Chapter 15

Steiner forest

In this chapter, we will obtain a factor 2 approximation algorithm for the Steiner forest problem using the primal-dual schema. An interesting feature of this algorithm is the manner in which dual complementary slackness conditions will be relaxed. The Steiner forest problem generalizes the metric Steiner tree problem, for which a factor 2 algorithm was presented in Chapter 3. Recall, however, that we had postponed giving the lower bounding method behind that algorithm; we will clarify this as well.

Problem 15.1 (Steiner forest) Given a graph $G = (V, E)$, a cost function on edges $c : E \to \mathbf{Q}^+$ (not necessarily satisfying the triangle inequality), and a collection of disjoint subsets of V, $S_1, \ldots S_k$, find a minimum cost subgraph in which each pair of vertices belonging to the same set S_i is connected.

Exercise 15.2 Show that there is no loss of generality in requiring that the edge costs satisfy the triangle inequality for the above problem. (The reasoning is the same as that for the Steiner tree problem.)

Let us restate the problem; this will also help generalize it later. Define a *connectivity requirement function* r that maps unordered pairs of vertices to $\{0, 1\}$ as follows:

$$r(u, v) = \begin{cases} 1 & \text{if } u \text{ and } v \text{ belong to the same set } S_i \\ 0 & \text{otherwise} \end{cases}$$

Now, the problem is to find a minimum cost subgraph F that contains a u–v path for each pair (u, v) with $r(u, v) = 1$. The solution will be a forest, in general.

In order to give an integer programming formulation for this problem, let us define a function on all cuts in G, $f : 2^V \to \{0, 1\}$, which specifies the minimum number of edges that must cross each cut in any feasible solution.

$$f(S) = \begin{cases} 1 & \text{if } \exists\ u \in S \text{ and } v \in \overline{S} \text{ such that } r(u, v) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Let us also introduce a 0/1 variable x_e for each edge $e \in E$; x_e will be set to 1 iff e is picked in the subgraph. The integer program is:

$$\text{minimize} \quad \sum_{e \in E} c_e x_e \tag{15.1}$$

$$\begin{array}{lll} \text{subject to} & \displaystyle\sum_{e:\ e\in\delta(S)} x_e \geq f(S), & S \subseteq V \\ & x_e \in \{0,1\}, & e \in E \end{array}$$

where $\delta(S)$ denotes the set of edges crossing the cut $(S, \overline{S})$.

Exercise 15.3 Show, using the max-flow min-cut theorem, that a subgraph has all the required paths iff it does not violate any of the cut requirements. Use this fact to show that (15.1) is an integer programming formulation for the Steiner forest problem.

Following is the LP-relaxation of (15.1); once again, we have dropped the redundant conditions $x_e \leq 1$.

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{e\in E} c_e x_e & \\ \text{subject to} & \displaystyle\sum_{e:\ e\in\delta(S)} x_e \geq f(S), & S \subseteq V \\ & x_e \geq 0, & e \in E \end{array} \tag{15.2}$$

The dual program is:

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{S\subseteq V} f(S)\cdot y_S & \\ \text{subject to} & \displaystyle\sum_{S:\ e\in\delta(S)} y_S \leq c_e & e \in E \\ & y_S \geq 0 & S \subseteq V \end{array} \tag{15.3}$$

Notice that the primal and dual programs form a covering and packing pair of LP's; see Chapter 12 for this notion. Some figurative terminology will help describe the algorithm more easily. Let us say that edge e *feels* dual y_S if $y_S > 0$ and $e \in \delta(S)$. Say that set S has been *raised* in a dual solution if $y_S > 0$. Clearly, raising S or $\overline{S}$ has the same effect. So, sometimes we will also say that we have raised the cut $(S, \overline{S})$. Further, there is no advantage in raising set S with $f(S) = 0$, since this does not contribute to the dual objective function. So, we may assume that such cuts are never raised. Say that edge e is *tight* if the total amount of dual it feels equals its cost. The dual program is trying to maximize the sum of the dual variables y_S subject to the condition that no edge feels more dual than its cost, i.e., is not *over-tight*.

Next, let us state the primal and relaxed dual complementary slackness conditions. The algorithm will pick edges integrally only. Define the *degree of set* S to be the number of picked edges crossing the cut $(S, \overline{S})$.

Primal conditions: For each $e \in E$, $x_e \neq 0 \Rightarrow \sum_{i:\ e\in\delta(S)} y_S = c_e$.
Equivalently, *every picked edge must be tight.*

Relaxed dual conditions: The following relaxation of the dual conditions would have led to a factor 2 algorithm:
For each $S \subseteq V, y_S \neq 0 \Rightarrow \sum_{e:\ e\in\delta(S)} x_e \leq 2\cdot f(S)$, i.e., every raised cut has degree at most 2. Clearly, each cut $(S, \overline{S})$ with $f(S) = 1$ must have degree at least one, just to ensure feasibility. We do not know how to enforce this relaxation of the dual condition. Interestingly enough, we can still obtain a factor 2 algorithm – by relaxing this condition further! Raised sets will be allowed to

have high degree; however, we will ensure that *on average, raised duals have degree at most 2.* The exact definition of "on average" will be given later.

The algorithm starts with no edges picked and no cuts raised. In the spirit of the primal-dual schema, the current primal solution indicates which cuts need to be raised, and in turn, the current dual solution indicates which edge needs to be picked. Thus, the algorithm iteratively improves the feasibility of the primal, and the optimality of the dual, until a feasible primal is obtained.

Let us describe what happens in an iteration. In any iteration, the picked edges form a forest. Say that set S is *unsatisfied* if $f(S) = 1$, but there is no picked edge crossing the cut $(S, \overline{S})$. Set S is said to be *active* if it is a minimal (w.r.t. inclusion) unsatisfied set in the current iteration.

Lemma 15.4 *Set S is active iff it is a connected component in the currently picked forest and $f(S) = 1$.*

Proof : Let S be an active set. Now, S cannot cross a connected component because otherwise there will already be a picked edge in the cut $(S, \overline{S})$. So, S is a union of connected components. Since $f(S) = 1$, there is a vertex $u \in S$ and $v \in \overline{S}$ such that $r(u, v) = 1$. Let S' be the connected component containing u. Clearly, S' is also unsatisfied, and by minimality of S, $S = S'$. □

By the the characterization of active sets given in Lemma 15.4, it is easy to find all active sets in the current iteration. The dual variables of these sets are raised simultaneously until some edge goes tight. Any one of the newly tight edges is picked, and the current iteration terminates.

When a primal feasible solution is found, say F, the edge augmentation step terminates. However, F may contain redundant edges, which need to be pruned for achieving the desired approximation factor; this is illustrated in Example 15.6. Formally, edge $e \in F$ is said to be *redundant* if $F - \{e\}$ is also a feasible solution. All redundant edges can be dropped simultaneously from F. Equivalently, only non-redundant edges are retained.

This algorithm is presented below. We leave its efficient implementation as an exercise.

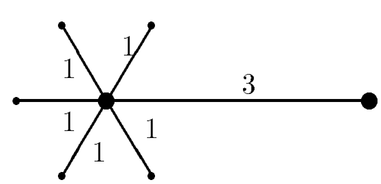
Algorithm 15.5 (Steiner forest)

1. **(Initiallization)** $F \leftarrow \emptyset$; for each $S \subseteq V$, $y_S \leftarrow 0$.
2. **(Edge augmentation)** while there exists an unsatisfied set do:

   simultaneously raise y_S for each active set S, until some edge e goes tight;

   $F \leftarrow F \cup \{e\}$.
3. **(Pruning)** return $F' = \{e \in F |\ F - \{e\}$ is primal infeasible$\}$

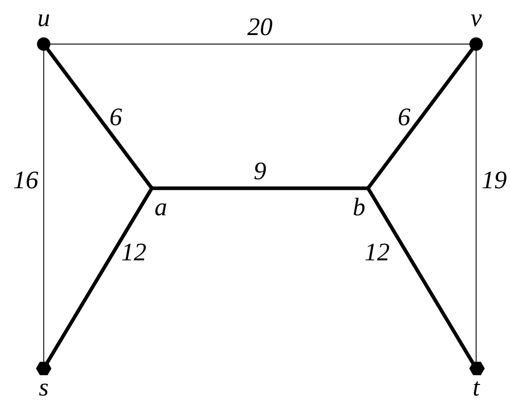
Example 15.6 Consider a star in which all edges have cost 1, except one edge whose cost is 3.



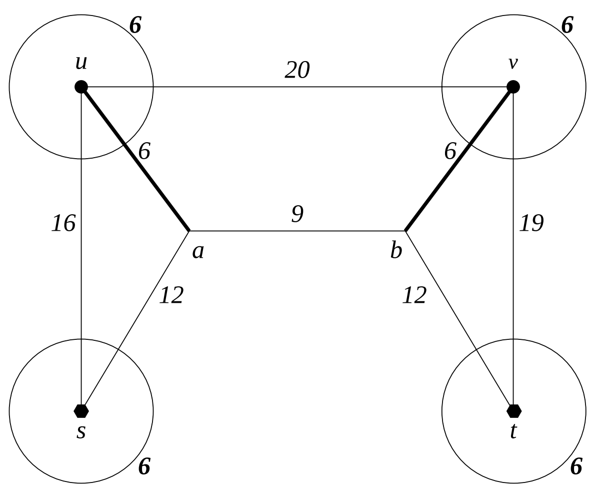
The only requirement is to connect the end vertices of the edge of cost 3. The algorithm will add to F all edges of cost 1 before adding the edge of cost 3. Clearly, at this point, F is not within twice the optimal. However, this will be corrected in the pruning step when all edges of cost 1 will be removed. □

Let us run the algorithm on a non-trivial example to illustrate its finer points.

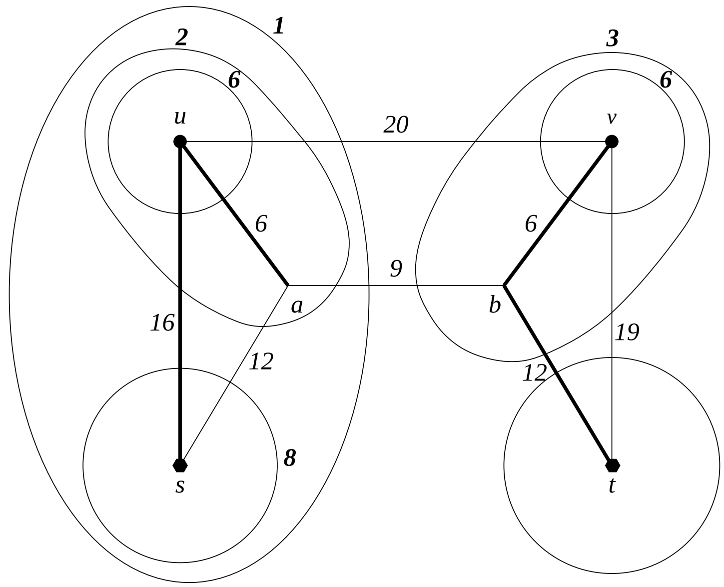
Example 15.7 Consider the following graph. Costs of edges are marked, and the only non-zero connectivity requirements are $r(u,v) = 1$ and $r(s,t) = 1$. The thick edges indicate an optimal solution, of cost 45.



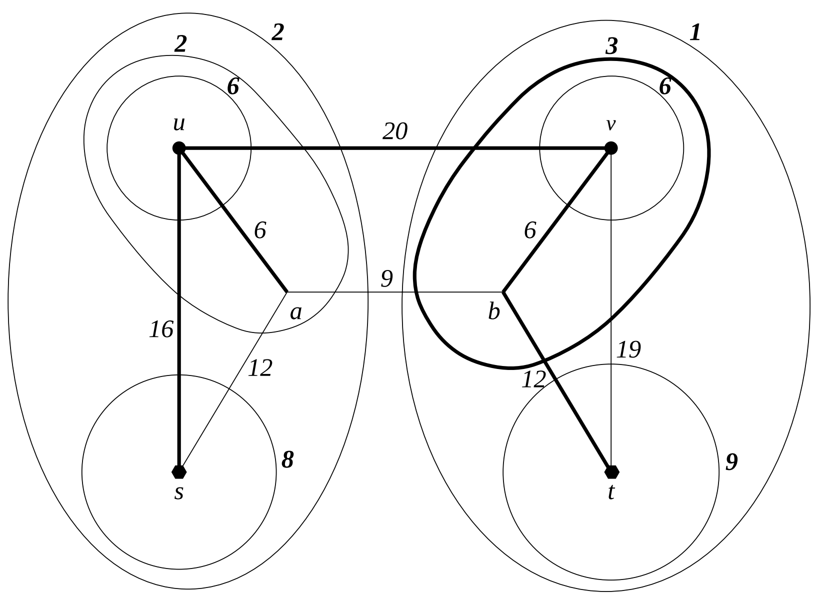
In the first iteration, the following four singleton sets are active: $\{s\}$, $\{t\}$, $\{u\}$, and $\{v\}$. When their dual variables are raised to 6 each, edges (u,a) and (v,b) go tight. One of them, say (u,a) is picked, and the iteration ends. In the second iteration, $\{u,a\}$ replaces $\{u\}$ as an active set. However, in this iteration there is no need to raise duals, since there is already a tight edge, (v,b). This edge is picked, and the iteration terminates. The primal and dual solutions at this point are shown below, with picked edges marked thick:



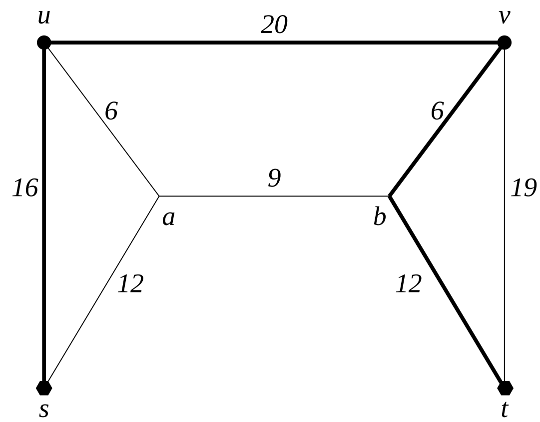
In the third iteration, $\{v,b\}$ replaces $\{v\}$ as an active set. When the active sets are raised by 2 each, edge (u,s) goes tight and is picked. In the fourth iteration, the active sets are $\{u,s,a\}$, $\{v\}$ and $\{t\}$. When they are raised by 1 each, edge (b,t) goes tight and is picked. The situation now is:



In the fifth iteration, the active sets are $\{a, s, u\}$ and $\{b, v, t\}$. When they are raised by 1 each, (u, v) goes tight, and we now have a primal feasible solution:



In the pruning step, edge (u, a) is deleted, and we obtain the following solution of cost 54:



□

In Lemma 15.8 we will show that *simultaneously* deleting all redundant edges still leaves us with a primal feasible solution, i.e., it is never the case that two edges e and f are both redundant individually, but on deletion of e, f becomes non-redundant.

Lemma 15.8 *At the end of the algorithm, F' and $\boldsymbol{y}$ are primal and dual feasible solutions respectively.*

Proof : At the end of Step 2, F satisfies all connectivity requirements. In each iteration, dual variables of connected components only are raised. Therefore, no edge running within the same component can go tight, and so F is acyclic, i.e., it is a forest. Therefore, if $r(u,v)=1$, there is a *unique* u–v path in F. So, each edge on this path in non-redundant and is not deleted in Step 3. Hence F' is primal feasible.

Since whenever an edge goes tight the current iteration ends and active sets are redefined, no edge is overtightened. Hence $\boldsymbol{y}$ is dual feasible. □

Let $\deg_{F'}(S)$ denote the number of edges of F' crossing the cut $(S,\overline{S})$. The characterization of degrees of satisfied components established in the next lemma will be used crucially in proving the approximation guarantee of the algorithm.

Lemma 15.9 *Consider any iteration of the algorithm, and let C be a component w.r.t. the currently picked edges. If $f(C)=0$ then $\deg_{F'}(C)\neq 1$.*

Proof : Suppose $\deg_{F'}(C)=1$, and let e be the unique edge of F' crossing the cut $(C,\overline{C})$. Since e is non-redundant (every edge in F' is non-redundant), there is a pair of vertices, say u,v, such that $r(u,v)=1$ and e lies on the unique u–v path in F'. Since this path crosses the cut $(C,\overline{C})$ exactly once, one of these vertices must lie in C and the other in $\overline{C}$. But since $r(u,v)=1$, we get that $f(C)=1$, thus leading to a contradiction. □

Lemma 15.10 $$\sum_{e\in F'} c_e \leq 2\sum_{S\subseteq V} y_S$$

Proof : Since every picked edge is tight,

$$\sum_{e\in F'} c_e = \sum_{e\in F'}\ \sum_{S:\ e\in\delta(S)} y_S$$

Changing the order of summation we get:

$$\sum_{e \in F'} c_e = \sum_{S \subseteq V} \sum_{e \in \delta(S) \cap F'} y_S = \sum_{S \subseteq V} \deg_{F'}(S) \cdot y_S,$$

So, we need to show that

$$\sum_{S \subseteq V} \deg_{F'}(S) \cdot y_S \leq 2 \sum_{S \subseteq V} y_S.$$

We will prove a stronger claim: that in each iteration, the increase in the l.h.s. of this inequality is bounded by the increase in the r.h.s. Consider an iteration, and let Δ be the extent to which active sets were raised in this iteration. Then, we need to show:

$$\Delta \times \left(\sum_{S \text{ active}} \deg_{F'}(S) \right) \leq 2\Delta \times (\text{\# of active sets})$$

Notice that the degree w.r.t. F' of any active set S is due to edges that will be picked *during or after* the current iteration. Let us rewrite this inequality as follows:

$$\frac{\sum_{S \text{ active}} \deg_{F'}(S)}{\text{\# of active sets}} \leq 2. \tag{15.4}$$

Thus we need to show that in this iteration, the average degree of active sets with respect to F' is at most 2. The mechanics of the argument lies in the fact that in a tree, or in general in a forest, the average degree of vertices is at most 2.

Let H be a graph on vertex set V and edge set F'. Consider the set of connected components w.r.t. F at the beginning of the current iteration. In H, shrink the set of vertices of each of these components to a single node, to obtain graph H' (we will call the vertices of H' as nodes for clarity). Notice that in going from H to H', all edges picked in F before the current iteration have been shrunk. Clearly, the degree of a node in H' is equal to the degree of the corresponding set in H. Let us say that a node of H' corresponding to an active component is an *active node*; any other node will be called *inactive*. Each active node of H' has non-zero degree (since there must be an edge incident to it to satisfy its requirement), and H' is a forest. Now, remove all isolated nodes from H'. The remaining graph is a forest with average degree at most 2. By Lemma 15.9 the degree of each inactive node in this graph is at least 2, i.e., the forest has no inactive leaves. Hence, the average degree of active nodes is at most 2. □

Observe that the proof given above is essentially a charging argument: for each active node of degree greater than 2, there must be correspondingly many active nodes of degree one, i.e., leaves, in the forest. The exact manner in which the dual conditions have been relaxed must also be clear now: in each iteration, the duals being raised have average degree at most 2. Lemmas 15.8 and 15.10 give:

Theorem 15.11 *Algorithm 15.5 achieves an approximation guarantee of factor 2 for the Steiner forest problem.*

The tight example given for the metric Steiner tree problem, Example 3.3, is also a tight example for this algorithm.