Prediction with function approximation

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Machine and Reinforcement Learning in Control Applications

Introduction

- If the state space is large, then tabular methods are not implementable.
- Use function approximation
  - introduce weight vector $\mathbf{w} \in \mathbb{R}^d$, with $d \ll |\mathcal{S}|$;
  - approximate $v_\pi(s)$ with $\hat{v}(s, \mathbf{w})$.
- Adjusting $\mathbf{w}$ changes the value to several states.
- Makes RL applicable to partially observable MDPs.

Molto più piccola dello spazio di stato

da ricostruire tramite approssimazioni

cambiando una componente si cambiano molti più stati. Quindi abbiamo una maggiore generalizzazione

Target update

Tabellari eprché associano il ritoro specifico ad uno stato specifico. Si aggiorna l'assegnazione del valore uno stato per volta

- Several types of updates
  - MC update
  $$S_t \mapsto G_t;$$
  - TD(0) update
  $$S_t \mapsto R_{t+1} + \gamma V(S_{t+1});$$
  - n-step TD update
  $$S_t \mapsto G_{t:t+n};$$
  - DP update
  $$S_t \mapsto \mathbb{E}_\pi[R_{t+1} + \gamma V(S_{t+1})|S_t].$$
- With function approximation $V(S_t)$ is substituted by

Perché cerchiamo di far tendere la funzione ad un valore.

$$\hat{v}(S_t, \mathbf{w}_t).$$

L'aggiornamento non sarà sullo stato ma sui pesi, in maniera che sia il più vicina possivbile alla funzione valore

- Use *supervised learning* to mimic input-output relations
  - need to deal with non-stationary relations.

Prediction objective

Dobbiamo quantificare l'errore che stiamo commettendo per il risolvere un rpbolema di ottimizzazione. Il vettore di pesi cambia l'approssimazione ovunque. Se si rende migliore la stima per uno stato, la stima per gli altri stati potrebbe diventare peggiore.

Potenzialmente lo sp. di stato è infinito, siccome la funzione è finita non possiamo approssimare perfettamente gli altri stati. La approssima per gli stati che mi interessano di più (i oiù visitati) deve essere rispetto all'esperienza che ho avuto.

- In tabular methods, an update at one state affected no other.
- Making a state's estimate better makes other less accurate.
- We need to specify which state we care most about ON-POLICY DISTRIBUTION
  - use a distribution μ such that $\mu(s) \geqslant 0$, $\sum_s \mu(s) = 1$; (o anche l'integrale)
  - often $\mu(s)$ is the fraction of time spent in s.

Misura il tempo speso in un determinato stato

- The objective is to minimize the mean square value error

$$\overline{\mathrm{VE}}(\mathbf{w}) = \sum_s \mu(s) \left(v_\pi(s) - \hat{v}(s, \mathbf{w})\right)^2 .$$

PROBLEMA NON CONVESSO

- Often just a local optimum is achievable.

Gradient descent

- $\mathbf{w}$ is a vector of real valued numbers.
- $\hat{v}(s, \mathbf{w})$ is a real valued differentiable function.
- Assume that
  - states S_t are presented with distribution μ;
  - you can measure the true value $v_\pi(S_t)$. (per assurdo)
- Update $\mathbf{w}$ using stochastic (one-sample) gradient descent Poiché basato su un solo campione

Deve essere molto piccolo 1e-7

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \tfrac{1}{2}\alpha\nabla\left(v_\pi(S_t) - \hat{v}(S_t, \mathbf{w}_t)\right)^2$$ Errore sul singolo campione nell'istante t

effettuo un passo nella direzione del gradiente

$$= \mathbf{w}_t + \alpha\left(v_\pi(S_t) - \hat{v}(S_t, \mathbf{w}_t)\right)\nabla\hat{v}(S_t, \mathbf{w}_t),$$

where α is a small parameter and

$$\nabla\hat{v}(S_t, \mathbf{w}) = \left[\begin{array}{ccc} \frac{\partial\hat{v}(S_t,\mathbf{w})}{\partial w_1} & \cdots & \frac{\partial\hat{v}(S_t,\mathbf{w})}{\partial w_d} \end{array}\right]^\top.$$

Approximation of stochastic gradient descent

- Let U_t be a stochastic approximation of $v_\pi(S_t)$.
- The general SGD is then

$$\mathbf{w}_{t+1} = \mathbf{w}_t + \alpha(U_t - \hat{v}(S_t, \mathbf{w}_t))\nabla\hat{v}(S_t, \mathbf{w}_t).$$

- If $\mathbb{E}[U_t|S_t = s] = v_\pi(s)$ and α satisfies the usual conditions
  - $\mathbf{w}_t$ converges to a local optimum.
- In general α has to be small since moving toward a single sample may be detrimental for others.

Gradient Monte Carlo algorithm

Gradient Monte Carlo algorithm

Input: $\alpha > 0$, $\hat{v}$

Output: approximate of v_π

Initialization

$\mathbf{w} \leftarrow$arbitrarily

Loop

generate an episode following π: $S_0, A_0, R_1, S_1, A_1, \ldots, R_T, S_T$

$G \leftarrow 0$

for each step $t = T-1, T-2, \ldots, 0$ **do**

$G \leftarrow \gamma G + R_{t+1}$

$\mathbf{w} \leftarrow \mathbf{w} + \alpha(G - \hat{v}(S_t, \mathbf{w}))\nabla\hat{v}(S_t, \mathbf{w})$

Bootstrapping with function approximation

Il problema è che dobbiamo aspetare la fine dell'episodio. Tramite il bootstrapping (stima corrente per predire il futuro)

- MC SGD converges since U_t is an unbiased estimate of v_π.
- By bootstrapping the target depends on $\mathbf{w}_t$
  - the target is biased;
  - it is not a true SGD.
- Account for changes on the estimate, but ignore changes on target caused by varied $\mathbf{w}$
  - usually referred to as *semi-gradient* methods.

MC : $U_t \leftarrow G_t$

↳ ritorno

TD : $U_t \leftarrow R_{t+1} + \gamma \hat{v}(S_{t+1}, w_t)$

Problematica di Bias (preconcetto)

$\mathbb{E}[R_{t+1} + \gamma \hat{v}(S_{t+1}, w_t)] \neq v_\pi(S_t)$

Problematica del calcolo del gradiente

$$\frac{1}{2} \alpha \nabla \left(v_\pi(S_t) - \hat{v}(S_t, w) \right) \neq \alpha \left(R_{t+1} + \gamma \hat{v}(S_{t+1}, w_t) - \hat{v}(S_t, w_t) \right) \nabla \hat{v}(S_t, w)$$

$R_{t+1} + \gamma \hat{v}(S_{t+1}, w)$

Per risolvere queste problematica, si considera come se queste problematiche non ci fossero. L'aggiornamento effettuato solo sui pesi del secondo termine

Semi-gradient TD(0) algorithm

Semi-gradient TD(0) algorithm

Input: $\alpha > 0$, $\hat{v}$ such that $\hat{v}(\text{terminal}, \cdot) = 0$
Output: approximate of v_π

Initialization

$\mathbf{w} \leftarrow$arbitrarily

Loop

initialize S
for each step of the episode **do**
$A \leftarrow \pi(\cdot|S)$
take action A and observe R, S'
$\mathbf{w} \leftarrow \mathbf{w} + \alpha(R + \gamma\hat{v}(S', \mathbf{w}) - \hat{v}(S, \mathbf{w}))\nabla\hat{v}(S, \mathbf{w})$
$S \leftarrow S'$
if S is terminal **then**
reinitialize the episode

Semi gradiente perché non considera la sua variazione (diepndenza dai pesi)

Linear methods

- The approximate function is **linear** with respect to $\mathbf{w}$.
- Associate to every $s \in \mathcal{S}$ a *feature vector*

$$\mathbf{x}(s) = \begin{bmatrix} x_1(s) & \cdots & x_d(s) \end{bmatrix}^\top.$$

- The approximate function is linear

$$\hat{v}(s, \mathbf{w}) = \mathbf{w}^\top \mathbf{x}(s).$$

- The SGD reduces to a simple form

$$\mathbf{w}_{t+1} = \mathbf{w}_t + \alpha(U_t - \hat{v}(S_t, \mathbf{w}_t))\mathbf{x}(S_t).$$

Convergence of linear methods

- The MC algorithm converges to the global optimum of $\overline{\text{VE}}$.
- The TD algorithm update is

Vettore delle feature nello stato successivo

$$\begin{aligned}\mathbf{w}_{t+1} &= \mathbf{w}_t + \alpha(R_{t+1} + \gamma \mathbf{w}_t^\top \mathbf{x}_{t+1} - \mathbf{w}_t^\top \mathbf{x}_t)\mathbf{x}_t \\ &= \mathbf{w}_t + \alpha(R_{t+1}\mathbf{x}_t - \mathbf{x}_t(\mathbf{x}_t - \gamma \mathbf{x}_{t+1})^\top \mathbf{w}_t).\end{aligned}$$

- The expected next weight is

=0 per avere convergenza

$$\mathbf{E}[\mathbf{w}_{t+1}|\mathbf{w}_t] = \mathbf{w}_t + \alpha(\mathbf{b} - \mathbf{A}\mathbf{w}_t),$$

where $\mathbf{b} = \mathbb{E}[R_{t+1}\mathbf{x}_t]$ and $\mathbf{A} = \mathbb{E}[\mathbf{x}_t(\mathbf{x}_t - \gamma \mathbf{x}_{t+1})^\top]$.

- If TD converges, it converges to

$$\mathbf{w}_{\text{TD}} = \mathbf{A}^{-1}\mathbf{b}.$$

Notes on linear TD methods

- It can be proved that

$$\overline{\mathrm{VE}}(\mathbf{w}_{\mathrm{TD}}) \leqslant \frac{1}{1-\gamma} \min_{\mathbf{w}} \overline{\mathrm{VE}}(\mathbf{w}).$$

- Estimate may be biased.
- Variance is usually reduced with respect to MC.
- Can be extended to the n-step case.

Least squares temporal-difference

- Compute estimates of $\mathbf{A}$ and $\mathbf{b}$ and directly compute $\mathbf{w}_{\mathrm{TD}}$

$$\begin{aligned}\hat{\mathbf{A}}_t &= \textstyle\sum_{k=0}^{t-1} \mathbf{x}_k(\mathbf{x}_k - \gamma\mathbf{x}_{k+1})^\top + \epsilon I,\\ \hat{\mathbf{b}}_t &= \textstyle\sum_{k=0}^{t-1} R_{k+1}\mathbf{x}_k,\\ \hat{\mathbf{w}}_t &= \hat{\mathbf{A}}_t^{-1}\hat{\mathbf{b}}_t.\end{aligned}$$

- Note that, letting $\hat{\mathbf{A}}_0^{-1} = \epsilon^{-1}\mathbf{I}$,

$$\hat{\mathbf{A}}_t^{-1} = \hat{\mathbf{A}}_{t-1}^{-1} - \frac{\hat{\mathbf{A}}_{t-1}^{-1}\mathbf{x}_{t-1}(\mathbf{x}_{t-1} - \gamma\mathbf{x}_t)^\top \hat{\mathbf{A}}_{t-1}^{-1}}{1 + (\mathbf{x}_{t-1} - \gamma\mathbf{x}_t)^\top \hat{\mathbf{A}}_{t-1}^{-1}\mathbf{x}_{t-1}}.$$

- Complexity $O(d^2)$ vs $O(d)$.

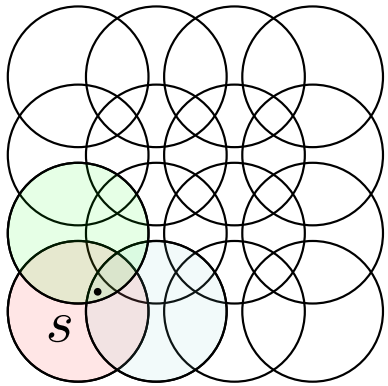
- Does not require a step size.

Feature selection

- Choosing features appropriate to the task is an important way of adding prior domain knowledge.
- The features should correspond to the aspects of the state space along which generalization may be appropriate.
- Linear forms cannot take into account any interactions between features if coded separately.

Features

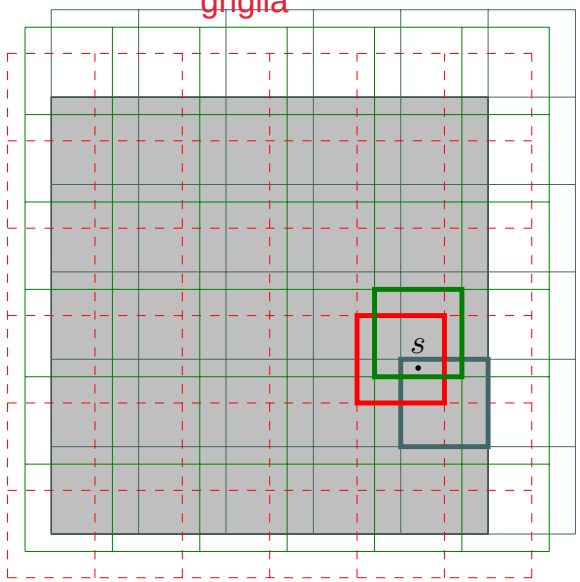
- Polynomials: $s_1, s_2, s_1^2, s_2^2, s_1 s_2, \ldots$
  - grows exponentially with the dimension of the state space.
- Fourier basis: $\cos(i\pi s)$, $s \in [0, 1]$
  - useful for periodic functions or bounded domains;
  - if approximating over $[0, \frac{\tau}{2}]$, you can just use cosines;
  - grows exponentially with the dimension of the state space.
- Coarse coding: features $\mathbf{x} \in \{0, 1\}$
  - each circle has a corresponding weight;
  - the weights of all circles intersecting a state are affected;
  - large feature give broad generalizations;
  - acuity is controlled by the total number of features.



Tile coding

- Use partitions of the state space
  - *e.g.*, grids.
- Overlapping receptive fields
  - tiles of a partition do not overlap.
- Exactly one feature in each tiling.
- Best generalization if offset asymmetrically.
- It may be desirable to use different shaped tiles in different tilings.
- Hashing may reduce the curse of dimensionality.

Una feature per ogni casella di ogni griglia



L'intersezione rappresenta la generalizzazione

OFFSET ASIMMETRICO di numeri dispari lungo gli assi

Radial basis functions

- Generalization of coarse coding.
- Feature can be anything in the interval $[0, 1]$.
- Usually Gaussian shaped functions

$$x_i(s) = \exp\left(-\frac{\|s - c_i\|^2}{\sigma_i^2}\right).$$

- Differentiable approximate functions.
- Greater computational complexity.

Step-size

- Theoretically the step-size should decrease slowly

$$\alpha_t = \frac{1}{t}.$$

- A good rule of thumb for setting the step-size parameter is

$$\alpha = (\tau \mathbb{E}[\mathbf{x}^\top \mathbf{x}])^{-1},$$

tau valore piccolo

where $\mathbf{x} \simeq \mu$.

- In tile coding, a reasonable choice is

$$\alpha = \frac{1}{10n},$$

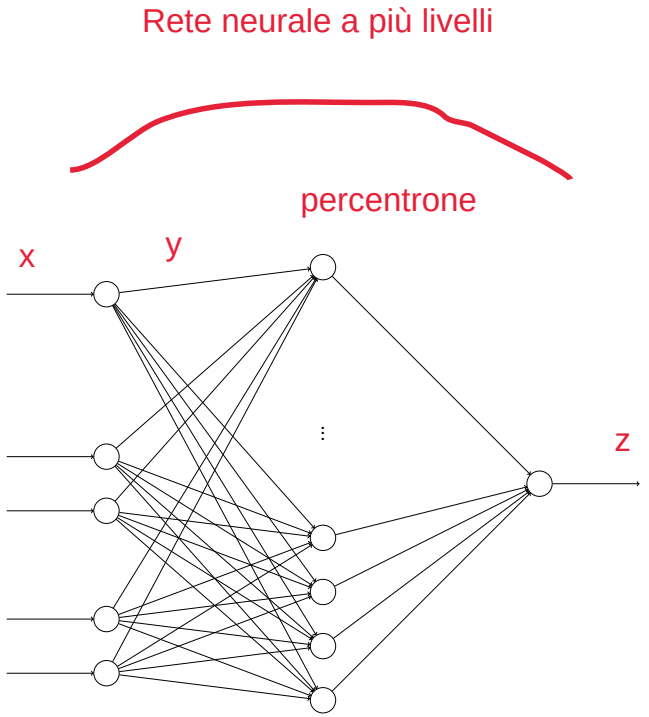
where n is the number of tiles.

griglie

In pratica si sceglie un alpha costante

Feedforward neural networks

- No loops in the network.
- A real-valued weight is associated with each link.
- The units compute a weighted sum of their inputs and then apply their *activation function*.
- With one hidden layer, universal approximation.
- Deeper architectures are capable of more general abstraction.
- Gradient computed via back-propagation.



Inter player (non si vede ne da ingresso ne da uscits)

Memory-based function approximation

- Non-parametric approach.
- Store samples as they are gathered
  - without updating any parameter.
- Combine samples to output estimated values for query state
  - retrieve samples from memory whose states are judged to be the most relevant to the query state;
  - local approximation discarded after evaluation.
- The simplest forms are
  - nearest neighborhood: return value of closest state;
  - weighted average: locally fits a function.
- Approximation improves as more samples are gathered
  - but requires particular data structures.
- Do not rely on particular functional forms.

Niente bias perché non limitiamo la rete neurale a un certo numero di laye

Kernel-based methods

basati sulla memoria

- In weighted average, distance is used to measure influence.
- We can use a *kernel* $k(s, s')$ to measure the strength of generalization from s to s', *e.g.*,

$$k(s, s') = \exp\left(-\frac{\|s - s'\|^2}{\sigma^2}\right).$$

- Letting $\mathcal{D}$ be the stored data and g the corresponding value

va,lore memorizzato

$$\hat{v}(s, \mathcal{D}) = \sum_{s' \in \mathcal{D}} g(s')k(s, s').$$

- The kernel $k(s, s') = \mathbf{x}^\top(s)\mathbf{x}(s')$ produces the same approximation of the corresponding linear method.

Adapting eligibility traces to function approximation

Poiché è iun metodo per trasmettee la conoscenza allìindietro.

- Eligibility traces can be used with function approximation.
- In the off-line λ-return algorithm, at the end of the episode, a sequence of off-line updates are made according to

$$\mathbf{w}_{t+1} = \mathbf{w}_t + \alpha(G_t^\lambda - \hat{v}(S_t, \mathbf{w}_t))\nabla\hat{v}(S_t, \mathbf{w}_t).$$

- Eligibility traces are vectors $\mathbf{z} \in \mathbb{R}^d$ that constitute a short-term memory.

Pesi cambiati maggiormente nei passi precedenti. Deve modificare tutti quanti i pesi

- They affect the weight vector, and then the weight vector determines the estimated value.

Vogliamo un metodo che ci permetta di capire quali pesi aggiornare.

TD(λ) with function approximation

- In TD(λ), $\mathbf{z}$ is updated as

$$\mathbf{z}_{-1} = 0,$$
$$\mathbf{z}_t = \gamma\lambda\mathbf{z}_{t-1} + \nabla\hat{v}(S_t, \mathbf{w}_t).$$

Mi devo ricordare la direzione in cui si muovo i pesi; gradiente rispetto ai pesi dell'approssimatore fnzionale

- The TD error for state-value prediction is

$$\delta_t = R_{t+1} + \gamma\hat{v}(S_{t+1}, \mathbf{w}_t) - \hat{v}(S_t, \mathbf{w}_t).$$

- The weight vector is updated as

$$\mathbf{w}_{t+1} = \mathbf{w}_t + \alpha\delta\mathbf{z}_t.$$

- It can be proved that linear TD(λ) converges to $\mathbf{w}_\infty$ such that

$$\overline{\mathrm{VE}}(\mathbf{w}_\infty) \leqslant \frac{1-\gamma\lambda}{1-\gamma} \min_{\mathbf{w}} \overline{\mathrm{VE}}(\mathbf{w}).$$

Truncated λ-return

Metodo Off-Line

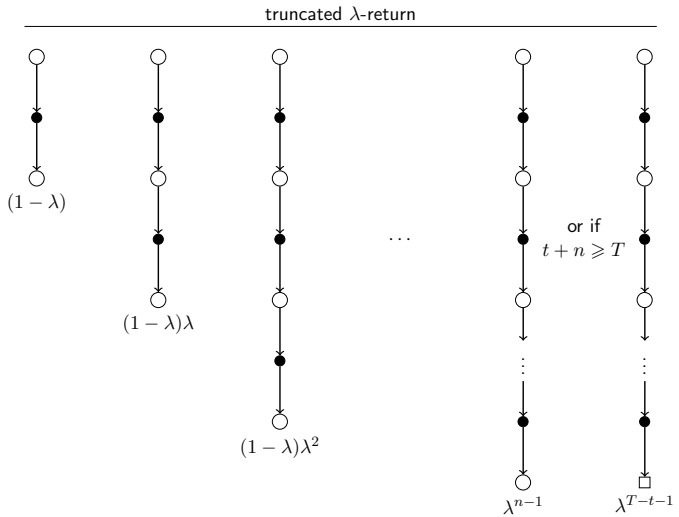
- The λ-return is not known until the end of the episode.
- Dependence is weaker for delayed rewards, falling by $\gamma\lambda$.

Penalizzazione

- Define the *truncated λ-return*

$$G^{\lambda}_{t:h} = (1-\lambda) \sum_{n=1}^{t-h-1} \lambda^{n-1} G_{t:t+n} + \lambda^{t-h-1} G_{t:h}.$$

Il ritorno troncato vedo fino all'h-esimo passo e non considero nulla da li in poi. valido anche per task continuativi



Notes on the truncated λ-return

- Updates are delayed by n steps and only take into account the first n rewards.
- All the k-step returns are included for $1 \leqslant k \leqslant n$.
- The longest component update is at most n steps.
- TTD(λ) is then defined as

$$\mathbf{w}_{t+n} = \mathbf{w}_{t+n-1} + \alpha \left(G^{\lambda}_{t:t+n} + \hat{v}(S_t, \mathbf{w}_{t+n-1}) \right) \nabla \hat{v}(S_t, \mathbf{w}_{t+n-1}),$$

where

$$G^{\lambda}_{t:t+k} = \hat{v}(S_t, \mathbf{w}_{t-1}) + \sum_{i=t}^{t+k-1} (\gamma\lambda)^{i-t} \delta'_i,$$

$$\delta'_t = R_{t+1} + \gamma \hat{v}(S_{t+1}, \mathbf{w}_t) - \hat{v}(S_t, \mathbf{w}_{t-1}).$$

Redoing updates

- In order to obtain an on-line algorithm, we must redo updates.
- Always use the latest horizon
  - in each pass over the episode, use a slightly longer horizon.
- Given h steps of the episode, the update at $0 \leqslant t \leqslant h \leqslant T$ is

$$\mathbf{w}_{t+1}^h = \mathbf{w}_t^h + \alpha(G_{t:h}^\lambda - \hat{v}(S_t, \mathbf{w}_t^h))\nabla\hat{v}(S_t, \mathbf{w}_t)$$

- This update, together with

$$\mathbf{w}_t = \mathbf{w}_t^t$$

defines the *online λ-return algorithm*.

Updates in the online λ-return algorithm

- $\mathbf{w}_0^h$ are the weights from the end of the previous episode.
- The updates at each step of the episode are

$$
\begin{aligned}
h = 1: \quad & \mathbf{w}_1^1 = \mathbf{w}_0^1 + \alpha(G_{0:1}^\lambda - \hat{v}(S_0, \mathbf{w}_0^1))\nabla\hat{v}(S_0, \mathbf{w}_0^1), \\
h = 2: \quad & \mathbf{w}_1^2 = \mathbf{w}_0^2 + \alpha(G_{0:2}^\lambda - \hat{v}(S_0, \mathbf{w}_0^2))\nabla\hat{v}(S_0, \mathbf{w}_0^2), \\
& \mathbf{w}_2^2 = \mathbf{w}_1^2 + \alpha(G_{1:2}^\lambda - \hat{v}(S_1, \mathbf{w}_1^2))\nabla\hat{v}(S_1, \mathbf{w}_1^2), \\
h = 2: \quad & \mathbf{w}_1^3 = \mathbf{w}_0^3 + \alpha(G_{0:3}^\lambda - \hat{v}(S_0, \mathbf{w}_0^3))\nabla\hat{v}(S_0, \mathbf{w}_0^3), \\
& \mathbf{w}_2^3 = \mathbf{w}_1^3 + \alpha(G_{1:3}^\lambda - \hat{v}(S_1, \mathbf{w}_1^3))\nabla\hat{v}(S_1, \mathbf{w}_1^3) \\
& \mathbf{w}_3^3 = \mathbf{w}_2^3 + \alpha(G_{2:3}^\lambda - \hat{v}(S_2, \mathbf{w}_2^3))\nabla\hat{v}(S_2, \mathbf{w}_2^3).
\end{aligned}
$$

True online TD(λ)

- The online λ-return algorithm generates the weights

$$\begin{array}{lccccc}
h=0 & \mathbf{w}_0^0 & & & & \\
h=1 & \mathbf{w}_0^1 & \mathbf{w}_1^1 & & & \\
h=2 & \mathbf{w}_0^2 & \mathbf{w}_1^2 & \mathbf{w}_2^2 & & \\
h=3 & \mathbf{w}_0^3 & \mathbf{w}_1^3 & \mathbf{w}_2^3 & \mathbf{w}_3^3 & \\
 & \vdots & \vdots & \vdots & \vdots & \ddots \\
h=T & \mathbf{w}_0^T & \mathbf{w}_1^T & \mathbf{w}_2^T & \mathbf{w}_3^T & \cdots \quad \mathbf{w}_T^T
\end{array}$$

- With linear approximation, the *true online TD(λ)* algorithm

$$\mathbf{w}_{t+1} = \mathbf{w}_t + \alpha\delta_t\mathbf{z}_t + \alpha(\mathbf{w}_t^\top\mathbf{x}_t - \mathbf{w}_{t-1}^\top\mathbf{x}_t)(\mathbf{z}_t - \mathbf{x}_t),$$

where

$$\mathbf{z}_t = \gamma\lambda\mathbf{z}_{t-1} + (1 - \alpha\gamma\lambda\mathbf{z}_{t-1}^\top\mathbf{x}_t)\mathbf{x}_t,$$

produce exactly the same sequence of weight vectors.

True online TD(λ) algorithm

True online TD(λ) algorithm

Input: $\alpha > 0$, $\lambda > 0$, basis $\mathbf{x}$ such that $\mathbf{x}(\text{terminal}) = 0$, policy π
Output: approximate of v_π

Initialization

$\mathbf{w} \leftarrow$arbitrarily

Loop

initialize S and obtain initial feature vector $\mathbf{x}$
$\mathbf{z} \leftarrow 0$
$V_{\text{old}} \leftarrow 0$
for each step of the episode **do**
  $A \leftarrow \pi(\cdot|S)$
  take action A and observe $R, \mathbf{x}'$
  $V \leftarrow \mathbf{w}^\top \mathbf{x}$
  $V' \leftarrow \mathbf{w}^\top \mathbf{x}'$
  $\delta \leftarrow R + \gamma V' - V$
  $\mathbf{z} \leftarrow \gamma\lambda\mathbf{z} + (1 - \alpha\gamma\lambda\mathbf{z}^\top \mathbf{x})\mathbf{x}$
  $\mathbf{w} \leftarrow \mathbf{w} + \alpha(\delta + V - V_{\text{old}})\mathbf{z} - \alpha(V - V_{\text{old}})\mathbf{x}$
  $V_{\text{old}} \leftarrow V'$
  $\mathbf{x} \leftarrow \mathbf{x}'$
  if $\mathbf{x}' = 0$ **then**
    reinitialize the episode