Nonlinear Systems and Control
Lecture # 32

Robust Stabilization

Sliding Mode Control

Example

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = h(x) + g(x)u, \quad g(x) \geq g_0 > 0$$

Sliding Manifold (Surface):

$$s = a_1 x_1 + x_2 = 0$$

$$s(t) \equiv 0 \;\; \Rightarrow \;\; \dot{x}_1 = -a_1 x_1$$

$$a_1 > 0 \;\; \Rightarrow \;\; \lim_{t \to \infty} x_1(t) = 0$$

How can we bring the trajectory to the manifold $s = 0$?

How can we maintain it there?

$$\dot{s} = a_1\dot{x}_1 + \dot{x}_2 = a_1x_2 + h(x) + g(x)u$$

Suppose

$$\left|\frac{a_1x_2 + h(x)}{g(x)}\right| \leq \varrho(x)$$

$$V = \tfrac{1}{2}s^2$$

$$\dot{V} = s\dot{s} = s[a_1x_2 + h(x)] + g(x)su \leq g(x)|s|\varrho(x) + g(x)su$$

$$\beta(x) \geq \varrho(x) + \beta_0, \quad \beta_0 > 0$$

$$s > 0, \quad u = -\beta(x)$$

$$\dot{V} \leq g(x)|s|\varrho(x) - g(x)\beta(x)|s|$$

$$\dot{V} \leq g(x)|s|\varrho(x) - g(x)(\varrho(x) + \beta_0)|s| = -g(x)\beta_0|s|$$

$$s < 0, \quad u = \beta(x)$$

$$\dot{V} \le g(x)|s|\varrho(x) + g(x)su = g(x)|s|\varrho(x) - g(x)\beta(x)|s|$$

$$\dot{V} \le g(x)|s|\varrho(x) - g(x)(\varrho(x) + \beta_0)|s| = -g(x)\beta_0|s|$$

$$\mathrm{sgn}(s) = \begin{cases} 1, & s > 0 \\ -1, & s < 0 \end{cases}$$

$$u = -\beta(x)\,\mathrm{sgn}(s)$$

$$\dot{V} \le -g(x)\beta_0|s| \le -g_0\beta_0|s|$$

$$\dot{V} \le -g_0\beta_0\sqrt{2V}$$

$$\dot{V} \le -g_0\beta_0\sqrt{2V}$$

$$\frac{dV}{\sqrt{V}} \le -g_0\beta_0\sqrt{2}\,dt$$

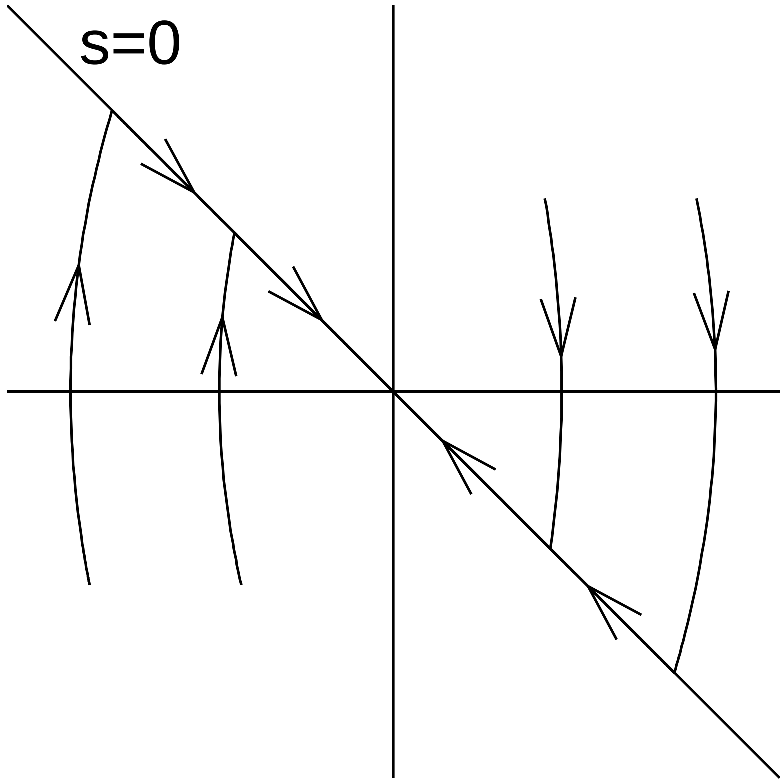
$$2\,\sqrt{V}\,\Big|_{V(s(0))}^{V(s(t))} \le -g_0\beta_0\sqrt{2}\,t$$

$$\sqrt{V(s(t))} \le \sqrt{V(s(0))} - g_0\beta_0\frac{1}{\sqrt{2}}\,t$$

$$|s(t)| \le |s(0)| - g_0\beta_0\,t$$

$s(t)$ reaches zero in finite time

Once on the surface $s = 0$, the trajectory cannot leave it



What is the region of validity?

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = h(x) - g(x)\beta(x)\mathrm{sgn}(s)$$

$$\dot{x}_1 = -a_1 x_1 + s \qquad \dot{s} = a_1 x_2 + h(x) - g(x)\beta(x)\mathrm{sgn}(s)$$

$$s\dot{s} \le -g_0\beta_0|s|, \quad \text{if } \beta(x) \ge \varrho(x) + \beta_0$$

$$V_1 = \tfrac{1}{2}x_1^2$$

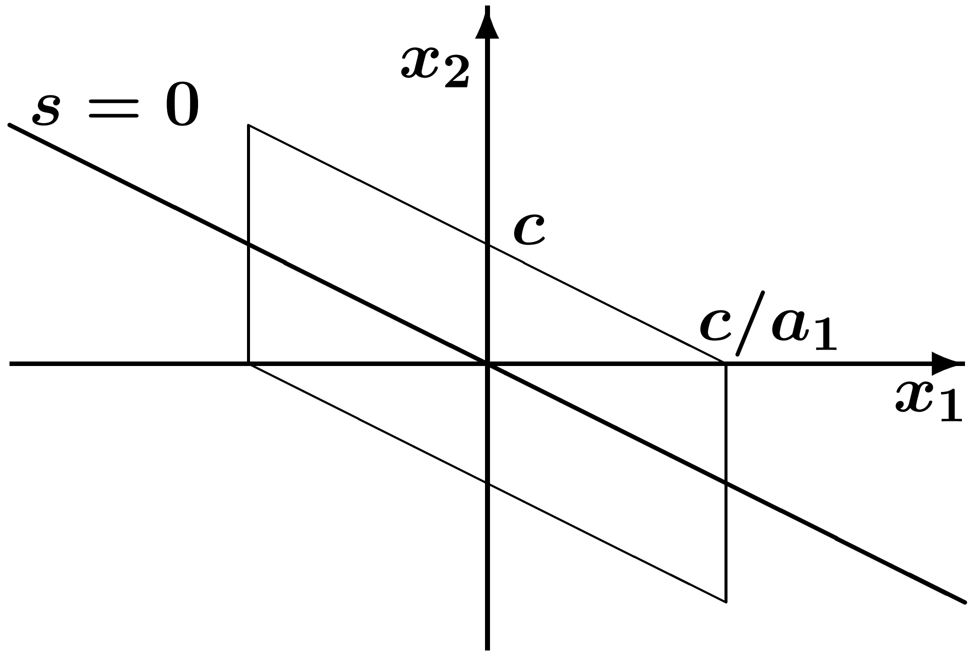
$$\dot{V}_1 = x_1\dot{x}_1 = -a_1 x_1^2 + x_1 s \le -a_1 x_1^2 + |x_1|c \le 0$$

$$\forall\ |s| \le c \text{ and } |x_1| \ge \frac{c}{a_1}$$

$$\Omega = \left\{ |x_1| \le \frac{c}{a_1},\ |s| \le c \right\}$$

Ω is positively invariant if $\left|\dfrac{a_1 x_2 + h(x)}{g(x)}\right| \le \varrho(x)$ over Ω

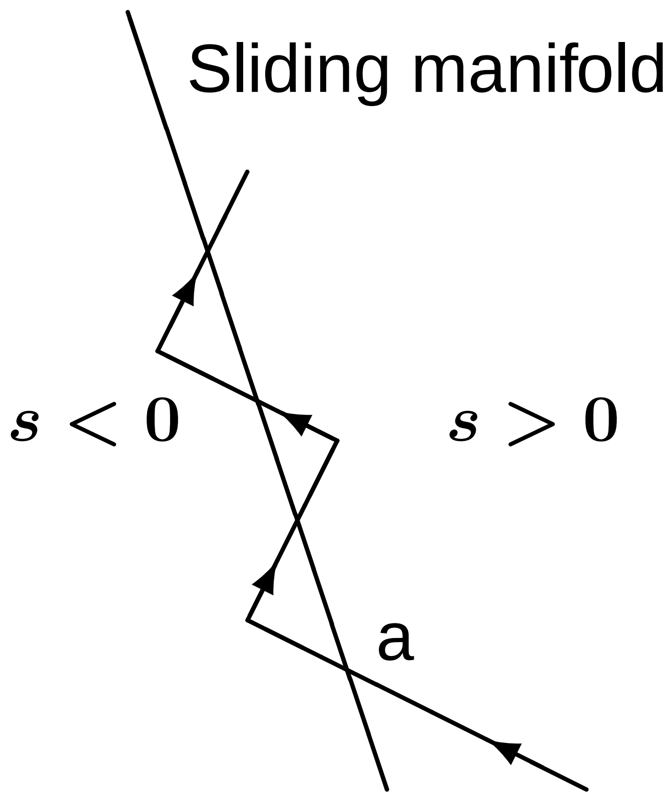
$$\Omega = \left\{ |x_1| \le \frac{c}{a_1},\ |s| \le c \right\}$$



$$\left| \frac{a_1 x_2 + h(x)}{g(x)} \right| \le k_1 < k, \ \ \forall\, x \in \Omega$$

$$u = -k \,\mathrm{sgn}(s)$$

Chattering



How can we reduce or eliminate chattering?

Reduce the amplitude of the signum function

$$\dot{s} = a_1 x_2 + h(x) + g(x)u$$

$$u = -\frac{[a_1 x_2 + \hat{h}(x)]}{\hat{g}(x)} + v$$

$$\dot{s} = \delta(x) + g(x)v$$

$$\delta(x) = a_1 \left[1 - \frac{g(x)}{\hat{g}(x)}\right] x_2 + h(x) - \frac{g(x)}{\hat{g}(x)}\hat{h}(x)$$

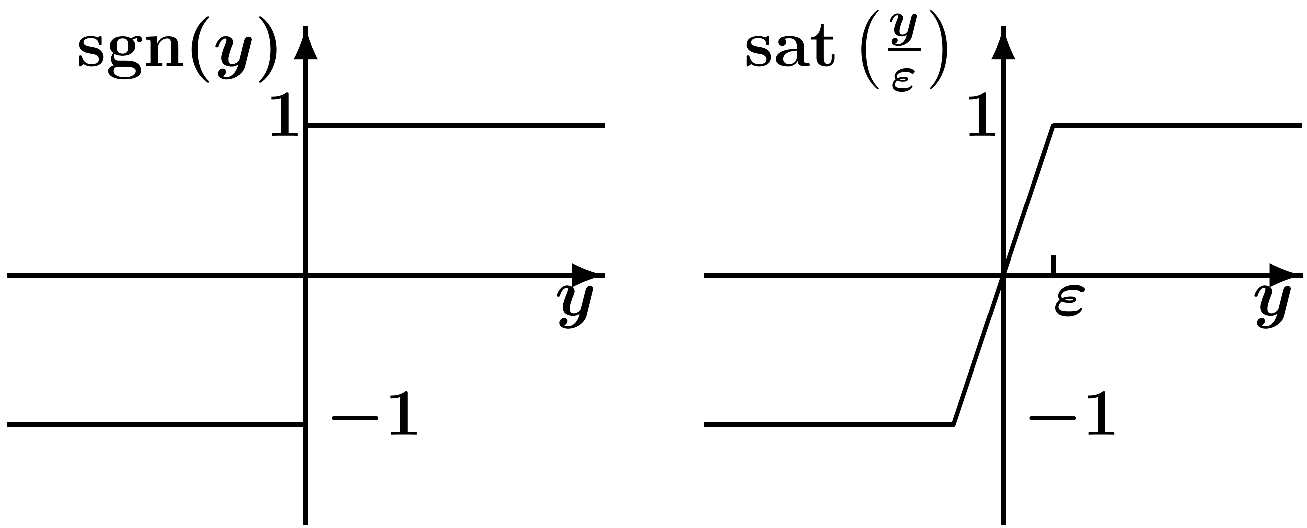
$$\left|\frac{\delta(x)}{g(x)}\right| \le \varrho(x), \qquad \beta(x) \ge \varrho(x) + \beta_0$$

$$v = -\beta(x)\,\mathrm{sgn}(s)$$

Replace the signum function by a high-slope saturation function

$$u = -\beta(x)\ \mathrm{sat}\left(\frac{s}{\varepsilon}\right)$$

$$\mathrm{sat}(y) = \begin{cases} y, & \text{if } |y| \le 1 \\ \mathrm{sgn}(y), & \text{if } |y| > 1 \end{cases}$$



How can we analyze the system?

$$\text{For } |s| \geq \varepsilon, \quad u = -\beta(x)\, \text{sgn}(s)$$

With $c \geq \varepsilon$

- $\Omega = \left\{ |x_1| \leq \frac{c}{a_1},\ |s| \leq c \right\}$ is positively invariant
- The trajectory reaches the boundary layer $\{|s| \leq \varepsilon\}$in finite time
- The boundary layer is positively invariant

Inside the boundary layer:

$$\dot{x}_1 = -a_1 x_1 + s \qquad \dot{s} = a_1 x_2 + h(x) - g(x)\beta(x)\frac{s}{\varepsilon}$$

$$x_1 \dot{x}_1 \leq -a_1 x_1^2 + |x_1|\varepsilon$$

$$0 < \theta < 1$$

$$x_1 \dot{x}_1 \leq -(1-\theta) a_1 x_1^2, \;\; \forall\, |x_1| \geq \frac{\varepsilon}{\theta a_1}$$

The trajectories reach the positively invariant set

$$\Omega_\varepsilon = \{|x_1| \leq \frac{\varepsilon}{\theta a_1}, \; |s| \leq \varepsilon\}$$

in finite time

What happens inside Ω_ε?

Find the equilibrium points

$$0 = -a_1 x_1 + s = x_2, \qquad 0 = a_1 x_2 + h(x) - g(x)\beta(x)\frac{s}{\varepsilon}$$

$$\phi(x_1) = \left.\frac{h(x)}{a_1 g(x)\beta(x)}\right|_{x_2=0}$$

$$x_1 = \varepsilon\phi(x_1)$$

Suppose $x_1 = \varepsilon\phi(x_1)$ has an isolated root $\bar{x}_1 = \varepsilon k_1$

$$h(0) = 0 \;\Rightarrow\; \bar{x}_1 = 0$$

$$z_1 = x_1 - \bar{x}_1, \qquad z_2 = s - a_1\bar{x}_1$$

$$x_2 = -a_1x_1 + s = -a_1(x_1 - \bar{x}_1) + s - a_1\bar{x}_1 = -a_1z_1 + z_2$$

$$\dot{z}_1 = -a_1x_1 + s = -a_1z_1 + z_2$$

$$\begin{aligned} \dot{z}_2 &= a_1x_2 + h(x) - g(x)\beta(x)\frac{s}{\varepsilon} \\ &= a_1(z_2 - a_1z_1) + h(x) - g(x)\beta(x)\frac{z_2 + a_1\bar{x}_1}{\varepsilon} \end{aligned}$$

$$\dot{z}_2 = \ell(z) - g(x)\beta(x)\frac{z_2}{\varepsilon}$$

$$\ell(z) = a_1(z_2 - a_1z_1) + a_1g(x)\beta(x)\left[\frac{h(x)}{a_1g(x)\beta(x)} - \frac{\bar{x}_1}{\varepsilon}\right]$$

$$\dot{z}_1 = -a_1 z_1 + z_2, \qquad \dot{z}_2 = \ell(z) - g(x)\beta(x)\frac{z_2}{\varepsilon}$$

$$\ell(0) = 0, \;\; |\ell(z)| \le \ell_1|z_1| + \ell_2|z_2|$$

$$g(x)\beta(x) \ge g_0\beta_0$$

$$V = \tfrac{1}{2}z_1^2 + \tfrac{1}{2}z_2^2$$

$$\dot{V} = z_1(-a_1 z_1 + z_2) + z_2\left[\ell(z) - g(x)\beta(x)\frac{z_2}{\varepsilon}\right]$$

$$\dot{V} \le -a_1 z_1^2 + (1+\ell_1)|z_1|\,|z_2| + \ell_2 z_2^2 - \frac{g_0\beta_0}{\varepsilon}z_2^2$$

$$\dot{V} \le -a_1 z_1^2 + (1+\ell_1)|z_1|\,|z_2| + \ell_2 z_2^2 - \frac{g_0\beta_0}{\varepsilon} z_2^2$$

$$\dot{V} \le - \begin{bmatrix} |z_1| \\ |z_2| \end{bmatrix}^T \underbrace{\begin{bmatrix} a_1 & -\frac{1}{2}(1+\ell_1) \\ -\frac{1}{2}(1+\ell_1) & \left(\frac{g_0\beta_0}{\varepsilon} - \ell_2\right) \end{bmatrix}}_{Q} \begin{bmatrix} |z_1| \\ |z_2| \end{bmatrix}$$

$$\det(Q) = a_1 \left(\frac{g_0\beta_0}{\varepsilon} - \ell_2\right) - \tfrac{1}{4}(1+\ell_1)^2$$

$$h(0) = 0 \;\Rightarrow\; \lim_{t\to\infty} x(t) = 0$$

$$h(0) \ne 0 \;\Rightarrow\; \lim_{t\to\infty} x(t) = \begin{bmatrix} \bar{x}_1 \\ 0 \end{bmatrix}$$

Read Section 14.1.1