Nonlinear Systems and Control
Lecture # 3
Second-Order Systems

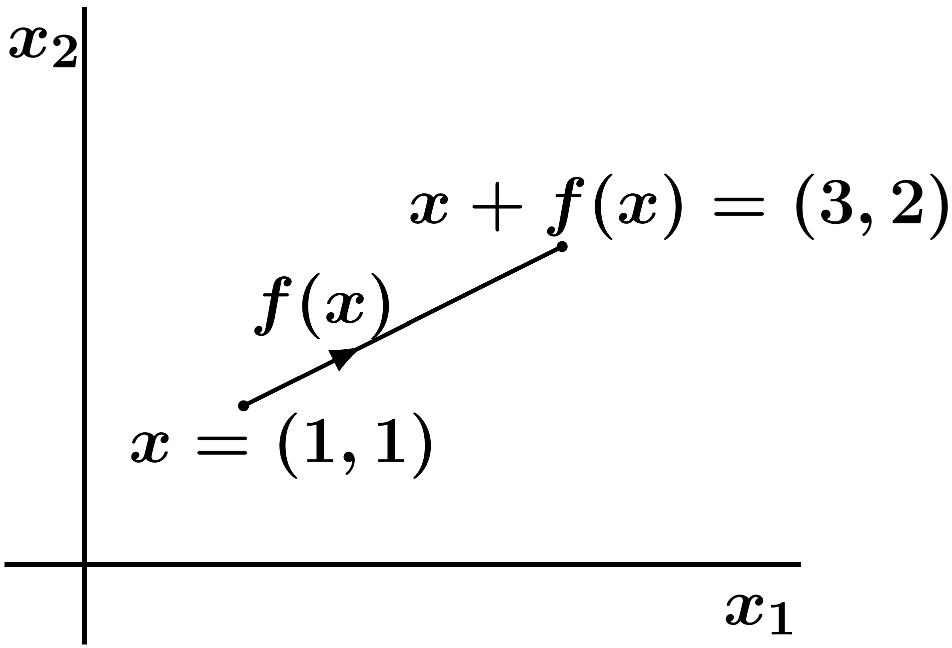
$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2) = f_1(x) \\
\dot{x}_2 &= f_2(x_1, x_2) = f_2(x)
\end{aligned}
$$

Let $x(t) = (x_1(t), x_2(t))$ be a solution that starts at initial state $x_0 = (x_{10}, x_{20})$. The locus in the x_1–x_2 plane of the solution $x(t)$ for all $t \geq 0$ is a curve that passes through the point x_0. This curve is called a *trajectory* or *orbit*
The x_1–x_2 plane is called the *state plane* or *phase plane*
The family of all trajectories is called the *phase portrait*
The *vector field* $f(x) = (f_1(x), f_2(x))$ is tangent to the trajectory at point x because

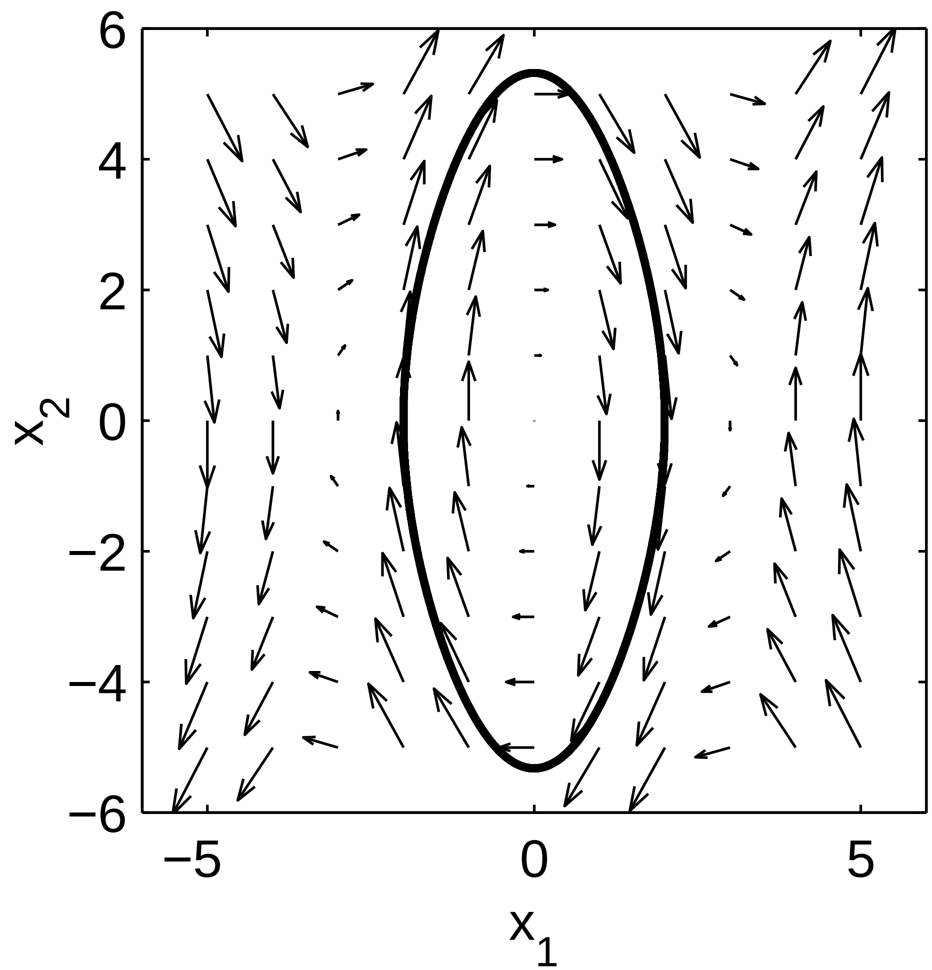
$$
\frac{dx_2}{dx_1} = \frac{f_2(x)}{f_1(x)}
$$

Vector Field diagram

Represent $f(x)$ as a vector based at x; that is, assign to x the directed line segment from x to $x + f(x)$



Repeat at every point in a grid covering the plane



$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -10\sin x_1$$

Numerical Construction of the Phase Portrait:

- Select a bounding box in the state plane
- Select an initial point x_0 and calculate the trajectory through it by solving

$$\dot{x} = f(x), \quad x(0) = x_0$$

in forward time (with positive t) and in reverse time (with negative t)

$$\dot{x} = -f(x), \quad x(0) = x_0$$

- Repeat the process interactively

Use Simulink or pplane

Qualitative Behavior of Linear Systems

$$\dot{x} = Ax, \quad A \text{ is a } 2 \times 2 \text{ real matrix}$$

$$x(t) = M \exp(J_r t) M^{-1} x_0$$

$$J_r = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \text{ or } \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \text{ or } \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \text{ or } \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

$$x(t) = Mz(t)$$

$$\dot{z} = J_r z(t)$$

Case 1. Both eigenvalues are real: $\lambda_1 \neq \lambda_2 \neq 0$

$$M = [v_1, v_2]$$

v_1 & v_2 are the real eigenvectors associated with λ_1 & λ_2

$$\dot{z}_1 = \lambda_1 z_1, \qquad \dot{z}_2 = \lambda_2 z_2$$

$$z_1(t) = z_{10} e^{\lambda_1 t}, \qquad z_2(t) = z_{20} e^{\lambda_2 t}$$

$$z_2 = c z_1^{\lambda_2/\lambda_1}, \quad c = z_{20}/(z_{10})^{\lambda_2/\lambda_1}$$

The shape of the phase portrait depends on the signs of λ_1 and λ_2

$$\boldsymbol{\lambda_2 < \lambda_1 < 0}$$

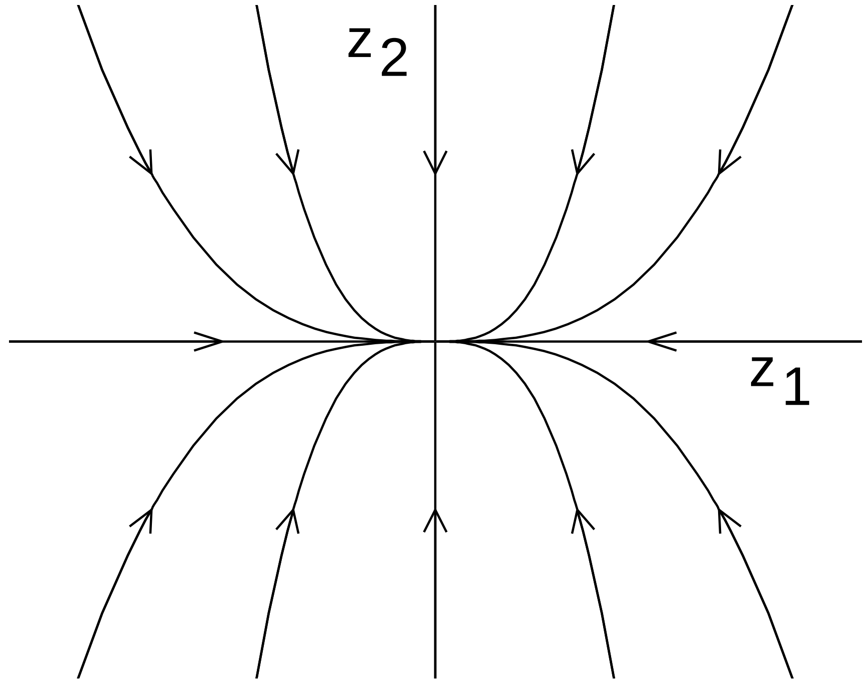
$e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ tend to zero as $t \to \infty$

$e^{\lambda_2 t}$ tends to zero faster than $e^{\lambda_1 t}$

Call λ_2 the fast eigenvalue (v_2 the fast eigenvector) and λ_1 the slow eigenvalue (v_1 the slow eigenvector)

The trajectory tends to the origin along the curve $z_2 = c z_1^{\lambda_2/\lambda_1}$ with $\lambda_2/\lambda_1 > 1$

$$\frac{dz_2}{dz_1} = c \frac{\lambda_2}{\lambda_1} z_1^{[(\lambda_2/\lambda_1)-1]}$$

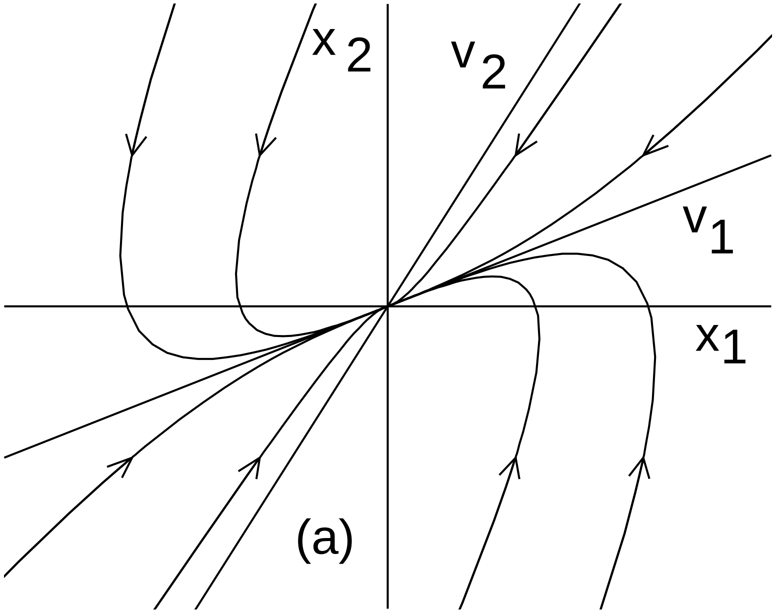


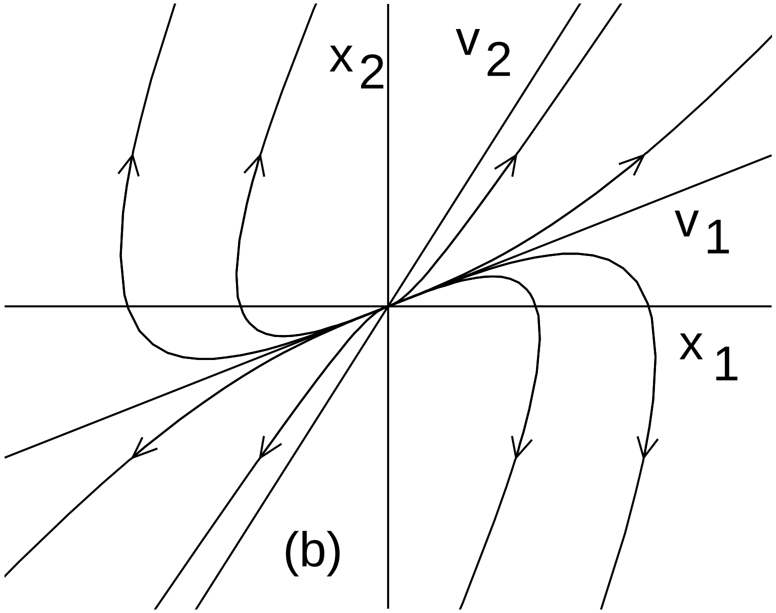
Stable Node

$$\lambda_2 > \lambda_1 > 0$$

Reverse arrowheads

Reverse arrowheads $\Longrightarrow$ Unstable Node





Stable Node

Unstable Node

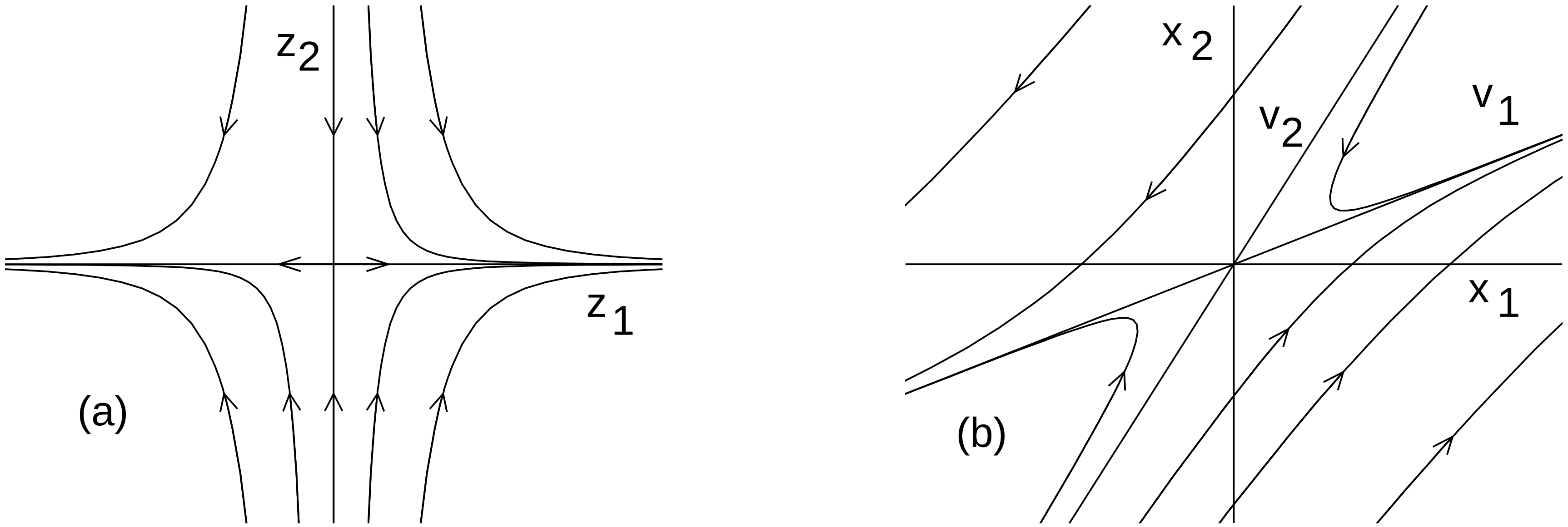
$$\lambda_2 < 0 < \lambda_1$$

$$e^{\lambda_1 t} \longrightarrow \infty, \text{ while } e^{\lambda_2 t} \longrightarrow 0 \text{ as } t \longrightarrow \infty$$

Call λ_2 the stable eigenvalue (v_2 the stable eigenvector) and λ_1 the unstable eigenvalue (v_1 the unstable eigenvector)

$$z_2 = c z_1^{\lambda_2/\lambda_1}, \quad \lambda_2/\lambda_1 < 0$$

Saddle



Phase Portrait of a Saddle Point

Case 2. Complex eigenvalues: $\lambda_{1,2} = \alpha \pm j\beta$

$$\dot{z}_1 = \alpha z_1 - \beta z_2, \qquad \dot{z}_2 = \beta z_1 + \alpha z_2$$

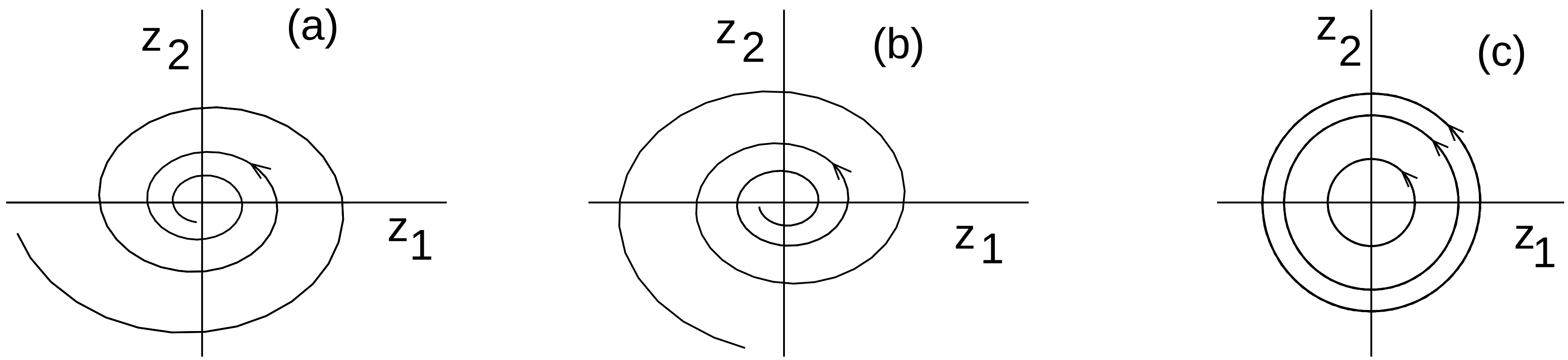
$$r = \sqrt{z_1^2 + z_2^2}, \qquad \theta = \tan^{-1}\left(\frac{z_2}{z_1}\right)$$

$$r(t) = r_0 e^{\alpha t} \quad \text{and} \quad \theta(t) = \theta_0 + \beta t$$

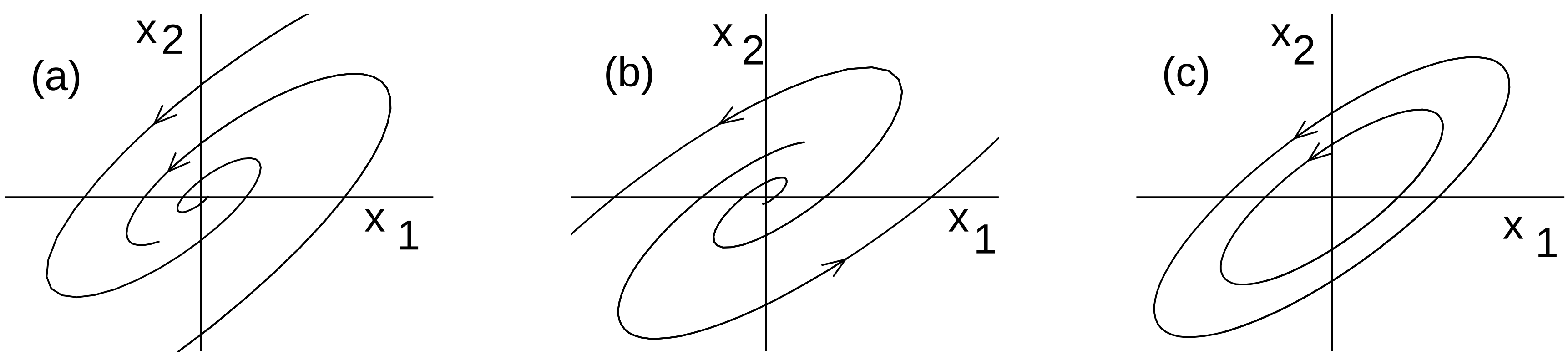
$$\alpha < 0 \quad \Rightarrow \quad r(t) \to 0 \text{ as } t \to \infty$$

$$\alpha > 0 \quad \Rightarrow \quad r(t) \to \infty \text{ as } t \to \infty$$

$$\alpha = 0 \quad \Rightarrow \quad r(t) \equiv r_0 \; \forall \; t$$



$\alpha < 0$	$\alpha > 0$	$\alpha = 0$
Stable Focus	Unstable Focus	Center



Effect of Perturbations

$$\boldsymbol{A} \longrightarrow \boldsymbol{A} + \boldsymbol{\delta A} \quad (\boldsymbol{\delta A} \text{ arbitrarily small})$$

The eigenvalues of a matrix depend continuously on its parameters

A node (with distinct eigenvalues), a saddle or a focus is structurally stable because the qualitative behavior remains the same under arbitrarily small perturbations in $\boldsymbol{A}$

A stable node with multiple eigenvalues could become a stable node or a stable focus under arbitrarily small perturbations in $\boldsymbol{A}$

A center is not structurally stable

$$\begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}$$

$$\text{Eigenvalues} = \mu \pm j$$

$$\mu < 0 \;\Rightarrow\; \text{Stable Focus}$$

$$\mu > 0 \;\Rightarrow\; \text{Unstable Focus}$$