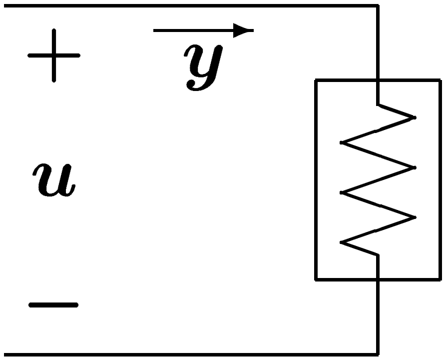
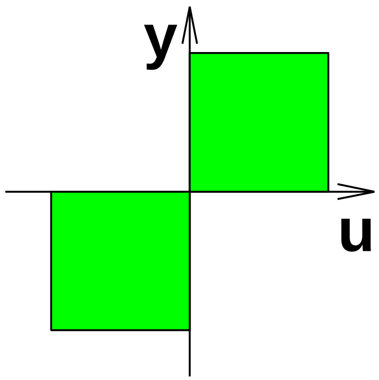
Nonlinear Systems and Control
Lecture # 14
Passivity
Memoryless Functions
&
State Models

Memoryless Functions



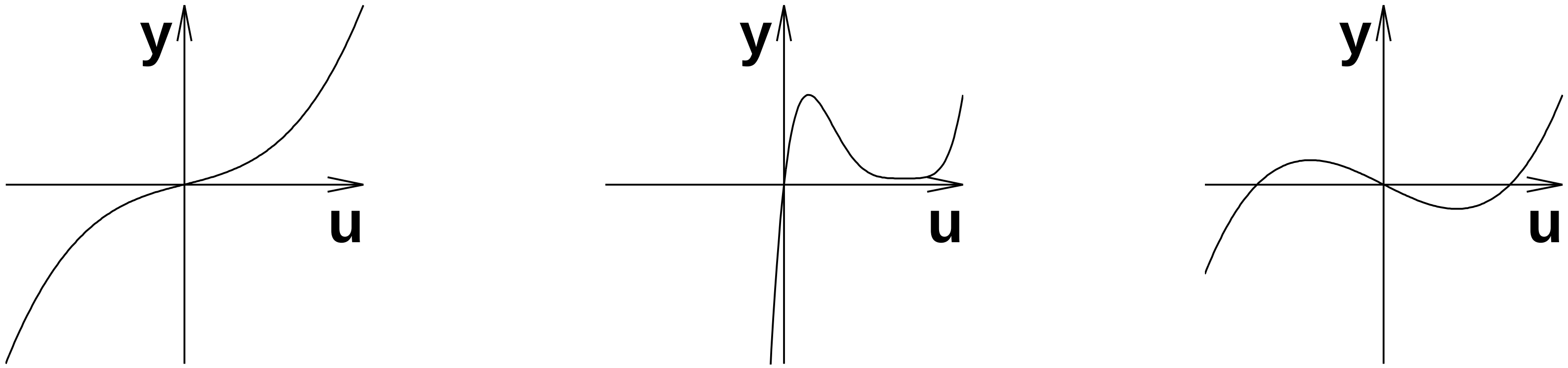
(a)



(b)

power inflow $= uy$

Resistor is passive if $uy \geq 0$



(a) (b) (c)

Passive Passive Not passive

$$y = h(t,u), \quad h \in [0,\infty]$$

Vector case:

$$y = h(t,u), \quad h^T = \left[\begin{array}{cccc} h_1, & h_2, & \cdots, & h_p \end{array}\right]$$

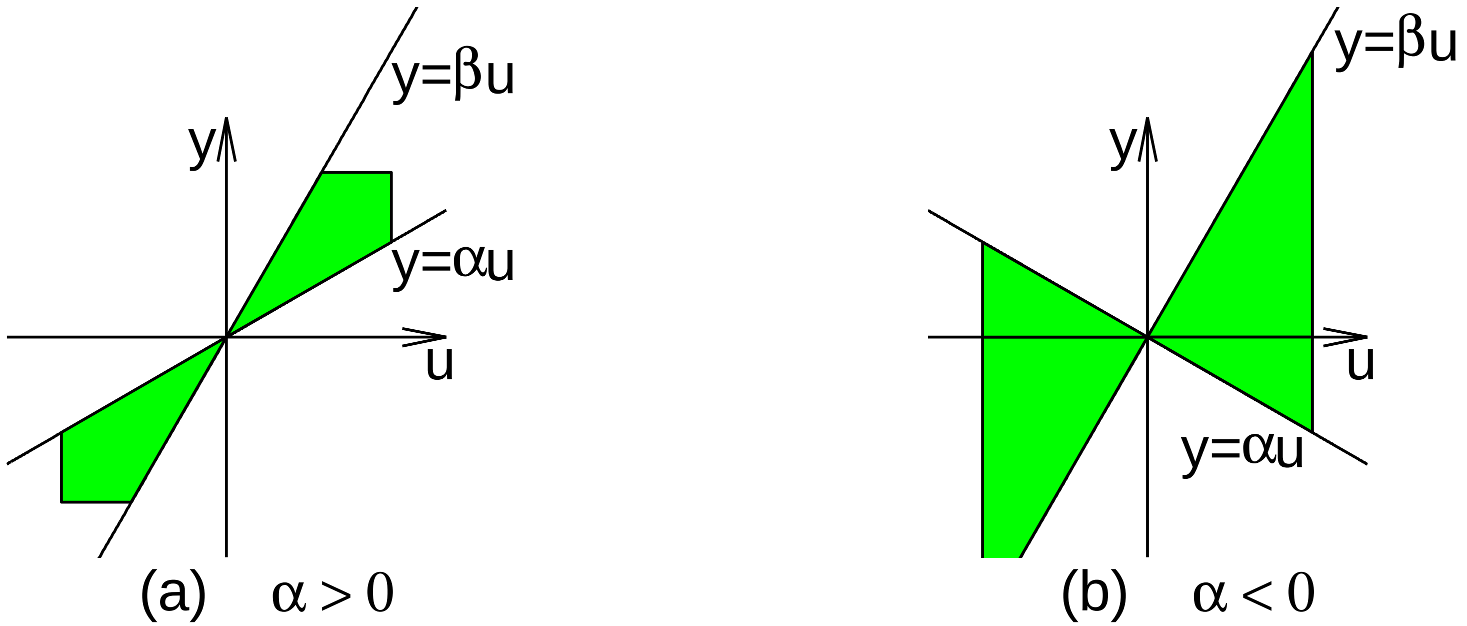
power inflow $= \Sigma_{i=1}^{p} u_i y_i = u^T y$

Definition: $y = h(t, u)$ is

- passive if $u^T y \geq 0$
- lossless if $u^T y = 0$
- input strictly passive if $u^T y \geq u^T \varphi(u)$ for some function φ where $u^T \varphi(u) > 0$, $\forall\ u \neq 0$
- output strictly passive if $u^T y \geq y^T \rho(y)$ for some function ρ where $y^T \rho(y) > 0$, $\forall\ y \neq 0$

Sector Nonlinearity: h belongs to the sector $[\alpha, \beta]$ ($h \in [\alpha, \beta]$) if

$$\alpha u^2 \leq u h(t, u) \leq \beta u^2$$



Also, $h \in (\alpha, \beta]$, $h \in [\alpha, \beta)$, $h \in (\alpha, \beta)$

$$\alpha u^2 \leq uh(t,u) \leq \beta u^2 \Leftrightarrow [h(t,u) - \alpha u][h(t,u) - \beta u] \leq 0$$

Definition: A memoryless function $h(t,u)$ is said to belong to the sector

- $[0, \infty]$ if $u^T h(t,u) \geq 0$
- $[K_1, \infty]$ if $u^T [h(t,u) - K_1 u] \geq 0$
- $[0, K_2]$ with $K_2 = K_2^T > 0$ if $h^T(t,u)[h(t,u) - K_2 u] \leq 0$
- $[K_1, K_2]$ with $K = K_2 - K_1 = K^T > 0$ if

$$[h(t,u) - K_1 u]^T [h(t,u) - K_2 u] \leq 0$$

Example

$$\boldsymbol{h(u)} = \begin{bmatrix} \boldsymbol{h_1(u_1)} \\ \boldsymbol{h_2(u_2)} \end{bmatrix}, \quad \boldsymbol{h_i \in [\alpha_i, \beta_i],\ \ \beta_i > \alpha_i\ \ i = 1, 2}$$

$$\boldsymbol{K_1} = \begin{bmatrix} \boldsymbol{\alpha_1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\alpha_2} \end{bmatrix}, \quad \boldsymbol{K_2} = \begin{bmatrix} \boldsymbol{\beta_1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\beta_2} \end{bmatrix}$$

$$\boldsymbol{h \in [K_1, K_2]}$$

$$\boldsymbol{K = K_2 - K_1} = \begin{bmatrix} \boldsymbol{\beta_1 - \alpha_1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\beta_2 - \alpha_2} \end{bmatrix}$$

Example

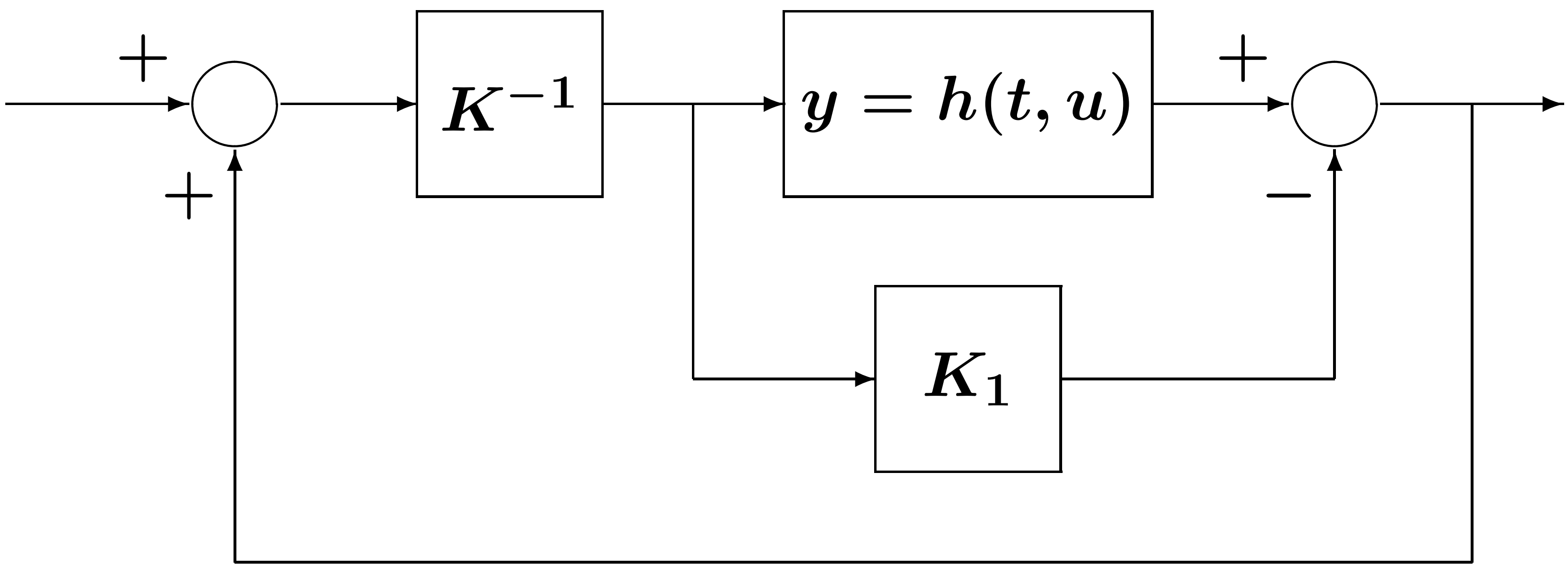
$$\|h(u) - Lu\| \leq \gamma \|u\|$$

$$K_1 = L - \gamma I, \quad K_2 = L + \gamma I$$

$$[h(u) - K_1 u]^T [h(u) - K_2 u] = \\ \|h(u) - Lu\|^2 - \gamma^2 \|u\|^2 \leq 0$$

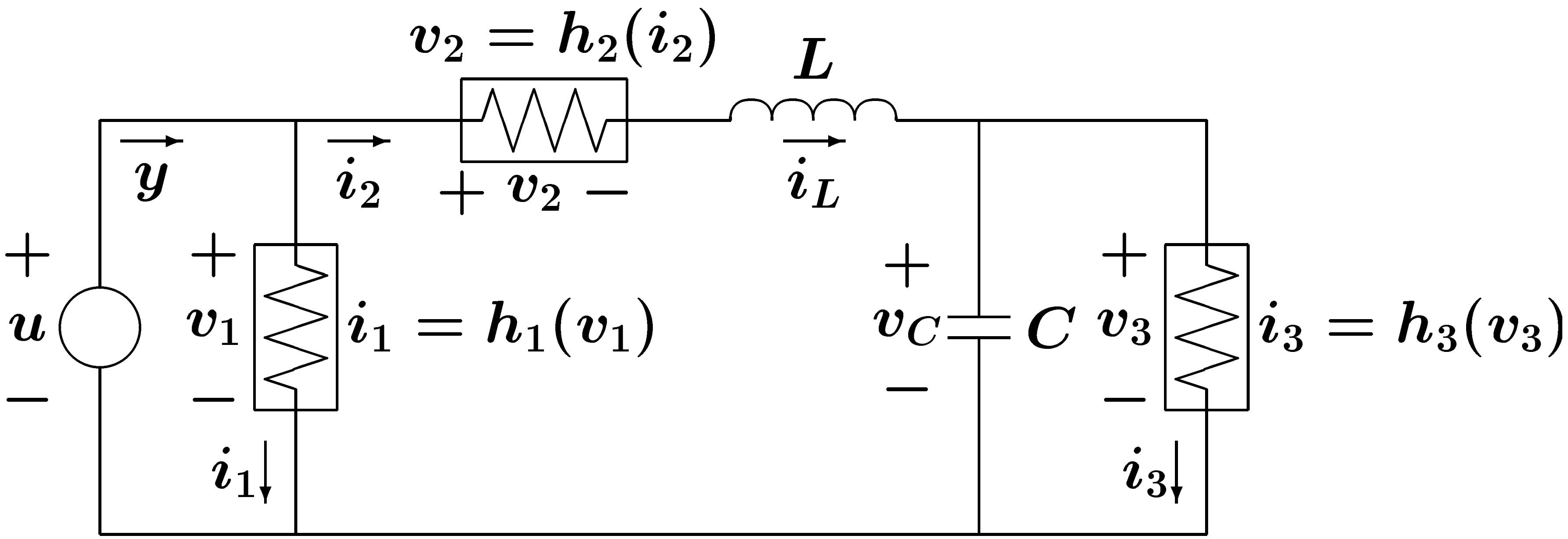
$$K = K_2 - K_1 = 2\gamma I$$

A function in the sector $[K_1, K_2]$ can be transformed into a function in the sector $[0, \infty]$ by input feedforward followed by output feedback



$$[K_1, K_2] \xrightarrow{\text{Feedforward}} [0, K] \xrightarrow{K^{-1}} [0, I] \xrightarrow{\text{Feedback}} [0, \infty]$$

State Models



$$
\begin{aligned}
L\dot{x}_1 &= u - h_2(x_1) - x_2 \\
C\dot{x}_2 &= x_1 - h_3(x_2) \\
y &= x_1 + h_1(u)
\end{aligned}
$$

$$V(x) = \tfrac{1}{2}Lx_1^2 + \tfrac{1}{2}Cx_2^2$$

$$\int_0^t u(s)y(s)\,ds \ge V(x(t)) - V(x(0))$$

$$u(t)y(t) \ge \dot{V}(x(t), u(t))$$

$$\begin{aligned}
\dot{V} &= Lx_1\dot{x}_1 + Cx_2\dot{x}_2 \\
&= x_1[u - h_2(x_1) - x_2] + x_2[x_1 - h_3(x_2)] \\
&= x_1[u - h_2(x_1)] - x_2h_3(x_2) \\
&= [x_1 + h_1(u)]u - uh_1(u) - x_1h_2(x_1) - x_2h_3(x_2) \\
&= uy - uh_1(u) - x_1h_2(x_1) - x_2h_3(x_2)
\end{aligned}$$

$$uy = \dot{V} + uh_1(u) + x_1h_2(x_1) + x_2h_3(x_2)$$

If h_1, h_2, and h_3 are passive, $uy \geq \dot{V}$ and the system is passive

Case 1: If $h_1 = h_2 = h_3 = 0$, then $uy = \dot{V}$; no energy dissipation; the system is lossless

Case 2: If $h_1 \in (0, \infty]$ $(uh_1(u) > 0$ for $u \neq 0)$, then

$$uy \geq \dot{V} + uh_1(u)$$

The energy absorbed over $[0, t]$ will be greater than the increase in the stored energy, unless the input $u(t)$ is identically zero. This is a case of input strict passivity

Case 3: If $h_1 = 0$ and $h_2 \in (0, \infty]$, then

$$y = x_1 \text{ and } uy \geq \dot{V} + yh_2(y)$$

The energy absorbed over $[0, t]$ will be greater than the increase in the stored energy, unless the output y is identically zero. This is a case of output strict passivity

Case 4: If $h_2 \in (0, \infty)$ and $h_3 \in (0, \infty)$, then

$$uy \geq \dot{V} + x_1 h_2(x_1) + x_2 h_3(x_2)$$

$x_1 h_2(x_1) + x_2 h_3(x_2)$ is a positive definite function of x. This is a case of state strict passivity because the energy absorbed over $[0, t]$ will be greater than the increase in the stored energy, unless the state x is identically zero

Definition: The system

$$\dot{x} = f(x,u), \qquad y = h(x,u)$$

is passive if there is a continuously differentiable positive semidefinite function $V(x)$ (the storage function) such that

$$u^T y \geq \dot{V} = \frac{\partial V}{\partial x} f(x,u), \;\; \forall \; (x,u)$$

Moreover, it is said to be

- lossless if $u^T y = \dot{V}$
- input strictly passive if $u^T y \geq \dot{V} + u^T \varphi(u)$ for some function φ such that $u^T \varphi(u) > 0, \; \forall \; u \neq 0$

- output strictly passive if $u^T y \geq \dot{V} + y^T \rho(y)$ for some function ρ such that $y^T \rho(y) > 0, \ \forall \ y \neq 0$

- strictly passive if $u^T y \geq \dot{V} + \psi(x)$ for some positive definite function ψ

Example

$$\dot{x} = u, \qquad y = x$$

$$V(x) = \tfrac{1}{2}x^2 \ \Rightarrow \ uy = \dot{V} \ \Rightarrow \ \text{Lossless}$$

Example

$$\dot{x} = u, \qquad y = x + h(u), \quad h \in [0, \infty]$$

$$V(x) = \frac{1}{2}x^2 \;\Rightarrow\; uy = \dot{V} + uh(u) \;\Rightarrow\; \text{Passive}$$

$$h \in (0, \infty] \;\Rightarrow\; uh(u) > 0 \; \forall \; u \neq 0$$

$$\Rightarrow \; \text{Input strictly passive}$$

Example

$$\dot{x} = -h(x) + u, \qquad y = x, \quad h \in [0, \infty]$$

$$V(x) = \frac{1}{2}x^2 \;\Rightarrow\; uy = \dot{V} + yh(y) \;\Rightarrow\; \text{Passive}$$

$$h \in (0, \infty] \;\Rightarrow\; \text{Output strictly passive}$$

Example

$$\dot{x} = u, \qquad y = h(x), \quad h \in [0, \infty]$$

$$V(x) = \int_0^x h(\sigma)\, d\sigma \;\Rightarrow\; \dot{V} = h(x)\dot{x} = yu \;\Rightarrow\; \text{Lossless}$$

Example

$$a\dot{x} = -x + u, \qquad y = h(x), \quad h \in [0, \infty]$$

$$V(x) = a\int_0^x h(\sigma)\, d\sigma \;\Rightarrow\; \dot{V} = h(x)(-x+u) = yu - xh(x)$$

$$yu = \dot{V} + xh(x) \;\Rightarrow\; \text{Passive}$$

$$h \in (0, \infty] \;\Rightarrow\; \text{Strictly passive}$$