Control in The Presence of Uncertainty

Robust Control

Dispense del Corso di Controllo Robusto e Adattativo

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Robust control is a control method in which uncertainties, both static and dynamic, are considered in the design stage.

This approach allows designing controllers with guaranteed stability and performance for a wide set of (bounded in some sense) *perturbations*.

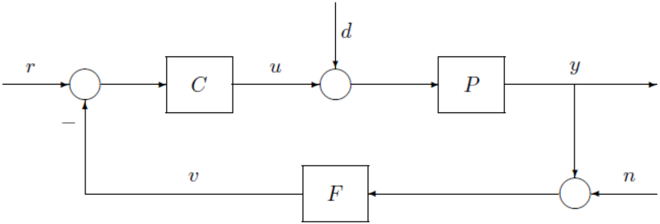
The point of departure of robust control is the use of parametric models in which the uncertainty has been extracted and represented in a convenient form, for example using a variable gain or a phase shift.

Once a parameterized model is available, one has to design a controller able to guarantee nominal properties, such as stability and performance, and robust properties, again in terms of stability and performance.

While there are several ways in which robust synthesis can be carried out, we focus on the so-called H_∞ design paradigm, which heavily exploits the notion of L_2-gain. Recall, in fact, that for linear systems the L_2-gain of a system coincides with the H_∞ norm of its transfer function.

In what follows we consider the basic feedback system (often referred to as the feedback system) depicted in the figure, if which r is the reference input, v is the sensor output, u is the control input, d is an external disturbance, y is the plant output, and n is the sensor noise. The signals r, d and n are exogenous signals.

The blocks P, C and F describe the plant (to be controlled), the controller (to be designed) and the sensor or communication infrastructure. For simplicity we assume that these blocks are described by means of transfer functions (and we often omit the argument "s"), although we could consider more general systems, and that the transfer functions are SISO, although all conclusions can be extended to the MIMO case. Finally, we assume that in each of these transfer functions there is no pole-zero cancellation (that is the numerator and denominator polynomials are coprime).



To study the feedback system we define the variables

$$x_1 = r - Fx_3, \qquad x_2 = d + Cx_1, \qquad x_3 = n + Px_2,$$

which describe the signals just after the three summation blocks.

These equations can be rewritten in matrix form as

$$\begin{bmatrix} 1 & 0 & F \\ -C & 1 & 0 \\ 0 & -P & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} r \\ d \\ n \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{1+PCF} \begin{bmatrix} 1 & -PF & -F \\ C & 1 & -CF \\ PC & P & 1 \end{bmatrix} \begin{bmatrix} r \\ d \\ n \end{bmatrix}$$

We say that the feedback system is well-posed if $1 + PCF$ is not identically zero, which is the case under the natural assumptions that P is strictly proper and C and F are proper. In fact, if this assumptions hold $1 + PCF(\infty) = 1$, hence $1 + PCF$ is not identically zero.

Note that these assumptions rule out, for example, that C be an ideal PD/PID controller.

The relation that we have established between the exogenous input and the x_i variables identifies nine transfer functions.

If all these transfer functions are stable then the feedback system is said to be *internally stable*: all internal signals are bounded provided the exogenous signals are bounded.

The advantage of this definition is that it requires studying only input-output properties.

Theorem

The feedback system is internally stable if and only if it has no closed-loop poles in $\mathbb{C}^{\geq 0}$.

Proof (Sufficiency).

Assume for simplicity that $F = 1$.

Let Z_P/R_P and Z_C/R_C be the numerators/denominators of P and C, respectively.

The characteristic polynomial of the feedback system is therefore $Z_P Z_C + R_P R_C$: the feedback system is therefore internally stable if the characteristic polynomial have no zeroes in $\mathbb{C}^{\geq 0}$.

□

Proof (Necessity).

Necessity is more subtle.

Suppose the feedback system is internally stable. Then all nine transfer functions are stable, that is have no poles in $\mathbb{C}^{\geq 0}$.

This does not allow to conclude directly that $Z_P Z_C + R_P R_C$ has no zeroes in $\mathbb{C}^{\geq 0}$, since this polynomial may have a zero in $\mathbb{C}^{\geq 0}$ which is also a zero of all nine numerators: $R_P R_C$, $Z_P R_C$, ...

This is however impossible under the assumption that the transfer functions P and C do not have any cancellation, that is Z_P and R_P (resp. Z_C and R_C) are coprime. □

One could also provide an alternative characterization of internal stability.

Theorem

The feedback system is internally stable if and only if

- *the transfer function* $1 + PCF$ *has not zeros in* $\mathbb{C}^{\geq 0}$*;*
- *there is no pole-zero cancellation in* $\mathbb{C}^{\geq 0}$ *in the product* PCF*.*

Internal stability of the feedback system can be also assessed using Nyquist criterion, which we recall for convenience.

Theorem (Nyquist Criterion)

Consider the feedback system and construct the Nyquist plot of PCF, indenting to the left around poles on the imaginary axis.

Let n denote the number of poles of P, C, and F in $\mathbb{C}^{\geq 0}$.

Then the feedback system is internally stable if and only if the Nyquist plot does not pass through the point $-1+0j$ and encircles it exactly n times counterclockwise.

Performance

We now consider the problem of tracking a set of reference signals and to provide a bound on the steady-state error. While the exact tracking problem for specific reference signals generated by an exosystem can be solved using the regulator theory, in the current context we aim at establishing an error bound in terms of a weighted norm bound.

Assume for simplicity that $F = 1$ and let $L = PC$ denote the loop transfer function (also known as the loop gain). The transfer function from r to e is

$$S = \frac{1}{1+L},$$

which is known as the sensitivity function. Note that if the feedback system is internally stable then S is stable. In addition, since L is strictly proper (by the assumption that P is strictly proper) then $S(\infty) = 1$.

The name sensitivity function comes from the observation that the transfer function from r to y is (this transfer function is called complementary sensitivity function)

$$T = \frac{PC}{1+PC} = \frac{L}{1+L} = 1 - S$$

and its sensitivity to small variations ΔP in P is given by

$$\lim_{\Delta P \to 0} \frac{\Delta T/T}{\Delta P/P} = \frac{dT}{dP}\,\frac{P}{T} = S,$$

that is S is the sensitivity of the transfer function T to infinitesimal variations of P.

Performance

To decide on a performance specification we need to consider what prior knowledge is available on the class of reference signals r and how we want to measure the tracking error.

To illustrate this perspective suppose that r can be any sinusoidal input with amplitude not larger than one, and that we would like the tracking error (in steady-state) to have amplitude not larger than ϵ, for some $\epsilon > 0$. This specification can be expressed by the condition

$$\|S\|_\infty \leq \epsilon.$$

This bound can be re-written, by defining the trivial weighting function $W_1 = 1/\epsilon$ as

$$\|W_1 S\|_\infty \leq 1.$$

This perspective can be made more realistic and useful by defining a frequency dependent weighting function.

Suppose that the family of reference signal of interest in composed of all signals of the form $r = W_1 r_{pf}$, where r_{pf}, a pre-filtered input, is any sinusoidal signal of amplitude not larger than one. Then the maximum amplitude of e is $\|W_1 S\|_\infty$.

Suppose now that the tracking error measure is the 2-norm of e and that r_{pf} is a signal with finite 2-norm. Then

$$\sup_{r_{pf}} \|e\|_2 = \sup\{\|SW_1 r_{pf}\|_2 \ : \ \|r_{pf}\|_2 \leq 1\} = \|W_1 S\|_\infty.$$

Consider now the case in which the designer of a control system has concluded, by means of experiments, that the control systems has a satisfactory behavior provided the Bode magnitude plot of S has a particular shape. In particular, good behavior is achieved if $|S(j\omega)|$ lies under some curve. This specification can be rewritten as

$$|S(j\omega)| < |W_1(j\omega)|^{-1},$$

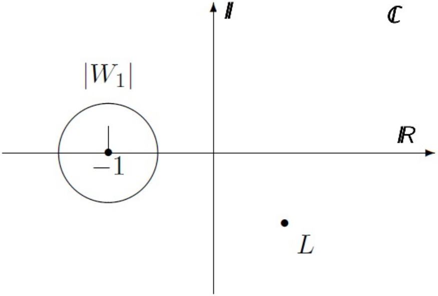
which is equivalent to $\|W_1 S\|_\infty < 1$.

Performance

The norm bound $\|W_1 S\|_\infty < 1$ has a very simple graphical interpretation. To see this note that

$$\|W_1 S\|_\infty < 1 \quad \Leftrightarrow \quad \left|\frac{W_1(j\omega)}{1 + L(j\omega)}\right| < 1, \quad \forall \omega \in \mathbb{R} \quad \Leftrightarrow \quad |W_1(j\omega)| < |1 + L(j\omega)|, \quad \forall \omega \in \mathbb{R}$$

which means that for every ω the point $L(j\omega)$ on the Nyquist plot of the loop gain lies outside the disk of center -1 and radius $|W_1(j\omega)|$, as indicated in the figure.



Performance

Other performance problems can be posed by focusing on the response to the other two exogenous signals d and n, yielding the matrix of transfer functions

$$\begin{bmatrix} e \\ u \end{bmatrix} = -\begin{bmatrix} PS & S \\ T & CS \end{bmatrix} \begin{bmatrix} d \\ n \end{bmatrix}.$$

One could therefore consider performance specifications in terms of weighted version of these transfer functions. Note, however, that a performance specification with weight W on PS is equivalent to the weight WP in S. Similarly, a performance specification with weight W on $CS = T/P$ is equivalent to the weight W/P on T. As a result, performance specifications involving e result in weights on S and performance specifications involving u result in weights on T.

Essentially, an elementary perspective of robust control boils down to weighting S, T or some combination of S and T, and the trade-off between making S small and T small is the main objective of robust control.

Exercise

Explain why S and T cannot be made simultaneously arbitrarily small for a given ω.

Robustness

Up to now we have used the words robust or robustness without providing a formal definition, that is using its colloquial meaning of "unlikely to break or fail". We would like to provide now a formal and quantitative definition.

To this end, suppose that the plant transfer function P belongs to a set $\mathcal{P}$, as discussed for example in the introductory part of the course.

Consider now some characteristic of the feedback system, such as stability or performance. A controller C is robust with respect to this characteristic if it holds for every plant in $\mathcal{P}$.

The notion of robustness therefore requires three ingredients: a set of plants, a controller, and a characteristic.

One could say that a controller C provide a robust characteristic if it provide such characteristic for all $P \in \mathcal{P}$. In addition, if $\mathcal{P}$ has some sort of size, the maximum size for which a characteristic holds is a measure of robustness.

The most important characteristics are stability and performance, which we aim to characterize in what follows.

The gain margin and the phase margin are elementary measures of robust stability, since they specify the maximum permissible size of very specific perturbations.

The gain margin considers the family of plants $\mathcal{P}$ parameterized by kP, with k constant. For $k = 1$ the feedback system is internally stable, hence (by continuity) there exists a set (k_{min}, k_{max}) such that the feedback system remains internally stable for all $k \in (k_{min}, k_{max})$. k_{min} is called the lower gain margin and k_{max} is called the upper gain margin. k_{max} may be equal to ∞ if the feedback system remains internally stable for all $k \geq 1$. Note, finally, that the set of all k's such that the feedback system is stable may be the union of several open subsets of the real axis.

The phase margin considers the family of plants $\mathcal{P}$ parameterized by $e^{-j\phi}P$, with ϕ a positive constant. The phase margin is the largest positive number ϕ_{max} (usually expressed in degrees) such that internal stability holds for all $\phi \in [0, \phi_{max})$.

The gain and phase margins can be read on the Nyquist plot of the loop gain. Note that they do not provide any robustness guarantee for simultaneous perturbations in *gain* and *phase*, that is for the family of plants $ke^{-j\phi}P$.

Robust Stability

Consider now the family of plants $\mathcal{P}$ parameterized using the multiplicative perturbation model, that is the family of plants described by

$$(1 + \Delta W_2)P, \qquad \|\Delta\|_\infty \leq 1.$$

Theorem (Robust Stability)

The controller C guarantees robust stability if and only if $\|W_2 T\|_\infty < 1$.

Proof (Sufficiency).

Assume that $\|W_2 T\|_\infty < 1$. Since the nominal feedback system is internally stable, from the Nyquist criterion we can conclude that the Nyquist plot of L does not touch the point $-1 + j0$ and the number of its counterclockwise encirclements of $-1 + j0$ equals the number of poles of PC in $\mathbb{C}^{\geq 0}$.

Consider now a perturbation Δ such that $\|\Delta\|_\infty \leq 1$ and construct the Nyquist plot of $\tilde{P}C = (1 + \Delta W_2)L$. Note that the *factor* $1 + \Delta W_2$ does not introduce additional poles on the imaginary axis.

We have therefore to show that the Nyquist plot of $\tilde{P}C$ does not touch the point $-1 + j0$ and the number of its counterclockwise encirclements of $-1 + j0$ equals the number of poles of $\tilde{P}C$ in $\mathbb{C}^{\geq 0}$, that is the perturbation does not change the number of encirclements.

Proof (Sufficiency).

To show that this is the case note that

$$1 + (1 + \Delta W_2)L = (1 + L) + \Delta W_2 L = (1 + L)\left(1 + \Delta W_2 \frac{L}{1 + L}\right) = (1 + L)(1 + \Delta W_2 T)$$

and that

$$\|\Delta W_2 T\|_\infty \leq \|W_2 T\|_\infty < 1.$$

As a result, the point $1 + \Delta W_2 T$ lies on some closed disk with center 1 and radius smaller than 1 for every s on the Nyquist path.

Thus, as s *travels* along the Nyquist path, the net change in angle of $1 + (1 + \Delta W_2)L$ equals the net change in angle of $1 + L$, which gives the claim.

□

Proof (Necessity).

Suppose that the feedback system is internally stable and that $\|W_2 T\|_\infty \geq 1$.

Then there exists a Δ, with $\|\Delta\|_\infty \leq 1$, which destabilizes the feedback system.

Since T is strictly proper there exists ω such that

$$|W_2(j\omega)T(j\omega)| = 1.$$

Suppose, for simplicity, that $\omega = 0$, although similar considerations applies to any ω.

Then $|W_2(0)T(0)| = 1$. Let $\Delta = -W_2(0)T(0)$, which gives a feasible perturbation, and note that

$$1 + \Delta W_2(0)T(0) = 0,$$

that is the Nyquist plot of $(1 + \Delta W_2)L$ goes through the point $-1 + j0$, hence the feedback system is not internally stable. □

Robust Stability

This theorem can be used to quantify the stability margin of the feedback system.

To this end consider the family of perturbed plants (for some $\beta > 0$)

$$\begin{aligned} \{\tilde{P} = (1+\Delta W_2)P \ : \ \|\Delta\|_\infty \le \beta\} &= \{\tilde{P} = (1+\beta^{-1}\Delta\beta W_2)P \ : \ \|\beta^{-1}\Delta\|_\infty \le 1\} \\ &= \{\tilde{P} = (1+\tilde{\Delta}\beta W_2)P \ : \ \|\tilde{\Delta}\|_\infty \le 1\} \end{aligned}$$

This implies that the maximum allowable size of the perturbation is

$$\beta_{sup} = \sup_{\beta} \{\beta \ : \ \|\beta W_2 T\|_\infty < 1\} = \frac{1}{\|W_2 T\|_\infty}.$$

The robust stability condition has also a graphical interpretation. To see this note that

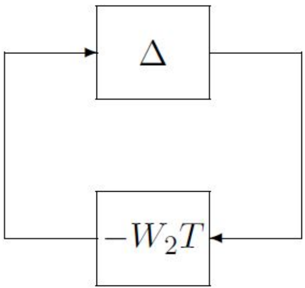
$$\begin{aligned} \|W_2 T\|_\infty < 1 \quad &\Leftrightarrow \quad \left|\frac{W_2(j\omega)L(j\omega)}{1+L(j\omega)}\right| < 1, \quad \forall\omega \\ &\Leftrightarrow \quad |W_2(j\omega)L(j\omega)| < |1+L(j\omega)|, \quad \forall\omega. \end{aligned}$$

This inequality shows that for every ω the critical point $-1 + j0$ lies outside the disk of center $L(j\omega)$ and radius $|W_2(j\omega)L(j\omega)|$.

The robust stability condition can be finally interpreted using the small gain theorem.

To this end, consider the feedback system without input and note that it can be represented by the block diagram below, which can be studied using the small gain theorem.

This yields readily the result of the Robust Stability Theorem and allows generalizing the theorem to MIMO systems.



Exercise

Let Δ be such that $\|\Delta\|_\infty \leq 1$.

- *Consider the family of plants described by (feedback perturbation)*

  $$\tilde{P} = \frac{P}{1 + \Delta W_2 P}.$$

  Show that the robust stability condition is $\|W_2 PS\|_\infty < 1$.

- *Consider the family of plants described by*

  $$\tilde{P} = P + \Delta W_2.$$

  Show that the robust stability condition is $\|W_2 CS\|_\infty < 1$.

- *Consider the family of plants described by*

  $$\tilde{P} = \frac{P}{1 + \Delta W_2}.$$

  Show that the robust stability condition is $\|W_2 S\|_\infty < 1$.

Explain why only T, CS, PS, and S are involved in the robust stability tests.

The Robust Stability Theorem can be used as an analysis and as a design tool.

To see this let

$$\begin{array}{rcl} \dot{x} & = & Ax + B\omega, \\ y & = & Cx, \end{array} \qquad \begin{array}{rcl} \dot{\xi} & = & F\xi + Gv, \\ u & = & H\xi, \end{array} \qquad \begin{array}{rcl} \dot{\alpha} & = & L\alpha + M\delta, \\ z & = & N\alpha + Q\delta, \end{array}$$

be the state-space realizations of P, C and W_2, respectively. Note that these systems are interconnected, in the multiplicative perturbation model with perturbation Δ, via the equations

$$\omega = u + w, \qquad v = -y, \qquad \delta = u, \qquad w = \Delta z.$$

Removing the perturbation one has a system with input w and output z, with a state-space realization given by

$$\left[\begin{array}{c} \dot{x} \\ \dot{\xi} \\ \dot{\alpha} \end{array}\right] = \left[\begin{array}{ccc} A & BH & 0 \\ -GC & F & 0 \\ 0 & MH & L \end{array}\right]\left[\begin{array}{c} x \\ \xi \\ \alpha \end{array}\right] + \left[\begin{array}{c} B \\ 0 \\ 0 \end{array}\right] w, \qquad z = \left[\begin{array}{ccc} 0 & QH & N \end{array}\right]\left[\begin{array}{c} x \\ \xi \\ \alpha \end{array}\right],$$

or, with obvious notation,

$$\dot{X} = A_{W_2T}X + B_{W_2T}w, \qquad z = C_{W_2T}X.$$

Corollary (Analysis)

The robust stability condition holds if and only if all eigenvalues of A_{W_2T} have negative real part and the Hamiltonian matrix

$$H_{W_2T} = \begin{bmatrix} A_{W_2T} & B_{W_2T}B_{W_2T}^\top \\ -C_{W_2T}^\top C_{W_2T} & -A_{W_2T}^\top \end{bmatrix}$$

has no eigenvalues on the imaginary axis.

Equivalently, the robust stability condition holds if and only if there exists a matrix $P = P^\top > 0$ such that

$$A_{W_2T}^\top P + PA_{W_2T} + PB_{W_2T}B_{W_2T}^\top P + C_{W_2T}^\top C_{W_2T} < 0.$$

Corollary (Design)

The feedback system can be rendered robustly stable if and only if there exist matrices F, G and H such that all eigenvalues of A_{W_2T} have negative real part and the Hamiltonian matrix

$$H_{W_2T} = \begin{bmatrix} A_{W_2T} & B_{W_2T}B_{W_2T}^\top \\ -C_{W_2T}^\top C_{W_2T} & -A_{W_2T}^\top \end{bmatrix}$$

has no eigenvalues on the imaginary axis.

Equivalently, the feedback system can be rendered robustly stable if and only if there exist matrices F, G, H, and $P = P^\top > 0$ such that

$$A_{W_2T}^\top P + PA_{W_2T} + PB_{W_2T}B_{W_2T}^\top P + C_{W_2T}^\top C_{W_2T} < 0.$$

Note that in the above statement one has either three unknowns, that is F, G and H, or four unknowns, that is F, G, H and P.

These statements are not limited to the SISO case, that is are valid in the general MIMO case, hence generalize the Robust Stability Theorem that we have proved using Nyquist criterion.

While the analysis conditions are fairly easy to use, that is one has either to compute the eigenvalues of two matrices or to solve an ARI, the design conditions are not easy to exploit in practice.

Nevertheless, one could proceed as follows. Removing the controller and the perturbation from the feedback loop yields a system with inputs u and w and outputs z and $-y$ described by the state-space realization

$$\begin{bmatrix} \dot{x} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} x \\ \alpha \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} w + \begin{bmatrix} B \\ M \end{bmatrix} u,$$

$$z = \begin{bmatrix} 0 & N \end{bmatrix} \begin{bmatrix} x \\ \alpha \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} w + \begin{bmatrix} Q \end{bmatrix} u,$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ \alpha \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} w + \begin{bmatrix} 0 \end{bmatrix} u.$$

This system, with some abuse of notation and obvious definitions, can be written as

$$\begin{aligned} \dot{x} &= A\,x + B_1\,w + B_2\,u, \\ z &= C_1\,x + D_{11}\,w + D_{12}\,u, \\ y &= C_2\,x + D_{21}\,w + D_{22}\,u. \end{aligned}$$

The robust stabilization problem therefore boils down to the design of a feedback controller, with input y and output u, guaranteeing that the resulting closed-loop system, with input w and output z has H_∞ norm, that is L_2-gain, smaller than 1.

A modification of this problem, known as the *regular* measurement feedback H_∞ control problem, can be solved using the theory of dissipative systems.

The robust stabilization problem is more difficult to solve because it has the form of the so-called *singular* measurement feedback H_∞ control problem. It is however not difficult to regularize singular problems, provided that one can tolerate a more conservative design.

To study the robust stability, and in general robust design, problems in full generality we formulate the so-called measurement feedback H_∞ control problem.

Consider again the system

$$\begin{array}{ccccccc} \dot{x} & = & A\,x & + & B_1\,w & + & B_2\,u, \\ z & = & C_1\,x & + & D_{11}\,w & + & D_{12}\,u, \\ y & = & C_2\,x & + & D_{21}\,w & + & D_{22}\,u. \end{array}$$

The first equation describes a plant with state $x(t) \in \mathbb{R}^n$, control input $u(t) \in \mathbb{R}^m$, and exogenous input $w(t) \in \mathbb{R}^r$.

The second equation defines a penalty variable $z(t) \in \mathbb{R}^s$, which may include a tracking error as well as a cost on the input required to achieve the desired control goal.

The third equation defines a set of measured variables.

The control action is to be provided by a dynamic controller which processes the measured variable y and generates the control input u.

The purpose of the control is twofold:

- to achieve closed-loop stability;
- to attenuate the effect of the exogenous input w on the penalty variable z in the L_2 sense, that is

$$\|z\|_2 \leq \gamma \|w\|_2 + \beta$$

for all $w \in L_2$, for some constant $\gamma > 0$ and for some positive $\beta = \beta(x(0))$.

For simplicity, we assume that $x(0) = 0$, hence $\beta = 0$. Note, also, that the L_2 condition is equivalent to

$$\int_0^T z^\top(s) z(s) ds \leq \gamma^2 \int_0^T w^\top(s) w(s) ds, \qquad \forall T > 0.$$

Recall, finally, that the L_2 condition is equivalent to a condition on the H_∞ norm of the closed-loop system, that is the L_2 condition is equivalent to the condition

$$\|\Sigma_{cl}\|_\infty \leq \gamma,$$

where Σ_{cl} denotes the closed-loop system.

The measurement feedback H_∞ control problem is said to be regular if the matrices defining the problems are such that

$$D_{12}^\top D_{12} > 0, \qquad D_{21} D_{21}^T > 0.$$

The first condition implies that all control signals *contribute* to the norm of the penalty variable, that is there is no control that is not penalized.

The second condition does not have an intuitive interpretation: it is essential to construct a *robust* observer.

These two conditions are often strengthened to

$$D_{11} = 0, \qquad C_1^\top D_{12} = 0, \qquad D_{12}^\top D_{12} = I, \qquad D_{21} B_1^\top = 0, \qquad D_{21} D_{21}^T = I, \qquad D_{22} = 0.$$

The first condition implies that the disturbance does not affect the penalty variable; the second condition means that in the norm of the penalty variable there are no cross terms in x and u; the forth condition implies that the process and measurement disturbances are uncorrelated; and the sixth condition imply that the control does not contribute to the measured variable.

The H_∞ Control Problem – Regularization

The robust stability condition that we have seen does not satisfy both *regularity* conditions, since $D_{21} = 0$. One could regularize the problem introducing a *dummy* disturbance, that is a measurement noise n and redefining the variable y as

$$y = -Cx + n.$$

With this redefinition one has a new exogenous signal $w_{ex} = [w^\top, \ n^\top]^\top$ and

$$D_{21} = \begin{bmatrix} 0 & I \end{bmatrix}, \qquad B_1 = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} \qquad \Rightarrow \qquad D_{21} D_{21}^\top = I > 0, \qquad D_{21} B_1^\top = 0.$$

Suppose now that there exists a controller such that

$$\|z\|_2 \leq \gamma \left\| \begin{array}{c} w \\ n \end{array} \right\|_2,$$

for all w and n. Then trivially

$$\|z\|_2 \leq \gamma \left\| \begin{array}{c} w \\ 0 \end{array} \right\|_2,$$

that is the solution of the regularized problem solves the original problem.

A similar strategy can be used to regularize problems in which $D_{12}^\top D_{12} \not> 0$.

We now consider the problem of characterizing the performance of the perturbed plant.

Suppose that the plant transfer function belongs to a set $\mathcal{P}$. The general notion of robust performance stipulates that internal stability and performance should hold for all plants in the set $\mathcal{P}$.

In the case of a family of plants parameterized using a multiplicative perturbation we have shown that, provided the feedback system is internally stable, the nominal performance condition in $\|W_1 S\|_\infty < 1$ and the robust stability condition is $\|W_2 T\|_\infty < 1$.

If P is perturbed to $(1 + \Delta W_2)P$ then S is perturbed to

$$\frac{1}{1 + (1 + \Delta W_2)L} = \frac{S}{1 + \Delta W_2 T}.$$

Therefore, the robust performance requirement is captured by the conditions

$$\|W_2 T\|_\infty < 1, \qquad \left\| \frac{W_1 S}{1 + \Delta W_2 T} \right\|_\infty < 1, \qquad \forall \Delta \ : \ \|\Delta\|_\infty \leq 1.$$

Theorem

The controller C guarantees robust performance if and only if $\| |W_1 S| + |W_2 T| \|_\infty < 1$.

Proof (Sufficiency).

Assume the conditions of the statement hold and note that these are equivalent to

$$\|W_2 T\|_\infty < 1, \qquad \left\| \frac{W_1 S}{1 - |W_2 T|} \right\|_\infty < 1.$$

Pick now an admissible Δ and consider all functions evaluated at some arbitrary point $j\omega$.

The following implications hold:

$$1 = |1 + \Delta W_2 T - \Delta W_2 T| \leq |1 + \Delta W_2 T| + |W_2 T|$$

$$\Downarrow$$

$$1 - |W_2 T| \leq |1 + \Delta W_2 T|$$

□

Proof (Sufficiency).

This implies that

$$\left\| \frac{W_1 S}{1 - |W_2 T|} \right\|_\infty \geq \left\| \frac{W_1 S}{1 + \Delta W_2 T} \right\|_\infty ,$$

which yields

$$\left\| \frac{W_1 S}{1 + \Delta W_2 T} \right\|_\infty < 1,$$

hence the claim.

$\square$

Proof (Necessity).

Assume the feedback system satisfies the robust performance requirements.

Select a value of ω such that $j\omega$ maximizes

$$\frac{|W_1 S|}{1 - |W_2 T|}$$

and pick Δ such that (such a Δ always exists)

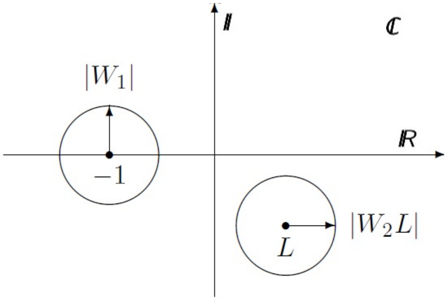
$$1 - |W_2 T| = |1 + \Delta W_2 T| < 1.$$

The proof is completed noting that

$$\left\| \frac{W_1 S}{1 - |W_2 T|} \right\|_\infty = \frac{|W_1 S|}{1 - |W_2 T|} = \frac{|W_1 S|}{|1 + \Delta W_2 T|} \leq \left\| \frac{W_1 S}{1 + \Delta W_2 T} \right\|_\infty < 1.$$

$\square$

The robust performance condition has, similarly to the robust stability one, a simple graphical interpretation, as indicated in the figure below: for each ω consider the two closed disks with center -1 and radius $|W_1(j\omega)|$, and center $L(j\omega)$ and radius $|W_2(j\omega)L(j\omega)|$, respectively. The feedback system satisfies the robust performance requirements if and only if these two disks are disjoint.



The robust performance condition states that *robust performance level* 1 is achieved.

One may wish to consider more stringent requirements, hence we say that the robust performance level α is achieved if

$$\|W_2 T\|_\infty < 1, \qquad \left\| \frac{W_1 S}{1 + \Delta W_2 T} \right\|_\infty < \alpha, \qquad \forall \Delta \ : \ \|\Delta\|_\infty \le 1.$$

Note now that, for each $j\omega$,

$$\max_{|\Delta| \le 1} \left| \frac{W_1 S}{1 + \Delta W_2 T} \right| = \frac{|W_1 S|}{1 - |W_2 T|},$$

hence the minimum achievable robust performance level α_{min} is

$$\alpha_{min} = \max_\omega \frac{|W_1 S|}{1 - |W_2 T|} = \left\| \frac{W_1 S}{1 - |W_2 T|} \right\|_\infty .$$

Robust Performance

Similarly, one may wish to quantify the largest possible uncertainty that can be tolerated while the robust performance conditions hold. To this end we scale the uncertainty and allow Δ to be such that $\|\Delta\|_\infty \leq \beta$, for some $\beta > 0$.

A direct application of the results established yields that internal stability of the feedback system is robust if and only if $\|W_2 T\|_\infty < \dfrac{1}{\beta}$.

We therefore say that the *uncertainty level* β is permissible if

$$\|\beta W_2 T\|_\infty < 1, \qquad \left\| \frac{W_1 S}{1 + \Delta W_2 T} \right\|_\infty < 1, \qquad \forall \Delta \ : \ \|\Delta\|_\infty \leq \beta.$$

By replacing Δ with $\beta\Delta$, scaling the bound on Δ, and noting that

$$\max_{|\Delta| \leq 1} \left| \frac{W_1 S}{1 + \beta \Delta W_2 T} \right| = \frac{|W_1 S|}{1 - \beta |W_2 T|},$$

we conclude that the maximum value of β is given by

$$\beta_{max} = \left\| \frac{W_2 T}{1 - |W_1 S|} \right\|_\infty^{-1}.$$

Exercise

Consider the family of plants $\mathcal{P}$ described by the model

$$\tilde{P} = P\frac{1 + \Delta_2 W_2}{1 + \Delta_1 W_1},$$

with W_1 and W_2 internally stable, $\|\Delta_1\|_\infty \leq 1$ and $\|\Delta_2\|_\infty \leq 1$.

Show that the robust stability condition for this family of plants is

$$\| \, |W_1 S| + |W_2 T| \, \|_\infty < 1,$$

that is it coincides with the robust performance condition for the family of plants with multiplicative perturbation.

(Hint: Fix Δ_2 and determine a robust stability condition with respect to Δ_1 and show that this holds if the given condition holds.)

Exercise

Consider the family of plants $\mathcal{P}$ described by the model

$$\tilde{P} = P + W_2\Delta.$$

Suppose that the nominal performance condition is $\|W_1 S\|_\infty < 1$*. Show that the robust performance condition is*

$$\| \, |W_1 S| + |W_2 CS| \, \|_\infty < 1.$$

Exercise

Consider the family of plants $\mathcal{P}$ described by the model

$$\tilde{P} = \frac{P}{1 + \Delta W_2 P}.$$

Suppose that the nominal performance condition is $\|W_1 T\|_\infty < 1$*. Show that the robust performance condition is*

$$\| \, |W_1 T| + |W_2 PS| \, \|_\infty < 1.$$

Consider a feedback system with multiplicative perturbation and assume it is internally stable. The nominal performance condition is $\|W_1 S\|_\infty < 1$ and the robust stability condition is $\|W_2 T\|_\infty < 1$.

The condition for simultaneously achieving nominal performance and robust stability is

$$\| \max(|W_1 S|, |W_2 T|) \|_\infty < 1,$$

whereas the robust performance condition is

$$\| \, |W_1 S| + |W_2 T| \, \|_\infty < 1.$$

Note that

$$\max(|W_1 S|, |W_2 T|) \leq |W_1 S| + |W_2 T| \leq 2 \max(|W_1 S|, |W_2 T|),$$

hence robust performance is achieved if nominal performance and robust stability are achieved with *level* 1/2.

Exercise

Show that robust performance is achieved if

$$\sqrt{|W_1 S|^2 + |W_2 T|^2} < 1, \qquad \forall \omega.$$

Exercise (Assignment 6)

Consider the family of plants

$$\tilde{P} = P(1 + \Delta W_2),$$

with

$$P(s) = \frac{1}{s-1}, \qquad W_2(s) = \frac{2}{s+10}, \qquad C(s) = k, \qquad W_1(s) = \frac{1}{s+1}.$$

Assume that Δ *is such that* $\|\Delta\|_\infty \leq 2$.

Determine the range of values of k *for which robust stability is achieved.*

Determine the value of k *which gives robust stability and minimizes the robust performance level* α.

The foregoing discussion highlights the fact that robust control problems can be formulated as H_∞ control problems and have, at least in the SISO case, very simple and intuitive interpretations in the frequency domain or in terms of the Nyquist plot.

While the stated robustness conditions can be readily exploited in robustness analysis, that is in quantifying the robustness of a feedback system for which the controller has already been designed, they are much more difficult to translate into design guidelines and/or algorithm.

There are several ways, therefore, in which robust control design can be undertaken.

The so-called loop-shaping technique allows shaping the loop gain in the frequency domain to satisfy given robustness bounds.

This technique is an enhancement of the classical technique based on the design of lead/lag compensators and it is essentially a graphical technique, suitable for SISO (stable and minimum phase) plants, which relies on a trial and error approach.

Loop-shaping is therefore a non-systematic approach and it can be applied almost exclusively to SISO plants. On the positive side, it may generate very simple (that is low dimensional) compensators even for very complex plant models.

Because of the limitations highlighted above, we do not study in detail the loop-shaping design method.

We stress however that this is often used in practice and its use is simplified by the availability of ad hoc software packages.

One alternative approach to robust control design is based on the parameterization of all stabilizing controllers for a given plant.

Within this framework robust control design can be accomplished in two steps:

- obtain an explicit parameterization of all stabilizing controllers;
- identify a member of this parameterized set to achieve the desired performance specification (such as robust stability or robust performance).

This approach, applicable to SISO and (with some limitations) MIMO systems, relies on an algorithmic procedure to parameterize all stabilizing controllers, but requires the solution of complex optimization problems, or of a series of trial and error steps, to achieve additional performance specifications.

It is, however, applicable to a vast range of problems, including the so-called strong stabilization or simultaneous stabilization problems, that is the problems of determining a stabilizing controller which is itself stable or which stabilizes multiple plants.

One additional disadvantage of this approach is that it may generate controllers of large dimension.

Since most robust design specifications are naturally expressed using H_∞-type bounds, it makes sense to approach robust control design as an H_∞ control problem.

This yields a systematic, algorithmic, procedure to design robust controllers with an a priori known dimension.

The H_∞ synthesis is performed on the basis of *state space data* and linear-algebra-based algorithms, hence it is equally applicable to SISO and MIMO plants.

The main disadvantage of the H_∞ design method is that often robust control problems are non-standard H_∞ control problem, hence one has to undertake additional steps (such as the regularization step) to transform the underlying problem in a form amenable to the application of standard design procedures. These additional steps often generate conservative solutions.

Controller Parameterization – Stable Plants

The so-called controller parameterization approach to robust control consists in explicitly parameterizing all stabilizing controllers for a given plant and then in identifying the elements of the parameterized family which satisfy the given robustness requirements.

To derive an explicit controller parameterization consider, to begin with, the case of a stable SISO plant P.

Denote with $\mathcal{S}$ the set of all stable SISO transfer functions.

The set $\mathcal{S}$ is closed under addition (parallel interconnection) and multiplication (series interconnection), that is

$$F \in \mathcal{S}, \; G \in \mathcal{S} \Rightarrow F + G \in \mathcal{S}, \; FG \in \mathcal{S}.$$

Theorem

Consider the feedback system with $F = 1$ and assume that $P \in \mathcal{S}$. The set of all C's for which the closed-loop system is internally stable is given by

$$C = \frac{Q}{1 - PQ}, \qquad Q \in \mathcal{S}.$$

Proof.

We begin by proving that if C yields internal stability then there exists $Q \in \mathcal{S}$ such that C can be parameterized in the indicated way.

Suppose that C achieves internal stability. Let Q be the transfer function from r to u, that is

$$Q = \frac{C}{1 + PC}.$$

Then $Q \in \mathcal{S}$ and

$$C = \frac{Q}{1 - PQ},$$

hence the claim.

□

Proof.

Conversely, suppose that $Q \in \mathcal{S}$ and define $C = \dfrac{Q}{1 - PQ}$.

To prove that the controller achieves internal stability we need to consider the nine transfer functions (recall that $F = 1$)

$$\frac{1}{1+PC}\begin{bmatrix} 1 & -P & -1 \\ C & 1 & -C \\ PC & P & 1 \end{bmatrix}$$

and show that these are stable and proper.

Replacing the equation of the controller, and clearing fractions, yields

$$\begin{bmatrix} 1-PQ & -P(1-PQ) & -(1-PQ) \\ Q & 1-PQ & -Q \\ PQ & P(1-PQ) & 1-PQ \end{bmatrix}$$

hence the claim. □

Note that all closed-loop transfer functions are affine functions of Q, that is are of the form $T_1 + T_2 Q$ for some $T_1 \in \mathcal{S}$ and $T_2 \in \mathcal{S}$.

Exercise

Consider a plant P and assume that $P \in \mathcal{S}$. Consider the problem of designing a controller C achieving internal stability and asymptotic tracking of step references.

Show that any controller C which achieves these objectives can be parameterized as

$$C = \frac{Q}{1 - PQ}, \qquad Q \in \mathcal{S}, \qquad Q(0) = \frac{1}{P(0)}.$$

Conclude, consistently with the necessary and sufficient conditions resulting from the Regulator Theory, that the problem has a solution if and only if $P(0) \neq 0$.

Exercise

Consider a plant P and assume that $P \in \mathcal{S}$. Derive the robust stability condition with weight W_2 and show that it is always possible to select $Q \in \mathcal{S}$ such that the resulting controller is robustly stabilizing for any given weight W_2.

Exercise

Consider the plant

$$P(s) = \frac{1}{(s+1)(s+2)}$$

and the problem of designing a controller achieving internal stability and asymptotic tracking of ramp references.

Let

$$Q(s) = k\frac{s+a}{s+b}, \qquad b > 0.$$

Determine, if possible, k*,* a *and* b *such that the resulting controller solves the considered problem. Show that the obtained* Q *is such that* $Q \in \mathcal{S}$*.*

To avoid imposing a stability constraint on Q *one could consider the parameterization*

$$Q(s) = \frac{as+b}{s+p},$$

with $p > 0$ *a priori selected, and* a *and* b *to be determined.*

The parameterization of all controllers in the case of unstable, SISO, plants has been independently developed by Youla and Kucera, and it is often denoted as the YK parameterization.

To obtain such a parameterization we describe the plant is a special form: we write the transfer function P as

$$P = \frac{N}{M},$$

with $N \in \mathcal{S}$ and $M \in \mathcal{S}$, that is N and M are not polynomial, but transfer functions.

The construction of N and M is trivial. For example, recall that $P = Z_P/R_P$ and define

$$N = \frac{Z_P}{(s+1)^k}, \qquad M = \frac{R_P}{(s+1)^k},$$

with k the maximum of the degrees of Z_P and R_P (we will justify this selection shortly).

Definition

The transfer functions $M \in \mathcal{S}$ and $N \in \mathcal{S}$ are coprime if there exist $X \in \mathcal{S}$ and $Y \in \mathcal{S}$ such that

$$NX + MY = 1.$$

Theorem

Two functions $M \in \mathcal{S}$ and $N \in \mathcal{S}$ are coprime if and only if they do not have any common zero in $\mathbb{C}^{\geq 0}$ nor at $s = \infty$.

Proof.

We only prove the necessity. Suppose that M and N are coprime and have a common zero, s_0, in $\mathbb{C}^{\geq 0}$ or at $s = \infty$. Then

$$N(s_0)X(s_0) + M(s_0)Y(s_0) = 0,$$

which gives a contradiction. $\square$

This theorem demonstrates that the selection of the integer k in the construction of the functions M and N is instrumental to rule out common zeros at $s = \infty$.

Controller Parameterization – General Plants

The construction of the functions $X \in \mathcal{S}$ and $Y \in \mathcal{S}$ can be undertaken

- using the so-called Euclid's algorithm;
- with a direct computation, that is parameterizing the numerator and denominator polynomials of X and Y in terms of their coefficients and then deriving a system of equations by equating terms of equal powers;
- using the state-space formulae

  $$X(s) = F(sI - (A + HC))^{-1}H, \qquad Y(s) = -F(sI - (A + HC))^{-1}B,$$

  where A, B, and C are such that

  $$P(s) = C(sI - A)^{-1}B,$$

  and F and H are such that $A + BF$ and $A + HC$ have all their eigenvalues with negative real part.

Note that the last equations are valid also in the MIMO case.

Theorem (YK Parameterization)

Consider the feedback system with $F = 1$. The set of all C's for which the closed-loop system is internally stable is given by

$$C = \frac{X + MQ}{Y - NQ}, \qquad Q \in \mathcal{S}.$$

Note that if P is stable one could select $N = P$, $M = 1$, $X = 0$ and $Y = 1$ and obtain the result for stable plants.

Lemma

Let $C = \frac{N_C}{M_C}$, with $N_C \in \mathcal{S}$ and $M_C \in \mathcal{S}$, be a coprime factorization of C. Then the feedback system is internally stable if and only if $(NN_C + MM_C)^{-1} \in \mathcal{S}$.

Proof of the YK parameterization.

Suppose that $Q \in \mathcal{S}$ and let

$$C = \frac{X + MQ}{Y - NQ}.$$

Define

$$N_C = X + MQ, \qquad M_C = Y - NQ$$

and note that $N_C \in \mathcal{S}$, $M_C \in \mathcal{S}$ and that $NN_C + MM_C = 1$. Hence M_C and N_C are a coprime factorization of C with the property that $(NN_C + MM_C)^{-1} = 1 \in \mathcal{S}$, which proves the claim.

The proof of the converse statement, that is of the fact that for any controller C achieving internal stability there is a $Q \in \mathcal{S}$ such that C can be written in the indicated form, is constructive, that is one explicitly determine Q from C.

The details can be found in the literature.

□

Exercise

Let $P(s) = \dfrac{1}{(s-1)(s-2)}$ *and select*

$$N(s) = \frac{1}{(s+1)^2}, \qquad M(s) = \frac{(s-1)(s-2)}{(s+1)^2}.$$

Show that

$$X(s) = \frac{19s-11}{s+1}, \qquad Y(s) = \frac{s+6}{s+1}$$

are such that $NX + MY = 1$.

Hence determine a stabilizing controller selecting $Q = 0$.

Suppose that the stabilizing controller has to be such that the closed-loop system is of type 1*, that is the controller has a pole at* $s = 0$*. Determine* Q *such that this requirement is satisfied.*

We now consider the H_∞ control problem and provide a systematic solution based on the solution of two coupled Algebraic Riccati Inequalities (ARIs): the first to solve the state feedback H_∞ control problem and the second to solve a robust estimation problem.

The state feedback H_∞ control problem is the problem of designing a state feedback controller such that the system (note that $D_{11} = 0$)

$$\dot{x} = Ax + B_1 w + B_2 u, \qquad z = C_1 x + D_{12} u$$

in closed-loop with $u = Kx$, for some K to be found, has the following properties:

(S) the matrix $A + B_2 K$ has all eigenvalues with negative real part;

(H$_\gamma$) the closed-loop system with input w and output z has H_∞ norm, that is L_2-gain, smaller than a given $\gamma > 0$.

In what follows we assume, for simplicity, that

$$C_1^\top D_{12} = 0, \qquad D_{12}^\top D_{12} = I.$$

Theorem

Consider the system

$$\dot{x} = Ax + B_1 w + B_2 u, \qquad z = C_1 x + D_{12} u.$$

Assume that $C_1^\top D_{12} = 0$ *and* $D_{12}^\top D_{12} = I$.

Let $\gamma > 0$ *be given. Suppose there exists a state feedback control law* $u = Kx$ *such that (S) and* (H_γ) *hold.*

Then there exists a matrix $P = P^\top > 0$ *such that*

$$A^\top P + PA + P\frac{B_1 B_1^\top}{\gamma^2}P - PB_2 B_2^\top P + C_1^\top C_1 < 0.$$

Proof.

By assumption the eigenvalues of $A + B_2K$ have negative real part and there exists a matrix $\tilde{P} = \tilde{P}^\top > 0$ such that

$$(A + B_2K)^\top \tilde{P} + \tilde{P}(A + B_2K) + \tilde{P}\frac{B_1B_1^\top}{\gamma^2}\tilde{P} + C_1^\top C_1 + K^\top K < 0.$$

Note now that the ARI can be re-written as

$$A^\top \tilde{P} + \tilde{P}A + \tilde{P}\frac{B_1B_1^\top}{\gamma^2}\tilde{P} - PB_2B_2^\top P + C_1^\top C_1 + (K + B_2\tilde{P})^\top(K + B_2\tilde{P}) < 0,$$

hence

$$A^\top \tilde{P} + \tilde{P}A + \tilde{P}\frac{B_1B_1^\top}{\gamma^2}\tilde{P} - \tilde{P}B_2B_2^\top \tilde{P} + C_1^\top C_1 < -(K + B_2^\top \tilde{P})^\top(K + B_2^\top \tilde{P}) \leq 0,$$

which proves the claim with $P = \tilde{P}$.

□

Theorem

Consider the system

$$\dot{x} = Ax + B_1 w + B_2 u, \qquad z = C_1 x + D_{12} u.$$

Assume that $C_1^\top D_{12} = 0$, $D_{12}^\top D_{12} = I$ *and that the system is stabilizable.*

Let $\gamma > 0$ *be given.*

Suppose there exists a matrix $P = P^\top > 0$ *such that*

$$A^\top P + PA + P\frac{B_1 B_1^\top}{\gamma^2}P - PB_2 B_2^\top P + C_1^\top C_1 < 0.$$

Then the state feedback $u = -B_2^\top Px$ *is such that the closed-loop system satisfies the conditions (S) and* (H_γ).

Proof.

The ARI can be written as

$$(A + B_2K)^\top P + P(A + B_2K) + P\frac{B_1B_1^\top}{\gamma^2}P + C_1^\top C_1 + K^\top K < 0,$$

with $K = -B_2^\top P$. Hence

$$(A + B_2K)^\top P + P(A + B_2K) < -P\frac{B_1B_1^\top}{\gamma^2}P - C_1^\top C_1 - K^\top K \leq 0,$$

which directly implies (S) and (H_γ). □

The foregoing discussion suggests that the function $S(x) = x^\top P x$ can be used as a storage function with respect to the supply rate $\gamma^2\|w\|^2 - \|z\|^2$ for a specific selection of u.

(Note that there is a factor $\frac{1}{2}$ missing in the definitions of the supply rate and of the storage function when compared with the general case, which does not affect the results.)

The dissipative systems perspective allow understanding the role of the term $P\dfrac{B_1 B_1^\top}{\gamma^2}P$ in the ARI. In fact, to have dissipativity with respect to the given supply rate one must have

$$\dot{S} = x^\top P(Ax + B_1 w + B_2 u) + (Ax + B_1 w + B_2 u)^\top P x \le \gamma^2 w^\top w - (u^\top u + x^\top C_1^\top C_1 x)$$

or, equivalently,

$$x^\top P(Ax + B_1 w + B_2 u) + (Ax + B_1 w + B_2 u)^\top P x - \gamma^2 w^\top w + u^\top u + x^\top C_1^\top C_1 x \le 0,$$

for all w and some selection of u.

This latter equation is convex in u and concave in w. Since we are looking for the existence of a u such that the inequality holds for all w, we can rewrite the inequality as

$$x^\top \left(A^\top P + PA + P\frac{B_1 B_1^\top}{\gamma^2}P + PB_2B_2^\top P + C_1^\top C_1\right)x - \gamma^2(w-w^\star)^\top(w-w^\star) + (u-u^\star)^\top(u-u^\star) \leq 0,$$

with

$$(w^\star, u^\star) = \left(\frac{B_1^\top P}{\gamma^2}x, -B_2 P x\right)$$

the saddle point of the considered function in terms of w and u.

This equation reveals that the H_∞ control problem can be interpreted as a differential game, in which the disturbance acts to maximize the cost and the control to minimize it.

It also reveals that while the state feedback controller solving the problem is $u = u^\star$, the disturbance $w = w^\star$ is the worst case disturbance.

The points of departure for the solution of the more realistic measurement feedback H_∞ control problem are the design of a state observer, to replace the unmeasured state with an asymptotic estimate, and the redefinition of the control signal in the spirit of the separation principle.

The design of the observer is, however, made difficult by the presence of the exogenous input w in the dynamic equation and in the measured variable y.

To deal with this problem suppose that the observer is designed assuming that the exogenous signal always *plays* in the worst possible way, that is $w = w^\star$ and that the separation principle can be used.

As a result the dynamics of the observer is described by the equation

$$\dot{\xi} = \left(A + \frac{B_1 B_1^\top}{\gamma^2} P \right) \xi + B_2 u + G(y - C_2 \xi),$$

with G an injection gain to be designed and the control signal is given by

$$u = -B_2^\top P \xi.$$

To study the resulting closed-loop system define the variable $e = x - \xi$ and note that (assuming $D_{22} = 0$)

$$\dot{e} = \left(A + \frac{B_1 B_1^\top}{\gamma^2} P - G C_2 \right) e - \frac{B_1 B_1^\top}{\gamma^2} P x + (B_1 - G D_{21}) w,$$

$$u = -B_2^\top P x + B_2^\top P e, \qquad z = (C_1 + D_{12} B_2^\top P) x - D_{12} B_2^\top P e,$$

hence the closed-loop system with state (x, e), input w and output z is described by

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \overbrace{\begin{bmatrix} A - B_2 B_2^\top P & B_2 B_2^\top P \\ -\frac{B_1 B_1^\top}{\gamma^2} P & A + \frac{B_1 B_1^\top}{\gamma^2} P - G C_2 \end{bmatrix}}^{A_{cl}} \begin{bmatrix} x \\ e \end{bmatrix} + \overbrace{\begin{bmatrix} B_1 \\ B_1 - G D_{21} \end{bmatrix}}^{B_{cl}} w,$$

$$z = \overbrace{\begin{bmatrix} C_1 + D_{12} B_2^\top P & -D_{12} B_2^\top P \end{bmatrix}}^{C_{cl}} \begin{bmatrix} x \\ e \end{bmatrix}.$$

The objective is therefore to derive conditions on G such that the closed-loop system has L_2-gain smaller than γ, that is conditions for the existence of G and $\mathcal{P} = \mathcal{P}^\top > 0$ such that

$$A_{cl}^\top \mathcal{P} + \mathcal{P} A_{cl} + \mathcal{P} \frac{B_{cl} B_{cl}^\top}{\gamma^2} \mathcal{P} + C_{cl}^\top C_{cl} < 0.$$

To begin with assume that the matrix $\mathcal{P}$ is defined as

$$\mathcal{P} = \begin{bmatrix} P & 0 \\ 0 & Z \end{bmatrix},$$

where $P = P^\top > 0$ is the solution of the ARI arising in the state feedback problem and $Z = Z^\top > 0$ is to be found (as a solution of some *filtering* ARI).

Suppose, finally, that in addition to the structural assumptions considered so far one has

$$D_{21} B_1^\top = 0, \qquad D_{21} D_{21}^T = I.$$

A direct computation yields the following terms.

$\underline{A_{cl}^\top \mathcal{P} + \mathcal{P} A_{cl}}$:

$$\begin{bmatrix} A^\top P + PA - PB_2B_2^\top P - PB_2B_2^\top P & PB_2B_2^\top P - P\dfrac{B_1B_1^\top}{\gamma^2}Z \\ PB_2B_2^\top P - Z\dfrac{B_1B_1^\top}{\gamma^2}P & \left(A + \dfrac{B_1B_1^\top}{\gamma^2}P - GC_2\right)^\top Z + Z\left(A + \dfrac{B_1B_1^\top}{\gamma^2}P - GC_2\right) \end{bmatrix}$$

$\underline{\mathcal{P}\dfrac{B_{cl}B_{cl}^\top}{\gamma^2}\mathcal{P}}$:

$$\frac{1}{\gamma^2}\begin{bmatrix} PB_1B_1^\top P & PB_1B_1^\top Z \\ ZB_1B_1^\top P & ZB_1B_1^\top Z + ZGG^\top Z \end{bmatrix}$$

$\underline{C_{cl}^\top C_{cl}}$:

$$\begin{bmatrix} C_1C_1^\top + PB_2B_2^\top P & -PB_2B_2^\top P \\ -PB_2B_2^\top P & PB_2B_2^\top P \end{bmatrix}$$

As a result

$$A_{cl}^\top \mathcal{P} + \mathcal{P} A_{cl} + \mathcal{P} \frac{B_{cl} B_{cl}^\top}{\gamma^2} + C_{cl}^\top C_{cl} = \begin{bmatrix} A^\top P + PA + \frac{B_1 B_1^\top}{\gamma^2} P - P B_2 B_2^\top P + C_1 C_1^\top & 0 \\ 0 & \star \end{bmatrix},$$

where

$$\star = \left(A + \frac{B_1 B_1^\top}{\gamma^2} P - G C_2 \right)^\top Z + Z \left(A + \frac{B_1 B_1^\top}{\gamma^2} P - G C_2 \right) + Z \frac{B_1 B_1^\top}{\gamma^2} Z + Z \frac{G G^\top}{\gamma^2} Z + P B_2 B_2^\top P.$$

Note that the $(1,1)$ block is negative definite, since $P = P^\top > 0$ is a solution of the ARI arising in the state feedback problem. We therefore have to provide a condition on G and $Z = Z^\top > 0$ such that the $(2,2)$ block is also negative definite.

To this end note that

$$\left(A+\frac{B_1B_1^\top}{\gamma^2}P-GC_2\right)^\top Z+Z\left(A+\frac{B_1B_1^\top}{\gamma^2}P-GC_2\right)+Z\frac{B_1B_1^\top}{\gamma^2}Z+Z\frac{GG^\top}{\gamma^2}Z+PB_2B_2^\top P=$$

$$\left(A+\frac{B_1B_1^\top}{\gamma^2}P\right)^\top Z+Z\left(A+\frac{B_1B_1^\top}{\gamma^2}P\right)+Z\frac{B_1B_1^\top}{\gamma^2}Z+PB_2B_2^\top P-\gamma^2C_2C_2^\top+$$

$$+\left(\frac{ZG}{\gamma}-\gamma C_2^\top\right)\left(\frac{G^\top Z}{\gamma}-\gamma C_2\right),$$

which suggests selecting $G=\gamma^2Z^{-1}C_2^\top$.

Theorem

Consider the system

$$\dot{x} = Ax + B_1 w + B_2 u, \qquad z = C_1 x + D_{12} u, \qquad y = C_2 x + D_{21} w.$$

Assume that $C_1^\top D_{12} = 0$, $D_{12}^\top D_{12} = I$, $D_{21} B_1^\top = 0$, *and* $D_{21} D_{21}^\top = I$.

Let $\gamma > 0$ *be given. Suppose there exist positive definite matrices* P *and* Z *such that*

$$A^\top P + PA + P\frac{B_1 B_1^\top}{\gamma^2}P - PB_2 B_2^\top P + C_1^\top C_1 < 0$$

and

$$\left(A + \frac{B_1 B_1^\top}{\gamma^2}P\right)^\top Z + Z\left(A + \frac{B_1 B_1^\top}{\gamma^2}P\right) + Z\frac{B_1 B_1^\top}{\gamma^2}Z + PB_2 B_2^\top P - \gamma^2 C_2 C_2^\top < 0.$$

Then the measurement feedback controller

$$\dot{\xi} = \left(A + \frac{B_1 B_1^\top}{\gamma^2}P - B_2 B_2^\top P - GC_2\right)\xi + Gy, \qquad u = -B_2^\top P\xi,$$

with $G = \gamma^2 Z^{-1} C_2^\top$, *is such that (S) and* (H_γ) *hold.*

The second ARI in the unknown Z is coupled to the first one, that is the equation depends on the solution P of the *control* ARI.

It is possible to decouple the *filtering* ARI defining the matrix $Y = Y^\top > 0$ such that

$$YA^\top + AY + B_1 B_1^\top - YC_2^\top C_2 Y + Y \frac{C_1^\top C_1}{\gamma^2} Y \leq 0,$$

and noting that

$$Z = \gamma^2 Y^{-1} - P$$

is a solution of the *filtering* ARI provided $\rho(PY) < \gamma^2$, where $\rho(\cdot)$ denotes the spectral radius, that is the largest eigenvalue, of a matrix.

The sufficient condition that we have derived is actually also necessary, under suitable assumptions. In addition, the ARIs can be replaced by AREs.

Theorem

Consider the system

$$\dot{x} = Ax + B_1 w + B_2 u, \qquad z = C_1 x + D_{12} u, \qquad y = C_2 x + D_{21} w.$$

Assume that $C_1^\top D_{12} = 0$*,* $D_{12}^\top D_{12} = I$*,* $D_{21} B_1^\top = 0$*, and* $D_{21} D_{21}^\top = I$*. Let* $\gamma > 0$*. Suppose that the following holds.*

(L1) *The pair* (A, B_1) *is stabilizable.*

(L2) *The pair* (A, C_1) *is detectable.*

(L3) *There exists a matrix* $P = P^\top > 0$ *solving the ARE*

$$A^\top P + PA + P\frac{B_1 B_1^\top}{\gamma^2}P - PB_2 B_2^\top P + C_1^\top C_1 = 0.$$

(L4) *There exists a matrix* $Y = Y^\top > 0$ *solving the ARE*

$$YA^\top + AY + B_1 B_1^\top - YC_2^\top C_2 Y + Y\frac{C_1^\top C_1}{\gamma^2}Y = 0.$$

(L5) $\rho(PY) < \gamma^2$.

Then there exists a measurement feedback controller such that the closed-loop system has L_2*-gain from* w *to* z *smaller or equal to* γ*.*

Theorem

Consider the system

$$\dot{x} = Ax + B_1 w + B_2 u, \qquad z = C_1 x + D_{12} u, \qquad y = C_2 x + D_{21} w.$$

Assume that $C_1^\top D_{12} = 0$, $D_{12}^\top D_{12} = I$, $D_{21} B_1^\top = 0$, *and* $D_{21} D_{21}^\top = I$. *Let* $\gamma > 0$.

Suppose (L1) and (L2) hold and there exists a linear controller such that the closed-loop system has L_2*-gain smaller than* γ.

Then conditions (L3), (L4) and (L5) hold.

The proposed controller is not the only one solving the measurement feedback H_∞ control problem. All controllers solving the problem can be parameterized in terms of a *parameter*, that is a system with L_2-gain smaller or equal to γ.

Letting $\gamma \to \infty$ one relaxes the *disturbance attenuation* requirement and the P equation reduces to the ARE arising in the LQR problem (with P related to the optimal cost), whereas the Y equation is the ARE arising in the stationary Kalman filter (with Y related to the steady-state covariance of the state estimation error).

The P and Z equations have a direct nonlinear counterpart and allows solving the measurement feedback H_∞ control problems for a class of nonlinear affine, in w and u, systems.

Conclusions

We have introduced the basic tools for the analysis and design of robust control systems. In particular, we have presented conditions for robust stability and robust performance which, in the SISO case, have simple graphical interpretations based on the use of the Nyquist plot of the loop gain. For MIMO systems the use of the small gain theorem allows deriving equivalent conditions.

We have highlighted that robust control design may be undertaken from various perspective, to be selected on the basis of the structure and complexity of the problem and of the design specifications.

Regardless of the design approach that is pursued there are, however, *fundamental* design constraints that have to be taken into consideration, otherwise one may pose an unsolvable problems.

The design constraints limit the level of achievable performance and arise from two sources: algebraic relations among the various transfer functions and the requirement that the closed-loop system be stable.

The first constraint is due to the relation between S and T, that is

$$S + T = 1.$$

In particular, for any ω, $|S(j\omega)|$ and $|T(j\omega)|$ cannot be simultaneously smaller than $1/2$.

The second constraint is expressed by the fact that the condition

$$\min\{|W_1(j\omega)|, |W_2(j\omega)|\} < 1, \qquad \forall\omega$$

is necessary for robust performance. To see this, fix ω and assume $|W_1(j\omega)| \leq |W_2(j\omega)|$. Then (dropping arguments)

$$|W_1| = |W_1(S+T)| \leq |W_1 S| + |W_1 T| \leq \| |W_1 S| + |W_2 T| \|_\infty .$$

Hence, robust performance, that is $\| |W_1 S| + |W_2 T| \|_\infty < 1$ implies $|W_1| < 1$ and $\min\{|W_1(j\omega)|, |W_2(j\omega)|\} < 1$. The same conclusion can be drawn when $|W_2| \leq |W_1|$.

This means that at every frequency either $|W_1(j\omega)|$ or $|W_2(j\omega)|$ must be less than 1. Typically, $|W_1(j\omega)|$ is monotonically decreasing, for good tracking at low frequency, and $|W_2(j\omega)|$ is monotonically increasing, since uncertainty increases with frequency.

The third constraint relates to the values of S and T at specific points of $\mathbb{C}^{\geq 0}$.

Let p be a pole of L in $\mathbb{C}^{\geq 0}$ and z be a zero of L in $\mathbb{C}^{\geq 0}$.

Then

$$S(p) = 0, \qquad S(z) = 1, \qquad T(p) = 1, \qquad T(z) = 0.$$

These conditions are often called interpolation constraints and yield conditions on the weights W_1 and W_2. In fact

$$|W_1(z)| = |W_1(z)S(z)| \leq \sup_{s \geq 0} |W_1(s)S(s)| = \|W_1 S\|_\infty,$$

where the inequality is a consequence of the so-called *maximum modulus theorem* and of the fact that the system is internally stable, and, similarly,

$$|W_2(p)| \leq \|W_2 T\|_\infty.$$

As a result, a necessary condition for the performance criterion $\|W_1 S\|_\infty < 1$ is that $|W_1(z)| < 1$ and a necessary condition for the robust stability criterion $\|W_2 T\|_\infty < 1$ is that $|W_2(p)| < 1$.

Conclusions

One could also obtain (with some additional work) more accurate constraints, such as (recall that p is a pole of L in $\mathbb{C}^{\geq 0}$ and z is a zero of L in $\mathbb{C}^{\geq 0}$)

$$\|W_1 S\|_\infty \geq \left| W_1(z) \frac{z+p}{z-p} \right|, \qquad \|W_2 T\|_\infty \geq \left| W_2(p) \frac{p+z}{p-z} \right|.$$

Thus if there are a pole and a zero in $\mathbb{C}^{\geq 0}$ close to each other they *amplify* their effect on the performance/robust stability criterion and dictate bounds on the weights W_1 and W_2.

Finally, let $p > 0$ and consider the plant

$$P(s) = \frac{s-1}{(s+1)(s-p)}.$$

Then, for any $0 \leq \omega_1 < \omega_2$ there exist $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \log_{10} \max_{\omega_1 \leq \omega \leq \omega_2} |S(j\omega)| + c_2 \log_{10} \|S\|_\infty \geq \log_{10} \left| \frac{1+p}{1-p} \right|.$$

This means that if one wishes to reduce S is a frequency band for good tracking performance, then S is forced to be large, in the H_∞ sense, affecting the stability margin. This conclusion, illustrated on a specific example, is valid in general and constitutes what is known as the *waterbed effect*.

We conclude this discussion with the introduction of an integral constrain.

Let p_i be the poles of L in $\mathbb{C}^{\geq 0}$ and assume the relative degree of L is at least 2.

Then

$$\int_0^\infty \log_{10} |S(j\omega)| d\omega = \pi \log_{10} e \sum_i \mathrm{Re}(p_i)$$

This implies that the *negative area* of $|S(j\omega)|$, that is the area below the line $|S| = 1$, has to be compensated by the *positive area* of $|S(j\omega)|$, that is the area above the line $|S| = 1$.

The consideration of all the listed constraints may render the formulation of a specific robust control problem challenging. Such a consideration is, however, necessary to formulate a feasible and meaningful problem.