Integrator Forwarding

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Mathematical backgorund

Consider the system in **strict-feedforward form** (*upper-triangular structure*)

$$\dot{\eta} = h(x), \tag{1}$$
$$\dot{x} = f(x) + g(x)u, \tag{2}$$

with $\eta \in \mathbb{R}$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, where $f(\cdot), h(\cdot) \in \mathcal{C}^q$ with $q \in \mathbb{N}_{\geq 1}$ are zero at zero, $g(\cdot)$ is continuous and $g(0) \neq 0$.

Assumption (*Stabilizability* $-$ $x = 0$ GAS+LES)

The Jacobian linearization of (1)-(2) at $(\eta, x) = (0, 0)$ is stabilizable (controllable). The origin of the system (2), with $u \equiv 0$, is GAS and LES with Lyapunov function $V(x)$.

$$\Downarrow$$

There exists a class-$\mathcal{K}$ function $\rho(\cdot)$ that is locally quadratic around the origin and such that

$$\dot{V} = \frac{\partial V(x)}{\partial x} f(x) \leq -\rho(\|x\|), \text{ with } u(t) \equiv 0. \tag{3}$$

Mathematical background

Assumption 1 and the properties of $h(x)$, with $u \equiv 0$, are sufficient conditions for the existence of a map $M(\cdot) : \mathbb{R}^n \to \mathbb{R}$ such that

$$h(x) = \frac{\partial M(x)}{\partial x} f(x), \quad M(0) = 0. \tag{4}$$

Equivalently

$$M(x) = \int_0^\infty h(\hat{x}(\tau))d\tau, \; \hat{x}(0) = x \tag{5}$$

The map $M(\cdot)$ defines implicitly the stable invariant manifold given by the graph $\eta = M(x)$ that is equal to $\lim_{t\to\infty} \eta(t)$ where the auxiliary variable $\hat{x}$ satisfies

$$\dot{\hat{x}} = f(\hat{x}), \; \hat{x}(0) = x. \tag{6}$$

How to compute $M(x)$: solving the PDE with boundary conditions in (4) or the integral in (5) and the differential equation (6).

The algorithm

Define the global change of co-ordinates

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = T(\eta, x) = \begin{bmatrix} \eta - M(x) \\ x \end{bmatrix}, \tag{7}$$

with $z_1 \in \mathbb{R}$, $z_2 \in \mathbb{R}^n$. Note that

$$\frac{\partial T}{\partial(\eta, x)} = \begin{bmatrix} 1 & -\frac{\partial M(x)}{\partial x} \\ 0 & 1 \end{bmatrix}$$

is non singular for all $(\eta, x) \in \mathbb{R}^{n+1}$ (global diffeomorphism).

The algorithm

In the new co-ordinates the system (2)-(1) is rewritten as

$$\dot{z}_1 = h(z_2) - \frac{\partial M(z_2)}{\partial z_2}(f(z_2) + g(z_2)u) = -\frac{\partial M(z_2)}{\partial z_2}g(z_2)u, \tag{8}$$

$$\dot{z}_2 = f(z_2) + g(z_2)u. \tag{9}$$

It is possible to analyze the stability properties of the origin $z = 0$ by the Lyapunov *composed* function

$$W(z_1, z_2) = V(z_2) + z_1^2/2.$$

Note that $u \equiv 0$ yields

$$\dot{W} = \frac{\partial V(z_2)}{\partial z_2}f(z_2) \leq -\rho(\|z_2\|),$$

proving global stability of $z = 0$. Furthermore, since $\rho(\|z_2\|)$ is locally quadratic around the origin, we obtain also that $z_2 \in \mathcal{L}_2$.

The algorithm

To state asymptotic stability of the origin a possible control law u can be retrieved by noting that

$$\begin{aligned}\dot{W} &= \frac{\partial V(z_2)}{\partial z_2}(f(z_2)+g(z_2)u) - z_1\frac{\partial M(z_2)}{\partial z_2}g(z_2)u,\\ &= \frac{\partial V(z_2)}{\partial z_2}f(z_2) + \left(\frac{\partial V(z_2)}{\partial z_2} - z_1\frac{\partial M(z_2)}{\partial z_2}\right)g(z_2)u,\end{aligned}$$

then a possible selection is

$$u = -\left(\frac{\partial V(z_2)}{\partial z_2} - z_1\frac{\partial M(z_2)}{\partial z_2}\right)g(z_2),$$

The algorithm

Theorem (1 (Single-Step Integrator Forwarding))

Consider the system (2)-(1), let Assumption 1 hold and suppose that the mapping $M(x)$ is known. Then the control law

$$u(\eta, x) = -\left(\frac{\partial V(x)}{\partial x} - (\eta - M(x))\frac{\partial M(x)}{\partial x}\right) g(x) \tag{10}$$

renders the origin of the system GAS+LES.

Proof: As first, perform the change of co-ordinates $z = T(\eta, x)$ proposed in (7) that transforms (2)-(1) into (8)-(9).

Consider the Lyapunov function $W(z_1, z_2) = V(z_2) + z_1^2/2$, then

$$\dot{W} = \frac{\partial V(z_2)}{\partial z_2} f(z_2) + \left(\frac{\partial V(z_2)}{\partial z_2} - z_1 \frac{\partial M(z_2)}{\partial z_2}\right) g(z_2) u, \tag{11}$$

$$\leq -\rho(\|z_2\|) - \left(\left(\frac{\partial V(z_2)}{\partial z_2} - z_1 \frac{\partial M(z_2)}{\partial z_2}\right) g(z_2)\right)^2 \tag{12}$$

The algorithm: proof...

Then

$$\begin{aligned}
\dot{W} \leq & -\rho(\|z_2\|) - \left(\frac{\partial V(z_2)}{\partial z_2} g(z_2)\right)^2 + 2\frac{\partial V(z_2)}{\partial z_2} g(z_2) z_1 \frac{\partial M(z_2)}{\partial z_2} g(z_2) + \\
& - \left(z_1 \frac{\partial M(z_2)}{\partial z_2} g(z_2)\right)^2, \\
\leq & -\rho(\|z_2\|) - \left(\frac{\partial V(z_2)}{\partial z_2} g(z_2)\right)^2 + \sigma \left(\frac{\partial V(z_2)}{\partial z_2} g(z_2)\right)^2 + \\
& \frac{1}{\sigma}\left(z_1 \frac{\partial M(z_2)}{\partial z_2} g(z_2)\right)^2 - \left(z_1 \frac{\partial M(z_2)}{\partial z_2} g(z_2)\right)^2, \qquad \text{with } \sigma > 0, \\
\leq & -\rho(\|z_2\|) - (1-\sigma)\left(\frac{\partial V(z_2)}{\partial z_2} g(z_2)\right)^2 - \left(1 - \frac{1}{\sigma}\right)\left(z_1 \frac{\partial M(z_2)}{\partial z_2} g(z_2)\right)^2, \\
\leq & -\rho(\|z_2\|) + \varepsilon \left(\frac{\partial V(z_2)}{\partial z_2} g(z_2)\right)^2 - \frac{\varepsilon}{1+\varepsilon}\left(z_1 \frac{\partial M(z_2)}{\partial z_2} g(z_2)\right)^2, \ (\sigma = 1 + \varepsilon > 1)
\end{aligned}$$

The algorithm: proof...

Since $V(z_2)$ is locally quadratic around the origin $z_2 = 0$, $g(\cdot)$ is continuous and $\varepsilon > 0$ can be taken arbitrarily small, then there exists a locally quadratic $\tilde{\rho} \in \mathcal{K}$ such that

$$-\rho(\|z_2\|) + \varepsilon \left(\frac{\partial V(z_2)}{\partial z_2} g(z_2) \right)^2 \leq -\tilde{\rho}(\|z_2\|) \qquad (13)$$

holds around the origin $z_2 = 0$ yielding

$$\dot{W} \leq -\tilde{\rho}(\|z_2\|) - \frac{\varepsilon}{1+\varepsilon} \left(z_1 \frac{\partial M(z_2)}{\partial z_2} g(z_2) \right)^2 .$$

The algorithm: proof...

Note that a global diffeomorphism does not alter the stabilizability nor controllability properties of the Jacobian linearization around the origin, then since in the new co-ordinates the system is

$$\begin{aligned} \dot{z}_1 &= -\frac{\partial M(z_2)}{\partial z_2} g(z_2) u, \\ \dot{z}_2 &= f(z_2) + g(z_2) u. \end{aligned}$$

it has to hold that

$$\left.\frac{\partial M(z_2)}{\partial z_2}\right|_{z_2=0} g(0) \neq 0. \tag{14}$$

The proof is concluded exploiting La Salle's Invariance Principle yielding the global asymptotic stability of the origin $z = 0$ and then of $(\eta, x) = (0, 0)$. Property (14) yields local exponential stability around the origin since

$$\dot{W} \leq -\tilde{\rho}(\|z_2\|) - z_1^2 \frac{\varepsilon}{1+\varepsilon} \left(\frac{\partial M(z_2)}{\partial z_2} g(z_2) \right)^2 .$$

The algorithm

Remark

Why local exponential stability is important?
If x exponentially goes to zero around the origin, the regularity of $h(x)$ (around the origin) allows to conclude that the integral

$$\int_0^\infty h(\hat{x}(t))\, dt \tag{15}$$

which defines the map $M(x)$, is finite for any value of x (initial condition of $\dot{\hat{x}} = f(\hat{x})$).
However, this requirement is only a sufficient condition to state the boundedness of the above integral (and the existence of such $M(x)$...).

There is no necessity of the subsystem x to be LES if it is known that (15) is bounded. Nevertheless, in case of recursive Integrator Forwarding, boundedness of (15) have to be satisfied at each step.

The algorithm: single-step example

Consider the system

$$\dot{x}_1 = \sin(x_2), \tag{16}$$
$$\dot{x}_2 = x_1 + u, \tag{17}$$

and note that it is not in *strict-feedforward* form as (2)-(1). However, a preliminary control

$$u = -x_1 - x_2 + u_1$$

yields

$$\dot{x}_1 = \sin(x_2), \tag{18}$$
$$\dot{x}_2 = -x_2 + u_1, \tag{19}$$

which is in *strict-feedforward* form and the Laypunov function yielding LES (GES) of the x_2 subsystem can be taken as $V(x_2) = x_2^2/2$. To just apply the formula (22) of the single-step Integrator Forwarding, then set $\eta = x_1$ and $x = x_2$, hence

$$u_1 = -\left(x_2 - (x_1 - M(x_2))\frac{\partial M(x)}{\partial x}\right).$$

The algorithm: single-step example

To evaluate the map $M(x)$ we can proceed solving the PDE

$$i) \qquad h(x_2) = \frac{\partial M(x_2)}{\partial x_2} f(x_2), \quad M(0) = 0.$$

or the integral

$$ii) \qquad M(x_2) = \int_0^{\infty} h(\hat{x}(\tau)) d\tau, \quad \dot{\hat{x}} = -\hat{x},\ \hat{x}(0) = x_2.$$

As first, consider $i)$ that rewrites as

$$\sin(x_2) = -\frac{\partial M(x_2)}{\partial x_2} x_2, \quad M(0) = 0,$$

the solution of which is obtained (via symbolic solver) and is

$$M(x_2) = -\int_0^{x_2} \frac{\sin(t)}{t}\, dt.$$

The algorithm: single-step example

Evaluating the integral $ii)$, that is rewritten as

$$M(x_2) = \int_0^{\infty} \sin(x_2\, e^{-t})\, dt \quad \Leftarrow \quad \hat{x}(t) = x_2\, e^{-t},$$

whose solution lead to

$$M(x_2) = -\int_0^{x_2 e^{-t}} \frac{\sin(\tau)}{\tau}\, d\tau \,\Bigg|_{t=0}^{t=\infty} = -\int_0^{x_2} \frac{\sin(t)}{t}\, dt.$$

Then, since

$$\frac{\partial M(x_2)}{\partial x_2} = -\frac{\sin(x_2)}{x_2},$$

the final control law u_1 is

$$u_1(x_1, x_2) = -\left(x_2 + \left(x_1 + \int_0^{x_2} \frac{\sin(s)}{s}\, ds\right) \frac{\sin(x_2)}{x_2}\right).$$

The algorithm: single-step example

An apparently "trivial" system

$$\dot{x}_1 = \sin(x_2), \quad (20)$$
$$\dot{x}_2 = x_1 + u, \quad (21)$$

has got a non trivial (even for computation) control

$$u(x_1, x_2) = -x_1 - x_2 - \left(x_2 + \left(x_1 + \int_0^{x_2} \frac{\sin(s)}{s}\, ds\right) \frac{\sin(x_2)}{x_2}\right),$$

yielding GAS+LES of the origin.

What if we **now** try to analyze the stability property of the origin?

High gain control?
Suggestion: tray with $u = -x1 - k^2 x_1 - k x_2$ and analyze the stability property of the origin with the function $V(X) = X'PX$, with $X = [x_1, x_2]'$ and

$$P = \begin{bmatrix} k^2 & \frac{k}{2} \\ \frac{k}{2} & 1 \end{bmatrix}.$$

The algorithm: saturated control

Corollary (Saturated control)

In place of the control law (22) in Theorem 1, it is possible to consider the saturated control law

$$u(\eta, x) = -\sigma\left(\left(\frac{\partial V(x)}{\partial x} - (\eta - M(x))\frac{\partial M(x)}{\partial x}\right)g(x)\right), \tag{22}$$

where the nonlinear saturation function $\sigma(\cdot) : \mathbb{R}^p \Rightarrow \mathbb{R}^p$ is continuous and such that

$$\sigma(s)s > 0\,, \forall s \neq 0, \quad \text{and } \sigma(s)s = \|s\|^2 \text{ in a neighbor of } s = 0. \tag{23}$$

Then, the origin of (2)-(1) is GAS+LES.

Proof:

Directly note that

$$\dot{W} = \frac{\partial V(z_2)}{\partial z_2} f(z_2) + \\ - \left(\frac{\partial V(z_2)}{\partial z_2} - z_1 \frac{\partial M(z_2)}{\partial z_2}\right) g(z_2)\sigma\left(\left(\frac{\partial V(z_2)}{\partial z_2} - z_1 \frac{\partial M(z_2)}{\partial z_2}\right) g(z_2)\right),$$

The algorithm: saturated control - proof

Far from the origin it holds that

$$\dot{W} \leq -\rho(\|z_2\|),$$

yielding convergence to zero of z_2, furthermore

$$\left.\frac{\partial V(z_2)}{\partial z_2}\right|_{z_2=0} = 0$$

$$\Downarrow$$

$$-\left(\frac{\partial V(z_2)}{\partial z_2} - z_1 \frac{\partial M(z_2)}{\partial z_2}\right) g(z_2)\sigma\left(\left.\left(\frac{\partial V(z_2)}{\partial z_2} - z_1 \frac{\partial M(z_2)}{\partial z_2}\right) g(z_2)\right)\right|_{z_2=0} =$$

$$= -\left(z_1 \left.\frac{\partial M(z_2)}{\partial z_2}\right|_{z_2=0}\right) g(0)\sigma\left(\left(z_1 \left.\frac{\partial M(z_2)}{\partial z_2}\right|_{z_2=0}\right) g(0)\right) \leq -\gamma(|z_1|)$$

by Assumption 1 for some class-$\mathcal{K}$ function $\gamma(\cdot)$ which implies (by regularity...) that z_1 goes to zero. Then, the trajectories of the z-system enters in finite time within a sufficiently small neighbor of the origin such that $\sigma(s)s = \|s\|^2$ and the same arguments of Theorem 1's proof hold allows to conclude the proof. $\square$

Recursive algorithm: an example

Consider the system in *strict-feedforward form* described by

$$\dot{x}_1 = x_2 + (x_2 - x_3)^2, \tag{24a}$$
$$\dot{x}_2 = x_3, \tag{24b}$$
$$\dot{x}_3 = -2x_3 + u. \tag{24c}$$

STEP 1: Consider the subsystem

$$\dot{x}_2 = x_3, \tag{25a}$$
$$\dot{x}_3 = -2x_3 + u, \tag{25b}$$

define $h(x_3) = x_3$, $f(x_3) = -2x_3$ and find the map $M(x)$ such that

$$h(x_3) = \frac{\partial M(x_3)}{\partial x_3} f(x_3) \to x_3 = -2\frac{\partial M(x_3)}{\partial x_3} x_3,$$

and $M(0) = 0$.
The solution is $M(x_3) = -x_3/2$. Define $z_3 = x_3$, $V(z_3) = z_3^2/2$, $u = u_2$, $z_2 = x_2 - M(x_3) = x_2 + x_3/2$, then

Recursive algorithm: an example

In the new $[z_2, z_3]$ co-ordinates the dynamics of the (x_2, x_3)-subsystem are

$$\dot{z}_2 = -\frac{\partial M(z_3)}{\partial z_3} u_2 = \frac{u_2}{2}, \tag{26a}$$

$$\dot{z}_3 = -2z_3 + u_2, \tag{26b}$$

and the control law of Theorem 1 yields

$$u_2 = -\left(z_3 + \frac{z_2}{2}\right).$$

Let's check which is the derivative of the aggregate Lyapunov function $W_2(z_2, z_3) = z_3^2/2 + z_2^2/2$,

$$\begin{aligned} \dot{W} &= z_3(-2z_3 + u_2) + z_2 \frac{u_2}{2}, \\ &= -2z_3^2 + (z_3 + \frac{z_2}{2})u_2, \\ &= -z_3^2 - \left(z_3 + \frac{z_2}{2}\right)^2, \end{aligned}$$

yielding $(z_2, z_3) = (0, 0)$ GAS+LES.

Recursive algorithm: an example

STEP 2: Add a new row at the top of the previous subsystem performing the partial change of co-ordinates

$$\begin{bmatrix} z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} x_2 + x_3/2 \\ x_3 \end{bmatrix}$$

and letting

$$u_2 = -\left(z_3 + \frac{z_2}{2}\right) + u_1,$$

then

$$\dot{x}_1 = \underbrace{z_2 - \frac{z_3}{2}}_{x_2} + \underbrace{\left(z_2 - \frac{3z_3}{2}\right)^2}_{(x_2 - x_3)^2}, \tag{27a}$$

$$\dot{z}_2 = -\frac{2z_3 + z_2}{4} + u_1, \tag{27b}$$

$$\dot{z}_3 = -2z_3 - \left(z_3 + \frac{z_2}{2}\right) + u_1. \tag{27c}$$

Recursive algorithm: an example

Let

$$h(z_2, z_3) = z_2 - \frac{z_3}{2} + \left(z_2 - \frac{3z_3}{2}\right)^2, \qquad f(z_2, z_3) = \begin{bmatrix} -\frac{2z_3+z_2}{4} \\ -3z_3 - \frac{z_2}{2} \end{bmatrix},$$

and find the map $M(z_2, z_3)$ such that $M(0,0) = 0$ and

$$h(z_2, z_3) = \frac{\partial M(z_2, z_3)}{\partial(z_2, z_3)} f(z_2, z_3),$$

$$\Downarrow$$

$$z_2 - \frac{z_3}{2} + \left(z_2 - \frac{3z_3}{2}\right)^2 = -\frac{\partial M(z_2, z_3)}{\partial z_2} \frac{2z_3 + z_2}{4} - \frac{\partial M(z_2, z_3)}{\partial z_3} \left(3z_3 + \frac{z_2}{2}\right),$$

for which it is really difficult to find out the solution.... however, since (27b)-(27c) is a linear system when $u_1 = 0$, we can tray to evaluate the integral form of $M(z_2, z_3)$.

Recursive algorithm: an example

It holds

$$f(z_2, z_3) = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & -3 \end{bmatrix} \begin{bmatrix} z_2 \\ z_3 \end{bmatrix} := A\, z$$

Then

$$\hat{z}_{23}(t) = e^{At}\, z_{23} \to M(z_{23}) = \int_0^\infty h(\hat{x}(t))dt$$

where $z_{23} = [z_2, z_3]'$, which is a mess but can be computed in closed form....

Integrator forwarding without PDEs, Carnevale and Astolfi, CDC 2009

Let $m(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$, $m(x) = [m_1(x), m_2(x), \ldots, m_n(x)]^\top$

$$h(x) - m(x)^\top f(x) = 0, \quad \text{such that} \quad m(0)^\top g(0) \neq 0.$$

Instead of finding $M(x)$ such that

$$h(x) = L_f M(x),$$

define the map $\mathcal{M}(\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as

$$\mathcal{M}(x, \xi) = \sum_{i=1}^{n} \int_0^{x_i} m_i(\xi) \Big|_{\xi_i = s} ds.$$

Let $e = \xi - x$, then

$$\frac{\partial \mathcal{M}(x, \xi)}{\partial x} = m(x)^\top + e^\top \Delta(x, e),$$

$$e^\top \Delta(x, e) = \frac{\partial \mathcal{M}(x, \xi)}{\partial x} - m(x)^\top = [m_1(x_1, \xi_2, \ldots, \xi_n) - m_1(x_1, x_2, \ldots, x_n), \ldots],$$

$$= \left[\sum_{j=1}^{n} e_j \delta_{1j}(x, e), \sum_{j=1}^{n} e_j \delta_{2j}(x, e), \ldots, \sum_{j=1}^{n} e_j \delta_{nj}(x, e) \right],$$

$$\delta_{ij}(\cdot) : \mathbb{R}^{2n} \to \mathbb{R}, \delta_{ii}(\cdot) \equiv 0.$$

Necessary assumption

Note that

$$\left.\frac{\partial \mathcal{M}(x,0)}{\partial x}\right|_{x=0} g(0) = m(0)^\top g(0) \neq 0.$$

Let $z = y - \mathcal{M}(x,\xi)$, $(y = \eta)$

$$\begin{aligned}\dot{z} &= h(x) - \frac{\partial \mathcal{M}(x,\xi)}{\partial x}(f(x) + g(x)u) - \frac{\partial \mathcal{M}(x,\xi)}{\partial \xi}\dot{\xi}, \\ &= -e^\top \Delta(x,e) f(x) - \frac{\partial \mathcal{M}(x,\xi)}{\partial \xi}\dot{\xi} - \frac{\partial \mathcal{M}(x,\xi)}{\partial x} g(x)u.\end{aligned}$$

The next Assumption is instrumental to prove the main theorems.

Assumption 2: There exist a positive definite function $L(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a function $\gamma(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ such that

$$\text{i)}\ \frac{\partial L(V)}{\partial V}\frac{\partial V(x)}{\partial x} f(x) + \frac{||f(x)||^2}{2\gamma(x)} \leq -\rho(||x||),$$

$$\text{ii)}\ \frac{\partial L(V)}{\partial V}\frac{\partial V(x)}{\partial x} f(x) + \frac{||x||^2}{2\gamma(x)} \leq -\rho(||x||),$$

for some non-decreasing and locally quadratic at the origin function $\rho(\cdot)$.

The main Theorem

Theorem 1 *(only dynamic scaling)*: Consider the system (1)-(2) and assume Assumptions 1 and 2.i hold. Define the change of coordinates $[z, x] = [y - \mathcal{M}(x, \xi), x]$. Select $\dot{\xi} = 0$ and $\xi(0) = 0$ and

$$\dot{r} = \gamma(x)||x^\top \Delta(x, -x)||^2 - \frac{r^2 - 1}{1 + z^2}\rho(||x||), \tag{28}$$

with $r(0) \geq 1$, $\rho(\cdot)$ and $\gamma(x)$ as in the Assumption 2.i, and the control law

$$u = -\left(\frac{\partial L}{\partial V}\frac{\partial V}{\partial x} - \frac{z}{r}\frac{\partial \mathcal{M}(x, 0)}{\partial x}\right) g(x)v, \tag{29}$$

with $v > 0$. Then the origin $(x, z) = (0, 0)$ of the closed-loop system is globally asymptotically stable, $(x, z, u) \in \mathcal{L}^2$ and $r \in \mathcal{L}_\infty$. Moreover, if $L(\cdot)$ is locally quadratic, the origin is locally exponentially stable.

Proof of the main Theorem 1

To avoid burden of notation we assume that $V(x)$ in Assumption 1 satisfies also Assumption 2.i with $L(V) = V$ for some $\gamma(x)$. To analyse the stability property of the origin $(x, z) = (0, 0)$ of the closed-loop system we select the composite Lyapunov function

$$W(x,z)_r = V(x) + \frac{z^2}{2r},$$

through the dynamic scaling[1] r, time derivative along the system trajectories given by

$$\dot{W}_r = \left(\frac{\partial V}{\partial x} + \frac{z}{r} x^\top \Delta(x, -x)\right) f(x) + \left(\frac{\partial V}{\partial x} - \frac{z}{r}\frac{\partial \mathcal{M}(x,0)}{\partial x}\right) g(x)u - \frac{z^2 \dot{r}}{2r^2}. \tag{30}$$

[1] Note that the second term in the rhs of (28) avoids drifting of r in case of measurement noise and, with $r(0) \geq 1$, yields $r \geq 1$.

Proof of Theorem 1 (cont'd)

Using Young's inequality as

$$\frac{z}{r}e^\top \Delta(x,-x)f(x) \leq \frac{1}{2}\left(\frac{\gamma(x)z^2}{r^2}||x^\top \Delta(x,-x)||^2 + \frac{||f(x)||^2}{\gamma(x)}\right),$$

we have

$$\begin{aligned}\dot{W}_r \leq & \frac{\partial V}{\partial x}f(x) + \frac{||f(x)||^2}{2\gamma(x)} + \frac{z^2}{2r^2}\left(\gamma(x)||x^\top \Delta(x,-x)||^2 - \dot{r}\right) \\ & + \left(\frac{\partial V}{\partial x} - \frac{z}{r}\frac{\partial \mathcal{M}(x,0)}{\partial x}\right)g(x)u.\end{aligned}$$

The choice (28) and (29) yield

$$\dot{W}_r \leq -\frac{\rho(||x||)}{2} - \left(\left(\frac{\partial V}{\partial x} - \frac{z}{r}\frac{\partial \mathcal{M}(x,0)}{\partial x}\right)g(x)\right)^2 v, \tag{31}$$

proving that $\rho(||x||) \in \mathcal{L}_1$ and $z/r \in \mathcal{L}_\infty$ (with $r \geq 1$), and $u \in \mathcal{L}^2$.

Proof of Theorem 1 (cont'd)

The boundedness of the scaling factor r follows by

$$\begin{aligned} \dot{r} &= \gamma(x)||x^\top \Delta(x,-x)||^2 - \frac{r^2-1}{1+z^2}\rho(||x||), \\ &\le ||x||^2 \gamma(x)||\Delta(x,-x)||^2, \end{aligned} \tag{32}$$

by the fact that $\rho(||x||)$ is locally quadratic at the origin yields $x \in \mathcal{L}^2$, and by the comparison principle [Khalil, Lemma 3.4] $r \in \mathcal{L}_\infty$.
We conclude that the origin of the closed loop system is globally asymptotically stable given that $\dot{W}_r < 0$ for all $(x, z/r) \neq (0,0)$, *i.e.* , by boundedness of r, $(x,z) \neq (0,0)$ and $z\mathcal{L}\infty$, $z \in \mathcal{L}_2$. When $V(\cdot)$ (or in general $L(\cdot)$) is locally quadratic, locally exponential stability of the origin can be proved using recursively the Young's inequality in (31). ∎

Theorem 2

Theorem 2: Consider the system (1)-(2) and assume Assumptions 1 and 2.ii hold. Define the change of coordinates $[z, x] = [y - \mathcal{M}(x, \xi), x]$. Select the dynamics of r and ξ as

$$\dot{r} = \gamma(x) \left|\left| \frac{\partial \overline{\mathcal{M}}(x, \xi)}{\partial \xi} \dot{\xi} \right|\right|^2 - \frac{r^2 - 1}{1 + z^2} k_r, \tag{33}$$

$$\dot{\xi} = -K_e \xi + \dot{x} + K_e x + \frac{z}{r} \Delta(x, e) f(x), \tag{34}$$

with $r(0) \geq 1$, $k_r > 0$, K_e positive definite, $\rho(\cdot)$ and $\gamma(x)$ as in Assumption 2.ii and the control law

$$u = -\left(\frac{\partial L}{\partial V} \frac{\partial V}{\partial x} - \frac{z}{r} \frac{\partial \mathcal{M}(x, \xi)}{\partial x} \right) g(x), \tag{35}$$

with $v > 0$. Then the origin $(x, z, e) = (0, 0, 0)$ of the closed-loop system is globally asymptotically stable, $(x, e, u) \in \mathcal{L}_2$, and $r \in \mathcal{L}_\infty$. Moreover, if $L(\cdot)$ is locally quadratic, the origin is locally exponentially stable.

Proof of Theorem 2

As in the first Theorem's proof, to avoid burden of notation we assume that $V(x)$ in Assumption 1 satisfies also Assumption 2 with $L(V) = V$ for some $\gamma(x)$. The stability analysis of the origin $(x, z, e) = (0, 0, 0)$ is pursued with the composite Lyapunov function

$$W(x, z, \xi) = V(x) + \frac{z^2}{2r} + \frac{e^\top e}{2},$$

with scaling r, $r \geq 1$ by (28), and with time derivative along the system trajectories

$$\begin{aligned} \dot{W} = &\left(\frac{\partial V}{\partial x} - \frac{z}{r} e^\top \Delta(x, e) \right) f(x) + e^\top \dot{e} - \frac{z^2 \dot{r}}{2r^2} + \\ &\left(\frac{\partial V}{\partial x} - \frac{z}{r} \frac{\partial \mathcal{M}(x, \xi)}{\partial x} \right) g(x) u - \frac{z}{r} \frac{\partial \mathcal{M}(x, \xi)}{\partial \xi} \dot{\xi}. \end{aligned} \tag{36}$$

Proof of Theorem 2 (cont'd)

Using the fact $\mathcal{M}(x,\xi) = x^\top \partial\overline{\mathcal{M}}(x,\xi)/\partial\xi$ and Young's inequality as

$$\begin{aligned} \frac{z}{r}\frac{\partial\mathcal{M}(x,\xi)}{\partial\xi}\dot{\xi} &= x^\top \frac{z}{r}\frac{\partial\overline{\mathcal{M}}(x,\xi)}{\partial\xi}\dot{\xi} \\ &\leq \frac{||x||^2}{2\gamma(x)} + \frac{z^2}{2r^2}\left|\left|\frac{\partial\overline{\mathcal{M}}(x,\xi)}{\partial\xi}\dot{\xi}\right|\right|^2\gamma(x), \end{aligned}$$

yielding

$$\dot{e} = -K_e e + \frac{z}{r}\Delta(x,e)f(x), \tag{37}$$

and

$$\begin{aligned} \dot{W} \leq \frac{\partial V}{\partial x}f(x) + \frac{||x||^2}{2\gamma(x)} + \left(\frac{\partial V}{\partial x} - \frac{z}{r}\frac{\partial\mathcal{M}(x,\xi)}{\partial x}\right)g(x)u \\ + \frac{c_1 k_r}{2} - e^\top K_e e, \end{aligned} \tag{38}$$

with $c_1 \leq 1$.

Proof of Theorem 2 (cont'd)

The selection of u as in (29), K_e positive definite, and selecting

$$k_r = \rho(||x||) + e^\top K_e e + u^2/v, \tag{39}$$

yield

$$\dot{W} \leq -\frac{\left(\rho(||x||) + \left(\left(\frac{\partial V}{\partial x} - \frac{z}{r}\frac{\partial \mathcal{M}(x,\xi)}{\partial x}\right) g(x)\right)^2 v + e^\top K_e e\right)}{2} \tag{40}$$

proving that $\rho(||x||) \in \mathcal{L}_1$, $(e,u) \in \mathcal{L}^2$, and $z/r \in \mathcal{L}_\infty$ (with $r \geq 1$). By (37) and $\rho(||x||)$ locally quadratic at the origin,

$$||\dot{\xi}||^2 \leq 2\left(||\dot{x}||^2 + \frac{z^2}{r^2}||\Delta(x,e)F(x)||^2||x||^2 + ||K_e||^2||e||^2\right)$$

is integrable, yielding $r \in \mathcal{L}_\infty$. We conclude that the origin of the closed loop system is globally asymptotically stable given that $\dot{W} < 0$ for all $(x,e,z/r) \neq (0,0,0)$, *i.e.* , by boundedness of r, $(x,e,z) \neq (0,0,0)$. Local exponential stability of the origin can be proved as in Theorem 1. ■

Saturated control

Remark 1: Within the settings of Theorems 1 and 2, there exists a positive definite function $\sigma(\cdot) : \mathbb{R}^n \to \mathbb{R}_{>}0$ such that with

$$u = \frac{z}{r}\frac{\partial \mathcal{M}(x, \xi)}{\partial x} g(x)\sigma(x)v, \tag{41}$$

the results of Theorems 1 and 2 hold, respectively.

To meet actuator constraints it is also possible to implement the control law

$$u = \text{sat}\left(\frac{z}{r}\frac{\partial \mathcal{M}(x, \xi)}{\partial x} g(x)\sigma(x)\right), \tag{42}$$

The benchmark example

[P. V. Kokotovic, I. Kanellakopoulos and A. S. Morse. *Foundations of Adaptive Control*, Springer-Verlag, 1991] (with a preliminary control $u_p = -x_1 - 2x_2$ as in Sepulchre et all, AUT97). Consider the system

$$\begin{cases} \dot{y} & = x_1 + (x_1 - x_2)^2, \\ \dot{x}_1 & = x_2, \\ \dot{x}_2 & = -x_1 - 2x_2 + u. \end{cases} ,$$

and the quadratic Lyapunov function $V = x^\top P x$,

$$P = \frac{1}{2} \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix},$$

with $\dot{V} = -x_1^2 - x_2^2$. The manifold $\mathcal{M}(x)$ is given by

$$x_1 + (x_1 - x_2)^2 = \frac{\partial \mathcal{M}(x)}{\partial x} [x_2, \, -x_1 - 2x_2]^\top .$$

We select the approximated (algebraic) solution $m(x)$ as (inconsistent PDE)!

$$m(x)^\top = [-2 + x_2 - 4x_1, \, -1 - x_1], \qquad (m(0)^\top g(0) \neq 0)$$

The benchmark example (cont'd)

Then $\mathcal{M}(x, \xi) = (\xi_2 - 2)x_1 - 2x_1^2 - (\xi_1 + 1)x_2$ and

$$\frac{\partial \mathcal{M}(x, \xi)}{\partial x} = [\xi_2 - 2 - 4x_1, -\xi_1 - 1], \quad -x^\top \Delta(x, -x) = [-x_2, x_1].$$

Let $L(V) = (7/5 + \mu)V$ in Assumption 2.i, with $\mu > 0$, and $\gamma(x) = 1$, then $\rho(||x||) = \mu\,(x_1^2 + x_2^2)$. By Theorem 1, $z = y - (-2x_1 - 2x_1^2 - x_2)$ and

$$\begin{aligned} \dot{r} &= x_1^2 + x_2^2 - \frac{r^2 - 1}{1 + z^2}\rho(||x||), \\ u &= -\left((7/5 + \mu)\,(x_1 + 2x_2) + \frac{z}{r}\right) v. \end{aligned}$$

Results are compared with the control law

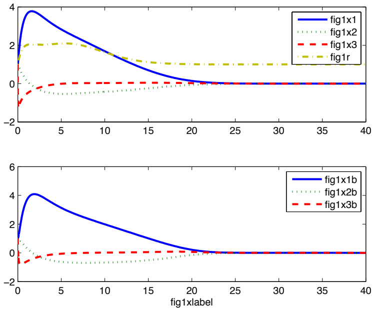
$$u = -(1 + 2x_2 + x_1)(y + 2x_1 + x_2 + (x_1 + x_2)^2/2 + 2x_1^2). \qquad (43)$$

The benchmark example (cont'd)

Note that

$$M(x) - \mathcal{M}(x,0) = (x_1 + x_2)^2/2.$$

To steer the system closer to the one given by the SJK feedback the following parameters have been chosen: $v = 10$, $\mu = 0.5$, with $\xi_i(0) = 0$ and $r(0) = 1$.



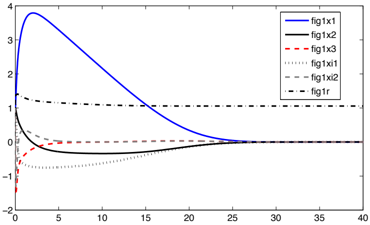
Simulation results for the benchmark system: control law of Theorem 1 (top), and (43) (bottom).

The benchmark example (cont'd)

Theorem 2: Initial conditions has been selected as $\xi_i(0) = 1$, $\gamma(x) = 0.01$, $K_e = 5I_{2\times 2}$, $k_r = \rho(||x||) + e^\top K_e e + u^2/v$ (suggested by the proof of the Theorem),

$$\frac{\partial \mathcal{M}(x,\xi)}{\partial \xi} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \Delta(x,e) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

In this case there are not improvements considering the control law of Theorem 2 with respect to the previous one.



Simulation results for the benchmark system: the control law of Theorem 2.