Nonlinear Systems and Control
Lecture # 41

Integral Control

$$
\begin{aligned}
\dot{x} &= f(x, u, w) \\
y &= h(x, w) \\
y_m &= h_m(x, w)
\end{aligned}
$$

$x \in R^n$ state, $u \in R^p$ control input

$y \in R^p$ controlled output, $y_m \in R^m$ measured output

$w \in R^l$ unknown constant parameters and disturbances

Goal:

$$y(t) \to r \ \text{ as } \ t \to \infty$$

$r \in R^p$ constant reference, $\quad v = (r, w)$

$$e(t) = y(t) - r$$

Assumption: e can be measured

Steady-state condition: There is a unique pair $(x_{\rm ss}, u_{\rm ss})$ that satisfies the equations

$$\begin{aligned} 0 &= f(x_{\rm ss}, u_{\rm ss}, w) \\ 0 &= h(x_{\rm ss}, w) - r \end{aligned}$$

Stabilize the system at the equilibrium point $x = x_{ss}$

Can we reduce this to a stabilization problem by shifting the equilibrium point to the origin via the change of variables

$$x_\delta = x - x_{ss}, \quad u_\delta = u - u_{ss} \ ?$$

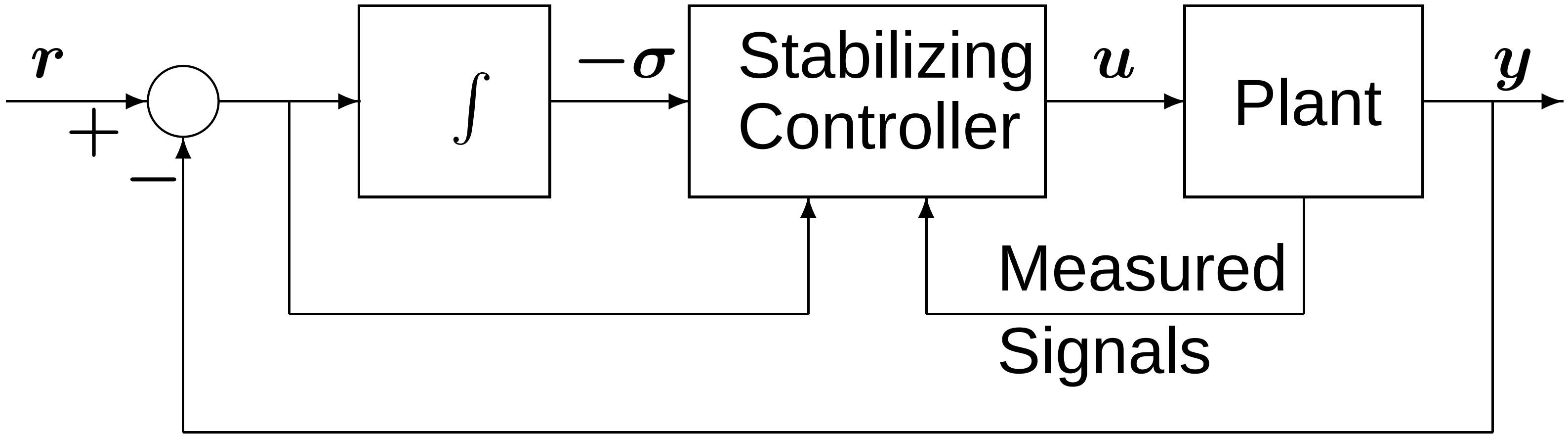
Integral Action:

$$\dot{\sigma} = e$$

Augmented System:

$$\begin{aligned} \dot{x} &= f(x, u, w) \\ \dot{\sigma} &= h(x, w) - r \end{aligned}$$

Task: Stabilize the augmented system at (x_{ss}, σ_{ss}) where σ_{ss} produces u_{ss}



Integral Control via Linearization

State Feedback:

$$u = -K_1 x - K_2 \sigma - K_3 e$$

Closed-loop system:

$$\begin{aligned} \dot{x} &= f(x, -K_1 x - K_2 \sigma - K_3 (h(x, w) - r), w) \\ \dot{\sigma} &= h(x, w) - r \end{aligned}$$

Equilibrium points:

$$\begin{aligned} 0 &= f(\bar{x}, \bar{u}, w) \\ 0 &= h(\bar{x}, w) - r \\ \bar{u} &= -K_1 \bar{x} - K_2 \bar{\sigma} \end{aligned}$$

Unique equilibrium point at $x = x_{ss}, \ \sigma = \sigma_{ss}, \ u = u_{ss}$

Linearization about $(\boldsymbol{x}_{\mathrm{ss}}, \boldsymbol{\sigma}_{\mathrm{ss}})$:

$$\boldsymbol{\xi}_\delta = \begin{bmatrix} \boldsymbol{x} - \boldsymbol{x}_{\mathrm{ss}} \\ \boldsymbol{\sigma} - \boldsymbol{\sigma}_{\mathrm{ss}} \end{bmatrix}$$

$$\dot{\boldsymbol{\xi}}_\delta = (\mathcal{A} - \mathcal{B}\mathcal{K})\boldsymbol{\xi}_\delta$$

$$\mathcal{A} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \; A = \frac{\partial f}{\partial x}(x, u, w)\bigg|_{\mathrm{eq}}, \; C = \frac{\partial h}{\partial x}(x, w)\bigg|_{\mathrm{eq}}$$

$$\mathcal{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \; B = \frac{\partial f}{\partial u}(x, u, w)\bigg|_{\mathrm{eq}}$$

$$\mathcal{K} = \begin{bmatrix} K_1 + K_3 C & K_2 \end{bmatrix}$$

$(\mathcal{A}, \mathcal{B})$ is controllable if and only if (A, B) is controllable and

$$\textbf{rank} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = n + p$$

Task: Design $\mathcal{K}$, independent of v, such that $(\mathcal{A} - \mathcal{B}\mathcal{K})$ is Hurwitz for all v

$(x_{\text{ss}}, \sigma_{\text{ss}})$ is an exponentially stable equilibrium point of the closed-loop system. All solutions starting in its region of attraction approach it as t tends to infinity

$$e(t) \to 0 \text{ as } t \to \infty$$

Pendulum Example:

$$\ddot{\theta} = -a \sin\theta - b\dot{\theta} + cT$$

Regulate θ to δ

$$x_1 = \theta - \delta, \quad x_2 = \dot{\theta}, \quad u = T$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin(x_1 + \delta) - bx_2 + cu \end{aligned}$$

$$x_{ss} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad u_{ss} = \frac{a}{c}\sin\delta$$

$$\dot{\sigma} = x_1$$

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 \\ -a\cos\delta & -b & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0 \\ c \\ 0 \end{bmatrix}$$

$$K_1 = [k_1\ k_2], \quad K_2 = k_3, \quad K_3 = 0$$

$(\mathcal{A} - \mathcal{B}\mathcal{K})$ will be Hurwitz if

$$b + k_2 c > 0, \quad (b + k_2 c)(a\cos\delta + k_1 c) - k_3 c > 0, \quad k_3 c > 0$$

Suppose $\dfrac{a}{c} \le \rho_1, \quad \dfrac{1}{c} \le \rho_2$

$$k_2 > 0, \quad k_3 > 0, \quad k_1 > \rho_1 + \rho_2 \frac{k_3}{k_2}$$

Output Feedback: We only measure e and y_m

$$\begin{aligned} \dot{\sigma} &= e = y - r \\ \dot{z} &= Fz + G_1\sigma + G_2 y_m \\ u &= Lz + M_1\sigma + M_2 y_m + M_3 e \end{aligned}$$

Task: Design F, G_1, G_2, L, M_1, M_2, and M_3, independent of v, such that $\mathcal{A}_c$ is Hurwitz for all v

$$\mathcal{A}_c = \begin{bmatrix} A + BM_2C_m + BM_3C & BM_1 & BL \\ C & 0 & 0 \\ G_2C_m & G_1 & F \end{bmatrix}$$

$$C_m = \frac{\partial h_m}{\partial x}(x, w)\bigg|_{\mathrm{eq}}$$

Integral Control via Sliding Mode Design

$$
\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi, w) \\
\dot{\xi}_1 &= \xi_2 \\
\vdots \quad & \quad \vdots \\
\dot{\xi}_{\rho-1} &= \xi_\rho \\
\dot{\xi}_\rho &= b(\eta, \xi, u, w) + a(\eta, \xi, w)u \\
y &= \xi_1 \\
& \quad a(\eta, \xi, w) \ge a_0 > 0
\end{aligned}
$$

Goal:

$$y(t) \to r \ \text{ as } \ t \to \infty$$

$$\xi_{\text{ss}} = [r, 0, \ldots, 0]^T$$

Steady-state condition: There is a unique pair $(\eta_{\rm ss}, u_{\rm ss})$ that satisfies the equations

$$\begin{aligned} 0 &= f_0(\eta_{\rm ss}, \xi_{\rm ss}, w) \\ 0 &= b(\eta_{\rm ss}, \xi_{\rm ss}, u_{\rm ss}, w) + a(\eta_{\rm ss}, \xi_{\rm ss}, w)u_{\rm ss} \end{aligned}$$

$$\dot{e}_0 = y - r$$

$$z = \eta - \eta_{\rm ss}, \quad e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_\rho \end{bmatrix} = \begin{bmatrix} \xi_1 - r \\ \xi_2 \\ \vdots \\ \xi_\rho \end{bmatrix}$$

$$
\begin{aligned}
\dot{z} &= f_0(\eta, \xi, w) \stackrel{\text{def}}{=} \tilde{f}_0(z, e, w, r) \\
\dot{e}_0 &= e_1 \\
\dot{e}_1 &= e_2 \\
\vdots \quad & \quad \vdots \\
\dot{e}_{\rho-1} &= e_\rho \\
\dot{e}_\rho &= b(\eta, \xi, u, w) + a(\eta, \xi, w)u
\end{aligned}
$$

Partial State Feedback: $\{e_1, \ldots, e_\rho\}$ are measured

$$s = k_0 e_0 + k_1 e_1 + \cdots + k_{\rho-1} e_{\rho-1} + e_\rho$$

k_0 to $k_{\rho-1}$ are chosen such that the polynomial

$$\lambda^\rho + k_{\rho-1}\lambda^{\rho-1} + \cdots + k_1\lambda + k_0 \quad \text{is Hurwitz}$$

$$\dot{s} = k_0 e_1 + \cdots + k_{\rho-1} e_\rho + b(\eta, \xi, u, w) + a(\eta, \xi, w) u$$

$$\dot{s} = \Delta(\eta, \xi, u, w, r) + a(\eta, \xi, w) u$$

$$\left| \frac{\Delta(\eta, \xi, u, w, r)}{a(\eta, \xi, w)} \right| \le \varrho(e) + \kappa_0 |u|, \quad 0 \le \kappa_0 < 1$$

$$u = -\beta(e) \,\mathrm{sat}\left(\frac{s}{\mu}\right)$$

$$\beta(e) \ge \frac{\varrho(e)}{(1 - \kappa_0)} + \beta_0, \quad \beta_0 > 0$$

For $|s| \ge \mu, \quad s\dot{s} \le -a_0 (1 - \kappa_0) \beta_0$

What about the other state variables?

$$
\begin{aligned}
\dot{z} &= \tilde{f}_0(z, e, w, r) \\
\dot{\zeta} &= A\zeta + Bs \qquad \text{(A is Hurwitz)} \\
\dot{s} &= -a(\cdot)\beta(e)\,\mathrm{sat}\left(\frac{s}{\mu}\right) + \Delta(\cdot)
\end{aligned}
$$

$$\zeta = [e_0, \ldots, e_{\rho-1}]^T$$

$$\tilde{\alpha}_1(\|z\|) \le V_1(z, w, r) \le \tilde{\alpha}_2(\|z\|)$$

$$\frac{\partial V_1}{\partial z}\tilde{f}_0(z, e, w, r) \le -\tilde{\alpha}_3(\|z\|), \quad \forall\ \|z\| \ge \tilde{\gamma}(\|e\|)$$

$$V_2(\zeta) = \zeta^T P \zeta, \qquad PA + A^T P = -I$$

$$\Omega = \{|s| \le c\} \cap \{V_2 \le c^2 \rho_1\} \cap \{V_1 \le c_0\}$$

$$\Omega_\mu = \{|s| \le \mu\} \cap \{V_2 \le \mu^2 \rho_1\} \cap \{V_1 \le \tilde{\alpha}_2(\tilde{\gamma}(\mu \rho_2))\}$$

All trajectories starting in Ω enter Ω_μ in finite time and stay in thereafter

Inside Ω_μ there is a unique equilibrium point at

$$(z = 0,\ e = 0,\ e_0 = \bar{e}_0), \quad \bar{s} = k_0 \bar{e}_0, \quad u_{ss} = -\beta(0)\ \frac{\bar{s}}{\mu}$$

Under additional conditions (the origin of $\dot{z} = \tilde{f}_0(z, 0, w, r)$ is exponentially stable), local analysis inside Ω_μ shows that for sufficiently small μ all trajectories converge to the equilibrium point as time tends to infinity

Output Feedback: Only e_1 is measured

High-gain Observer:

$$
\begin{aligned}
\dot{e}_0 &= e_1 \\
u &= -\beta \,\mathrm{sat}\left(\frac{k_0 e_0 + k_1 e_1 + k_2 \hat{e}_2 + \cdots + \hat{e}_\rho}{\mu}\right) \\
\dot{\hat{e}}_i &= \hat{e}_{i+1} + \left(\frac{\alpha_i}{\varepsilon^i}\right)(e_1 - \hat{e}_1), \quad 1 \le i \le \rho - 1 \\
\dot{\hat{e}}_\rho &= \left(\frac{\alpha_\rho}{\varepsilon^\rho}\right)(e_1 - \hat{e}_1)
\end{aligned}
$$

$$\beta = \beta(e_1, \hat{e}_2, \ldots, \hat{e}_\rho)$$