Temporal-difference learning

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Machine and Reinforcement Learning in Control Applications

Introduction

- Learn directly from experience without a model of the environment.
- As DP, use learned estimates to update the prediction
  - learns from incomplete episodes;
  - bootstrap.
- Updates a guess towards a guess.
- Can be used both for prediction and control.

Monte Carlo vs temporal-difference prediction

- Given a policy π, the goal is to estimate v_π.
- MC methods wait until the return following the visit is known
  - use the return G_t as a target for $V(S_t)$

$$V(S_t) \leftarrow V(S_t) + \alpha(G_t - V(S_t)).$$

- The simplest TD method (TD(0)) uses the immediate reward
  - use the reward R_{t+1}
  - and the expected return $\gamma V(S_{t+1})$ as a target for $V(S_t)$

$$V(S_t) \leftarrow V(S_t) + \alpha(R_{t+1} + \gamma V(S_{t+1}) - V(S_t)).$$

One-step temporal-difference

Tabular TD(0)

Input: π
Output: v_π

Initialization

$V(s) \leftarrow$ arbitrary, $\forall s \in \mathcal{S}$
$V(\text{terminal}) \leftarrow 0$

Loop

initialize S
for each step of the episode **do**
  $A \leftarrow \pi(S)$
  take action A and observe R, S'
  $V(S) \leftarrow V(S) + \alpha(R + \gamma V(S') - V(S))$
  $S \leftarrow S'$
  if S is terminal **then**
    reinitialize the episode

Comparison of DP, MC, and TD

- DP estimates $v_\pi(s)$ by bootstrapping

$$V(s) \leftarrow \mathbb{E}_\pi[R_{t+1} + \gamma \underbrace{V(S_{t+1})}_{\text{estimate}} | S_t = s].$$

- MC estimates $v_\pi(s)$ by sample mean

$$V(s) \leftarrow \underbrace{\mathbb{E}_\pi}_{\text{sample}}[G_t | S_t = s].$$

- TD estimates $v_\pi(s)$ by estimating both

$$V(s) \leftarrow \underbrace{\mathbb{E}_\pi}_{\text{sample}}[R_{t+1} + \gamma \underbrace{V(S_{t+1})}_{\text{estimate}} | S_t = s].$$

Temporal-difference error

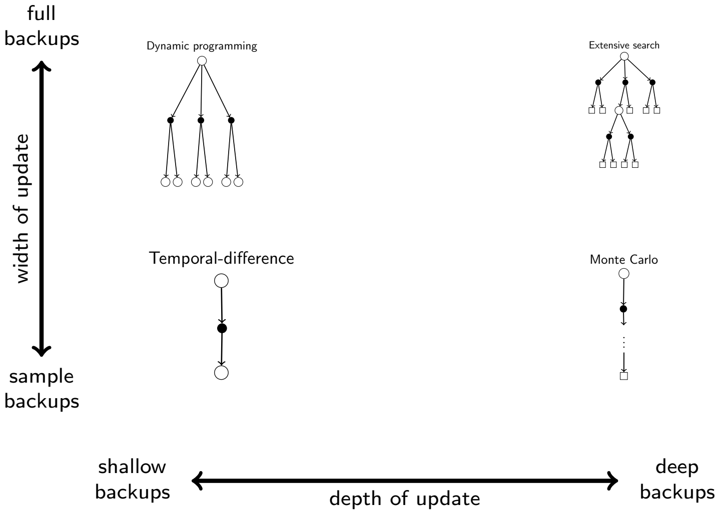
- The *TD error* is

$$\delta_t = R_{t+1} + \gamma V(S_{t+1}) - V(S_t).$$

- This error is used to update $V(S_t)$ towards a better estimate.
- It is not available until $t+1$.
- If V does not change, the MC error satisfies

$$\begin{aligned} G_t - V(S_t) &= R_{t+1} + \gamma G_{t+1} - V(S_t) \\ &= \delta_t + \gamma(G_{t+1} - V(S_{t+1})) \\ &= \delta_t + \gamma\delta_{t+1} + \gamma^2(G_{t+1} - V(S_{t+1})) \\ &= \sum_{k=t}^{T-1} \gamma^{k-t}\delta_k. \end{aligned}$$

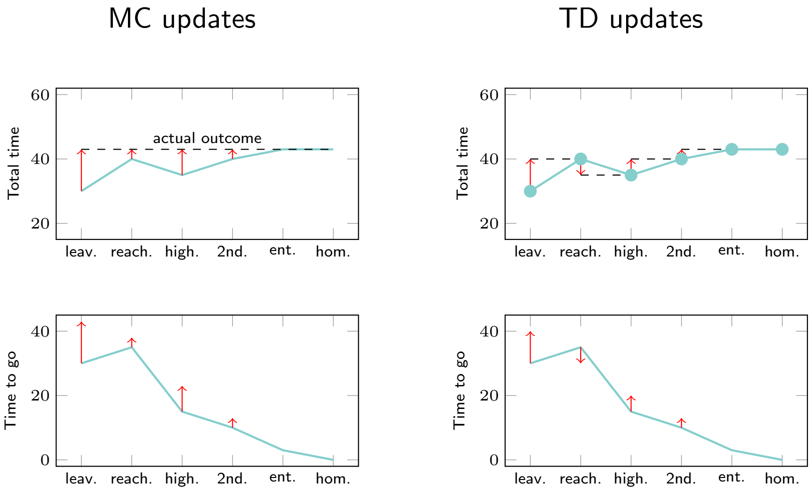
Unified view



Driving home example

State	Elapsed time	Predicted time to go	Predicted total time
Leaving university	0	30	30
Reaching car	5	35	40
Exiting highway	20	15	35
Secondary road	30	10	40
Entering home	40	3	43
Home	43	0	43

MC vs TD updates



Advantages of TD

- TD can be implemented online.

- MC waits until the end of the episode.
- TD require just a single step.

- MC requires complete sequences.
- TD works also for incomplete sequences.

- MC can be applied just for episodic tasks.
- TD can be used for continuous and episodic tasks.

Bias and variance of estimators

- MC target

$$G_t = R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{T-1} R_T$$

is an unbiased estimate of $v_\pi(S_t)$.

- TD true target

$$R_{t+1} + \gamma v_\pi(S_{t+1})$$

is an unbiased estimate of $v_\pi(S_t)$.

- TD target

$$R_{t+1} + \gamma V(S_{t+1})$$

is a biased estimate of $v_\pi(S_t)$.

- TD target has lower variance than MC target
  - return depends on *many* random actions, transitions, rewards;
  - TD target depends on *one* random action, transition, reward.

Advantages and disadvantages of MC and TD

- MC has high variance and zero bias
  - good convergence properties;
  - not very sensitive to initial value;
  - simple to understand and use;
  - more efficient in non-Markov environments.

- TD has low variance and nonzero bias
  - usually more efficient than MC;
  - TD(0) proved to converge to v_π
    - in the mean for constant (small) α;
    - almost surely if $\sum_k \alpha_k = \infty$ and $\sum_k \alpha_k^2 < \infty$;
  - sensitive to initial value;
  - more efficient in Markov environments.

Batch updating

- Suppose to have a finite amount of experience.
- We can present the experience repeatedly until convergence.
- Given V, update it only once for each batch
  - compute the updates $\alpha(\text{target}_t - V(S_t))$ at each time step;
  - change the value function once with the sum of all increments.
- Constant α TD converges deterministically.
- Constant α MC converges deterministically.
- The two results may be different.

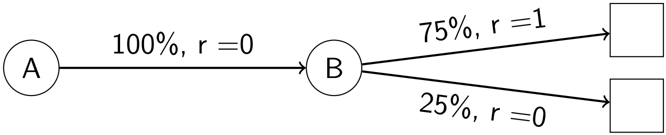
You are the predictor (1)

- Suppose you observe the following eight episodes
  1. A, 0, B, 0;
  2. B, 1;
  3. B, 1;
  4. B, 1;
  5. B, 1;
  6. B, 1;
  7. B, 1;
  8. B, 0.
- We want to estimate $V(A)$ and $V(B)$.

You are the predictor (2)

- The optimal value for B is $V(B) = \frac{6}{8} = 0.75$.

- Modeling experience via an MP



  - $V(A) = 0.75$;
  - same answer given by TD(0).

- We observed the return from A once and it was 0
  - $V(A) = 0$;
  - same answer given by MC;
  - minimum squared error on training data.

Certainty equivalence

- Batch MC converges to solution with minimum MS error
  - best fit to the observed returns

$$\sum_{k=1}^{K}\sum_{t=1}^{T_k}\left(G_t^k - V(S_t^k)\right)^2.$$

- Batch TD(0) converges to solution of max likelihood MDP
  - solution to the MDP that best fits the data

$$\hat{P}_{s,s',a} = \frac{1}{N(s,a)}\sum_{k=1}^{K}\sum_{t=1}^{T_k}\mathbf{1}(s_{t+1}^k = s', s_t^k = s, a_t^k = a),$$

$$\hat{R}_{s,a} = \frac{1}{N(s,a)}\sum_{k=1}^{K}\sum_{t=1}^{T_k} r_t^k\mathbf{1}(s_t^k = s, a_t^k = a);$$

  - equivalent to assuming that the process estimate was known.

Off-policy TD prediction

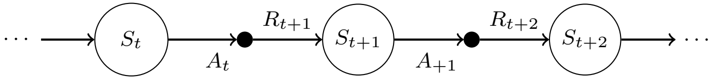
- Use TD targets generated from b to evaluate π.
- Weight TD target by importance sampling.
- Only need a single importance sampling correction

$$V(S_t) \leftarrow V(S_t)+\alpha \left(\frac{\pi(A_t|S_t)}{b(A_t|S_t)} \underbrace{(R_{t+1} + \gamma V(S_{t+1}))}_{\text{TD target}} - V(S_t) \right).$$

- Lower variance than Monte-Carlo importance sampling.

On-policy TD control

- We follow the path of GPI.
- Use TD for the evaluation part.
- Estimate q_π rather than v_π.
- Transitions from a state–action pair to a state–action pair.



- This is still an MDP.

SARSA

- Apply TD(0) to action values

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha \underbrace{(\underbrace{R_{t+1} + \gamma Q(S_{t+1}, A_{t+1})}_{\text{TD target}} - Q(S_t, A_t))}_{\text{TD error } \delta_t}.$$

- If S_{t+1} is terminal

$$Q(S_{t+1}, A_{t+1}) = 0, \quad \forall A_t.$$

- The update is carried out on the basis of the quintuple

$$(S_t, A_t, R_{t+1}, S_{t+1}, A_{t+1}).$$

Greedy in the Limit with Infinite Exploration

GLIE policies

A policy π is GLIE if

1. All state-action pairs are explored infinitely many times

$$N_k(s,a) \to \infty, \quad \forall a \in \mathcal{A}(s), \forall s \in \mathcal{S}.$$

2. The policy converges on a greedy policy

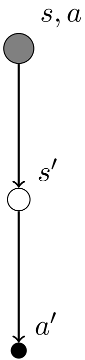
$$\pi(a|s) = \mathbf{1}(a = \arg\max_a Q(s,a)).$$

- For instance, ε-greedy policies are GLIE if $\varepsilon_k = \frac{1}{k}$.

Notes on SARSA

- If Q does not change, the MC error satisfies

$$\begin{aligned} G_t - Q(S_t, A_t) &= R_{t+1} + \gamma G_{t+1} - Q(S_t, A_t) \\ &= \delta_t + \gamma(G_{t+1} - Q(S_{t+1}, A_{t+1})) \\ &= \delta_t + \gamma\delta_{t+1} + \gamma^2(G_{t+2} - Q(S_{t+2}, A_{t+2})) \\ &= \sum_{k=t}^{T-1} \gamma^{k-t}\delta_k. \end{aligned}$$



- If the policy is GLIE and

$$\sum_k \alpha_k = \infty, \quad \sum_k \alpha_k^2 < \infty$$

  - SARSA converges with probability 1.

On-policy TD control algorithm

SARSA algorithm

Input: $\alpha > 0$, $\varepsilon > 0$
Output: q_*, π_*

Initialization

$Q(s, a) \leftarrow \text{arbitrary}, \forall a \in \mathcal{A}(s), \forall s \in \mathcal{S}$
$Q(\text{terminal}, \cdot) \leftarrow 0$

Loop

initialize S
$A \leftarrow$action derived by $Q(S, \cdot)$ (e.g., ε-greedy)
for each step of the episode **do**
  take action A and observe R, S'
  $A' \leftarrow$action derived by $Q(S', \cdot)$ (e.g., ε-greedy)
  $Q(S, A) \leftarrow Q(S, A) + \alpha(R + \gamma Q(S', A') - Q(S, A))$
  $S \leftarrow S'$
  $A \leftarrow A'$
  if S is terminal **then**
    reinitialize the episode

Q-learning

- Independent of the policy being followed.
- Directly approximate q_*.
- Update Q as

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha(R_{t+1} + \gamma \max_a Q(S_{t+1}, a) - Q(S_t, A_t)).$$

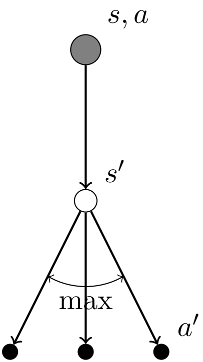
- No importance sampling is required.

Notes on Q-learning

- Next action is chosen using behavior policy, e.g., ε-greedy.
- We consider alternative successor action following the greedy target policy π.
- Both behavior and target policies improve.
- If all pairs continue to be updated and

$$\sum_k \alpha_k(x_k, a_k) = \infty, \quad \sum_k \alpha_k^2(x_k, a_k) < \infty$$

  - Q-learning converges with probability 1.



Off-policy TD control algorithm

Q-learning algorithm

Input: $\alpha > 0$, $\varepsilon > 0$
Output: q_*, π_*

Initialization

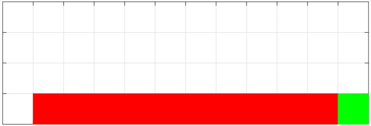
$Q(s, a) \leftarrow \text{arbitrary}, \forall a \in \mathcal{A}(s), \forall s \in \mathcal{S}$
$Q(\text{terminal}, \cdot) \leftarrow 0$

Loop

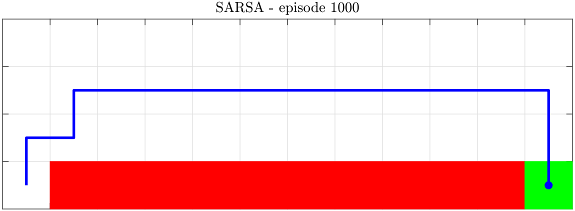
initialize S
for each step of the episode **do**
  $A \leftarrow$action derived by $Q(S, \cdot)$ (e.g., ε-greedy)
  take action A and observe R, S'
  $Q(S, A) \leftarrow Q(S, A) + \alpha(R + \gamma \max_a \gamma Q(S', a) - Q(S, A))$
  $S \leftarrow S'$
  if S is terminal **then**
    reinitialize the episode

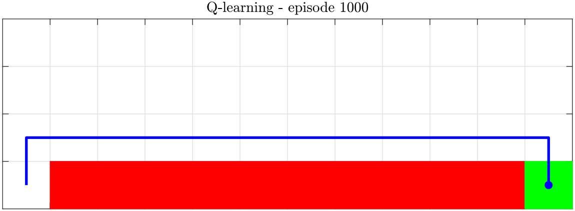
Cliff walking

- Consider the gridworld on the right.
- Terminal states are those in red and green.
- Reward is -1 on all transitions except those into the red area.
- Stepping into this region incurs a reward of -100.



Comparison of SARSA and Q-learning on cliff walking





Expected SARSA

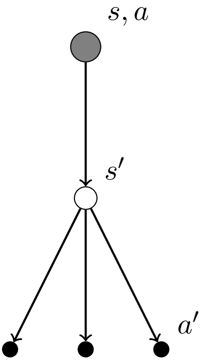
- Consider the basic update of Q-learning.
- Rather than maximizing, take expectation

$$\begin{aligned} Q(S_t, A_t) &\leftarrow Q(S_t, A_t) + \alpha(R_{t+1} + \gamma \mathbb{E}_\pi[Q(S_{t+1}, A_{t+1})|S_t] - Q(S_t, A_t)) \\ &= Q(S_t, A_t) + \alpha\left(R_{t+1} + \gamma\left(\sum_a \pi(a|S_{t+1})Q(S_{t+1}, a)\right) - Q(S_t, A_t)\right). \end{aligned}$$

- This moves deterministically as SARSA moves in expectation.
- It eliminates the variance due to the random selection of A_{t+1}.

Notes on expected SARSA

- Retains advantages of SARSA.
- Consistent empirical advantage of expected SARSA over SARSA.
- Can select large values of α ($\simeq 1$) when the environment is non-stochastic.
- Can be used off-policy
  - π is the greedy policy
  - b is the behavior policy
    - expected SARSA is Q-learning.



Maximization bias

- All control algorithms discussed so far involve maximization
  - in Q-learning the target policy is the greedy policy on Q;
  - in SARSA in Sarsa the policy is often ε-greedy.
- A maximization over estimates may lead to a (positive) bias.
- Consider, e.g., the case
  - $q(s,a) = 0$ for all $a \in \mathcal{A}(s)$;
  - $Q(s,a)$ are uncertain and thus distributed some above and some below zero.
- The problem is due to the fact that the same data are used to estimate both optimal actions and their values.

Double learning

- Divide data in two sets and used them to learn two independent estimates: Q_1 and Q_2.

- Q_1 can be used to determine the maximizing action

$$A_t^* = \arg\max_a Q_1(S_t, a).$$

- Q_2 can be used to estimate its value

$$Q_2(S_t, A_t^*) = Q_2(S_t, \arg\max_a Q_1(S_t, a)).$$

- This is equivalent to

$$Q_2(s, a) \leftarrow Q_2(s, a) + \alpha(R + \gamma Q_1(s', \arg\max_{a'} Q_2(s', a')) - Q_2(S, A)).$$

- The estimate will be unbiased

$$\mathbb{E}[Q_2(S_t, A_t^*)] = q(S_t, A_t^*).$$

Double Q-learning

Double Q-learning algorithm

Input: $\alpha > 0$, $\varepsilon > 0$
Output: q_*, π_*

Initialization

$Q_1(s,a) \leftarrow \text{arbitrary}, \forall a \in \mathcal{A}(s), \forall s \in \mathcal{S}$, $Q_1(\text{terminal}, \cdot) \leftarrow 0$
$Q_2(s,a) \leftarrow \text{arbitrary}, \forall a \in \mathcal{A}(s), \forall s \in \mathcal{S}$, $Q_2(\text{terminal}, \cdot) \leftarrow 0$

Loop

initialize S
for each step of the episode **do**
  $A \leftarrow$action derived by $Q_1(S,\cdot) + Q_2(S,\cdot)$ (e.g., ε-greedy)
  take action A and observe R, S'
  with probability 0.5
    $Q_1(S,A) \leftarrow Q_1(S,A) + \alpha(R + \gamma Q_2(S', \arg\max_a Q_1(S',a)) - Q_1(S,A))$
  else
    $Q_2(S,A) \leftarrow Q_2(S,A) + \alpha(R + \gamma Q_1(S', \arg\max_a Q_2(S',a)) - Q_2(S,A))$
  $S \leftarrow S'$
  if S is terminal **then**
    reinitialize the episode