Nonlinear Systems and Control
Lecture # 21

$\mathcal{L}_2$ Gain

&

The Small-Gain theorem

Theorem 5.4: Consider the linear time-invariant system

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du$$

where A is Hurwitz. Let $G(s) = C(sI - A)^{-1}B + D$.
Then, the $\mathcal{L}_2$ gain of the system is $\sup_{\omega \in R} \|G(j\omega)\|$

Lemma: Consider the time-invariant system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

where f is locally Lipschitz and h is continuous for all $x \in R^n$ and $u \in R^m$. Let $V(x)$ be a positive semidefinite function such that

$$\dot{V} = \frac{\partial V}{\partial x} f(x, u) \leq a(\gamma^2 \|u\|^2 - \|y\|^2), \quad a, \gamma > 0$$

Then, for each $x(0) \in R^n$, the system is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to γ. In particular

$$\|y_\tau\|_{\mathcal{L}_2} \leq \gamma \|u_\tau\|_{\mathcal{L}_2} + \sqrt{\frac{V(x(0))}{a}}$$

Proof

$$V(x(\tau)) - V(x(0)) \le a\gamma^2 \int_0^\tau \|u(t)\|^2 \, dt - a \int_0^\tau \|y(t)\|^2 \, dt$$

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$$V(x) \ge 0$$

$$\int_0^\tau \|y(t)\|^2\, dt \le \gamma^2 \int_0^\tau \|u(t)\|^2\, dt + \frac{V(x(0))}{a}$$

Proof

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$$V(x) \geq 0$$

$$\int_0^\tau \|y(t)\|^2 \, dt \leq \gamma^2 \int_0^\tau \|u(t)\|^2 \, dt + \frac{V(x(0))}{a}$$

$$\|y_\tau\|_{\mathcal{L}_2} \leq \gamma \|u_\tau\|_{\mathcal{L}_2} + \sqrt{\frac{V(x(0))}{a}}$$

Lemma 6.5: If the system

$$\dot{x} = f(x,u), \qquad y = h(x,u)$$

is output strictly passive with

$$u^T y \ge \dot{V} + \delta y^T y, \quad \delta > 0$$

then it is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to $1/\delta$

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Proof

$$\begin{aligned} \dot{V} &\leq u^T y - \delta y^T y \\ &= -\frac{1}{2\delta}(u - \delta y)^T (u - \delta y) + \frac{1}{2\delta} u^T u - \frac{\delta}{2} y^T y \\ &\leq \frac{\delta}{2}\left(\frac{1}{\delta^2} u^T u - y^T y\right) \end{aligned}$$

Example

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -ax_1^3 - kx_2 + u, \quad y = x_2, \quad a, k > 0$$

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$$\begin{aligned}\dot{V} &= ax_1^3 x_2 + x_2(-ax_1^3 - kx_2 + u) \\ &= -kx_2^2 + x_2 u \;=\; -ky^2 + yu\end{aligned}$$

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The system is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to $1/k$

Theorem 5.5: Consider the time-invariant system

$$\dot{x} = f(x) + G(x)u, \qquad y = h(x)$$

$$f(0) = 0, \qquad h(0) = 0$$

where f and G are locally Lipschitz and h is continuous over R^n. Suppose $\exists\ \gamma > 0$ and a continuously differentiable, positive semidefinite function $V(x)$ that satisfies the Hamilton–Jacobi inequality

$$\frac{\partial V}{\partial x} f(x) + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G(x) G^T(x) \left(\frac{\partial V}{\partial x} \right)^T + \frac{1}{2} h^T(x) h(x) \leq 0$$

$\forall\ x \in R^n$. Then, for each $x(0) \in R^n$, the system is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain $\leq \gamma$

Proof

$$
\begin{aligned}
\frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} G(x) u = \\
& - \frac{1}{2}\gamma^2 \left\| u - \frac{1}{\gamma^2} G^T(x) \left(\frac{\partial V}{\partial x} \right)^T \right\|^2 + \frac{\partial V}{\partial x} f(x) \\
& + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G(x) G^T(x) \left(\frac{\partial V}{\partial x} \right)^T + \frac{1}{2}\gamma^2 \|u\|^2
\end{aligned}
$$

Proof

$$\frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} G(x) u =$$

$$- \frac{1}{2}\gamma^2 \left\| u - \frac{1}{\gamma^2} G^T(x) \left(\frac{\partial V}{\partial x}\right)^T \right\|^2 + \frac{\partial V}{\partial x} f(x)$$

$$+ \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G(x) G^T(x) \left(\frac{\partial V}{\partial x}\right)^T + \frac{1}{2}\gamma^2 \|u\|^2$$

$$\dot{V} \leq \frac{1}{2}\gamma^2 \|u\|^2 - \frac{1}{2}\|y\|^2$$

Example

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \qquad \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$$

Example

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

Suppose there is $P = P^T \geq 0$ that satisfies the Riccati equation

$$PA + A^T P + \frac{1}{\gamma^2} PBB^T P + C^T C = 0$$

for some $\gamma > 0$.

Example

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Example

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

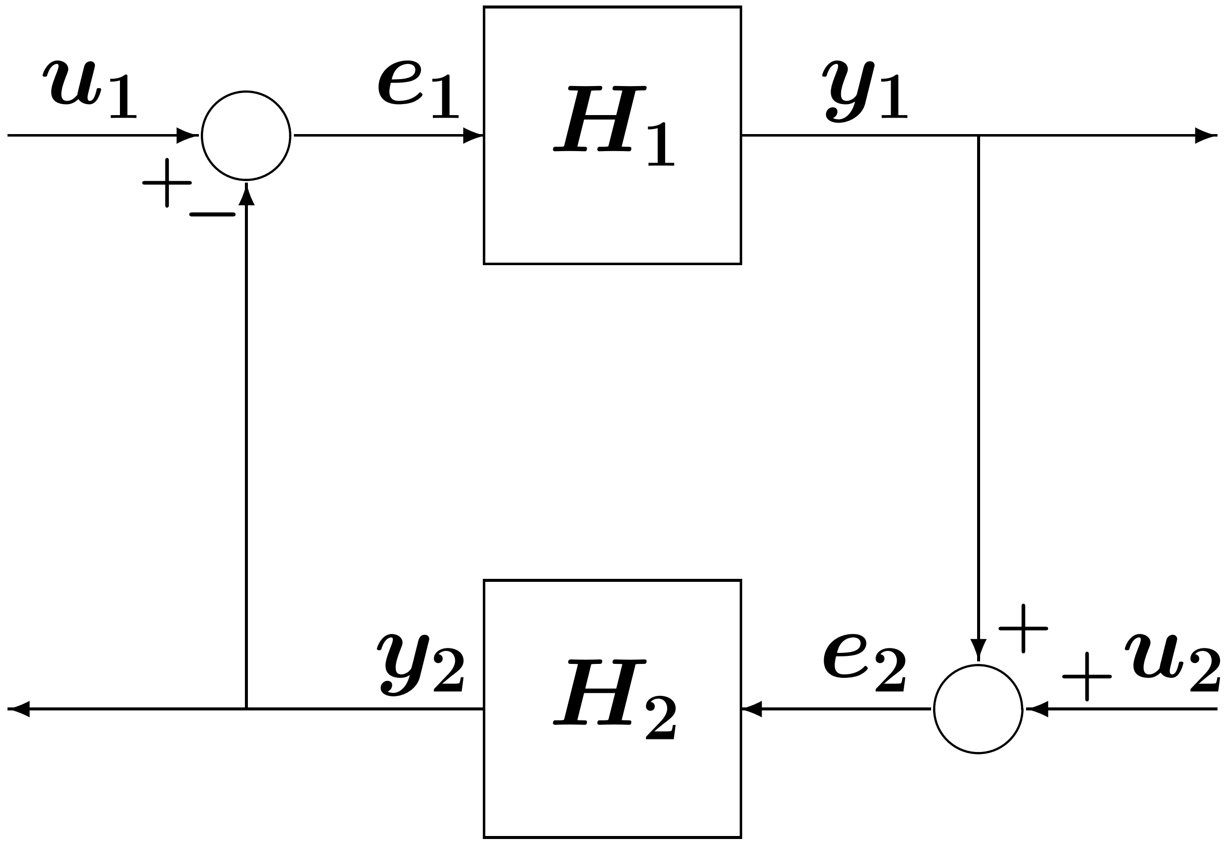
Suppose there is $P = P^T \geq 0$ that satisfies the Riccati equation

$$PA + A^TP + \frac{1}{\gamma^2}PBB^TP + C^TC = 0$$

for some $\gamma > 0$. Verify that $V(x) = \frac{1}{2}x^TPx$ satisfies the Hamilton-Jacobi equation

The system is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to γ

The Small-Gain Theorem



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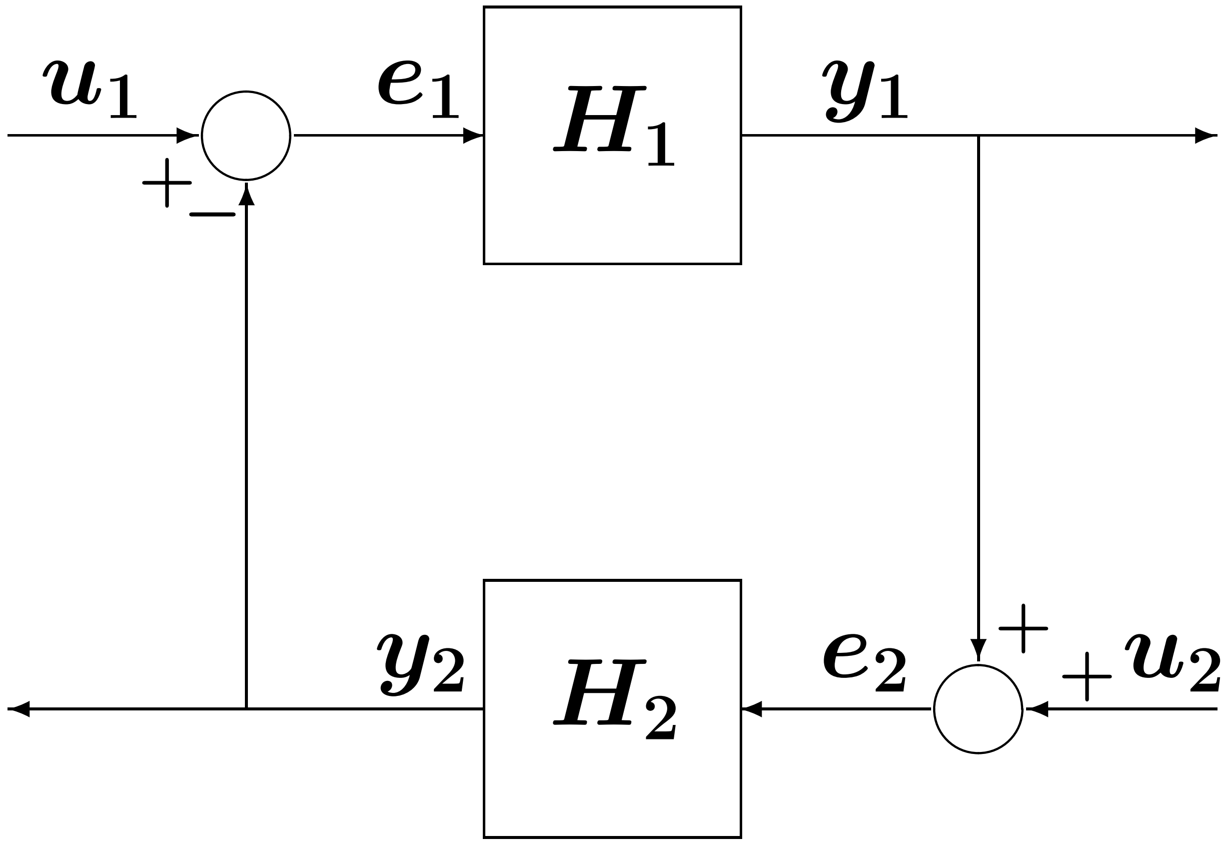

$$\|y_{1\tau}\|_{\mathcal{L}} \leq \gamma_1 \|e_{1\tau}\|_{\mathcal{L}} + \beta_1, \;\; \forall \; e_1 \in \mathcal{L}_e^m, \; \forall \; \tau \in [0, \infty)$$

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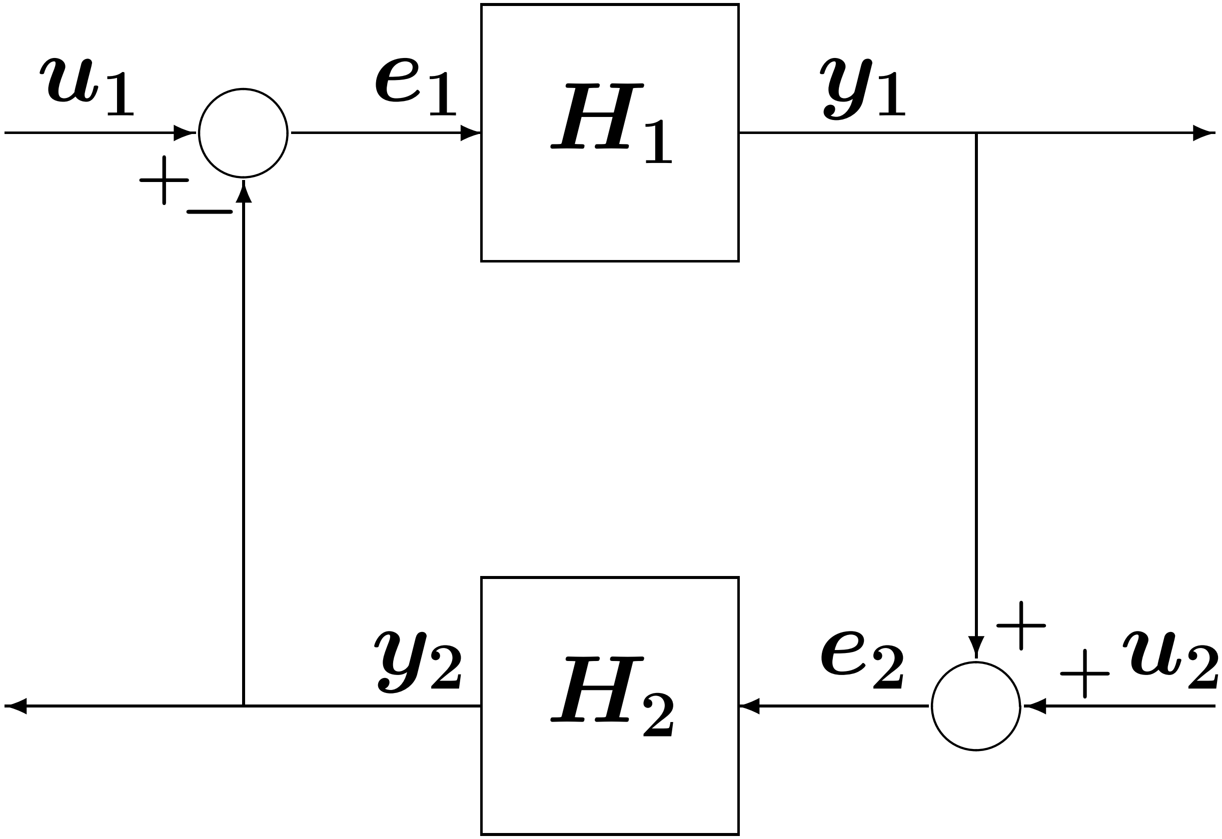

$$\|y_{1\tau}\|_{\mathcal{L}} \leq \gamma_1 \|e_{1\tau}\|_{\mathcal{L}} + \beta_1, \;\; \forall\, e_1 \in \mathcal{L}_e^m, \; \forall\, \tau \in [0, \infty)$$

$$\|y_{2\tau}\|_{\mathcal{L}} \leq \gamma_2 \|e_{2\tau}\|_{\mathcal{L}} + \beta_2, \;\; \forall\, e_2 \in \mathcal{L}_e^q, \; \forall\, \tau \in [0, \infty)$$

$$\boldsymbol{u} = \begin{bmatrix} \boldsymbol{u_1} \\ \boldsymbol{u_2} \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} \boldsymbol{y_1} \\ \boldsymbol{y_2} \end{bmatrix}, \quad \boldsymbol{e} = \begin{bmatrix} \boldsymbol{e_1} \\ \boldsymbol{e_2} \end{bmatrix}$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

Theorem: The feedback connection is finite-gain $\mathcal{L}$ stable if $\gamma_1 \gamma_2 < 1$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

Theorem: The feedback connection is finite-gain $\mathcal{L}$ stable if $\gamma_1\gamma_2 < 1$

Proof

$$e_{1\tau} = u_{1\tau} - (H_2 e_2)_\tau, \quad e_{2\tau} = u_{2\tau} + (H_1 e_1)_\tau$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

Theorem: The feedback connection is finite-gain $\mathcal{L}$ stable if $\gamma_1\gamma_2 < 1$

Proof

$$e_{1\tau} = u_{1\tau} - (H_2 e_2)_\tau, \quad e_{2\tau} = u_{2\tau} + (H_1 e_1)_\tau$$

$$\begin{aligned} \|e_{1\tau}\|_{\mathcal{L}} &\leq \|u_{1\tau}\|_{\mathcal{L}} + \|(H_2 e_2)_\tau\|_{\mathcal{L}} \\ &\leq \|u_{1\tau}\|_{\mathcal{L}} + \gamma_2\|e_{2\tau}\|_{\mathcal{L}} + \beta_2 \end{aligned}$$

$$
\begin{aligned}
\|e_{1\tau}\|_{\mathcal{L}} &\leq \|u_{1\tau}\|_{\mathcal{L}} + \gamma_2 \left(\|u_{2\tau}\|_{\mathcal{L}} + \gamma_1 \|e_{1\tau}\|_{\mathcal{L}} + \beta_1\right) + \beta_2 \\
&= \gamma_1\gamma_2\|e_{1\tau}\|_{\mathcal{L}} \\
&\quad + \left(\|u_{1\tau}\|_{\mathcal{L}} + \gamma_2\|u_{2\tau}\|_{\mathcal{L}} + \beta_2 + \gamma_2\beta_1\right)
\end{aligned}
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\end{aligned}
$$

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\|e_{1\tau}\|_{\mathcal{L}} \leq \frac{1}{1-\gamma_1\gamma_2}\left(\|u_{1\tau}\|_{\mathcal{L}} + \gamma_2\|u_{2\tau}\|_{\mathcal{L}} + \beta_2 + \gamma_2\beta_1\right)
$$

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&= \gamma_1\gamma_2 \|e_{1\tau}\|_{\mathcal{L}} \\
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&= \gamma_1\gamma_2\|e_{1\tau}\|_{\mathcal{L}} \\
&\quad + \left(\|u_{1\tau}\|_{\mathcal{L}} + \gamma_2\|u_{2\tau}\|_{\mathcal{L}} + \beta_2 + \gamma_2\beta_1\right)
\end{aligned}
$$

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$$

$$
\|e_{\tau}\|_{\mathcal{L}} \leq \|e_{1\tau}\|_{\mathcal{L}} + \|e_{2\tau}\|_{\mathcal{L}}
$$