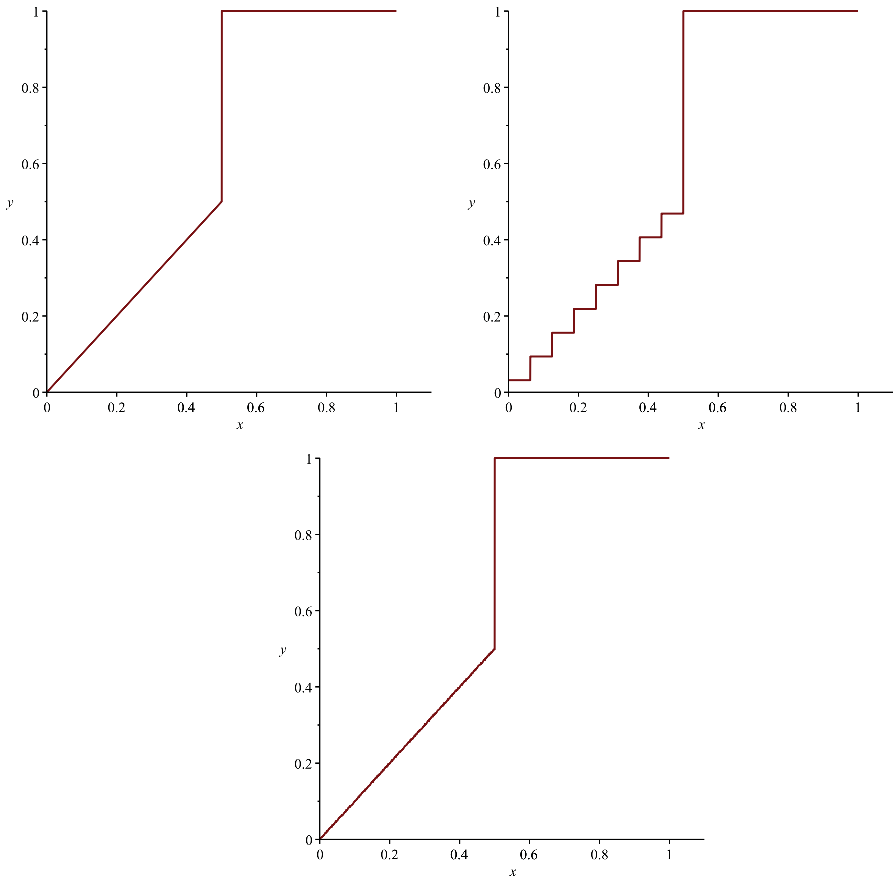



**FIGURE 6.14** $f(x)$, $G_3(x)$, and $G_7(x)$.

But for $k = 7$ the sum in (6.52) has 256 terms. We could send a list of these 256 values to the remote computer, which could then use them to reconstruct the series in (6.52) and plot the result. The display would be the graph on the right in Figure 6.14, which looks very much like the graph of $f(x)$. We have succeeded in reducing the amount of data we need to send (compressed the data) by 75%.

Note that in Figure 6.14, the main difference between the graph of $f(x)$ and the graphs of $G(5)$ and $G(7)$ is that the latter two are jagged and the first is straight. This is due to the discontinuity of the Haar functions. One suspects that we would obtain better approximations if we could replace $H_{0,0}$ with a continuous function. It turns out that this is indeed possible and correct.

Let $L^2(\mathbb{R})$ be the set of integrable functions $f(x)$ defined on all of $\mathbb{R}$ such that

$$\|f\| = \sqrt{(f,f)} < \infty$$

where the scalar product is defined in (6.45) on page 345. Then $L^2(\mathbb{R})$ is a scalar product space.

**Definition 6.9** *Let $\psi_{0,0} \in L^2(\mathbb{R})$. Then $\psi_{0,0}$ is is an* ***orthogonal wavelet*** *if the functions*

$$\psi_{j,k}(x) = \psi_{0,0}(2^k(x - 2^{-k}j)),\ j,k \in \mathbb{Z}$$

*forms an orthonormal basis for $L^2(\mathbb{R})$. (See Definition 6.8 on page 340.)*

As noted previously, the set of Haar functions is an orthonormal set. Thus the following theorem, whose proof we omit, adds to our knowledge only in that it implies that any function in $L^2(\mathbb{R})$ can be approximated arbitrarily close in norm by using sufficiently many terms of its Haar series.

**Theorem 6.17** *The function $H_{0,0}$ defined by (6.42) on page 344 is an orthogonal wavelet.*

Producing other examples of wavelets is difficult, despite the fact that there are an infinite number of them. The best we can do here is to give a graph of one, the Daubechies wavelet $\psi = \mathcal{D}_4$. (Figure 6.15).

Note that near $x = 0.5$ the graph of $f(x)$ and the graphs $G(3)$ and $G(7)$ in Figure 6.14 on page 349 rise suddenly from 0.5 to 1. This illustrates an important feature of the Haar series: large changes in a function near a particular value produce large changes in its Haar approximations near the same value. To understand why this happens note that from (6.50) on page 347, the value of $h(f)_{j,k}$ reflects the values of $f(x)$ on the interval $I_{j,k}$ while the term $h(f)_{j,k}H_{j,k}$ in the Haar series affects the values of the Haar series on the same interval. Similar comments would hold for approximations based on any wavelet $\psi$ for which the set of $x$ such that $\psi(x) \neq 0$ is contained in a finite interval $[a, b]$, such as the wavelet graphed in Figure 6.15. This makes wavelets useful for image compression.

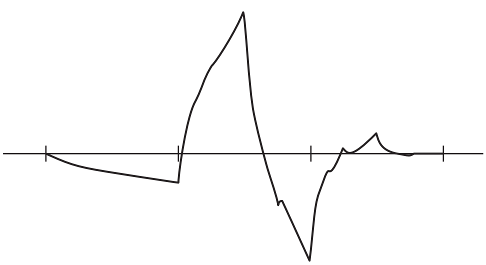


**FIGURE 6.15** A Daubechies Wavelet.

Before discussing this, we need to discuss Haar functions on $\mathbb{R}^2$. For $(x, y) \in \mathbb{R}^2$ and pairs of integers $j = (j_1, j_2)$ and $k = (k_1, k_2)$ let

$$H_{j,k}(x, y) = H_{j_1,k_1}(x)H_{j_2,k_2}(y)$$

Then, from Exercise 6.67 on page 353 these functions form an orthonormal subset of functions on $\mathbb{R}^2$ relative to the scalar product

$$(f, g) = \int_{\mathbb{R}^2} f(x, y)g(x, y)\, dxdy \tag{6.54}$$

For $f(x, y)$ a function on $\mathbb{R}^2$ and $j, k$ as above we define

$$h(f)_{j,k} = (f, H_{j,k})$$

whenever the integral in question exists. We can approximate functions of two variable in exactly the same way we approximated functions of one variable previously.

Suppose, for example, we want to send the image in Figure 6.16 over the internet to be displayed on the computer screen described on page 351. Let $f_{j_1,j_2}$ be the intensity of the pixel at $(j_1 2^{-10}, j_2 2^{-10})$, $0 \le j_1, j_2 < 2^{10}$ and let $f(x, y)$ be the function on the rectangle $[0, 1) \times [0, 1)$ defined by

$$f(x, y) = f_{j_1,j_2},\ j_1 2^{-10} \le x < (j_1 + 1)2^{-10},\ j_2 2^{-10} \le y < (j_2 + 1)2^{-10}$$

In analogy to (6.52) on page 348 we define successive approximations to $f(x)$ by

$$G_m(x, y) = (f, 1_{0,0}) + \sum_{k_1=0}^{m} \sum_{k_2=0}^{m} \sum_{j_1=0}^{2^{k_1}-1} \sum_{j_1=0}^{2^{k_2}-1} h(f)_{j,k} H_{j,k}(x, y)$$

where $1_{0,0}(x, y) = 1$ for all $(x, y) \in [0, 1) \times [0, 1)$.

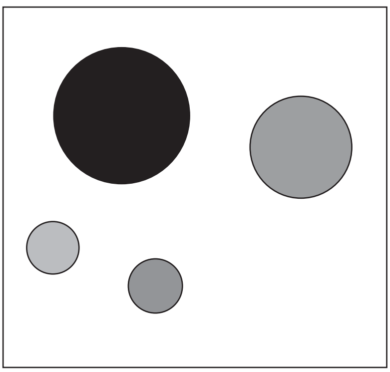


**FIGURE 6.16** Image.

The coefficients $(f, 1_{0,0})$ and $h(f)_{j,k}$ could be computed by, say, numerical integration, although techniques similar to those described in Exercise 6.66 would be better. Then, for a sufficiently large value of $m$, we send only the coefficients of $G(m)$ from which the remote computer can compute $G(m)$ and use it to produce a facsimile of Figure 6.16.

## EXERCISES

**6.61** Find $h(f)_{j,k}$ where $f(x) = x^2$.

**6.62** Find $h(g)_{j,k}$ and $h(g)_0$ for the function $g \in \mathcal{F}([0, 1))$defined below. State the values of $j$ and $k$ for which the formula is meaningful.

$$g(x) = x^2, 0 \leq x < 1$$

**6.63** Prove formula (6.44) on page 345.

**6.64** Prove formula (6.47) on page 346.

**6.65** Let $f(x) \in \mathcal{F}([0, 1))$ and $g(x) = f(2^{k_0}x)$ where $k_0 \in Z$. Then $g(x) \in \mathcal{F}([0, 2^{-k_0}))$. Prove that for $0 \leq j < 2^{k-k_0}$

$$\begin{aligned} h(g)_{j,k} &= 2^{-\frac{3k_0}{2}} h(f)_{j,k+k_0} \\ h(g)_0 &= 2^{-\frac{k_0}{2}} h(f)_0 \end{aligned}$$

[*Hint:* Make a change of variables in (6.50) on page 347.]

**6.66** For $f \in \mathcal{F}([0, 1))$ and integers $j, k$ where $k \geq 0$ and $0 \leq j < 2^k$ let

$$c(f)_{j,k} = 2^{k/2} \int_{2^{-k}j}^{2^{-k}(j+1)} f(x)\, dx$$

Let $C(f)_k$ and $H(f)_k$ be the $1 \times 2^k$ matrices defined by respectively

$$\begin{aligned} C(f)_k &= [c(f)_{0,k}, \dots, c(f)_{2^k-1,k}]^t \\ H(f)_k &= [h(f)_{0,k}, \dots, h(f)_{2^k-1,k}]^t \end{aligned}$$

**(a)** Show that

$$C(f)_k = S_{2^k}(C(f)_{k+1})$$

where $S_n : \mathbb{R}^{2n} \mapsto \mathbb{R}^n$ is defined by

$$S_n([x_1, x_2, \ldots, x_{2n}]^t) = \frac{1}{\sqrt{2}}[x_1 + x_2, x_3 + x_4, \ldots, x_{2n-1} + x_{2n}]^t$$

**(b)** Show that

$$H(f)_k = D_{2^{k+1}}(C(f)_{k+1})$$

where $D_n : \mathbb{R}^{2n} \mapsto \mathbb{R}^n$ is defined by

$$D_n([x_1, x_2, \ldots, x_{2n}]^t) = \frac{1}{\sqrt{2}}[x_1 - x_2, x_3 - x_4, \ldots, x_{2n-1} - x_{2n}]^t \tag{6.55}$$

**(c)** In Example 6.12 on page 348, find $C(f)_4$.

**(d)** Use the result of Exercise 6.66 to find $H(f)_3$ and $C(f)_3$, then $H(f)_2$ and $C(f)_2$, $H(f)_1$ and $C(f)_1$, and finally $H(f)_0$ and $C(f)_0$.

**(e)** Find $h_0(f)$ by noting that $h_0(f) = C_0(f)$.

**(f)** Show that the values of $H(f)_k$ found in parts (e) and (f) agree with the values found in Example 6.12.

*Remark.* This exercise illustrates the **Fast Haar Transform**: given $C(f)_{k+1}$, we use 6.66 and 6.66 to find $H(f)_k$ and $C(f)_k$. Repeating this process we can find all of the $H(f)_\ell$ and $C(f)_\ell$ for $0 \leq \ell < k$. Many wavelets admit a **Fast Wavelet Transform**.

**6.67** For $i = 1, 2$, let $f_i$ and $g_i$ belong to $\mathcal{F}((0, 1])$. Let $k_i(x, y) = f_i(x)g_i(y)$, $i = 1, 2$. Prove that in the scalar product (6.54) on page 351, $(k_1, k_2) = (f_1, f_2)(g_1, g_2)$.

### 6.3.2 Computer Projects

The following "program" can be used to generate the graphs of the Fourier sine series for the rasp from the text. Enter each line in MATLAB *exactly as shown*. Note in particular the period after the "y" in the last line.

```
hold on; axis([-1,4,-2.5,2.5]); grid on;
x=-1:.01:1; t=-1:.01:4;
n=1; sm=0; y=x;
b=trapz(x,y.*sin(n*pi*x)); sm=b*sin(n*pi*t)+sm; n=n+1;
```

The first time you enter the last line, you compute the first Fourier approximation to the rasp. [The command trapz(t,y.∗sin(n∗pi∗t)); tells MATLAB to integrate $x \sin(\pi x)$

over the interval $-1 \le t \le 1$ using a trapezoidal approximation.] If you press the up-arrow key and reenter this line you will compute the second Fourier approximation. (Note that the last term in this line advances the value of $n$ each time the line is entered.)

To plot your approximation and compare it with the rasp, enter the following line. The rasp will plot in red over three periods.

```
cla;plot(t,sm);plot(x,y,'r');plot(x+2,y,'r');plot(x+4,y,'r');
```

## EXERCISES

**1.** Plot the first, fourth, and tenth approximations to the rasp. (If you need to restart the program, just reenter the second-to-last line. If you have closed the Figure window, you will also need to reenter the first line to reinitialize the graph.)

**2.** Use the up-arrow key to move up to the next to last line and change the expression y=x; to y=-1∗(x<0)+1∗(x>=0);. This represents a function $y(x)$ that equals $-1$ if $x < 0$ and equals 1 if $x \ge 0$. Compute and graph the best approximations to $y$ using 4, 8, and 20 sine functions.

The wave you are approximating is called a "square wave." Notice the "ear-like" peaks on your graph at the discontinuities of the wave. These peaks are referred to as the "Gibbs phenomenon." They are quite pronounced, even after 20 terms of the Fourier series. Their existence shows that it takes a very high fidelity amplifier to accurately reproduce a square wave. For this reason, square waves are sometimes used to test the fidelity of an amplifier.

**3.** Use the up-arrow key to move up to the next to last line in the program and change the expression y=x; to y=(1+x.^ 2).^ (-1);. (Again, note the placement of the periods.) Compute and graph the fourth approximation to this function. Explain why your graphs do not agree. (See Exercises 6.42 and 6.43 on page 341.)

**4.** Modify the above program to approximate for the function in Exercise 3 using cosine functions. Plot the approximations using one, four, and eight cosine functions. (*Note:* Be sure to reenter the line that defines $y$ before running your program as this line also initializes $n$ and $sm$.)

**5.** You might have wondered about the placement of the periods in the above program. To understand their meaning, let A $= [1, 2, 3, 4, 5]$ and ask MATLAB to compute the following quantities (one at a time): A∗A, B=A.∗A, and C=B.^ (1/2). Try the same operations on a $2 \times 3$ matrix of your choice. Describe what you think these operations do.

This type of multiplication of matrices is called "Hadamard multiplication." It is necessary in our program because $x$ is a vector. (Ask MATLAB to display $x$ by entering x at the prompt. What size is $x$? What do its entries represent?) The quantities y, sin(n∗pi∗x), and sm are also vectors, while b is a scalar.

## 6.4 ORTHOGONAL MATRICES

In Section 3.1, we saw that rotation about the origin by a fixed angle $\theta$ in $\mathbb{R}^2$ defines a linear transformation. This transformation is described by the matrix

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{6.56}$$

One of the most striking features of rotation is that it does not change lengths of vectors; if $X$ is rotated into $X'$, then $X$ and $X'$ have the same length (Figure 6.17, left).

Another example of a linear transformation that does not change length is the transformation defined by the matrix

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{6.57}$$

Multiplication by $M$ transforms $[x, y]^t$ into $[x, -y]^t$. Geometrically, this describes a reflection in the $x$ axis (Figure 6.17, right). We shall show that the only linear transformations of $\mathbb{R}^2$ into itself that do not change length are compositions of reflections and rotations.

There also exist transformations of $\mathbb{R}^3$ into itself that do not change length. For example, multiplication by

$$R_\theta^z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.58}$$

rotates by $\theta$ radians around the $z$ axis (Figure 6.18). (Notice that multiplication of $[x, y, z]^t$ by this matrix multiplies $[x, y]^t$ by the two-dimensional rotation matrix $R_\theta$ without changing $z$.)

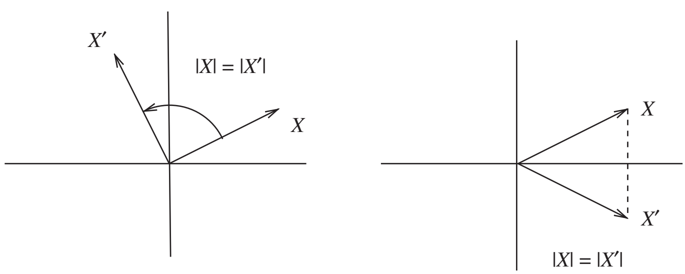



**FIGURE 6.17** Rotation and reflection.

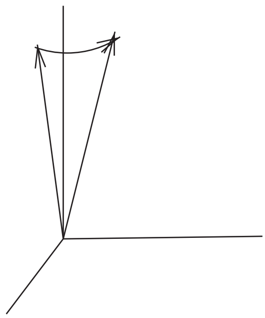


**FIGURE 6.18** A three-dimensional rotation.

This certainly should not change lengths of the vectors. Similarly, multiplication by either of the following matrices should not change the vectors' lengths, since each describes rotation about a different axis in $\mathbb{R}^3$.

$$R_\theta^x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad R_\theta^y = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{6.59}$$

In general, an $n \times n$ matrix $A$ is said to be **orthogonal** if $T_A$ does not change lengths. More formally:

**Definition 6.10** *An $n \times n$ matrix $A$ is orthogonal if*

$$|AX| = |X|$$

*for all $X \in \mathbb{R}^n$.*

We think of $n \times n$ orthogonal matrices as defining transformations analogous to rotations and reflections in $\mathbb{R}^2$ and $\mathbb{R}^3$.

*Remark.* Recall that points in $\mathbb{R}^n$ represent vectors whose initial point is at the origin. Thus, our definition says only that multiplication by an orthogonal matrix preserve the lengths of such vectors. However, it follows that multiplication by an orthogonal matrix, in fact, preserves the lengths of all vectors. To see this, let $X$ and $Y$ be points in $\mathbb{R}^n$ and let $A$ be an $n \times n$ orthogonal matrix. Multiplication by $A$ transforms the vector from $X$ to $Y$ into the vector from $AX$ to $AY$. The length of this latter vector is

$$|AX - AY| = |A(X - Y)| = |X - Y|$$

which is the length of the former vector.

In $\mathbb{R}^2$, rotation preserves angles as well as lengths. There is a sense in which multiplication by general orthogonal matrices also preserve angles. From formula (6.8) on page 310, the angle $\phi$ between two nonzero vectors $X$ and $Y$ in $\mathbb{R}^n$ is defined by

$$\cos\phi = \frac{X \cdot Y}{|X|\,|Y|} \tag{6.60}$$

Let $A$ be an $n \times n$ orthogonal matrix and let $X' = AX$ and $Y' = AY$. The angle $\phi'$ between $X'$ and $Y'$ satisfies

$$\cos\phi' = \frac{X' \cdot Y'}{|X'|\,|Y'|}$$

Since $|X| = |X'|$ and $|Y| = |Y'|$, we see that $\cos\phi' = \cos\phi$ if and only if $X \cdot Y = X' \cdot Y'$. The following theorem proves this equality.

**Theorem 6.18** (Preservation of Angles Theorem). *Suppose that A is an orthogonal $n \times n$ matrix. Then, for all X and Y in $\mathbb{R}^n$,*

$$AX \cdot AY = X \cdot Y$$

*Proof.* From the law of cosines [formula (6.6) on page 310]

$$2(X \cdot Y) = |X|^2 + |Y|^2 - |X - Y|^2 \tag{6.61}$$

Hence,

$$\begin{aligned} 2(AX \cdot AY) &= |AX|^2 + |AY|^2 - |AX - AY|^2 \\ &= |X|^2 + |Y|^2 - |X - Y|^2 = 2(X \cdot Y) \end{aligned}$$

proving our theorem. □

In the $2 \times 2$ case, the orthogonal matrices are easy to describe. Let $A$ be some $2 \times 2$ orthogonal matrix and let $I_1 = [1, 0]^t$ and $I_2 = [0, 1]^t$. The vectors $I_1$ and $I_2$ have length 1 and are perpendicular to each other. The vectors $A_1 = AI_1$ and $A_2 = AI_2$ are the columns of $A$. From the orthogonality of $A$ and the preservation of angles theorem, $A_1$ and $A_2$ also have length 1 and are perpendicular to each other. In particular, the $A_i$ represent points on the unit circle.

Let

$$A_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

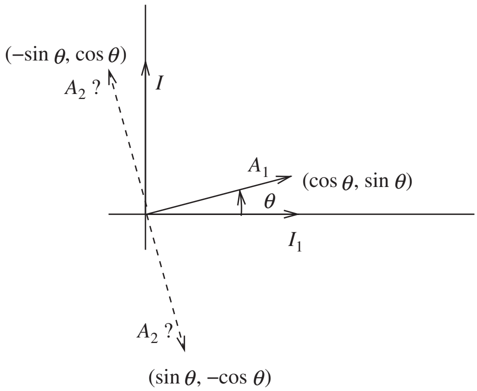



**FIGURE 6.19** Orthogonal matrices in $R^2$.

where $\theta$ is the angle between $A_1$ and the $x$ axis. (See Figure 6.19.) Since $A_2$ is perpendicular to $A_1$, there are only two possibilities for $A_2$:

$$A_2 = \pm \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

In the first case,

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

which is just the rotation matrix $R_\theta$.

In the second case,

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

This is $R_\theta M$, where $M$ is the reflection matrix from formula (6.57). Thus, the transformation defined by multiplication by a $2 \times 2$ orthogonal matrix is either a rotation, a reflection (if $\theta = 0$ in the above formula), or a composition of a rotation and a reflection.

In higher dimensions, the description of the general orthogonal transformation is less simple. However, some of what we have said carries over to the general case.

**Proposition 6.3** *Let A be an $n \times n$ orthogonal matrix. The set $\mathcal{B} = \{A_1, A_2, \dots, A_n\}$, where $A_k$ is the kth column of A, is an orthonormal basis of $\mathbb{R}^n$.*

*Proof.* The standard basis $\{I_1, I_2, \dots, I_n\}$ of $\mathbb{R}^n$ is an orthonormal set. Our proposition follows from the preservation of angles theorem together with the observation that, for $1 \le k \le n$, $A_k = AI_k$. □

*Remark.* Mathematical notation is not always consistent or logical. Proposition 6.3 says that the columns of an *orthogonal matrix* form an orthonormal set of vectors. One would think that such matrices would be called "orthonormal" matrices and that orthogonal matrices would have perpendicular columns that are not necessarily of length 1. Unfortunately, this is not "standard" terminology.

There is a very useful way of stating Proposition 6.3 using matrix multiplication. Let $A = [A_1, A_2, \dots, A_n]$ be an $n \times n$ orthogonal matrix with columns $A_i$ and consider $B = A^t A$. Since the $i$th row of $A^t$ is $A_i^t$,

$$b_{ij} = A_i^t A_j = A_i \cdot A_j$$

From the orthogonality of the $A_i$, $b_{ij} = 0$ if $i \neq j$. Also $b_{ii} = A_i \cdot A_i = |A_i|^2 = 1$. This exactly describes the identity matrix. Thus, Proposition 6.3 may be summarized in the matrix equality

$$A^t A = I$$

Hence (using Theorem 3.10 on page 188), $A^t = A^{-1}$. This makes orthogonal matrices wonderful: to invert them, just take the transpose!

Actually, Proposition 6.3 characterizes orthogonal matrices.

**Theorem 6.19** *Let $A$ be an $n \times n$ matrix. Then $A$ is orthogonal if and only if either of the following statements holds:*

*(a)* $A^t A = I$.
*(b) The columns of $A$ form an orthnormal subset of* $\mathbb{R}^n$.

*Proof.* We noted above that both (a) and (b) hold for orthogonal matrices and that (a) and (b) are equivalent statements. Thus, we need only prove that if $A^t A = I$, then $A$ is orthogonal. However, using formula (6.9) on page 311, we see that for all $X \in \mathbb{R}^n$

$$\begin{aligned} |AX|^2 &= AX \cdot AX = (AX)^t AX \\ &= X^t A^t AX = X^t IX \\ &= X^t X = X \cdot X = |X|^2 \end{aligned}$$

proving the orthogonality. □

The following examples demonstrate the use of parts (a) and (b) of Theorem 6.19 in proving orthogonality.

**EXAMPLE 6.13**

Prove that the matrix $A$ is orthogonal:

$$A = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{2} & 1 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix}$$

*Solution.* It is easily checked that $A^tA = I$.

**EXAMPLE 6.14**

Find numbers $a$, $b$, $c$, and $d$ such that the matrix $A$ is orthogonal:

$$A = d\begin{bmatrix} 2 & -1 & a \\ -1 & 2 & b \\ 2 & 2 & c \end{bmatrix}$$

*Solution.* The first column of $A$ is $A_1 = [2d, -d, 2d]^t$. Its length is

$$\sqrt{(2d)^2 + (-d)^2 + (2d)^2} = 3|d|$$

Since the length must be 1, we see that $d = \pm 1/3$. This also makes $A_2$ have length 1.

We note that $A_1 \cdot A_2 = 0$. This is good since otherwise, the problem would not be solvable. For the third column to be perpendicular to the first two, we require

$$\begin{aligned} 2a - \phantom{2}b + 2c &= 0 \\ -a + 2b + 2c &= 0 \end{aligned}$$

The general solution is $[a, b, c]^t = [-2, -2, 1]^t c$. Hence, the third column of $A$ is

$$\pm\frac{c}{3}[-2, -2, 1]^t$$

which has length

$$1 = \frac{|c|}{3}\sqrt{(-2)^2 + (-2)^2 + 1} = |c|$$

Thus, $c = \pm 1$ and $a = b = -2c$. Our final answer is then

$$A = \pm\frac{1}{3}\begin{bmatrix} 2 & -1 & -2(\pm 1) \\ -1 & 2 & -2(\pm 1) \\ 2 & 2 & \pm 1 \end{bmatrix}$$

where $\pm 1$ terms in the third column all have the same sign while the sign of the $\pm\frac{1}{3}$ term is independent of the sign of the $\pm 1$ terms.

### Householder Matrices

At the beginning of this section we noted that in $\mathbb{R}^2$ reflection about the $x$ axis, parallel to the $y$ axis, is described by multiplication by the orthogonal matrix $M$ from formula (6.57). We may in fact reflect parallel to any vector $P$ in $\mathbb{R}^n$, about the hyperplane $\mathcal{W} = P^\perp$ of all vectors perpendicular to $P$. In $\mathbb{R}^2$ or $\mathbb{R}^3$, the reflection $X'$ of a given vector $X$ is defined by:

1. $X$ and $X'$ lie on the opposite side of $\mathcal{W}$.
2. $X - X'$ is parallel to $P$.
3. The distance from $X'$ to $\mathcal{W}$ equals the distance from $X$ to $\mathcal{W}$.

(See Figure 6.20.)

It is possible to define reflections in any dimension. Let $P \in \mathbb{R}^n$. From Theorem 6.9 on page 324, every $X \in \mathbb{R}^n$ may be written as

$$X = X_1 + X_2$$

where $X_1 = \mathrm{Proj}_P(X)$ lies in the line through $P$ and $X_2 = \mathrm{Orth}_P(X)$ belongs to $\mathcal{W} = P^\perp$.

From Figure 6.21 and formula (6.15) on page 319, the reflection of $X$ about $\mathcal{W}$ is

$$\begin{aligned} X' &= -X_1 + X_2 \\ &= -\mathrm{Proj}_P(X) + \mathrm{Orth}_P(X) \\ &= -\frac{X \cdot P}{P \cdot P}P + \left(X - \frac{X \cdot P}{P \cdot P}P\right) \\ &= X - 2P\frac{P \cdot X}{P \cdot P} \end{aligned}$$

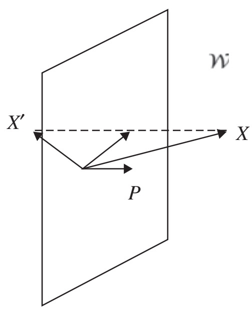



**FIGURE 6.20** Reflection in $\mathbb{R}^3$.

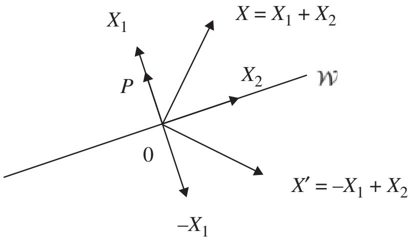



**FIGURE 6.21** Formula for Reflections.

Note that

$$P(P \cdot X) = P(P^t X) = (PP^t)X$$

Hence,

$$X' = \left(I - \frac{2}{P \cdot P} PP^t\right) X$$

Thus, the reflection is described by multiplication by the matrix

$$M_P = I - \frac{2}{|P|^2} PP^t \tag{6.62}$$

The matrix $M_P$ is referred to as the **Householder matrix** defined by the vector $P$.

## EXAMPLE 6.15

Find the matrix $M$ that describes reflection about the plane $2x + 3y + z = 0$ in $\mathbb{R}^3$.

*Solution.* The vector $P = [2, 3, 1]^t$ is perpendicular to the plane. From formula (6.62) the desired reflection is

$$\begin{aligned} M_P &= I - \frac{2}{2^2 + 3^2 + 1^2} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} [2, 3, 1] \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{14} \begin{bmatrix} 4 & 6 & 2 \\ 6 & 9 & 3 \\ 2 & 3 & 1 \end{bmatrix} \\ &= \frac{1}{7} \begin{bmatrix} 3 & -6 & -2 \\ -6 & -2 & -3 \\ -2 & -3 & 6 \end{bmatrix} \end{aligned}$$

which finishes the example.

Geometric considerations suggest that if we reflect $X$'s reflection, we should get $X$. We also expect reflection to preserve lengths and, hence, the Householder matrices should be orthogonal matrices. Thus, we expect that

$$\begin{aligned} M_P^2 &= I \\ M_P^t M_P &= I \end{aligned} \tag{6.63}$$

The first equation implies that $M_P = M_P^{-1}$, which, along with the second equation, implies $M_P = M_P^t$—that is, $M_P$ is a symmetric matrix. The following proposition proves that this geometric intuition is correct.

**Proposition 6.4** *For all $P \in \mathbb{R}^n$, the Householder matrix $M_P$ from formula (6.62) is orthogonal and satisfies $M_P^2 = I$ and $M_P^t = M_P$.*

*Proof.* The symmetry holds because

$$\begin{aligned} M_P^t &= I^t - \frac{2}{|P|^2}(PP^t)^t \\ &= I - \frac{2}{|P|^2}(P^t)^t P^t = M_P \end{aligned}$$

Next we prove that $M_P^2 = I$. Note first that

$$M_P = I - 2UU^t$$

where $U = P/|P|$. Since $1 = U \cdot U = U^t U$, we see

$$\begin{aligned} M_P^2 &= (I - 2UU^t)^2 \\ &= I - 4UU^t + 4(UU^t)(UU^t) \\ &= I - 4UU^t + 4U(U^t U)U^t \\ &= I - 4UU^t + 4UU^t = I \end{aligned}$$

as desired. The orthogonality also follows since, from the above computations, $M_P^t = M_P = M_P^{-1}$. □

*Remark.* The Householder matrices are important computational tools. For example, they form the basis for one of the most efficient and accurate methods for computing the $QR$ factorization discussed in Section 6.2 (see Exercise 6.86 below). They also play a key role in one of the more common algorithms for computing the eigenvalues of large matrices. Finally, they occur prominently in the first steps of one common algorithm for computing the singular value decomposition, which is discussed in Section 6.7. (See Exercises 6.133 and 6.137 on page 406.)

**True-False Questions**: Justify your answers.

**6.15** There exist $4 \times 4$ orthogonal matrices with rank 3.

**6.16** If $A$ is an $n \times n$ orthogonal matrix and $B$ is an $n \times 1$ matrix, then the equation $AX = B$ has a unique solution.

**6.17** Multiplication by an orthogonal matrix transforms congruent triangles into congruent triangles.

**6.18** If multiplication by an orthogonal matrix transforms a given parallelogram into a square, then the parallelogram was a square to begin with.

**6.19** The following matrix is orthogonal:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

**6.20** Suppose that $A = [A_1, A_2, A_3]$ is an orthogonal matrix where $A_i$ are the columns of $A$. Then $|A_1 + A_2 + A_3| = \sqrt{3}$.

**6.21** Let $Q$ be an $n \times n$ orthogonal matrix. Then

$$Q^2(Q^t)^3 Q^{-1}(Q^t)^{-1}(Q^{-1})^t = Q$$

**6.22** The following matrix is orthogonal:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [1, 1, 1]$$

**6.23** The matrix $A$ from question 8 satisfies $A^2 - A = I$.

## EXERCISES

**6.68** ✓✓Let $X = [4, 2\sqrt{2}, 6]^t$. Show by direct computation that $|AX| = |X|$, where $A$ is as in Example 6.13 on page 360.

**6.69** ✓✓In the text, we used geometric intuition to justify the orthogonality of the matrices from formulas (6.56) and (6.57) on page 355. Use Theorem 6.19 on page 359 to prove that they are orthogonal.

**6.70** Change just one column of each of the following matrices to make them orthogonal:

**(a)** $\frac{1}{5}\begin{bmatrix} 3 & 4 & 3 \\ -4 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

**(b)** ✓ $\frac{1}{25}\begin{bmatrix} 16 & -12 & -15 \\ -12 & 9 & -20 \\ -15 & -20 & 1 \end{bmatrix}$

**(c)** $\frac{1}{9}\begin{bmatrix} 8 & 1 & -4 \\ -4 & 1 & -7 \\ 1 & 1 & 4 \end{bmatrix}$

**6.71** In each part, find numbers $C$ or $a$, $b$, $c$, and $d$ such that the given matrix $A$ is orthogonal:

**(a)** ✓✓

$$A = C\begin{bmatrix} 9 & -12 & -8 \\ -12 & -1 & -12 \\ -8 & -12 & 9 \end{bmatrix}$$

**(b)**

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} & a \\ 1 & -1 & 0 & b \\ 1 & 1 & -\sqrt{2} & c \\ 1 & -1 & 0 & d \end{bmatrix}$$

**6.72** Check your work in Exercise 6.71.a by choosing a specific vector $X$ and showing (by direct computation) that $|CAX| = |X|$. Choose your $X$ so that none of its entries are zero.

**6.73** Show that the matrix $M_P$ in Example 6.15 on page 362 is orthogonal and satisfies $M_P^2 = I$.

**6.74** ✓Let $A$ and $B$ be $n \times n$ orthogonal matrices. Prove that $AB$ is orthogonal by showing that for all $X \in \mathbb{R}^n$, $|(AB)X| = |X|$.

**6.75** ✓✓Redo Exercise 6.74 by showing that $(AB)^t(AB) = I$.

**6.76** Prove that the inverse of an orthogonal matrix is orthogonal.

**6.77** Suppose that the set of rows of an $n \times n$ matrix $A$, considered as vectors in $\mathbb{R}^n$, is an orthonormal set. Prove that $A$ is orthogonal. [*Hint:* Consider $A^t$.]

**6.78** Prove that permutation matrices are orthogonal matrices. (See Definition 3.9 on page 210.) [*Hint:* Use the result from Exercise 6.77.]

**6.79** Use the result from Exercise 6.78 together with the remark at the end of Exercise 3.113.a on page 213 to prove that the transpose of a permutation matrix $P$ is a permutation matrix.

**6.80** Let $P$ be an $n \times n$ permutation matrix and let $A$ be an $m \times n$ matrix. Prove that $AP$ is $A$ with its columns permuted according to the permutation that defines $P^{-1}$. (See the remark at the end of Exercise 3.113.a on page 213.) [*Hint:* Use the result from Exercise 6.79 together with $(AP)^t = P^tA^t$.]

**6.81** Is it possible to find a $3 \times 2$ matrix with orthonormal *rows*? Explain.

**6.82** Give an example of a $3 \times 2$ matrix $A$ with all entries nonzero that has orthonormal columns. Compute $AA^t$ and $A^tA$. Which is the identity? Prove that the similar product equals $I$ for any $A$ that has orthonormal columns.

**6.83** In parts (a)-(-c), find the matrix $M$ such that multiplication by $M$ describes reflection about the given line, plane, or hyperplane. (Use formula (6.62) on page 362.)

**(a)** $2x - 5y = 0$ **(b)** ✓✓ $x + y - 3z = 0$

**(c)** $2x + y - z - 3w = 0$

**6.84** For each given vector $X \in \mathbb{R}^n$, find a scalar $k$, a vector $P \in \mathbb{R}^n$, and a Householder matrix $M_P$ [formula (6.62) on page 362] such that $M_P X = kI_1$ where $I_1$ is the first standard basis element of $\mathbb{R}^n$ [formula (6.62) on page 362].

**(a)** ✓ $X = [3, 4]^t$ **(b)** $X = [1, 1]^t$

**(c)** $X = [1, 2, 2]^t$

**6.85** The following exercise generalizes Exercise 6.84.

**(a)** Use formula (6.62) on page 362 to prove that if $P \in \mathbb{R}^n$ and $X = cP$, then $M_P X = -X$.

**(b)** Use formula (6.62) on page 362 to prove that if $P$ and $X$ are elements of $\mathbb{R}^n$ and $X \cdot P = 0$, then $M_P X = X$.

**(c)** Let $X$ and $Y$ be nonzero elements of $\mathbb{R}^n$ with $|X| = |Y|$. Let $P = X - Y$. Show that $M_P X = Y$, where $M_P$ is the Householder matrix from formula (6.62) on page 362. [*Hint:* First show that $(X - Y) \cdot (X + Y) = 0$. Then use the equality $X = \frac{1}{2}[(X + Y) - (X - Y)]$ together with the result from parts (a) and (b).]

**(d)** Prove that for $X$ and $Y$ in $\mathbb{R}^n$ there is a $P$ in $\mathbb{R}^n$ such that $M_P X = Y$ if and only if $|X| = |Y|$.

**6.86** ✓Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

**(a)** Find a $P \in \mathbb{R}^3$ and scalars $a$, $b$, $c$, $d$, $e$, and $f$ such that

$$M_P A = \begin{bmatrix} \sqrt{2} & a & b \\ 0 & c & d \\ 0 & e & f \end{bmatrix}$$

where $M_P$ is the $3 \times 3$ Householder matrix from formula (6.62) on page 362. [*Hint:* Use Exercise 6.85.c to find a $P$ such that $M_P[1, 0, 1]^t = [\sqrt{2}, 0, 0]^t$. This $P$ exists since $\left|[1, 0, 1]^t\right| = \sqrt{2}$.]

**(b)** Find a $Q \in \mathbb{R}^2$ and scalars $u, v, w$ such that

$$M_Q \begin{bmatrix} c & d \\ e & f \end{bmatrix} = \begin{bmatrix} u & v \\ 0 & w \end{bmatrix}$$

where $c$, $d$, $e$, and $f$ are as in part (a) and $M_Q$ is a $2 \times 2$ Householder matrix as in formula (6.62) on page 362.

**(c)** Let $M_P$ and $M_Q$ be the matrices from parts (a) and (b). Show that

$$\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & M_Q \end{bmatrix} M_P A = R$$

where $R$ is an upper triangular matrix and the partitioned matrix on the left is $3 \times 3$.

*Remark.* It follows from Exercise 6.86.c and the orthogonality of the Householder matrices that

$$A = QR$$

where

$$Q = M_P^t \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & M_Q^t \end{bmatrix}$$

From Exercise 6.74 (or Exercise 6.75) $Q$ is an orthogonal matrix. Thus, we have obtained the $QR$ factorization of $A$. The reader should compare the work here with that from Example 6.9 on page 326 where the $QDU$ factorization was computed for this same matrix. The Householder matrices may be used in a similar manner to that demonstrated here to compute the $QR$ factorization for any $m \times n$ rank $n$ matrix. The resulting algorithm is one of the most common and most efficient for computing $QR$ factorizations.

## Discrete Wavelet Transform

When sending large data sets over the internet it is often necessary to compress the data. Often one transforms a large data set into a smaller data set and then sends the smaller data set. The remote computer then attempts to use this smaller set to reconstruct the larger set. Since we have lost information, it is impossible to reconstruct the larger set exactly. The goal of data compression is to find compression and decompression algorithms that minimize the inaccuracy. In the following sequence of exercises we describe one such technique: The discrete Wavelet Transform.

**6.87** Let $S_n$ and $D_n$ be the following $n \times 2n$ matrices

$$S_n = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix},$$

$$D_n = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix} \tag{6.64}$$

**(a)** Show that the $n \times n$ matrix $W_{2n} = \begin{bmatrix} S_n \\ D_n \end{bmatrix}$ is an orthogonal matrix. [*Hint:* Consider rows.]

**(b)** We wish to send the data represented by the vector $X$ over the internet where

$$X = [0.988, 0.987, 0.543, 0.544, 0.541, 0.443, 0.444, 0.736]^t$$

Compute $H_4 = D_4X$ and $C_4 = S_4X$, rounded to 2 decimal places. [Ans. :$H_4 \approx H'_4 = [0, 0, 0.07, -0.21]^t$, $C_4 \approx C'_4 = [1.40, 0.77, 0.70, 0.84]^t$.]
We send $H'_4$ and $C'_4$.

**(c)** Compute $X' = W_8^{-1}[H'_4, C'_4]^t$. This is the value the remote computer uses as an approximation to $X$. *Hint* $W_8^{-1} = W_8^t$.

*Remark.* This exercise illustrates the **Discrete Haar Transform**: given $X \in \mathbb{R}^{2n}$, we compute $H_n = D_nX$ and and $C_n = S_nX$. We round this data to a prespecified tolerance, letting the rounded vectors be denoted by $H'_n$ and $C'_n$. The remote computer then sets $X' = W_n[H'_n, C'_n]^t$. Note that from (6.64), if $k$ is odd and $x_k$ is sufficiently close to $x_{k+1}$ then the corresponding entry of $H'_n$ will be zero. Hence we send less data where the original data is not changing rapidly.

**6.88** One can replace $S_n$ and $D_n$ with other matrices. For $a_\pm = 1 \pm \sqrt{3}$ and $b_\pm = 3 \pm \sqrt{3}$ let

$$S = \frac{1}{4\sqrt{2}} \begin{bmatrix} a_+ & b_+ & b_- & a_- & 0 & 0 & 0 & 0 \\ 0 & 0 & a_+ & b_+ & b_- & a_- & 0 & 0 \\ 0 & 0 & 0 & 0 & a_+ & b_+ & b_- & a_- \\ b_- & a_- & 0 & 0 & 0 & 0 & a_+ & b_+ \end{bmatrix}$$

$$D = \frac{1}{4\sqrt{2}} \begin{bmatrix} a_- & -b_- & b_+ & -a_+ & 0 & 0 & 0 & 0 \\ 0 & 0 & a_- & -b_- & b_+ & -a_+ & 0 & 0 \\ 0 & 0 & 0 & 0 & a_- & -b_- & b_+ & -a_+ \\ b_+ & -a_+ & 0 & 0 & 0 & 0 & a_- & -b_- \end{bmatrix}$$

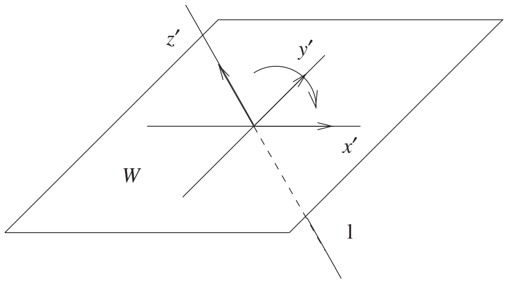



**FIGURE 6.22** Rotation in a plane.

**(a)** It turns out that the matrix $W = \begin{bmatrix} S \\ D \end{bmatrix}$ is orthogonal. For example, let $S_i$ and $D_i$, $i = 1, 2, 3, 4$, be respectively the rows of $S$ and $D$. Show that (i) for all $i$, $|S_i|^2 = |D_i|^2 = 1$, (ii) $S_1 \cdot D_1 = S_2 \cdot S_4 = D_3 \cdot D_2 = 0$.

**(b)** Repeat Exercises 6.87.b and 6.87.c for the data in Exercise 6.87.b with $D_4$, $S_4$, and $W_8$ replaced by $D$, $S$, and $W$.

*Remark.* The wavelet transform described in this exercise is one of a large class of wavelet transforms defined in 1988 by the mathematician Ingrid Daubechies. This class is very important in applications.

### 6.4.1 Computer Projects

Let $R^x_\theta$ and $R^y_\theta$ be as defined on page 356 and let

$$A = R^x_{\pi/6} R^y_{\pi/4}$$

Since the product of two orthogonal matrices is orthogonal, $A$ is orthogonal. The purpose of this set of exercises is to demonstrate that $A$ defines a rotation around a fixed axis by a particular angle as in Figure 6.22.

Points on this axis remain fixed under the rotation. Thus, if $X$ is on the axis of rotation, it satisfies $AX = X$. Notice that this equation is equivalent to

$$(A - I)X = \mathbf{0}$$

## EXERCISES

1. Use MATLAB to compute $A$.
2. We can find an X on the axis of rotation with the command

```
X=null(A-eye(3))
```

Note that eye(n) is MATLAB's notation for the $n \times n$ identity matrix and null produces an orthonormal basis for the nullspace of the matrix.

**3.** Plot the line segment from $-X$ to $X$. For this set W=[X,-X] and enter

```
plot3(W(1,:),W(2,:),W(3,:),'r')
```

This tells MATLAB to plot the columns of $W$ (in red), using the first row as the $x$ coordinates, the second row as the $y$ coordinates, and the third row as $z$ coordinates. MATLAB connects the columns with a line.

**4.** The plane $\mathcal{W} = X^{\perp}$ is called the "plane of rotation." It consists of all vectors $Y$ such that $X \cdot Y = 0$. Since $X \cdot Y = X^t Y$, we can find an orthonormal basis for $\mathcal{W}$ by entering Q=null(X') into MATLAB. The columns of $Q$ form the basis. To see this basis, we must plot the line segment from the first column of Q to the origin and the line segment from the origin to the second column of Q. We begin by entering Z=[0;0;0], followed by

```
P=[Q(:,1),Z,Q(:,2)]
hold on; axis equal;
plot3(P(1,:),P(2,:), P(3,:),'y')
```

From this angle, the basis elements appear neither orthogonal nor normal. To select a better viewing angle, enter view([1,3,10]). This causes us to look down toward the origin in the direction of the vector $[1, 3, 10]$. Now it is at least plausible that the basis is orthonormal.

**5.** We expect that multiplication by $A$ rotates elements in $P$ around the axis of rotation by a fixed angle. To test this, enter the following line. Explain what you see.

```
P=A*P;plot3(P(1,:),P(2,:),P(3,:),'y')
```

**6.** Enter the line given in Exercise 5 fifteen times and explain what you see. (Use the up arrow to avoid retyping.)

**7.** Determine the angle of the rotation in $\mathcal{W}$. (*Hint:* Use formula (6.8), page 310.)

## 6.5 LEAST SQUARES

Suppose that you are employed by a statistical consulting firm that has been hired by a large Midwestern university to help make admissions decisions. Each applicant provides the university with three pieces of information: the score on the SAT exam, the score on the ACT exam, and a high school GPA (0–4 scale). The college wishes to know what weight to put on each of these numbers.

You begin by collecting data from the previous year's freshman class. In addition to the admissions data, you collect the student's current (college) GPA (0–4 scale). A partial listing of your data might look like this:

| *SAT* | *ACT* | *GPA* | *C-GPA* |
|---|---|---|---|
| 600 | 30 | 3.0 | 3.2 |
| 500 | 28 | 2.9 | 3.0 |
| 750 | 35 | 3.9 | 3.5 |
| 650 | 30 | 3.5 | 3.5 |
| 550 | 25 | 2.8 | 3.2 |
| 800 | 35 | 3.7 | 3.7 |

Ideally, you would like numbers $x_1$, $x_2$, and $x_3$ such that, for all students,

$$(\mathrm{SAT})x_1 + (\mathrm{ACT})x_2 + (\mathrm{GPA})x_3 = \text{C-GPA}$$

These numbers would tell you exactly what weight to put on each each piece of data. Finding such numbers would be equivalent to solving the system $AX = B$, where $X = [x_1, x_2, x_3]^t$,

$$A = \begin{bmatrix} 600 & 30 & 3.0 \\ 500 & 28 & 2.9 \\ 750 & 35 & 3.9 \\ 650 & 30 & 3.5 \\ 550 & 25 & 2.8 \\ 800 & 35 & 3.7 \end{bmatrix}, \quad B = \begin{bmatrix} 3.2 \\ 3.0 \\ 3.5 \\ 3.5 \\ 3.2 \\ 3.7 \end{bmatrix} \tag{6.65}$$

Statistically, it is highly unlikely that the solution $X$ exists: we do not expect to be able to predict a student's college GPA with certainty on the basis of admissions data. Thus, we feel confident that the system $AX = B$ is inconsistent. More mathematically, from Theorem 1.8 on page 76, the set of $B$ in $\mathbb{R}^6$ for which $AX = B$ is solvable is the column space of $A$. The dimension of the column space is at most 3 since $A$ has only three columns. The probability that a randomly selected vector in $\mathbb{R}^6$ lies in a given three-dimensional subspace of $\mathbb{R}^6$ is practically nil. (Think about the probability of a randomly selected vector in $\mathbb{R}^3$ lying in some given plane.)

The mathematical interpretation, however, suggests a "geometric" solution to our problem using the concept of projections discussed in Section 6.2. Specifically, let $B_0 = \mathrm{Proj}_{\mathcal{W}}(B)$ be the projection of $B$ to the column space $\mathcal{W}$ of $A$. $B$ as possible. Any such $X$ is referred to as a **least-squares solution** to the system $AX = B$.

In Section 6.2 we used orthogonal bases to compute projections. There is, however, a way of finding $B_0$ that does not require first finding an orthogonal basis for $\mathcal{W}$. We first note the following proposition. Recall that if $\mathcal{W}$ is a subspace of $\mathbb{R}^n$, then $\mathcal{W}^{\perp}$ is the subspace of $\mathbb{R}^n$ consisting of all vectors orthogonal to every element of $\mathcal{W}$. (See Definition 6.6 on page 320.)

**Proposition 6.5** *Let $C$ be an $m \times n$ matrix and let $\mathcal{W} \subset \mathbb{R}^n$ be the column space of $C$. A vector $Y \in \mathbb{R}^n$ belongs to $\mathcal{W}^{\perp}$ if and only if $C^t Y = \mathbf{0}$.*

*Proof.* Let $C = [C_1, C_2, \dots, C_n]$, where the $C_i$ are the columns of $C$. From Exercise 6.34 on page 330, a vector $Y \in \mathbb{R}^n$ is orthogonal to the column space of $C$ if and only if

$$0 = C_i \cdot Y = C_i^t Y$$

for all $1 \le i \le n$. Our proposition follows since the $C_i^t$ are precisely the rows of $C^t$. □

It follows from Proposition 6.5 that $B_0 = \mathrm{Proj}_{\mathcal{W}}(B)$ is the unique vector in the column space of $A$, satisfying

$$A^t(B - B_0) = \mathbf{0}$$

A least-squares solution $X$ then will be found by solving

$$A^t(B - AX) = \mathbf{0}$$

for $X$—that is,

$$A^t AX = A^t B$$

Specifically, for $A$ and $B$ as in formula (6.65), we compute (using MATLAB)

$$A^t A = \begin{bmatrix} 2{,}537{,}500.0 & 119{,}500.0 & 12{,}950.00 \\ 119{,}500.0 & 5{,}659.0 & 612.20 \\ 12{,}950.0 & 612.2 & 66.40 \end{bmatrix} \qquad A^t B = \begin{bmatrix} 13{,}040.00 \\ 617.00 \\ 66.85 \end{bmatrix} \tag{6.66}$$

Next, we ask our computer to solve $A^t AX = A^t B$. We get

$$X = \begin{bmatrix} -0.0002 \\ 0.0461 \\ 0.6227 \end{bmatrix}$$

As a check on our work, we compute

$$B_0 = AX = \begin{bmatrix} 3.1251 \\ 2.9917 \\ 3.8846 \\ 3.4260 \\ 2.7806 \\ 3.7495 \end{bmatrix} \tag{6.67}$$

This matrix represents the "expected GPA" of each student on the basis of the admissions data. It compares favorably with the actual GPA, although the third student did not do quite as well as we might have expected, while the fifth was definitely an overachiever.

The ideas used in solving our admissions problem generalize directly to any $m \times n$ matrix $A$. The following theorem summarizes our conclusions. What this theorem says is quite remarkable; if we are given an inconsistent system

$$AX = B$$

where $A$ is an $m \times n$ matrix, then multiplying both sides of the above equality by $A^t$ yields the system

$$A^t AX = A^t B$$

which is *automatically consistent*. Furthermore, if $X$ solves the latter equation, then $AX$ is as close to equaling $B$ as possible!

**Theorem 6.20** (Least-Squares Theorem). *Let $A$ be an $m \times n$ matrix. Then for all $B \in \mathbb{R}^m$ there exists an $X \in \mathbb{R}^n$ such that*

$$A^t B = (A^t A)X \tag{6.68}$$

*For any such $X$, $AX = \mathrm{Proj}_{\mathcal{W}}(B)$, where $\mathcal{W}$ is the column space of $A$. In particular, $X$ has the property that it minimizes $|B - AX|$.*

*Proof.* Let $\mathcal{W}$ be the column space of $A$ and let $B_0 = \mathrm{Proj}_{\mathcal{W}}(B)$. Then $B - B_o \in \mathcal{W}^\perp$ and, hence, from Proposition 6.5,

$$A^t(B - B_0) = \mathbf{0}$$

On the other hand, since $B_0$ belongs to the column space of $A$, there is an $X$ such that $AX = B_0$. Substitution into the preceding equation shows that $A^t(B - AX) = \mathbf{0}$, which is equivalent to equation (6.68). Hence, $X$ exists.

Conversely, suppose $X$ is a solution to equation (6.68). Then $A^t(B - AX) = \mathbf{0}$, which, according to Proposition 6.5, implies that $B - AX \in \mathcal{W}^\perp$. It follows that $AX = \mathrm{Proj}_{\mathcal{W}}(B)$ since $\mathrm{Proj}_{\mathcal{W}}(B)$ is the unique element in $\mathcal{W}$ for which $B - \mathrm{Proj}_{\mathcal{W}}(B) \in \mathcal{W}^\perp$ (Theorem 6.7 on page 321). This proves our theorem. □

The following example illustrates a typical application of the least squares technique.

## EXAMPLE 6.16

Suppose that we are studying a chemical reaction that generates heat. The following table shows our measurements of the temperature $T$ as a function of time $t$:

| | | | | | |
|---|---|---|---|---|---|
| $t$(seconds) | 0.5 | 1.1 | 1.5 | 2.1 | 2.3 |
| $T$(celsius) | 32.0 | 33.0 | 34.2 | 35.1 | 35.7 |

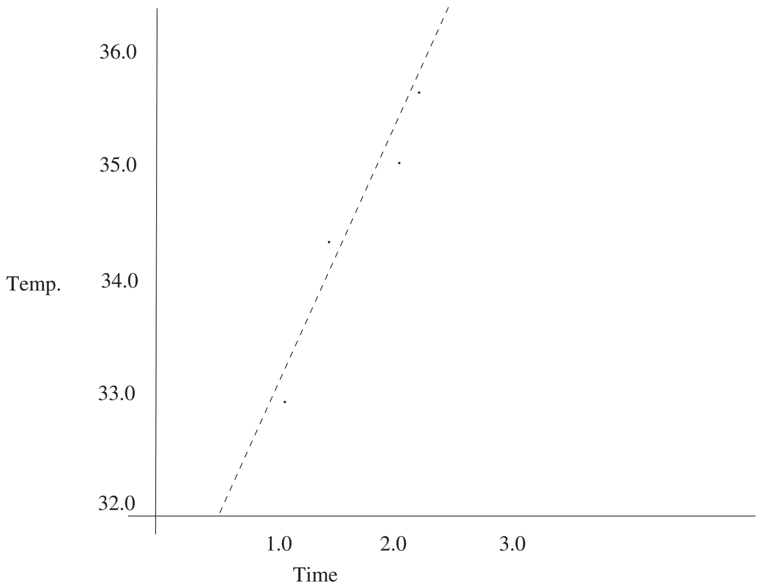



**FIGURE 6.23** Fitting a straight line.

Upon plotting the data points (Figure 6.23), we discover that they are roughly collinear. This suggests a linear relation between $T$ and $t$—that is, a relation of the form

$$T = a + bt \tag{6.69}$$

We wish to find values for $a$ and $b$ such that the above equation represents the observed data as closely as possible.

We could, of course, simply use a ruler to draw a line onto Figure 6.23 that appears to be close to the data points. Then $a$ would be the slope of this line and $b$ the intercept. This seems, however, rather imprecise; different investigators might draw quite different lines. Instead, we will use the least squares method to find $a$ and $b$.

We begin with the observation that if all our data points satisfied equation (6.69), then each of the following five equations would be satisfied:

$$\begin{aligned}
32.0 &= a + 0.5b \\
33.0 &= a + 1.1b \\
34.2 &= a + 1.5b \\
35.1 &= a + 2.1b \\
35.7 &= a + 2.3b
\end{aligned}$$

As a matrix equation, this system may be written $B = AX$, where $X = [a, b]^t$, $B = [32.0, 33.0, 34.2, 35.1, 35.7]^t$, and

$$A = \begin{bmatrix} 1 & 0.5 \\ 1 & 1.1 \\ 1 & 1.5 \\ 1 & 2.1 \\ 1 & 2.3 \end{bmatrix} \tag{6.70}$$

Since the points do not all lie on a line, we know that this equation does not have a solution. However, we can proceed to find an approximate solution just as before. Specifically, we replace the system $B = AX$ by the system $A^tB = A^tAX$ and solve this system for $X = [a, b]^t$. We leave the computation of $X$ as an exercise.

*Remark.* This kind of analysis of data is also called **regression analysis**, since one of the early applications of least squares was to genetics to study the well-known phenomenon that children of abnormally tall or abnormally short parents tend to be more normal in height than their parents. In more technical language, the children's height tends to "regress toward the mean."

Theorem 6.20 provides a technique for computing projections without using orthogonal bases. Before explaining this, we note the following proposition that is key to this technique.

**Proposition 6.6** *Let $C$ be an $m \times n$ matrix. Then $C$ and $C^tC$ have the same rank. In particular, if $C$ has rank $n$, then $C^tC$ is invertible.*

*Proof.* We will show that $C$ and $C^tC$ have the same nullspace. Our result then follows from the rank-nullity theorem (Theorem 2.17 on page 139) since $C$ and $C^tC$ both have the same number of columns.

To see that $C$ and $C^tC$ have the same nullspace, let $X \in \mathbb{R}^n$. We need to show that $C^tCX = \mathbf{0}$ if and only if $CX = \mathbf{0}$. It is clear that if $CX = \mathbf{0}$, then $C^tCX = \mathbf{0}$. Thus, suppose that $C^tCX = \mathbf{0}$. Then

$$\begin{aligned} 0 &= X^tC^tCX \\ &= (CX)^tCX \\ &= CX \cdot CX = |CX|^2 \end{aligned} \tag{6.71}$$

Hence, $CX = \mathbf{0}$, proving the equality of the nullspaces. □

We can now describe the technique for computing projections mentioned above. Let $\mathcal{W}$ be a $n$-dimensional subspace of $\mathbb{R}^m$ and let $\mathcal{B} = \{A_1, A_2, \dots, A_n\}$ be a basis for $\mathcal{W}$. Then $\mathcal{W}$ is the column space of the $m \times n$ matrix $A = [A_1, A_2, \dots, A_n]$. Note that the linear independence of $\mathcal{B}$ implies that $A$ has rank $n$.

According to Theorem 6.20, for all $B \in \mathbb{R}^m$,

$$\mathrm{Proj}_{\mathcal{W}}(B) = AX \tag{6.72}$$

where $X \in \mathbb{R}^n$ is any solution to

$$A^tB = A^tAX$$

From Proposition 6.6, $A^tA$ is invertible, allowing us to write

$$X = (A^tA)^{-1}A^tB$$

Then, from formula (6.72)

$$\mathrm{Proj}_{\mathcal{W}}(B) = AX = A(A^tA)^{-1}A^tB$$

We have proved the following theorem. The matrix $P_{\mathcal{W}}$ in Theorem 6.21 is referred to as the **projection matrix** for $\mathcal{W}$.

**Theorem 6.21** *Let $\mathcal{B} = \{A_1, A_2, \dots, A_n\}$ be a basis for an n-dimensional subspace $\mathcal{W}$ of $\mathbb{R}^m$. Then, for all $B \in \mathbb{R}^m$, $\mathrm{Proj}_{\mathcal{W}}(B) = P_{\mathcal{W}}B$, where*

$$P_{\mathcal{W}} = A(A^tA)^{-1}A^t \tag{6.73}$$

*Remark.* The notation $P_{\mathcal{W}}$ suggests that the projection matrix is independent of the choice of the basis $\mathcal{B}$ used to compute it. This is true since from Theorem 6.21 above $P_{\mathcal{W}}$ is the matrix of the linear transformation $\mathrm{Proj}_{\mathcal{W}}$. Exercises 6.95–6.98 explore an algebraic proof of the fact that $P_{\mathcal{W}}$ does not depend on the choice of basis of $\mathcal{W}$.

### EXAMPLE 6.17

Let $\mathcal{W}$ be the subspace of $\mathbb{R}^4$ spanned by $A_1 = [1, 2, 1, 1]^t$ and $A_2 = [1, 0, 1, 0]^t$. Compute $\mathrm{Proj}_{\mathcal{W}}(B)$ for $B = [1, 2, 3, 4]^t$.

*Solution.* The subspace in question is the column space of the matrix $A$:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

We compute

$$A^tA = \begin{bmatrix} 7 & 2 \\ 2 & 2 \end{bmatrix}$$

$$(A^tA)^{-1} = \frac{1}{10}\begin{bmatrix} 2 & -2 \\ -2 & 7 \end{bmatrix}$$

$$P_{\mathcal{W}} = A(A^tA)^{-1}A^t = \frac{1}{10}\begin{bmatrix} 5 & 0 & 5 & 0 \\ 0 & 8 & 0 & 4 \\ 5 & 0 & 5 & 0 \\ 0 & 4 & 0 & 2 \end{bmatrix}$$

Finally,

$$\text{Proj}_{\mathcal{W}}(B) = P_{\mathcal{W}}B = \begin{bmatrix} 2.0 \\ 3.2 \\ 2.0 \\ 1.6 \end{bmatrix}$$

We may verify our work by checking that $B_1 = B - B_0 = [-1, -1.2, 1, 2.4]^t$ is perpendicular to both $A_1$ and $A_2$. It is easily seen that this is valid.

## EXERCISES

**6.89** ✓✓Compute $X$ in Example 6.16 on page 373 by solving the normal equation. Then compute $B_0 = AX$. The answer should agree closely with $B$. Check that $B - B_0$ is perpendicular to the columns of $A$.

**6.90** Suppose that in the context of Example 6.16, it is proposed that our data would be better described by an equation of the form

$$T = a + b \ln t$$

Use the techniques of this section to find approximate values for $a$ and $b$. How would you go about deciding which equation better fits the given data: the logarithmic equation proposed in this exercise or the linear equation proposed in Example 1? (*Hint:* Think in terms of distance in $\mathbb{R}^5$.)

**6.91** ✓✓Suppose that in the context of Example 6.16 it is proposed that our data would be better described by an equation of the form

$$T = a + bt + ct^2$$

**(a)** Explain how you would use the techniques of this section to find approximate values for $a$, $b$, and $c$. Specifically, in the normal equation $A^tB = A^tAX$, what are appropriate choices for $A$ and $B$?

**(b)** If you have appropriate software available, compute the solution to the normal equation. Note the size of the constant $c$. What is your interpretation of this result?

**6.92** The data in the chart below is the estimated population of the United States (in millions), rounded to three digits, from 1980 to 2000.[2] Your goal in this exercise is to predict the U.S. population in the year 2010.

| Year | 1980 | 1985 | 1990 | 1995 | 2000 |
|---|---|---|---|---|---|
| Population | 227 | 238 | 249 | 263 | 273 |

For this, let $t$ denote "years after 1980" and $I$ represent the increase in population over the 1980 level (see the chart below). Use the method of least squares to find constants $a$ and $b$ such that $I$ is approximately equal to $at + b$. Then use your formula to predict the 2010 population.

| Years after 1980 | 0 | 5 | 10 | 15 | 20 |
|---|---|---|---|---|---|
| Increase over 1980 | 0 | 11 | 22 | 36 | 46 |

**6.93** ✓✓In population studies, exponential models are much more commonly used than linear models. This means that we hope to find constants $a$ and $b$ such that the population $P$ is given approximately by the equation $P = ae^{bt}$. To convert this into a linear equation, we take the natural logarithm of both sides, producing

$$\ln P = \ln a + bt$$

Use the method of least squares to find values for $\ln a$ and $b$ that make this equation approximate the data from Exercise 4. You can let $t$ denote "years after 1980" but you should use the actual population for $P$, not the increase in population over 1980. Finally, use the formula $P = ae^{bt}$ to predict the population in the year 2010.

**6.94** Let $B_0$ be as in formula (6.67) on page 372 and $A$ and $B$ be as in (6.65) on page 371. Check that $B_1 = B - B_0$ is orthogonal to the columns of the matrix $A$.

**6.95** ✓✓Let $\mathcal{W}$ be the subspace of $\mathbb{R}^4$ spanned by $A_1 = [1, 2, 0, 1, 0]^t$ and $A_2 = [1, 1, 1, 1, 1]^t$. Use Theorem 6.21 on page 376 to find the projection matrix $P_{\mathcal{W}}$, $\text{Proj}_{\mathcal{W}}(B)$ and $\text{Orth}_{\mathcal{W}}(B)$, where $B = [1, 2, 3, 4, 5]^t$. Show by direct calculation that $\text{Orth}_{\mathcal{W}}(B)$ is orthogonal to $A_1$ and $A_2$.

**6.96** Let $\mathcal{B}' = \{B_1, B_2\}$, where $B_1 = [0, 1, -1, 0, -1]^t$ and $B_2 = [2, 3, 1, 2, 1]^t$.

**(a)** Let $A_1$ and $A_2$ be as in Exercise 6.95. Find scalars $a$, $b$, $c$, and $d$ such that

$$B_1 = aA_1 + bA_2$$
$$B_2 = cA_1 + dA_2$$

[2]*Source:* U.S. Census Bureau.

How does it follow that $\mathcal{B}'$ forms a basis for the subspace $\mathcal{W}$ of $\mathbb{R}^5$ from Exercise 6.95.

**(b)** Use $\mathcal{B}'$ to compute the projection matrix $P_{\mathcal{W}}$ for $\mathcal{W}$. If you work carefully, it should be the same as the projection matrix found in Exercise 6.95.

**6.97** Use Theorem 6.21 on page 376 to find the projections requested in Exercise 6.19.

**6.98** Suppose that $A$ is an $m \times n$ matrix with rank $n$ and $C$ is an invertible $n \times n$ matrix.

**(a)** Let $B = AC$. Prove that

$$A(A^tA)^{-1}A^t = B(B^tB)^{-1}B^t$$

**(b)** Let $A = [A_1, A_2]$, $B = [B_1, B_2]$ be $4 \times 2$ matrices, where $A_1$ and $A_2$ are as in Exercise 6.95 and $B_1$ and $B_2$ are as in Exercise 6.96. Show that $B = AC$, where

$$C = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

with $a$, $b$, $c$, and $d$ being the scalars computed in Exercise 6.96.a. Note that the equality of the projection matrices found in Exercises 6.95 and 6.96 now follows from part (a).

**(c)** Let $\mathcal{A} = \{A_1, A_2, \dots, A_n\}$ be a basis for a subspace $\mathcal{W}$ of $\mathbb{R}^m$ and let $A$ be the $m \times n$ matrix whose $i$th column is $A_i$, $1 \leq i \leq n$. Let $\mathcal{B} = \{B_1, B_2, \dots, B_n\}$ be another basis for $\mathcal{W}$ and let $B$ be the corresponding $m \times n$ matrix. Show that there is an $n \times n$ matrix $C$ such that $A = BC$. Show that rank $C = n$. It now follows that $C$ is invertible. Hence, $\mathcal{A}$ and $\mathcal{B}$ define the same projection matrices.

**6.99** Let $\mathcal{W}$ be the plane in $\mathbb{R}^3$ defined by $2x - y + 3z = 0$. Use Theorem 6.21 to find $\mathrm{Proj}_{\mathcal{W}}(B)$. Use your answer to compute the distance from $B$ to $\mathcal{W}$. [*Hint:* Begin by finding a basis for the solution set to the equation $2x - y + 3z = 0$. The distance from $B$ to $\mathcal{W}$ is $|\mathrm{Orth}_{\mathcal{W}}(B)|$.]

**6.100** ✓✓From Exercises 3.80–3.82, $(AB)^{-1} = B^{-1}A^{-1}$. What is wrong with the following calculation?

$$P_{\mathcal{W}} = A(A^tA)^{-1}A^t = AA^{-1}(A^t)^{-1}A^t = II = I$$

**6.101** Let $P_{\mathcal{W}}$ be the projection matrix corresponding to an $m$-dimensional subspace $\mathcal{W}$ of $\mathbb{R}^n$.

**(a)** Show that $X \in \mathcal{W}$ if and only if $P_{\mathcal{W}}X = X$. How does it follow that rank $P_{\mathcal{W}} = m$? [*Hint:* According to Theorem 6.21 on page 376, any $X \in \mathbb{R}^n$ may be written in the form $X = X_1 + X_2$, where $X_1 \in \mathcal{W}$ and $X_2 \in \mathcal{W}^{\perp}$.]

**(b)** Show that the kernel of $P_{\mathcal{W}}$ is $\mathcal{W}^{\perp}$.

**(c)** Use the rank-nullity theorem (Theorem 2.17 on page 139) to conclude that $\dim \mathcal{W} + \dim \mathcal{W}^{\perp} = n$.

**6.102** This exercise studies projection matrices.

**(a)** Compute $P_{\mathcal{W}}$ for the subspace $\mathcal{W}$ of $\mathbb{R}^3$ in Exercise 6.99. Then compute $(P_{\mathcal{W}})^2$. If you are careful, it should be the case that $P_{\mathcal{W}} = (P_{\mathcal{W}})^2$. Why, geometrically, should this not be a surprise?

**(b)** Use formula (6.73) on page 376 to prove that if $P_{\mathcal{W}}$ is the projection matrix corresponding to a subspace $\mathcal{W}$ of $\mathbb{R}^n$, then $(P_{\mathcal{W}})^2 = P_{\mathcal{W}}$.

**(c)** Use formula (6.73) on page 376 to prove that if $P_{\mathcal{W}}$ is the projection matrix corresponding to a subspace $\mathcal{W}$ of $\mathbb{R}^n$, then $(P_{\mathcal{W}})^t = P_{\mathcal{W}}$.

**6.103** Let $P$ be an $n \times n$ matrix such that (i) $P^2 = P$ and (ii) $P = P^t$. Let $\mathcal{W}$ be the column space of $P$. In this exercise, we outline a proof that $P = P_{\mathcal{W}}$.

**(a)** Prove that, for all $X \in \mathcal{W}$, $PX = X$. [*Hint:* Since $X \in \mathcal{W}$, $X = PY$ for some $Y \in \mathbb{R}^n$.]

**(b)** Prove that, for all $X \in \mathcal{W}^{\perp}$, $PX = 0$. [*Hint:* Show that for all $Y \in \mathbb{R}^n$, $(PX)^t Y = 0$ and, hence, $(PX)^t PX = 0$.]

**(c)** Explain how it follows that $P = P_{\mathcal{W}}$. [*Hint:* The matrix of a linear transformation is unique.]

**6.104** Let $P$ be an $n \times n$ matrix such that (i) $P^2 = P$ and (ii) $P = P^t$. Let $Q = I - P$.

**(a)** Show that (i) $Q^2 = Q$ and (ii) $Q = Q^t$.

**(b)** Show that $QP = PQ = \mathbf{0}$.

**(c)** According to Exercise 6.103, $P = P_{\mathcal{W}}$, where $\mathcal{W}$ is the column space of $P$. Show that $Q = P_{\mathcal{W}^{\perp}}$. [*Hint:* Use the result of 6.103.a and 6.103.c to show that $Q$ and $P_{\mathcal{W}^{\perp}}$ define the same linear transformation of $\mathbb{R}^n$ into $\mathbb{R}^n$.]

**(d)** Prove that $(\mathcal{W}^{\perp})^{\perp} = \mathcal{W}$. [*Hint:* According to the result of 6.103.c, $I - Q$ is the projection matrix for which subspace?]

### 6.5.1 Computer Projects

### EXERCISES

**1.** Imagine that you are an astronomer investigating the orbit of a newly discovered asteroid. You want to determine (a) how close the asteroid comes to the sun and (b) how far away from the sun the asteroid gets. To solve your problem, you will make use of the following facts:

**(a)** Asteroids' orbits are approximately elliptical, with the sun as one focus.

**(b)** In polar coordinates, an ellipse with one focus at the origin can be described by a formula of the following form, where $a$, $b$, and $c$ are constants:

$$r = \frac{c}{1 + a\sin\theta + b\cos\theta}$$

You have also collected the data tabulated below, where $r$ is the distance from the sun in millions of miles and $\theta$ is the angle in radians between the vector from the sun to the asteroid and a fixed axis through the sun. This data is, of course, subject to experimental error.

Your strategy is to use the given data to find values of $a$, $b$, and $c$ that cause the formula to agree as closely as possible with the given data. This involves setting up a system of linear equations and solving the normal equation. (Give the augmented matrices for both the original system and the normal equation.) You will then graph the given formula and measure the desired data from the graph. Good luck!

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\theta$ | 0.00 | 0.60 | 1.80 | 1.40 | 2.10 | 3.20 | 5.40 |
| $r$ | 329.27 | 313.80 | 319.49 | 310.91 | 327.88 | 374.91 | 367.49 |

*Remark.* Once you have found values of $a$, $b$, and $c$, you will need to plot the orbit of the asteroid. For this, you should create a vector of $\theta$ values and a vector of the corresponding $r$ values. Thus, you might enter T=0:.1:2∗pi followed by

```
r=c*(1+a*sin(T)+b*cos(T)).^(-1)
```

(Note the period.) Then you enter polar(T,r) to construct the plot.

**2.** MATLAB has a way of solving least squares problems that is easier (and better) than solving the normal equation. If the (inconsistent) system is expressed in the form $AX = B$, then the least-squares solution is $A\backslash B$. Try it!

## 6.6 QUADRATIC FORMS: ORTHOGONAL DIAGONALIZATION

In analytic geometry, one often studies the set $V$ of points $[x, y]^t$ in $\mathbb{R}^2$ that solve an equation of the form

$$ax^2 + bxy + cy^2 = d \tag{6.74}$$

where $a$, $b$, $c$, and $d$ are constants. If $b = 0$, then the equation takes the form

$$ax^2 + cy^2 = d$$

which we refer to as **standard form**. In this case, $V$ is typically either an ellipse or a hyperbola. For example, the solution set to

$$x^2 + \frac{y^2}{9} = 1$$

is an ellipse, while the solution set to

$$x^2 - \frac{y^2}{9} = 1$$

is a hyperbola.

In the general ($b \neq 0$) case, $V$ is (usually) an ellipse or a hyperbola, rotated by some fixed angle in the $xy$ plane. In fact, it is possible to find new coordinates for $\mathbb{R}^2$, defined by perpendicular axes, such that $V$ is the solution set of an equation in standard form in the new coordinates. In this section we describe how linear algebra can be used to find such coordinates.

To make the connection with linear algebra, let $X = [x, y]^t$ and

$$A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \tag{6.75}$$

Then, we compute that

$$\begin{aligned} X^tAX &= [x, y] \begin{bmatrix} ax + b/2y \\ b/2x + cy \end{bmatrix} \\ &= ax^2 + bxy + cy^2 \end{aligned}$$

Thus, equation (6.74) is equivalent to the matrix equality

$$d = X^tAX$$

It is important to note that equation (6.74) is in standard form if and only if $b = 0$, which is equivalent to $A$ being a diagonal matrix.

Now, let new coordinates be defined by an ordered *orthonormal* basis $\mathcal{B} = \{Q_1, Q_2\}$ for $\mathbb{R}^2$. Let $Q = [Q_1, Q_2]$ be the $2 \times 2$ point matrix for this basis. Note that the orthonormality of the basis is equivalent to $Q$ being an orthogonal matrix.

Let $X = [x, y]^t$ have $\mathcal{B}$ coordinate vector $X' = [x', y']^t$ so that $X = QX'$. In $\mathcal{B}$ coordinates, our curve, then, is described by

$$\begin{aligned} d &= X^tAX \\ &= (QX')^tAQX' \\ &= (X')^tQ^tAQX' \\ &= (X')^tQ^{-1}AQX' \end{aligned}$$

From the preceding comments, *the equation that describes our curve in the new coordinates will be in standard form if and only if*

$$D = Q^{-1}AQ$$

*is a diagonal matrix.* In the language of Section 5.2, $D$ is then a diagonalization of $A$. Furthermore, Theorem 5.4 on page 287 says that if the columns of $Q$ are eigenvectors for $A$, then the curve will be described by an equation in standard form relative to the $\mathcal{B}$ coordinates.

### EXAMPLE 6.18

Graph the solution set $V$ in $\mathbb{R}^2$ of the following equation:

$$3x^2 + 2xy + 3y^2 = 4$$

*Solution.* From equation (6.75), this is the same as $4 = X^t AX$, where

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Diagonalizing $A$ is simple. In fact, in Example 5.2, page 273, it was found that $X = [1, -1]^t$ and $Y = [1, 1]^t$ are eigenvectors for $A$ corresponding to the eigenvalues 2 and 4, respectively. The corresponding diagonal matrix is

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

It is easily computed that $X \cdot Y = 0$ and, hence, these vectors form an orthogonal basis for $\mathbb{R}^2$. We normalize this basis, setting

$$Q_1 = \frac{[1, -1]^t}{\sqrt{2}}, \qquad Q_2 = \frac{[1, 1]^t}{\sqrt{2}}$$

We use the basis $\mathcal{B} = \{Q_1, Q_2\}$ to define coordinates for $\mathbb{R}^n$.

From the preceding discussion, $X \in V$ if and only if its $\mathcal{B}$ coordinate vector $X' = [x', y']^t$ satisfies

$$4 = (X')^t DX' = [x', y'] \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 2(x')^2 + 4(y')^2$$

which describes an ellipse with intercepts $x' = \pm\sqrt{2}$ and $y' = \pm 1$. The graph of $V$ is indicated in Figure 6.24.

In the above discussion, we seem to have been very lucky in that the eigenvectors $[1, -1]^t$ and $[1, 1]^t$ turned out to be perpendicular to each other. If this had not been the case, we would not have been able to find an *orthonormal* basis of eigenvectors

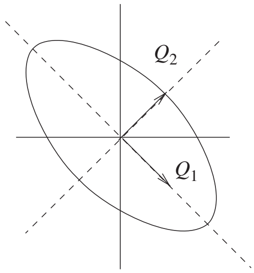



**FIGURE 6.24** Example 6.18.

for $A$ and, hence, we would not have been able to solve our problem. This actually was not luck at all. Notice that the matrix $A$ of formula (6.75) satisfies

$$A = A^t$$

Matrices that satisfy this property are called **symmetric**. Theorem 6.22 below, which is almost immediate from the following proposition, explains our "stroke of luck." Despite its simplicity, Proposition 6.7 is key to most of the deeper properties of symmetric matrices. Its proof is left as an exercise (Exercise 6.123).

**Proposition 6.7** *Let $A$ be an $n \times n$ matrix. Then for all $X, Y \in \mathbb{R}^n$*

$$AX \cdot Y = X \cdot A^t Y \tag{6.76}$$

*In particular, if $A$ is symmetric, then*

$$AX \cdot Y = X \cdot AY \tag{6.77}$$

**Theorem 6.22** *Let $A$ be an $n \times n$ symmetric matrix and let $X$ and $Y$ be eigenvectors for $A$ corresponding, respectively, to eigenvalues $\lambda$ and $\beta$, where $\lambda \neq \beta$. Then $X \cdot Y = 0$.*

*Proof.* From (6.77)

$$\begin{aligned} \lambda(X \cdot Y) &= (\lambda X) \cdot Y \\ &= AX \cdot Y \\ &= X \cdot AY \\ &= X \cdot \beta Y \\ &= \beta(X \cdot Y) \end{aligned}$$

and thus,

$$0 = (\lambda - \beta)(X \cdot Y)$$

Since $\lambda \neq \beta$, this implies $X \cdot Y = 0$, proving the theorem. □

This theorem implies that if an $n \times n$ symmetric matrix $A$ has $n$ distinct eigenvalues, then any diagonalizing basis for $A$ consists of mutually perpendicular vectors and, hence, forms an orthogonal basis. By normalizing each basis element, we may construct an orthonormal basis, and hence, an orthogonal matrix $Q$ such that $Q^{-1}AQ$ is diagonal. Actually, we can find such a matrix even if $A$ has fewer than $n$ eigenvalues as long as $A$ is symmetric. The following is one of the most important theorems of linear algebra. It has applications far beyond the sphere of quadratic curves. We prove it at the end of this section.

## The Spectral Theorem

**Theorem 6.23** (Spectral Theorem). *Let $A$ be an $n \times n$, real symmetric matrix. Then there is an orthonormal basis $\mathcal{B} = \{Q_1, Q_2, \dots, Q_n\}$ for $\mathbb{R}^n$ consisting of (real) eigenvectors for $A$. In particular, all the eigenvalues for $A$ are real. The matrix $Q = [Q_1, Q_2, \dots, Q_n]$ is then an orthogonal matrix such that $D = Q^{-1}AQ$ is diagonal.*

This theorem tells us that the analysis we did in Example 6.18 generalizes to higher dimensions with almost no modification. Let $A = [a_{ij}]$ be an $n \times n$ matrix. The function $F_A : \mathbb{R}^n \to \mathbb{R}$ defined by

$$Q_A(X) = X^tAX = \sum a_{ij}x_ix_j \qquad (1 \le i \le n,\ 1 \le j \le n) \tag{6.78}$$

is referred to as the **quadratic form** defined by $A$. The matrix $A$ is referred to as a **defining matrix** for the quadratic form. For each $d \in \mathbb{R}$, the set $V$ of all $X = [x_1, x_2, \dots, x_n]^t$ in $\mathbb{R}^n$ satisfying

$$X^tAX = d$$

is said to be a **quadratic variety**.

Since one-by-one matrices are symmetric

$$X^tAX = (X^tAX)^t = X^tA^tX \tag{6.79}$$

Hence

$$\begin{aligned} Q_A(X) &= \tfrac{1}{2}(X^tAX + X^tA^tX) \\ &= \tfrac{1}{2}X^t(A + A^t)X \end{aligned}$$

Hence $Q_A(X) = Q_B(X)$ where $B = \frac{1}{2}(A + A^t)$. Note that $B = B^t$ and, hence, *every quadratic form has a symmetric defining matrix.*

The quadratic form $Q_A$ is said to be in **standard form** if $A$ is a diagonal matrix. In this case, the corresponding variety is described by an equation of the form

$$a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 = d$$

The following result is a direct consequence of the spectral theorem.

### The Principal Axis Theorem

**Theorem 6.24** (Principal Axis Theorem). *Let the notation be as in the statement of the spectral theorem (Theorem 6.23 on page 385). For $X \in \mathbb{R}^n$, let $X' = Q^{-1}X$ be the $\mathcal{B}$ coordinates of $X$. Then $X$ satisfies*

$$X^t A X = d \tag{6.80}$$

*if and only if*

$$(X')^t D X' = d$$

*Thus, the quadratic variety $V$ described by equation* (6.80) *is the set of points $X$ in $\mathbb{R}^n$ whose $\mathcal{B}$ coordinate vector $X' = [x_1', x_2', \cdots, x_n']^t$ satisfies*

$$\lambda_1(x_1')^2 + \lambda_2(x_2')^2 + \cdots + \lambda_n(x_n')^2 = d \tag{6.81}$$

*where the $\lambda_i$ are the eigenvalues of $A$ (listed in an order consistent with the ordering of $\mathcal{B}$). Thus, in $\mathcal{B}$ coordinates, $V$ is described by a quadratic form in standard form.*

*Proof.* From the spectral theorem, there is an $n \times n$ orthogonal matrix $Q$ such that $D = Q^t A Q$ is diagonal. The columns of $Q$ form an orthonormal basis for $\mathbb{R}^n$. If $X \in \mathbb{R}^n$ has coordinates $X'$ with respect to this basis, then $X = QX'$. Hence,

$$X^t A X = (QX')^t A Q X' = (X')^t Q^t A Q X' = (X')^t D X'$$

This is equivalent to formula (6.81). □

#### EXAMPLE 6.19

What is the nature of the surface described by the following equation?

$$2x^2 + 2y^2 + 2z^2 + 6\sqrt{2}xy - 6\sqrt{2}yz = 8$$

*Solution.* We first write the equation in symmetric form:

$$2x^2 + 2y^2 + 2z^2 + 3\sqrt{2}xy + 3\sqrt{2}yx - 3\sqrt{2}yz - 3\sqrt{2}zy = 8$$

This may be expressed as

$$8 = [x, y, z]\begin{bmatrix} 2 & 3\sqrt{2} & 0 \\ 3\sqrt{2} & 2 & -3\sqrt{2} \\ 0 & -3\sqrt{2} & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

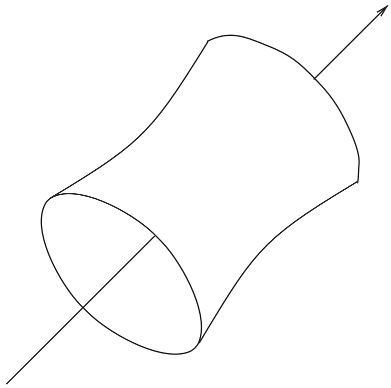


**FIGURE 6.25** A hyperboloid.

Let $A$ be the $3 \times 3$ matrix in the equation. We find that

$$\det(A - \lambda I) = (2 - \lambda)(\lambda^2 - 4\lambda - 32) = (2 - \lambda)(\lambda + 4)(\lambda - 8)$$

Thus, our eigenvalues are 2, −4, and 8. From the spectral theorem, our variety is described by

$$2(x')^2 - 4(y')^2 + 8(z')^2 = 8$$

Our equation describes a hyperboloid of one sheet opening along the $y'$ axis. Roughly, it would appear as shown in Figure 6.25.

Notice that we did not need to find the eigenvectors in order to identify the surface. Had we been asked for coordinates that produce standard form, then we would have had to compute an orthonormal diagonalizing basis. In this particular case, this would not have been difficult. Since $A$ has distinct eigenvalues, any diagonalizing basis for $A$ automatically forms an orthogonal basis (from Theorem 6.22). Thus, we need only to find a diagonalizing basis and then normalize it. If $A$ had not had distinct eigenvalues, then we could have used the Gram-Schmidt process from Section 6.2 to produce an orthogonal diagonalizing basis, as the next example demonstrates.

### EXAMPLE 6.20

Find a diagonal matrix $D$ and an orthogonal matrix $Q$ such that $A = QDQ^t$, where $A$ is as follows:

$$A = \begin{bmatrix} 2 & 2 & 6 & -2 \\ 2 & -1 & 3 & -1 \\ 6 & 3 & 7 & -3 \\ -2 & -1 & -3 & -1 \end{bmatrix}$$

*Solution.* Using a computer, we discover that

$$\det(A - \lambda I) = -104 - 148\lambda - 66\lambda^2 - 7\lambda^3 + \lambda^4$$

which factors as $(\lambda - 13)(\lambda + 2)^3$. Thus, the eigenvalues are 13 and $-2$. We find bases for the eigenspaces as usual, finding that a basis for the $-2$ eigenspace $\mathcal{W}_{-2}$ is formed by the vectors

$$X_1 = [-1, 2, 0, 0]^t, \quad X_2 = [-3, 0, 2, 0]^t, \quad X_3 = [1, 0, 0, 2]^t$$

while the 13 eigenspace $\mathcal{W}_{13}$ is spanned by $Y = [-2, -1, -3, 1]^t$. Direct computation shows that the vector $Y$ is orthogonal to each of $X_1$, $X_2$, and $X_3$, as predicted by Theorem 6.22.

The set $\mathcal{B} = \{X_1, X_2, X_3\}$ is not an orthogonal set. We can, however, use the Gram-Schmidt process to produce an orthogonal basis for $\mathcal{W}_{-2}$. Actually, $X_1$, $X_2$, and $X_3$ were considered in Example 6.8 on page 324. The computations from this example show that $\mathcal{B}' = \{P_1, P_2, P_2\}$ is an orthogonal basis for $\mathcal{W}_{-2}$ where

$$P_1 = [-1, 2, 0, 0]^t, \quad P_2 = \tfrac{2}{5}[-6, -3, 5, 0]^t, \quad \tfrac{1}{7}P_3 = [2, 1, 3, 14]^t$$

The $P_i$ are all orthogonal to $Y$ since, again, eigenvectors corresponding to different eigenvalues are perpendicular. This means that the set $\mathcal{B}'' = \{P_1, P_2, P_3, Y\}$ forms an orthogonal, diagonalizing basis for $A$. We obtain an orthonormal basis by normalizing each element of this basis. The basis elements then become the columns of $Q$. We find that

$$Q = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 & -6/\sqrt{14} & 2/\sqrt{42} & -2/\sqrt{3} \\ 2 & -3/\sqrt{14} & 1/\sqrt{42} & -1/\sqrt{3} \\ 0 & 5/\sqrt{14} & 3/\sqrt{42} & -3/\sqrt{3} \\ 0 & 0 & 14/\sqrt{42} & 1/\sqrt{3} \end{bmatrix}$$

The diagonal matrix in question is

$$D = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

We end this section by proving two fundamental theorems, the spectral theorem and the following theorem.

**Theorem 6.25** *Let $A$ be a real, symmetric, $n \times n$ matrix and let $p(x)$ be the characteristic polynomial for $A$. Then every root $\lambda$ of $p(x)$ is real and, hence, every eigenvalue of $A$ is real.*

*Proof.* Suppose that $\lambda$ is a complex root of $p$. Then there is a complex eigenvector $X$ for $A$ corresponding to $\lambda$. Then $\overline{X}$ is also an eigenvector for $A$ corresponding to $\overline{\lambda}$. Assume that $\lambda$ is not real so that $\lambda \neq \overline{\lambda}$.

In Theorem 6.22, we proved that for a symmetric matrix, if $X$ and $Y$ are eigenvectors corresponding to different eigenvalues, then $X \cdot Y = 0$. An inspection of the proof shows that the proof is valid for complex eigenvectors as well. It follows that $X \cdot \overline{X} = 0$. If $X = [z_1, z_2, \ldots, z_n]^t$, then

$$0 = X \cdot \overline{X} = z_1\overline{z}_1 + z_2\overline{z}_2 + \cdots + z_n\overline{z}_n \tag{6.82}$$

If $z_k = x_k + iy_k$, then (as the reader may verify)

$$z_k\overline{z}_k = x_k^2 + y_k^2 \geq 0$$

Thus, the only way for equation (6.82) to hold is for all $x_k$ and $y_k$ to be zero, proving that $X = \mathbf{0}$. But, eigenvectors must be nonzero. Thus, $\lambda$ must be real. □

Next, we prove the spectral theorem. First, however, we note the following definition and prove Proposition 6.8, which is needed for the proof of the spectral theorem.

**Definition 6.11** *Let $A$ be an $n \times n$ matrix. We say that a subspace $\mathcal{W}$ of $\mathbb{R}^n$ is **invariant** under multiplication by $A$ if, for all $X \in \mathcal{W}$, $AX \in \mathcal{W}$—that is, $A\mathcal{W} \subset \mathcal{W}$. We also say that $\mathcal{W}$ is "A-invariant."*

**Proposition 6.8** *Let $A$ be an $n \times n$ matrix that has only real eigenvalues and let $\mathcal{W}$ be an $A$-invariant subspace of $\mathbb{R}^n$. Then there is a real eigenvector $X$ of $A$ such that $X \in \mathcal{W}$.*

*Proof.* Let $\mathcal{B} = \{X_1, X_2, \ldots, X_m\}$ be an ordered basis for $\mathcal{W}$. Since multiplication by $A$ defines a linear transformation of $\mathcal{W}$ into itself, Theorem 3.14 on page 220 implies that there is an $m \times m$ matrix $M$ such that for all $X \in \mathcal{W}$

$$(AX)' = M(X') \tag{6.83}$$

where $X'$ and $(AX)'$ are, respectively, the $\mathcal{B}$ coordinates of $X$ and $AX$. From Theorem 5.7 on page 302, $M$ has at least one (possibly complex) eigenvalue $\lambda$. We will prove that $\lambda$ is real. Granted this, there is then a nonzero vector $X \in \mathcal{W}$ such that

$$MX' = \lambda X'$$

It follows from formula (6.83) that

$$(AX)' = MX' = \lambda X' = (\lambda X)'$$

Hence, $AX = \lambda X$, proving our proposition.

To see that $\lambda$ is real, let $\lambda = c + id$ and let $X' \in \mathbb{C}^m$ be an eigenvector for $M$ corresponding to $\lambda$. We may write

$$X' = U' + iV'$$

where $U'$ and $V'$ both belong to $\mathbb{R}^m$. From Exercise 5.54 on page 305,

$$\begin{aligned} MU' &= cU' - dV' \\ MV' &= dU' + cV' \end{aligned}$$

Let $U$ and $V$ be the unique elements of $\mathcal{W}$ having $\mathcal{B}$ coordinates $U'$ and $V'$, respectively. It follows from equation (6.83) that

$$\begin{aligned} AU &= cU - dV \\ AV &= dU + cV \end{aligned}$$

which, according to Exercise 5.54 on page 305, implies that $AX = \lambda X$ where $X = U + iV$. Thus, $\lambda$ is an eigenvalue for $A$. Since, by hypothesis, $A$ has only real eigenvalues, it follows that $\lambda$ is real, finishing our proof. □

We are now ready to prove the spectral theorem. It is somewhat easier to prove the following generalization of the spectral theorem. The spectral theorem is the case where $\mathcal{W} = \mathbb{R}^n$.

**Theorem 6.26** (Generalized Spectral Theorem). *Let $A$ be an $n \times n$, real symmetric matrix and let $\mathcal{W}$ be an $A$-invariant, $m$-dimensional subspace of $\mathbb{R}^n$. Then there is an orthonormal basis $\mathcal{B} = \{Q_1, Q_2, \ldots, Q_m\}$ for $\mathcal{W}$ consisting of eigenvectors for $A$.*

*Proof.* Suppose first that $m = 1$ so that $\mathcal{W}$ is one-dimensional. Let $Q$ be a nonzero element of $\mathcal{W}$. Then $Q$ spans $\mathcal{W}$. Since $AQ \in \mathcal{W}$, there is a $\lambda \in \mathbb{R}$ such that $AQ = \lambda Q$. Hence, $Q$ is an eigenvector for $A$. Our theorem follows by normalizing $Q$ to have length 1.

Now, assume by mathematical induction that our theorem has been proved for all $A$-invariant subspaces of $\mathbb{R}^n$ of dimension less than $m$. Let $\mathcal{W}$ be an $m$-dimensional, $A$-invariant subspace of $\mathbb{R}^n$.

From Proposition 6.8, $A$ has an eigenvector $Q_1 \in \mathcal{W}$ corresponding to an eigenvalue $\lambda_1$. We normalize $Q_1$ so that $|Q_1| = 1$. Let

$$\mathcal{W}' = \{W \in \mathcal{W} \mid W \cdot Q_1 = 0\}$$

Then $\mathcal{W}'$ is a subspace of $\mathcal{W}$ that does not contain $Q_1$. Hence, the dimension of $\mathcal{W}'$ is less than $m$. We will show that $\mathcal{W}'$ is $A$-invariant. Granted this, it follows from the induction hypothesis that there is an orthonormal basis $\{Q_2, Q_3, \dots, Q_k\}$ of $\mathcal{W}'$ consisting of eigenvectors for $A$.

We claim that $\mathcal{B} = \{Q_1, Q_2, Q_3, \dots, Q_k\}$ is an orthonormal basis for $\mathcal{W}$. Since orthogonal sets are linearly independent, it suffices to show that $\mathcal{B}$ spans $\mathcal{W}$. To show the spanning, let $X \in \mathcal{W}$. From formula (6.15) on page 319, the vector

$$\mathrm{Orth}_{Q_1}(X) = X - c_1 Q_1$$

where $c_1 = X \cdot Q_1$, is both orthogonal to $Q_1$ and an element of $\mathcal{W}$ (since both $X$ and $Q_1$ belong to $\mathcal{W}$) and, hence, an element of $\mathcal{W}'$. Thus, there are scalars $c_2, \dots, c_k$ such that

$$X - c_1 Q_1 = c_2 Q_2 + \cdots + c_k Q_k$$

proving that $\mathcal{B}$ spans $\mathcal{W}$. The spectral theorem now follows by mathematical induction since $\mathcal{B}$ is clearly a basis for $\mathcal{W}$ of the desired form.

To prove the $A$-invariance of $\mathcal{W}'$, let $W \in \mathcal{W}'$. Then, from formula (6.77) on page 384,

$$\begin{aligned} AW \cdot Q_1 &= W \cdot AQ_1 \\ &= W \cdot \lambda_1 Q_1 \\ &= \lambda_1 (W \cdot Q_1) = 0 \end{aligned}$$

proving that $AW \in \mathcal{W}'$. The spectral theorem follows. □

**True-False Questions**: Justify your answers.

**6.24** The matrix $A$ is symmetric and has the characteristic polynomial $p(\lambda) = \lambda^3(\lambda - 1)^2(\lambda + 3)$. Then the nullspace of $A$ might have dimension 2.

**6.25** It is possible for a symmetric matrix $A$ to have $2 + 4i$ as an eigenvalue.

**6.26** The polynomial $p(\lambda) = (\lambda - 1)(\lambda - 2)^3(\lambda^2 + 1)$ could be the characteristic polynomial of a symmetric matrix.

**6.27** It is impossible for a symmetric $3 \times 3$ matrix $A$ to have the vectors $[1, 2, 3]^t$ and $[1, 1, 1]^t$ as eigenvectors corresponding to the eigenvalues 3 and 5, respectively.

**6.28** It is impossible for a symmetric $3 \times 3$ matrix $A$ to have the vectors $[1, 2, 3]^t$ and $[1, 1, 1]^t$ as eigenvectors corresponding to the eigenvalue 3.

**6.29** Given that the matrix $A$ below has $-5$ and $-10$ as eigenvalues, it follows that the quadratic curve described by $-9x^2 + 4xy - 6y^2 = 1$ is an ellipse.

$$A = \begin{bmatrix} -9 & 2 \\ 2 & -6 \end{bmatrix}$$

**6.30** The following polynomial $p(\lambda) = (\lambda^2 - 1)(\lambda^2 + 4)^2$ could not be the characteristic polynomial of a symmetric matrix.

**6.31** The graph of $ax^2 + bxy + cy^2 = 1$ is either an ellipse or a hyperbola.

**6.32** Suppose that $A$ is a $3 \times 3$ symmetric matrix with eigenvalues 1, 2, and 3. Then there is only one orthogonal matrix $Q$ such that $A = QDQ^t$ where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

**6.33** Suppose that $A$ is a $3 \times 3$ symmetric matrix with eigenvalues 1,2 and 3. Then there are exactly eight different orthogonal matrices $Q$ such that $A = QDQ^t$ is a diagonal matrix.

**6.34** Suppose that $A$ is a $3 \times 3$ symmetric matrix with eigenvalues 1 and 2. Then there is at most a finite number of different orthogonal matrices $Q$ such that $A = QDQ^t$, where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

## EXERCISES

**6.105** ✓✓Let $A$ be the matrix in Example 6.18 and let $A'$ be as below. Show that for all $X \in \mathbb{R}^2$, $Q_A(X) = Q_{A'}(X)$, where $Q_A$ is as defined in equation (6.78). What are all matrices $B$ such that $Q_A = Q_B$? Show that if $B$ is also symmetric, then $B = A$.

$$A' = \begin{bmatrix} 3 & 0 \\ 2 & 3 \end{bmatrix}$$

**6.106** ✓✓Let

$$A = \begin{bmatrix} 10 & 2 & 2 \\ 2 & 13 & 4 \\ 2 & 4 & 13 \end{bmatrix}$$

Find an orthogonal matrix $Q$ and a diagonal matrix $D$ such that $A = QDQ^t$, given that the eigenvalues of $A$ are $\lambda = 9$ and $\lambda = 18$.

**6.107** For each quadratic variety, find an orthonormal basis $\mathcal{B}$ for the corresponding $\mathbb{R}^n$ for which the variety is in standard form in the $\mathcal{B}$ coordinates and give a defining equation for the variety in the $\mathcal{B}$ coordinates. (See Example 6.19.)

**(a)** ✓✓ $2x^2 + 4xy - y^2 = 1$

**(b)** ✓✓ $x_1^2 + x_2^2 + x_3^2 + \sqrt{2}x_1x_2 + \sqrt{2}x_2x_3 = 4$

**(c)** $11x^2 - 8xy - 8xz + 17y^2 + 16yz + 17z^2 = 3$

**(d)** $18x^2 + 5y^2 + 12z^2 - 12xy - 12yz = 7$

**(e)** $-5x^2 + 2y^2 - 11z^2 - 12xy + 12yz = -14$

**6.108** For each variety find a quadratic form in standard form that describes the variety relative to the $\mathcal{B}$ coordinates for some orthonormal basis $\mathcal{B}$. (You need not find the basis $\mathcal{B}$.) What, geometrically, does each variety represent?

**(a)** ✓✓ $x^2 + xy + 2y^2 = 1$ **(b)** $x^2 + 4xy + 2y^2 = 1$

**(c)** $x^2 + 4xy + 4y^2 = 1$ **(d)** ✓ $14x^2 + 4xy + 11y^2 = 1$

**6.109** For each variety in parts (a)–(c) in Exercise 6.108, find an orthonormal basis $\mathcal{B}$ for $\mathbb{R}^2$ for which the variety is described by a quadratic form in standard form in the $\mathcal{B}$ coordinates.✓[(a)]

**6.110** Give an equation in the standard coordinates for $\mathbb{R}^2$ that describes an ellipse centered at the origin with a length 4 major cord parallel to the vector $[3, 4]^t$ and a length 2 minor axis. (The major cord is the longest line segment that can be inscribed in the ellipse.)

**6.111** ✓In Example 6.18 on page 383, what type of geometric object would $V$ be if the eigenvalues had been (a)✓✓2 and $-4$? (b) How about $-2$ and $-4$? (c) How about 2 and 0? State a general theorem relating the signs of the eigenvalues of a $2 \times 2$ matrix $A$ to the type of figure described by the equation $X^tAX = 4$.

**6.112** ✓✓Find $a$, $b$, and $c$ such that the curve defined by the equation below describes an ellipse that has its major axis of length 6 on the line spanned by the vector $[3, 4]$ and its minor axis of length 4. [*Hint:* If you can find $Q$ and $D$, then you can let $A = QDQ^t$.]

$$ax^2 + bxy + cy^2 = 1$$

**6.113** ✓✓In Exercise 6.112, find $a$, $b$, and $c$ such that the curve defined by the given equation represents a hyperbola opening on the stated line. What is the formula for your hyperbola in rotated coordinates?

**6.114** Let $a$, $b$, and $c$ be numbers with $a + c > 0$. Prove that the formula $ax^2 + 2bxy + cy^2 = 1$ represents an ellipse if and only if $b^2 < ac$. What does the curve represent if $b^2 = ac$?

**6.115** Let $A$ be as shown. Find an orthogonal matrix $Q$ and a diagonal matrix $D$ such that $Q^tAQ = D$. To save you time, we have provided the eigenvalues (there are only two).

$$A = \begin{bmatrix} 3 & 1 & 1 & 2 \\ 1 & 3 & 1 & 2 \\ 1 & 1 & 3 & 2 \\ 2 & 2 & 2 & 6 \end{bmatrix}, \quad \lambda_1 = 9, \quad \lambda_2 = 2$$

**6.116** ✓✓A symmetric matrix $A$ is said to be positive definite if $X^tAX > 0$ for all $X \neq \mathbf{0}$. Show that the matrix $A$ from Example 6.18 on page 383 is positive definite.

**6.117** Prove that a symmetric matrix $A$ is positive definite if and only if all its eigenvalues are positive.

**6.118** ✓✓Find a $2 \times 2$ matrix that has only positive entries and is not positive definite.

**6.119** Show that the matrix $A$ in Example 6.19 is not positive definite by explicitly exhibiting a vector $X$ such that $X^tAX < 0$. [*Hint:* Try letting $X$ be an appropriately chosen eigenvector for $A$.]

**6.120** ✓✓Let $B$ be an $m \times n$ matrix. Let $A = B^tB$. Prove that $A$ is symmetric. Prove that $X^tAX \geq 0$ for all $X$. This condition is called "positive semidefinite." What does this condition tell you about the eigenvalues of $A$?

**6.121** Under the hypotheses of Exercise 6.120, prove that $A = B^tB$ is positive definite if and only if rank $B = n$. [*Hint:* Show that if $X^tAX = 0$, then $|BX|^2 = 0$.]

**6.122** ✓✓Let $A$ be an $n \times n$ matrix. Suppose that there is an *orthogonal* matrix $Q$ and a diagonal matrix $D$ such that $A = QDQ^{-1}$. Prove that $A$ must be symmetric. Thus, symmetric matrices are the only matrices that may be diagonalized using an orthogonal diagonalizing basis.

**6.123** Prove Proposition 6.7 on page 381. [*Hint:* Use Theorem 3.4 on page 170.]

**6.124** Let $B$ be an $n \times n$ matrix such that for all $X$ and $Y$ in $\mathbb{R}^n$

$$BX \cdot Y = X \cdot BY$$

Prove that $B$ must be symmetric. (*Hint:* What does the above equation say if $X = I_j$ and $Y = I_k$, where $\{I_1, I_2, \ldots, I_n\}$ is the standard basis for $\mathbb{R}^n$?)

**6.125** Let $A = [a_{ij}]$ be an $n \times n$ symmetric matrix. For $X$ and $Y$ in $\mathbb{R}^n$, let

$$G_A(X, Y) = X^tAY$$

**(a)** Prove that $G_A(X, Y) = G_A(Y, X)$.

**(b)** Prove that

$$G_A(X, Y) = \tfrac{1}{2}(Q_A(X + Y) - Q_A(X) - Q_A(Y))$$

where $Q_A$ is as in equation (6.78) on page 385.

**(c)** Suppose that $A$ and $B$ are $n \times n$ symmetric matrices that define the same quadratic form—that is, $Q_A(X) = Q_B(X)$ for all $X \in \mathbb{R}^n$. It follows from part (b) then, that for all $X$ and $Y$ in $\mathbb{R}^n$, $G_A(X, Y) = G_B(X, Y)$. Use this to prove that $A = B$. [*Hint:* Show that $a_{ij} = G_A(I_i, I_j)$, where $I_i$ is the $i$ standard basis element for $\mathbb{R}^n$.]

### 6.6.1 Computer Projects

The purpose of this exercise set is to check graphically a few of the general principles described above. We begin by graphing the quadratic variety

$$2y^2 + xy + x^2 = 1$$

To graph it, we first must express $y$ as a function of $x$. We do so by writing the formula for the variety as

$$2y^2 + xy + (x^2 - 1) = 0$$

We then solve for $y$ using the quadratic equation:

$$y = \frac{-x \pm \sqrt{x^2 - 8(x^2 - 1)}}{4} = \frac{-x \pm \sqrt{8 - 7x^2}}{4}$$

Notice that the square root is real for $-\sqrt{8/7} \leq x \leq \sqrt{8/7}$.

We plot our curve in two pieces with

```
axis equal; grid on; hold on;
fplot('(-x+sqrt(8-7*x^2))/4',[-1.08,1.08]);
fplot('(-x-sqrt(8-7*x^2))/4',[-1.08,1.08]);
```

We see that the graph appears to be a rotated ellipse.

## EXERCISES

**1.** According to the general theory, the major and minor axes of the ellipse should lie along the coordinate axes determined by the eigenvectors for the matrix $A$ that describes the ellipse. To verify this in MATLAB, enter the matrix A for the ellipse and then set [Q,D]=eig(A). The matrix $Q$ is then a matrix whose columns are eigenvectors of $A$. They are even normalized to have length 1. We plot the corresponding basis vectors (in red) with the commands

```
V(:,1)=Q(:,1); V(:,3)=Q(:,2)
plot(V(1,:),V(2,:), 'r')
```

Get this plot printed.

**2.** On the printed plot from Exercise 1, draw in the axes determined by the eigenvectors. On these axes, put tick marks on every quarter unit, noting that each eigenvector is one unit long. Use a ruler to guarantee the accuracy of your tick marks. According to the general theory, the ellipse should cross the new axes at the points $(\pm\lambda_1, 0)$ and $(0, \pm\lambda_2)$, where $\lambda_1$ and $\lambda_2$ are eigenvalues of $A$. Verify this by reading the appropriate coordinates off of your graph.

**3.** Find a formula for the ellipse centered at $[0, 0]^t$ whose major axis is four units long and lies along the line determined by the vector $[3, 4]^t$ and whose minor axis is two units long.

[*Hint:* If you can figure out what $Q$ and $D$ should be, then you can set $A = QDQ^t$.]

Get MATLAB to graph the ellipse as well as the unit eigenvectors for the corresponding symmetric matrix. Draw in the axes determined by the eigenvectors and put tick marks on them as before to demonstrate that your answer is correct.

## 6.7 THE SINGULAR VALUE DECOMPOSITION (SVD)

Let $B$ be an $n \times n$ symmetric matrix. According to the spectral theorem, there is an $n \times n$ orthogonal matrix $Q$ and an $n \times n$ diagonal matrix $D$ such that

$$B = QDQ^t \tag{6.84}$$

More explicitly, the set $\mathcal{B} = \{Q_1, Q_2, \dots, Q_n\}$ of columns of $Q$ forms an ordered orthonormal eigenbasis for $B$, and $D$ is the $n \times n$ diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \dots, \lambda_n$, where, for each $i$, $\lambda_i$ is the eigenvalue of $B$ corresponding to $Q_i$.

Suppose that $B$ has rank $r$. The zero eigenspace of $B$ is its nullspace, which, from the rank-nullity theorem, has dimension $n - r$. We assume that the $Q_i$ corresponding to the eigenvalue 0 are listed last in $\mathcal{B}$. Then we may write the $n \times n$ matrix $D$ as a partitioned matrix

$$D = \begin{bmatrix} E & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{6.85}$$

where $E$ is the $r \times r$ diagonal matrix with the nonzero eigenvalues $\lambda_1, \dots, \lambda_r$ on its main diagonal and the zero matrix in the lower right-hand corner has size $(n - r) \times (n - r)$.

Let

$$Q = [U, V]$$

where

$$\begin{aligned} U &= [Q_1, Q_2, \dots, Q_r] \\ W &= [Q_{r+1}, \dots, Q_n] \end{aligned} \tag{6.86}$$

Then equation (6.84) implies

$$\begin{aligned} B &= [U, W] \begin{bmatrix} E & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} U^t \\ W^t \end{bmatrix} \\ &= UEU^t \end{aligned} \tag{6.87}$$

We refer to this equation as the **reduced spectral decomposition** of $B$.

If $r \neq n$, the matrices $U$ and $V$ in formula (6.86) are not orthogonal matrices since they are not square. Rather, they are "partially orthogonal."

**Definition 6.12** *An $n \times m$ matrix $U$ is **partially orthogonal** if the set $\{U_1, U_2, \dots, U_m\}$ of columns of $U$ is an orthonormal subset of $\mathbb{R}^n$.*

Orthogonal matrices $Q$ are characterized by the property that $Q^t = Q^{-1}$. Partially orthogonal matrices are characterized by the property that $U^tU = I$. We leave the proof of the following proposition as an exercise.

**Proposition 6.9** *An $n \times m$ matrix $U$ is partially orthogonal if and only if $U^tU = I$.*

An immediate consequence of this proposition is that under the notation in formula (6.87)

$$\begin{aligned} BUU^t &= (UEU^t)UU^t \\ &= UE(U^tU)U^t \\ &= UEU^t \qquad (\text{since } U^tU = I) \\ &= B \end{aligned} \tag{6.88}$$

The reduced spectral decomposition is valid only for $n \times n$ symmetric matrices. There is, however, a similar result, valid for any $m \times n$ matrix, which is referred to as the "reduced singular value decomposition.".

**Theorem 6.27** (Reduced Singular Value Theorem). *Let $A$ be a rank $r$, $m \times n$ matrix. Then there is an $r \times r$ diagonal matrix $D$ with strictly positive entries on the main diagonal, and partially orthogonal matrices $V$ and $U$ of sizes $m \times r$ and $n \times r$, respectively, such that*

$$A = VDU^t \tag{6.89}$$

*Proof.* Note that if $A$ can be decomposed as in formula (6.89), then

$$\begin{aligned} A^tA &= (VDU^t)^tVDU^t \\ &= UD(V^tV)DU^t \\ &= UD^2U^t \qquad (\text{since } V^tV = I) \end{aligned} \tag{6.90}$$

This appears to be similar to formula (6.87), where $B = A^tA$ and $E = D^2$. In fact, it follows from Exercise 6.120 on page 394 that $B = A^tA$ is a symmetric matrix. Furthermore, from Proposition 6.6 on page 375, rank $B$ = rank $A = r$. Thus, equation (6.90) *is* a reduced spectral decomposition of $B$.

These calculations suggest that we *define* $U$ to be the matrix from formula (6.87) corresponding to $B = A^tA$. The columns $U_i$ of $U$ are then eigenvectors for $B$

corresponding to nonzero eigenvalues $\lambda_i$ of $B$. The following computation shows that these $\lambda_i$ are in fact positive:

$$\begin{aligned}
\lambda_i &= \lambda_i |U_i|^2 \\
&= (\lambda_i U_i) \cdot U_i \\
&= (A^t A U_i)^t U_i \\
&= (AU_i)^t (A^t)^t U_i \\
&= (AU_i) \cdot (AU_i) = |AU_i|^2 \geq 0
\end{aligned}$$

Let $D$ be the $r \times r$ diagonal matrix whose main diagonal entries are $d_{ii} = \sqrt{\lambda_i}$, $1 \leq i \leq r$. Then

$$D^2 = E$$

in accordance with equation (6.90).

Solving equation (6.89) for $V$ using $U^t U = I$ suggests that we define

$$V = AUD^{-1} \tag{6.91}$$

Thus, to finish our proof of Theorem 6.27, we must show that for $V$, $D$, and $U$ as just defined

(A) $V^t V = I$
(B) $A = VDU^t$

Item (A) is simple. From formula (6.91) together with $U^t U = I$,

$$\begin{aligned}
V^t V &= (AUD^{-1})^t AUD^{-1} \\
&= D^{-1} U^t A^t AUD^{-1} \\
&= D^{-1} U^t BUD^{-1} \\
&= D^{-1} U^t (UEU^t) UD^{-1} \\
&= D^{-1} ED^{-1} = I
\end{aligned}$$

Item (B) is similar:

$$\begin{aligned}
VDU^t &= (AUD^{-1}) DU^t \\
&= AUU^t
\end{aligned}$$

Hence, we must show that $AUU^t = A$, which is equivalent to

$$A(U^t U - I) = \mathbf{0} \tag{6.92}$$

where $I$ is the $n \times n$ identity matrix and $\mathbf{0}$ is the $m \times n$ zero matrix. However, it follows from equation (6.88) that

$$B(U^tU - I) = \mathbf{0}$$

Equality (6.92) follows since $A$ and $B$ have the same nullspace from the proof of Proposition 6.6 on page 375, finishing our proof. □

The decomposition (6.89) is referred to as the **reduced singular value decomposition (RSVD)** of $A$ and diagonal entries of $D$ are the **singular values** of $A$. They yield important information concerning the matrix. For example, we have noted previously that round-off errors can cause the rank of a matrix $A$ to differ from the number of nonzero rows of a computed echelon form of $A$. Hence, finding the rank $A$ by counting the nonzero rows in a computed echelon form of $A$ might yield erroneous results.

Since the number of nonzero singular values equals the rank, we can use the singular values to find the rank. Of course, round-off errors can also change the number of nonzero singular values. However, hopefully, any spurious singular values would be very small in magnitude. Hence, we could use the number of singular values greater than some prescribed tolerance $\epsilon$ as the definition of what might be referred to as the "numerical" rank of $A$.

We can find the RSVD for an $m \times n$ matrix $A$ by repeating the steps in the proof of Theorem 6.27. Explicitly, the steps are:

1. Find an ordered orthonormal eigenbasis $\{Q_1, Q_2, \dots, Q_n\}$ for $B = A^tA$, where the $Q_i$ correspond to the eigenvalue $\lambda_i$ of $B$ and where the $Q_i$ are ordered so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. We let $U = [Q_1, Q_2, \dots, Q_r]$, where $\{\lambda_1, \lambda_2, \dots, \lambda_r\}$ is the set of nonzero $\lambda_i$.
2. Let $D$ be the $r \times r$ diagonal matrix whose main diagonal entries are $d_{ii} = \sqrt{\lambda_i}$, $1 \leq i \leq r$. and let $V = AUD^{-1}$.

### EXAMPLE 6.21

Find the RSVD for the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

*Solution.* We compute

$$A^tA = \begin{bmatrix} 10 & 2 & 6 \\ 2 & 2 & -2 \\ 6 & -2 & 10 \end{bmatrix}$$

The characteristic polynomial is

$$p(\lambda) = \lambda^3 - 22\lambda^2 + 96\lambda$$

which factors as

$$(\lambda - 16)(\lambda - 6)\lambda$$

yielding eigenvalues 16, 6, and 0. The corresponding eigenvectors are

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

Note that we have listed the eigenvectors in order of decreasing eigenvalue. Upon normalization, $X_1$ and $X_2$ become the columns of $U$:

$$U = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{3} & \sqrt{2} \\ 0 & \sqrt{2} \\ \sqrt{3} & -\sqrt{2} \end{bmatrix}$$

$D$'s main diagonal entries are the square roots of the nonzero eigenvalues. Hence,

$$D = \begin{bmatrix} 4 & 0 \\ 0 & \sqrt{6} \end{bmatrix}$$

Finally,

$$V = AUD^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

*Remark.* Example 6.21 worked out easily due to the small size of $A$ and a careful choice of its entries. There are considerably more efficient methods of computing the reduced singular value decomposition for large matrices that make use of the Householder matrices.

Theorem 6.27 is not as useful as it might be since $V$ and $U$ are not invertible unless $r = n$. We can, however, state a similar theorem in which $V$ and $U$ are replaced by orthogonal matrices. To explain this, let $A$ be an $m \times n$ rank $r$ matrix and let $U$, $V$, and $D$ be defined as in statements 1 and 2 that directly precede Example 6.21. By statement 2, the columns of $U$ are the first $r$ elements of the orthonormal eigenbasis $\{Q_1, Q_2, \ldots, Q_n\}$ for $B = A^tA$. Let

$$U' = [Q_{r+1}, Q_{r+2}, \ldots, Q_n]$$
$$Q = [Q_1, Q_2, \ldots, Q_n] = [U, U']$$

so that $Q$ is an $n \times n$ orthogonal matrix.

The set of columns $\{V_1, V_2, \dots, V_r\}$ of $V$ is an orthonormal set in $\mathbb{R}^m$. Using, for example, the Gram-Schmidt process, we may extend this set to an orthonormal basis $\{V_1, V_2, \dots, V_m\}$ for $\mathbb{R}^m$. Let

$$\begin{aligned} V' &= [V_{r+1}, V_{r+2}, \dots, V_m] \\ Q' &= [V_1, V_2, \dots, V_m] = [V, V'] \end{aligned}$$

Then $Q'$ is an orthogonal matrix. Finally, let

$$E = \begin{bmatrix} D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

where the sizes of the submatrices denoted by "$\mathbf{0}$" are chosen so that $E$ has size $m \times n$. (Recall that $D$ is $r \times r$.)

Thus, from formula (6.89),

$$\begin{aligned} Q'EQ^t &= [V, V'] \begin{bmatrix} D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} U^t \\ (U')^t \end{bmatrix} \\ &= [V, V'] \begin{bmatrix} DU^t \\ \mathbf{0} \end{bmatrix} \\ &= VDU^t = A \end{aligned}$$

This proves the following theorem which is usually referred to as the singular value theorem.

**Theorem 6.28** (Singular Value Theorem). *Let $A$ be a rank $r$, $m \times n$ matrix. Then there is an $r \times r$ diagonal matrix $D$, an $m \times m$ orthogonal matrix $Q'$, and an $n \times n$ orthogonal matrix $Q$ such that*

$$A = Q'EQ^t \tag{6.93}$$

*where $E$ is the $m \times n$ matrix*

$$E = \begin{bmatrix} D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

### EXAMPLE 6.22

Find the singular value decomposition for the matrix $A$ in Example 6.21.

*Solution.* The required computations were all done in the solution to Example 6.21. The matrix $V$ found in Example 6.21 is already orthogonal and, hence,

$$Q' = V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

The matrix $Q$ is $[U, U_3]$ where $U_3 = X_3/|X_3|$. Hence,

$$Q = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{3} & \sqrt{2} & 1 \\ 0 & \sqrt{2} & -2 \\ \sqrt{3} & -\sqrt{2} & -1 \end{bmatrix}$$

Thus, the singular value decomposition is

$$\begin{aligned} A &= Q'[D, \mathbf{0}]Q^t \\ &= \frac{1}{\sqrt{12}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3} & \sqrt{2} & 1 \\ 0 & \sqrt{2} & -2 \\ \sqrt{3} & -\sqrt{2} & -1 \end{bmatrix}^t \end{aligned}$$

### Application of the SVD to Least-Squares Problems

In Section 6.5 we discussed the least squares technique, which is a way of finding the "best possible" solution to an inconsistent system $AX = B$. We simply replace this system with $A^tAX = A^tB$ and solve. Remarkably, as shown in Section 6.5, this new system is consistent and any solution $X$ has the property that $|AX - B|$ is as small as possible. The unique $X$ that is also orthogonal to the nullspace of $A$ is "the least-squares solution to the system." (Other solutions are referred to as "a least-squares solution to the system.")

The least-squares solution $X$ is easily computed using the RSVD of $A$. Specifically, let

$$A = VDU^t$$

as in formula (6.89). Then the least-squares solution is determined by

$$\begin{aligned} A^tAX &= A^tB \\ UDV^tVDU^tX &= UDV^tB \\ UD^2U^tX &= UDV^tB \end{aligned} \tag{6.94}$$

The last equation is valid if

$$DU^tX = V^tB$$

This equation, in turn, is valid if

$$X = A^+B$$

where

$$A^+ = UD^{-1}V^t \tag{6.95}$$

From the definition of $U$, its column space is orthogonal to the 0-eigenspace of $A^tA$ which is the nullspace of $A$. Hence, $X = A^+B$ is orthogonal to the nullspace of $A$, making $X$ the least-squares solution to $AX = B$. The matrix $A^+$ referred to as the **pseudoinverse** of $A$.

### EXAMPLE 6.23

Find the least-squares solution to the equation $AX = B$, where $A$ is as in Example 6.21 and $B = [1, 2]^t$.

*Solution.* From the above discussion and the computations in Example 6.21,

$$
\begin{aligned}
X &= UD^{-1}V^tB \\
&= \frac{1}{\sqrt{2}\sqrt{6}}\begin{bmatrix} \sqrt{3} & \sqrt{2} \\ 0 & \sqrt{2} \\ \sqrt{3} & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/4 & 0 \\ 0 & 1/\sqrt{6} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{24}\begin{bmatrix} 13 \\ 4 \\ 5 \end{bmatrix}
\end{aligned}
$$

**True-False Questions**: Justify your answers.

**6.35** It is possible for a partially orthogonal matrix to have linearly dependent columns.

**6.36** If a partially orthogonal matrix $A$ has linearly independent rows, then $A$ is orthogonal.

**6.37** If $A$ and $A^t$ are both partially orthogonal, then $A$ is orthogonal.

**6.38** Given an $m \times n$ rank $r$ matrix $A$, the matrix $V$ from the RSVD (Theorem 6.27 on page 397) is uniquely determined by the choice of the matrices $D$ and $U$.

**6.39** Suppose that $A$ is a rank 10 matrix with size $13 \times 20$. Let $A = VDU^t$ be a RSVD for $A$. Then $D$ has size $13 \times 13$.

**6.40** Suppose that $A$ is a rank 10 matrix with size $13 \times 20$. Let $A = VDU^t$ be a RSVD for $A$. Then $V$ has size $13 \times 10$.

**6.41** Suppose that $A$ is a rank 10 matrix with size $13 \times 20$. Let $A = VDU^t$ be a RSVD for $A$. Then $U$ has size $10 \times 20$.

**6.42** Suppose that $A$ is a $3 \times 3$ matrix, where $A^tA$ has eigenvalues 1, 2, and 0. Then there is only one $3 \times 2$ partially orthogonal matrix $U$ and only one $2 \times 2$ partially orthogonal matrix $V$ such that $A = VDU^t$, where

$$
D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}
$$

[*Hint:* How are the columns of $Q$ chosen?]

**6.43** Suppose that $A$ is a $3 \times 4$ matrix where $A^tA$ has eigenvalues 1, 2 and 0. Assume that the 0 eigenspace has multiplicity 2. Then there is an infinite number of

$4 \times 2$ partially orthogonal matrices $U$ and $2 \times 2$ partially orthogonal matrices $V$ such that $A = VDU^t$, where

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

[*Hint:* How are the columns of $Q$ chosen?]

## EXERCISES

**6.126** In parts (a)–(d), find the matrices $U$ and $E$ from the reduced spectral decomposition [formula (6.87) on page 396] for the given symmetric matrices.

**(a)** ✓✓

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

**(b)**

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

**(c)** ✓✓

$$B = \begin{bmatrix} 4 & 2 & 6 & -2 \\ 2 & 1 & 3 & -1 \\ 6 & 3 & 9 & -3 \\ -2 & -1 & -3 & 1 \end{bmatrix}$$

[*Hint:* $B = A + 2I$, where $A$ is as in Example 6.20 on page 387.]

**(d)**

$$B = \begin{bmatrix} -11 & 2 & 6 & -2 \\ 2 & -14 & 3 & -1 \\ 6 & 3 & -6 & -3 \\ -2 & -1 & -3 & -14 \end{bmatrix}$$

[*Hint:* $B = A - 13I$, where $A$ is as in Example 6.20 on page 387.]

**6.127** For the following matrices $A$, find (i) the matrices $D$ and $U$ from Theorem 6.27 on page 397 and (ii) the matrices $Q$ and $E$ from Theorem 6.28 on page 401. If appropriate software is available, also find the matrix $V = AUD^{-1}$ and $A^+$ [formula (6.95) on page 402]. As an aid, in some cases we have provided the eigenvalues for $A^tA$.

**(a)** ✓✓ $A = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 3 & 1 \end{bmatrix}$

**(b)** $A = \frac{1}{\sqrt{3}} \begin{bmatrix} -6 & 9 & 6 \\ -8 & 2 & -2 \end{bmatrix}$, 60,15,0

**(c)** ✓✓ $A = \frac{1}{\sqrt{3}} \begin{bmatrix} -6 & -8 \\ 9 & 2 \\ 6 & -2 \end{bmatrix}$

**(d)** $A = \frac{1}{15} \begin{bmatrix} 19 & 8 \\ 14 & -2 \\ 10 & 20 \end{bmatrix}$

**(e)** ✓✓ $A = \frac{1}{15} \begin{bmatrix} 19 & 14 & 10 \\ 8 & -2 & 20 \end{bmatrix}$, $0, 1, 4$

**6.128** $A$ is a $5 \times 3$ matrix and $B = A^tA$ has eigenvalues and eigenvectors as indicated below. Find the matrices $D$ and $U$ from Theorem 6.27 on page 397.

$$\lambda = 0, \quad \begin{bmatrix} 6 \\ -2 \\ -3 \end{bmatrix} \quad \lambda = 2, \quad \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$$

**6.129** For the matrix $A$ described in Exercise 6.128, express the "pseudo inverse" $A^+$ of $A$ in terms of $V$, $D$, and $U$. For $Y$ in $\mathbb{R}^5$, let $X = A^+Y$. What is the "geometric" significance of $AX$? What is special about $X$?

**6.130** In each part below you are given matrices $V$, $D$, and $U$ and a vector $B$. In each case, (i) verify that $V$ and $U$ are partially orthogonal and (ii) find the least-squares solution $X$ of smallest magnitude to the (possibly inconsistent) system $AX = B$, where $A = VDU^t$. (*Note:* It is not necessary to compute $A$.) [*Hint:* In a hand computation, rather than first computing $A^+$, it is easier to compute $U(D^{-1}(V^tB))$.]

**(a)** ✓✓ $B = [2, 3]^t$

$$V = \frac{1}{13}\begin{bmatrix} 5 & -12 \\ -12 & -5 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad U = \frac{1}{3}\begin{bmatrix} 2 & -2 \\ -2 & -1 \\ -1 & -2 \end{bmatrix}$$

**(b)** $B = [1, -2, 1]^t$

$$V = \frac{1}{17}\begin{bmatrix} -1 & -12 \\ -12 & 9 \\ 12 & 8 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \quad U = \frac{1}{17}\begin{bmatrix} -15 & -8 \\ -8 & 15 \end{bmatrix}$$

**(c)** ✓✓ $B = [1, 1, 2]^t$

$$V = \frac{1}{3}\begin{bmatrix} 2 & -2 \\ -2 & -1 \\ -1 & -2 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad U = \frac{1}{13}\begin{bmatrix} 5 & -12 \\ -12 & -5 \end{bmatrix}$$

**(d)** $B = [3, -2]^t$

$$V = \frac{1}{17}\begin{bmatrix} -15 & -8 \\ -8 & 15 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \quad U = \frac{1}{17}\begin{bmatrix} -1 & -12 \\ -12 & 9 \\ 12 & 8 \end{bmatrix}$$

**6.131** Prove that an $m \times n$ matrix $A$ is partially orthogonal if and only if, for all $X \in \mathbb{R}^n$, $|AX| = |X|$. [*Hint:* See the proof of Theorem 6.19 on page 359 and the remark following Proposition 6.3 on page 358.]

**6.132** Prove that an $m \times n$ matrix $A$ is partially orthogonal if and only if for all vectors $X$ and $Y$ in $\mathbb{R}^n$, $X \cdot Y = (AX) \cdot (AY)$. What does this tell you about the angle between $AX$ and $AY$? [*Hint:* The result of Exercise 6.131 above might help in one part.]

**6.133** Suppose that $U$ and $V$ are partially orthogonal matrices for which the product $UV$ is defined. Use Proposition 6.9 on page 397 to prove that $UV$ is also partially orthogonal.

**6.134** Suppose that $U$ and $V$ are partially orthogonal matrices for which the product $UV$ is defined. Use the result of Exercise 6.131 to prove that $UV$ is also partially orthogonal.

**6.135** Let $A$ be an $m \times n$ matrix and $B = Q^tAP$ where $Q$ is an $m \times m$ orthogonal matrix and $P$ is an $n \times n$ orthogonal matrix. Suppose that a RSVD for $B$ is $B = VDU^t$. Show that then $A = (QV)D(PU)^t$ is a RSVD for $A$.

### Computing the SVD Using Householder Matrices

**6.136** In this exercise, we demonstrate the use of the Householder matrices from Section 6.4 in the computation of the RSVD.

In Exercise 6.85.c on page 366, it was shown that if $X$ and $Y$ are elements of $\mathbb{R}^n$ with $|X| = |Y|$, then $M_PX = Y$, where $P = X - Y$ and $M_P$ is the Householder matrix (6.62) on page 362. Recall that $M_P$ is both orthogonal and symmetric.

*Let $A' = \sqrt{3}A$ where $A$ is the matrix from Exercise 6.127.b on page 404.*

**(a)** Note that the first column of $A'$ is $[-6, -8]^t$. Find a $P \in \mathbb{R}^2$ such that $M_P[-6, -8]^t = [10, 0]^t$.

**(b)** Compute $B = M_PA'$, where $M_P$ is as in part (a). You should find

$$M_PA' = \begin{bmatrix} 10 & -7 & -2 \\ 0 & -6 & -6 \end{bmatrix}$$

**(c)** From part (b) the first row of $M_PA'$ is $[10, X]$, where $X = [-7, -2]$. Then $|X| = \sqrt{53} \approx 7.28$. Let $Y = [7.28, 0]$ and let $R = X - Y$. Check that, to three-decimal accuracy, $M_R[-7, -2]^t = [7.28, 0]^t$. How does it follow that, to three-decimal accuracy, $[-7, -2]M_R = [7.28, 0]$?

**(d)** Let $Q$ be the $3 \times 3$ matrix defined by

$$Q = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & M_R \end{bmatrix}$$

where $M_R$ is as in part (c). Let $M_P$ be as in part (a). Compute $C = M_PA'Q^t$ to three-decimal accuracy. You should get a matrix of the form

$$C = \begin{bmatrix} a & b & 0 \\ 0 & c & d \end{bmatrix}$$

*Remark.* An $m \times n$ matrix $B$ is a **band matrix** with **band width** $b$ if $b_{ij} = 0$ for all $i$ and $j$ with $|i - j| \geq b$. Thus, for example, the general $4 \times 6$ band width 2 matrix has the form

$$B = \begin{bmatrix} * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ 0 & * & * & * & 0 & 0 \\ 0 & 0 & * & * & * & 0 \end{bmatrix} \tag{6.96}$$

In the $b = 2$ case, $B$ can also be said to be **tridiagonal**.

The matrix $C$ from Exercise 6.136.d is an upper triangular tridagonal matrix for which $A' = M_P^t CQ$. If $C = VDU^t$ is $C$'s RSVD, then $A' = (M_P^t V)D(UQ^t)^t$ is a RSVD of $A'$. Hence, we used Householder matrices to reduce the computation of $A'$'s RSVD to that of an upper triangular tridiagonal matrix. A similar reduction is possible in general. Exercise 6.137 is another illustration of this reduction.

**6.137** This exercise is a continuation of Exercise 6.136.

**(a)** For the matrix $A$ on the left below, find a vector $P_1 \in \mathbb{R}^3$ and a scalar $a'$ such that $M_{P_1}A$ is a matrix of the form of $A'$ below on the right. *Do not compute either $M_{P_1}$ or $M_{P_1}A$. We only want $P_1$ and $a'$.*

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \quad A' = \begin{bmatrix} a' & b' & c' \\ 0 & e' & f' \\ 0 & h' & i' \end{bmatrix}$$

**(b)** For the matrix $B$ on the left below, find a vector $P_2 \in \mathbb{R}^2$ and a scalar $b''$ such that $BM_{P_2}$ is a matrix of the form of $B'$ below on the right. *Do not compute either $M_{P_2}$ or $BM_{P_2}$. We only want $P_2$ and $b''$.*

$$B = \begin{bmatrix} b' & c' \\ e' & f' \\ h' & i' \end{bmatrix}, \quad B' = \begin{bmatrix} b'' & 0 \\ e'' & f'' \\ h'' & i'' \end{bmatrix},$$

**(c)** With notation as in parts (a) and (b), show that there are scalars $a''$ and $b''$ such that

$$M_{P_1}A \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & M_{P_2} \end{bmatrix} = \begin{bmatrix} a'' & b'' & 0 \\ 0 & e'' & f'' \\ 0 & h'' & i'' \end{bmatrix}$$

**(d)** For the matrix $C$ on the left below, find a vector $P_3 \in \mathbb{R}^2$ and a scalar $e'''$ such that $M_{P_3}C$ is a matrix of the form of $C'$ below on the right. *Do not compute either $M_{P_3}$ or $M_{P_3}C$. We only want $P_3$ and $e'''$.*

$$B = \begin{bmatrix} e'' & f'' \\ h'' & i'' \end{bmatrix}, \quad \begin{bmatrix} e''' & f''' \\ 0 & i''' \end{bmatrix}$$

**(e)** With notation as in parts (a)–(d), show that there are scalars $a'''$ and $b'''$ such that

$$\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & M_{P_3} \end{bmatrix} M_{P_1} A \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & M_{P_2} \end{bmatrix} = \begin{bmatrix} a''' & b''' & 0 \\ 0 & e''' & f''' \\ 0 & 0 & i''' \end{bmatrix}$$

## Diagonalizing Matrices Using Householder Matrices

**6.138** Let

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 3 & 2 \\ 4 & 2 & 1 \end{bmatrix}$$

**(a)** Find a $P \in \mathbb{R}^2$ such that $M_P([3,4]^t) = [5,0]^t$ where $M_P$ is the Householder matrix (6.62) on page 362. *We only want $P$, not $M_P$*! (See Exercise 6.85 on page 366.)

**(b)** Let $Q$ be the $3 \times 3$ matrix

$$Q = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & M_P \end{bmatrix} \tag{6.97}$$

*Without computing $M_P$*, explain why $A' = QAQ^t$ has the form

$$A' = QAQ^t = \left[\begin{array}{c|cc} 1 & a & b \\ \hline 5 & & \\ & & E \\ 0 & & \end{array}\right] \tag{6.98}$$

where

$$E = M_P \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} M_P^t$$

**(c)** Prove that, in (6.98), $a = 5$ and $b = 0$. *Hint:* $A = A^t$.

**(d)** Suppose that $SA'S^{-1} = D$ where $D$ is diagonal. Show that $TAT^{-1} = D$ where $T = QS$.

*Remark.* The **first lower diagonal** of an $n \times n$ matrix $A$ consists of the entries $A_{i,i-1}$, $2 \le i \le n$. $A$ is said to be **upper Hessenberg** if all of its entries below the first lower diagonal equal zero—i.e., $A_{ij} = 0$, $1 \le j < i - 1$. For example the general $4 \times 7$ upper Hessenberg matrix would have the form

$$A = \begin{bmatrix} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * \end{bmatrix} \tag{6.99}$$

Exercise 6.138.b illustrates that Householder matrices may be used to reduce the diagonalization of a general $n \times n$ matrix $A$ to that of an upper Hessenberg matrix $A'$. Exercise 6.138.c illustrates that if $A$ is symmetric, then $A'$ may be chosen to be a symmetric band width 2 matrix. Exercise 6.139 illustrates that the same reduction works for larger matrices.

**6.139** Let

$$B = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 61 & 23 & 14 \\ 2 & 23 & -17 & -8 \\ 2 & 14 & -8 & 1 \end{bmatrix}$$

**(a)** Find an $R \in \mathbb{R}^3$ such that $M_R[1, 2, 2]^t = [3, 0, 0]^t$.

**(b)** Let $Q_1$ be the $4 \times 4$ matrix

$$Q_1 = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & M_R \end{bmatrix}$$

*Without computing* $M_R$, explain why $B_1 = Q_1 B Q_1^t$ has the form

$$B_1 = \left[\begin{array}{c:ccc} 1 & * & * & * \\ \hdashline 3 & & & \\ 0 & & C & \\ 0 & & & \end{array}\right]$$

where

$$C = M_R \begin{bmatrix} 61 & 23 & 14 \\ 23 & -17 & -8 \\ 14 & -8 & 1 \end{bmatrix} M_R^t \tag{6.100}$$

**(c)** It turns out that $C = \frac{1}{9}A$ where $A$ is as in Exercise 6.138. Let

$$Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}$$

where $Q$ is as in Exercise 6.138.b. Prove that $Q_2 B_1 Q_2^t = B_2$ where $B_2$ is an upper triangular Hessenberg matrix.

**(d)** Use the symmetry of $A$ to prove that $B_2$ is a symmetric, band width 1 matrix.

## 6.8 HERMITIAN SYMMETRIC AND UNITARY MATRICES

In Section 5.3, we discussed matrices having complex entries. In this section we study such matrices more deeply. We use "Hermitian geometry," which is named after the French mathematician Charles Hermite (1822–1901).

Recall that $\mathbb{C}^n$ is the set of all $n \times 1$ matrices with complex entries. Hence, the general element of $\mathbb{C}^n$ is

$$Z = [z_1, z_2, \ldots, z_n]^t \tag{6.101}$$

where

$$z_k = x_k + iy_k$$

with $i = \sqrt{-1}$. The space $\mathbb{C}^n$ is a complex vector space. (See page 303.)

In Section 5.3, we commented that $\mathbb{C}$ may be identified with $\mathbb{R}^2$. Specifically, $z = x + iy$ is identified with

$$z_r = \begin{bmatrix} x \\ y \end{bmatrix}$$

If $w$ is a complex number, then multiplication by $w$ may be described using a $2 \times 2$ matrix. Specifically, if $w = u + iv$ and $z = x + iy$, then

$$\begin{aligned} wz &= (u + iv)(x + iy) \\ &= (ux - vy) + i(vx + uy) \end{aligned}$$

Hence

$$\begin{aligned} (wz)_r &= \begin{bmatrix} ux - vy \\ vx + uy \end{bmatrix} = \begin{bmatrix} u & -v \\ v & u \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= < u + iv > z_r \end{aligned} \tag{6.102}$$

where

$$< u + iv >= \begin{bmatrix} u & -v \\ v & u \end{bmatrix} \tag{6.103}$$

It is left as an exercise to prove that for all complex numbers $z$ and $w$,

$$< zw >=< z >< w > \tag{6.104}$$

Similarly, we may think of $\mathbb{C}^n$ as being $\mathbb{R}^{2n}$ by identifying the point $Z$ above with

$$Z_r = [x_1, y_1, x_2, y_2, \dots , x_n, y_n]^t \tag{6.105}$$

which belongs to $\mathbb{R}^{2n}$. It follows from formula (6.102) that if $A$ is an $m \times n$ complex matrix, then

$$(AZ)_r =< A > Z_r \tag{6.106}$$

where $< A >$ is the $2m \times 2n$ real matrix defined by

$$< A >= \begin{bmatrix} < a_{11} > & \cdots & < a_{1n} > \\ \vdots & \vdots & \vdots \\ < a_{1m} > & \cdots & < a_{mn} > \end{bmatrix} \tag{6.107}$$

and the $< a_{ij} >$ are the $2 \times 2$ matrices defined by (6.103). The following proposition is an immediate consequence of (6.104).

**Proposition 6.10** *Let A and B be, respectively, complex $m \times n$ and $n \times p$ matrices. Then*

$$< AB >=< A >< B >$$

In light of (6.105), it is natural to define the length of $Z$ by

$$|Z| = |Z_r| = \sqrt{x_1^2 + y_1^2 + \cdots + x_n^2 + y_n^2} \tag{6.108}$$

Hence, for example,

$$|\ [3 - 4i, 1 + 2i]^t\ | = \sqrt{3^2 + (-4)^2 + 1^2 + 2^2} = \sqrt{30}$$

Similarly, if $W = [w_1, z_2, \dots , w_n]^t \in \mathbb{C}^n$ where $w_i = u_i + v_i$, we define

$$Z \cdot W = Z_r \cdot W_r = x_1u_1 + y_1v_1 + \cdots + x_nu_n + y_nv_n \tag{6.109}$$

Following Definition 6.10 on page 356, we say:

**Definition 6.13** *An $n \times n$ complex matrix $A$ is unitary if*

$$|AZ| = |Z|$$

*for all $Z \in \mathbb{C}^n$.*

It is clear from the preceding discussion that $A$ is a unitary matrix if and only if $< A >$ is an orthogonal matrix which, in turn, is equivalent with $< A >^t < A >= I$. (Theorem 6.19 on page 359.)

Notice that

$$< u + iv >^t = \begin{bmatrix} u & -v \\ v & u \end{bmatrix}^t = \begin{bmatrix} u & v \\ -v & u \end{bmatrix} = < u - iv >$$

Hence, for $w \in \mathbb{C}$, $< w >^t = < \overline{w} >$. More generally, it follows from formula (6.107) that for any $m \times n$ complex matrix $A$

$$< A >^t = < \overline{A}^t > \tag{6.110}$$

The matrix

$$A^* = \overline{A}^t \tag{6.111}$$

is referred to as the **Hermitian adjoint** of $A$. The following theorem follows immediately from Theorem 6.19 on page 359 and Proposition 6.10.

**Theorem 6.29** *Let $A$ be an $n \times n$ complex matrix. Then $A$ is unitary if and only $A^*A = I$.*

The columns of an orthogonal matrix form an orthonormal basis for $\mathbb{R}^n$. This is a consequence of formula (6.9) on page 311, which states that for all $X, Y \in \mathbb{R}^n$

$$X \cdot Y = Y^t X \tag{6.112}$$

*This formula does not hold in $\mathbb{C}^n$.*

For example, if $Z = [1 + 2i, 3 + 4i]^t$, $W = [5 - 6i, 7i + 8i]^t \in \mathbb{C}^2$, then

$$\begin{aligned} Z \cdot W &= 1 \cdot 5 - 2 \cdot 6 + 3 \cdot 7 + 4 \cdot 8 = 46 \\ W^t Z &= (1 + 2i)(5 - 6i) + (3 + 4i)(7 + 8i) = 6 + 56i \end{aligned}$$

The correct analogue of (6.112) involves the complex conjugate. Note that for $z = x + iy, w = u + iv \in \mathbb{C}$,

$$\overline{w}z = (u - iv)(x + iy) = xu + yv + (yu - xv)i$$

Thus

$$\text{re}\,(\overline{w}z) = xu + yv = z \cdot w$$

(If $u = a + ib$ is a complex number, then $a$ is called the **real part** of $u$ and is denoted re $u$.)

More generally, from (6.109),

$$\begin{aligned} Z \cdot W &= \text{re}\,(\overline{w}_1 z_1 + \overline{w}_2 z_2 + \cdots + \overline{w}_n z_n) \\ &= \text{re}\,\left([\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_n]([z_1, z_2, \vdots, z_n]^t)\right) \\ &= \text{re}\,(H(Z, W)) \end{aligned} \tag{6.113}$$

where

$$H(Z, W) = \overline{W}^t Z \tag{6.114}$$

The following proposition is left as an exercise (Exercise 6.144.e).

**Proposition 6.11** *Let $A$ be as $n \times n$ matrix. Then for all $Z, W \in \mathbb{C}^n$*

$$H(AZ, W) = H(Z, A^* W) \tag{6.115}$$

*In particular, if $A$ is unitary,*

$$H(AZ, AW) = H(Z, W) \tag{6.116}$$

The form $H$ plays the same role in $\mathbb{C}^n$ as the dot product does in $\mathbb{R}^n$. We make the following definition in analogy with Definition 6.5 on page 315. Note that it follows from formula (6.113) that Hermitian orthogonal sets are also orthogonal.

**Definition 6.14** *A set $\mathcal{B} = \{P_1, P_2, \ldots, P_k\}$ of vectors in $\mathbb{C}^n$ is Hermitian orthogonal if $|P_j| = 1$ for all $j$ and $H(P_i, P_j) = 0$ for $i \neq j$. A Hermitian orthogonal basis for a complex subspace $\mathcal{W}$ of $\mathbb{C}^n$ is an Hermitian orthogonal set of vectors in $\mathcal{W}$ that forms a complex basis for $\mathcal{W}$.*

The following results are, respectively, the Hermitian analogues of Theorem 6.5 on page 314 and Theorem 6.8 on page 323. The proofs are straightforward adaptations of the corresponding proofs in the real case.

**Theorem 6.30** *Any Hermitian orthogonal set of vectors in $\mathbb{C}^n$ is linearly independent.*

**Theorem 6.31** *Every nonzero complex subspace $\mathcal{W} \subset \mathbb{C}^n$ has a Hermitian orthogonal basis.*

The proof of the following proposition is the same as that of Proposition 6.3 on page 358.

**Proposition 6.12** *Let $A$ be an $n \times n$ unitary matrix. Then the set $\mathcal{B} = \{A_1, A_2, \ldots, A_n\}$, where $A_k$ is the kth column of $A$, is a Hermitian orthogonal basis of $\mathbb{C}^n$.*

We say that an $n \times n$ matrix $A$ is **Hermitian symmetric** if $A^* = A$. For example, if

$$A = \begin{bmatrix} 2 & 1+i \\ 1-i & -1 \end{bmatrix}$$

then

$$A^* = \overline{\begin{bmatrix} 2 & 1-i \\ 1+i & -1 \end{bmatrix}} = \begin{bmatrix} 2 & \overline{1+i} \\ \overline{1-i} & -1 \end{bmatrix} = A$$

From equation (6.110) on page 412, $A$ being Hermitian symmetric is equivalent with $< A >$ being a symmetric matrix. The following is the Hermitian analogue of the spectral theorem. (Theorem 6.23 on page 385).

**Theorem 6.32** (Hermitian spectral theorem). *Let $A$ be an $n \times n$, Hermitian-symmetric matrix. Then there is a Hermitian orthogonal basis $\mathcal{B} = \{Q_1, Q_2, \ldots, Q_n\}$ for $\mathbb{C}^n$ consisting of eigenvectors for A. The matrix $Q = [Q_1, Q_2, \ldots, Q_n]$ is then a unitary matrix such that $D = Q^{-1}AQ$ is a real diagonal matrix. Furthermore all of the eigenvalues of A are real.*

The proof uses the following theorem which is the Hermitian version of Theorem 6.22 on page 384. The proof is left as an exercise (Exercise 6.144).

**Theorem 6.33** *Let $A$ be an $n \times n$ Hermitian-symmetric matrix and let $X$ and $Y$ be eigenvectors for A corresponding, respectively, to eigenvalues $\lambda$ and $\beta$, where $\lambda \neq \beta$. Then $H(X, Y) = 0$.*

*Proof. (of Theorem 6.32)* The spectral theorem (Theorem 6.23 on page 385) implies that $\mathbb{R}^{2n}$ has a real $\tilde{\mathcal{B}} = \{\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_{2n}\}$ consisting of eigenvectors of $< A >$ corresponding to real eigenvalues $\beta_1 < \beta_2 < \cdots < \beta_k$.

Let $\tilde{X}_i = (X_i)_r$ where $X_i \in \mathbb{C}^n$. The $X_i$ are eigenvectors for $A$ since, from formula (6.106) on page 411 and Exercise 6.143 on page 417, for some index $j$,

$$(AX_i)_r =< A > \tilde{X}_i = \beta_j \tilde{X}_i = (\beta_j X_i)_r$$

Since $\tilde{\mathcal{B}}$ is a basis of $\mathbb{R}^{2n}$, every element of $\mathbb{C}^n$ is a linear combination (with real coefficients) of elements from $\mathcal{B} = \{X_1, X_2, \ldots, X_{2n}\}$. Eliminating dependent elements of $\tilde{\mathcal{B}}$ produces a complex basis $\mathcal{B}_o = \{Z_1, Z_2, \ldots, Z_n\}$ for $\mathbb{C}^n$ consisting of eigenvectors for $A$, each corresponding to a real eigenvalue $\beta_j$.

To prove the existence of a *Hermitian* orthogonal basis of eigenvectors, let, for $1 \le i \le k$,

$$\mathcal{W}_i = \text{span } \{Z_j \mid Z_j \text{ is a } \beta_i\text{-eigenvector for } A\}$$

where the span is in the sense of complex vector spaces.

From Proposition 6.12 each $\mathcal{W}_i$ has a Hermitian orthogonal basis $\{Z_1^i, \ldots, Z_{k_i}^i\}$. From Theorem 6.29, $i \neq j$ implies that $H(Z_p^i, Z_q^j) = 0$ for all $p$ and $q$. If $i = j$, Hermitian orthogonality in $\mathcal{W}_i$ implies that $H(Z_p^i, Z_q^i) = 0$ unless $p = q$. Hence $\mathcal{B}_1 = \{Z_j^i \mid 1 \le i \le k, 1 \le j \le k_i\}$ is a Hermitian orthogonal set. The span of $\mathcal{B}_1$ contains all of the $\mathcal{W}_i$ and, hence, all of the $Z_j$, proving that $\mathcal{B}_o$ also spans $\mathbb{C}^n$. Our theorem follows. □

## EXAMPLE 6.24

Let $A$ be as shown. Find a unitary matrix $U$ and a diagonal matrix $D$ such that $A = UDU^*$, given that $X = [5, 4i, -6 + 2i]^t$, $Y = [-4i, 5, -2 - 6i]^t$, and $Z = [-6 - 2i, -2 + 6i, -1]^t$ are eigenvectors for $A$.

$$A = \begin{bmatrix} 59 & 20\,i & 12 + 4\,i \\ -20\,i & 62 & -2 + 6\,i \\ 12 - 4\,i & -2 - 6\,i & 41 \end{bmatrix}$$

*Solution.* We compute

$$\begin{aligned} AX &= [5, 4i, -6 + 2i]^t = X \\ AY &= [-8i, 10, -4 - 12i]^t = 2Y \\ AZ &= [-18 - 6i, -6 + 18i, -3]^t = 3Z \end{aligned}$$

Hence the eigenvalues of $A$ are $1, 2, 3$ and

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

From Theorem 6.33, $\{X, Y, Z\}$ is a Hermitian orthogonal set. Our Hermitian orthogonal basis is obtained by dividing each of these vectors by its length. We find $|X| = |Y| = |Z| = 9$. Hence

$$Q = \left[\frac{X}{|X|}, \frac{Y}{|Y|}, \frac{Z}{|Z|}\right] = \frac{1}{9}\begin{bmatrix} 5 & -4i & -6 - 2i \\ 4i & 5 & -2 + 6i \\ -6 + 2i & -2 - 6i & -1 \end{bmatrix}$$

The following result is left as an exercise. (Exercise 6.144.d.)

**Theorem 6.34** (Hermitian Product Theorem). *Let $Z$, $W$, and $U$ be vectors in $\mathbb{C}^n$. Let $c \in \mathbb{C}$ be a scalar. Then*

*(a)* $H(Z, W) = \overline{H(W, Z)}$ (*Hermitian symmetry law*)
*(b)* $H(Z + W, U) = H(Z, U) + H(W, U)$ (*left-additive law*)
*(c)* $H(U, Z + W) = H(U, Z) + H(U, W)$ (*right-additive law*)
*(d)* $H(cZ, W) = cH(Z, W) = H(Z, \bar{c}\, W)$ (*scalar law*)
*(e)* $|Z|^2 = H(Z, Z)$

**True-False Questions**: Justify your answers.

**6.44** The Hermitian symmetric matrix $A$ has characteristic polynomial $p(\lambda) = \lambda^3(\lambda - 1)^2(\lambda + 3)$. The nullspace of $A$ might have dimension 2.

**6.45** It is possible for a Hermitian symmetric matrix $A$ to have $2 + 4i$ as an eigenvalue.

**6.46** The polynomial $p(\lambda) = (\lambda - 1)(\lambda - 2)^3(\lambda^2 + 1)$ could be the characteristic polynomial of a Hermitian symmetric matrix.

**6.47** It is impossible for the given vectors to be eigenvectors for a Hermitian symmetric matrix $A$ corresponding respectively to the given eigenvalues.

**(a)** $[1 + i, 1 + i, 2]^t$ and $[1 + i, 1 + i, -2]^t$, eigenvalues 3 and 5.

**(b)** $[1 + i, 1 + i, 2]^t$ and $[1 + i, 1 + i, -2]^t$, eigenvalues 3 and 5.

**(c)** $[1 + i, 2 - i, 3]^t$ and $[1 - i, 2 - i, 1]^t$, eigenvalue 3.

**6.48** The polynomial $p(\lambda) = (\lambda^2 - 1)(\lambda^2 + 4)^2$ could not be the characteristic polynomial of a Hermitian symmetric matrix.

**6.49** There exist $4 \times 4$ unitary matrices with rank 3.

**6.50** If $A$ is an $n \times n$ unitary matrix and $B$ is an $n \times 1$ matrix, then the equation $AX = B$ has a unique solution.

**6.51** The following matrix is unitary if and only if $|z| = 1$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & z & -z \\ 0 & z & z \end{bmatrix}$$

**6.52** All unitary matrices have at least one imaginary entry.

**6.53** Suppose that $A = [A_1, A_2, A_3]$ is a unitary matrix where $A_i$ are the columns of $A$. Then $|A_1 + 2iA_2 + 2A_3| = 3$.

**6.54** Let $Q$ be an $n \times n$ unitary matrix and let $a, b, c \in \mathbb{N}$. Then

$$Q^a(Q^*)^b Q^{-1}(Q^*)^c(Q^{-1})^* = Q^{a-b+c-2}$$

## EXERCISES

**6.140** Compute $H(Z, W)$ where $Z$ and $W$ are respectively

**(a)** ✓✓ $[1+i, 2i, 3]^t$, $[2-i, 1+i, -5i]^t$

**(b)** $[2, 2i, 2, 2i]^t$, $[2i, 2, 2i, 2]^t$.

**(c)** ✓✓ $[3+4i, 2i+3]^t$, $[1-i, 1+i]^t$.

**(d)** $[i, i, i, i]^t$, $[2-i, 2-i, 2-i, 2-i]^t$.

**6.141** This exercise studies the Hermitian analogue of formula (3.6) on page 170.

**(a)** Let $A$ and $B$ be as in Exercise 5.44 on page 304. Show by direct computation that $(AB)^* = B^*A^*$.

**(b)** ✓✓Prove that the property stated in part (a) is true for any complex matrices that can be multiplied.

**6.142** Let notation be as in (6.103) and (6.107) on page 411. Find

**(a)** ✓✓ $< 3-2i >$

**(b)** $< \begin{bmatrix} 3-2i & 2 \\ 4i & 1+i \end{bmatrix} >$

**(c)** $< \begin{bmatrix} 1+i & 2-i \\ 2-i & 3 \end{bmatrix} >$

**6.143** Prove that if $A$ is an $m \times n$ matrix and $c \in \mathbb{R}$, then $< cA >= c < A >$. Give an example to show that in general this is false for $c \in \mathbb{C}$.

**6.144** Prove the following results:

**(a)** Formula (6.104) on page 411.

**(b)** ✓✓Theorem 6.30 on page 413.

**(c)** Theorem 6.33 on page 414.

**(d)** Theorem 6.34 on page 416.

**(e)** Proposition 6.11 on page 413.

**6.145** For $Z_1, Z \in \mathbb{C}^n$, $|Z_1| = 1$, let $Z_0 = Z - H(Z_1, Z)Z_1$. Prove that the real part of $H(Z_1, Z_0)$ is zero.

**6.146** In each case change just one entry of the given matrix to make it Hermitian symmetric.

**(a)** $\begin{bmatrix} 2 & 1+i \\ 1+i & 3 \end{bmatrix}$

**(b)** $\begin{bmatrix} 2i & 1+i \\ 1-i & 3 \end{bmatrix}$

**(c)** $\begin{bmatrix} 2 & 1+i & 3-i \\ 2-i & 3 & 4+5i \\ 3+i & 4-5i & -2 \end{bmatrix}$

**6.147** Give an example of a $4 \times 4$ Hermitian-symmetric matrix having all entries nonzero. Use as few real entries as possible.

**6.148** Prove that all of the diagonal entries of an $n \times n$ Hermitian-symmetric matrix $A$ must be real.

**6.149** Prove that the following matrices are unitary.

**(a)**
$$\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ i+1 & -1 \end{bmatrix}$$

**(b)**
$$\frac{1}{9}\begin{bmatrix} 5 & -4i & -6-2i \\ 4i & 5 & -2+6i \\ -6+2i & -2-6i & -1 \end{bmatrix}$$

**(c)**
$$\frac{1}{11}\begin{bmatrix} 10 & -1-2i & -1-2i \\ -1+2i & 6 & -5 \\ -1+2i & -5 & 6 \end{bmatrix}$$

**(d)**
$$\frac{1}{18}\begin{bmatrix} 11+2i & -8i & -11-2i \\ 2-2i & 10 & -2+2i \\ -5+2i & -4-12i & 5-2i \end{bmatrix}$$

**6.150** Find a unitary matrix $U$ and a diagonal matrix $D$ such that $A = UDU^*$. In part (c) the given vectors are eigenvectors for $A$.

**(a)** ✓✓ $A = \begin{bmatrix} 2 & 1+i \\ 1-i & 1 \end{bmatrix}$ **(b)** $A = \begin{bmatrix} -1 & 1+3i \\ 1-3i & 1 \end{bmatrix}$

**(c)** $A = \frac{1}{8}\begin{bmatrix} 4 & 1-3i & -1+i \\ 1+3i & 11 & 2+i \\ -1-i & 2-i & 15 \end{bmatrix}$
$[2, -1-3i, 1+i]^t$, $[-1+3i, -1, 2-i]^t$, $[1-i, 2+i, 3]^t$

**6.151** ✓✓Let $A$ be an $m \times n$ complex matrix and let $c \in \mathbb{C}$. Prove that $(cA)^* = \bar{c}(A^*)$.

**6.152** ✓ $\lambda$ be a complex eigenvalue for an $n \times n$ unitary matrix $U$. Show that $|\lambda| = 1$.

**6.153** Prove that for all $X$ and $Y$ in $\mathbb{C}^n$,

$$H(X, Y) = \tfrac{1}{4}(|X+Y|^2 - |X-Y|^2 + i(|X+iY|^2 - |X-iY|^2)$$

**6.154** Use Exercise 6.153 together with Definition 6.13 on page 412 that if $A$ is unitary, then $H(AX, AY) = H(X, Y)$ for all $X, Y \in \mathbb{C}^n$.

**6.155** This exercise provides an alternate proof that $A$ is unitary, $A^*A = I$.

**(a)** ✓✓Suppose that $A$ is an $n \times n$ matrix such that $H(AZ, W) = 0$ for all $Z, W \in \mathbb{C}^n$. Prove that $A = \mathbf{0}$. *Hint:* Let $W = AZ$.

**(b)** Use part (a) together with Exercise 6.154 to prove directly from Definition 6.13 on page 412 that if $A$ is unitary, then $A^*A = I$.

**6.156** Let $A$ be a Hermitian symmetric matrix. Let $\mathcal{V}$ be the image of $T_A$ and $\mathcal{N}$ the nullspace of $A$. (See Definition 3.2 on page 152.) Show that $X \in \mathcal{N}$ if and only if $H(X, Y) = 0$ for all $Y \in \mathcal{V}$.

**6.157** State and prove a generalization of the result stated in Exercise 6.156 relating the image and nullspace of $A$ with those of $A^*$.

**6.158** ✓✓Let $P \in \mathbb{C}^n$. Prove that $M_P$ is both unitary and Hermitian symmetric where

$$M_P = I - \frac{2}{H(P,P)} PP^*$$

## CHAPTER SUMMARY

Chapter 6 studied distance and angle. A basis is (***orthogonal***) if its elements are mutually perpendicular and ***orthonormal*** if additionally every basis element has length 1. In this case Theorem 6.4 on page 313 provides a simple formula for the coordinates of a given point. The ***Gram-Schmidt process*** provides a technique for transforming nonorthogonal bases into orthogonal bases. It folllows that every subspace of $\mathbb{R}^n$ has an orthonormal basis. The ***QR factorization*** of matrices also follows.

The ***projection*** $\mathrm{Proj}_{\mathcal{W}}(X)$ of a given vector $X \in \mathbb{R}^n$ onto a subspace $\mathcal{W}$ of $\mathbb{R}^n$ is the closest point in $\mathcal{W}$ to $X$. Both Theorem 6.7 and Theorem 6.20 on page 373 (the least squares theorem) provide formulas for $\mathrm{Proj}_{\mathcal{W}}(X)$.

Projections are useful in the study of inconsistent systems $AX = B$. Given such a system, let $B_o = \mathrm{Proj}_{\mathcal{W}}(X)$ where $\mathcal{W}$ is the column space of $A$. Then there is an $X$ such that $AX = B_o$. Furthermore $|AX - B|$ is as small as possible.

In Section 6.3 we defined a ***scalar product*** for functions that is analogous to the dot product of vectors. We used this concept to compute projections in spaces of functions. We used these ideas to approximate periodic functions (waves) by combinations of “pure tones” (sine and cosines). We also used approximations by ***Wavelets*** to a compress data.

Another important topic from this chapter was ***orthogonal matrices*** (Section 6.4). An $n \times n$ matrix $A$ is ***orthogonal*** if multiplication of vectors by $A$ does not change their lengths. Orthogonal matrices are characterized by the property that their inverse equals their transpose ($A^tA = I$). They are also characterized by the fact that their columns are mutually orthogonal and have length 1. In $\mathbb{R}^2$ and $\mathbb{R}^3$, the transformation defined by an orthogonal matrix can be expressed as a product of rotations and reflections. An important class of $n \times n$ orthogonal matrices is the ***Householder*** matrices, which describe reflections about hyperplanes through the origin in $\mathbb{R}^n$. The Householder matrices are both orthogonal and symmetric.

In Section 6.6, we studied generalizations of curves defined by equations of the form $ax^2 + bxy + cy^2 = d$ (***quadratic curves***). Specifically, we studied the solution sets in $\mathbb{R}^n$ to equations of the form

$$X^tAX = d$$

where $A$ is an $n \times n$ ***symmetric*** matrix—that is, $A = A^t$. The solution to such an equation is a ***quadratic variety***. The function $Q_A(X) = X^tAX$ is said to be a ***quadratic form***. It turned out that the quadratic form $Q_A$ has a particularly simple form if we use coordinates defined by the eigenvectors of the matrix $A$ (the principal axis theorem in Section 6.6).

A consequence of the spectral theorem is that any $n \times n$ symmetric matrix $A$ may be expressed in the form $A = QDQ^t$, where $Q$ is an orthogonal $n \times n$ matrix and $D$ is an $n \times n$ diagonal matrix.

In Section 6.7, we proved the existence of the reduced singular value decomposition (RSVD) for an $m \times n$ matrix $A$, which is an analogue of the spectral theorem. This led to the full singular value decomposition. We discussed an application of this theory to least squares problems.

Finally, in Section 6.8, we studied **Hermitian symmetric** and **unitary** matrices, which are, respectively, the complex analogues of symmetric and orthogonal matrices. Explicitly, given an $m \times n$ matrix $A$, we defined $A^* = \overline{A}^t$. Then a square matrix $A$ is Hermitian symmetric if $A^* = A$ and orthogonal if $A^*A = I$. The key result in Section 6.8 is Theorem 6.32 on page 414 which states that the spectral theorem (Theorem 6.23 on page 385) is valid for complex matrices, with "symmetric" replaced by "Hermitian symmetric" and "orthogonal" replaced by "unitary."

# CHAPTER 7

# GENERALIZED EIGENVECTORS

## 7.1 GENERALIZED EIGENVECTORS

In this chapter we continue the discussion of eigenvectors begun in Chapter 5. As an introduction, we consider the following example which we were unable to solve in Section 5.1 (Example 5.4 on page 277).

### EXAMPLE 7.1

Compute $A^{100}C$ where $C = [1, 1, 1]^t$ and

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

In Section 5.1, we solved such problems by expressing $C$ as a linear combination of eigenvectors for $A$ and then making use of the observation that if $AX = \lambda X$, then $A^n X = \lambda^n X$. Unfortunately, $A$ is not diagonalizable. Specifically, the characteristic polynomial of $A$ is

$$p(\lambda) = (3 - \lambda)^2(5 - \lambda).$$

*Linear Algebra: Ideas and Applications*, Fourth Edition. Richard C. Penney.

Companion website: www.wiley.com/go/penney/linearalgebra

The $\lambda = 3$ and $\lambda = 5$ eigenspaces are both one-dimensional and are spanned by

$$X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } W_0 = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} \tag{7.1}$$

respectively. Since $C$ is not a linear combination of $X_0$ and $W_0$, the technique breaks down.

It turns out, however, that there exists a vector $X_1 \in \mathbb{R}^3$, a "generalized eigenvector," satisfying

$$(A - 3I)^2 X_1 = \mathbf{0}, \quad (A - 3I)X_1 \neq \mathbf{0} \tag{7.2}$$

Equation (7.2) is equivalent with $Z_1 = (A - 3I)X_1$ being a 3-eigenvector since $Z_1 \neq 0$ and $(A - 3I)Z_1 = \mathbf{0}$. Hence, we can find such a vector $X_1$ by solving the system $(A - 3I)X_1 = X_0$, which has augmented matrix

$$[A - 3I, X_0] = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

We reduce, obtaining

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The variable $x$ is free. The general solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{7.3}$$

We set $x = 0$ obtaining $X_1 = [0, 1, 0]^t$ as an order 2 generalized eigenvector.

To compute $A^n C$, note that

$$C = [1, 1, 1]^t = \tfrac{1}{4}(X_0 + 2X_1 + W_0) \tag{7.4}$$

It suffices to compute $A^n X_0$, $A^n X_1$, and $A^n W_0$. Since $X_0$ and $W_0$ are eigenvectors,

$$\begin{aligned} A^n X_0 &= 3^n X_0 = [3^n, 0, 0]^t \\ A^n W_0 &= 5^n W_0 = [3(5^n), 2(5^n), 4(5^n)]^t \end{aligned} \tag{7.5}$$

To compute $A^n X_1$ note that

$$(A - 3I)X_1 = X_0, \quad (A - 3I)X_0 = \mathbf{0} \tag{7.6}$$

Let $B = A - 3I$. Since $A = 3I + B$,

$$\begin{aligned}
AX_1 &= (3I + B)X_1 = 3X_1 + X_0 \\
A^2X_1 &= (3I + B)^2X_1 \\
&= (9I + 6B + B^2)X_1 = 9X_1 + 6X_0 \\
A^3X_1 &= (3I + B)^3X_1 \\
&= (27I + 27B + 9B^2 + B^3)X_1 = 27X_1 + 27X_0
\end{aligned}$$

To find the general formula, recall that the binomial theorem says that if $B$ and $C$ are numbers, then

$$(B + C)^n = \sum_{k=0}^{n} \binom{n}{k} B^k C^{n-k} \tag{7.7}$$

where

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} \tag{7.8}$$

Exercise 7.11 shows that (7.7) also holds if $B$ and $C$ are $n \times n$ matrices such that $BC = CB$.

Since

$$\binom{n}{1} = \frac{n!}{(n-1)!} = n$$

we see

$$\begin{aligned}
A^nX_1 &= (3I + B)^nX_1 \\
&= ((3I)^n + nB(3I)^{n-1} + \cdots + B^n)X_1 \\
&= 3^nX_1 + n3^{n-1}X_0 \\
&= [n3^{n-1}, 3^n, 0]^t
\end{aligned}$$

Finally,

$$\begin{aligned}
A^{100}C &= \tfrac{1}{4}(A^{100}X_0 + 2A^{100}X_1 + A^{100}W_0) \\
&= \tfrac{1}{4}[3^{100} + 200(3^{99}) + 3(5^{100}), 2(3^{100}) + 2(5^{100}), 4(5^{100})]^t
\end{aligned} \tag{7.9}$$

solving Example 7.1.

*Remark.* Notice that the spanning vector, $[1, 0, 0]^t$, in (7.3) is our original eigenvector $X_0$. This is to be expected since any two solutions to $(A - 3I)Y = Z$ will differ by an element of the nullspace of $A - 3I$ which is the 3-eigenspace.

The technique just illustrated is quite general, although we will need to make use of the imaginary eigenvalues when they exist. *In order to discuss the real and complex cases simultaneously, we let* $\mathbb{K}$ *denote either* $\mathbb{R}$ *or* $\mathbb{C}$.[1] When $\mathbb{K} = \mathbb{C}$, except when noted otherwise, vector space concepts such as span, dimension, and subspace are all taken in the sense of complex vector spaces. The reader who is uncomfortable with complex numbers can assume $\mathbb{K} = \mathbb{R}$ throughout.

**Definition 7.1** *Let $A$ be an $n \times n$ matrix and let $\lambda \in \mathbb{K}$. A nonzero vector $X \in \mathbb{K}^n$ is said to be a **$\lambda$-generalized eigenvector** for $A$ if there is a natural number $k$ such that*

$$(A - \lambda I)^k X = \mathbf{0}$$

*The $\lambda$-**order** of $X$ is the smallest $k$ for which such an equality holds. The nullspace of $(A - \lambda I)^k$ is denoted $\mathbb{K}^n(A, \lambda, k)$. It is the space of generalized $\lambda$-eigenvectors of order at most $k$ together with the zero vector. $\mathbb{K}^n(A, \lambda)$ denotes the space of vectors belonging to $\mathbb{K}^n(A, \lambda, k)$ for some $k$.*

*The **$\lambda$-chain** generated by an $X \in \mathbb{R}^n$ is the set*

$$C(X, \lambda) = \{(A - \lambda I)^k X \mid k = 0, 1, \dots\}$$

The set $\mathcal{B} = \{X_0, X_1, W_1\}$ from Example 7.1 is a basis for $\mathbb{R}^3$. Equation (7.6) implies that $\{X_0, X_1\}$ is the $\lambda = 3$ chain generated by $X_1$ while $\{W_1\}$ is the $\lambda = 5$ chain generated by $W_1$. Thus $\mathcal{B}$ is a union of disjoint $\lambda$-chains. Such bases are called "chain bases."

**Definition 7.2** *Let $A$ be an $n \times n$ matrix and let $\mathcal{W}$ be a subspace of $\mathbb{K}^n$. A basis $\mathcal{B}$ of $\mathcal{W}$ is said to be a **chain basis** for $\mathcal{W}$ with respect to $A$ if there are eigenvalues $\lambda_1, \dots, \lambda_k$ for $A$ and $\lambda_i$ generalized eigenvectors $Y_i$ for $A$ such that* (1) *$\mathcal{B} = C(Y_1, \lambda_1) \cup \dots \cup C(Y_k, \lambda_k)$ and* (2) *for $i \neq j$, $C(Y_i, \lambda_i) \cap C(Y_j, \lambda_j) = \phi$.*

*Remark.* Let $\mathcal{W}$ be a subspace of $\mathbb{K}^n$ and let $B$ be an $n \times n$ matrix. We say that $\mathcal{W}$ is $B$-invariant if $BX \in \mathcal{W}$ for all $X \in \mathcal{W}$. It is clear that in the preceding definition $\mathcal{W}_i =$ span $C(Y_i, \lambda_i)$ is $(A - \lambda_i I)$-invariant. Since $A = (A - \lambda_i I) + \lambda_i I$, $\mathcal{W}_i$ is also $A$-invariant. Thus, only $A$-invariant subspaces can have chain bases. The converse statement, that if all of the eigenvalues of $A$ belong to $\mathbb{K}$ then all $A$-invariant subspaces have a chain basis, is a major result. We prove this result for $\mathcal{W} = \mathbb{K}^n$ in the next section.

[1] In fact, $\mathbb{K}$ could be any system which satisfies the field properties (a)–(i) on page 301, that is, $\mathbb{K}$ could be any field. In particular, we could let $\mathbb{K} = \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers.

In Example 7.1, we found an order 2 generalized eigenvector by solving the equation $(A - \lambda I)X_1 = X_0$, where $X_0$ is an eigenvector. In general, we can find order $k$ generalized eigenvectors $X$ by solving $(A - \lambda)X = Y$ where $Y$ is an order $k - 1$ eigenvector. We say that $X$ is a **pull back** of $Y$.

### EXAMPLE 7.2

Find a chain basis of $\mathbb{R}^4$ for

$$A = \begin{bmatrix} 2 & -2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

*Solution.* The characteristic polynomial is $p(\lambda) = (\lambda - 2)^3(\lambda - 1)$. We compute the bases for the eigenspaces as usual. We discover that both the $\lambda = 2$ and the $\lambda = 1$ eigenspaces are one dimensional and are spanned by $X_0 = [1, 0, 0, 0]^t$ and $W_0 = [2, 1, 0, 0]^t$, respectively. Since the multiplicity of $\lambda = 2$ in $p(\lambda)$ is 3, we pull $X_0$ back twice. Specifically, we first solve the equation

$$(A - 2I)X_1 = X_0 \tag{7.10}$$

Since

$$B = A - 2I = \begin{bmatrix} 0 & -2 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

the system corresponding to (7.10) has augmented matrix

$$[B, X_0] = \begin{bmatrix} 0 & -2 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The reduced form of this matrix is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The only free variable is $x$, which we set equal to zero, producing the solution $X_1 = [0, -1, -1, 0]^t$, which is an order 2 generalized eigenvector.

Now we pull $X_1$ back, solving the system $(A - 2I)X_2 = X_1$, which has augmented matrix

$$[B, X_1] = \begin{bmatrix} 0 & -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now the reduced form is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Again, the only free variable is $x$, which we set equal to zero, producing the solution $X_2 = [0, -1, -1, -1]^t$, which is an order 3 generalized eigenvector. The set $\{X_0, X_1, X_2\}$ is a chain basis for $\mathbb{R}^4(A, 2)$. The set $\mathcal{B} = \{X_0, X_1, X_2, W_0\}$ is the desired chain basis for $\mathbb{R}^4$.

### EXAMPLE 7.3

Compute a formula for $A^n C$, where $A$ is as in Example 7.2 and $C = [2, -1, -2, -1]^t$.

*Solution.* We note that

$$C = X_1 + X_2 + W_0$$

where $X_1$, $X_2$, and $W_0$ are as in the solution to Example 7.2. Hence

$$A^n C = A^n X_1 + A^n X_2 + A^n W_0$$

Since $W_0$ is a 1-eigenvector,

$$A^n W_0 = W_0 = [2, 1, 0, 0]^t$$

To compute $A^n X_1$ and $A^n X_2$, let

$$B = A - 2I = \begin{bmatrix} 0 & -2 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so that $A = 2I + B$. Then from (7.7) on page 423, since $BX_1 = X_0$ and $BX_0 = \mathbf{0}$,

$$\begin{aligned} A^n X_1 &= (2^n I + n2^{n-1}B)X_1 \\ &= 2^n X_1 + n2^{n-1}X_0 \\ &= [n2^{n-1}, -2^n, -2^n, 0]^t \end{aligned}$$

Similarly, $BX_2 = X_1$, $BX_1 = X_0$, and $BX_0 = \mathbf{0}$. Hence

$$\begin{aligned}
A^n X_2 &= \left(2^n I + n2^{n-1}B + \frac{n(n-1)}{2}2^{n-2}B^2\right) X_2 \\
&= 2^n X_2 + n2^{n-1}X_1 + \frac{n(n-1)}{2}2^{n-2}X_0 \\
&= [n\,(n-1)\,2^{n-3}, -2^n - n2^{n-1}, -2^n - n2^{n-1}, -2^n]^t
\end{aligned}$$

Our final answer is

$$\begin{aligned}
A^n C &= A^n X_2 + A^n X_1 + A^n W_0 \\
&= [2 + n2^{n-1} + 2^{n-3}n\,(n-1)\,, 1 - 2^{1+n} - n2^{n-1}, -2^{1+n} - n2^{n-1}, -2^n]^t
\end{aligned}$$

The pulling back process can be used to produce chain bases. Unfortunately, this process can become complicated when the dimension of the eigenspace is greater than 1, as the next example shows.

### EXAMPLE 7.4

Find a chain basis of $\mathbb{R}^4$ for $A$ where

$$A = \begin{bmatrix} 2 & 0 & -1 & 0 \\ 1 & 2 & -1 & -1 \\ 0 & 0 & 2 & 0 \\ -2 & 0 & 2 & 4 \end{bmatrix}$$

*Solution.* Computation shows that (i) $p_A(\lambda) = (\lambda - 2)^3(\lambda - 4)$, (ii) the 4-eigenspace is spanned by

$$W_0 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

and (iii) the 2-eigenspace is spanned by

$$X_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad Y_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

However, neither $X_0$ nor $Y_0$ have pull backs: the system $(A - 2I)X_1 = X_0$ has augmented matrix

$$[A - 2I, X_0] = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 2 & 0 \end{bmatrix}$$

The reduced form of this matrix is

$$\begin{bmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which represents an inconsistent system. Similarly, $[A - 2I, Y_0]$ represents an inconsistent system.

But there still is an order 2, $\lambda = 2$ generalized eigenvector $Z_1$. If $Z_1$ exists, $(A - \lambda I)Z_1$ must be an eigenvector. Hence, there should exist scalars $u$ and $v$ such that

$$(A - 2I)Z_1 = uX_0 + vY_0$$

We find $Z_1 = [x, y, z, w]^t$, $u$, and $v$ simultaneously by solving this system. The augmented matrix is $[A - 2I, -X_0, -Y_0, \mathbf{0}]$, which is

$$\begin{bmatrix} 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 2 & 0 & -1 & 0 \end{bmatrix}$$

We row reduce, getting

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The variables $y$, $w$, and $v$ are free. We set $y = w = 0$ and $v = -2$, yielding $x = 3, y = 0, z = 2, w = 0, u = 1$. Hence

$$Z_1 = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

is an order 2 generalized eigenvector and $AZ_1 = Z_0$ where

$$Z_0 = uX_0 + vY_0 = X_0 - 2Y_0 = [-2, 1, 0, -2]^t$$

The set $\{X_0, Y_0, Z_0\}$ is dependent. To obtain a chain basis we eliminate either $X_0$ or $Y_0$. Hence, $\mathcal{B} = \{X_0, Z_0, Z_1, W_0\}$ is a chain basis as is $\mathcal{B}' = \{Y_0, Z_0, Z_1, W_0\}$.

## EXERCISES

**7.1** Find a chain basis for each of the following matrices. *Note:* The work from this exercise is used in Exercise 7.2 below and Exercise 7.12 on page 443.

**(a)** ✓✓ $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

**(b)** $A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

**(c)** ✓✓ $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

**(d)** $A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & -1 & 0 \\ -2 & -2 & 4 \end{bmatrix}$ *Hint:* The eigenvalues are 3 and 2.

**(e)** ✓✓ $A = \begin{bmatrix} 2 & -2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

**(f)** $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

**(g)** ✓✓ $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

**(h)** $A = \begin{bmatrix} -1 & 0 & 2 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

**7.2** ✓✓For each matrix in Exercise 7.1, compute a formula for $A^nB$ where $B$ is either $[1, 1, 1]^t$ or $[1, 1, 1, 1]^t$, as appropriate. Express your answer as a linear combination of the vectors found in the solution to Exercise 7.1.

**7.3** Suppose that $A$ is a $4 \times 4$ matrix. Use the following information to compute a formula for $A^nB$ where $B = Z + X + W$. Your answer will be expressed as a linear combination of $X$, $Y$, $Z$, and $W$.

**(a)** The characteristic polynomial of $A$ is $p(x) = (x - 3)^3(x + 4)$.

**(b)** The space of generalized eigenvectors corresponding to $\lambda = 3$ is spanned by $X$, $Y$, and $Z$ where $(A - 3I)Z = Y$, $(A - 3I)Y = \mathbf{0}$, $(A - 3I)X = \mathbf{0}$.

**(c)** The $\lambda = -4$ eigenspace of $A$ is spanned by $W$.

**7.4** ✓✓Let

$$A = \begin{bmatrix} 4 & a & b & c & d \\ 0 & 4 & e & f & g \\ 0 & 0 & 4 & h & i \\ 0 & 0 & 0 & 5 & j \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \qquad X = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

**(a)** Show by direct computation that $X$, $Y$, and $Z$ are generalized eigenvectors corresponding to the eigenvalue 4 of order at most 1, 2, and 3, respectively.

**(b)** ✓✓Show that if $W = [x, y, z, u, v]^t$ is a $\lambda = 4$ generalized eigenvector for $A$, then $u = v = 0$. [*Hint:* Write $A - 4I$ as a partitioned matrix.]

**(c)** How does it follow that the $\lambda = 4$ generalized eigenspace is just the set of vectors of the form $[x, y, z, 0, 0]^t$.

**(d)** For which values of the parameters will $Y$ have order 1? Show by direct computation that in this case $Z$ will have order at most 2.

**7.5** Let $\lambda$ be an eigenvalue of an $n \times n$ matrix $A$. Prove that $\mathbb{K}^n(A, \lambda)$ is a subspace of $\mathbb{K}^n$.

**7.6** ✓✓For the following matrix, find two generalized eigenvectors $X$ and $Y$ for $A$ corresponding to the eigenvalue 5, both of order 2, such that $X + Y$ is a generalized eigenvector of order 1.

$$A = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

**7.7** Let $\lambda$ be a generalized eigenvector for an $n \times n$ matrix $A$ and let $k \in \mathbb{N}$. Prove that the set of generalized eigenvectors corresponding to $\lambda$ of order less than or equal to $k$ together with $\{\mathbf{0}\}$ is a subspace of $\mathbb{K}^n$.

**7.8** Suppose that $A$ and $B$ are $n \times n$ matrices such that $AB = BA$. Show that $\mathbb{K}^n(A, \lambda, k)$ is $B$-invariant, that is, show that if $X \in \mathbb{K}^n(A, \lambda, k)$ then the same is true for $BX$.

**7.9** ✓✓Let $A$ be an $n \times n$ matrix. Show that if $X$ belongs to both $\mathbb{K}^n(A, \lambda)$ and $\mathbb{K}^n(A, \beta)$, where $\lambda \neq \beta$, then $X = \mathbf{0}$.

**7.10** Suppose that $A$, $B$, and $P$ are $n \times n$ matrices such that $P^{-1}AP = B$.

**(a)** ✓✓Prove that $X \in \mathbb{K}^n(A, \lambda)$ if and only if $P^{-1}X \in \mathbb{K}^n(B, \lambda)$.

**(b)** Let $\{X_1, \ldots, X_m\}$ be a basis for $\mathbb{K}^n(A, \lambda)$. Prove that $\{P^{-1}X_1, \ldots, P^{-1}X_m\}$ is a basis for $\mathbb{K}^n(B, \lambda)$. Hence $\dim(\mathbb{K}^n(A, \lambda)) = \dim(\mathbb{K}^n(B, \lambda))$

**7.11** Let $B$ and $C$ be $n \times n$ matrices.

**(a)** Prove that $(B + C)^2 = B^2 + 2BC + C^2$ holds if and only if $CB = BC$. Show that this implies formula (7.7) on page 423 for $n = 2$.

**(b)** ✓✓Suppose that $BC = CB$. *Use part (a)* to prove the following formula. Show that this implies formula (7.7) for $n = 3$.

$$(B + C)^3 = B^3 + 3B^2C + 3BC^2 + C^3$$

**(c)** Let notation be as in (7.8) on page 423. Prove that for $1 \leq k \leq n$

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

**(d)** Suppose that $BC = CB$. Assume that formula (7.7) on page 423 holds for some $n$. Use $(B + C)^{n+1} = B(B + C)^n + C(B + C)^n$ together with part (d) to prove

$$(B + C)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} B^k C^{n-k}$$

*Remark.* It follows from Exercise 7.11(d) and mathematical induction that formula (7.7) on page 423 holds for all $n$. Specifically, 7.11(d) is formula (7.7) with $n$ replaced by $n + 1$. Hence, 7.11(d) says that if (7.7) holds for some $n$, it also holds for $n + 1$. Since (7.7) holds for $n = 1$, 7.11(d) implies that it holds for $n = 2$. Since it holds for $n = 2$, 7.11(d) implies that it holds for $n = 3$, etc.

## 7.2 CHAIN BASES

Let $A$ be an $n \times n$ matrix. We assume that its characteristic polynomial $p_A(x)$ factors as a product of powers of linear terms

$$p_A(x) = \pm(x - \lambda_1)^{n_1}(x - \lambda_2)^{n_2} \dots (x - \lambda_k)^{n_k} \tag{7.11}$$

where $\lambda_i \in \mathbb{K}$. From the Fundamental Theorem of Algebra, the existence of such a factorization is automatic if $\mathbb{K} = \mathbb{C}$.

We commented (without proof) in Section 5.1 that the dimension of the $\lambda_i$ eigenspace of $A$ is at most $n_i$. It could, however, be less. The following theorem explains this phenomenon as well as proves our comment.

**Theorem 7.1** *Let $A$ be an $n \times n$ matrix whose characteristic polynomial $p$ factors as in* (7.11) *with all of the $\lambda_i \in \mathbb{K}$. Then, for each $i$,* $\dim(\mathbb{K}^n(A, \lambda_i)) = n_i$. *Furthermore, if $X \in \mathbb{K}^n(A, \lambda_i)$, then $(A - \lambda_i I)^{n_i} X = \mathbf{0}$—i.e., $\mathbb{K}^n(A, \lambda_i) = \mathbb{K}^n(A, \lambda_i, n_i)$.*

If $A$ is an upper triangular matrix, then its diagonal entries are its eigenvalues. Let $\lambda_1, \dots, \lambda_k$ be a listing of these eigenvalues. We say that $A$ is in **standard form** relative to the given listing, if its eigenvalues occur on the diagonal of $A$ in the stated order as we scan down the diagonal. Thus, the following matrix $A$ is in standard form with respect to the listing 5, 3:

$$A = \begin{bmatrix} 5 & 1 & 2 & 3 & 4 \\ 0 & 5 & 1 & 2 & 3 \\ 0 & 0 & 5 & 1 & 2 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \tag{7.12}$$

The next two propositions show that in the proof of Theorem 7.1 we may assume that $A$ is an upper triangular matrix in standard form relative to any given listing of its eigenvalues. The first proposition follows from Exercise 7.10 on page 430.

**Proposition 7.1** *Suppose that $A$, $B$, and $P$ are $n \times n$ matrices, with $P$ invertible, such that $P^{-1}AP = B$. Then $X \in \mathbb{K}^n(A, \lambda, k)$ if and only if $P^{-1}X \in \mathbb{K}^n(B, \lambda, k)$, and hence* $\dim(\mathbb{K}^n(A, \lambda)) = \dim(\mathbb{K}^n(B, \lambda))$.

**Proposition 7.2** *Let the assumptions be as stated in Theorem 7.1 and let $\lambda_1, \dots, \lambda_k$ be a listing of the eigenvalues of $A$. Then there is an $n \times n$ invertible matrix $P$ and an $n \times n$ upper triangular matrix $B$ in standard form relative to $\lambda_1, \dots, \lambda_k$ such that*

$$P^{-1}AP = B$$

*Proof.* Since $\lambda_1$ is a root of the characteristic polynomial, there is a nonzero element $Q_1 \in \mathbb{K}^n$ such that

$$AQ_1 = \lambda_1 Q_1$$

Choose elements $Q_i \in \mathbb{K}^n$, $2 \le i \le n$, such that $\{Q_1, Q_2, \dots, Q_n\}$ forms a basis for $\mathbb{K}^n$. Let $Q = [Q_1, \dots, Q_n]$ be the $n \times n$ matrix having the $Q_i$ as columns.

Since the $Q_i$ form a basis, there are scalars $c_{ij}$ such that

$$\begin{aligned} AQ_1 &= \lambda_1 Q_1 \\ AQ_i &= c_{1i}Q_1 + c_{2i}Q_2 + \dots + c_{ni}Q_n, \quad 2 \le i \le n \end{aligned}$$

The above equalities are equivalent with

$$A[Q_1, Q_2, \dots, Q_n] = [Q_1, Q_2, \dots, Q_n]C$$

where

$$C = \begin{bmatrix} \lambda_1 & c_{12} & \dots & c_{1n} \\ 0 & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & c_{n2} & \dots & c_{nn} \end{bmatrix} \tag{7.13}$$

Hence $Q^{-1}AQ = C$.

Write $C$ as a partitioned matrix

$$C = \left[\begin{array}{c|ccc} \lambda_1 & c_{12} & \dots & c_{1n} \\ \hline 0 & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & c_{n2} & \dots & c_{nn} \end{array}\right] = \begin{bmatrix} \lambda_1 & C_{12} \\ \mathbf{0} & C_{22} \end{bmatrix}$$

where $C_{22}$ is an $(n-1)\times(n-1)$ matrix.

Assume by mathematical induction that our proposition holds for all $m \times m$ matrices where $m < n$. (Note that it holds trivially for $1 \times 1$ matrices.) We will show that the characteristic polynomial of $C_{22}$ is

$$\tilde{p}(x) = \pm(x-\lambda_1)^{n_1-1}(x-\lambda_2)^{n_2}\dots(x-\lambda_k)^{n_k} \tag{7.14}$$

Granted this, it follows from the inductive hypothesis that there is an $(n-1)\times(n-1)$ invertible matrix $P_o$ such that

$$P_o^{-1}C_{22}P_o = B_{22}$$

where $B_{22}$ is an upper triangular $(n-1)\times(n-1)$ matrix in standard form with respect to either $\lambda_1,\dots,\lambda_n$ if $n_1 > 1$ or $\lambda_2,\dots,\lambda_n$ if $n_1 = 1$.

Let $Q_o$ be the $n \times n$ matrix

$$Q_o = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & P_o \end{bmatrix}$$

and let $P = Q_oQ$. Then

$$\begin{aligned} PAP^{-1} &= Q_oQAQ^{-1}Q_o^{-1} \\ &= Q_oCQ_o^{-1} \\ &= \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & P_o \end{bmatrix}\begin{bmatrix} \lambda_1 & C_{12} \\ \mathbf{0} & C_{22} \end{bmatrix}\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & P_o^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & C_{12}P_o^{-1} \\ \mathbf{0} & P_oC_{22}P_o^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & D \\ \mathbf{0} & B_{22} \end{bmatrix} \end{aligned}$$

which is an upper triangular matrix with respect to $\lambda_2,\dots,\lambda_n$. Hence, our proposition will follow once we prove formula (7.14).

However, expanding determinants along the first column in (7.13) shows that the characteristic polynomial of $C$ is

$$p_C(x) = (\lambda_1 - x)p_2(x)$$

where $p_2(x)$ is the characteristic polynomial of $C_{22}$. On the other hand, it follows from Exercise 5.39 on page 292, that $P_C(x) = p_A(x)$. Hence, from formula (7.11),

$$(\lambda_1 - x)p_2(x) = \pm(x - \lambda_1)^{n_1}(x - \lambda_2)^{n_2} \cdots (x - \lambda_k)^{n_k}$$

Formula (7.14) follows. □

The idea behind the proof of Theorem 7.1 on page 431 in the case of a matrix in triangular form is now fairly simple to explain with an example:

### EXAMPLE 7.5

Prove Theorem 7.1 (except for the existence of the chain basis) for the matrix $A$ in (7.12) on page 431.

*Proof.* The eigenvalues of $A$ are 5 and 3 with multiplicities 3 and 2, respectively.
$\mathbb{K}^5(A,5)$ is easily described. As a partitioned matrix

$$A = \left[\begin{array}{ccc|cc} 5 & 1 & 2 & 3 & 4 \\ 0 & 5 & 1 & 2 & 3 \\ 0 & 0 & 5 & 1 & 2 \\ \hline 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ \mathbf{0} & A_{22} \end{bmatrix}$$

We claim that $\mathbb{K}^3(A,5)$ is just the set of vectors of the form

$$Z = \begin{bmatrix} X \\ \mathbf{0} \end{bmatrix}, \quad X \in \mathbb{K}^3, \mathbf{0} \in \mathbb{K}^2 \tag{7.15}$$

If so, $\dim \mathbb{K}^3(A,5) = 3$, which is the multiplicity of the $\lambda = 5$ eigenvalue.
To see this, let

$$Z = \begin{bmatrix} X \\ Y \end{bmatrix}, \quad X \in \mathbb{K}^3, Y \in \mathbb{K}^2 \tag{7.16}$$

belong to $\mathbb{K}^3(A,5)$. Then

$$\begin{aligned}(A - 5I)Z &= \begin{bmatrix} A_{11} - 5I & A_{12} \\ \mathbf{0} & A_{22} - 5I \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} (A_{11} - 5I)X + A_{12}Y \\ (A_{22} - 5I)Y \end{bmatrix} \\ &= \begin{bmatrix} X_1 \\ (A_{22} - 5I)Y \end{bmatrix}\end{aligned}$$

where $X_1 \in \mathbb{K}^3$. Repeating this computation $k$ times shows that

$$(A-5I)^k Z = \begin{bmatrix} X_k \\ (A_{22}-5I)^k Y \end{bmatrix} \tag{7.17}$$

where $X_k \in \mathbb{K}^3$. If $(A-5I)^k Z = \mathbf{0}$, then $(A_{22}-5I)^k Y = \mathbf{0}$ as well. But $A_{22}-5I$ is invertible since 5 is not an eigenvalue of $A_{22}$, and hence $Y = \mathbf{0}$, proving that $Z$ is as in (7.15).

Conversely, suppose that $Z$ is as in (7.15). Then

$$(A-5I)Z = \begin{bmatrix} A_{11}-5I & A_{12} \\ \mathbf{0} & A_{22}-5I \end{bmatrix} \begin{bmatrix} X \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} (A_{11}-5I)X \\ \mathbf{0} \end{bmatrix}$$

Repeating this computation $k$ times shows that

$$(A-5I)^k Z = \begin{bmatrix} (A_{11}-5I)^k X \\ \mathbf{0} \end{bmatrix} \tag{7.18}$$

But

$$A_{11}-5I = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{7.19}$$

It is easily checked that $(A_{11}-5I)^3 = \mathbf{0}$, showing that $Z$ is a generalized eigenvector of order at most 3.

$\mathbb{K}^5(A,3)$ is more difficult to analyze; it is NOT the space of vectors of the form $[0,0,0,x_3,x_4]^t$. However, applying Proposition 7.2 to $A$ with its eigenvalues listed in the order 3, 5 shows that there is a matrix $Q$ such that

$$Q^{-1}AQ = \begin{bmatrix} 3 & a & b & c & d \\ 0 & 3 & e & f & g \\ 0 & 0 & 5 & e & f \\ 0 & 0 & 0 & 5 & g \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = B$$

$\mathbb{K}^5(B,3)$ IS the set of vectors of the form $[x_1,x_2,0,0,0]^t$. Proposition 7.1 shows that therefore $\dim(\mathbb{K}^3(A,3)) = 2$. Furthermore, it is easily seen that each element of $\mathbb{K}^5(B,3)$ has order at most 2, and hence the same is true for $A$. Since we already know that $\dim(\mathbb{K}^3(A,5)) = 3$ and each element of this space has order at most 3, Theorem 7.1 is proved for $A$, except for the statement about chain bases.

Below is the general proof based on the example just given.

*Proof of Theorem 7.1 on page 432.*

Let $A$ be as in the statement of Theorem 7.1. By renaming the $\lambda_i$ if necessary, it suffices to prove the result for $\lambda_1$. From Proposition 7.1 on page 432 and Proposition 7.2 on page 432, we may assume that $A$ is in triangular form with first eigenvalue $\lambda_1$.

We write $A$ as a partitioned matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ \mathbf{0} & A_{22} \end{bmatrix}$$

where $A_{11}$ has size $n_1 \times n_1$ ($n_1$ is the multiplicity of $\lambda_1$) and $A_{22}$ is square. Note that (i) $A_{11}$ and $A_{22}$ are upper triangular, (ii) all of the diagonal entries of $A_{11}$ equal $\lambda_1$, and (iii) none of the diagonal entries of $A_{22}$ equal $\lambda_1$. It follows from (i) and (iii) that $A_{22} - \lambda_1 I$ is invertible and from (i) and (ii), together with Exercise 3.53 on page 177, that $(A_{11} - \lambda_1 I)^{n_i} = \mathbf{0}$.

Let $Z \in \mathbb{K}^n(A, \lambda_1)$ be written as in (7.16) on page 434 where now $X \in \mathbb{K}^{n_1}$. Reasoning as in Example 7.5 and using formula (7.19) on page 435 with 5 replaced by $\lambda_1$, we discover that $Y = \mathbf{0}$.

Conversely, if $Y = \mathbf{0}$, then formula (7.19), with 5 replaced by $\lambda_1$, shows that $(A - \lambda_1 I)^{n_1} Z = \mathbf{0}$. Hence, $Z$ has order at most $n_1$ and $\mathbb{K}^n(A, \lambda_1)$ is the space of all vectors in $\mathbb{K}^n$ of the form (7.15) on page 435 where $X \in \mathbb{K}^{n_1}$. In particular, $\dim \mathbb{K}^n(A, \lambda_1) = n_1$.

To finish the proof of Theorem 7.1 we need only prove the existence of a chain basis for $\mathbb{K}^n(A, \lambda_1)$. From formula (7.18) on page 435, with 5 replaced by $\lambda_1$, it suffices to prove the existence of a chain basis for $A_{22} - \lambda_1 I$ in $\mathbb{K}^{n_1}$. In other words, it suffices to assume that $A$ is upper triangular with main diagonal equal to $\mathbf{0}$—that is, $A$ is an upper triangular nilpotent matrix. In particular, $A^n = \mathbf{0}$. Note that in this case every element $X$ of $\mathbb{K}^n$ is a generalized eigenvector corresponding to the eigenvalue 0. Theorem 7.1 follows from the following proposition. □

**Proposition 7.3** *Let $A$ be an $n \times n$ upper triangular nilpotent matrix and let $\mathcal{W}$ be a nonzero $A$-invariant subspace of $\mathbb{K}^n$. Then $\mathcal{W}$ has a chain basis for $A$.*

*Proof.* Suppose Proposition 7.3 fails. Then there is an $A$-invariant subspace $\mathcal{W}$ of $\mathbb{K}^n$ of minimal dimension for which this proposition fails. Proposition 7.3 is clear for any subspace of null $A$, and hence, $\mathcal{W} \not\subset$ null $A$. The space $\mathcal{W}_1 = \{AX \mid X \in \mathcal{W}\}$ is then a nonzero $A$-invariant subspace of $\mathcal{W}$ and multiplication by $A$ defines a linear transformation of $\mathcal{W}$ onto $\mathcal{W}_1$. Let $Y \in \mathcal{W}$, $AY \neq \mathbf{0}$. From the nilpotence of $A$, there is a smallest $k$ such that $A^k Y = \mathbf{0}$. Since $X = A^{k-1} Y \in \mathcal{W}$, there is a nonzero element $X \in \mathcal{W}$ such that $AX = \mathbf{0}$. It then follows from the rank-nullity theorem (Theorem 3.16 on page 229) that $\dim \mathcal{W}_1 < \dim \mathcal{W}$. Hence, Proposition 7.3 holds for $\mathcal{W}_1$, implying that there are elements $\tilde{Y}_i \in \mathcal{W}_1$, $1 \leq i \leq k$, that generate a chain basis for $\mathcal{W}_1$. Let $\tilde{Y}_i = AY_i$ where $Y_i \in \mathcal{W}$.

Let

$$\begin{aligned}\mathcal{B}_0 &= \mathcal{C}(Y_1, 0) \cup \cdots \cup \mathcal{C}(Y_k, 0)\\ \mathcal{W}_0 &= \text{span } \mathcal{B}_0\\ \mathcal{Z}_0 &= (\text{null } A) \cap \mathcal{W}_0\\ \mathcal{Z}_1 &= (\text{null } A) \cap \mathcal{W}\end{aligned}$$

Since $\mathcal{Z}_0 \subset \mathcal{Z}_1$, we may choose a basis $\{Z_1, \ldots, Z_m\}$ for $\mathcal{Z}_1$ for which $\{Z_{l+1}, \ldots, Z_m\}$ is a basis for $\mathcal{Z}_0$. The following lemma proves that $\mathcal{W}$ has a chain basis, contrary to the assumption that Proposition 7.3 fails. Theorem 7.1 follows. □

**Lemma 7.1** $\mathcal{B} = \{Z_1, \ldots, Z_l\} \cup \mathcal{B}_0$ *is a chain basis for* $\mathcal{W}$.

*Proof.* Let $n_i$ be the order of $Y_i$ as a generalized eigenvector of $A$. Then the order of $\tilde{Y}_i$ is $n_i - 1$.

We claim that $\mathcal{B}$ spans $\mathcal{W}$. To see this let $X \in \mathcal{W}$. Since $AX \in \mathcal{W}_1$, we may write

$$AX = \sum_{i=1}^{k} \sum_{n=0}^{n_i-1} c_{i,n} A^n \tilde{Y}_i$$

for some scalars $c_{i,n}$. Let

$$X_o = \sum_{i=1}^{m} \sum_{n=0}^{n_i-1} c_{i,n} A^n Y_i$$

Then

$$AX_o = \sum_{i=1}^{m} \sum_{n=0}^{n_i-1} c_{i,n} A^n \tilde{Y}_i = AX \tag{7.20}$$

Hence, $X - X_o \in (\text{null } A) \cap \mathcal{W}$ so

$$\begin{aligned}X - X_o &= d_1 Z_1 + \cdots + d_m Z_m\\ X &= d_1 Z_1 + \cdots + d_l Z_l + (d_{l+1} Z_{l+1} + \cdots + d_m Z_m) + X_o\end{aligned}$$

By definition, $Z_i \in \text{span } \mathcal{B}_0$ for $i > l$. Since $X_o$ also belongs to this span, we see that $X$ is a linear combination of elements of $\mathcal{B}$, as desired.

The dependency relation for $\mathcal{B}$ is

$$d_1 Z_1 + \cdots + d_l Z_l + \sum_{i=1}^{k} \sum_{n=0}^{n_i-1} c_{i,n} A^n (Y_i) = \mathbf{0} \tag{7.21}$$

Multiplying by $A$, and using $A^{n_i}(Y_i) = \mathbf{0}$, shows

$$\sum_{i=1}^{k} \sum_{n=0}^{n_i-2} c_{i,n} A^n(\tilde{Y}_i) = \mathbf{0}$$

Since the $\tilde{Y}_i$ generate a chain basis of $\mathcal{W}_1$, it follows that $c_{i,n} = 0$ for $n \le n_i - 2$. Thus (7.21) implies

$$d_1 Z_1 + \cdots + d_l Z_l + \sum_{i=1}^{m} c_{i,n_i-1} A^{n_i-1}(Y_i) = \mathbf{0}$$

But $A^{n_i-1}(Y_i) \in (\text{null}\, A) \cap \mathcal{W}_0$. It follows from the linear independence of $\{Z_1, \ldots, Z_m\}$ that the $d_i$ are all 0. It then follows from the independence of the chain basis generated by the $\tilde{Y}_i$ that the $c_{i,n_i-1}$ are also zero as well, proving independence. It also follows that the chains $\mathcal{C}(Y_i)$ are nonoverlapping, since otherwise we would have a relation of the form $A^k(Y_i) - A^l(Y_j) = \mathbf{0}$. This finishes the proof of Proposition 7.3, and hence of Theorem 7.1. □

## Jordan Form

Recall that we said in Section 5.2 that an eigenbasis for an $n \times n$ matrix $A$ is a basis $\mathcal{B} = \{Q_1, Q_2, \ldots, Q_n\}$ for $\mathbb{K}^n$ consisting of eigenvectors for $A$. In Theorem 5.4 on page 287 we saw that in this case, if $Q = [Q_1, Q_2, \ldots, Q_n]$ is the $n \times n$ matrix having the $Q_i$ as columns, then $Q^{-1}AQ$ is a diagonal matrix having the eigenvalues of $A$ as the diagonal entries. In the nondiagonalizable case, we use chain bases instead of eigenbases.

**Definition 7.3** *Let $A$ be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be a listing of the eigenvalues of $A$. For each $i$, let $\mathcal{B}_i$ be a chain basis for $\mathbb{K}^n(A, \lambda_i)$. Then the set*

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_m \tag{7.22}$$

*is the **chain basis** for $A$ defined by the $\mathcal{B}_i$.*

*Remark.* Corollary 7.1 on page 443 proves that $\mathcal{B}$ actually is a basis for $\mathbb{K}^n$.

In the diagonalizable case, the diagonal matrix $D = Q^{-1}AQ$ depends on the ordering of the basis elements; if $Q_i$ corresponds to the eigenvalue $\lambda_i$, then $D_{ii} = \lambda_i$, and hence permuting the $Q_i$ permutes the diagonal entries of $D$. In the nondiagonalizable case, $Q^{-1}AQ$ has its simplest form only for specific orderings of the chain basis. We say first that in (7.22) the elements of $\mathcal{B}_i$ come before those of $\mathcal{B}_j$ for $i < j$. Within $\mathcal{B}_i$, we group of elements from the same chain together, listing the chains in order of increasing length and listing the elements in each chain in order of increasing degree. Thus, if $Y$ generates a chain length $n$ corresponding to the eigenvalue $\lambda$, then $(A - \lambda I)^{n-1}Y$ is considered to be the first element of the chain and $Y$ is the last.

**Definition 7.4** *Let* $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$ *be a listing of the eigenvalues of an* $n \times n$ *matrix* $A$. *Let* $\mathcal{B} = \{Q_1, \dots, Q_n\}$ *be a chain basis for an* $n \times n$ *matrix* $A$ *ordered as described and let* $Q = [Q_1, \dots, Q_n]$. *Then* $M = Q^{-1}AQ$ *is the* ***Jordan canonical form*** *for* $A$ *corresponding to the given listing of the eigenvalues.*

Before describing the Jordan form in general (Theorem 7.2 below) we consider an example.

**EXAMPLE 7.6**

Let

$$A = \begin{bmatrix} 2 & -2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

be the matrix from Example 7.2 on page 425. Find the Jordan form for $A$ where the eigenvalues are listed from largest to smallest.

*Solution.* On page 429, we noted that if

$$X = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ Y = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \ Z = \begin{bmatrix} 0 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \ W = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

then

$$(A - 2I)Z = Y, \quad (A - 2I)Y = X, \quad (A - 2I)X = \mathbf{0}, \quad (A - I)W = \mathbf{0}$$

Thus $Z$ and $W$ generate chains that together define a chain basis. If we list the eigenvalues as 1, 2, then we should order our basis $\{X, Y, Z, W\}$. Hence

$$Q = [X, Y, Z, W] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & -1 \\ 2 & 1 & 0 & 0 \end{bmatrix}$$

Note that $AX = 2X$, $AY = X + 2Y$, $AZ = Y + 2Z$ and $AW = 4W$. Hence

$$A[X, Y, Z, W] = [X, Y, Z, W] \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

The $4 \times 4$ matrix $M$ on the right is a Jordan canonical form for $A$.

Note that we may write the matrix $M$ from Example 7.6 as a partitioned matrix

$$M = \left[\begin{array}{ccc:c} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right] \tag{7.23}$$

The $3 \times 3$ matrix in the upper left corner of $M$ is an example of a "Jordan block." In general, a Jordan block is an $n \times n$ upper triangular matrix of the form

$$J = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ 0 & 0 & \lambda & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{bmatrix} \tag{7.24}$$

Thus $J$ has all of the main diagonal entries equal to a single value $\lambda$, all the entries on the second diagonal equal to 1, and all other entries 0. The $1 \times 1$ matrix in the lower right hand corner of (7.23) is also a Jordan block.

The Jordan canonical form of any $n \times n$ matrix $A$ is describable in terms of Jordan blocks as described in the next theorem. We assume that

$$p_A(x) = \pm(x - \lambda_1)^{n_1}(x - \lambda_2)^{n_2} \cdots (x - \lambda_k)^{n_k}$$

Note that this formula defines both an ordering of the eigenvalues as well as $k$ (the number of distinct factors in $p_A$) and $n_i$ (the multiplicity of $\lambda_i$).

**Theorem 7.2** *For $A$ as in the preceding paragraph, let $\mathcal{B}$ be a chain basis of $\mathbb{K}^n$ for $A$, ordered as described above Definition 7.3 on page 438, and let $M$ be the corresponding Jordan canonical form for $A$. Then, as partitioned matrices,*

$$M = \begin{bmatrix} M_{11} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & M_{22} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & M_{33} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & M_{kk} \end{bmatrix}$$

*where $M_{ii}$ is an $n_i \times n_i$ matrix. Furthermore*

$$M_{ii} = \begin{bmatrix} J^i_{11} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & J^i_{22} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & J^i_{33} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & J^i_{k_i,k_i} \end{bmatrix}$$

*where each $J^i_{jj}$ is a Jordan block having $\lambda_i$ as its diagonal entry. The size of $J^i_{jj}$ is the length of the jth chain in the chain basis for $\mathbb{K}^n(A, \lambda_i)$ and $k_i$ is the number of distinct chains occurring in this chain basis. In particular, the sizes of the $J^i_{jj}$ are nondecreasing as j increases.*

*Proof.* Let

$$X_i^k = (A - \lambda_i I)^{n_i - k} Y_i, \quad 1 \leq k \leq n_i$$

Then, as an ordered basis,

$$\mathcal{B} = \left\{X_1^1, \dots, X_1^{n_1}, X_2^1, \dots, X_2^{n_2}, \dots, X_k^1, \dots, X_k^{n_k}\right\}$$

Let $Q_i$ be the $n \times n_i$ matrix $Q_i = [X_i^1, \dots, X_i^{n_i}]$ so that, as a partitioned matrix,

$$Q = [Q_1, \dots, Q_k]$$

For $M$ as above,

$$QM = [Q_1 J_{11}, \dots, Q_k J_{kk}]$$

From (7.24),

$$\begin{aligned} Q_i J_{ii} &= \left[X_i^1, X_i^2, \dots, X_i^{n_i}\right] J_{ii} \\ &= \left[\lambda_i X_i^1, \lambda_i X_i^2 + X_i^1, \dots, \lambda_i X_i^{n_i} + X_i^{n_i - 1}\right] \end{aligned} \tag{7.25}$$

On the other hand $AQ = [AQ_1, \dots, AQ_k]$ and

$$\begin{aligned} AQ_i &= \left[AX_i^1, AX_i^2, \dots, AX_i^{n_i}\right] \\ &= \left[\lambda_i X_i^1, \lambda_i X_i^2 + X_i^1, \dots, \lambda_i X_i^{n_i} + X_i^{n_i - 1}\right] \end{aligned} \tag{7.26}$$

Comparing (7.25) and (7.26) proves $AQ = QM$, proving our theorem. □

*Remark.* From Theorem 7.2, changing the order of the $\lambda_i$ will permute the $M_i$. The canonical form is otherwise unique; for a given ordering of the eigenvalues, any two chain bases, when ordered as stated, will yield the same canonical form. We will not prove this fact.

Once we know the eigenvalues of $A$ and the degrees of the generators of the chain basis, we can write down the Jordan form with essentially no work.

### EXAMPLE 7.7

The matrix $A$ has the following properties:

1. The characteristic polynomial of $A$ is $p_A(x) = (x-2)^3(x+1)^4(x-5)$.
2. The dimensions of the $\lambda = 2, -1, 5$ eigenspaces are, respectively, 1, 2, and 1.
3. All elements of $\mathbb{K}^n(A, -1)$ have degree at most 2.

Find the Jordan form $M$ of $A$ where the eigenvalues are listed in the order $2, -1, 5$.

*Solution.* Note that the degree of $p_A$ is $3 + 1 + 4 = 8$, implying that $A$ has size $8 \times 8$. To find $M$, we consider each eigenvalue separately:

$\boldsymbol{\lambda = 2}$**:** From Theorem 7.1 on page 431, $\dim \mathbb{K}^8(A, 2) = 3$. Since every chain ends in an eigenvector, and the dimension of the eigenspace is given as 1, there is only one chain, and hence the first Jordan block is

$$J_{11} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

From Theorem 7.1 on page 431, $\dim \mathbb{K}^8(A, -1) = 4$. Since every chain ends in an eigenvector, and the dimension of the eigenspace is given as 2, there are two chains in the basis each of which has length at most 2. (The maximum degree is 2.) Hence, we have two chains of length 2, implying that we have two Jordan blocks of size $2 \times 2$:

$$J_{22} = J_{33} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

$\boldsymbol{\lambda = 5}$**:** The eigenspace is one-dimensional and the multiplicity is 1. Hence there is one chain of length 1 and $J_{44} = [5]$.

Our final answer is

$$M = \begin{bmatrix} J_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J_{22} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & J_{33} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & J_{44} \end{bmatrix} = \left[\begin{array}{ccc:cc:cc:c} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{array}\right]$$

We require two additional results to finish this section, both of which are left as exercises.

**Proposition 7.4** *Let the assumptions be as in Theorem 7.1 on page 431. Suppose $X_i \in \mathbb{K}^n(A, \lambda_i)$, $1 \le i \le m$, where the $\lambda_i$ are distinct—i.e., $\lambda_i \neq \lambda_j$ if $i \neq j$. Then*

$$X_1 + \cdots + X_m = \mathbf{0} \tag{7.27}$$

*implies that $X_i = \mathbf{0}$ for all $i$.*

**Corollary 7.1** *The set $\mathcal{B}$ from* (7.22) *on page 438 is a basis for $\mathbb{K}^n$.*

**True-False Questions**: Justify your answers.

**7.1** $A$ has characteristic polynomial $p_A = (\lambda - 1)(\lambda - 2)^3$. If $A$ is not diagonalizable, then $A$ has an order 3 generalized eigenvector.

**7.2** $A$ has characteristic polynomial $p_A = -(\lambda - 1)^2(\lambda - 2)^3$. If $A$ has no order 3 generalized eigenvectors, then $A$ has at least three linearly independent eigenvectors.

**7.3** $A$ has characteristic polynomial $p_A = \lambda^6(\lambda - 1)(\lambda - 2)^3$. Then the dimension of the nullspace of $A^6$ can be at most 3.

**7.4** $A$ has characteristic polynomial $p_A = \lambda^6(\lambda - 1)(\lambda - 2)^3$. Then the rank of $A^6$ is 4.

**7.5** $A$ has characteristic polynomial $p_A = \lambda^2(\lambda - 1)(\lambda - 2)$. Then $A^2$ is diagonalizable.

**7.6** $A$ has characteristic polynomial $p_A = (\lambda - 2)^2$. Then $A^2$ is diagonalizable.

**7.7** $A$ has characteristic polynomial $p_A = -\lambda^2(\lambda - 1)^3(\lambda - 2)^2$. If $A$ has an order 3 generalized eigenvector, then the $\lambda = 1$ eigenspace is one-dimensional.

## EXERCISES

**7.12** For each of the matrices $A$ in Exercise 7.1 on page 429 find a Jordan canonical form $J$ and a matrix $Q$ such that $A = QJQ^{-1}$. ✓✓[(a), (e), (g)]

**7.13** Find a Jordan canonical form $M$ and a matrix $Q$ such that $Q^{-1}AQ = M$ for the matrix $A$ in Example 7.1 on page 421.

**7.14** Let $A$ be an $8 \times 8$ matrix with $p_A(x) = (x - 3)^3(x - 4)(x - 6)^4$. Below, we provide the orders $n_i$ for the generators of the chain bases for the $\lambda = 3$ and $\lambda = 6$ eigenspaces. Use this data to find a Jordan canonical form for $A$.

**(a)** ✓✓ $\lambda = 3$: order $n_1 = 3$, $\lambda = 6$: orders $n_1 = 3, n_2 = 1$.

**(b)** $\lambda = 3$: orders $n_1 = 1, n_2 = 2$, $\lambda = 6$: orders $n_1 = 2$, $n_2 = 1, n_3 = 1$.

**(c)** $\lambda = 3$: orders $n_1 = 2$, $n_2 = 1$, $\lambda = 6$: order $n_1 = 4$.

**(d)** $\lambda = 3$: orders $n_1 = n_2 = n_3 = 1$, $\lambda = 6$: orders $n_1 = 3$, $n_2 = 1$.

**7.15** Let $A$ be a matrix with $p_A(x) = -(x - 2)^5$. Write all possible Jordan canonical forms for $A$.

**7.16** Let $A$ be a $5 \times 5$ matrix such that $A^3 = 0$ but $A^2 \neq \mathbf{0}$. Write all possible Jordan canonical forms for $A$.

**7.17** Let $A$ be a matrix with $p_A(x) = (x-2)^2(x-4)^2$. Write all possible Jordan canonical forms for $A$ where the eigenvalues are ordered 2, 4.

**7.18** ✓✓Prove Proposition 7.4 on page 443.

**7.19** ✓✓Use Proposition 7.4 on page 443 to prove Corollary 7.1 on page 443.

**7.20** The purpose of the following exercise is to demonstrate that computing powers of matrices in Jordan form is relatively simple. Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**(a)** Compute $A^n$ for $n = 2, 3, 4, 5$.

**(b)** Use formula (7.7) on page 423 to derive the following equality:

$$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^5 = \begin{bmatrix} \lambda^5 & 5\lambda^4 & 10\lambda^3 \\ 0 & \lambda^5 & 5\lambda^4 \\ 0 & 0 & \lambda^5 \end{bmatrix}$$

**(c)** Compute $B^n$ for $n = 2, 3, 4, 5, 6$.

**(d)** Use formula (7.7) on page 423 to derive the following equality:

$$\begin{bmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix}^5 = \begin{bmatrix} \lambda^5 & 5\lambda^4 & 10\lambda^3 & 10\lambda^2 & 5\lambda \\ 0 & \lambda^5 & 5\lambda^4 & 10\lambda^3 & 10\lambda^2 \\ 0 & 0 & \lambda^5 & 5\lambda^4 & 10\lambda^3 \\ 0 & 0 & 0 & \lambda^5 & 5\lambda^4 \\ 0 & 0 & 0 & 0 & \lambda^5 \end{bmatrix}$$

**(e)** ✓✓Use the answers to parts (a) and (b) to compute $C^5$ where $C$ is as below.

$$C = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

### The Cayley-Hamilton Theorem

**7.21** Suppose that the characteristic polynomial of $A$ is $P_A(\lambda) = \lambda^3(\lambda - 1)^5$.

**(a)** Prove that $A(A - I) = (A - I)A$.

**(b)** Prove that $A^3(A - I)^5 X = \mathbf{0}$ for all $X \in \mathbb{K}^8(A, 0)$.

**(c)** Prove that $A^3(A - I)^5 X = \mathbf{0}$ for all $X \in \mathbb{K}^8(A, 1)$.

**(d)** Prove that $A^3(A - I)^5 = \mathbf{0}$.

**7.22** Suppose that $A$ is an $n \times n$ matrix $A$ and $P_A(\lambda) = \pm(\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \dots (\lambda - \lambda_k)^{n_k}$. Prove that

$$\pm(A - \lambda_1 I)^{n_1}(A - \lambda_2 I)^{n_2} \cdots (A - \lambda_k I)^{n_k} = \mathbf{0}$$

*Remark.* The result just proved is the Cayley-Hamilton theorem, which states that $p_A(A) = \mathbf{0}$.

## CHAPTER SUMMARY

In this chapter we completed, and extended, the discussions of eigenvectors and diagonalization begun in Chapter 5. Let $A$ be an $n \times n$ matrix and $\lambda \in \mathbb{K}$ where $\mathbb{K}$ can be either $\mathbb{R}$ or $\mathbb{C}$. A $\lambda$-eigenvector $X$ for $A$ is an element of $\mathbb{K}^n$ such that $(A - \lambda I)X = \mathbf{0}$; $X$ is a $\lambda$ generalized eigenvector if there is an $n$ such that $(A - \lambda I)^n X = \mathbf{0}$. For $\lambda$ generalized eigenvectors to exist, $\lambda$ must be an eigenvalue of $A$.

The main result of this chapter is Theorem 7.1 on page 431 which asserts, among other things, that if all of the eigenvalues of $A$ belong to $\mathbb{K}$, then there is a basis of $\mathbb{K}^n$ with each basis element being a generalized eigenvector for $A$. One important consequence is that we can use such a basis to compute $A^n$. This computation is particularly simple if we use a ***chain basis***. [See formula (7.9) on page 423 and the supporting work.]

In the diagonalizable case, there exists $n \times n$ matrices $Q$ and $J$,$Q$ invertible and $J$ diagonal, such that $A = QJQ^{-1}$. In the general case, according to Theorem 7.2 on page 440, the same is true except that now $J$ is upper triangular, bidiagonal, with the second diagonal entries equal to 0 or 1. In either case, the diagonal entries of $D$ are the eigenvalues of $A$. The matrix $Q$ is just the matrix whose columns are the elements of some chain basis for $A$. The matrix $J$ is the ***Jordan canonical form*** for $A$.

# CHAPTER 8

# NUMERICAL TECHNIQUES

## 8.1 CONDITION NUMBER

Computational techniques, which is the subject of this chapter, are discussed in many other places in this text, including Sections 1.3, 2.3, 3.2, and 3.3. This chapter adds to this discussion the concept of a *norm*, which is used to measure error in vector calculations, the *condition number*, which measures how ill behaved a system of equations can be, and iterative methods for computing eigenvalues. We merely "scratch the surface" of these topics. Our goal is more to give the reader a feel for some of the issues involved than to present a comprehensive discussion. A reader who wants to know more is advised to consult any good text on numerical analysis.

### Norms

Any computer, no matter how sophisticated, can carry only a fixed number of digits. Hence the result of every computation is rounded off, producing **round-off error**; the more calculations done, the greater the round-off error. Different techniques for computing the same quantity can produce vastly different amounts of such error. For example, we saw on page 61 that the amount of round off error in the row reduction process depends on the pivoting strategy.

To discuss error, we need to measure it. If the answer is a single number, then the error in the approximation $X \approx X'$ is just $|X - X'|$. Hence, for example, the error in $10.00 \approx 10.02$ is $|10.00 - 10.02| = 0.02$. Quantifying error when the quantities are

*Linear Algebra: Ideas and Applications*, Fourth Edition. Richard C. Penney.

Companion website: www.wiley.com/go/penney/linearalgebra

vectors is more problematic. What, for instance, should the error in the approximation $X \approx X'$ be if

$$X = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \quad X' = \begin{bmatrix} 2.01 \\ 2.98 \\ -1.05 \end{bmatrix} \tag{8.1}$$

There are at least three natural answers:

1. $X$ and $X'$ are points in $\mathbb{R}^3$. The error is the distance between them, which is

$$|X - X'| = \sqrt{(-0.01)^2 + (-0.02)^2 + 0.05^2} \approx 0.055$$

In general, if $X \in \mathbb{R}^n$, $|X|$ is called the **Euclidean norm** of $X$. If $X' \in \mathbb{R}^n$, the **Euclidean norm error** in the approximation $X \approx X'$ is $|X - X'|$.

2. The error in the approximation $X \approx X'$ is the error in the least accurate entry. For the vectors in (8.1), the least accurate entry is the last, which has an error of $|-1-(-1.05)| = 0.05$. In general, if $X = [x_1, \dots, x_n]^t \in \mathbb{R}^n$, we define

$$|X|_\infty = \max\{|x_1|, |x_2|, \dots, |x_n|\} \tag{8.2}$$

which is referred to as the **infinity-norm** of $X$. If $X' \in \mathbb{R}^n$, the **infinity-norm error** in the approximation $X \approx X'$ is $|X - X'|_\infty$.

3. The error is the sum of the errors in each entry which, for the vectors in (8.1) is

$$0.01 + 0.02 + 0.05 = 0.08$$

In general, if $X = [x_1, \dots, x_n]^t \in \mathbb{R}^n$, we define

$$|X|_1 = |x_1| + |x_2| + \cdots + |x_n| \tag{8.3}$$

which is referred to as the **one-norm** of $X$. If $X' \in \mathbb{R}^n$, the **one-norm error** in the approximation $X \approx X'$ is $|X - X'|_1$.

In general, a "norm" on a vector space $\mathcal{V}$ is any function fulfilling the requirements of the following definition. Theorem 6.3 on page 311 proves these properties for the Euclidean norm while Exercise 8.2 proves them for our other two norms. Property 4 is referred to as the "triangle inequality."

**Definition 8.1** *Let $\mathcal{V}$ be a vector space. A norm on $\mathcal{V}$ is a real valued function*[1] $|\cdot|\tilde{}$ *on $\mathcal{V}$ satisfying:*

[1]It is common to denote general norms by $\|\cdot\|$ rather than $|\cdot|\tilde{}$. We reserve the double bar notation for operator norms, using subscripts, such as $|\cdot|_1$ and $|\cdot|_\infty$, to distinguish between norms.

1. *for all* $X \in \mathcal{V}$, $|X|^{\tilde{}} \geq 0$,
2. $|X|^{\tilde{}} = 0$ *if and only if* $X = \mathbf{0}$,
3. *for all scalars* $c$, $|cX|^{\tilde{}} = |c|\,|X|^{\tilde{}}$, *and*
4. *for all* $X, Y \in \mathcal{V}$, $|X + Y|^{\tilde{}} \leq |X|^{\tilde{}} + |Y|^{\tilde{}}$.

We use the one-norm here. There is an entirely parallel theory for the other norms.

## Condition Number

Suppose we wish to solve the system $AX = B$ where

$$A = \begin{bmatrix} 1 & 0.99 \\ 1 & 1.00 \end{bmatrix}, \quad B = \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix} \tag{8.4}$$

Then it is easily computed that

$$A^{-1} = \begin{bmatrix} 100 & -99 \\ -100 & 100 \end{bmatrix}, \quad X = A^{-1}B = \begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix} \tag{8.5}$$

Suppose, however, that the entries of $B$ are subject to a measurement error of up to $\pm.1$ so that the measured value of $B$ could be, say, $B' = [1.1, 1]^t$. In this case, the computed value of $X$ is

$$X' = \begin{bmatrix} 100 & -99 \\ -100 & 100 \end{bmatrix} \begin{bmatrix} 1.1 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 11.0 \\ -10.0 \end{bmatrix}$$

The percentage error in $B'$ is

$$100\frac{|B - B'|_1}{|B|_1} = 100\frac{|1 - 1.1| + |1 - 1|}{1 + 1} = 5\%$$

while the percentage error in $X'$ is

$$100\frac{|X - X'|_1}{|X|_1} = 100\frac{|1 - 11| + |0 - (-10)|}{1 + 0} = 2000\%$$

Hence, a 5% error in $B$ resulted in a 2000% error in $X$; the percentage of error increased by a factor of 400!

To consider this situation in general, we make the following definition:

**Definition 8.2** *Let* $A = [A_1, \ldots, A_n]$ *be an* $m \times n$ *matrix. Then*

$$\|A\| = \max\{|A_1|_1, \ldots, |A_n|_1\} \tag{8.6}$$

Less formally, $\|A\|$ is the one-norm length of the longest column of $A$.

## EXAMPLE 8.1

Compute $\|A\|$ and $\|A^{-1}\|$ for the matrix $A$ in formula (8.4).

*Solution.* From (8.4) and (8.5), the first columns of both $A$ and $A^{-1}$ have the greatest one-norm. Hence

$$\begin{aligned}\|A\| &= |A_1|_1 = 1 + 1 = 2\\ \|A^{-1}\| &= |\left(A^{-1}\right)_1|_1 = 100 + 100 = 200\end{aligned}$$

$\|A\|$ measures the tendency of multiplication by $A$ to expand or contract lengths:

**Proposition 8.1** *Let $A$ be an $n \times n$ matrix. Then, for all $X \in \mathbb{R}^n$,*

$$|AX|_1 \leq \|A\|\,|X|_1 \tag{8.7}$$

*Furthermore, there is at least one $X \in \mathbb{R}^n$ such that*

$$|AX|_1 = \|A\|\,|X|_1$$

*Proof.* Let $X = [x_1, \ldots, x_n]^t \in \mathbb{R}^n$. The triangle inequality implies that

$$\begin{aligned}|AX|_1 &= |A_1x_1 + \cdots + A_nx_n|_1\\ &\leq |A_1|_1\,|x_1| + \cdots + |A_n|_1\,|x_n|\\ &\leq \|A\|\,|x_1| + \cdots + \|A\|\,|x_n|\\ &= \|A\|\,|X|_1\end{aligned} \tag{8.8}$$

proving the first claim in the proposition. The second claim is Exercise 8.6. □

*Remark.* Proposition 8.1 implies that $\|A\|$ is the smallest number satisfying inequality (8.7) for all $X \in \mathbb{R}^n$. It can be shown that if $|\cdot|^{\sim}$ is a norm on a finite-dimensional vector space $\mathcal{V}$ and $T : \mathcal{V} \to \mathcal{V}$ is a linear transformation, then there is a smallest number $\|T\|_{op}$ satisfying

$$|T(X)|^{\sim} \leq \|T\|_{op}\,|X|^{\sim}$$

for all $X \in \mathcal{V}$. This number is referred to as the operator norm for $T$ corresponding to the norm $|\cdot|^{\sim}$. Hence, Proposition 8.1 says that $\|A\|$ is the operator norm for the linear transformation defined by $A$ relative to the one-norm.

Proposition 8.1 helps explain the magnification of the error observed previously. Suppose that we are given a system $AX = B$ where $A$ is an $n \times n$ invertible matrix and $B \in \mathbb{R}^n$. Our solution is then

$$X = A^{-1}B$$

Assume that $B$ was reported, inaccurately, to be $B + \Delta B$, where $\Delta B \neq \mathbf{0}$. The computed value of $X$ is then

$$\begin{aligned} X' &= A^{-1}(B + \Delta B) \\ &= X + \Delta X, \quad \Delta X = A^{-1}\Delta B \end{aligned}$$

and, using Proposition 8.1, the error is

$$|\Delta X|_1 = |A^{-1}\Delta B|_1 \leq \|A^{-1}\| \, |\Delta B|_1 \tag{8.9}$$

On the other hand, from Proposition 8.1 again,

$$|B|_1 = |AX|_1 \leq \|A\| \, |X|_1 \tag{8.10}$$

which implies that

$$\frac{1}{|X|_1} \leq \frac{\|A\|}{|B|_1} \tag{8.11}$$

Thus (8.9) and (8.11) together imply

$$\frac{|\Delta X|_1}{|X|_1} \leq \|A^{-1}\| \, \|A\| \frac{|\Delta B|_1}{|B|_1} \tag{8.12}$$

The fraction on the right is the fractional error in the reported value of $B$ and the fraction on the left is the fractional error in the computed value of $X$. Thus, *the percentage error can increase by a factor of at most* $\|A^{-1}\| \, \|A\|$.

Exercise 8.7 shows that there exist choices of $B$ and $\Delta B$ for which inequality (8.12) becomes an equality, and hence magnification of the percentage of error by a factor of $\|A^{-1}\| \, \|A\|$ **will** be observed for some choice of $B$ and $\Delta B$. Note that if $A$ is as in Example 8.1, $\|A^{-1}\| \, \|A\| = 400$, implying the potential of a 400-fold increase in in the percentage of error, which we observed.

**Definition 8.3** *If $A$ is an $n \times n$ invertible matrix, then* $\operatorname{cond}(A) = \|A^{-1}\| \, \|A\|$ *is the **condition number** for A in the one-norm.*

As the above discussion shows, systems having coefficient matrices with large condition numbers require special care.

**Proposition 8.2** *For all $n \times n$ matrices $A$ and $B$,* $\|AB\| \leq \|A\| \, \|B\|$. *In particular, if $A$ is invertible,* $1 \leq \operatorname{cond}(A)$.

*Proof.* Let $B = [B_1, \dots, B_n]$ where the $B_i$ are the columns of $B$. Then $AB = [AB_1, \dots, AB_n]$. It follows from Proposition 8.1 that

$$|AB_i|_1 \le \|A\|\,|B_i|_1 \le \|A\|\,\|B\|$$

The first statement of the proposition follows. The second statement follows as well since

$$1 = \|I\| = \|A^{-1}A\| \le \|A^{-1}\|\,\|A\| = \mathrm{cond}(A)$$

□

## Least Squares

In Section 6.5 we constructed the least-squares solution to the system $AX = X$ where $A$ and $B$ are defined in formula (6.65) on page 371. We did so by solving the normal equations $CX = A^tB$ where $C = A^tA$. From formula (6.66) on page 372,

$$C = \begin{bmatrix} 2{,}537{,}500.0 & 119{,}500.0 & 12{,}950.00 \\ 119{,}500.0 & 5659.0 & 612.20 \\ 12{,}950.0 & 612.2 & 66.40 \end{bmatrix}$$

The longest column of $C$ is the first. Hence

$$\|C\| = |C_1|_1 = (2.669950)10^6$$

For $C$ to have a small condition number, $\|C^{-1}\|$ would need to be *very* small. However

$$C^{-1} = \begin{bmatrix} 0.0000900 & -0.000631 & -0.011690 \\ -0.0006311 & 0.072977 & -0.549764 \\ -0.0116900 & -0.549764 & 7.363775 \end{bmatrix} \quad \text{so} \quad \|C^{-1}\| = 7.925229$$

Hence

$$\mathrm{cond}(C) = (2.669950)10^6 \cdot 7.925229 = (2.115997)10^7$$

which is, of course, quite large! In general, the techniques from Section 6.5 tend to produce ill conditioned systems. For this reason in least-squares computations, we usually use the SVD method demonstrated in Example 6.23 on page 403.

## EXERCISES

**8.1** Compute $|X|_1$, $|X|_\infty$, $|Y|_1$, and $|Y|_\infty$ for each vector $X$ and $Y$ in Exercise 6.1 on page 316. ✓[(a), (c), (e), (g)]

**8.2** Prove that $|\cdot|_1$ and $|\cdot|_\infty$ are norms on $\mathbb{R}^n$-that is, the properties 1–4 of Definition 8.1 on page 447 hold for both $|\cdot|_1$ and $|\cdot|_\infty$. You may assume the triangle inequality for real numbers.

**8.3** Prove that for all $X \in \mathbb{R}^n$:

**(a)** $|X|_\infty \leq |X|$. **(b)** $|X| \leq \sqrt{n}\,|X|_\infty$.

**(c)** ✓ $|X|_1 \leq \sqrt{n}|X|$. **(d)** $|X|_1 \leq n|X|_\infty$.

**8.4** Compute $\|A\|$ for the matrices in parts (a), (b), (d), (g), (i) Exercise 3.64 on page 190. ✓✓[(a), (d), (g)]

**8.5** Compute cond($A$) for the matrices in parts (a), (b), (d), (g), and (h) of Exercise 3.64 on page 190. Note that $A^{-1}$ is given on page 473 in the Answers, Exercise 3.64. ✓✓[(a), (d), (g)]

**8.6** Let $A$ be an $n \times n$ matrix and let $I_j$ be the $j$th standard basis element of $\mathbb{R}^n$. Prove that there is a $j$ such that $|AI_j|_1 = \|A\|$. Note that this completes the proof of Proposition 8.1 on page 449.

**8.7** ✓Let $A$ be an invertible $n \times n$ matrix. Prove that there exists $B$ and $B'$ in $\mathbb{R}^n$ for which inequality (8.12) on page 450 becomes an equality. Give a rule for finding suitable values of $B$ and $\Delta B$ from $A$ and $A^{-1}$.

**8.8** Let $A$ be as in (6.70) on page 375. Compute $\|A^tA\|$, $\|(A^tA)^{-1}\|$, and cond($A^tA$).

**8.9** For the following $n \times n$ matrices $A$ compute cond $A$. Then find explicit $B$ and $\Delta B$ in $\mathbb{R}^n$, with $|B|_1 = 1$ and $|\Delta B|_1 = .001$ such that inequality (8.12) on page 450 becomes an equality. We have provided $A^{-1} = C$ to simplify computations.

**(a)** $A = \begin{bmatrix} 15.00 & 19.00 & -3.30 \\ 19.00 & 30.00 & -10.00 \\ -3.30 & -10.00 & 6.44 \end{bmatrix}$ $C = \begin{bmatrix} 202.61 & -194.26 & -197.83 \\ -194.26 & 186.33 & 189.78 \\ -197.83 & 189.78 & 193.48 \end{bmatrix}$

**(b)** ✓ $A = \begin{bmatrix} 30 & 21 & 77 \\ 21 & 18 & 61 \\ 77 & 61 & 213 \end{bmatrix}$ $C = \begin{bmatrix} 12.56 & 24.89 & -11.67 \\ 24.89 & 51.22 & -23.67 \\ -11.67 & -23.67 & 11.00 \end{bmatrix}$

**8.10** ✓Show that, for an orthogonal matrix $A$, cond($A$) $\leq n$.

## 8.2 COMPUTING EIGENVALUES

The technique described in Section 5.1 for finding the eigenvectors of an $n \times n$ matrix $A$ is not practical for large matrices. It requires: (a) computing all $n$ of the coefficients of the characteristic polynomial $p_A(x)$, (b) computing the (possibly imaginary) roots $\lambda_i$ of $p_A(x)$, and finally (c) solving each of the systems $A - \lambda_i I$. This is all very "expensive" in terms of computation time. There are several much better algorithms for computing eigenvalues. Below we describe three such techniques.

## Iteration

In Example 5.2 on page 273 we computed $A^{10}B$, where

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

finding

$$A^{10}B = [1572352, 1573376]^t$$

It is striking that the two components of the answer are almost equal. It is easy to understand why. We saw that $[1, 1]^t$ and $[1, -1]^t$ are eigenvectors of $A$ corresponding to the eigenvalues 4 and 2, respectively. Since

$$[1, 2]^t = \tfrac{3}{2}[1, 1]^t + \tfrac{1}{2}[1, -1]^t$$

we see that

$$\begin{aligned} A^kB &= \frac{3(4^k)}{2}[1, 1]^t - \frac{2^k}{2}[1, -1]^t \\ \frac{A^kB}{4^k} &= \frac{3}{2}\,[1, 1]^t - \frac{2^k}{2(4^k)}[1, -1]^t \end{aligned}$$

(See formula (5.5) on page 275.) Hence

$$\lim_{k\to\infty} \frac{A^kB}{4^k} = \frac{3}{2}[1, 1]^t$$

showing that we approach a multiple of $[1, 1]^t$.[2] This particular eigenvector is singled out because 4 is an eigenvalue with largest absolute value—a "dominant eigenvalue."

**Definition 8.4** *Let $A$ be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_m$. Then $\lambda_i$ is said to be dominant if $|\lambda_i| \geq |\lambda_j|$ for all $j$. We say that $\lambda_i$ is subordinate if $|\lambda_i| \leq |\lambda_j|$ for all $j$.*

In general, we have the following theorem:

**Theorem 8.1** *Suppose that $A$ is an $n \times n$ diagonalizable matrix having a unique dominant eigenvalue $\lambda$. Then for all $B \in \mathbb{R}^n$*

$$\lim_{k\to\infty} \frac{A^kB}{\lambda^k} = X_1 \tag{8.13}$$

*exists and, if nonzero, is a $\lambda$-eigenvector of $A$.*

[2]See equation (3.22) on page 201 for the definition of limits of matrices.

*Remark.* The assumption that there is a unique dominant eigenvalue $\lambda$ implies that $\lambda$ is real, since $|\lambda| = |\bar{\lambda}|$.

*Proof.* Let the eigenvalues of $A$ be $\lambda_1, \ldots, \lambda_m$ where $\lambda = \lambda_1$. Since $A$ is diagonalizable we may write

$$B = X_1 + \cdots + X_m \tag{8.14}$$

where $AX_j = \lambda_j X_j$.

Then

$$\begin{aligned} A^k B &= \lambda_1^k X_1 + \cdots + \lambda_m^k X_1 \\ \frac{A^k B}{\lambda_1^k} &= X_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^k X_2 + \cdots + \left(\frac{\lambda_m}{\lambda_1}\right)^k X_n \end{aligned}$$

For $i > 1$, $|\lambda_i|/|\lambda_1| < 1$, and hence $\lim_{k\to\infty}(\lambda_i/\lambda_1)^k = 0$, proving the theorem. □

*Remark.* The rate of convergence of (8.13) depends upon $|\lambda_i|/|\lambda|$ where $\lambda_i$ is the second most dominant eigenvalue. The closer this ratio is to 1, the slower the convergence.

**Theorem 8.2** *Let the notation be as in Theorem 8.1 and let $X_1$ be as in* (8.14). *Then if $X_1 \neq \mathbf{0}$,*

$$\lim_{k\to\infty} \frac{A^k B}{|A^k B|} = Z_1 \tag{8.15}$$

*where $Z_1 = X_1/|X_1|$. Furthermore*

$$\lambda = \lim_{k\to\infty} \frac{(A^{k+1}B)\cdot(A^k B)}{(A^k B)\cdot(A^k B)} \tag{8.16}$$

*Proof.* From Theorem 8.1,

$$\begin{aligned} \lim_{k\to\infty} \frac{A^k B}{|A^k B|} &= \lim_{k\to\infty} \frac{A^k B}{\lambda^k} \frac{\lambda^k}{|A^k B|} \\ &= \lim_{k\to\infty} \frac{A^k B}{\lambda^k} \lim_{k\to\infty} \left|\frac{A^k B}{\lambda^k}\right|^{-1} \\ &= X_1 \, |X_1|^{-1} = Z_1 \end{aligned}$$

To prove (8.16), let $Y_k = A^k B/|A^k B|$. Then, division of both the numerator and denominator in (8.16) by $|A^k B|^2$ shows

$$\begin{aligned}\frac{(A^{k+1}B)\cdot(A^kB)}{(A^kB)\cdot(A^kB)} &= \frac{(AY_k)\cdot Y_k}{Y_k\cdot Y_k}\\ &= (AY_k)\cdot Y_k\end{aligned}$$

From (8.15), the limit as $k$ tends to infinity is

$$AZ_1 \cdot Z_1 = \lambda(Z_1 \cdot Z_1) = \lambda$$

as claimed. □

*Remark.* The ratio in (8.16) is called the **Rayleigh quotient**.

*Remark.* The assumption in Theorem 8.2 that $X_1 \neq \mathbf{0}$ is difficult to verify. However, the set of $B$ for which $X_1 = \mathbf{0}$ is the span $\mathcal{W}$ of the $\lambda_i \neq \lambda_1$ eigenspaces, which has dimension less than $n$. It follows that the probability of a randomly chosen vector $B$ having $X_1 = \mathbf{0}$ is almost zero. In fact, if $\{B_1, \dots, B_n\}$ is a basis for $\mathbb{R}^n$, then there is at least one $i$ such that $B_i \notin \mathcal{W}$. Hence to compute the dominant eigenvalue we can either choose $B$ using a random number generator, in which case the limits in (8.16) and (8.15) are "almost certainly" a dominant eigenvalue and the corresponding eigenvector respectively, or we can approximate the limit in (8.16) for each of the $B_i$ in our basis. The answer with the largest absolute value will be an approximation to the dominant eigenvalue. In practice, one needs better ways of choosing $B$. Discussion of this issue is beyond the scope of this text.

### EXAMPLE 8.2

Use Theorem 8.2 with $k = 5$ and $B = [1, 1, 1]^t$ to approximate an eigenvector for

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$$

*Solution.* We compute that, to three decimal places,

$$\begin{aligned}\frac{A^5B}{|A^5B|} &= \frac{[364, 1, 727]^t}{29\sqrt{786}}\\ &= [0.4478, 0.0012, 0.8942]^t\\ \frac{(A^6B)\cdot(A^5B)}{(A^5B)\cdot(A^5B)} &= \frac{[1093, 1, 2185]^t \cdot [364, 1, 727]^t}{[364, 1, 727]^t \cdot [364, 1, 727]^t}\\ &= 3.005\end{aligned}$$

Hence $[0.4478, 0.0012, 0.8942]^t$ approximates a $\lambda$-eigenvector with $\lambda \approx 3.005$.

From Example 5.3 on page 276, the eigenvalues of $A$ are 1 and 3 and the 3-eigenspace is spanned by $Y = [1, 0, 2]^t$. Hence $(0.4478)Y = [0.4478, 0.8956]^t$ is a 3-eigenvector, which agrees closely with our work.

It is of course a serious limitation to the preceding method that we only approximate the dominant eigenvalue. If $A$ is invertible and has a unique subordinate eigenvalue, we can approximate it by applying iteration to $A^{-1}$, since the inverse of the dominant eigenvalue of $A^{-1}$ is the subordinate eigenvalue of $A$. Note that if $X_k = (A^{-1})^k B$, then

$$\begin{aligned} AX_k &= A(A^{-1})^k B \\ &= (A^{-1})^{k-1} B = X_{k-1} \end{aligned} \tag{8.17}$$

Typically $X_k$ is computed by solving the preceding equation, making use of the $LU$ factorization from Section 3.4 on page 203, rather than finding $A^{-1}$. Formula (8.16) now implies that if $A$ is invertible and has a unique subordinate eigenvalue $\beta$, then

$$\beta^{-1} = \lim_{k\to\infty} \frac{X_{k+1} \cdot X_k}{X_k \cdot X_k} \tag{8.18}$$

This technique is called **inverse iteration**.

### EXAMPLE 8.3

Use formula (8.18) with $k = 5$ along with MATLAB to approximate the subordinate eigenvalue for the matrix $A$ in Example 8.2 on page 455.

*Solution.* We arbitrarily set $X_1 = [1, 0, 0]^t$. After entering `X1=[1;0;0]` and $A$ into MATLAB, we issue the commands `X2=A\X1`, `X3=A\X2, ...,` `X6=A\X5` yielding, ultimately, $X_5 = [0.0123, 0, -1.9753]^t$ and $X_6 = [0.0041, 0, -1.9918]^t$. Next we enter `dot(X6,X5)/dot(X5,X5)` obtaining 1.0083 as our approximation.

Actually, being able to approximate subordinate eigenvalues allows us to approximate any eigenvalue, provided we can find a sufficiently accurate first approximation. Let the eigenvalues of $A$ be $\lambda_1, \ldots, \lambda_m$ and suppose that $c$ is close enough to $\lambda_i$ to guarantee that

$$|\lambda_i - c| < |\lambda_j - c|$$

for all $j \neq i$. The eigenvalues of $A - cI$ are $\lambda_j - c$, and hence $\lambda_i - c$ is the unique subordinate eigenvalue of $A - cI$. We may therefore use inverse iteration applied to $A - cI$ to approximate $\lambda_i - c$, and thus $\lambda_i$, as closely as desired.

As commented previously, the smaller the value of $|\lambda_i|/|\lambda|$, the faster the convergence of the limit in (8.15), where $\lambda_i$ is the second most dominant eigenvalue. Hence, when using inverse iteration on $A - cI$, the limit will converge fastest for small values of $|\lambda_i - c|/|\lambda_j - c|$ where $\lambda_j$ is the second closest eigenvalue to $c$. Thus, the closer $c$

is to $\lambda_i$, the faster the convergence. As we obtain better approximations to $\lambda_i$, we can update the value of $c$ to improve the rate of convergence.

### The *QR* Method

There is another quite remarkable computational technique, the "*QR* method," that yields approximations to all of the eigenvalues at once. This is the most common method for approximating eigenvalues.

Let $A$ be an $n \times n$ invertible matrix. According to Theorem 6.10 on page 326 and Theorem 6.11 on page 327, there are unique $n \times n$ matrices $P_k$ and $S_k$ with $P_k$ orthogonal and $S_k$ upper triangular with all of its diagonal entries positive such that

$$A^k = P_k S_k, \quad k = 0, 1, 2, \ldots \tag{8.19}$$

This is the "*QR* decomposition" of $A^k$. The following result, which we prove at the end of this section, is the basis of the *QR* method.

**Theorem 8.3** *Assume that A is diagonalizable with all of its eigenvalues positive and distinct. Let $P_k$ and $S_k$ be as in* (8.19). *Then*

$$\begin{aligned} P_\infty &= \lim_{k\to\infty} P_k \\ R_\infty &= \lim_{k\to\infty} S_k (S_{k-1})^{-1} \end{aligned} \tag{8.20}$$

*both exist. Furthermore, $P_\infty$ is orthogonal, $R_\infty$ is upper triangular, and*

$$A = P_\infty R_\infty P_\infty^t \tag{8.21}$$

*In particular, the diagonal entries of $R_\infty$ are the eigenvalues of A.*

*Remark.* For a matrix with real eigenvalues, the positivity assumption on the eigenvalues is not a serious limitation. If $c > |\lambda|$ where $\lambda$ is the dominant eigenvalue of $A$, then the eigenvalues of $B = A + cI$ are positive and we can compute the eigenvalues of $A$ by computing those of $B$ and subtracting $c$. The assumption that the eigenvalues are real is, however, a serious limitation. There are versions of the *QR* method, which we do not discuss, that can approximate complex eigenvalues as well.

It follows from Theorem 8.3 that we can approximate the eigenvalues of $A$ by computing the diagonal entries of $S_k(S_{k-1})^{-1}$ for sufficiently large $n$. This appears to require computing $A^k$, $S_k$, $(S_{k-1})^{-1}$, and finally $S_k(S_{k-1})^{-1}$. There is, however, a beautiful technique, the ***QR* method**, for computing $S_k(S_{k-1})^{-1}$ without computing any of these matrices.

We first compute the *QR* decomposition of $A$:

$$A = Q_1 R_1$$

We next compute the $QR$ decomposition of $R_1Q_1$:

$$R_1Q_1 = Q_2R_2$$

In general, we define $Q_k$ and $R_k$ inductively as the $QR$ decomposition of $R_{k-1}Q_{k-1}$:

$$R_{k-1}Q_{k-1} = Q_kR_k \tag{8.22}$$

**Theorem 8.4** *Let $Q_k$ and $R_k$ be as defined above and let $P_k$ and $A^k = P_kS_k$ be the $QR$ factorization of $A^k$. Then*

$$\begin{aligned} P_k &= Q_1Q_2 \dots Q_k \\ S_k &= R_kR_{k-1} \dots R_1 \end{aligned} \tag{8.23}$$

*In particular*

$$\begin{aligned} S_k(S_{k-1})^{-1} &= R_k \\ (P_{k-1})^{-1}P_k &= Q_k \end{aligned} \tag{8.24}$$

*Proof.* Let

$$\begin{aligned} A_k &= Q_kR_k \\ &= R_{k-1}Q_{k-1} \\ &= Q_{k-1}^{-1}Q_{k-1}R_{k-1}Q_{k-1} \\ &= Q_{k-1}^{-1}A_{k-1}Q_{k-1} \end{aligned} \tag{8.25}$$

Applying the same equality to $A_{k-1}$, and then to $A_{k-2}, \dots$, shows that

$$A_k = Q_{k-1}^{-1}Q_{k-2}^{-1} \dots Q_1^{-1}AQ_1Q_2 \dots Q_{k-1} \tag{8.26}$$

Since $A_k = Q_kR_k$, this implies that

$$\begin{gathered} Q_1 \dots Q_{k-1}Q_kR_k = AQ_1Q_2 \dots Q_{k-1} \\ Q_1 \dots Q_kR_kR_{k-1} \dots R_1 = AQ_1 \dots Q_{k-1}R_{k-1} \dots R_1 \end{gathered}$$

For $k = 2$, this equality implies

$$Q_1Q_2R_2R_1 = AQ_1R_1 = A^2$$

For $k = 3$ we conclude

$$Q_1Q_2Q_3R_3R_2R_1 = AQ_1Q_2R_2R_1 = A^3$$

In general, it follows from mathematical induction that

$$A^k = Q_1 \dots Q_kR_k \dots R_1 \tag{8.27}$$

Our theorem follows from the uniqueness of the $QR$ decomposition and the observations that $P_k = Q_1 \dots Q_k$ is an orthogonal matrix and $S_k = R_k \dots R_1$ is upper triangular with positive diagonal entries. □

*Remark.* The $QR$ factorization is unique only if we insist that the diagonal entries of $R$ be positive. The $QR$ algorithms in many software packages, such as the ones included in MATLAB and in Maple, do not so insist. In this case, formula (8.27) continues to hold. However, (8.23) holds if and only if the diagonal entries of $\tilde{S}_k = R_k \dots R_1$ are all positive.

This is not a major difficulty. There is a diagonal matrix $D_k$, with all of its diagonal entries equal to $\pm 1$, such that $D_k \tilde{S}_k$ has all positive diagonal entries. Let $\tilde{P}_k = Q_1 \dots Q_{k-1} Q_k D_k$. Then the uniqueness of the positive diagonal $QR$ factorization implies that $P_k = \tilde{P}_k D_k$ and $S_k = D_k \tilde{S}_k$. Hence

$$S_k (S_{k-1})^{-1} = D_k \tilde{S}_k (\tilde{S}_{k-1})^{-1} D_{k-1}^{-1} = D_k R_k D_{k-1}$$

It follows that *each diagonal entry of* $S_k(S_{k-1})^{-1}$ *is the absolute value of the corresponding entry of* $R_k$. Hence, the absolute values of the diagonal entries of $R_k$ approximate the eigenvalues of $A$.

## EXAMPLE 8.4

Use the $QR$ method, along with MATLAB, to approximate the eigenvalues of

$$A = \begin{bmatrix} 0 & 3 & -3 \\ 2 & 2 & -2 \\ -4 & -1 & 1 \end{bmatrix}$$

*Solution.* We first use formula (8.16) on page 454 with $k = 4$ and $B = [1, 0, 0]^t$ to approximate the dominant eigenvalue. We find

$$\frac{(A^5 B) \cdot (A^4 B)}{(A^4 B) \cdot (A^4 B)} = 5.972$$

suggesting that the dominant eigenvalue is around 6. To be assured of positive eigenvalues, we work with

$$C = A + 8I = \begin{bmatrix} 8 & 3 & -3 \\ 2 & 10 & -2 \\ -4 & -1 & 9 \end{bmatrix}$$

In MATLAB we enter the matrix $A$ followed by `C=A+8*eye(3)`.

We find a $QR$ factorization of $C$ by entering `[Q1,R1]=qr(C)`, yielding

$$Q_1 = \begin{bmatrix} -0.9595 & 0.1126 & -0.2583 \\ -0.1615 & -0.9710 & 0.1763 \\ -0.2310 & 0.2109 & 0.9498 \end{bmatrix}, \quad R_1 = \begin{bmatrix} -12.7055 & -3.6360 & -3.6552 \\ 0.0000 & -8.4941 & 0.9567 \\ 0.0000 & 0.0000 & 5.1889 \end{bmatrix}$$

Next, we enter `C1=R1*Q1` producing

$$C_1 = \begin{bmatrix} 12.1905 & 2.5318 & 2.2743 \\ 2.0515 & 8.8349 & 0.5761 \\ 2.9346 & -0.9514 & 5.9746 \end{bmatrix}$$

We continue, entering `[Q2,R2]=qr(C1)`, `C2=R2*Q2`, `[Q3,R3]=qr(C2)`, `C2=R2*Q2`, ... .

Note that, from (8.24), $\lim_{k\to\infty} Q_k = I$. Hence, for large $k$, $Q_k R_k \approx R_k$, an upper triangular matrix. We iterate until the entries of $C_k$ below the main diagonal equal 0 to some predetermined tolerance—e.g., $\pm.05$. In this case 10 iterations are required, yielding

$$C_{10} = \begin{bmatrix} 13.9998 & 0.0313 & -0.0082 \\ 0.0311 & 8.0217 & 1.6937 \\ 0.0008 & -0.0384 & 4.9785 \end{bmatrix}$$

and

$$Q_{11} = \begin{bmatrix} -1.0000 & 0.0022 & -0.0001 \\ -0.0022 & -1.0000 & 0.0048 \\ -0.0001 & 0.0048 & 1.0000 \end{bmatrix}, \quad R_{11} = \begin{bmatrix} -13.9999 & -0.0491 & 0.0042 \\ 0.0000 & -8.0217 & -1.6699 \\ 0.0000 & 0.0000 & 4.9865 \end{bmatrix}$$

Rounding to one decimal, and taking the absolute value, we conclude that the eigenvalues of $C$ are approximately 14.0, 8.0, and 5.0. Subtracting 8 produces the eigenvalues of $A$: 6.0, 2.0, and $-3.0$. These answers are in fact exact, as the reader may easily verify.

*Remark.* Without further simplifications, the $QR$ method requires far too much computation to be practical. Typically one uses the techniques of Exercise 6.138 on page 408 to first reduce to either the Hessenberg or, in the case of a symmetric matrix, the symmetric tridiagonal case. It is an important fact that if $A$ is in one of either of these two forms, then so are the $A_k$. (See Exercises 8.14 and 8.15.)

## Proof of Theorem 8.3 on page 457

The following result is the key to the proof of Theorem 8.3. Recall that an $n \times n$ lower triangular matrix is nilpotent if all of its diagonal entries equal 0. A matrix $L$ is said to be unipotent of $L = I + N$ where $N$ is nilpotent.

**Lemma 8.1** *Let $D$ be an $n \times n$ diagonal matrix whose diagonal entries $\lambda_i$ satisfy $\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0$ and let $L$ be an $n \times n$ lower triangular unipotent matrix. Set $L_k = D^k L D^{-k}$. Then*

$$\lim_{k\to\infty} L_k = I$$

*Proof.* Write $L = I + N$ where $N$ is nilpotent. Then

$$D^k L D^{-k} = I + D^k N D^{-k}$$

The reader can check that

$$(D^k N D^{-k})_{ij} = \left(\frac{\lambda_i}{\lambda_j}\right)^k n_{ij}$$

Since $N$ is lower triangular and nilpotent, this is nonzero only for $j > i$, in which case $\lambda_i/\lambda_j < 1$. Hence, $\lim_{k\to\infty} D^k N D^{-k} = \mathbf{0}$, proving our lemma. □

We also need the following "continuity" theorem for the $QR$ factorization, which follows from the proof of the existence of the $QR$ factorization (Theorem 6.10 on page 326).

**Theorem 8.5** *Suppose that $X_k$ is a sequence of invertible $n \times n$ matrices such that $X = \lim_{k\to\infty} X_k$ exists and is invertible. Let $X_k = Q_k R_k$ and $X = QR$ be, respectively, the $QR$ factorizations of $X_k$ and $X$. Then*

$$\lim_{k\to\infty} Q_k = Q, \quad \lim_{k\to\infty} R_k = R$$

Theorem 8.3 follows, we claim, from the convergence of the $P_k$. In fact

$$\begin{aligned} A &= A^k (A^{k-1})^{-1} \\ &= P_k S_k (S_{k-1})^{-1} (P_{k-1})^{-1} \end{aligned}$$

Hence, from the orthogonality of the $P_k$,

$$P_k^t A P_{k-1} = S_k (S_{k-1})^{-1}$$

Given the convergence of the $P_k$, and taking limits, we see that $R_\infty$ exists and

$$P_\infty^t A P_\infty = R_\infty$$

The upper triangularity of $R_\infty$ follows from the upper triangularity of $S_k(S_{k-1})^{-1}$. The orthogonality of $P_\infty$ is clear since, for all $X \in \mathbb{R}^n$, $|X| = |P_k X| = \lim_{k\to\infty} |P_k X| = |P_\infty X|$. (See Definition 6.10 on page 356.)

To prove convergence, we note that, by hypothesis, there is an $n \times n$ invertible matrix $X$ and an $n \times n$ diagonal matrix such that

$$A = X^{-1}DX$$

where $D$ may be chosen so as to satisfy the hypotheses of Lemma 8.1 on page 461.

Assume for the moment that $X = LU$, where $U$ is an $n \times n$ upper triangular matrix and $L$ is an $n \times n$ lower triangular unipotent matrix. Then

$$\begin{aligned} A^k &= X^{-1}D^kX \\ &= X^{-1}D^kLU \\ &= X^{-1}L_kD^kU, \quad L_k = D^kLD^{-k} \end{aligned} \tag{8.28}$$

Let $X^{-1}L_k = \tilde{Q}_k\tilde{R}_k$ be the $QR$ factorization of $X^{-1}L_k$. From (8.28) and the uniqueness of the $QR$ factorization, $P_k = \tilde{Q}_k$. The convergence of the $P_k$ follows from Lemma 8.1 on page 461 and Theorem 8.5 on page 461.

Of course, not every matrix has such an $LU$ factorization. However, $X$ has a "modified Bruhat decomposition,"

$$X = L\Pi U$$

where $L$ and $U$ are as before and and $\Pi$ is an $n \times n$ permutation matrix (Definition 3.9 on page 210). (For a proof, see Eugene E. Tyrtyshnikov, *A Brief Introduction to Numerical Analysis*, p. 90.)

Instead of formula (8.28), we have

$$\begin{aligned} A^k &= X^{-1}L_kD^k\,\Pi\, U \\ &= X^{-1}L_k\,\Pi\, D_\Pi^kU, \quad D_\Pi = \Pi^{-1}D\,\Pi \end{aligned} \tag{8.29}$$

It is easily seen that $D_\Pi$ is a diagonal matrix. (It is just $D$ with its diagonal entries permuted.) Our theorem follows as before, using $X^{-1}L_k\,\Pi$ in place of $X^{-1}L_k$.

## EXERCISES

**8.11** *(Requires technology)* For the matrices in Exercise 5.3 on page 280:

**(a)** Use formulas (8.15) and (8.16) on page 454 with $k = 5$ and $B = [1, 0, 0]^t$ to approximate the dominant eigenvalue and eigenvector.

**(b)** ✓Use formulas (8.15) and (8.16) on page 454 with $k = 5$, $B = [1, 0, 0]^t$, and $A$ replaced by $A^{-1}$ to approximate the subordinate eigenvalue and eigenvector for $A$. *Problem:* The matrix $A$ in Exercise 5.3(a) is not invertible. What can you do?

**(c)** Reason as in Example 8.4 on page 459 with $k = 5$ to approximate all of the eigenvalues.

*Remark.* The exact values for the eigenvalues may be found in the solution to Exercise 5.27 on page 290.

**8.12** *(Requires technology)* Repeat Exercise 8.11 for the matrices from (a)–(c) in Exercise 5.29 on page 290 with $B = [1, 0, 0]^t$ in parts (a) and (b) and $B = [1, 0, 1, 0]^t$ in part (c). [*Note:* You will discover that the technique breaks down in part (c). Explain why this is expected in terms of the eigenvalues of $A$.]

**8.13** *(Requires technology)* Let

$$A = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 2 & -4 \end{bmatrix}$$

**(a)** Use the eigenvalue command [in MATLAB it is `eig(A)`] to approximate the dominant eigenvalue $\lambda$ of $A$.

**(b)** Let $B = [1, 0, 0, 0]^t$. Find (by "trial and error") the smallest value of $k$ so that the value produced by formula (8.16) on page 454 approximates $\lambda$ to within $\pm.001$.

**(c)** Reason as in Example 8.4 on page 459 with $k = 5$ to approximate all of the eigenvalues. (You will discover that the approximations are not extremely accurate.)

**8.14** Note that the matrix $A$ in Exercise 8.13 is upper Hessenberg [equation (6.99) on page 409]. The reader perhaps noted that the matrices $Q_i$, $R_i$ and $R_iQ_i$ found in Exercise 8.13 share this property. This exercise proves that this is a general phenomenon.

**(a)** Let $\mathbb{R}^n_i = \{X \in \mathbb{R}^n \mid x_j = 0, j > i\}$. Prove that an $n \times n$ matrix $A$ is upper triangular if and only if, for all $X \in \mathbb{R}^n_i$, $AX \in \mathbb{R}^n_i$.

**(b)** Prove that an $n \times n$ matrix $A$ is upper Hessenberg if and only if, for all $X \in \mathbb{R}^n_i$, $AX \in \mathbb{R}^n_{i+1}$.

**(c)** Prove that if $R$ and $A$ are respectively upper triangular and upper Hessenberg $n \times n$ matrices, then both $AR$ and $RA$ are upper Hessenberg.

**(d)** Prove that if $A = QR$ is the $QR$ factorization of an invertible $n \times n$ upper Hessenberg matrix $A$, then both $Q$ and $RQ$ are upper Hessenberg.

**(e)** Prove that each of the matrices $A_k$ from formula (8.25) on page 458 is upper Hessenberg.

**8.15** Let $A$ be an $n \times n$ symmetric tridiagonal matrix—that is, a band width 2 matrix [Formula (6.96) on page 407]. Prove that each of the matrices $A_k$ from formula (8.25) on page 458 are symmetric tridiagonal matrices. [*Hint:* Use formula (8.26) on page 458 together with Exercise 8.14.]

**8.16** Let $A$ satisfy the hypotheses of Theorem 8.3 on page 457.

**(a)** Use formula (8.21) on page 457 to prove that the first column $P^1_\infty$ of $P_\infty$ is an eigenvector for $A$ with eigenvalue $\lambda_1$ where the diagonal entries of $R_\infty$ are $\lambda_1, \dots, \lambda_n$.

**(b)** Let $P^1_k$ be the first column of $P_k$ from formula (8.19) on page 457. Prove that $P^1_k = A^k B/|A^k B|$ where $B = [1, 0, \dots, 0]^t \in \mathbb{R}^n$. Hence, $P^1_\infty = Z_1$ where $Z_1$ is as in (8.15) on page 454.

## CHAPTER SUMMARY

Section 8.1 began by introducing three norms on $\mathbb{R}^n$, $|\cdot|_1$, $|\cdot|_\infty$, and $|\cdot|$, that can be used to measure error in vector calculations. Corresponding to any given norm, there is a matrix norm, $\|A\|_{op}$, defined for all $n \times n$ matrices $A$. For sake of simplicity our discussion focused on $|\cdot|_1$, and the corresponding matrix norm, denoted $\|\cdot\|$. This led to the crucial concept of condition number, which for an $n \times n$ matrix $A$ is defined by $\text{cond}(A) = \|A\|\,\|A^{-1}\|$. In general, $\text{cond}(A) \geq 1$. If $\text{cond}(A)$ is considerably greater than 1, then the system $AX = B$ is “numerically unstable” in that solution $X$ is highly susceptible to small errors in the computation of $B$.

Section 8.2 discussed iterative methods for computing eigenvalues: forward iteration, reverse iteration, and the $QR$ method. The first technique yields the dominant eigenvalue, the second yields the subordinate eigenvalue, and the third yields all eigenvalues.

# ANSWERS AND HINTS

## Section 1.1 on page 17

**1.1**

**(a)** $$\begin{bmatrix} -1 & -4 & -7 & -10 \\ 1 & -2 & -5 & -8 \\ 3 & 0 & -3 & -6 \end{bmatrix}, [3, 0, -3, -6], \begin{bmatrix} -4 \\ -2 \\ 0 \end{bmatrix},$$

**(c)** $$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \\ -1 & 0 \end{bmatrix}, [-1, 0], \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$

**1.3** $C = A + B$.

**1.5** **(a)** $[1, 1, 4] = [1, 1, 2] + 2[0, 0, 1]$. **(c)** The second matrix is a linear combination of the other three. **(d)** Sum the third and fourth matrices.

**1.6** $P_2 = P_5 - P_1 - P_3 - P_4$, where $P_i$ is the $i$th row of $P$.

**1.9** Each vector has a nonzero entry in the positions where the other two vectors have zeros.

**1.13** **(a)** $-2X + Y = [1, 1, 4]$ (other answers are possible). **(b)** Let $[x, y, z] = aX + bY = [-a - b, a + 3b, -a + 2b]$ and substitute into $5x + 3y - 2z$. You should get 0. **(c)** Any point $[x, y, z]$ that does not solve the equation $5x + 3y - 2z = 0$ will work—for example $[1, 1, 1]$.

*Linear Algebra: Ideas and Applications*, Fourth Edition. Richard C. Penney.

Companion website: www.wiley.com/go/penney/linearalgebra

**1.23** For the first part, use various values of $a$, $b$, and $c$ in $aX + bY + cZ$. For the second part, consider the (2, 1) entry of the general element of the span?

**1.25** Let $V$ and $W$ be elements of the span. Then $V = aX + bY$ and $W = cX + dY$. Simplify $xV + yW$ until you obtain an expression of the form $eX + fY$, where $e$ and $f$ are scalars. This proves that $xV + yW$ is in the span of $X$ and $Y$.

**1.26** Let the columns of $A$ be $A_i$, $i = 1, 2, 3$. Then $3A_3 - A_2 = A_1$.

**1.29** You could, say, make the fourth row be the sum of the first three or the second a multiple of the first.

**1.33** **(b)** $\sinh x = \frac{1}{2}(e^x - e^{-x}) = \frac{1}{4}(2e^x) - \frac{1}{6}(3e^{-x})$.

**(d)** Use the double angle formula for the cosine function.

**(f)** $(x + 3)^2 = \ ?$

**(h)** Use the angle addition formulas for $\sin x$ and $\cos x$.

**(i)** Use $\ln(a/b) = \ln a - \ln b$ and $\ln a^b = b \ln a$.

**1.36** **(a)** Let $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ and solve $A + B = A$. **(b)** Solved similarly to (a).

**1.40** Here are the four steps, but you still must put in the reasons.

$$\begin{aligned} -(aX) + (aX + (bY + cZ)) &= -(aX) + 0 \\ (-(aX) + aX) + (bY + cZ) &= -(aX) \\ 0 + (bY + cZ) &= -1(aX) \\ bY + cZ &= (-a)X \end{aligned}$$

Next, multiply both sides by $(-a)^{-1}$ and simplify some more.

## Section 1.2 on page 38

**1.49** $X$ is a solution. $Y$ is not.

**1.50** $b = 0$, all $a$.

**1.55** (Roman numerals are equation numbers.)

**(b)** Solution: $[-\frac{59}{9}, \frac{20}{9}, \frac{8}{9}]^t$, spanning: 0, translation: $[-\frac{59}{9}, \frac{20}{9}, \frac{8}{9}]^t$.

**(c)** Solution: the line $[1, 0, 0]^t + t[-\frac{17}{2}, \frac{5}{2}, 1]^t$, spanning: $[-\frac{17}{2}, \frac{5}{2}, 1]^t$, translation: $[1, 0, 0]^t$, 2I+II=III.

**(d)** Inconsistent: 2I + II contradicts III.

**(f)** Solution: the line $[\frac{11}{7}, \frac{6}{7}, \frac{1}{7}, 0]^t + w[-\frac{10}{7}, -\frac{1}{7}, \frac{23}{14}, 1]$.

**(j)** Solution: the plane $[\frac{5}{4}, -\frac{1}{4}, 0, 0]^t + r[-\frac{3}{4}, -\frac{1}{4}, 1, 0] + s[-1, 0, 0, 1]$, spanning: $[-\frac{3}{4}, -\frac{1}{4}, 1, 0]^t$ and $[-1, 0, 0, 1]^t$, translation: $[\frac{5}{4}, -\frac{1}{4}, 0, 0]^t$.

**1.58** A point $(x, y)$ solves the system if and only if it lies on both lines. Since the lines are parallel, there is no solution to the system.

## Section 1.2.2 on page 46

**1.60** **(a)** $i_1 = -\frac{85}{16}$ amperes from C to B, $i_2 = -\frac{5}{6}$ amperes from E to D, $i_3 = 4$ amperes from F to C. c) $i_1 = \frac{34}{15}$ amperes from C to B, $i_2 = \frac{5}{6}$ amperes from H to C, $i_3 = -\frac{49}{10}$ amperes from D to C, $i_4 = -\frac{9}{10}$ amperes from G to D, $i_5 = -4$ amperes from E to D, $i_6 = -\frac{49}{10}$ amperes from H to G.

## Section 1.3 on page 63

**1.63** **(a)** Neither, **(b)** echelon, **(c)** neither, **(d)** echelon **(e)** reduced echelon.

**1.65** **(a)** $\begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$, **(c)** $\begin{bmatrix} 1 & 0 & \frac{10}{3} \\ 0 & 1 & \frac{1}{3} \end{bmatrix}$, **(e)** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

**(g)** $\begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 & 5 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

**1.67** **(a)** $c = a + 2b$, **(c)** $a = c$.

**1.75** **(b)** $\begin{bmatrix} 1 & 0 & 3 & -1 & 0 \\ 0 & 1 & -4 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

**1.78** **(a)** In the span, **(b)** in the span, **(c)** not in the span.

**1.80** This is not hard. Just pick a vector at random. The chances are that it will not be in the span. To prove it, reason as in Exercise 7.

**1.83**
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

**1.85** **(a)** No. **(b)** *Hint:* Can the system have any free variables? **(c)** No.

**1.87** $s[1, 0, 1, -2]^t + t[0, 1, -1, 1]^t$.

## Section 1.4 on page 86

**1.93** **(a)** $[0, 5, -11]^t$, **(b)** $[7, 10, 7, 5]^t$, **(c)** $[x_1 + 2x_2 + 3x_3, 4x_1 + 5x_2 + 6x_3]^t$.

**1.95** For (c), for example, you might choose

$$\begin{aligned} x + 2y + 3z &= 17 \\ 4x + 5y + 6z &= -4 \end{aligned}$$

**1.97** The nullspace is spanned by: **(a)** $\{[-1, 1, 1, 0, 0]^t, [-3, -1, 0, 0, 1]^t\}$ **(c)** $\{[-10, -1, 3]^t\}$, **(e)** $\{[0, 0]\}$, **(g)** $\{[\frac{1}{2}, -1, 1, 0, 0]^t, [-5, 1, 0, -2, 1]^t\}$.

**1.99** Why does the following matrix work?

$$\begin{bmatrix} 1 & 2 & -1 & 5 \\ 2 & 4 & -2 & 10 \\ 3 & 6 & -3 & 15 \end{bmatrix}$$

Find another example yourself.

**1.103** *Hint:* What is the simplest solution to the system below that you can think of? How does this prove that the system is consistent?

$$\begin{aligned} x + 2y + 3z + 4w + 5t + 6u + 7v &= 0 \\ x - 3y + 7z - 3w + 3t + 9u - 6v &= 0 \end{aligned}$$

**1.105** **(c)** From the translation theorem, my claim will be true if $[-1, 1, 2, 1, 1, 1]^t$ solves the nonhomogeneous equation and the vectors multiplied by $s$, $t$, and $u$ span the nullspace of $A$. Both statements are true. Why?

**1.106** **(a)** Let the equation be $ax + by + cz = d$. The zero vector must satisfy this equation. Why? What does this tell you about $d$? Also $[1, 2, 1]^t$ and $[1, 0, -3]^t$ satisfy this equation. Why? What does this tell you about $a$, $b$, and $c$? (b) This is easy once you have done (a)

**1.108** True. For your explanation, begin by showing that $Y_1$ and $Y_2$ both belong to the plane spanned by $X_1$ and $X_2$. How does it follow that all linear combinations of $Y_1$ and $Y_2$ also belong to this plane? Next show that $X_1$ and $X_2$ belong to the span of the $Y_i$.

**1.110** No, the two answers are not consistent.

**1.112** **(a)** If $W$ belongs to span $\{X, Y, Z\}$, then $W = aX + bY + cZ = aX + bY + c(2X + 3Y) = (a + 2c)X + (b + 3c)Y$, which belongs to span $\{X, Y\}$. **(b)** Conversely, if $W$ belongs to span $\{X, Y\}$, then $W = aX + bY = aX + bY + 0Z$, which belongs to span $\{X, Y, Z\}$. Thus, the two sets have the same elements and are therefore equal.

**1.116** $\mathcal{W}$ is the first quadrant in $\mathbb{R}^2$. No: $\mathcal{W}$ is not closed under scalar multiplication. (Explain.)

**1.119** Start by noting that the general upper triangular matrix is

$$A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

and then proceed as in the solution to part (i) of Exercise 1.111.1 on page 89 of the text.

**1.121** *Hint:* Does $\left(\frac{2}{3}\right)X + \left(\frac{1}{3}\right)Y$ work? Does $2X + 3Y$ work?

**1.122** **(b)** Suppose that $y$ and $z$ are two solutions. Then

$$y'' + 3y' + 2y = 0$$
$$z'' + 3z' + 2z = 0$$

If we add these two equations, we get $(y + z)'' + (y + z)' + 2(y + z) = 0$, showing that $y + z$ is a solution. Multiplying by $c$ yields $(cy)'' + (cy)' + 2cy = 0$ showing that $cy$ is a solution.

**1.123** b) The sum of any two solutions will solve the equation $y'' + 3y' + 2y = 2t$, not $y'' + 3y' + 2y = t$. (But you must show this!) What equation does $2y$ solve? Why is 0 not in the solution set?

**1.132** *Hint:* Suppose first that $\mathcal{S}$ and $\mathcal{T}$ are lines through the origin. Under what circumstance is $\mathcal{S} \cup \mathcal{T}$ closed under addition? What if $\mathcal{S}$ is a line and $\mathcal{T}$ is a plane?

### Section 2.1 on page 108

**2.1** **(a)** Independent, **(d)** independent, **(g)** dependent: The third matrix is $-3$ times the first plus 4 times the second. **( j)** Independent.

**2.3** **(a)** Pivot columns: $A_1$, $A_2$. $A_3 = 3A_1 - 3A_2$, $A_4 = \frac{5}{3}A_1 + \frac{2}{3}A_2$. **(c)** Pivot columns: $A_1$, $A_2$. $A_3 = -\frac{1}{3}A_1 - \frac{8}{3}A_2$. **(e)** Pivot columns: $A_1$, $A_2$, $A_3$. $A_4 = \frac{2}{5}A_1 + \frac{23}{10}A_2 + \frac{19}{5}A_3$.

**2.5** Let $A_i$ be the columns of $A$. Then $A_1$ and $A_2$ are a basis of the column space, and $A_3 = A_1 + A_2$ and $A_4 = A_1 - 2A_2$.

**2.6** Each vector has a nonzero entry in the positions, where the other two vectors have zeros.

**2.7** Let the rows of $A$ be $A_1$, $A_2$, and $A_3$. Then $xA_1 + yA_2 + zA_3 = 0$ yields the system

$$\begin{aligned} x &= 0 & xc + yf &= 0 \\ xa &= 0 & xd + yg + z &= 0 \\ xb + y &= 0 & xe + yh + zk &= 0 \end{aligned}$$

The first, third, and fifth equations show that $x = y = z = 0$, proving independence.

**2.11** *Hint:* The dependency equation for the $Y_i$ is $x_1Y_1 + x_2Y_2 = 0$. Substitute the given expressions for $Y_1$ and $Y_2$ into this equation, obtaining a linear combination of the $X_i$ that equals 0. Explain why the coefficients of $X_1$ and $X_2$ both must equal 0. Solve the resulting system to see that $x_1 = x_2 = 0$.

**2.20** *Hint:* Consider formula (1.42) on page 78.

**2.23** Let your matrices be $A$, $B$, $C$, $D$, and $E$. Show that the equation $xA + yB + zC + wD + uE = 0$ results in a system of four homogeneous equations in five unknowns. How does the more unknowns theorem prove that this system has a nonzero solution?

**2.24** *Hints:* **(b)** and **(d)** look up the double angle formula for the cosine. **(e)** $\ln ab = \ln a + \ln b$.

**2.25** **(b)** The dependency equation is $ae^x + be^{2x} + ce^{3x} = 0$. Differentiating twice, we see that $ae^x + 2be^{2x} + 3ce^{3x} = 0$ and $ae^x + 4be^{2x} + 9ce^{3x} = 0$. Setting $x = 0$ yields a system that you should solve to find $a = b = c = 0$, showing independence. Part (g) is similar to (a), except, after differentiating, set $x = 1$ instead of $x = 0$.

## Section 2.2 on page 123

**2.28** Let $B = [2, 25]^t$. Then $B = 8A_1 - 3A_2$. (Other answers are possible.)

**2.30** $B = [1, 1]^t$, $C = [4, 8]^t$ (or any other vector on the line spanned by $[1, 2]^t$).

**2.31** **(a)** Yes. **(b)** No.

**2.33** **(b)** $\mathcal{W}$ is a subspace because it is a span. A basis is

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

**2.34** **(b)** *Hint:* See Example 2.5 on page 116.

**2.35** **(d)** The dimension is 6.

**2.37** **(a)** The first, second, and fourth vectors form a basis for the span that has dimension 3. **(c)** These vectors are independent and form a basis for the span that has dimension 3.

**2.38** (b) $[-1, 0, 1, 0]^t$, $[2, 1, 0, -5]^t$.

**2.41** **(a)** Any vector that is independent of $X_1$ and $X_2$ will work. (Why?) **(b)** Yes, it is always possible, but why?

**2.45** $\{A, B, C\}$ is dependent. $\{B, C\}$ need not be a basis of $\mathcal{W}$.

**2.49** *Hint:* The proof is similar to the argument used to prove Theorem 2.8.

## Section 2.2.2 on page 131

**2.59** **(a)** $\{e^{-t}, e^{-2t}\}$ span the solution space. **(b)** There are no solutions of the form $e^{rt}$, where $r$ is real. **(c)** $\{e^t, e^{-t}\}$. They do not span the solution space.

**2.62** $e^{-2t} \cos 3t + \frac{4}{3} e^{-2t} \sin 3t$.

## Section 2.3 page 143

**2.64** $B$ is in the row space but $C$ is not.

**2.66** **(a)** Rank 3, row space: $\{[1, 0, 0, -2], [0, 1, 0, 0], [0, 0, 1, 7/2]\}$, column space: the first three columns of $A$. **(c)** Rank 2, row space: $\{[1, 0, 1/5], [0, 1, 3/5]\}$, column space: the first two columns of $A$.

**2.70** **(a)** $\{[1, 0, \frac{1}{11}, \frac{2}{11}]^t,\ [0, 1, \frac{3}{11}, \frac{6}{11}]^t\}$.

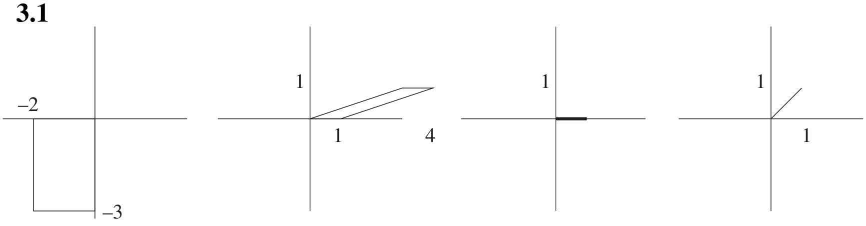
**2.71** $\{[1, 0, 2]^t\ [0, 1, -1]^t\}$.

**2.78** **(a)** Look for relations between the rows. **(b)** What is the dimension of the column space? **(c)** What is the dimension of the nullspace? **(d)** Note that $[1, 1, 2, 3]^t$ is the first column of $A$. **(e)** Think about **(c)** and **(d)** and the translation theorem.

**2.80** The dimension is $m - n + d$. Why?

## Section 3.1 on page 157

**3.1**

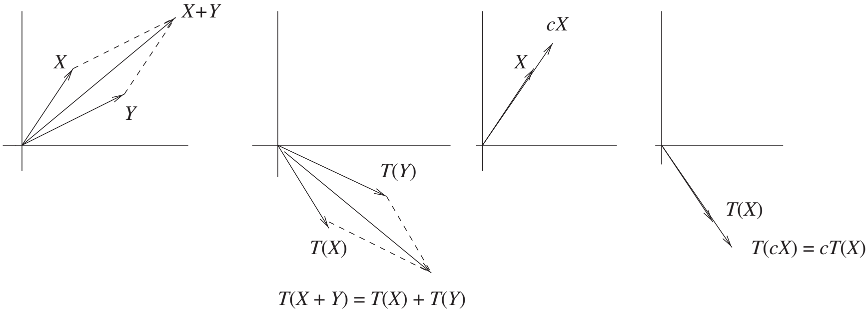



(a) (b) (c) (d)

**3.3** $u^2/4 + v^2/9 = (2x)^2/4 + (3y)^2/9 = x^2 + y^2 = 1.$

**3.5** Either $\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ or $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$. But how did you find it?

**3.8** $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.



**3.9** **(c)** When viewed from the positive $z$ axis, the transformation rotates $\psi$ radians counterclockwise about the $z$ axis.

**3.10** No.

**3.12** **(a)** $[1, -5]^t$, **(c)** $A = \begin{bmatrix} 0 & -2 & 3 \\ 3 & 0 & -5 \end{bmatrix}$. But how did you get this?

**3.15** **(a)** Let $X = [x_1, y_1, z_1]^t$ and $Y = [x_2, y_2, z_2]^t$. Then

$$\begin{aligned} T(X) + T(Y) &= [2x_1 + 3y_1 - 7z_1, 0]^t + [2x_2 + 3y_2 - 7z_2, 0]^t \\ &= [2(x_1 + x_2) + 3(y_1 + y_2) - 7(z_1 + z_2), 0]^t = T(X + Y) \end{aligned}$$

Also,

$$\begin{aligned} T(c[x, y, z]^t) &= T([cx, cy, cz]^t) = [2cx + 3cy - 7cz, 0]^t = c[2x + 3y - 7z, 0]^t \\ &= cT([x, y, z]) \end{aligned}$$

The matrix that describes $T$ is

$$\begin{bmatrix} 2 & 3 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$

**3.22** *Hint:* Suppose that $X_1 = aX_2 + bX_3$. What can you say about the relationship of $T(X_1)$ with $T(X_2)$ and $T(X_3)$? What if $X_2$ is a linear combination of the others? How about $X_3$? *Question:* Can you find a more efficient proof using the test for independence and Exercise 3.23?

**3.23** You need to show that the nullspace is closed under linear combinations. Suppose that $X$ and $Y$ belong to the nullspace. What does this say about $T(X)$ and $T(Y)$? What can you say about $T(aX + bY)$, where $a$ and $b$ are scalars?

## Section 3.2 on page 173

**3.26** **(a)** $BC = \begin{bmatrix} 8 & 13 & -3 \\ -2 & -3 & 1 \end{bmatrix}$, $AB = \begin{bmatrix} 2 & 1 \\ 4 & 4 \end{bmatrix}$

$$A(BC) = (AB)C = \begin{bmatrix} 4 & 7 & -1 \\ 12 & 20 & -4 \end{bmatrix}$$

**3.27** **(a)** $AB = \begin{bmatrix} 2 & 1 \\ 4 & 4 \end{bmatrix}$, $BA = \begin{bmatrix} 8 & 10 \\ -2 & -2 \end{bmatrix}$, **(c)** $B^tA^t = \begin{bmatrix} 2 & 4 \\ 1 & 4 \end{bmatrix} = (AB)^t$.

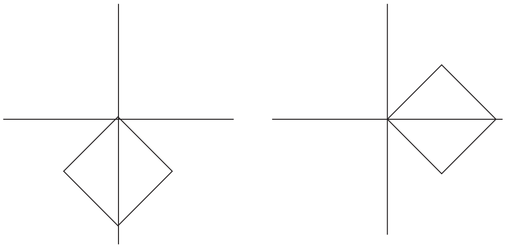
**3.28** $(ABC)^t = C^tB^tA^t$

**3.30** **(a)** $U = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$, $V = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 2 & 3 & 1 & 2 & 0 \\ 0 & -1 & 2 & 3 & -1 \end{bmatrix}$

**3.35** **(a)** The quadrilateral with vertices $[0,0]^t$, $[\sqrt{2}, \sqrt{2}/2]^t$, $[\sqrt{2}, \sqrt{2}]^t$, $[0, \sqrt{2}/2]^t$. **(b)** $\begin{bmatrix} \sqrt{2} & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$

**3.39** **(a)** The ellipse $u^2/16 + v^2/81 = 1$.
**(b)** $C = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$

**3.40** **(a)** See below. **(b)** $M = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ -\sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix}$, $Y = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix}$



My Answer Your Answer

**3.45** *Hint:* Try making one of the matrices equal to $I$.

**3.47** *Hint:* Almost any randomly chosen $A$ and $B$ will work. For the condition, expand $(A + B)(A - B)$ carefully using the distributive laws and set it equal to $A^2 - B^2$.

**3.51** One example is $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

**3.59** *Hint:* To prove additivity, begin with $S \circ T(X + Y) = S(T(X + Y)) = \dots$ .

## Section 3.3 on page 190

**3.63** **(a)** $\begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}$

**3.64** **(c)** and **(e)** are not invertible. The inverses of the others are:

**(a)** $\dfrac{1}{10}\begin{bmatrix} -26 & 15 & -12 \\ 20 & -10 & 10 \\ 12 & -5 & 4 \end{bmatrix}$, **(b)** $\dfrac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

**(d)** $\frac{1}{4}\begin{bmatrix} 2 & -1 & 1 \\ -2 & 5 & -1 \\ 0 & -2 & 2 \end{bmatrix}$, **(g)** $\frac{1}{3}\begin{bmatrix} 23 & 17 & -17 & -13 \\ 2 & 2 & -2 & -1 \\ 9 & 9 & -6 & -6 \\ -15 & -12 & 12 & 9 \end{bmatrix}$

**(h)** $\frac{1}{12}\begin{bmatrix} 12 & -6 & 0 & 0 \\ 0 & 6 & -4 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ **(i)** $\begin{bmatrix} a^{-1} & 0 & 0 & 0 \\ 0 & b^{-1} & 0 & 0 \\ 0 & 0 & c^{-1} & 0 \\ 0 & 0 & 0 & d^{-1} \end{bmatrix}$

**3.65** **(a)** $[-\frac{16}{5}, 3, -\frac{7}{5}]^t$, **(b)** $[2, 1, 0]^t$, **(c)** not invertible, **(d)** $[\frac{3}{4}, \frac{5}{4}, \frac{1}{2}]^t$, **(e)** not invertible, **(g)** $[-\frac{46}{3}, -\frac{4}{3}, -5, 11]^t$, **(h)** $[0, 0, 0, 1]^t$, **(i)** $[1/a, 2/b, 3/c, 4/d]^t$.

**3.71** $(ad - bc)^{-1}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

**3.72** $\begin{bmatrix} 1 & -a & ac - b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$

**3.74** **(b)** *Hint:* Simplify $(I - N)(I + N + N^2)$

**3.76** $\begin{bmatrix} 3 & 4 & 7 \\ 2 & 2 & 4 \\ 6 & 8 & 14 \end{bmatrix}$

**3.77** *Hint:* $I = -A^2 - 3A$.

**3.80** *Hint:* Begin by multiplying both sides of $ABX = Y$ on the left by $A^{-1}$.

**3.84** Another hint: $B^tC^t = (CB)^t$.

**3.86** **(a)** *Hint:* $(Q^{-1}DQ)^2 = (Q^{-1}DQ)(Q^{-1}DQ)$.

**3.88** **(a)** $B = \begin{bmatrix} -1 & 2 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}$ (Other answers are possible.)
**(c)** *Hint:* Use the rank of products theorem (Theorem 3.3 on page 169) as in the proof of Theorem 3.10 on page 188.

## Section 3.4 on page 212

**3.100** **(a)** $[-8, 3, -7, 6]^t$, **(b)** $[5, -7 - x_4, 3 - x_4, x_4]^t$

**3.101** **(a)** $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & \frac{1}{2} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 4 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

**(d)** $L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 4 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

**3.102** For your explanation, think about the steps in reducing $[U|I]$, where $U$ is unipotent. Which entries of $I$ would change and which would not?

**3.103** If $[U, B]$ is produced by reducing $[A, I]$, then $BA = U$.

**3.104** *Hint:* Write $U = DU'$, where $U'$ is unipotent and

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & \frac{2}{3} \end{bmatrix}$$

**3.106** *Hint:* $P = PI$.

**3.110** You must interchange rows 2 and 3 of $A$ for it to work. The $LU$ decomposition for the resulting matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix}$$

## Section 3.5 on page 230

**3.114** Let $X = [1, 1]^t$ and $Y = [1, -1]^t$. Then $[x, y]^t = x'X + y'Y = [x' + y', x' - y']^t$ so $1 = xy = (x' + y')(x' - y') = (x')^2 - (y')^2$, which is of the desired form.

**3.117** **(b)** The following matrices are respectively the point matrix, the coordinate matrix, and the coordinate vector for $[1, 2, 3]^t$

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & 3 & 1 \\ 1 & 2 & 0 \end{bmatrix}, \quad \frac{1}{7}\begin{bmatrix} 2 & -2 & 1 \\ -1 & 1 & 3 \\ 7 & 0 & -7 \end{bmatrix}, \quad \frac{1}{7}\begin{bmatrix} 1 \\ 10 \\ -14 \end{bmatrix}$$

**3.118** Call the polynomials $p_i$. **(b)** $p_3 = -p_1 + 2p_2$.

**3.119** **(a)** $\begin{bmatrix} 5 & 0 \\ 0 & 15 \end{bmatrix}$, **(c)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

**3.121** **(a)** $\begin{bmatrix} 4 & -11 & -5 \\ 6 & -5 & -3 \end{bmatrix}$

**3.126** **(a)** $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}$, **(c)** $\begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 4 & 6 & 8 \\ 3 & 6 & 4 & 8 \\ 9 & 12 & 12 & 16 \end{bmatrix}$, **(e)** $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

**3.130** **(a)** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, **(c)** $\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$, **(e)** $\begin{bmatrix} -7 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**3.133** **(a)** $\begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, **(c)** $\begin{bmatrix} -1 & 2 & -3 \\ 0 & -1 & 4 \\ 0 & 0 & -1 \end{bmatrix}$, **(e)** $\begin{bmatrix} -7 & 7 & -7 \\ 0 & -4 & 8 \\ 0 & 0 & 1 \end{bmatrix}$

## Section 4.1 on page 249

**4.1** **(a)** 14, **(c)** 0, **(e)** −16, **(g)** 0, **(i)** 34.

**4.3** For your proof, fully expand your formula and formula (4.2) to see that you get the same result.

**4.5**

$$\alpha = \begin{vmatrix} -13 & 7 & 9 & 5 \\ 16 & -37 & 99 & 64 \\ -42 & 78 & 55 & -3 \\ 47 & 29 & -14 & -8 \end{vmatrix} - 4 \begin{vmatrix} 24 & -7 & 9 & 5 \\ 11 & -37 & 99 & 64 \\ 31 & 78 & 55 & -3 \\ 62 & 29 & -14 & -8 \end{vmatrix}$$
$$+ 2 \begin{vmatrix} 24 & -13 & 9 & 5 \\ 11 & 16 & 99 & 64 \\ 31 & -42 & 55 & -3 \\ 62 & 47 & -14 & -8 \end{vmatrix} - 2 \begin{vmatrix} 24 & -13 & 7 & 5 \\ 11 & 16 & -37 & 64 \\ 31 & -42 & 78 & -3 \\ 62 & 47 & 29 & -8 \end{vmatrix}$$
$$- 3 \begin{vmatrix} 24 & -13 & 7 & 9 \\ 11 & 16 & -37 & 99 \\ 31 & -42 & 78 & 55 \\ 62 & 47 & 29 & -14 \end{vmatrix}$$

**4.7** **(b)** *Hint:* Split into three cases, the first case being that the last two rows are equal. In this case, expand along the first row. What can you say about the resulting $2 \times 2$ determinants? Next consider the case that the first and third rows are equal, etc. c) Again, expand along a row other than the two that are assumed equal. What can you say about the resulting $2 \times 2$ determinants?

## Section 4.2 on page 258

**4.12** **(a)** 0, **(c)** 34, **(e)** 0.

**4.14** **(a)** 14, **(c)** 0, **(e)** −16, **(g)** 0, **(i)** 34.

**4.15** 120 and −5.

**4.18** *Hint:* Do Exercise 4.17 first.

**4.20** *Hint:* Do Exercise 4.19 first.

**4.25** *Hint:* Use the product theorem.

### Section 4.3 on page 268

**4.34** $x = -\frac{9}{4}$, $y = -\frac{5}{4}$, $z = 2$, $w = \frac{15}{4}$.

**4.37** $c_1(x) = e^{-x}(1 + x^2)\cos x - e^{-x}x\sin x$, $c_2(x) = x\cos x + (1 + x^2)\sin x$.

**4.39** −1.

**4.40** You get 1. Why?

### Section 5.1 on page 279

**5.1** **(a)** $X$ is an eigenvector with eigenvalue 3, $Y$ is not an eigenvector. **(b)** $X$ is an eigenvector with eigenvalue 0. $Y$ is not an eigenvector. **(c)** $X$ is an eigenvector with eigenvalue −1 and $Y$ is not an eigenvector.

**5.2** **(a)** $p(\lambda) = -\lambda^2(\lambda - 3)$, eigenvalues and corresponding basis: $\lambda = 0$: $[1, 0, -1]^t$ and $[1, -1, 0]^t$, $\lambda = 3$, $[1, 1, 1]^t$, $A^nB = 5[3^{n-1}, 3^{n-1}, 3^{n-1}]^t$ for $n \geq 1$.

**5.4** *Hint:* Nonzero linear combinations of eigenvectors corresponding to the same eigenvalue are eigenvectors.

**5.5** **(a)** $\lambda = -3$, $[2, 1]^t$; $\lambda = 2$, $[3, 2]^t$, diagonalizable. **(c)** $\lambda = -3$, $[7, 3]^t$; $\lambda = 2$, $[3, 2]^t$, diagonalizable. **(e)** $\lambda = 2$, $[-2, 1, 0]^t, [1, 0, 1]^t$; $\lambda = 3$, $[0, 2, 5]^t$, diagonalizable. **(g)** $\lambda = 1$, $[3, 2, 3]^t$, deficient over the real numbers (but not over the complex numbers). (i) $\lambda = 1$, basis: $[1, 0, 1]^t$ and $\lambda = 2$, basis: $[2, 1, 2]^t$. This matrix is deficient.

**5.7** One eigenspace is the $z$ axis. What is the other?

**5.8** *Hint:* $X$ is an eigenvector with eigenvalue $\lambda$ if and only if $AX = \lambda X$.

**5.13** *Hint:* Multiply both sides of $AX = \lambda X$ by $A^{-1}$.

**5.14** *Hint:* See Exercise 5.8 and Theorem 5.2 on page 278.

### Section 5.1.2 on page 285

**5.17** **(a)** $[0.4167, 0.5833]^t$, (b) $[0.2195, 0.4146, 0.3659]^t$.

**5.18** **(a)** Takes 4 products, **(c)** takes 8 products.

**5.21** **(a)** Tuesday: $[0.4667, 0.3333, 0.2]^t$, Wednesday: $[0.4933, 0.3067, 0.2]^t$. **(b)** In two weeks the state vector is $[0.519993, 0.280007, 0.2]^t$. **(c)** The equilibrium distribution is $[0.52, 0.28, 0.2]^t$. After 50 days, there is no noticeable change. (We used logarithms to find our answer.) **(d)** *Hint:* Let $V_0$ be the initial state, $X$ the equilibrium state, and $Y_1$ and $Y_2$ bases for the other two eigenspaces. Show that $V_0 = X + cY_1 + dY_2$. For this note that there is an $a$ such that $V_0 = aX + cY_1 + dY_2$. Show that $a = 1$ by considering the sum of the entries in $V_0$, as compared to the sum of the entries on the right side of the equality. **(e)** The entries of the equilibrium state vector give the expected fractions.

**5.22** **(a)** $\begin{bmatrix} 0.6 & 0.15 & 0.10 \\ 0.2 & 0.7 & 0.10 \\ 0.2 & 0.15 & 0.8 \end{bmatrix}$

## Section 5.2 on page 290

**5.27** **(a)** $Q = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A^n = Q\begin{bmatrix} 6^n & 0 & 0 \\ 0 & (-3)^n & 0 \\ 0 & 0 & 0 \end{bmatrix}Q^{-1}$

**(b)** $Q = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -2 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, $A^n = Q\begin{bmatrix} 2^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 4^n \end{bmatrix}Q^{-1}$

**5.29** **(a)** Not diagonalizable over $\mathbb{R}$. **(b)** Not diagonalizable.

**(c)** $Q = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$

**(d)** $Q = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

**(f)** $Q = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

**5.33** *Hint:* Write $A = QDQ^{-1}$, where $D$ is diagonal. What can you say about $D$?

**5.35** Same hint as in Exercise 5.33.

**5.37** *Hint:* What must the rank of $A - 2I$ be?

## Section 5.3 on page 304

**5.44** $AB = \begin{bmatrix} -1+3i & -5+3i \\ -3+4i & -6 \end{bmatrix}$, $BA = \begin{bmatrix} 7-i & 2+9i \\ 1+11i & -14+4i \end{bmatrix}$

**5.46** $-7, 6.5 \pm (3\sqrt{3}/2)I$.

**5.48** $M = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Eigenvalues: $a \pm bi$

**5.50** With $z = a + bi$, $w = c + di$, compute $\overline{zw}$ and $\bar{z}\,\overline{w}$. Then use formula (3.4) on page 167.

**5.51** *Hint:* In $\mathbb{C}$, $a + bi = 0$ implies $a = b = 0$.

**5.52** *Hint:* Let $\mathcal{B} = \{I_j \mid 1 \le j \le n\}$ be the standard basis for $\mathbb{R}^n$. Prove that the set $\mathcal{B}' = \mathcal{B} \cup \{iI_j \mid 1 \le j \le n\}$ is a basis for $\mathbb{C}^n$ considered as a real vector space.

**5.53** *Hint:* Let $\mathcal{B} = \{X_j \mid 1 \le j \le n\}$ a complex basis for $\mathcal{V}$. Prove that the set $\mathcal{B}' = \mathcal{B} \cup \{iX_j \mid 1 \le j \le n\}$ is a basis for $\mathcal{V}$ considered as a real vector space.

**5.55** See the argument after equation (5.24) on page 303.

## Section 6.1 page 316

**6.1** **(a)** $|X - Y| = 2\sqrt{5}$, $|X| = 5$, $|Y| = \sqrt{5}$, $X \cdot Y = 5$, $\theta = 1.1071$ radians., **(c)** $|X - Y| = \sqrt{6}$, $|X| = \sqrt{14}$, $|Y| = \sqrt{6}$, $X \cdot Y = 7$, $\theta = 0.7017$ radians.

**6.2** **(a)** Not perpendicular, **(c)** perpendicular, **(e)** perpendicular.

**6.5** The cone of all vectors $W$ such that the measure of the angle between $X$ and $W$ is at most 30 degrees.

**6.6** **(a)** $A = [-1, 1, 0]^t$, $B = [-1, -1, -1]^t$, $C = [2, -1, 0]^t$ work.

**(b)** Compute $[2, 3, -6]^t \cdot (A - B)$ and $[2, 3, -6]^t \cdot (C - B)$. You should get 0.

**(c)** $[x, y, z]^t - B = [x + 1, y + 1, z + 1]^t$, which is perpendicular to $[2, 3, -6]^t$ if and only if $0 = [x + 1, y + 1, z + 1]^t \cdot [2, 3, -6]^t = 2x + 3y - 6z - 1$. So the set of solutions is the plane through $B$ perpendicular to $[2, 3, -6]^t$.

**6.7** **(a)** The third basis.

**(b)** $X' = \left[\frac{13}{14}, -\frac{2}{3}, -\frac{5}{42}\right]^t$.

**(c)** $X' = \left[\frac{1}{14}x + \frac{3}{14}y + \frac{1}{7}z, \frac{-1}{3}x + \frac{1}{3}y - \frac{1}{3}z, \frac{5}{42}x + \frac{1}{42}y - \frac{2}{21}z\right]^t$

**(d)** $\begin{bmatrix} \frac{1}{14} & \frac{3}{14} & \frac{1}{7} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{5}{42} & \frac{1}{42} & -\frac{2}{21} \end{bmatrix}$

**6.9** $X' = \left[\frac{4}{7}, \frac{16}{35}, 0, -\frac{3}{5}\right]^t$

## Section 6.2 on page 328

**6.19** **(a)** Projection: $\frac{1}{5}[8, 4, 4, -12]^t$.

**6.20** **(c)** Projection: $[\frac{47}{42}, \frac{85}{42}, \frac{40}{21}]^t$.

**6.21** **(e)** $((x+2y+z-w)/7)\,[1,2,1,-1]^t + ((-y+z-w)/3)\,[0,-1,1,-1]^t$.

**6.22** **(a)** $\{[0,1,1]^t, [1,0,0]^t\}$ **(c)** $\{[1,2,1,1]^t, [-2,1,1,-1]^t, \frac{1}{7}[-6,-2,0,10]^t\}$.

**6.25** **(a)**

$$Q = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{3} \end{bmatrix}$$

**6.29** **(a)** $\{[-1,0,3,2]^t, \frac{1}{14}[-39,14,-9,-6]^t\}$ **(b)** $\frac{1}{1310}[570,1710,-70,390]^t$

**6.30** **(a)** See the answer to 6.29 (a).

**6.31** Basis: $\{[1,3,1,-1]^t, [5,15,-19,31]^t\}$

## Section 6.3 on page 341

**6.41** $f_o(x) = \sum_{k=1}^{n}[12/(k^3\pi^3) - 2/(k\pi)](-1)^k \sin k\pi x$.

**6.43** **(a)** *Hint:* $(\cos a)(\cos b) = \frac{1}{2}(\cos(a+b) + \cos(a-b))$, **(b)** $(p_n, p_n) = 1$ if $n \neq 0$ and $(p_0, p_0) = 2$, **(c)** $f_o(x) = 1/3 + \sum_{k=1}^{n}[4/(k^2\pi^2)](-1)^k \cos k\pi x$.

**6.45** **(a)** *Hint:* $(\sin a)(\cos b) = \frac{1}{2}(\sin(a+b) + \sin(a-b))$.
**(b)**

$$f_o(x) = \frac{1}{4} + \sum_{k=1}^{n} \frac{(-1)^k - 1}{k^2\pi^2} \cos k\pi x + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k\pi} \sin k\pi x$$

**6.47** $c_k = ((-1)^{k+1}(0.002))/(k\pi)$.

**6.48** **(a)** 18. **(b)** *Hint:* A degree 2 polynomial $p(x)$ that is zero at three different values of $x$ must be the zero polynomial. **(d)** $c_1 = 0$, $c_2 = -1$, $c_3 = 2$.

**6.52** To prove the first equality in (c),

$$(cf, g) = \int_{-1}^{1} cf(x)g(x)\,dx = c\int_{-1}^{1} f(x)g(x)\,dx = c(f, g)$$

**6.54** *Hint:* $||V+W||^2 = (V+W, V+W)$.

**6.56** *Hint:* Try it first with only two $V_i$ so $W = c_1V_1 + C_2V_2$.

## Section 6.4 on page 364

**6.68** $|X| = |AX| = \sqrt{60}$.

**6.69** Compute $(R^x_\theta)^t R^x_\theta$, $(R^y_\theta)^t R^y_\theta$, and $(R^z_\theta)^t R^z_\theta$ to see that you do get $I$.

**6.71** **(a)** $C = \pm\frac{1}{17}$.

**6.75** Note that $(AB)^t = B^t A^t$.

**6.83** **(b)** $(1/11)\begin{bmatrix} 9 & -2 & 6 \\ -2 & 9 & 6 \\ 6 & 6 & -7 \end{bmatrix}$

## Section 6.5 on page 377

**6.89** $X = [30.9306, 2.0463]^t$, $B_0 = [31.9537, 33.1815, 34.0, 35.2278, 35.637]^t$.

**6.91** **(a)**

$$A = \begin{bmatrix} 1 & 0.5 & 0.25 \\ 1 & 1.1 & 1.21 \\ 1 & 1.5 & 2.25 \\ 1 & 2.1 & 4.41 \\ 1 & 2.3 & 5.29 \end{bmatrix}, \quad B = \begin{bmatrix} 32.0, 33.0, 34.2, 35.1, 35.7 \end{bmatrix}$$

**(b)** $[a, b, c]^t = [30.9622, 1.98957, 0.019944]^t$.

**6.93** $P = 227.1e^{.009378t}$. In 2010, $P = 300.89$.

**6.85** $B_0 = \frac{1}{14}[39, 24, 54, 39, 54]^t$.

**6.100** *Hint:* Try to compute $A^{-1}$.

## Section 6.6 on page 392

**6.105** For the first part of the exercise, let $X = [x, y]^t$ and compute $X^t A' X$. For the second part, let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and compute $X^t B X$ and compare it with the desired equation.

**6.106** $D = \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 9 \end{bmatrix}$, $\quad Q = \frac{1}{30}\begin{bmatrix} -4\sqrt{5} & -4\sqrt{5} & 10\sqrt{6} \\ 10\sqrt{5} & -8\sqrt{5} & 5\sqrt{6} \\ -8\sqrt{5} & 10\sqrt{5} & 5\sqrt{6} \end{bmatrix}$

**6.107** Equations: (a) $3(x')^2 - 2(y')^2 = 1$, (b) $(x_1')^2 + 2(x_2')^2 = 4$. Orthogonal matrices (the columns are the bases):

$$\frac{1}{5}\begin{bmatrix} \sqrt{5} & 2\sqrt{5} \\ -2\sqrt{5} & \sqrt{5} \end{bmatrix}, \quad \frac{1}{2}\begin{bmatrix} 1 & \sqrt{2} & 1 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix}$$

**6.108** **(a)** $((3 + \sqrt{2})/2)x^2 + ((3 - \sqrt{2})/2)y^2 = 1$.

**6.111** **(a)** A hyperbola. *Hint:* In attempting to cover all cases, do not forget to consider the possibility that one or both of the eigenvalues might be 0.

**6.112** $36x^2 - 24xy + 29y^2 = 180$.

**6.113** *Hint:* Use the quadratic equation to find the roots of the characteristic polynomial for the matrix of the quadratic form. Do not simplify your result too much!

**6.116** *Hint:* Look at the form of the quadratic form in the new coordinates.

**6.118** *Hint:* See Exercise 6.117.

**6.120** *Hint:* $X^tB^t = (BX)^t$.

**6.122** *Hint:* The proof is very similar to that of one of the results from this section.

## Section 6.7 on page 404

**6.126** **(a)** $U = \dfrac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad E = [2]$

**(c)** $U = \dfrac{1}{\sqrt{15}}\begin{bmatrix} -2 \\ -1 \\ -3 \\ 1 \end{bmatrix}, \quad E = [15]$

**6.127**

**(a)** $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 4 & 0 \\ 0 & \sqrt{6} \end{bmatrix}, V = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} \end{bmatrix}$

**(c)** $U = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}, D = \begin{bmatrix} 2\sqrt{15} & 0 \\ 0 & \sqrt{15} \end{bmatrix}, V = \frac{1}{3}\begin{bmatrix} -2 & 2 \\ 2 & 1 \\ 1 & 2 \end{bmatrix}$

**(e)** $U = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & -2 \\ 2 & 2 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, V = \frac{1}{5}\begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$

**6.130** **(a)** $X = [0, 1, 1]^t$, **(c)** $X = [23/9, 71/27]$.

## Section 6.8 on page 417

**6.140** **(a)** $3 + 20\,i$, **(c)** $4 + 6\,i$.

**6.141** **(b)** *Hint:* Use formulas (3.6) on page 170 and (5.24) on page 303.

**6.142** **(a)** $\begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}$, **(b)** $\begin{bmatrix} 3 & 2 & 2 & 0 \\ -2 & 3 & 0 & 2 \\ 0 & -4 & 1 & -1 \\ 4 & 0 & 1 & 1 \end{bmatrix}$

**6.144** **(b)** *Hint:* See the hint for Exercise 6.17 on page 318. (c)*Hint:* Use the linearity properties of matrix multiplication and, for the Hermitian symmetry, Exercise 6.141.b. (d)*Hint:* See the proof of Theorem 6.22 on page 384.

**6.150** **(a)** $(1/\sqrt{6})\begin{bmatrix} 1+i & \sqrt{2}(1+i) \\ -2 & \sqrt{2} \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$

**6.151** *Hint:* Write $A = [a_{ij}]$.

**6.155** **(a)** *Hint:* Let $W = AZ$.

**6.158** *Hint:* See the proof of Proposition 6.4 on page 363.

## Section 7.1 on page 429

**7.1** **(a)** $X_0 = [1, 0, 0]^t, X_1 = [0, 1, 1]^t, W_0 = [1, -1, 0]^t$

**(e)** $X_0 = [1, 0, 0, 0]^t, X_1 = [0, -1, -1, 0]^t, U_0 = [2, 1, 0, 0],$
$U_0 = [0, 0, -1, 1]^t$

**(g)** $X_0 = [1, 0, 0, 0]^t, X_1 = [0, \frac{5}{9}, -\frac{2}{9}, \frac{1}{9}]^t, W_0 = [2, 1, 0, 0]^t,$
$W_1 = [-\frac{5}{3}, 0, \frac{1}{3}, 0]^t$

**7.2** **(a)** $(3^n + 3^{n-1}n)X_0 + 3^n X_1$

**(e)** $-W_0 + (3 - n)2^n X_0 - 2^n 2X_1 - n2^n X_0 + 2^n U_0$

**(g)** $(24 + n)X_0 + 9X_1 + (3 - 2n)2^n W_0 - 2^n 4W_1$

**7.4** **(b)** *Hint:* Write $A - 4I$ and $X$ as partitioned matrices.

**7.6** *Hint:* Note the remark on page 424.

**7.9** *Hint:* First prove the result if $X$ is a $\lambda$-eigenvector for $A$. For the general case suppose that $X \neq 0$. Let $k$ be the order of $X$ as a generalized $\lambda$-eigenvector.

Then $X_o = (A - \lambda I)^{k-1}X$ is a nonzero $\lambda$-eigenvector. From Exercise 7.8 $X_o \in \mathbb{R}^n(A, \beta)$.

**7.10** **(a)** *Hint:* Show $B - \lambda I = P^{-1}(A - \lambda I)P$. **(b)** *Hint:* Show that $\{P^{-1}X_1, \dots, P^{-1}X_m\}$ is a basis for $\mathbb{R}^n(B, \lambda)$.

**7.11** **(b)** *Hint:* $(B + C)^3 = B(B + C)^2 + C(B + C)^2$. **(c)** *Hint:* $(B + C)^4 = B(B + C)^3 + C(B + C)^3$.

## Section 7.2 on page 443

**7.12** **(a)** $J = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

**(e)** $J = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

**(g)** $J = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 2 & -\frac{5}{3} \\ 0 & \frac{5}{9} & \frac{9}{1} & 0 \\ 0 & -\frac{2}{9} & 0 & \frac{1}{3} \\ 0 & \frac{1}{9} & 0 & 0 \end{bmatrix}$

**7.14** **(a)** The diagonal blocks in $J$ are

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}, [4], [6], \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & 6 \end{bmatrix}$$

**7.15** *Hint:* If the block sizes are 1,2,2, then $J$ is as shown. What are all possible block sizes? Certainly not 2,1,2.

$$J = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

**7.18** *Hint:* Assume that the result fails. Let $m$ be the minimal value for which equation (7.27) holds. In this case, none of the $X_i$ are zero. (Explain.) Multiply both sides of equation (7.27) by $(A - \lambda I)^k$ where $k$ is the order of $X_1$ as a generalized eigenvector. Then use Exercises 7.8 and 7.9 on page 430 to show that this results in a sum of $m - 1$ nonzero generalized eigenvectors satisfying the hypotheses of the proposition.

**7.19** *Hint:* From Theorem 5.1 on page 275, $n_1 + \cdots + n_k = n$. Hence, from Theorem 2.8 on page 119, it suffices to show that $\mathcal{B}$ is an independent set of elements in $\mathbb{R}^n$.

**7.20** **(e)** *Hint:* Write $C$ as a partitioned matrix.

### Section 8.1 on page 451

**8.1** For $|X|$ and $|Y|$, see the answers to Section 6.1, Exercise 6.1. $|X|_\infty$, $|Y|_\infty$, $|X|_1$, and $|Y|_1$ are respectively: (a) 4, 2, 7, 3. (c) Same as (a). (e) 6, 2, 16, 9.

**8.3** **(c)** *Hint:* Use Theorem 6.3 on page 311 with $X$ and $Y$ appropriately chosen.

**8.4** **(a)** 9, **(d)** 4, **(g)** 12.

**8.5** **(a)** $\frac{261}{5}$, **(d)** 8, **(g)** 196.

**8.7** *Hint:* We want both formulas (8.9) and (8.10) on page 450 to be equalities. See Exercise 8.6 as well as the answer to Exercise 8.9.b.

**8.9** **(b)** cond $A = 35,022$, $B = [77, 61, 213]^t$, $\Delta B = [0, .001, 0]^t$.

**8.10** *Hint:* Use Exercise 8.3.

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