Nonlinear Systems and Control Lecture # 33

Robust Stabilization

Sliding Mode Control

Regular Form:

$$egin{array}{ll} \dot{\eta} &=& f_a(\eta, \xi) \ \dot{\xi} &=& f_b(\eta, \xi) + g(\eta, \xi) u + \delta(t, \eta, \xi, u) \ && \eta \in R^{n-1}, \; \xi \in R, \; u \in R \ && f_a(0, 0) = 0, \; f_b(0, 0) = 0, \; g(\eta, \xi) \geq g_0 > 0 \end{array}$$

Sliding Manifold:

$$s = \xi - \phi(\eta) = 0, \quad \phi(0) = 0$$

$$s(t) \equiv 0 \implies \dot{\eta} = f_a(\eta, \phi(\eta))$$

Design ϕ s.t. the origin of $\dot{\eta} = f_a(\eta, \phi(\eta))$ is asymp. stable

$$egin{aligned} \dot{s} &= f_b(\eta, \xi) - rac{\partial \phi}{\partial \eta} f_a(\eta, \xi) + g(\eta, \xi) u + \delta(t, \eta, \xi, u) \ u &= -rac{1}{\hat{g}} \left(\hat{f}_b - rac{\partial \phi}{\partial \eta} \hat{f}_a
ight) + v \quad \emph{or} \quad u = v \ u &= -L \left(\hat{f}_b - rac{\partial \phi}{\partial \eta} \hat{f}_a
ight) + v, \quad L = rac{1}{\hat{g}} \quad \emph{or} \quad L = 0 \ \dot{s} &= g(\eta, \xi) v + \Delta(t, \eta, \xi, v) \ \Delta &= f_b - rac{\partial \phi}{\partial \eta} f_a + \delta - gL \left(\hat{f}_b - rac{\partial \phi}{\partial \eta} \hat{f}_a
ight) \ \left| rac{\Delta(t, \eta, \xi, v)}{g(\eta, \xi)}
ight| \leq arrho(\eta, \xi) + \kappa_0 |v| \end{aligned}$$

$$egin{aligned} \left| rac{\Delta(t,\eta,\xi,v)}{g(\eta,\xi)}
ight| &\leq arrho(\eta,\xi) + \kappa_0 |v| \ & arrho(\eta,\xi) \geq 0, \quad 0 \leq \kappa_0 < 1 \quad (Known) \ & s\dot{s} = sgv + s\Delta \leq sgv + |s| \mid \Delta \mid \ & s\dot{s} \leq g[sv + |s|(arrho + \kappa_0 |v|)] \ & v = -eta(\eta,\xi) \, ext{sgn}(s) \ & eta(\eta,\xi) \geq rac{arrho(\eta,\xi)}{1-\kappa_0} + eta_0, \quad eta_0 > 0 \ & s\dot{s} \leq g[-eta|s| + arrho|s| + \kappa_0 eta|s|] = g[-eta(1-\kappa_0)|s| + arrho|s| \,] \ & s\dot{s} \leq g[-arrho|s| - (1-\kappa_0)eta_0|s| + arrho|s| \,] \end{aligned}$$

$$s\dot{s} \le -g(\eta, \xi)(1 - \kappa_0)\beta_0|s| \le -g_0\beta_0(1 - \kappa_0)|s|$$

$$v=-eta(x) ext{ sat } \left(rac{s}{arepsilon}
ight), ~~arepsilon>0$$

$$s\dot{s} \leq -g_0\beta_0(1-\kappa_0)|s|, \quad \text{for } |s| \geq \varepsilon$$

The trajectory reaches the boundary layer $\{|s| \leq \varepsilon\}$ in finite time and remains inside thereafter

Study the behavior of η

$$\dot{\eta} = f_a(\eta,\phi(\eta)+s)$$

What do we know about this system and what do we need?

$$lpha_1(\|\eta\|) \leq V(\eta) \leq lpha_2(\|\eta\|)$$

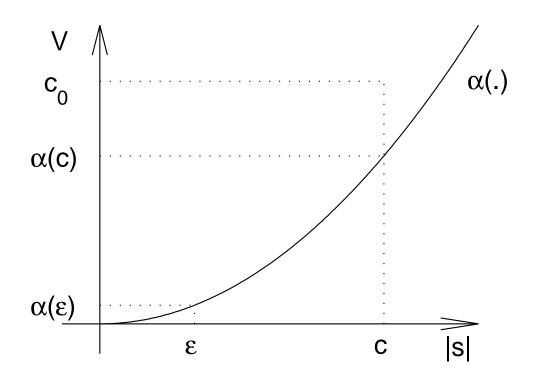
$$rac{\partial V}{\partial \eta} f_a(\eta,\phi(\eta)+s) \leq -lpha_3(\|\eta\|), \quad orall \, \|\eta\| \geq \gamma(|s|)$$

$$|s| \leq c \ \Rightarrow \ \dot{V} \leq -lpha_3(\|\eta\|), \ ext{ for } \|\eta\| \geq \gamma(c)$$
 $lpha(r) = lpha_2(\gamma(r))$

$$V(\eta) \ge \alpha(c) \Leftrightarrow V(\eta) \ge \alpha_2(\gamma(c)) \Rightarrow \alpha_2(\|\eta\|) \ge \alpha_2(\gamma(c))$$
$$\Rightarrow \|\eta\| \ge \gamma(c) \Rightarrow \dot{V} \le -\alpha_3(\|\eta\|) \le -\alpha_3(\gamma(c))$$

The set $\{V(\eta) \leq c_0\}$ with $c_0 \geq \alpha(c)$ is positively invariant

$$\Omega = \{V(\eta) \le c_0\} \times \{|s| \le c\}, \text{ with } c_0 \ge \alpha(c)$$



$$\Omega = \{V(\eta) \le c_0\} \times \{|s| \le c\}, \text{ with } c_0 \ge \alpha(c)$$

is positively invariant and all trajectories starting in Ω reach $\Omega_{\varepsilon} = \{V(\eta) \leq \alpha(\varepsilon)\} \times \{|s| \leq \varepsilon\}$ in finite time

Theorem 14.1: Suppose all the assumptions hold over Ω . Then, for all $(\eta(0), \xi(0)) \in \Omega$, the trajectory $(\eta(t), \xi(t))$ is bounded for all $t \geq 0$ and reaches the positively invariant set Ω_{ε} in finite time. If the assumptions hold globally and $V(\eta)$ is radially unbounded, the foregoing conclusion holds for any initial state

Theorem 14.2: Suppose all the assumptions hold over Ω

- $\varrho(0) = 0, \, \kappa_0 = 0$
- The origin of $\dot{\eta} = f_a(\eta, \phi(\eta))$ is exponentially stale

Then there exits $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, the origin of the closed-loop system is exponentially stable and Ω is a subset of its region of attraction. If the assumptions hold globally, the origin will be globally uniformly asymptotically stable

Example

$$\dot{x}_1 = x_2 + heta_1 x_1 \sin x_2, \qquad \dot{x}_2 = heta_2 x_2^2 + x_1 + u$$
 $| heta_1| \le a, \quad | heta_2| \le b$
 $x_2 = -k x_1 \quad \Rightarrow \quad \dot{x}_1 = -k x_1 + heta_1 x_1 \sin x_2$
 $V_1 = \frac{1}{2} x_1^2 \quad \Rightarrow \quad x_1 \dot{x}_1 \le -k x_1^2 + a x_1^2$
 $s = x_2 + k x_1, \quad k > a$
 $\dot{s} = heta_2 x_2^2 + x_1 + u + k (x_2 + heta_1 x_1 \sin x_2)$
 $u = -x_1 - k x_2 + v \quad \Rightarrow \quad \dot{s} = v + \Delta(x)$
 $\Delta(x) = heta_2 x_2^2 + k heta_1 x_1 \sin x_2$

$$egin{align} \Delta(x) &= heta_2 x_2^2 + k heta_1 x_1 \sin x_2 \ &|\Delta(x)| \leq a k |x_1| + b x_2^2 \ η(x) &= a k |x_1| + b x_2^2 + eta_0, \quad eta_0 > 0 \ &u &= -x_1 - k x_2 - eta(x) \operatorname{sgn}(s) \ \end{pmatrix}$$

Will

$$u = -x_1 - kx_2 - \beta(x) \operatorname{sat}\left(rac{s}{arepsilon}
ight)$$

stabilize the origin?

Example: Normal Form

$$egin{array}{lll} \dot{\eta} &=& f_0(\eta,\xi) \ \dot{\xi}_i &=& \xi_{i+1}, & 1 \leq i \leq
ho - 1 \ \dot{\xi}_
ho &=& L_f^
ho h(x) + L_g L_f^{
ho - 1} h(x) \ y &=& \xi_1 \end{array}$$

View ξ_{ρ} as input to the system

$$egin{array}{lll} \dot{\eta} &=& f_0(\eta, \xi_1, \cdots, \xi_{
ho-1}, \xi_
ho) \ \dot{\xi}_i &=& \xi_{i+1}, & 1 \leq i \leq
ho-2 \ \dot{\xi}_{
ho-1} &=& \xi_
ho \end{array}$$

Design $\xi_{\rho} = \phi(\eta, \xi_1, \dots, \xi_{\rho-1})$ to stabilize the origin

$$s=\xi_{
ho}-\phi(\eta,\xi_1,\cdots,\xi_{
ho-1})$$

Minimum Phase Systems: The origin of $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable

$$s=\xi_
ho+k_1\xi_1+\cdots+k_{
ho-1}\xi_{
ho-1}$$
 $\dot{\eta}=f_0(\eta,\xi_1,\cdots,\xi_{
ho-1},-k_1\xi_1-\cdots-k_{
ho-1}\xi_{
ho-1})$ $egin{bmatrix}\dot{\xi}_1\ dots\ \dot{\xi}_{
ho-1}\ \end{bmatrix}=egin{bmatrix}1\ -k_1\ -k_{
ho-1}\ \end{bmatrix}egin{bmatrix}\xi_1\ dots\ -k_{
ho-1}\ \end{bmatrix}egin{bmatrix}\xi_1\ dots\ \xi_{
ho-1}\ \end{bmatrix}$

Multi-Input Systems

$$egin{array}{ll} \dot{\eta}&=f_a(\eta,\xi)\ \dot{\xi}&=f_b(\eta,\xi)+G(\eta,\xi)E(\eta,\xi)u+\delta(t,\eta,\xi,u) \end{array}$$
 $\eta\in R^{n-p},\ \xi\in R^p,\ u\in R^p \end{array}$

$$f_a(0,0) = 0, \ f_b(0,0) = 0, \ \det(G) \neq 0, \ \det(E) \neq 0$$

$$G = \text{diag}[g_1, g_2, \dots, g_m], \ g_i(\eta, \xi) \ge g_0 > 0$$

Design ϕ s.t. the origin of $\dot{\eta} = f_a(\eta, \phi(\eta))$ is asymp. stable

$$s = \xi - \phi(\eta)$$

$$\dot{s} = f_b(\eta, \xi) - rac{\partial \phi}{\partial \eta} f_a(\eta, \xi) + G(\eta, \xi) E(\eta, \xi) u + \delta(t, \eta, \xi, u)$$

$$egin{aligned} \dot{s} &= f_b(\eta, \xi) - rac{\partial \phi}{\partial \eta} f_a(\eta, \xi) + G(\eta, \xi) E(\eta, \xi) u + \delta(t, \eta, \xi, u) \ u &= E^{-1} \left\{ -L \left[\hat{f}_b - rac{\partial \phi}{\partial \eta} \hat{f}_a
ight] + v
ight\}, \quad L = \hat{G}^{-1} \quad ext{or} \quad L = 0 \ \dot{s}_i &= g_i(\eta, \xi) v_i + \Delta_i(t, \eta, \xi, v), \quad 1 \leq i \leq p \ \left| rac{\Delta_i(t, \eta, \xi, v)}{g_i(\eta, \xi)}
ight| \leq arrho(\eta, \xi) + \kappa_0 \max_{1 \leq i \leq p} |v_i|, \quad orall \ 1 \leq i \leq p \ &arrho(\eta, \xi) \geq 0, \quad 0 \leq \kappa_0 < 1 \quad (Known) \ eta(x) \geq rac{arrho(x)}{1 - \kappa_0} + eta_0, \quad eta_0 > 0 \end{aligned}$$

$$s_i\dot{s}_i=s_ig_iv_i+s_i\Delta_i\leq g_i\{s_iv_i+|s_i|[arrho+\kappa_0\max_{1\leq i\leq p}|v_i|]\}$$

$$v_i = -\beta \operatorname{sgn}(s_i), \quad 1 \le i \le p$$

$$egin{array}{lll} s_i \dot{s}_i & \leq & g_i [-eta + arrho + \kappa_0 eta] |s_i| \ & = & g_i [-(1-\kappa_0)eta + arrho] |s_i| \ & \leq & g_i [-arrho - (1-\kappa_0)eta_0 + arrho] |s_i| \ & \leq & -g_0 eta_0 (1-\kappa_0) |s_i| \end{array}$$

Now use

$$v_i = -eta \operatorname{sat}\left(rac{s_i}{arepsilon}
ight), \quad 1 \leq i \leq p$$

Read Theorem 14.1 and 14.2 in the textbook