Assignment 7

Considera il modello del pendolo inverso su un cart descritto dalle equazioni:

$$M\,\ddot{s} + F\,\dot{s} - \mu = d_1 \quad \ddot{\phi} = \frac{g}{L}\,\sin(\phi) + \frac{1}{L}\,\ddot{s}\,\cos(\phi) = 0$$

con $M = 1$ kg, $L = 1\,m$, $F = 1$ kg s^{-1}, $g = 9.81\,m\,s^{-1}$.

A1) Calcolare tutte i punti di equilibrio del sistema per $\mu = d_1(t) = 0$,

(%i1) dx[1]:x[2]

(%o1) x_2

(%i2) dx[2]:(1/M)*(-F*x[2]+u[1]+u[2])

(%o2) $\frac{-x_2\,F + u_2 + u_1}{M}$

(%i3) dx[3]:x[4]

(%o3) x_4

(%i4) dx[4]:expand((1/L)*(g*sin(x[3])-dx[2]*cos(x[3])))

(%o4) $\frac{\sin(x_3)\,g}{L} + \frac{x_2\cos(x_3)\,F}{L\,M} - \frac{u_2\cos(x_3)}{L\,M} - \frac{u_1\cos(x_3)}{L\,M}$

A2) Scrivere le equazioni del sistema linearizzato attorno al punto di equilibrio $\phi = s = \dot{\phi} = 0$

```
(%i5) diffx(dx):=block(
      [res:zeromatrix(4,4)],
      for i:1 thru 4 do (
      for j:1 thru 4 do (
      res[i,j]:diff(dx[i],x[j])
      )
      ),
      return(res)
      )$
(%i6) diffu(dx):=block(
      [res:zeromatrix(4,2)],
      for i:1 thru 4 do (
      for j:1 thru 2 do (
      res[i,j]:diff(dx[i],u[j])
      )
      ),
      return(res)
      )$
(%i7) dx:[dx[1],dx[2],dx[3],dx[4]]$
(%i8) nablax:diffx(dx)
```

(%o8) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{F}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{\cos(x_3)\,F}{L\,M} & \frac{\cos(x_3)\,g}{L} - \frac{x_2\sin(x_3)\,F}{L\,M} + \frac{u_2\sin(x_3)}{L\,M} + \frac{u_1\sin(x_3)}{L\,M} & 0 \end{pmatrix}$

(%i9) nablau:diffu(dx)

(%o9) $\begin{pmatrix} 0 & 0 \\ \frac{1}{M} & \frac{1}{M} \\ 0 & 0 \\ -\frac{\cos(x_3)}{L\,M} & -\frac{\cos(x_3)}{L\,M} \end{pmatrix}$

(%i10) sub:[x[1]=0,x[2]=0,x[3]=0,x[4]=0,u[1]=0,u[2]=0]$

(%i11) A:subst(sub,nablax)

$$(\%o11)\quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{F}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{F}{L\,M} & \frac{g}{L} & 0 \end{pmatrix}$$

(%i12) B:subst(sub,nablau)

$$(\%o12)\quad \begin{pmatrix} 0 & 0 \\ \frac{1}{M} & \frac{1}{M} \\ 0 & 0 \\ -\frac{1}{L\,M} & -\frac{1}{L\,M} \end{pmatrix}$$

A3) Mostra che la coppia (A, B) è controllabile

(%i13) P:addcol(col(B,2),matrix([0],[0],[0],[0]),matrix([0],[0],[0],[0]))

$$(\%o13)\quad \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{M} & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{L\,M} & 0 & 0 \end{pmatrix}$$

(%i14) B:col(B,1)

$$(\%o14)\quad \begin{pmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{L\,M} \end{pmatrix}$$

(%i15) C:matrix([1,0,0,0],[0,0,1,0])

$$(\%o15)\quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

A4) Mostra che la coppia (A, B) è controllabile.

(%i16) A1B:A.B

$$(\%o16)\quad \begin{pmatrix} \frac{1}{M} \\ -\frac{F}{M^2} \\ -\frac{1}{L\,M} \\ \frac{F}{L\,M^2} \end{pmatrix}$$

(%i17) A2B:A.A1B

$$(\%o17)\quad \begin{pmatrix} -\frac{F}{M^2} \\ \frac{F^2}{M^3} \\ \frac{F}{L\,M^2} \\ -\frac{g}{L^2\,M} - \frac{F^2}{L\,M^3} \end{pmatrix}$$

(%i18) A3B:A.A2B

$$(\%o18)\quad \begin{pmatrix} \frac{F^2}{M^3} \\ -\frac{F^3}{M^4} \\ -\frac{g}{L^2\,M} - \frac{F^2}{L\,M^3} \\ \frac{F\,g}{L^2\,M^2} + \frac{F^3}{L\,M^4} \end{pmatrix}$$

```
(%i19) R:addcol(B,A1B,A2B,A3B)
```

$$(\%o19)\quad \begin{pmatrix} 0 & \frac{1}{M} & -\frac{F}{M^2} & \frac{F^2}{M^3} \\ \frac{1}{M} & -\frac{F}{M^2} & \frac{F^2}{M^3} & -\frac{F^3}{M^4} \\ 0 & -\frac{1}{L\,M} & \frac{F}{L\,M^2} & -\frac{g}{L^2\,M} - \frac{F^2}{L\,M^3} \\ -\frac{1}{L\,M} & \frac{F}{L\,M^2} & -\frac{g}{L^2\,M} - \frac{F^2}{L\,M^3} & \frac{F\,g}{L^2\,M^2} + \frac{F^3}{L\,M^4} \end{pmatrix}$$

```
(%i20) rank(R)
(%o20) 4
```

A5) Considera il sistema lineare A3). Supponi che d_1 non sia nota. La legge di controllo deve essere tale che l'effetto del disturbo d_1 sia asintoticamente respindo e la prima uscita $s(t)$ insegua asintoticamente il segnale di riferimento $d_2 = \alpha \sin(\omega t)$. Struttura il problema come un problema di regolazione.

```
(%i21) S:matrix([0,0,0],[0,0,omega],[0,-omega,0])
```

$$(\%o21)\quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & -\omega & 0 \end{pmatrix}$$

```
(%i22) Q:matrix([0,-1,0])
```

$$(\%o22)\quad \begin{pmatrix} 0 & -1 & 0 \end{pmatrix}$$

```
(%i23) Ce:row(C,1)
```

$$(\%o23)\quad \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$$

A6) Considera il problema di regolazione determinato in A5) e mostra che il problema è risolubile tramite la legge di controllo a full information

Lemma di Hautus:

```
(%i24) H:addcol(s*ident(4)-A,B)
```

$$(\%o24)\quad \begin{pmatrix} s & -1 & 0 & 0 & 0 \\ 0 & s + \frac{F}{M} & 0 & 0 & \frac{1}{M} \\ 0 & 0 & s & -1 & 0 \\ 0 & -\frac{F}{L\,M} & -\frac{g}{L} & s & -\frac{1}{L\,M} \end{pmatrix}$$

```
(%i25) H:addrow(H,addcol(row(C,1),matrix([0])))
```

$$(\%o25)\quad \begin{pmatrix} s & -1 & 0 & 0 & 0 \\ 0 & s + \frac{F}{M} & 0 & 0 & \frac{1}{M} \\ 0 & 0 & s & -1 & 0 \\ 0 & -\frac{F}{L\,M} & -\frac{g}{L} & s & -\frac{1}{L\,M} \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
(%i26) rank(H)
(%o26) 5
(%i27) rank(subst(s=0,H))
(%o27) 5
(%i28) rank(subst(s=%i*omega,H))
```

(%o28) 5

(%i29) rank(subst(s=-%i*omega,H))

(%o29) 5

A7) Considera il problema di regolazione determinato in A5). Mostra che il problema è risolubile tramite una legge di controllo in feedback dall'errore.

$$\begin{bmatrix} \dot{x} \\ \dot{d} \end{bmatrix} = \begin{bmatrix} A & P \\ 0 & S \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} \qquad e = \begin{bmatrix} C & Q \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix}$$

deve essere osservabile.

(%i30) Ao:addcol(A,P)

$$\text{(\%o30)} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{F}{M} & 0 & 0 & \frac{1}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{F}{L\,M} & \frac{g}{L} & 0 & -\frac{1}{L\,M} & 0 & 0 \end{pmatrix}$$

(%i31) Ao:addrow(Ao,addcol(zeromatrix(3,4),S))

$$\text{(\%o31)} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{F}{M} & 0 & 0 & \frac{1}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{F}{L\,M} & \frac{g}{L} & 0 & -\frac{1}{L\,M} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega \\ 0 & 0 & 0 & 0 & 0 & -\omega & 0 \end{pmatrix}$$

(%i32) Co:addcol(C,addrow(Q,zeromatrix(1,3)))

$$\text{(\%o32)} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(%i33) O:Co;

$$\text{(\%o33)} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(%i34) rank(O)

(%o34) 2

(%i35) CA1:Co.Ao$

(%i36) O:addrow(O,CA1)

$$\text{(\%o36)} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -\omega \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

(%i37) rank(O)

(%o37) 4

```
(%i38) CA2:CA1.Ao$
       O:addrow(O,CA2);
       rank(O);
```

(%o39) $$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -\omega \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{F}{M} & 0 & 0 & \frac{1}{M} & \omega^2 & 0 \\ 0 & \frac{F}{L\,M} & \frac{g}{L} & 0 & -\frac{1}{L\,M} & 0 & 0 \end{pmatrix}$$

(%o40) 6

```
(%i41) CA3:CA2.Ao$
       O:addrow(O,CA3);
       rank(O);
```

(%o42) $$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -\omega \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{F}{M} & 0 & 0 & \frac{1}{M} & \omega^2 & 0 \\ 0 & \frac{F}{L\,M} & \frac{g}{L} & 0 & -\frac{1}{L\,M} & 0 & 0 \\ 0 & \frac{F^2}{M^2} & 0 & 0 & -\frac{F}{M^2} & 0 & \omega^3 \\ 0 & -\frac{F^2}{L\,M^2} & 0 & \frac{g}{L} & \frac{F}{L\,M^2} & 0 & 0 \end{pmatrix}$$

(%o43) 7

```
(%i44) Ao2:Ao
```

(%o44) $$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{F}{M} & 0 & 0 & \frac{1}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{F}{L\,M} & \frac{g}{L} & 0 & -\frac{1}{L\,M} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega \\ 0 & 0 & 0 & 0 & 0 & -\omega & 0 \end{pmatrix}$$

```
(%i45) Co2:addcol(Ce,Q)
```

(%o45) $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$

```
(%i46) O2:Co2;
       rank(O2)
```

(%o46) $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$

(%o47) 1

```
(%i48) CA1_2:Co2.Ao2$
       O2:addrow(O2,CA1_2);
       rank(O2);
```

(%o49) $$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -\omega \end{pmatrix}$$

(%o50) 2

```
(%i51) CA2_2:CA1_2.Ao2$
       O2:addrow(O2,CA2_2);
       rank(O2);
```

(%o52) $$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -\omega \\ 0 & -\frac{F}{M} & 0 & 0 & \frac{1}{M} & \omega^2 & 0 \end{pmatrix}$$

(%o53) 3

```
(%i54) CA3_2:CA2_2.Ao2$
       O2:addrow(O2,CA3_2);
       rank(O2);
```

(%o55)
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -\omega \\ 0 & -\frac{F}{M} & 0 & 0 & \frac{1}{M} & \omega^2 & 0 \\ 0 & \frac{F^2}{M^2} & 0 & 0 & -\frac{F}{M^2} & 0 & \omega^3 \end{pmatrix}$$

(%o56) 4

```
(%i57) CA4_2:CA3_2.Ao2$
       O2:addrow(O2,CA4_2);
       rank(O2);
```

(%o58)
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -\omega \\ 0 & -\frac{F}{M} & 0 & 0 & \frac{1}{M} & \omega^2 & 0 \\ 0 & \frac{F^2}{M^2} & 0 & 0 & -\frac{F}{M^2} & 0 & \omega^3 \\ 0 & -\frac{F^3}{M^3} & 0 & 0 & \frac{F^2}{M^3} & -\omega^4 & 0 \end{pmatrix}$$

(%o59) 5

```
(%i60) CA5_2:CA4_2.Ao2$
       O2:addrow(O2,CA5_2);
       rank(O2);
```

(%o61)
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -\omega \\ 0 & -\frac{F}{M} & 0 & 0 & \frac{1}{M} & \omega^2 & 0 \\ 0 & \frac{F^2}{M^2} & 0 & 0 & -\frac{F}{M^2} & 0 & \omega^3 \\ 0 & -\frac{F^3}{M^3} & 0 & 0 & \frac{F^2}{M^3} & -\omega^4 & 0 \\ 0 & \frac{F^4}{M^4} & 0 & 0 & -\frac{F^3}{M^4} & 0 & -\omega^5 \end{pmatrix}$$

(%o62) 5

```
(%i63) CA6_2:CA5_2.Ao2$
       O2:addrow(O2,CA6_2);
       rank(O2);
```

(%o64)
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -\omega \\ 0 & -\frac{F}{M} & 0 & 0 & \frac{1}{M} & \omega^2 & 0 \\ 0 & \frac{F^2}{M^2} & 0 & 0 & -\frac{F}{M^2} & 0 & \omega^3 \\ 0 & -\frac{F^3}{M^3} & 0 & 0 & \frac{F^2}{M^3} & -\omega^4 & 0 \\ 0 & \frac{F^4}{M^4} & 0 & 0 & -\frac{F^3}{M^4} & 0 & -\omega^5 \\ 0 & -\frac{F^5}{M^5} & 0 & 0 & \frac{F^4}{M^5} & \omega^6 & 0 \end{pmatrix}$$

(%o65) 5

B1) Sia $d_1(t)$ un'onda quadra di ampiezza 0.5 e periodo 50s, $\alpha = 1$, $\omega = 0.1$. Progetta una legge di controllo a full information che risolve A5).

$$\Pi S = A\,\Pi + B\,\Gamma + P \quad 0 = C\,\Pi + Q$$

```
(%i66) Pi:matrix([p[1,1],p[1,2],p[1,3]],
       [p[2,1],p[2,2],p[2,3]],
       [p[3,1],p[3,2],p[3,3]],
       [p[4,1],p[4,2],p[4,3]])
```

(%o66) $$\begin{pmatrix} p_{1,1} & p_{1,2} & p_{1,3} \\ p_{2,1} & p_{2,2} & p_{2,3} \\ p_{3,1} & p_{3,2} & p_{3,3} \\ p_{4,1} & p_{4,2} & p_{4,3} \end{pmatrix}$$

```
(%i67) Gamma:matrix([g[1,1],g[1,2],g[1,3]])
```

(%o67) $$\begin{pmatrix} g_{1,1} & g_{1,2} & g_{1,3} \end{pmatrix}$$

```
(%i68) expr1:Pi.S-A.Pi-B.Gamma-P
```

(%o68) $$\begin{pmatrix} -p_{2,1}, & -p_{1,3}\,\omega - p_{2,2}, & p_{1,2}\,\omega - p_{2,3}; & \frac{p_{2,1}\,F}{M} - \frac{g_{1,1}}{M} - \frac{1}{M}, & -p_{2,3}\,\omega + \frac{p_{2,2}\,F}{M} - \frac{g_{1,2}}{M}, & p_{2,2}\,\omega + \frac{p_{2,3}\,F}{M} - \frac{g_{1,3}}{M}; & -p_{4,1}, & -p_{3,3}\,\omega - p_{4,2}, & p_{3,2}\,\omega - p_{4,3}; & -\frac{p_{3,1}\,g}{L} - \frac{p_{2,1}\,F}{L\,M} + \frac{g_{1,1}}{L\,M} + \frac{1}{L\,M}, & -p_{4,3}\,\omega - \frac{p_{3,2}\,g}{L} - \frac{p_{2,2}\,F}{L\,M} + \frac{g_{1,2}}{L\,M}, & p_{4,2}\,\omega - \frac{p_{3,3}\,g}{L} - \frac{p_{2,3}\,F}{L\,M} + \frac{g_{1,3}}{L\,M} \end{pmatrix}$$

```
(%i69) expr2:Ce.Pi+Q
```

(%o69) $$\begin{pmatrix} p_{1,1} & p_{1,2} - 1 & p_{1,3} \end{pmatrix}$$

```
(%i70) expr1
```

(%o70) $$\begin{pmatrix} -p_{2,1}, & -p_{1,3}\,\omega - p_{2,2}, & p_{1,2}\,\omega - p_{2,3}; & \frac{p_{2,1}\,F}{M} - \frac{g_{1,1}}{M} - \frac{1}{M}, & -p_{2,3}\,\omega + \frac{p_{2,2}\,F}{M} - \frac{g_{1,2}}{M}, & p_{2,2}\,\omega + \frac{p_{2,3}\,F}{M} - \frac{g_{1,3}}{M}; & -p_{4,1}, & -p_{3,3}\,\omega - p_{4,2}, & p_{3,2}\,\omega - p_{4,3}; & -\frac{p_{3,1}\,g}{L} - \frac{p_{2,1}\,F}{L\,M} + \frac{g_{1,1}}{L\,M} + \frac{1}{L\,M}, & -p_{4,3}\,\omega - \frac{p_{3,2}\,g}{L} - \frac{p_{2,2}\,F}{L\,M} + \frac{g_{1,2}}{L\,M}, & p_{4,2}\,\omega - \frac{p_{3,3}\,g}{L} - \frac{p_{2,3}\,F}{L\,M} + \frac{g_{1,3}}{L\,M} \end{pmatrix}$$

```
(%i71) transpose(flatten(args(expr1)=0))
```

(%o71) $$\mathrm{transpose}\left(\left[\left[-p_{2,1},\ -p_{1,3}\,\omega - p_{2,2},\ p_{1,2}\,\omega - p_{2,3}\right],\ \left[\frac{p_{2,1}\,F}{M} - \frac{g_{1,1}}{M} - \frac{1}{M},\ -p_{2,3}\,\omega + \frac{p_{2,2}\,F}{M} - \frac{g_{1,2}}{M},\ p_{2,2}\,\omega + \frac{p_{2,3}\,F}{M} - \frac{g_{1,3}}{M}\right],\ \left[-p_{4,1},\ -p_{3,3}\,\omega - p_{4,2},\ p_{3,2}\,\omega - p_{4,3}\right],\ \left[-\frac{p_{3,1}\,g}{L} - \frac{p_{2,1}\,F}{L\,M} + \frac{g_{1,1}}{L\,M} + \frac{1}{L\,M},\ -p_{4,3}\,\omega - \frac{p_{3,2}\,g}{L} - \frac{p_{2,2}\,F}{L\,M} + \frac{g_{1,2}}{L\,M},\ p_{4,2}\,\omega - \frac{p_{3,3}\,g}{L} - \frac{p_{2,3}\,F}{L\,M} + \frac{g_{1,3}}{L\,M}\right]\right] = 0\right)$$

```
(%i72) transpose(flatten(args(expr2)))
```

(%o72) $$\begin{pmatrix} p_{1,1} \\ p_{1,2} - 1 \\ p_{1,3} \end{pmatrix}$$

```
(%i73) toSubst:[p[1,1]=0,p[1,3]=0,p[1,2]=1,p[4,1]=0,p[2,1]=0]
```

(%o73) $$[p_{1,1} = 0,\ p_{1,3} = 0,\ p_{1,2} = 1,\ p_{4,1} = 0,\ p_{2,1} = 0]$$

```
(%i74) expr1:subst(toSubst,expr1)
```

(%o74) $$\begin{pmatrix} 0, & -p_{2,2}, & \omega - p_{2,3}; & -\frac{g_{1,1}}{M} - \frac{1}{M}, & -p_{2,3}\,\omega + \frac{p_{2,2}\,F}{M} - \frac{g_{1,2}}{M}, & p_{2,2}\,\omega + \frac{p_{2,3}\,F}{M} - \frac{g_{1,3}}{M}; & 0, & -p_{3,3}\,\omega - p_{4,2}, & p_{3,2}\,\omega - p_{4,3}; & -\frac{p_{3,1}\,g}{L} + \frac{g_{1,1}}{L\,M} + \frac{1}{L\,M}, & -p_{4,3}\,\omega - \frac{p_{3,2}\,g}{L} - \frac{p_{2,2}\,F}{L\,M} + \frac{g_{1,2}}{L\,M}, & p_{4,2}\,\omega - \frac{p_{3,3}\,g}{L} - \frac{p_{2,3}\,F}{L\,M} + \frac{g_{1,3}}{L\,M} \end{pmatrix}$$

```
(%i75) expr2:subst(toSubst,expr2)
```

(%o75) $$\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

(%i76) `transpose(flatten(args(expr1)))`

(%o76)
$$\begin{pmatrix} 0 \\ -p_{2,2} \\ \omega - p_{2,3} \\ -\frac{g_{1,1}}{M} - \frac{1}{M} \\ -p_{2,3}\,\omega + \frac{p_{2,2}\,F}{M} - \frac{g_{1,2}}{M} \\ p_{2,2}\,\omega + \frac{p_{2,3}\,F}{M} - \frac{g_{1,3}}{M} \\ 0 \\ -p_{3,3}\,\omega - p_{4,2} \\ p_{3,2}\,\omega - p_{4,3} \\ -\frac{p_{3,1}\,g}{L} + \frac{g_{1,1}}{L\,M} + \frac{1}{L\,M} \\ -p_{4,3}\,\omega - \frac{p_{3,2}\,g}{L} - \frac{p_{2,2}\,F}{L\,M} + \frac{g_{1,2}}{L\,M} \\ p_{4,2}\,\omega - \frac{p_{3,3}\,g}{L} - \frac{p_{2,3}\,F}{L\,M} + \frac{g_{1,3}}{L\,M} \end{pmatrix}$$

(%i77) `toSubst:append(toSubst,[p[2,2]=0,p[2,3]=omega,g[1,1]=-1])`

(%o77) $[p_{1,1}=0,\ p_{1,3}=0,\ p_{1,2}=1,\ p_{4,1}=0,\ p_{2,1}=0,\ p_{2,2}=0,\ p_{2,3}=\omega,\ g_{1,1}=-1]$

(%i78) `expr1:subst(toSubst,expr1)`

(%o78)
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\omega^2 - \frac{g_{1,2}}{M} & \frac{F\,\omega}{M} - \frac{g_{1,3}}{M} \\ 0 & -p_{3,3}\,\omega - p_{4,2} & p_{3,2}\,\omega - p_{4,3} \\ -\frac{p_{3,1}\,g}{L} & -p_{4,3}\,\omega - \frac{p_{3,2}\,g}{L} + \frac{g_{1,2}}{L\,M} & -\frac{F\,\omega}{L\,M} + p_{4,2}\,\omega - \frac{p_{3,3}\,g}{L} + \frac{g_{1,3}}{L\,M} \end{pmatrix}$$

(%i79) `toSubst:append(toSubst,[g[1,2]=-M*omega^2,g[1,3]=F*omega])`

(%o79) $[p_{1,1}=0,\ p_{1,3}=0,\ p_{1,2}=1,\ p_{4,1}=0,\ p_{2,1}=0,\ p_{2,2}=0,\ p_{2,3}=\omega,\ g_{1,1}=-1,\ g_{1,2}=-M\,\omega^2,\ g_{1,3}=F\,\omega]$

(%i80) `expr1:subst(toSubst,expr1)`

(%o80)
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -p_{3,3}\,\omega - p_{4,2} & p_{3,2}\,\omega - p_{4,3} \\ -\frac{p_{3,1}\,g}{L} & -\frac{\omega^2}{L} - p_{4,3}\,\omega - \frac{p_{3,2}\,g}{L} & p_{4,2}\,\omega - \frac{p_{3,3}\,g}{L} \end{pmatrix}$$

(%i81) `toSubst:append(toSubst,[p[3,1]=0])`

(%o81) $[p_{1,1}=0,\ p_{1,3}=0,\ p_{1,2}=1,\ p_{4,1}=0,\ p_{2,1}=0,\ p_{2,2}=0,\ p_{2,3}=\omega,\ g_{1,1}=-1,\ g_{1,2}=-M\,\omega^2,\ g_{1,3}=F\,\omega,\ p_{3,1}=0]$

(%i82) `expr1:subst(toSubst,expr1)`

(%o82)
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -p_{3,3}\,\omega - p_{4,2} & p_{3,2}\,\omega - p_{4,3} \\ 0 & -\frac{\omega^2}{L} - p_{4,3}\,\omega - \frac{p_{3,2}\,g}{L} & p_{4,2}\,\omega - \frac{p_{3,3}\,g}{L} \end{pmatrix}$$

(%i83) `linsol:transpose(flatten(args(expr1)))`

(%o83)
$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -p_{3,3}\,\omega - p_{4,2} \\ p_{3,2}\,\omega - p_{4,3} \\ 0 \\ -\frac{\omega^2}{L} - p_{4,3}\,\omega - \frac{p_{3,2}\,g}{L} \\ p_{4,2}\,\omega - \frac{p_{3,3}\,g}{L} \end{pmatrix}$$

(%i84) temp:[p[4,2]=-p[3,3]*omega,p[4,3]=p[3,2]*omega]

(%o84) $[p_{4,2} = -p_{3,3}\,\omega,\ p_{4,3} = p_{3,2}\,\omega]$

(%i85) tmp:subst(temp,expr1)

(%o85) $$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{\omega^2}{L} - p_{3,2}\,\omega^2 - \frac{p_{3,2}\,g}{L} & -p_{3,3}\,\omega^2 - \frac{p_{3,3}\,g}{L} \end{pmatrix}$$

(%i86) solve(tmp[4,3]=0,p[3,3])

(%o86) $[p_{3,3} = 0]$

(%i87) solve(tmp[4,2]=0,p[3,2])

(%o87) $$\left[p_{3,2} = -\frac{\omega^2}{L\,\omega^2 + g}\right]$$

(%i88) toSubst:append(toSubst,[p[3,3]=0,p[3,2]=-omega^2/(L*omega^2+g), p[4,2]=0,p[4,3]=-omega^2/(L*omega^2+g)*omega])

(%o88) $$\left[p_{1,1} = 0,\ p_{1,3} = 0,\ p_{1,2} = 1,\ p_{4,1} = 0,\ p_{2,1} = 0,\ p_{2,2} = 0,\ p_{2,3} = \omega,\ g_{1,1} = -1,\ g_{1,2} = -M\,\omega^2,\ g_{1,3} = F\,\omega,\ p_{3,1} = 0,\ p_{3,3} = 0,\ p_{3,2} = -\frac{\omega^2}{L\,\omega^2 + g},\ p_{4,2} = 0,\ p_{4,3} = -\frac{\omega^3}{L\,\omega^2 + g}\right]$$

(%i90) ratsimp(subst(toSubst,expr1))

(%o90) $$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(%i91) solPi:subst(toSubst,Pi)

(%o91) $$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ 0 & -\frac{\omega^2}{L\,\omega^2 + g} & 0 \\ 0 & 0 & -\frac{\omega^3}{L\,\omega^2 + g} \end{pmatrix}$$

(%i92) solGamma:subst(toSubst,Gamma)

(%o92) $\begin{pmatrix} -1 & -M\,\omega^2 & F\,\omega \end{pmatrix}$

(%i93) ratsimp(solPi.S-A.solPi-B.solGamma-P)

(%o93) $$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(%i94) ratsimp(Ce.solPi+Q)

(%o94) $\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$

(%i97) matsize(M):=[length(M),length(transpose(M))]$

(%i98) fbisol(A,B,C,S,P,Q):=block(

```
[dimA,dimS,dimB,XPi,XGamma,vars,eq1,eq2,exprs],
dimA:matsize(A),
dimS:matsize(S),
dimB:matsize(B),
XPi:zeromatrix(dimA[1],dimS[1]),
XGamma:zeromatrix(dimB[2],dimS[1]),
vars:[],
for r:1 thru dimA[1] do(
for c:1 thru dimS[1] do(
XPi[r,c]:p[r,c],
vars:append(vars,[p[r,c]])
)
),
for r:1 thru dimB[2] do(
for c:1 thru dimS[1] do(
XGamma[r,c]:g[r,c],
vars:append(vars,[g[r,c]])
)
),
eq1:XPi.S-A.XPi-B.XGamma-P,
eq2:C.XPi+Q,
exprs:[],
for r:1 thru dimA[1] do(
for c:1 thru dimS[1] do(
exprs:append(exprs,[eq1[r,c]])
)
),
for r:1 thru dimB[2] do(
for c:1 thru dimS[1] do(
exprs:append(exprs,[eq2[r,c]])
)
),
sol:solve(exprs,vars),
return([subst(sol[1],XPi),subst(sol[1],XGamma)])
)$
```

(%i99) sol:fbisol(A,B,Ce,S,P,Q)

(%o99)
$$\left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ 0 & -\frac{\omega^2}{L\,\omega^2+g} & 0 \\ 0 & 0 & -\frac{\omega^3}{L\,\omega^2+g} \end{pmatrix}, \begin{pmatrix} -1 & -M\,\omega^2 & F\,\omega \end{pmatrix}\right]$$

(%i100) sol[1].S

(%o100)
$$\begin{pmatrix} 0 & 0 & \omega \\ 0 & -\omega^2 & 0 \\ 0 & 0 & -\frac{\omega^3}{L\,\omega^2+g} \\ 0 & \frac{\omega^4}{L\,\omega^2+g} & 0 \end{pmatrix}$$

(%i101) A.sol[1]

$$(\%o101)\quad \begin{pmatrix} 0 & 0 & \omega \\ 0 & 0 & -\frac{F\,\omega}{M} \\ 0 & 0 & -\frac{\omega^3}{L\,\omega^2+g} \\ 0 & -\frac{g\,\omega^2}{L\,(L\,\omega^2+g)} & \frac{F\,\omega}{L\,M} \end{pmatrix}$$

(%i102) B.sol[2]

$$(\%o102)\quad \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{M} & -\omega^2 & \frac{F\,\omega}{M} \\ 0 & 0 & 0 \\ \frac{1}{L\,M} & \frac{\omega^2}{L} & -\frac{F\,\omega}{L\,M} \end{pmatrix}$$

(%i103) Ce.sol[1]

$$(\%o103)\quad \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$$