Nonlinear Systems and Control
Lecture # 25

Stabilization

Feedback Lineaization

Consider the nonlinear system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})\boldsymbol{u}$$

$$\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}, \quad \boldsymbol{x} \in \boldsymbol{R}^n, \quad \boldsymbol{u} \in \boldsymbol{R}^m$$

Suppose there is a change of variables $\boldsymbol{z} = \boldsymbol{T}(\boldsymbol{x})$, defined for all $\boldsymbol{x} \in \boldsymbol{D} \subset \boldsymbol{R}^n$, that transforms the system into the controller form

$$\dot{\boldsymbol{z}} = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{\gamma}(\boldsymbol{x})[\boldsymbol{u} - \boldsymbol{\alpha}(\boldsymbol{x})]$$

where $(\boldsymbol{A}, \boldsymbol{B})$ is controllable and $\boldsymbol{\gamma}(\boldsymbol{x})$ is nonsingular for all $\boldsymbol{x} \in \boldsymbol{D}$

$$\boldsymbol{u} = \boldsymbol{\alpha}(\boldsymbol{x}) + \boldsymbol{\gamma}^{-1}(\boldsymbol{x})\boldsymbol{v} \quad \Rightarrow \quad \dot{\boldsymbol{z}} = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{v}$$

$$v = -Kz$$

Design K such that $(A - BK)$ is Hurwitz

The origin $z = 0$ of the closed-loop system

$$\dot{z} = (A - BK)z$$

is globally exponentially stable

$$u = \alpha(x) - \gamma^{-1}(x)KT(x)$$

Closed-loop system in the x-coordinates:

$$\dot{x} = f(x) + G(x)\left[\alpha(x) - \gamma^{-1}(x)KT(x)\right]$$

What can we say about the stability of $x = 0$ as an equilibrium point of

$$\dot{x} = f(x) + G(x)\left[\alpha(x) - \gamma^{-1}(x) K T(x)\right]$$

$x = 0$ is asymptotically stable because $T(x)$ is a diffeomorphism. Show it!

Is $x = 0$ globally asymptotically stable? In general No

It is globally asymptotically stable if $T(x)$ is a global diffeomorphism (See page 508)

What information do we need to implement the control

$$u = \alpha(x) - \gamma^{-1}(x) K T(x) \text{ ?}$$

What is the effect of uncertainty in α, γ, and T?

Let $\hat{\alpha}(x)$, $\hat{\gamma}(x)$, and $\hat{T}(x)$ be nominal models of $\alpha(x)$, $\gamma(x)$, and $T(x)$

$$u = \hat{\alpha}(x) - \hat{\gamma}^{-1}(x) K \hat{T}(x)$$

Closed-loop system:

$$\dot{z} = (A - BK)z + B\delta(z)$$

$$\delta = \gamma[\hat{\alpha} - \alpha + \gamma^{-1} K T - \hat{\gamma}^{-1} K \hat{T}]$$

$$\dot{z} = (A - BK)z + B\delta(z) \qquad (*)$$

$$V(z) = z^T P z, \quad P(A - BK) + (A - BK)^T P = -I$$

Lemma 13.3

- If $\|\delta(z)\| \leq k\|z\|$ for all z, where

$$0 \leq k < \frac{1}{2\|PB\|}$$

  then the origin of (*) is globally exponentially stable

- If $\|\delta(z)\| \leq k\|z\| + \varepsilon$ for all z, then the state z is globally ultimately bounded by εc for some $c > 0$

Example (Pendulum Equation):

$$\ddot{\theta} = -a \sin\theta - b\dot{\theta} + cT$$

$$x_1 = \theta - \delta, \quad x_2 = \dot{\theta}, \quad u = T - T_{\rm ss} = T - \frac{a}{c}\sin\delta$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a[\sin(x_1 + \delta) - \sin\delta] - bx_2 + cu \end{aligned}$$

$$u = \frac{1}{c}\{a[\sin(x_1 + \delta) - \sin\delta] - k_1x_1 - k_2x_2\}$$

$$A - BK = \begin{bmatrix} 0 & 1 \\ -k_1 & -(k_2 + b) \end{bmatrix} \text{ is Hurwitz}$$

$$T = u + \frac{a}{c}\sin\delta = \frac{1}{c}\left[a\sin(x_1+\delta) - k_1x_1 - k_2x_2\right]$$

Let $\hat{a}$ and $\hat{c}$ be nominal models of a and c

$$T = \frac{1}{\hat{c}}\left[\hat{a}\sin(x_1+\delta) - k_1x_1 - k_2x_2\right]$$

$$\dot{x} = (A - BK)x + B\delta(x)$$

$$\delta(x) = \left(\frac{\hat{a}c - a\hat{c}}{\hat{c}}\right)\sin(x_1+\delta_1) - \left(\frac{c-\hat{c}}{\hat{c}}\right)(k_1x_1 + k_2x_2)$$

$$\delta(x) = \left(\frac{\hat{a}c - a\hat{c}}{\hat{c}}\right)\sin(x_1 + \delta_1) - \left(\frac{c - \hat{c}}{\hat{c}}\right)(k_1 x_1 + k_2 x_2)$$

$$|\delta(x)| \le k\|x\| + \varepsilon$$

$$k = \left|\frac{\hat{a}c - a\hat{c}}{\hat{c}}\right| + \left|\frac{c - \hat{c}}{\hat{c}}\right| \sqrt{k_1^2 + k_2^2}, \quad \varepsilon = \left|\frac{\hat{a}c - a\hat{c}}{\hat{c}}\right| \, |\sin\delta_1|$$

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}, \quad PB = \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix}$$

$$k < \frac{1}{2\sqrt{p_{12}^2 + p_{22}^2}}$$

$$\sin\delta_1 = 0 \;\Rightarrow\; \varepsilon = 0$$

Is feedback linearization a good idea?

Example

$$\dot{x} = ax - bx^3 + u, \quad a, b > 0$$

$$u = -(k + a)x + bx^3, \; k > 0, \;\; \Rightarrow \;\; \dot{x} = -kx$$

$-bx^3$ is a damping term. Why cancel it?

$$u = -(k + a)x, \; k > 0, \;\; \Rightarrow \;\; \dot{x} = -kx - bx^3$$

Which design is better?

Example

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -h(x_1) + u\\ & \quad h(0) = 0 \text{ and } x_1 h(x_1) > 0,\ \forall\, x_1 \neq 0\end{aligned}$$

Feedback Linearization:

$$u = h(x_1) - (k_1 x_1 + k_2 x_2)$$

With $y = x_2$, the system is passive with

$$V = \int_0^{x_1} h(z)\, dz + \tfrac{1}{2} x_2^2$$

$$\dot{V} = h(x_1)\dot{x}_1 + x_2 \dot{x}_2 = yu$$

The control

$$u = -\sigma(x_2), \quad \sigma(0) = 0, \; x_2\sigma(x_2) > 0 \; \forall \; x_2 \neq 0$$

creates a feedback connection of two passive systems with storage function V

$$\dot{V} = -x_2\sigma(x_2)$$

$$x_2(t) \equiv 0 \;\Rightarrow\; \dot{x}_2(t) \equiv 0 \;\Rightarrow\; h(x_1(t)) \equiv 0 \;\Rightarrow\; x_1(t) \equiv 0$$

Asymptotic stability of the origin follows from the invariance principle

Which design is better? (Read Example 13.20)