Nonlinear Systems and Control
Lecture # 38

Observers

Exact Observers

Observer with Linear Error Dynamics

Observer Form:

$$\dot{x} = Ax + \gamma(y, u), \qquad y = Cx$$

where (A, C) is observable, $x \in R^n$, $u \in R^m$, $y \in R^p$
From Lecture # 24: An n-dimensional SO system

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

is transformable into the observer form if and only if

$$\phi = \left[\begin{array}{cccc} h, & L_f h, & \cdots & L_f^{n-1} h \end{array}\right]^T, \;\; \text{rank}\left[\frac{\partial \phi}{\partial x}(x)\right] = n$$

$$b = \left[\begin{array}{cccc} 0, & \cdots & 0, & 1 \end{array}\right]^T, \;\; \frac{\partial \phi}{\partial x}\tau = b$$

$$[ad_f^i\tau, ad_f^j\tau] = 0, \quad 0 \le i,j \le n-1$$

$$[g, ad_f^j\tau] = 0, \quad 0 \le j \le n-2$$

Change of variables:

$$\tau_i = (-1)^{i-1} ad_f^{i-1}\tau, \quad 1 \le i \le n$$

$$\frac{\partial T}{\partial x}\begin{bmatrix} \tau_1, & \tau_2, & \cdots & \tau_n \end{bmatrix} = I$$

$$z = T(x)$$

$$\dot{x} = Ax + \gamma(y, u), \quad y = Cx$$

$$\dot{\hat{x}} = A\hat{x} + \gamma(y, u) + H(y - C\hat{x})$$

$$\tilde{x} = x - \hat{x}$$

$$\dot{\tilde{x}} = (A - HC)\tilde{x}$$

Design H such that $(A - HC)$ is Hurwitz

What about feedback control?

Let $u = \psi(x)$ be a globally stabilizing state feedback control

$$u = \psi(\hat{x})$$

$$\dot{\hat{x}} = A\hat{x} + \gamma(y, u) + H(y - C\hat{x})$$

How would you analyze the closed-loop system?

$$
\begin{aligned}
\dot{x} &= Ax + \gamma(Cx, \psi(x - \tilde{x})) \\
\dot{\tilde{x}} &= (A - HC)\tilde{x}
\end{aligned}
$$

We know that

- the origin of $\dot{x} = Ax + \gamma(Cx, \psi(x))$ is globally asymptotically stable
- the origin of $\dot{\tilde{x}} = (A - HC)\tilde{x}$ is globally exponentially stable

What additional assumptions do we need to show that the origin of the closed-loop system is globally asymptotically stable?

Circle Criterion Design

$$\dot{x} = Ax + \gamma(y,u) - L\beta(Mx), \quad y = Cx$$

where (A, C) is observable, $x \in R^n$, $u \in R^m$, $y \in R^p$, $Mx \in R^\ell$, $\beta(\eta) = [\ \beta_1(\eta_1), \ \ldots\ ,\ \beta_\ell(\eta_\ell)\ ]^T$

$$\dot{\hat{x}} = A\hat{x} + \gamma(y,u) - L\beta(M\hat{x} - N(y - C\hat{x})) + H(y - C\hat{x})$$

$$\tilde{x} = x - \hat{x}$$

$$\dot{\tilde{x}} = (A - HC)\tilde{x} - L[\beta(Mx) - \beta(M\hat{x} - N(y - C\hat{x}))]$$

$$\dot{\tilde{x}} = (A - HC)\tilde{x} - L[\beta(Mx) - \beta(Mx - (M + NC)\tilde{x})]$$

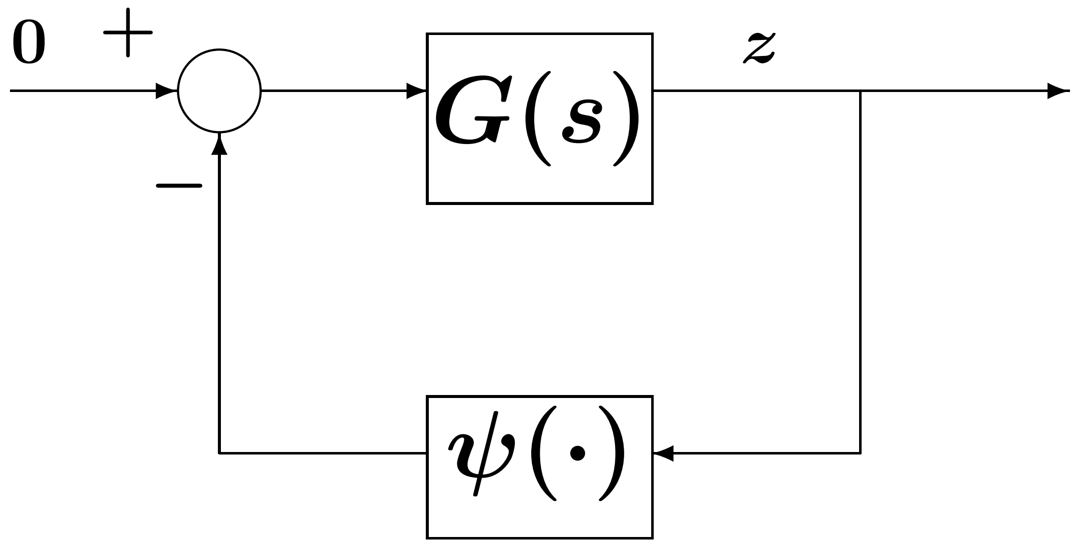
Define

$$z = (M + NC)\tilde{x}$$

$$\psi(t,z) = \beta(Mx(t)) - \beta(Mx(t) - z)$$

$$\begin{aligned} \dot{\tilde{x}} &= (A - HC)\tilde{x} - L\psi(t,z) \\ z &= (M + NC)\tilde{x} \end{aligned}$$

$$G(s) \stackrel{\text{def}}{=} (M + NC)[sI - (A - HC)]^{-1}L$$



$$\psi(t,z) = \left[\ \psi_1(t,z_1), \ldots, \psi_\ell(t,z_\ell)\ \right]^T$$

Main Assumption: $\beta_i(\cdot)$ is a nondecreasing function

$$(a-b)[\beta_i(a)-\beta_i(b)] \geq 0, \quad \forall\, a,b \in R$$

If $\beta_i(\eta_i)$ is continuously differentiable

$$\frac{d\beta_i}{d\eta_i} \geq 0, \;\; \forall \eta_i \in R$$

$$z_i\psi_i(t,z_i) = z_i[\beta_i((Mx)_i) - \beta_i((Mx)_i - z_i)] \geq 0$$

$$z^T\psi(t,z) \geq 0$$

By the circle criterion (Theorem 7.1) the origin of

$$\begin{aligned} \dot{\tilde{x}} &= (A - HC)\tilde{x} - L\psi(t, z) \\ z &= (M + NC)\tilde{x} \end{aligned}$$

is globally exponentially stable if

$$G(s) \stackrel{\text{def}}{=} (M + NC)[sI - (A - HC)]^{-1}L$$

is strictly positive real

Design Problem: Design H and N such that $G(s)$ is strictly positive real
Feasibility can be investigated using LMI (Arcak & Kokotovic, Automatica, 2001)

Example:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1^3 - x_2^3 + u, \qquad y = x_1$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \gamma = \begin{bmatrix} 0 \\ -y^3 + u \end{bmatrix},$$

$$L = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad \beta(\eta) = \eta^3, \quad \frac{d\beta}{d\eta} = 3\eta^2 \geq 0$$

$$h = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}, \qquad N$$

$$G(s) = (M + NC)[sI - (A - HC)]^{-1}L = \frac{s + N + h_1}{s^2 + h_1 s + h_2}$$

From Exercise 6.7, $G(s)$ is SPR if and only if

$$h_1 > 0, \quad h_2 > 0, \quad 0 < N + h_1 < h_1$$

$$h_1 = 2, \quad h_2 = 1, \quad N = -\tfrac{1}{2}$$

$$G(s) = \frac{s + \frac{3}{2}}{(s+1)^2}$$

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + 2(y - \hat{x}_1) \\ \dot{\hat{x}}_2 &= -y^3 + u - \left(\hat{x}_2 + \tfrac{1}{2}(y - \hat{x}_1)\right)^3 + (y - \hat{x}_1) \end{aligned}$$

What about feedback control?

Let $u = \phi(x)$ be a globally stabilizing state feedback control

Closed-loop system under output feedback:

$$
\begin{aligned}
\dot{x} &= Ax + \gamma(y, \phi(x - \tilde{x})) - L\beta(Mx) \\
\dot{\tilde{x}} &= (A - HC)\tilde{x} - L\psi(t, z) \\
z &= (M + NC)\tilde{x}
\end{aligned}
$$

How would you analyze the closed-loop system?

$\psi(t, z)$ depends on $x(t)$. How would you show that ψ is well defined?

What about the effect of uncertainty?