Nonlinear Systems and Control
Lecture # 24

Observer, Output Feedback
&
Strict Feedback Forms

Definition: A nonlinear system is in the observer form if

$$\dot{x} = Ax + \gamma(y, u), \quad y = Cx$$

where (A, C) is observable

Observer:

$$\dot{\hat{x}} = A\hat{x} + \gamma(y, u) + H(y - C\hat{x})$$

$$\tilde{x} = x - \hat{x}$$

$$\dot{\tilde{x}} = (A - HC)\tilde{x}$$

Design H such that $(A - HC)$ is Hurwitz

Theorem: An n-dimensional single-output (SO) system

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

is transformable into the observer form if and only if there is a domain D_0 such that

$$\text{rank}\left[\frac{\partial \phi}{\partial x}(x)\right] = n, \;\; \forall\, x \in D_0$$

where $\qquad \phi = \begin{bmatrix} h, & L_f h, & \cdots & L_f^{n-1} h \end{bmatrix}^T$

and the unique vector field solution τ of

$$\frac{\partial \phi}{\partial x}\tau = b, \quad \text{where } \; b = \begin{bmatrix} 0, & \cdots & 0, & 1 \end{bmatrix}^T$$

satisfies

$$[ad_f^i \tau, ad_f^j \tau] = 0, \quad 0 \le i, j \le n-1$$

and

$$[g, ad_f^j \tau] = 0, \quad 0 \le j \le n-2$$

The change of variables $z = T(x)$ is given by

$$\frac{\partial T}{\partial x} \begin{bmatrix} \tau_1, & \tau_2, & \cdots & \tau_n \end{bmatrix} = I$$

where

$$\tau_i = (-1)^{i-1} ad_f^{i-1} \tau, \quad 1 \le i \le n$$

Example

$$\dot{x} = \begin{bmatrix} \beta_1(x_1) + x_2 \\ f_2(x) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad y = x_1$$

$$\phi(x) = \begin{bmatrix} h(x) \\ L_f h(x) \end{bmatrix} = \begin{bmatrix} x_1 \\ \beta_1(x_1) + x_2 \end{bmatrix}$$

$$\frac{\partial \phi}{\partial x} = \begin{bmatrix} 1 & 0 \\ \frac{\partial \beta_1}{\partial x_1} & 1 \end{bmatrix}; \qquad \text{rank}\left[\frac{\partial \phi}{\partial x}(x)\right] = 2, \;\; \forall\, x$$

$$\frac{\partial \phi}{\partial x}\tau = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \tau = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$ad_f\tau = [f,\tau] = -\frac{\partial f}{\partial x}\tau = -\begin{bmatrix} * & 1 \\ * & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$$[\tau, ad_f\tau] = \frac{\partial(ad_f\tau)}{\partial x}\tau = -\begin{bmatrix} 0 & 0 \\ \frac{\partial^2 f_2}{\partial x_1 \partial x_2} & \frac{\partial^2 f_2}{\partial x_2^2} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[\tau, ad_f\tau] = 0 \Leftrightarrow \frac{\partial^2 f_2}{\partial x_2^2} = 0 \Leftrightarrow f_2(x) = \beta_2(x_1) + x_2\beta_3(x_1)$$

$[g,\tau] = 0$ (g and τ are constant vector fields)

All the conditions are satisfied

$$\tau_1 = (-1)^0 ad_f^0 \tau = \tau = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\tau_2 = (-1)^1 ad_f^1 \tau = -ad_f \tau = \begin{bmatrix} 1 \\ \beta_3(x_1) \end{bmatrix}$$

$$\frac{\partial T}{\partial x} \begin{bmatrix} \tau_1, & \tau_2 \end{bmatrix} = I$$

$$\begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} \\ \frac{\partial T_2}{\partial x_1} & \frac{\partial T_2}{\partial x_2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & \beta_3(x_1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\frac{\partial T_1}{\partial x_2} = 1, \quad \frac{\partial T_1}{\partial x_1} + \beta_3(x_1) \frac{\partial T_1}{\partial x_2} = 0$$

$$\frac{\partial T_2}{\partial x_2} = 0, \quad \frac{\partial T_2}{\partial x_1} + \beta_3(x_1) \frac{\partial T_2}{\partial x_2} = 1$$

$$\frac{\partial T_1}{\partial x_2} = 1 \quad \Rightarrow \quad \frac{\partial T_1}{\partial x_1} + \beta_3(x_1) = 0$$

$$T_1(x) = x_2 - \int_0^{x_1} \beta_3(\sigma)\, d\sigma$$

$$\frac{\partial T_2}{\partial x_2} = 0 \quad \Rightarrow \quad \frac{\partial T_2}{\partial x_1} = 1, \qquad T_2(x) = x_1$$

$$z_1 = x_2 - \int_0^{x_1} \beta_3(\sigma)\, d\sigma, \quad z_2 = x_1$$

$$y = z_2$$

$$\begin{aligned} \dot{z} &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} z + \begin{bmatrix} \beta_2(y) - \beta_1(y)\beta_3(y) + u \\ \int_0^y \beta_3(\sigma)\, d\sigma + \beta_1(y) \end{bmatrix} \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} z \end{aligned}$$

Definition: A nonlinear system is in the output feedback form if

$$
\begin{aligned}
\dot{x}_1 &= x_2 + \gamma_1(y) \\
\dot{x}_2 &= x_3 + \gamma_2(y) \\
&\vdots \\
\dot{x}_{\rho-1} &= x_\rho + \gamma_{\rho-1}(y) \\
\dot{x}_\rho &= x_{\rho+1} + \gamma_\rho(y) + b_m u, \qquad b_m > 0 \\
&\vdots \\
\dot{x}_{n-1} &= x_n + \gamma_{n-1}(y) + b_1 u \\
\dot{x}_n &= \gamma_n(y) + b_0 u \\
y &= x_1
\end{aligned}
$$

Show that

- The output feedback form is a special case of the observer form
- It has relative degree ρ
- It is minimum phase if the polynomial

$$b_m s^m + \cdots + b_1 s + b_0$$

is Hurwitz

Definition: A nonlinear system is in the strict feedback form if

$$
\begin{aligned}
\dot{x} &= f_0(x) + g_0(x)z_1 \\
\dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1)z_2 \\
\dot{z}_2 &= f_2(x, z_1, z_2) + g_2(x, z_1, z_2)z_3 \\
&\vdots \\
\dot{z}_{k-1} &= f_{k-1}(x, z_1, \ldots, z_{k-1}) + g_{k-1}(x, z_1, \ldots, z_{k-1})z_k \\
\dot{z}_k &= f_k(x, z_1, \ldots, z_k) + g_k(x, z_1, \ldots, z_k)u
\end{aligned}
$$

$x \in R^n$, z_1 to z_k are scalars

$$g_i(x, z_1, \ldots, z_i) \neq 0 \quad \text{for } 1 \leq i \leq k$$

- Find the relative degree if $y = z_1$
- Find the zero dynamics if $y = z_1$ and

$$f_i(x, 0) = 0, \quad \forall\, 1 \leq i \leq k$$