Nonlinear Systems and Control
Lecture # 25

Stabilization

Basic Concepts & Linearization

We want to stabilize the system

$$\dot{x} = f(x, u)$$

at the equilibrium point $x = x_{\rm ss}$

Steady-State Problem: Find steady-state control $u_{\rm ss}$ s.t.

$$0 = f(x_{\rm ss}, u_{\rm ss})$$

$$x_\delta = x - x_{\rm ss}, \qquad u_\delta = u - u_{\rm ss}$$

$$\dot{x}_\delta = f(x_{\rm ss} + x_\delta, u_{\rm ss} + u_\delta) \stackrel{\rm def}{=} f_\delta(x_\delta, u_\delta)$$

$$f_\delta(0, 0) = 0$$

$$u_\delta = \gamma(x_\delta) \;\;\Rightarrow\;\; u = u_{\rm ss} + \gamma(x - x_{\rm ss})$$

State Feedback Stabilization: Given

$$\dot{x} = f(x, u) \qquad [f(0, 0) = 0]$$

find

$$u = \gamma(x) \qquad [\gamma(0) = 0]$$

s.t. the origin is an asymptotically stable equilibrium point of

$$\dot{x} = f(x, \gamma(x))$$

f and γ are locally Lipschitz functions

Linear Systems

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

$(\boldsymbol{A}, \boldsymbol{B})$ is stabilizable (controllable or every uncontrollable eigenvalue has a negative real part)

Find $\boldsymbol{K}$ such that $(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})$ is Hurwitz

$$\boldsymbol{u} = -\boldsymbol{K}\boldsymbol{x}$$

Typical methods:

- Eigenvalue Placement
- Eigenvalue-Eigenvector Placement
- LQR

Linearization

$$\dot{x} = f(x, u)$$

$f(0,0) = 0$ and f is continuously differentiable in a domain $D_x \times D_u$ that contains the origin $(x = 0,\ u = 0)$ $(D_x \subset R^n, D_u \subset R^p)$

$$\dot{x} = Ax + Bu$$

$$A = \frac{\partial f}{\partial x}(x, u)\bigg|_{x=0,u=0}; \quad B = \frac{\partial f}{\partial u}(x, u)\bigg|_{x=0,u=0}$$

Assume (A, B) is stabilizable. Design a matrix K such that $(A - BK)$ is Hurwitz

$$u = -Kx$$

Closed-loop system:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, -\boldsymbol{K}\boldsymbol{x})$$

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \left[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}, -\boldsymbol{K}\boldsymbol{x}) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\boldsymbol{x}, -\boldsymbol{K}\boldsymbol{x})\ (-\boldsymbol{K})\right]_{\boldsymbol{x}=\boldsymbol{0}} \boldsymbol{x} \\
&= (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{x}
\end{aligned}$$

Since $(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})$ is Hurwitz, the origin is an exponentially stable equilibrium point of the closed-loop system

Example (Pendulum Equation):

$$\ddot{\theta} = -a\sin\theta - b\dot{\theta} + cT$$

Stabilize the pendulum at $\theta = \delta$

$$0 = -a\sin\delta + cT_{\text{ss}}$$

$$x_1 = \theta - \delta, \quad x_2 = \dot{\theta}, \quad u = T - T_{\text{ss}}$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a[\sin(x_1+\delta) - \sin\delta] - bx_2 + cu \end{aligned}$$

$$A = \left[\begin{array}{cc} 0 & 1 \\ -a\cos(x_1+\delta) & -b \end{array}\right]_{x_1=0} = \left[\begin{array}{cc} 0 & 1 \\ -a\cos\delta & -b \end{array}\right]$$

$$A = \begin{bmatrix} 0 & 1 \\ -a\cos\delta & -b \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ c \end{bmatrix}$$

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$A - BK = \begin{bmatrix} 0 & 1 \\ -(a\cos\delta + ck_1) & -(b + ck_2) \end{bmatrix}$$

$$k_1 > -\frac{a\cos\delta}{c}, \quad k_2 > -\frac{b}{c}$$

$$T = \frac{a\sin\delta}{c} - Kx = \frac{a\sin\delta}{c} - k_1(\theta - \delta) - k_2\dot{\theta}$$

Notions of Stabilization

$$\dot{x} = f(x,u), \quad u = \gamma(x)$$

Local Stabilization: The origin of $\dot{x} = f(x,\gamma(x))$ is asymptotically stable (e.g., linearization)

Regional Stabilization: The origin of $\dot{x} = f(x,\gamma(x))$ is asymptotically stable and a given region G is a subset of the region of attraction (for all $x(0) \in G$, $\lim_{t\to\infty} x(t) = 0$) (e.g., $G \subset \Omega_c = \{V(x) \leq c\}$ where Ω_c is an estimate of the region of attraction)

Global Stabilization: The origin of $\dot{x} = f(x,\gamma(x))$ is globally asymptotically stable

Semiglobal Stabilization: The origin of $\dot{x} = f(x, \gamma(x))$ is asymptotically stable and $\gamma(x)$ can be designed such that any given compact set (no matter how large) can be included in the region of attraction (Typically $u = \gamma_p(x)$ is dependent on a parameter p such that for any compact set G, p can be chosen to ensure that G is a subset of the region of attraction)

What is the difference between global stabilization and semiglobal stabilization?

Example

$$\dot{x} = x^2 + u$$

Linearization:

$$\dot{x} = u, \qquad u = -kx, \quad k > 0$$

Closed-loop system:

$$\dot{x} = -kx + x^2$$

Linearization of the closed-loop system yields $\dot{x} = -kx$. Thus, $u = -kx$ achieves local stabilization

The region of attraction is $\{x < k\}$. Thus, for any set $\{x \leq a\}$ with $a < k$, the control $u = -kx$ achieves regional stabilization

The control $u = -kx$ does not achieve global stabilization

But it achieves semiglobal stabilization because any compact set $\{|x| \le r\}$ can be included in the region of attraction by choosing $k > r$

The control

$$u = -x^2 - kx$$

achieves global stabilization because it yields the linear closed-loop system $\dot{x} = -kx$ whose origin is globally exponentially stable

Practical Stabilization

$$\dot{x} = f(x,u) + g(x,u,t)$$

$$f(0,0) = 0, \quad g(0,0,t) \neq 0$$

$$\|g(x,u,t)\| \leq \delta, \quad \forall\, x \in D_x,\ u \in D_u,\ t \geq 0$$

There is no control $u = \gamma(x)$, with $\gamma(0) = 0$, that can make the origin of

$$\dot{x} = f(x,\gamma(x)) + g(x,\gamma(x),t)$$

uniformly asymptotically stable because the origin is not an equilibrium point

Definition: The system

$$\dot{x} = f(x, u) + g(x, u, t)$$

is practically stabilizable if for any $\varepsilon > 0$ there is a control law $u = \gamma(x)$ such that the solutions of

$$\dot{x} = f(x, \gamma(x)) + g(x, \gamma(x), t)$$

are uniformly ultimately bounded by ε; i.e.,

$$\|x(t)\| \leq \varepsilon, \quad \forall\, t \geq T$$

Typically, $u = \gamma_p(x)$ is dependent on a parameter p such that for any $\varepsilon > 0$, p can be chosen to ensure that ε is an ultimate bound

With practical stabilization, we may have

- local practical stabilization
- regional practical stabilization
- global practical stabilization, or
- semiglobal practical stabilization

depending on the region of initial states

Example

$$\dot{x} = x^2 + u + d(t), \qquad |d(t)| \leq \delta, \; \forall\, t \geq 0$$

$$u = -kx, \;\; k > 0, \;\; \Rightarrow \;\; \dot{x} = x^2 - kx + d(t)$$

$$V = \tfrac{1}{2}x^2 \;\; \Rightarrow \;\; \dot{V} = x^3 - kx^2 + xd(t)$$

$$\dot{V} \leq -\tfrac{k}{3}x^2 - x^2\left(\tfrac{k}{3} - |x|\right) - |x|\left(\tfrac{k}{3}|x| - \delta\right)$$

$$\dot{V} \leq -\tfrac{k}{3}x^2, \quad \text{for } \frac{3\delta}{k} \leq |x| \leq \frac{k}{3}$$

$$\text{Take } \frac{3\delta}{k} < \varepsilon \;\; \Leftrightarrow \;\; k \geq \frac{3\delta}{\varepsilon}$$

By choosing k large enough we can achieve semiglobal practical stabilization

$$\dot{x} = x^2 + u + d(t)$$

$$u = -x^2 - kx, \quad k > 0, \quad \Rightarrow \quad \dot{x} = -kx + d(t)$$

$$V = \frac{1}{2}x^2 \quad \Rightarrow \quad \dot{V} = -kx^2 + xd(t)$$

$$\dot{V} \leq -\frac{k}{2}x^2 - |x| \left(\frac{k}{2}|x| - \delta \right)$$

By choosing k large enough we can achieve global practical stabilization