Nonlinear Systems and Control
Lecture # 38

Observers

High-Gain Observers
Motivating Example

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \phi(x,u), \qquad y = x_1$$

Let $u = \gamma(x)$ stabilize the origin of

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \phi(x,\gamma(x))$$

Observer:

$$\dot{\hat{x}}_1 = \hat{x}_2 + h_1(y - \hat{x}_1), \qquad \dot{\hat{x}}_2 = \phi_0(\hat{x},u) + h_2(y - \hat{x}_1)$$

$\phi_0(x,u)$ is a nominal model $\phi(x,u)$

$$\tilde{x}_1 = x_1 - \hat{x}_1, \qquad \tilde{x}_2 = x_2 - \hat{x}_2$$

$$\dot{\tilde{x}}_1 = -h_1\tilde{x}_1 + \tilde{x}_2, \qquad \dot{\tilde{x}}_2 = -h_2\tilde{x}_1 + \delta(x,\tilde{x})$$

$$\delta(x,\tilde{x}) = \phi(x,\gamma(\hat{x})) - \phi_0(\hat{x},\gamma(\hat{x}))$$

Design $H = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$ such that $A_o = \begin{bmatrix} -h_1 & 1 \\ -h_2 & 0 \end{bmatrix}$ is Hurwitz

Transfer function from δ to $\tilde{x}$:

$$G_o(s) = \frac{1}{s^2 + h_1 s + h_2} \begin{bmatrix} 1 \\ s + h_1 \end{bmatrix}$$

Design H to make $\sup_{\omega \in R} \|G_o(j\omega)\|$ as small as possible

$$h_1 = \frac{\alpha_1}{\varepsilon}, \quad h_2 = \frac{\alpha_2}{\varepsilon^2}, \qquad \varepsilon > 0$$

$$G_o(s) = \frac{\varepsilon}{(\varepsilon s)^2 + \alpha_1 \varepsilon s + \alpha_2} \begin{bmatrix} \varepsilon \\ \varepsilon s + \alpha_1 \end{bmatrix}$$

$$G_o(s) = \frac{\varepsilon}{(\varepsilon s)^2 + \alpha_1 \varepsilon s + \alpha_2} \begin{bmatrix} \varepsilon \\ \varepsilon s + \alpha_1 \end{bmatrix}$$

Observer eigenvalues are (λ_1/ε) and (λ_2/ε) where λ_1 and λ_2 are the roots of

$$\lambda^2 + \alpha_1 \lambda + \alpha_2 = 0$$

$$\sup_{\omega \in R} \|G_o(j\omega)\| = O(\varepsilon)$$

$$\eta_1 = \frac{\tilde{x}_1}{\varepsilon}, \qquad \eta_2 = \tilde{x}_2$$

$$\varepsilon\dot{\eta}_1 = -\alpha_1\eta_1 + \eta_2, \qquad \varepsilon\dot{\eta}_2 = -\alpha_2\eta_1 + \varepsilon\delta(x,\tilde{x})$$

Ultimate bound of η is $O(\varepsilon)$

η decays faster than an exponential mode $e^{-at/\varepsilon}$, $\quad a > 0$

Peaking Phenomenon:

$$x_1(0) \neq \hat{x}_1(0) \;\; \Rightarrow \;\; \eta_1(0) = O(1/\varepsilon)$$

The solution contains a term of the form $\dfrac{1}{\varepsilon}e^{-at/\varepsilon}$

$\dfrac{1}{\varepsilon}e^{-at/\varepsilon}$ approaches an impulse function as $\varepsilon \to 0$

Example

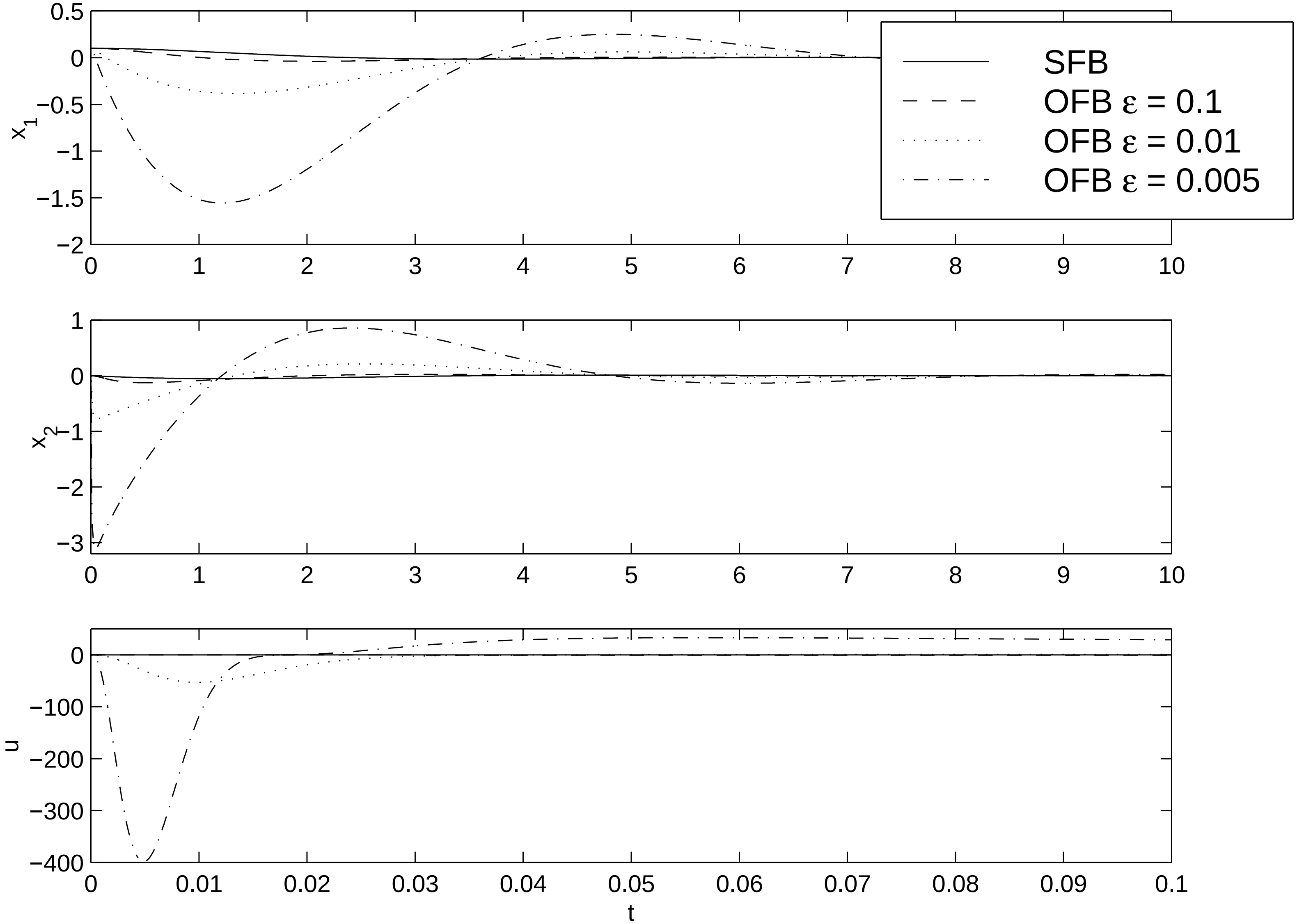
$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_2^3 + u, \qquad y = x_1$$

State feedback control:

$$u = -x_2^3 - x_1 - x_2$$

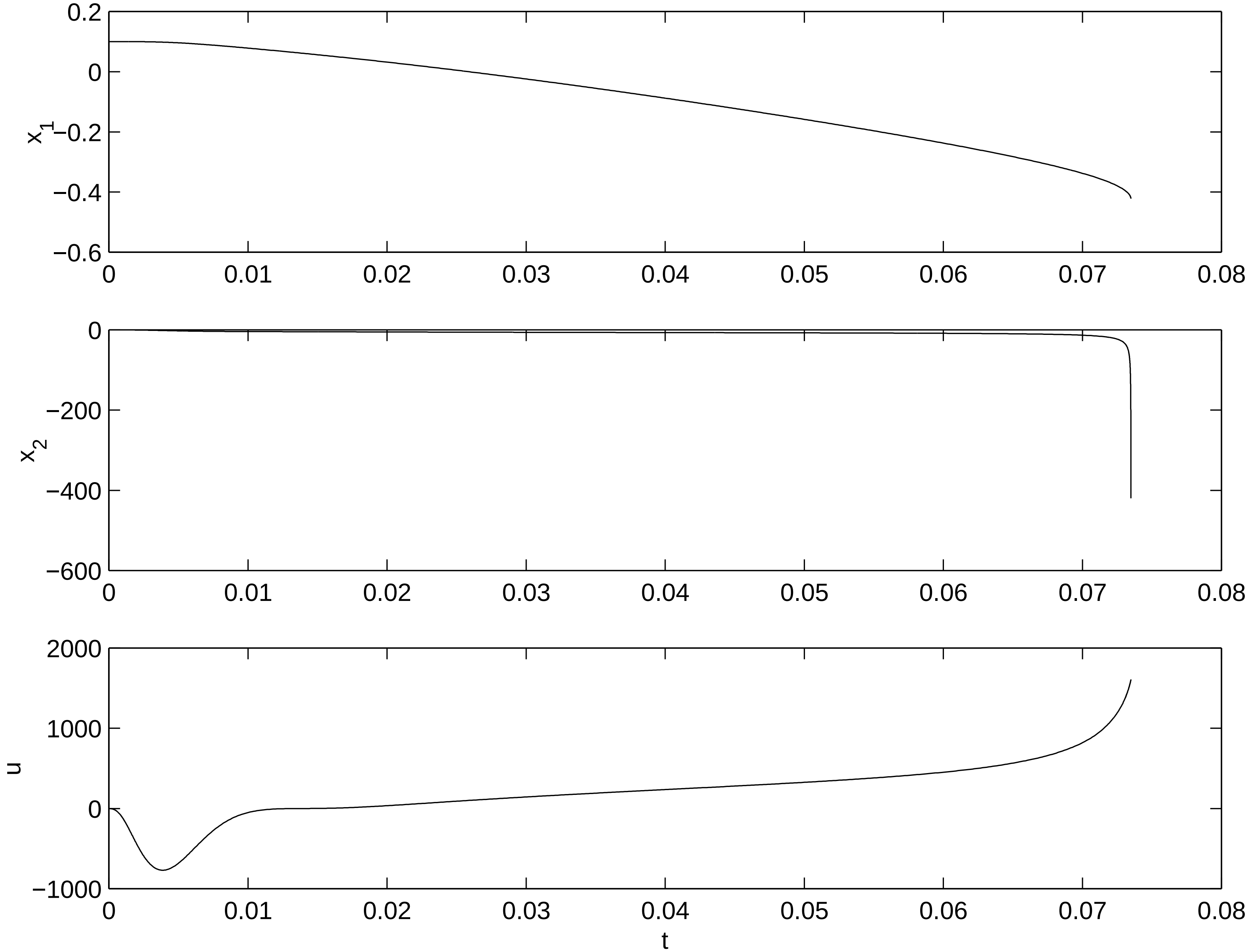
Output feedback control:

$$\begin{aligned} u &= -\hat{x}_2^3 - \hat{x}_1 - \hat{x}_2 \\ \dot{\hat{x}}_1 &= \hat{x}_2 + (2/\varepsilon)(y - \hat{x}_1) \\ \dot{\hat{x}}_2 &= (1/\varepsilon^2)(y - \hat{x}_1) \end{aligned}$$

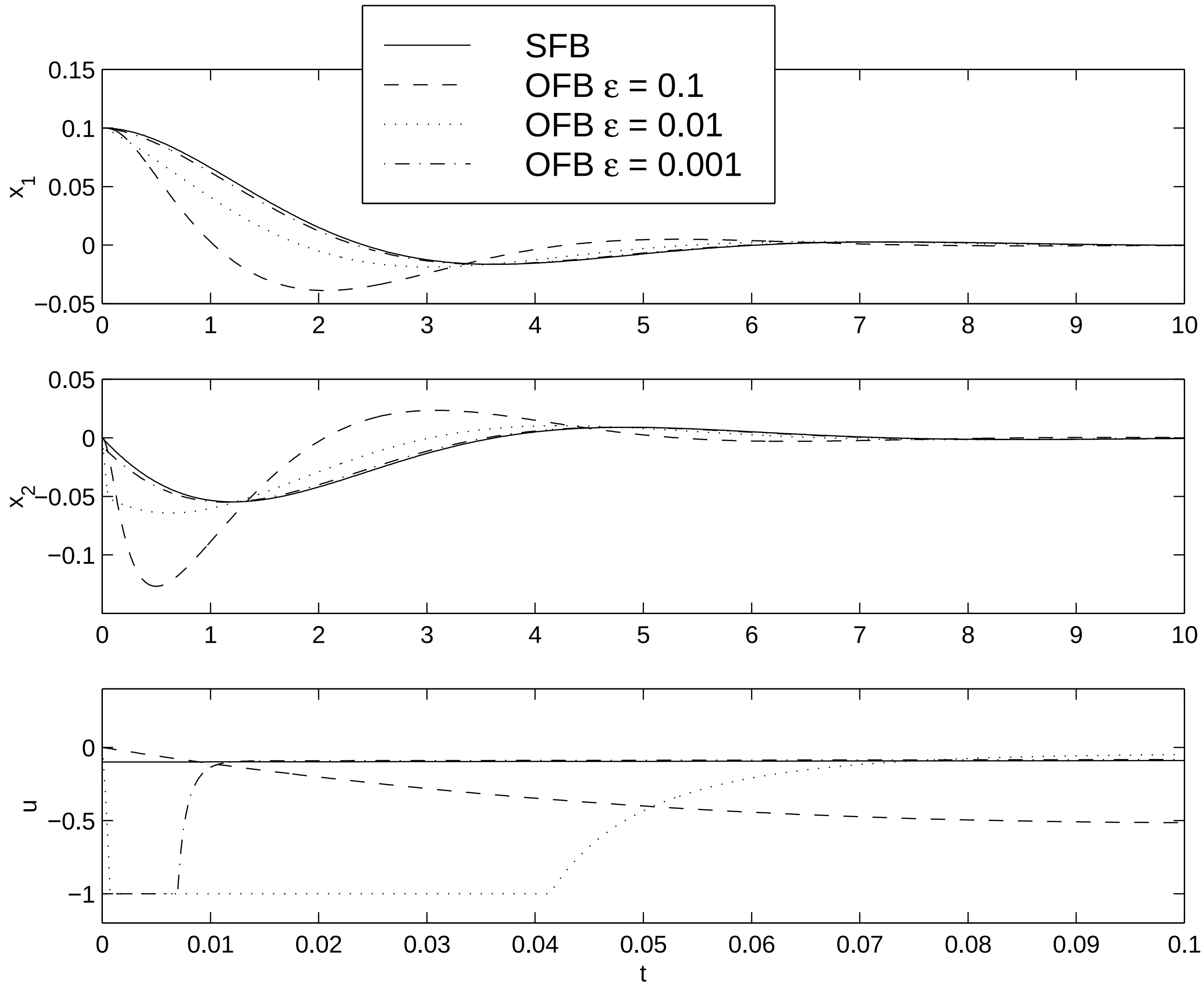


0.5
0
−0.5
−1
−1.5
−2
x_1
0
1
2
3
4
5
6
7
8
9
10
SFB
OFB ε = 0.1
OFB ε = 0.01
OFB ε = 0.005
1
0
−1
−2
−3
x_2
0
1
2
3
4
5
6
7
8
9
10
0
−100
−200
−300
−400
u
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1
t

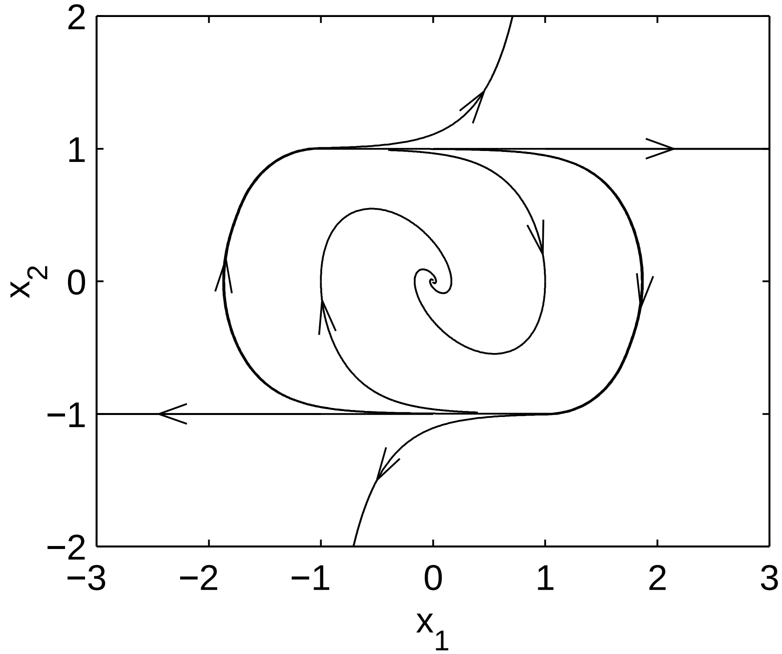
$\varepsilon = 0.004$



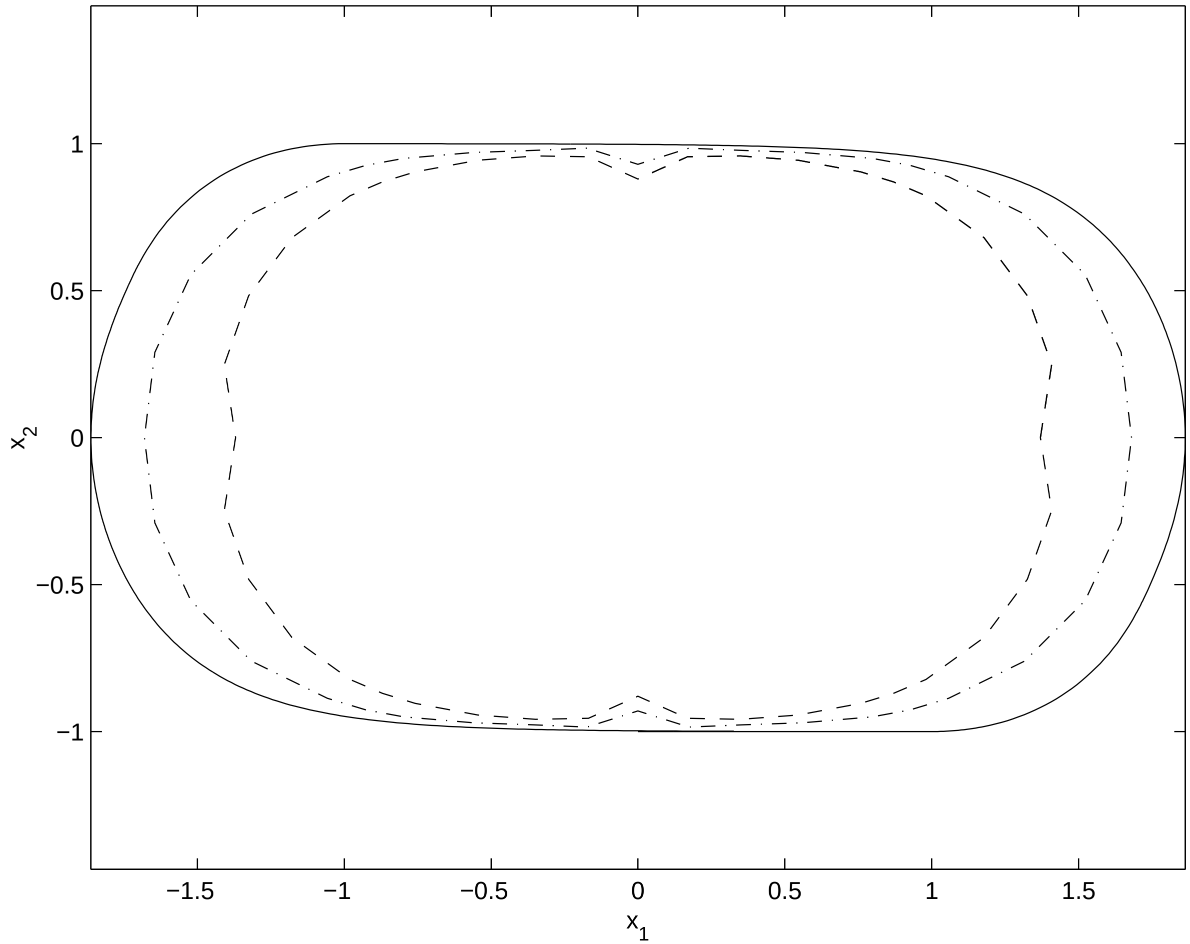
$$u = \mathrm{sat}(-\hat{x}_2^3 - \hat{x}_1 - \hat{x}_2)$$



Region of attraction under state feedback:



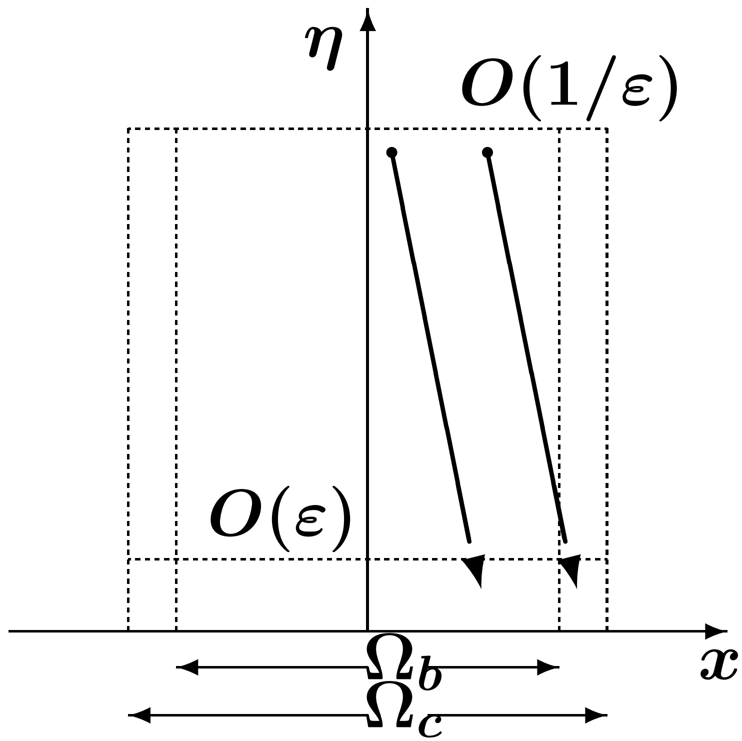
Region of attraction under outputfeedback:



$\varepsilon = 0.1$ (dashed) and $\varepsilon = 0.05$ (dash-dot)

Analysis of the closed-loop system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \varepsilon\dot{\eta}_1 &= -\alpha_1\eta_1 + \eta_2 \end{aligned} \qquad \begin{aligned} \dot{x}_2 &= \phi(x, \gamma(x - \tilde{x})) \\ \varepsilon\dot{\eta}_2 &= -\alpha_2\eta_1 + \varepsilon\delta(x, \tilde{x}) \end{aligned}$$



What is the effect of measurement noise?

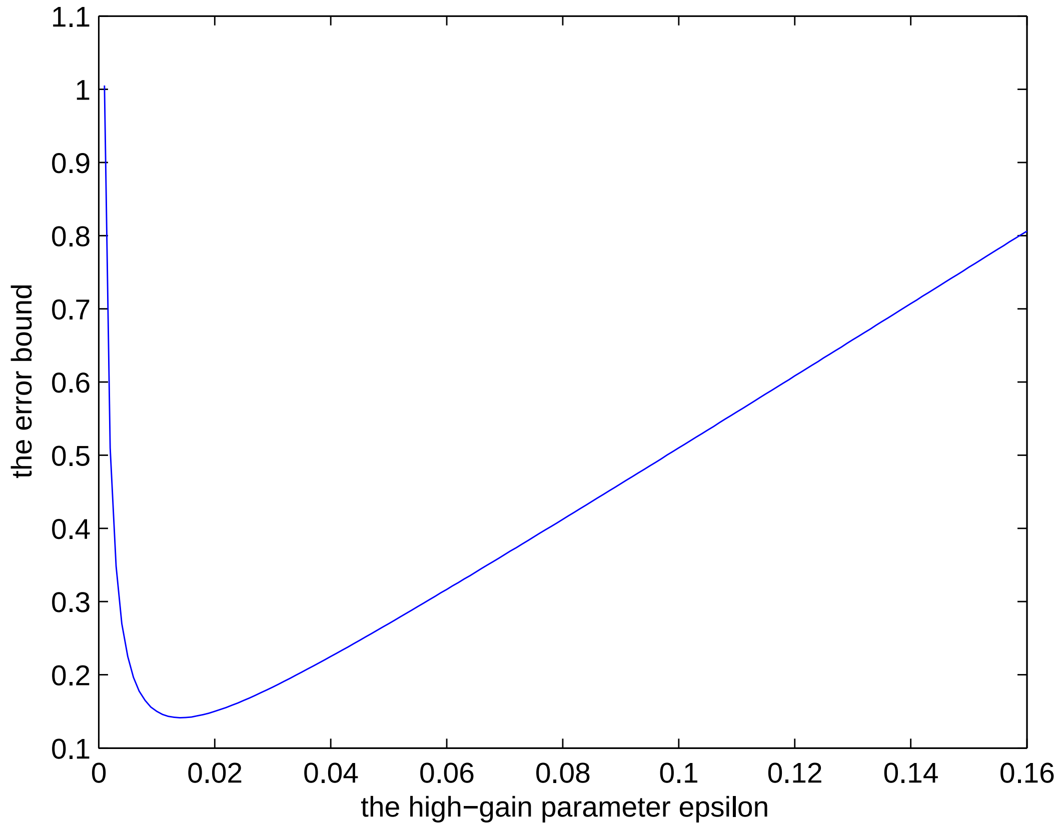
The high-gain observer is an approximate differentiator

Transfer function from y to $\hat{x}$ (with $\phi_0 = 0$):

$$\frac{\alpha_2}{(\varepsilon s)^2 + \alpha_1 \varepsilon s + \alpha_2} \begin{bmatrix} 1 + (\varepsilon \alpha_1/\alpha_2)s \\ s \end{bmatrix} \to \begin{bmatrix} 1 \\ s \end{bmatrix} \quad \text{as } \varepsilon \to 0$$

Differentiation amplifies the effect of measurement noise

$$y = x_1 + v, \qquad k_n = \sup_{t \geq 0} |v(t)| < \infty$$



$$\varepsilon_{opt} = O\left(\sqrt{\frac{k_n}{k_d}}\right), \quad k_d = \sup_{t \geq 0} |\ddot{x}_1(t)|, \quad k_n = \sup_{t \geq 0} |v(t)|$$