Nonlinear Systems and Control
Lecture # 34

Robust Stabilization

Lyapunov Redesign & Backstepping

Lyapunov Redesign (Min-max control)

$$\dot{x} = f(x) + G(x)[u + \delta(t, x, u)], \quad x \in R^n, \ u \in R^p$$

Nominal Model: $\dot{x} = f(x) + G(x)u$

Stabilizing Control: $u = \psi(x)$

$$\frac{\partial V}{\partial x}[f(x) + G(x)\psi(x)] \leq -W(x), \ \forall\, x \in D, \ W \text{ is p.d.}$$

$$u = \psi(x) + v$$

$$\|\delta(t, x, \psi(x) + v)\| \leq \rho(x) + \kappa_0 \|v\|, \quad 0 \leq \kappa_0 < 1$$

$$\dot{x} = f(x) + G(x)\psi(x) + G(x)[v + \delta(t, x, \psi(x) + v)]$$

$$\dot{V} = \frac{\partial V}{\partial x}(f + G\psi) + \frac{\partial V}{\partial x}G(v + \delta)$$

$$w^T = \frac{\partial V}{\partial x} G$$

$$\dot{V} \le -W(x) + w^T v + w^T \delta$$

$$w^T v + w^T \delta \le w^T v + \|w\| \, \|\delta\| \le w^T v + \|w\| [\rho(x) + \kappa_0 \|v\|]$$

$$v = -\eta(x) \, \frac{w}{\|w\|} \qquad \left(\frac{w}{\|w\|} = \mathrm{sgn}(w) \text{ for } p = 1 \right)$$

$$\begin{aligned} w^T v + w^T \delta &\le -\eta \|w\| + \rho \|w\| + \kappa_0 \eta \|w\| \\ &= -\eta(1 - \kappa_0) \|w\| + \rho \|w\| \end{aligned}$$

$$\eta(x) \ge \frac{\rho(x)}{(1 - \kappa_0)} \Rightarrow w^T v + w^T \delta \le 0 \Rightarrow \dot{V} \le -W(x)$$

$$v = \begin{cases} -\eta(x)\frac{w}{\|w\|}, & \text{if } \eta(x)\|w\| \geq \varepsilon \\ -\eta^2(x)\frac{w}{\varepsilon}, & \text{if } \eta(x)\|w\| < \varepsilon \end{cases}$$

$$\eta(x)\|w\| \geq \varepsilon \;\; \Rightarrow \;\; \dot{V} \leq -W(x)$$

For $\eta(x)\|w\| < \varepsilon$

$$\begin{aligned} \dot{V} &\leq -W(x) + w^T\left[-\eta^2 \cdot \frac{w}{\varepsilon} + \delta\right] \\ &\leq -W(x) - \frac{\eta^2}{\varepsilon}\|w\|^2 + \rho\|w\| + \kappa_0\|w\|\,\|v\| \\ &= -W(x) - \frac{\eta^2}{\varepsilon}\|w\|^2 + \rho\|w\| + \frac{\kappa_0\eta^2}{\varepsilon}\|w\|^2 \end{aligned}$$

$$\dot{V} \leq -W(x) + (1-\kappa_0)\left(-\frac{\eta^2}{\varepsilon}\|w\|^2 + \eta\|w\|\right)$$

$$-\frac{y^2}{\varepsilon} + y \leq \frac{\varepsilon}{4}, \quad \text{for } y \geq 0$$

$$\dot{V} \leq -W(x) + \varepsilon\frac{(1-\kappa_0)}{4}, \quad \forall\, x \in D$$

Theorem 14.3: $x(t)$ is uniformly ultimately bounded by a class $\mathcal{K}$ function of ε. If the assumptions hold globally and V is radially unbounded, then $x(t)$ globally uniformly ultimately bounded

Corollary 14.1: If $\rho(0) = 0$ and $\eta(x) \geq \eta_0 > 0$ we can recover uniform asymptotic stability

Example: Pendulum with horizontal acceleration of suspension point

$$m\left[\ell\ddot{\theta} + \mathcal{A}(t)\cos\theta\right] = T/\ell - mg\sin\theta$$

Stabilize the pendulum at $\theta = \pi$

$$x_1 = \theta - \pi, \; x_2 = \dot{\theta}, \; a = \frac{g}{\ell}, \; c = \frac{1}{m\ell^2}, \; h(t) = \frac{\mathcal{A}(t)}{\ell}$$

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = a\sin x_1 + cu + h(t)\cos x_1$$

Nominal Model: $\dot{x}_1 = x_2, \qquad \dot{x}_2 = \hat{a}\sin x_1 + \hat{c}u$

$$\psi(x) = -\left(\frac{\hat{a}}{\hat{c}}\right)\sin x_1 - \left(\frac{1}{\hat{c}}\right)(k_1x_1 + k_2x_2)$$

$$\underbrace{\begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}}_{\text{Hurwitz}}, \qquad V(x) = x^T P x$$

$$\begin{aligned}
\delta \;&=\; \frac{1}{\hat{c}} \left[\left(\frac{a\hat{c} - \hat{a}c}{\hat{c}} \right) \sin x_1 + h(t) \cos x_1 \right. \\
&\quad \left. - \left(\frac{c - \hat{c}}{\hat{c}} \right) (k_1 x_1 + k_2 x_2) \right] + \left(\frac{c - \hat{c}}{\hat{c}} \right) v
\end{aligned}$$

$$\left| \frac{c - \hat{c}}{\hat{c}} \right| \le \kappa_0, \; \left| \frac{\hat{a}c - a\hat{c}}{\hat{c}} \right| + \left| \frac{c - \hat{c}}{\hat{c}} \right| \sqrt{k_1^2 + k_2^2} \le k, \; |h(t)| \le H$$

$$|\delta| \le \frac{(k\|x\| + H)}{\hat{c}} + \kappa_0 \, |v| \stackrel{\text{def}}{=} \rho(x) + \kappa_0 \, |v|, \quad (\kappa_0 < 1)$$

$$\eta(x) = \frac{\rho(x)}{(1-\kappa_0)}, \qquad \eta(x) \geq \frac{H}{\hat{c}(1-\kappa_0)}$$

$$w = \frac{\partial V}{\partial x} G = 2x^T P \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2(p_{12}x_1 + p_{22}x_2)$$

$$v = \begin{cases} -\eta(x)\,\mathrm{sgn}(w), & \text{if } \eta(x)|w| \geq \varepsilon \\ -\eta^2(x)\frac{w}{\varepsilon}, & \text{if } \eta(x)|w| < \varepsilon \end{cases}$$

$$u = -\left(\frac{\hat{a}}{\hat{c}}\right)\sin x_1 - \left(\frac{1}{\hat{c}}\right)(k_1x_1 + k_2x_2) + v$$

Will this control stabilize the origin $x = 0$?

Backstepping

$$
\begin{aligned}
\dot{z}_1 &= f_1(z_1) + g_1(z_1)z_2 \\
\dot{z}_2 &= f_2(z_1, z_2) + g_2(z_1, z_2)z_3 \\
&\vdots \\
\dot{z}_{k-1} &= f_{k-1}(x, z_1, \ldots, z_{k-1}) + g_{k-1}(z_1, \ldots, z_{k-1})z_k \\
\dot{z}_k &= f_k(z_1, \ldots, z_k) + g_k(z_1, \ldots, z_k)u
\end{aligned}
$$

$$
g_i \neq 0, \quad 1 \leq i \leq k
$$

$$
\begin{aligned}
\dot{z}_1 &= f_1 + g_1 z_2 + \delta_1(z) \\
\dot{z}_2 &= f_2 + g_2 z_3 + \delta_2(z) \\
&\vdots \\
\dot{z}_{k-1} &= f_{k-1} + g_{k-1} z_k + \delta_{k-1}(z) \\
\dot{z}_k &= f_k + g_k u + \delta_k(z)
\end{aligned}
$$

$$
\begin{aligned}
|\delta_1(z)| &\le \rho_1(z_1) \\
|\delta_2(z)| &\le \rho_2(z_1, z_2) \\
&\vdots \\
|\delta_{k-1}| &\le \rho_{k-1}(z_1, \ldots, z_{k-1}) \\
|\delta_k| &\le \rho_k(z_1, \ldots, z_k)
\end{aligned}
$$

The virtual control $z_i = \phi_i(z_1, \ldots, z_{i-1})$ should be smooth

Example:

$$\dot{x}_1 = x_2 + \theta_1 x_1 \sin x_2, \qquad \dot{x}_2 = \theta_2 x_2^2 + x_1 + u$$

$$|\theta_1| \le a, \qquad |\theta_2| \le b$$

$$\delta_1 = \theta_1 x_1 \sin x_2, \qquad \delta_2 = \theta_2 x_2^2$$

$$\dot{x}_1 = x_2 + \theta_1 x_1 \sin x_2, \qquad |\theta_1 x_1 \sin x_2| \le a|x_1|$$

$$x_2 = -k_1 x_1$$

$$V_1 = \tfrac{1}{2} x_1^2, \quad \dot{V}_1 \le -(k_1 - a) x_1^2; \quad \text{Take } k_1 = 1 + a$$

$$z_2 = x_2 + (1 + a) x_1$$

$$\begin{aligned}\dot{x}_1 &= -(1+a)x_1 + \theta_1 x_1 \sin x_2 + z_2 \\ \dot{z}_2 &= \psi_1(x) + \psi_2(x,\theta) + u\end{aligned}$$

$$\psi_1 = x_1 + (1+a)x_2, \quad \psi_2 = (1+a)\theta_1 x_1 \sin x_2 + \theta_2 x_2^2$$

$$V_c = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}z_2^2$$

$$\dot{V}_c \le -x_1^2 + z_2[x_1 + \psi_1(x) + \psi_2(x,\theta) + u]$$

First Approach (Example 14.13):

$$u = -x_1 - \psi_1(x) - kz_2, \qquad k > 0$$

$$\dot{V}_c \le -x_1^2 - kz_2^2 + z_2\psi_2(x,\theta)$$

Restrict analysis to the compact set $\Omega_c = \{V_c(x) \le c\}$

$$\psi_2 = (1+a)\theta_1 x_1 \sin x_2 + \theta_2 x_2^2$$

$$|\psi_2| \le a(1+a)|x_1| + b\rho|x_2|, \qquad \rho = \max_{x\in\Omega_c} |x_2|$$

$$x_2 = z_2 - (1+a)x_1$$

$$|\psi_2| \le (1+a)(a+b\rho)|x_1| + b\rho|z_2|$$

$$\dot{V}_c \le -x_1^2 - kz_2^2 + (1+a)(a+b\rho)|x_1|\;|z_2| + b\rho z_2^2$$

We can make $\dot{V}$ neg. def. by choosing k large enough

Can this control achieve global stabilization?

Can it achieve semiglobal stabilization?

Second Approach (Example 14.14):

$$u = -x_1 - \psi_1(x) - kz_2 + v$$

$$\dot{V}_c \le -x_1^2 - kz_2^2 + z_2[\psi_2 + v]$$

$$|\psi_2| \le a(1+a)|x_1| + bx_2^2$$

$$v = \begin{cases} -\eta(x)\,\mathrm{sgn}(z_2), & \text{if } \eta(x)|z_2| \ge \varepsilon \\ -\eta^2(x) z_2/\varepsilon, & \text{if } \eta(x)|z_2| < \varepsilon \end{cases}$$

$$\eta(x) = \eta_0 + a(1+a)|x_1| + bx_2^2, \quad \eta_0 > 0, \quad \varepsilon > 0$$

$$\dot{V}_c \le -x_1^2 - kz_2^2 + \frac{\varepsilon}{4}$$

Show that this control is globally stabilizing