Nonlinear Systems and Control
Lecture # 28

Stabilization

Backstepping

$$
\begin{aligned}
\dot{\eta} &= f(\eta) + g(\eta)\xi \\
\dot{\xi} &= u, \quad \eta \in R^n, \ \xi, \ u \in R
\end{aligned}
$$

Stabilize the origin using state feedback

View ξ as "virtual" control input to

$$\dot{\eta} = f(\eta) + g(\eta)\xi$$

Suppose there is $\xi = \phi(\eta)$ that stabilizes the origin of

$$\dot{\eta} = f(\eta) + g(\eta)\phi(\eta)$$

$$\frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] \leq -W(\eta), \quad \forall \ \eta \in D$$

$$z = \xi - \phi(\eta)$$

$$\begin{aligned} \dot{\eta} &= [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)z \\ \dot{z} &= u - \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi] \end{aligned}$$

$$u = \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi] + v$$

$$\begin{aligned} \dot{\eta} &= [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)z \\ \dot{z} &= v \end{aligned}$$

$$V_c(\eta,\xi) = V(\eta) + \tfrac{1}{2}z^2$$

$$\begin{aligned}\dot{V}_c &= \frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] + \frac{\partial V}{\partial \eta}g(\eta)z + zv \\ &\le -W(\eta) + \frac{\partial V}{\partial \eta}g(\eta)z + zv\end{aligned}$$

$$v = -\frac{\partial V}{\partial \eta}g(\eta) - kz, \quad k > 0$$

$$\dot{V}_c \le -W(\eta) - kz^2$$

Example

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2, \qquad \dot{x}_2 = u$$

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2$$

$$x_2 = \phi(x_1) = -x_1^2 - x_1 \quad \Rightarrow \quad \dot{x}_1 = -x_1 - x_1^3$$

$$V(x_1) = \tfrac{1}{2}x_1^2 \quad \Rightarrow \quad \dot{V} = -x_1^2 - x_1^4, \quad \forall\, x_1 \in R$$

$$z_2 = x_2 - \phi(x_1) = x_2 + x_1 + x_1^2$$

$$\begin{aligned}
\dot{x}_1 &= -x_1 - x_1^3 + z_2 \\
\dot{z}_2 &= u + (1 + 2x_1)(-x_1 - x_1^3 + z_2)
\end{aligned}$$

$$V_c(x) = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2$$

$$\begin{aligned} \dot{V}_c &= x_1(-x_1 - x_1^3 + z_2) \\ &\quad + z_2[u + (1 + 2x_1)(-x_1 - x_1^3 + z_2)] \end{aligned}$$

$$\begin{aligned} \dot{V}_c &= -x_1^2 - x_1^4 \\ &\quad + z_2[x_1 + (1 + 2x_1)(-x_1 - x_1^3 + z_2) + u] \end{aligned}$$

$$u = -x_1 - (1 + 2x_1)(-x_1 - x_1^3 + z_2) - z_2$$

$$\dot{V}_c = -x_1^2 - x_1^4 - z_2^2$$

$$\dot{V}_c = -x_1^2 - x_1^4 - (x_2 + x_1 + x_1^2)^2$$

Example

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = u$$

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2, \qquad \dot{x}_2 = x_3$$

$$x_3 = -x_1 - (1 + 2x_1)(-x_1 - x_1^3 + z_2) - z_2 \stackrel{\text{def}}{=} \phi(x_1, x_2)$$

$$V(x) = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}z_2^2, \quad \dot{V} = -x_1^2 - x_1^4 - z_2^2$$

$$z_3 = x_3 - \phi(x_1, x_2)$$

$$\begin{aligned}
\dot{x}_1 &= x_1^2 - x_1^3 + x_2, \qquad \dot{x}_2 = \phi(x_1, x_2) + z_3 \\
\dot{z}_3 &= u - \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(\phi + z_3)
\end{aligned}$$

$$V_c = V + \tfrac{1}{2} z_3^2$$

$$\begin{aligned}
\dot{V}_c &= \frac{\partial V}{\partial x_1}(x_1^2 - x_1^3 + x_2) + \frac{\partial V}{\partial x_2}(z_3 + \phi) \\
&\quad + z_3 \left[u - \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(z_3 + \phi) \right]
\end{aligned}$$

$$\begin{aligned}
\dot{V}_c &= -x_1^2 - x_1^4 - (x_2 + x_1 + x_1^2)^2 \\
&\quad + z_3 \left[\frac{\partial V}{\partial x_2} - \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(z_3 + \phi) + u \right]
\end{aligned}$$

$$u = -\frac{\partial V}{\partial x_2} + \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) + \frac{\partial \phi}{\partial x_2}(z_3 + \phi) - z_3$$

$$
\begin{aligned}
\dot{\eta} &= f(\eta) + g(\eta)\xi \\
\dot{\xi} &= f_a(\eta,\xi) + g_a(\eta,\xi)u, \qquad g_a(\eta,\xi) \neq 0
\end{aligned}
$$

$$
u = \frac{1}{g_a(\eta,\xi)}[v - f_a(\eta,\xi)]
$$

$$
\begin{aligned}
\dot{\eta} &= f(\eta) + g(\eta)\xi \\
\dot{\xi} &= v
\end{aligned}
$$

Strict-Feedback Form

$$
\begin{aligned}
\dot{x} &= f_0(x) + g_0(x)z_1 \\
\dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1)z_2 \\
\dot{z}_2 &= f_2(x, z_1, z_2) + g_2(x, z_1, z_2)z_3 \\
&\vdots \\
\dot{z}_{k-1} &= f_{k-1}(x, z_1, \ldots, z_{k-1}) + g_{k-1}(x, z_1, \ldots, z_{k-1})z_k \\
\dot{z}_k &= f_k(x, z_1, \ldots, z_k) + g_k(x, z_1, \ldots, z_k)u
\end{aligned}
$$

$$
g_i(x, z_1, \ldots, z_i) \neq 0 \quad \text{for } 1 \leq i \leq k
$$

Example

$$\dot{\eta} = -\eta + \eta^2 \xi, \qquad \dot{\xi} = u$$

$$\dot{\eta} = -\eta + \eta^2 \xi$$

$$\xi = 0 \quad \Rightarrow \quad \dot{\eta} = -\eta$$

$$V_0 = \tfrac{1}{2}\eta^2 \quad \Rightarrow \quad \dot{V}_0 = -\eta^2, \ \forall\, \eta \in R$$

$$V = \tfrac{1}{2}(\eta^2 + \xi^2)$$

$$\dot{V} = \eta(-\eta + \eta^2 \xi) + \xi u = -\eta^2 + \xi(\eta^3 + u)$$

$$u = -\eta^3 - k\xi, \quad k > 0$$

$$\dot{V} = -\eta^2 - k\xi^2$$ Global stabilization