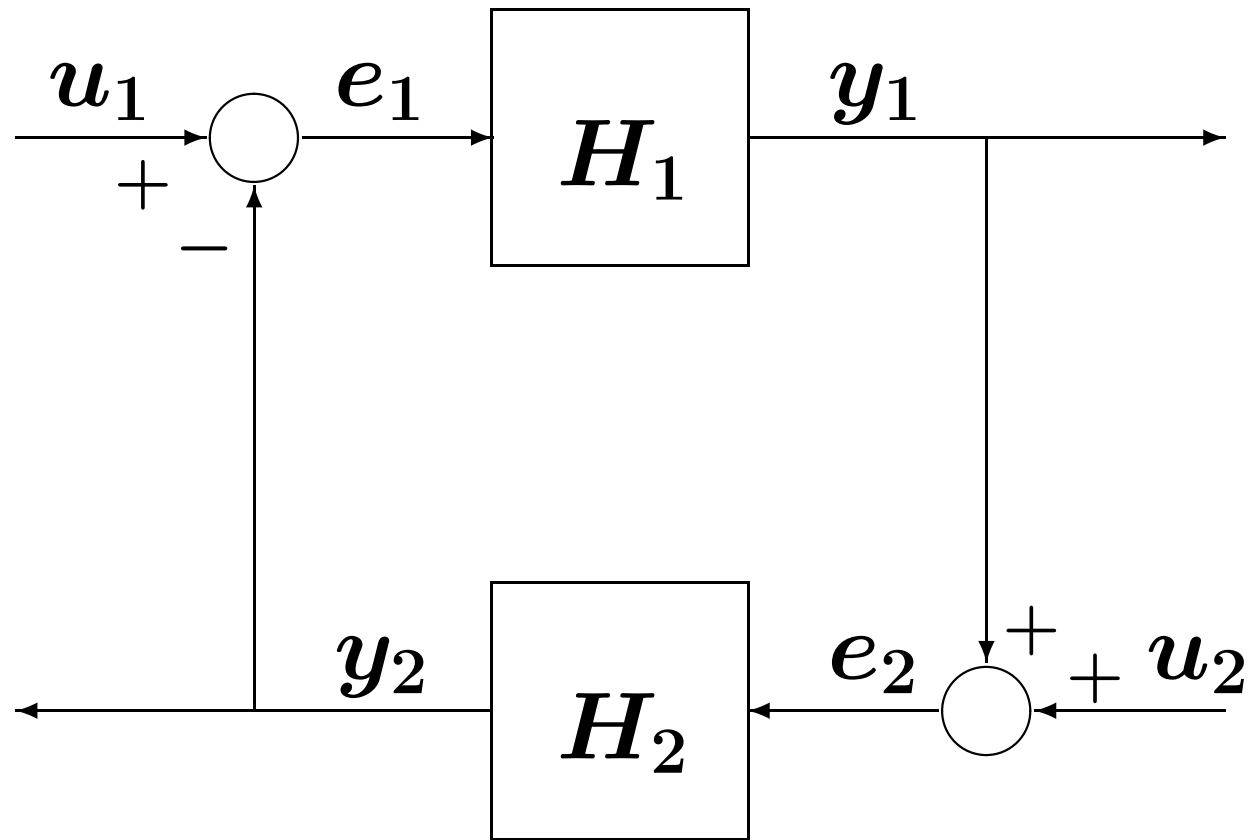


Nonlinear Systems and Control

Lecture # 16

Feedback Systems: Passivity Theorems



$$\dot{x}_i = f_i(x_i, e_i), \quad y_i = h_i(x_i, e_i)$$

$$y_i = h_i(t, e_i)$$

Passivity Theorems

Theorem 6.1: The feedback connection of two passive systems is passive

Theorem 6.3: Consider the feedback connection of two dynamical systems. When $u = 0$, the origin of the closed-loop system is asymptotically stable if each feedback component is either

- strictly passive, or
- output strictly passive and zero-state observable

Furthermore, if the storage function for each component is radially unbounded, the origin is globally asymptotically stable

Theorem 6.4: Consider the feedback connection of a strictly passive dynamical system with a passive memoryless function. When $u = 0$, the origin of the closed-loop system is uniformly asymptotically stable. if the storage function for the dynamical system is radially unbounded, the origin will be globally uniformly asymptotically stable

Prove using $V = V_1 + V_2$ as a Lyapunov function candidate

Proof of Theorem 6.3: H_1 is SP; H_2 is OSP & ZSO

$$e_1^T y_1 \geq \dot{V}_1 + \psi_1(x_1), \quad \psi_1(x_1) > 0, \quad \forall x_1 \neq 0$$

$$e_2^T y_2 \geq \dot{V}_2 + y_2^T \rho_2(y_2), \quad y_2^T \rho_2(y_2) > 0, \quad \forall y_2 \neq 0$$

$$e_1^T y_1 + e_2^T y_2 = (u_1 - y_2)^T y_1 + (u_2 + y_1)^T y_2 = u_1^T y_1 + u_2^T y_2$$

$$V(x) = V_1(x_1) + V_2(x_2)$$

$$\dot{V} \leq u^T y - \psi_1(x_1) - y_2^T \rho_2(y_2)$$

$$u = 0 \Rightarrow \dot{V} \leq -\psi_1(x_1) - y_2^T \rho_2(y_2)$$

$$\dot{V} = 0 \Rightarrow x_1 = 0 \text{ and } y_2 = 0$$

$$y_2(t) \equiv 0 \Rightarrow e_1(t) \equiv 0 \text{ (\& } x_1(t) \equiv 0) \Rightarrow y_1(t) \equiv 0$$

$$y_1(t) \equiv 0 \Rightarrow e_2(t) \equiv 0$$

By zero-state observability of H_2 : $y_2(t) \equiv 0 \Rightarrow x_2(t) \equiv 0$

Apply the invariance principle

Example

$$\underbrace{\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -ax_1^3 - kx_2 + e_1 \\ y_1 = x_2 + e_1 \end{array}}_{H_1} \quad \left| \quad \underbrace{\begin{array}{l} \dot{x}_3 = x_4 \\ \dot{x}_4 = -bx_3 - x_4^3 + e_2 \\ y_2 = x_4 \end{array}}_{H_2}$$

$$a, b, k > 0$$

$$V_1 = \frac{1}{4}ax_1^4 + \frac{1}{2}x_2^2$$

$$\dot{V}_1 = ax_1^3x_2 - ax_1^3x_2 - kx_2^2 + x_2e_1 = -ky_1^2 + y_1e_1$$

$$\text{With } e_1 = 0, \quad y_1(t) \equiv 0 \Leftrightarrow x_2(t) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

H_1 is output strictly passive and zero-state observable

$$V_2 = \frac{1}{2}bx_3^2 + \frac{1}{2}x_4^2$$

$$\dot{V}_2 = bx_3x_4 - bx_3x_4 - x_4^4 + x_4e_2 = -y_2^4 + y_2e_2$$

With $e_2 = 0$, $y_2(t) \equiv 0 \Leftrightarrow x_4(t) \equiv 0 \Rightarrow x_3(t) \equiv 0$

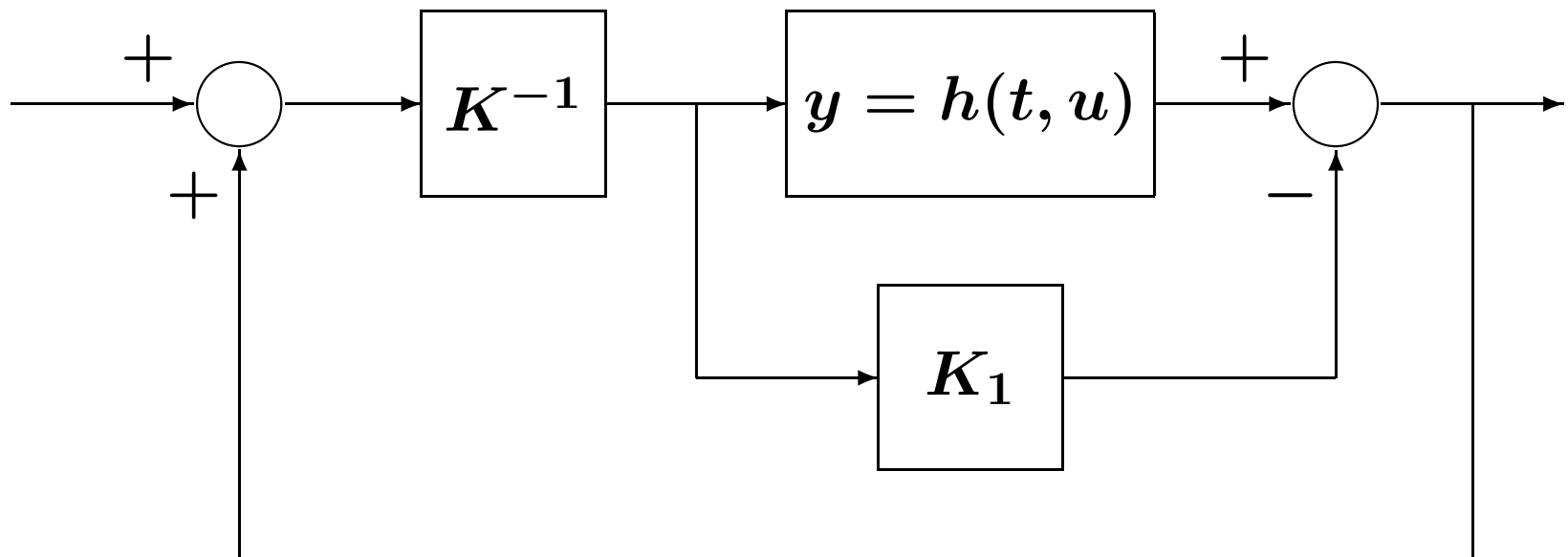
H_2 is output strictly passive and zero-state observable

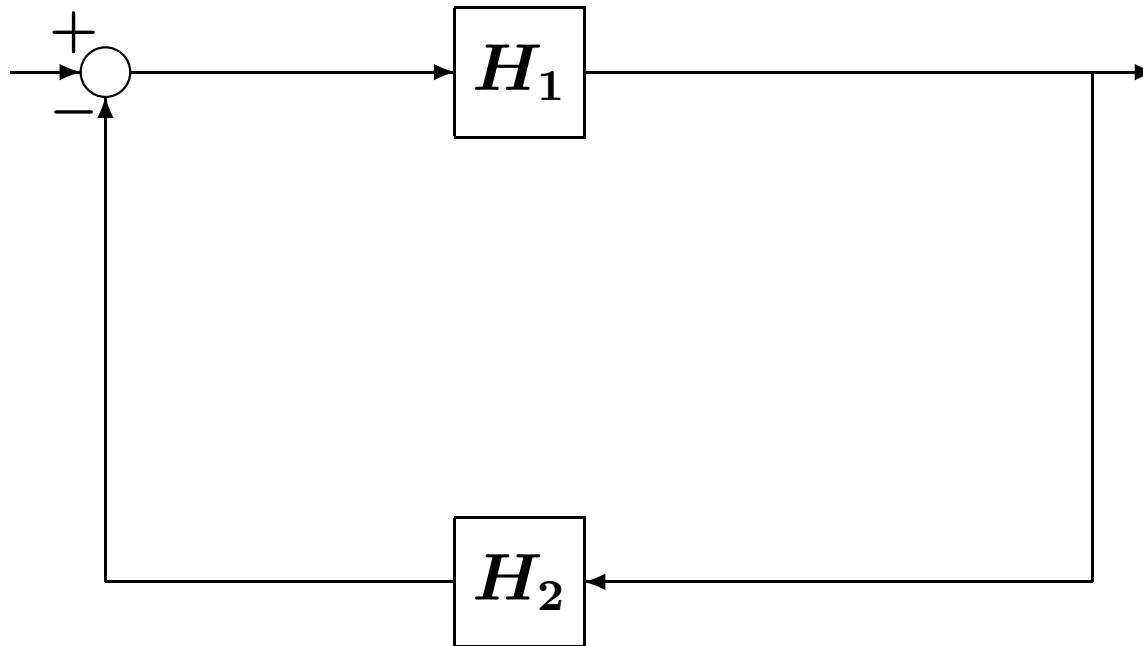
V_1 and V_2 are radially unbounded

The origin is globally asymptotically stable

Loop Transformations

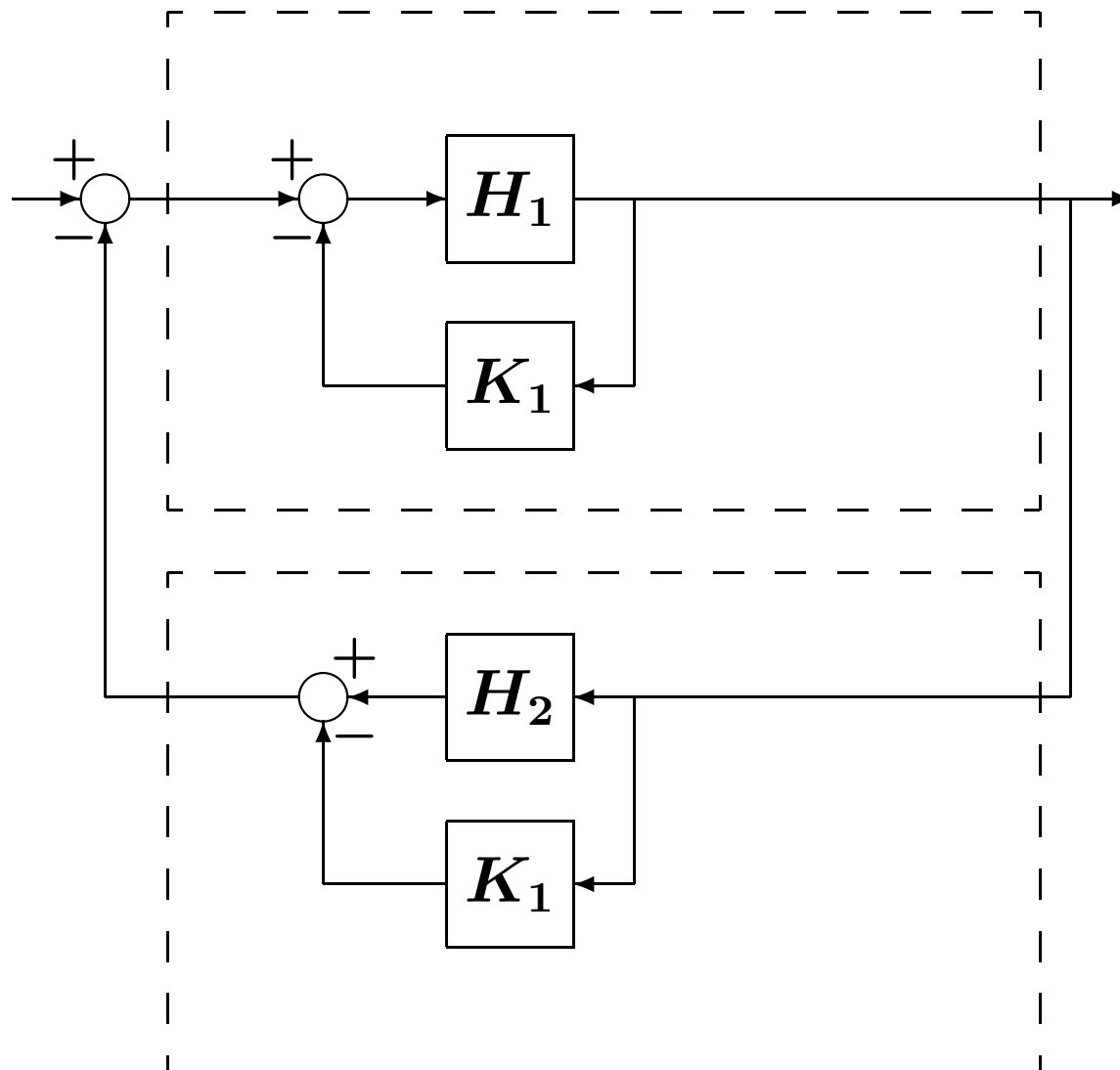
Recall that a memoryless function in the sector $[K_1, K_2]$ can be transformed into a function in the sector $[0, \infty]$ by input feedforward followed by output feedback

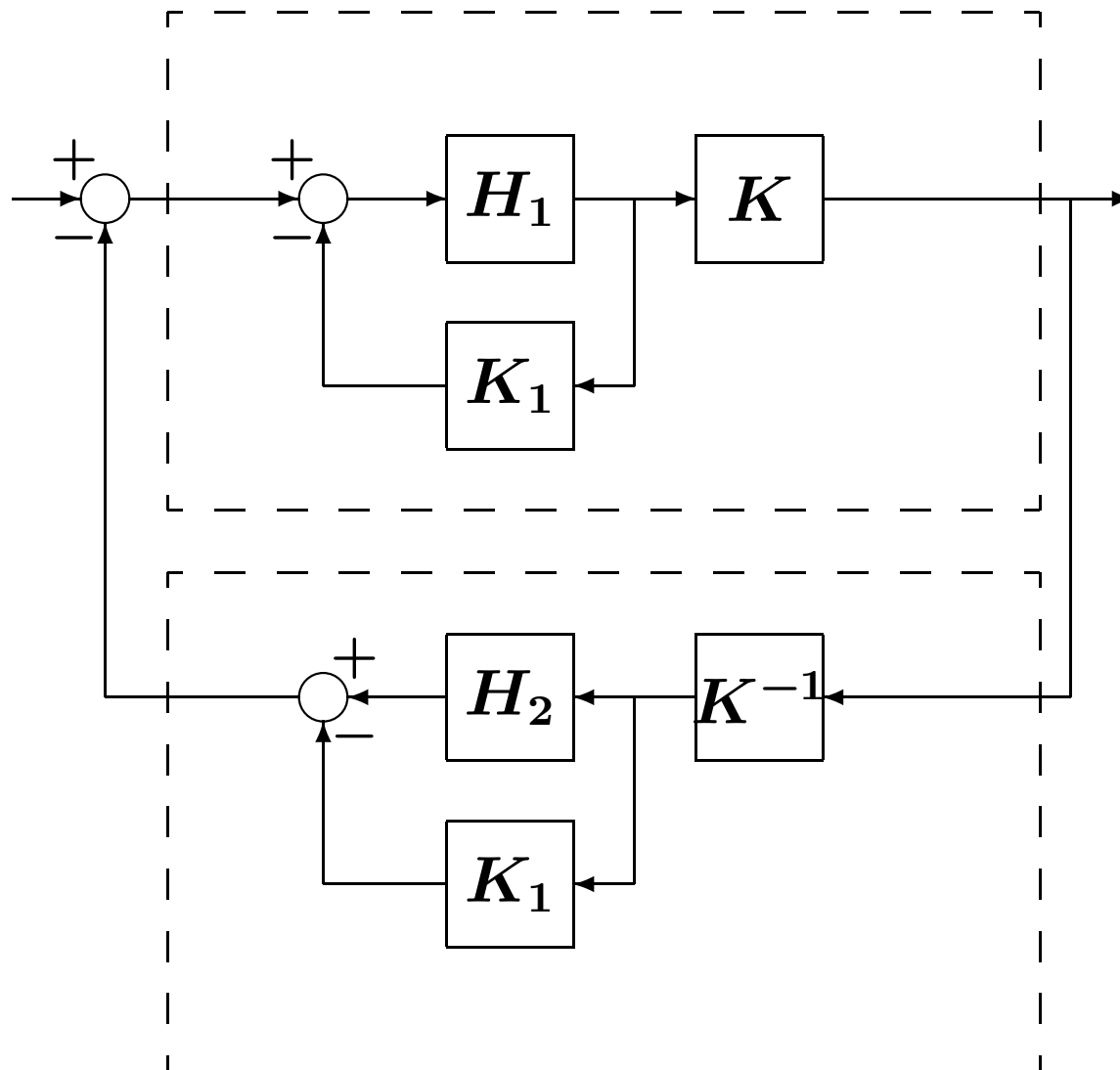


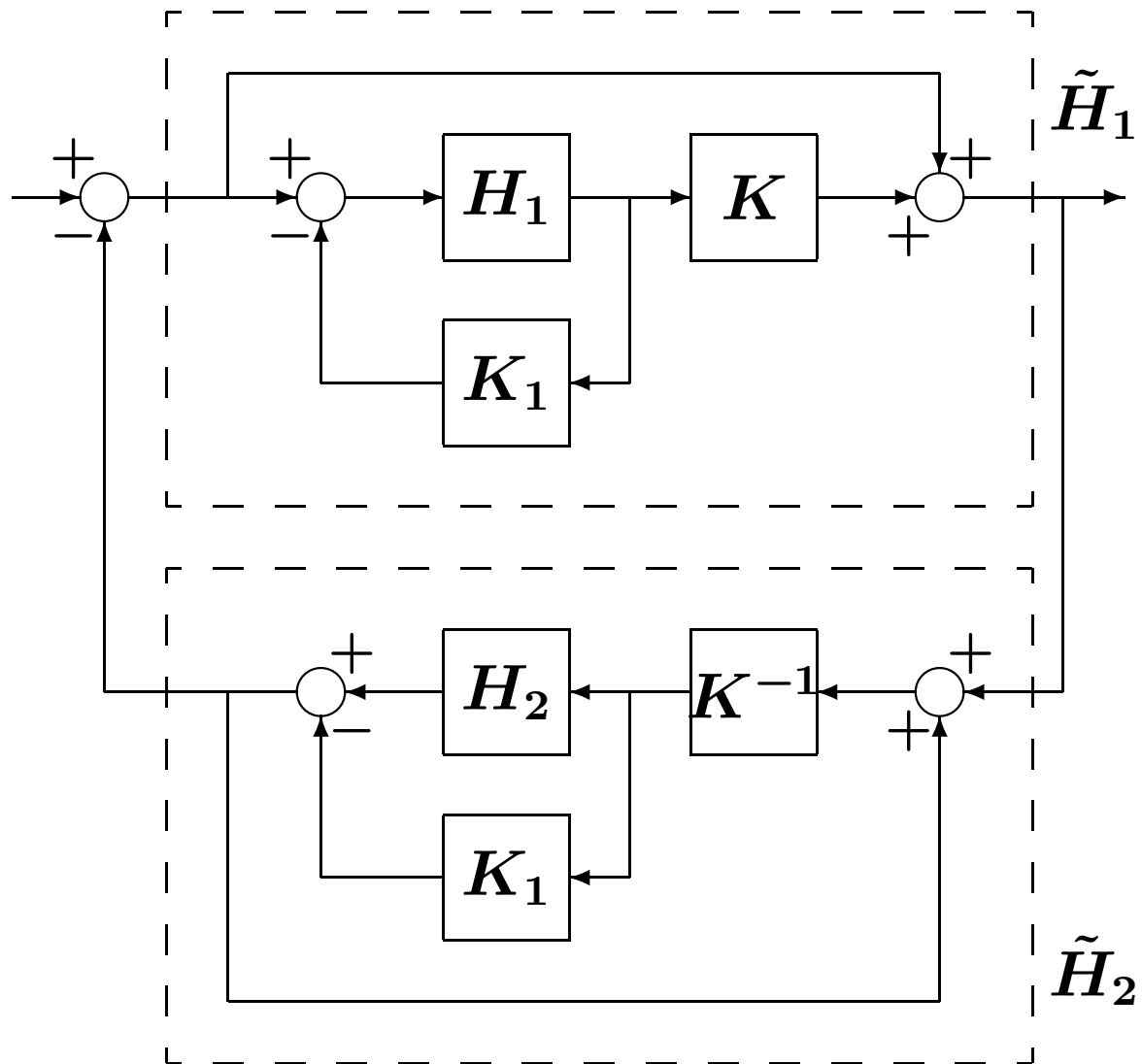


H_1 is a dynamical system

H_2 is a memoryless function in the sector $[K_1, K_2]$







Example

$$\underbrace{\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -h(x_1) + bx_2 + e_1 \\ y_1 = x_2 \end{array}}_{H_1} \quad \left| \quad \underbrace{y_2 = \sigma(e_2)}_{H_2}\right.$$

$$\sigma \in [\alpha, \beta], \quad h \in [\alpha_1, \infty], \quad b > 0, \quad \alpha_1 > 0, \quad k = \beta - \alpha > 0$$

$$\underbrace{\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -h(x_1) - ax_2 + \tilde{e}_1 \\ \tilde{y}_1 = kx_2 + \tilde{e}_1 \end{array}}_{\tilde{H}_1} \quad \left| \quad \underbrace{\tilde{y}_2 = \tilde{\sigma}(\tilde{e}_2)}_{\tilde{H}_2}\right.$$

$$\tilde{\sigma} \in [0, \infty], \quad a = \alpha - b$$

Assume $a = \alpha - b > 0$ and show that \tilde{H}_1 is strictly passive

$$V_1 = k \int_0^{x_1} h(s) \, ds + x^T P x$$

$$V_1 = k \int_0^{x_1} h(s) \, ds + p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2$$

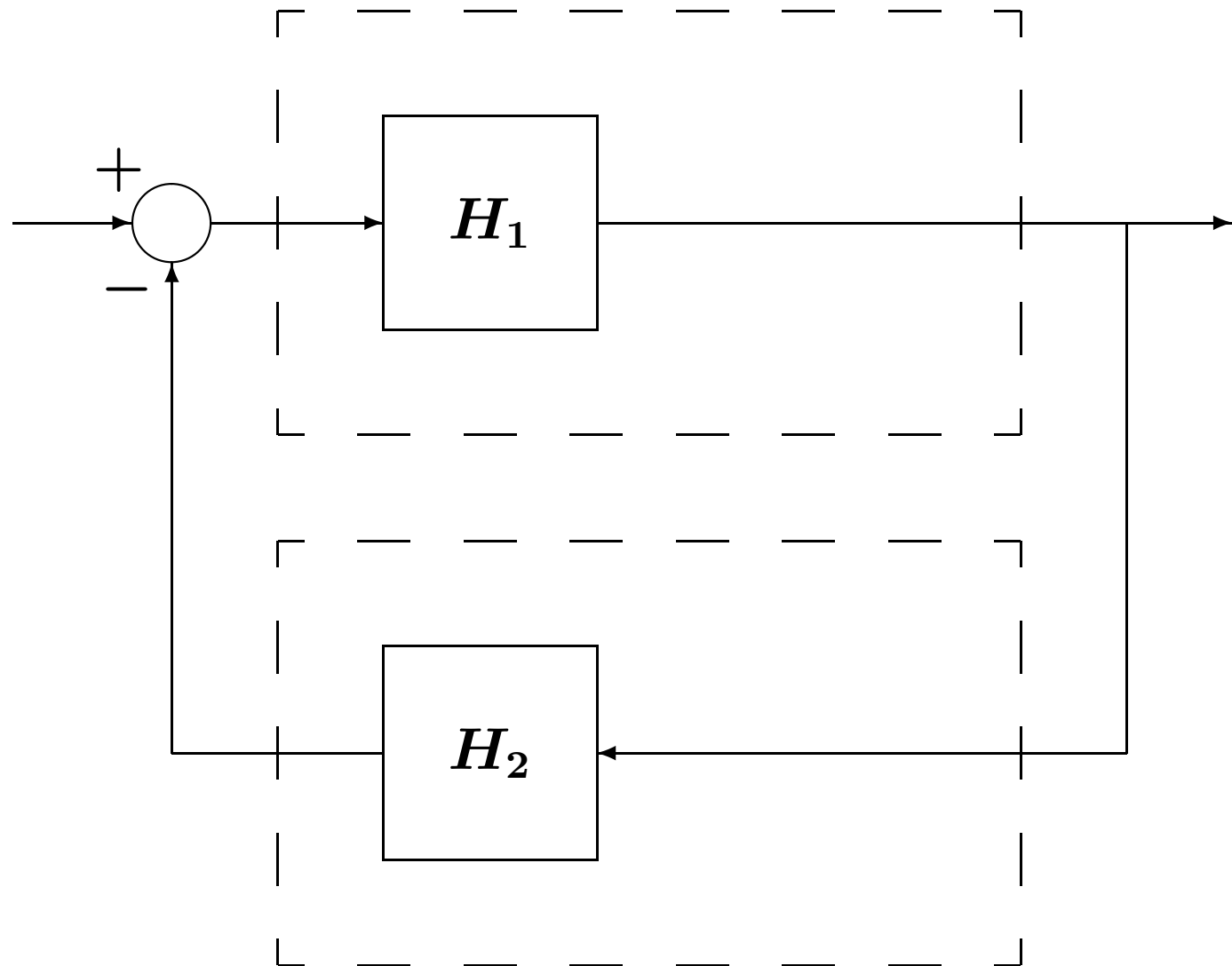
$$\begin{aligned} \dot{V} = & kh(x_1)x_2 + 2(p_{11}x_1 + p_{12}x_2)x_2 \\ & 2(p_{12}x_1 + p_{22}x_2)[-h(x_1) - ax_2 + \tilde{e}_1] \end{aligned}$$

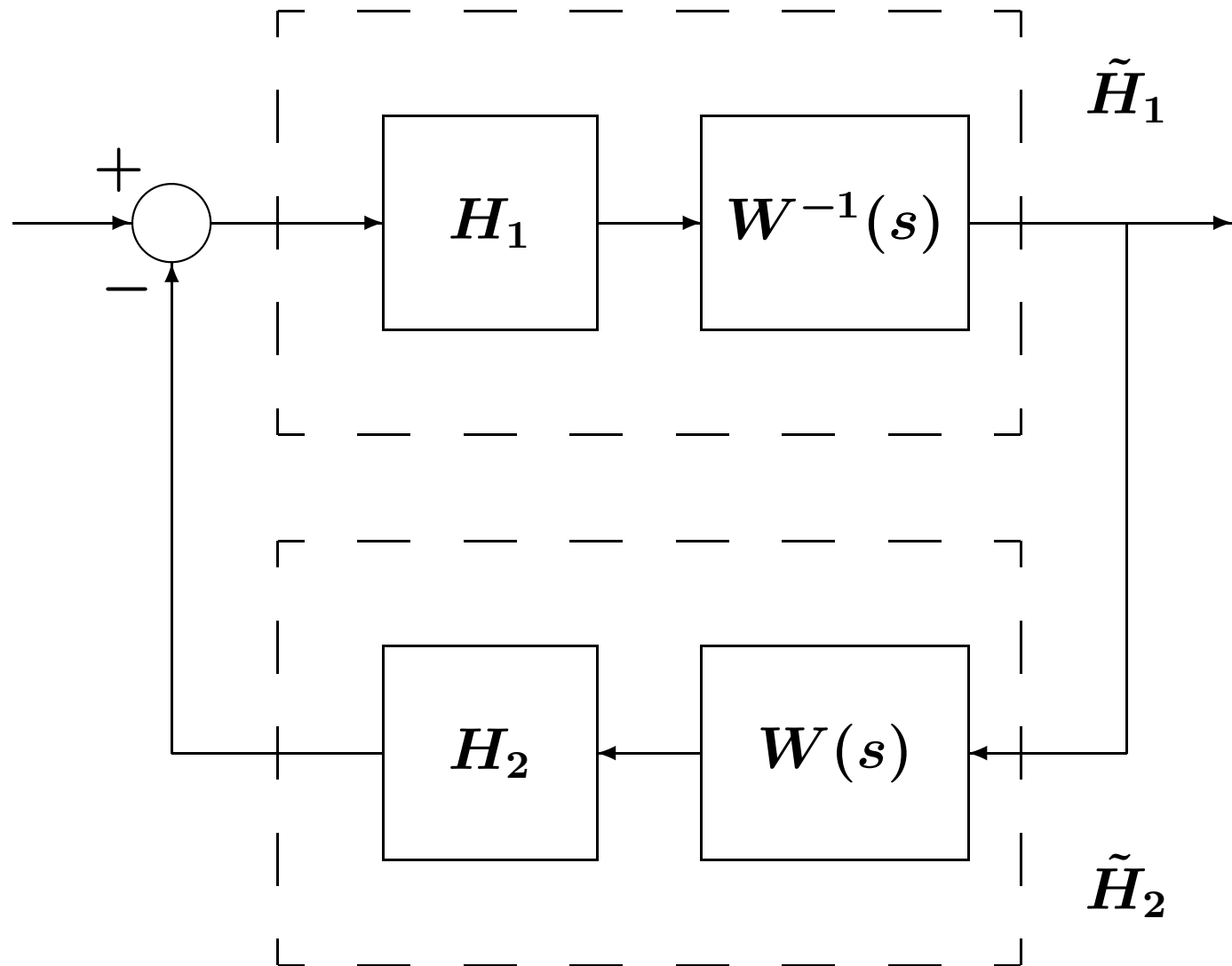
$$\text{Take } p_{22} = k/2, \quad p_{11} = ap_{12}$$

$$\begin{aligned}
\dot{V} &= -2p_{12}x_1h(x_1) - (ka - 2p_{12})x_2^2 \\
&\quad + kx_2\tilde{e}_1 + 2p_{12}x_1\tilde{e}_1 \\
&= -2p_{12}x_1h(x_1) - (ka - 2p_{12})x_2^2 \\
&\quad + (kx_2 + \tilde{e}_1)\tilde{e}_1 - \tilde{e}_1^2 + 2p_{12}x_1\tilde{e}_1 \\
\tilde{y}_1\tilde{e}_1 &= \dot{V} + 2p_{12}x_1h(x_1) + (ka - 2p_{12})x_2^2 \\
&\quad + (\tilde{e}_1 - p_{12}x_1)^2 - p_{12}^2x_1^2 \\
&\geq \dot{V} + p_{12}(2\alpha_1 - p_{12})x_1^2 + (ka - 2p_{12})x_2^2
\end{aligned}$$

Take $0 < p_{12} < \min \left\{ \frac{ak}{2}, 2\alpha_1 \right\} \Rightarrow p_{12}^2 < 2p_{12}\frac{k}{2} = p_{11}p_{22}$

\tilde{H}_1 is strictly passive. By Theorem 6.4 the origin is globally asymptotically stable (when $u = 0$)





Example

$$H_1 : \quad \dot{x} = Ax + Be_1, \quad y_1 = Cx$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$H_2 : \quad y_2 = h(e_2), \quad h \in [0, \infty]$$

$$C(sI - A)^{-1}B = \frac{1}{(s^2 + s + 1)} \quad \text{Not PR}$$

$$W(s) = \frac{1}{as + 1} \Rightarrow \frac{(as + 1)}{(s^2 + s + 1)}$$

$$\tilde{H}_1 : \quad \dot{x} = Ax + Be_1, \quad \tilde{y}_1 = \tilde{C}x = \begin{bmatrix} 1 & a \end{bmatrix} x$$

$$\frac{(as + 1)}{(s^2 + s + 1)}$$

$$\operatorname{Re} \left[\frac{1 + j\omega a}{1 - \omega^2 + j\omega} \right] = \frac{1 + (a - 1)\omega^2}{(1 - \omega^2)^2 + \omega^2}$$

$$\lim_{\omega \rightarrow \infty} \omega^2 \operatorname{Re} \left[\frac{1 + j\omega a}{1 - \omega^2 + j\omega} \right] = a - 1$$

$$a > 1 \Rightarrow \frac{(as + 1)}{(s^2 + s + 1)} \text{ is SPR}$$

$$V_1 = \frac{1}{2}x^T P x, \quad PA + A^T P = -L^T L - \varepsilon P, \quad PB = \tilde{C}^T$$

$$\tilde{H}_2 : \quad a\dot{e}_2 = -e_2 + \tilde{e}_2, \quad y_2 = h(e_2), \quad h \in [0, \infty]$$

\tilde{H}_2 is strictly passive with $V_2 = a \int_0^{e_2} h(s) ds$. Use

$$V = V_1 + V_2 = \frac{1}{2}x^T P x + a \int_0^{e_2} h(s) ds$$

as a Lyapunov function candidate for the original feedback connection

$$\begin{aligned}
\dot{V} &= \frac{1}{2}x^T P \dot{x} + \frac{1}{2}\dot{x}^T P x + ah(e_2)\dot{e}_2 \\
&= \frac{1}{2}x^T P[Ax - Bh(e_2)] + \frac{1}{2}[Ax - Bh(e_2)]^T P x \\
&\quad + ah(e_2)C[Ax - Bh(e_2)] \\
&= -\frac{1}{2}x^T L^T L x - (\varepsilon/2)x^T P x - x^T \tilde{C}^T h(e_2) \\
&\quad + ah(e_2)CAx \\
&= -\frac{1}{2}x^T L^T L x - (\varepsilon/2)x^T P x \\
&\quad - x^T [C + aCA]^T h(e_2) + ah(e_2)CAx \\
&= -\frac{1}{2}x^T L^T L x - (\varepsilon/2)x^T P x - e_2^T h(e_2) \\
&\leq -(\varepsilon/2)x^T P x
\end{aligned}$$

The origin is globally asymptotically stable