Nonlinear Systems and Control Lecture # 32

Robust Stabilization

Sliding Mode Control

Example

$$\dot{x}_1 = x_2$$
 $\dot{x}_2 = h(x) + g(x)u$, $g(x) \ge g_0 > 0$

Sliding Manifold (Surface):

$$egin{aligned} s &= a_1 x_1 + x_2 = 0 \ & \ s(t) \equiv 0 & \Rightarrow \dot{x}_1 = -a_1 x_1 \ & \ a_1 > 0 & \Rightarrow \lim_{t o \infty} x_1(t) = 0 \end{aligned}$$

How can we bring the trajectory to the manifold s=0?

How can we maintain it there?

$$\dot{s} = a_1 \dot{x}_1 + \dot{x}_2 = a_1 x_2 + h(x) + g(x)u$$

Suppose

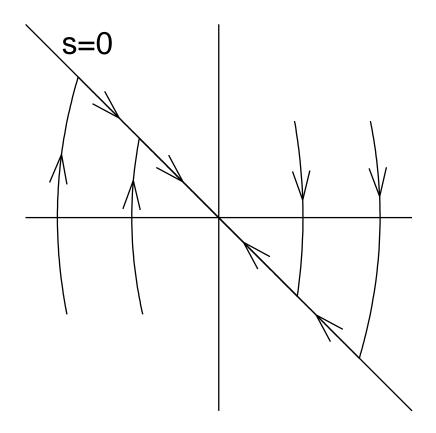
$$egin{aligned} \left| rac{a_1x_2 + h(x)}{g(x)}
ight| &\leq arrho(x) \ V = rac{1}{2}s^2 \ \dot{V} = s\dot{s} = s[a_1x_2 + h(x)] + g(x)su \leq g(x)|s|arrho(x) + g(x)su \ eta(x) &\geq arrho(x) + eta_0, \quad eta_0 > 0 \ s > 0, \quad u = -eta(x) \ \dot{V} &\leq g(x)|s|arrho(x) - g(x)eta(x) + g(x)|s| \ \dot{V} &\leq g(x)|s|arrho(x) - g(x)(arrho(x) + eta_0)|s| = -g(x)eta_0|s| \end{aligned}$$

$$egin{aligned} s < 0, \quad u = eta(x) \ \dot{V} \leq g(x)|s|arrho(x) + g(x)su &= g(x)|s|arrho(x) - g(x)eta(x)|s| \ \dot{V} \leq g(x)|s|arrho(x) - g(x)(arrho(x) + eta_0)|s| &= -g(x)eta_0|s| \ & ext{sgn}(s) = \left\{egin{aligned} 1, & s > 0 \ -1, & s < 0 \end{aligned}
ight. \ u &= -eta(x) ext{sgn}(s) \ \dot{V} \leq -g(x)eta_0|s| \leq -g_0eta_0|s| \ \dot{V} \leq -g_0eta_0\sqrt{2V} \end{aligned}$$

$$\dot{V} \leq -g_0eta_0\sqrt{2V}$$
 $\dfrac{dV}{\sqrt{V}} \leq -g_0eta_0\sqrt{2}\ dt$ $2\ \sqrt{V}igg|_{V(s(0))}^{V(s(t))} \leq -g_0eta_0\sqrt{2}\ t$ $\sqrt{V(s(t))} \leq \sqrt{V(s(0))} -g_0eta_0\dfrac{1}{\sqrt{2}}\ t$ $|s(t)| \leq |s(0)| -g_0eta_0\ t$

s(t) reaches zero in finite time

Once on the surface s=0, the trajectory cannot leave it



What is the region of validity?

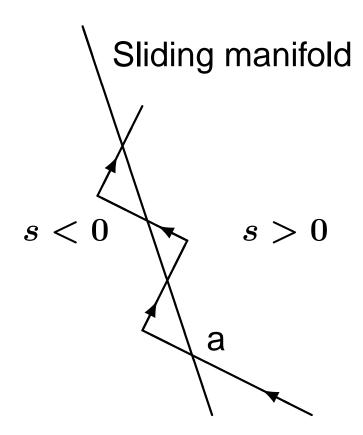
$$\dot{x}_1=x_2$$
 $\dot{x}_2=h(x)-g(x)eta(x) ext{sgn}(s)$ $\dot{x}_1=-a_1x_1+s$ $\dot{s}=a_1x_2+h(x)-g(x)eta(x) ext{sgn}(s)$ $s\dot{s}\leq -g_0eta_0|s|,$ if $eta(x)\geq arrho(x)+eta_0$ $V_1=rac{1}{2}x_1^2$ $\dot{V}_1=x_1\dot{x}_1=-a_1x_1^2+x_1s\leq -a_1x_1^2+|x_1|c\leq 0$ $orall\,|s|\leq c ext{ and }|x_1|\geq rac{c}{a_1}$ $\Omega=\left\{|x_1|\leq rac{c}{a_1},\;|s|\leq c
ight\}$

 Ω is positively invariant if $\left| rac{a_1 x_2 + h(x)}{g(x)}
ight| \leq arrho(x)$ over Ω

$$\Omega = \left\{ |x_1| \leq rac{c}{a_1}, \; |s| \leq c
ight\}$$
 $s = 0$
 c
 c/a_1
 x_1
 $\left| rac{a_1 x_2 + h(x)}{g(x)}
ight| \leq k_1 < k, \; orall \; x \in \Omega$

$$u = -k \operatorname{sgn}(s)$$

Chattering



How can we reduce or eliminate chattering?

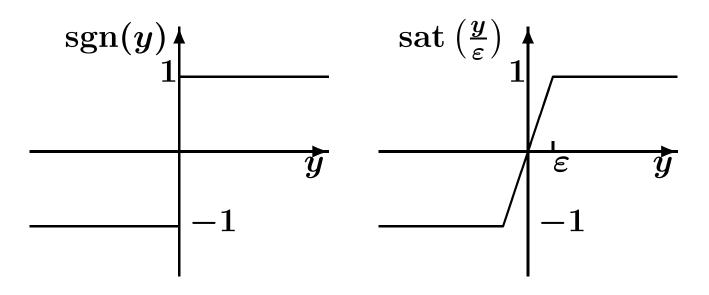
Reduce the amplitude of the signum function

$$egin{aligned} \dot{s} &= a_1 x_2 + h(x) + g(x) u \ u &= -rac{[a_1 x_2 + \hat{h}(x)]}{\hat{g}(x)} + v \ \dot{s} &= \delta(x) + g(x) v \ \delta(x) &= a_1 \left[1 - rac{g(x)}{\hat{g}(x)}
ight] x_2 + h(x) - rac{g(x)}{\hat{g}(x)} \hat{h}(x) \ \left|rac{\delta(x)}{g(x)}
ight| &\leq arrho(x), \quad eta(x) \geq arrho(x) + eta_0 \ v &= -eta(x) \ ext{sgn}(s) \end{aligned}$$

Replace the signum function by a high-slope saturation function

$$u = -\beta(x) \operatorname{sat}\left(\frac{s}{\varepsilon}\right)$$

$$\operatorname{sat}(y) = \left\{ egin{array}{ll} y, & ext{if } |y| \leq 1 \ \operatorname{sgn}(y), & ext{if } |y| > 1 \end{array}
ight.$$



How can we analyze the system?

For
$$|s| \geq arepsilon, \quad u = -\beta(x) \operatorname{sgn}(s)$$

With c>arepsilon

- $oldsymbol{\square}$ $\Omega = \left\{ |x_1| \leq rac{c}{a_1}, \; |s| \leq c
 ight\}$ is positively invariant
- The trajectory reaches the boundary layer $\{|s| \leq \varepsilon\}$ in finite time
- The boundary layer is positively invariant

Inside the boundary layer:

$$\dot{x}_1 = -a_1 x_1 + s$$
 $\dot{s} = a_1 x_2 + h(x) - g(x) eta(x) rac{s}{arepsilon}$ $x_1 \dot{x}_1 \le -a_1 x_1^2 + |x_1| arepsilon$ $0 < heta < 1$ $x_1 \dot{x}_1 \le -(1 - heta) a_1 x_1^2, \ \ orall \ |x_1| \ge rac{arepsilon}{ heta a_1}$

The trajectories reach the positively invariant set

$$\Omega_{\varepsilon} = \{|x_1| \le \frac{\varepsilon}{\theta a_1}, |s| \le \varepsilon\}$$

in finite time

What happens inside Ω_{ε} ?

Find the equilibrium points

$$0=-a_1x_1+s=x_2, \quad 0=a_1x_2+h(x)-g(x)eta(x)rac{s}{arepsilon}$$

$$\phi(x_1) = \left. rac{h(x)}{a_1 g(x) eta(x)}
ight|_{x_2 = 0}$$
 $x_1 = arepsilon \phi(x_1)$

Suppose $x_1 = arepsilon \phi(x_1)$ has an isolated root $ar{x}_1 = arepsilon k_1$

$$h(0) = 0 \Rightarrow \bar{x}_1 = 0$$

$$egin{aligned} z_1 &= x_1 - ar{x}_1, \quad z_2 = s - a_1 ar{x}_1 \ x_2 &= -a_1 x_1 + s = -a_1 (x_1 - ar{x}_1) + s - a_1 ar{x}_1 = -a_1 z_1 + z_2 \ \dot{z}_1 &= -a_1 x_1 + s = -a_1 z_1 + z_2 \ \dot{z}_2 &= a_1 x_2 + h(x) - g(x) eta(x) rac{s}{arepsilon} \ &= a_1 (z_2 - a_1 z_1) + h(x) - g(x) eta(x) rac{z_2 + a_1 ar{x}_1}{arepsilon} \ \dot{z}_2 &= \ell(z) - g(x) eta(x) rac{z_2}{arepsilon} \ \ell(z) &= a_1 (z_2 - a_1 z_1) + a_1 g(x) eta(x) \left[rac{h(x)}{a_1 g(x) eta(x)} - rac{ar{x}_1}{arepsilon}
ight] \end{aligned}$$

$$egin{aligned} \dot{z}_1 &= -a_1 z_1 + z_2, \qquad \dot{z}_2 = \ell(z) - g(x) eta(x) rac{z_2}{arepsilon} \ & \ell(0) = 0, \; \; |\ell(z)| \leq \ell_1 |z_1| + \ell_2 |z_2| \ & g(x) eta(x) \geq g_0 eta_0 \ & V = rac{1}{2} z_1^2 + rac{1}{2} z_2^2 \ & \dot{V} = z_1 (-a_1 z_1 + z_2) + z_2 \left[\ell(z) - g(x) eta(x) rac{z_2}{arepsilon}
ight] \ & \dot{V} \leq -a_1 z_1^2 + (1 + \ell_1) |z_1| \; |z_2| + \ell_2 z_2^2 - rac{g_0 eta_0}{arepsilon} z_2^2 \end{aligned}$$

$$\begin{split} \dot{V} &\leq -a_1 z_1^2 + (1+\ell_1)|z_1| \, |z_2| + \ell_2 z_2^2 - \frac{g_0 \beta_0}{\varepsilon} z_2^2 \\ \dot{V} &\leq - \left[\begin{array}{c} |z_1| \\ |z_2| \end{array} \right]^T \underbrace{\left[\begin{array}{c} a_1 & -\frac{1}{2}(1+\ell_1) \\ -\frac{1}{2}(1+\ell_1) & \left(\frac{g_0 \beta_0}{\varepsilon} - \ell_2 \right) \end{array} \right] \left[\begin{array}{c} |z_1| \\ |z_2| \end{array} \right]}_{Q} \\ \det(Q) &= a_1 \left(\frac{g_0 \beta_0}{\varepsilon} - \ell_2 \right) - \frac{1}{4}(1+\ell_1)^2 \\ h(0) &= 0 \quad \Rightarrow \quad \lim_{t \to \infty} x(t) = 0 \\ h(0) &\neq 0 \quad \Rightarrow \quad \lim_{t \to \infty} x(t) = \begin{bmatrix} \bar{x}_1 \\ 0 \end{bmatrix} \end{split}$$

Read Section 14.1.1