Nonlinear Systems and Control
Lecture # 33

Robust Stabilization

Sliding Mode Control

Regular Form:

$$\begin{aligned} \dot{\eta} &= f_a(\eta, \xi) \\ \dot{\xi} &= f_b(\eta, \xi) + g(\eta, \xi)u + \delta(t, \eta, \xi, u) \end{aligned}$$

$$\eta \in R^{n-1}, \ \xi \in R, \ u \in R$$

$$f_a(0,0) = 0, \ f_b(0,0) = 0, \ g(\eta, \xi) \geq g_0 > 0$$

Sliding Manifold:

$$s = \xi - \phi(\eta) = 0, \qquad \phi(0) = 0$$

$$s(t) \equiv 0 \ \Rightarrow \ \dot{\eta} = f_a(\eta, \phi(\eta))$$

Design ϕ s.t. the origin of $\dot{\eta} = f_a(\eta, \phi(\eta))$ is asymp. stable

$$\dot{s} = f_b(\eta,\xi) - \frac{\partial\phi}{\partial\eta} f_a(\eta,\xi) + g(\eta,\xi)u + \delta(t,\eta,\xi,u)$$

$$u = -\frac{1}{\hat{g}}\left(\hat{f}_b - \frac{\partial\phi}{\partial\eta}\hat{f}_a\right) + v \quad \textit{or} \quad u = v$$

$$u = -L\left(\hat{f}_b - \frac{\partial\phi}{\partial\eta}\hat{f}_a\right) + v, \quad L = \frac{1}{\hat{g}} \quad \textit{or} \quad L = 0$$

$$\dot{s} = g(\eta,\xi)v + \Delta(t,\eta,\xi,v)$$

$$\Delta = f_b - \frac{\partial\phi}{\partial\eta} f_a + \delta - gL\left(\hat{f}_b - \frac{\partial\phi}{\partial\eta}\hat{f}_a\right)$$

$$\left|\frac{\Delta(t,\eta,\xi,v)}{g(\eta,\xi)}\right| \le \varrho(\eta,\xi) + \kappa_0|v|$$

$$\left|\frac{\Delta(t,\eta,\xi,v)}{g(\eta,\xi)}\right| \le \varrho(\eta,\xi) + \kappa_0|v|$$

$$\varrho(\eta,\xi) \ge 0, \quad 0 \le \kappa_0 < 1 \quad (Known)$$

$$s\dot{s} = sgv + s\Delta \le sgv + |s|\,|\Delta|$$

$$s\dot{s} \le g[sv + |s|(\varrho + \kappa_0|v|)]$$

$$v = -\beta(\eta,\xi)\,\mathrm{sgn}(s)$$

$$\beta(\eta,\xi) \ge \frac{\varrho(\eta,\xi)}{1-\kappa_0} + \beta_0, \quad \beta_0 > 0$$

$$s\dot{s} \le g[-\beta|s| + \varrho|s| + \kappa_0\beta|s|] = g[-\beta(1-\kappa_0)|s| + \varrho|s|\,]$$

$$s\dot{s} \le g[-\varrho|s| - (1-\kappa_0)\beta_0|s| + \varrho|s|\,]$$

$$s\dot{s} \le -g(\eta,\xi)(1-\kappa_0)\beta_0|s| \le -g_0\beta_0(1-\kappa_0)|s|$$

$$v = -\beta(x)\ \mathrm{sat}\left(\frac{s}{\varepsilon}\right), \qquad \varepsilon > 0$$

$$s\dot{s} \le -g_0\beta_0(1-\kappa_0)|s|, \quad \text{for } |s| \ge \varepsilon$$

The trajectory reaches the boundary layer $\{|s| \le \varepsilon\}$ in finite time and remains inside thereafter

Study the behavior of η

$$\dot{\eta} = f_a(\eta, \phi(\eta) + s)$$

What do we know about this system and what do we need?

$$\alpha_1(\|\eta\|) \le V(\eta) \le \alpha_2(\|\eta\|)$$

$$\frac{\partial V}{\partial \eta} f_a(\eta, \phi(\eta) + s) \le -\alpha_3(\|\eta\|), \quad \forall\ \|\eta\| \ge \gamma(|s|)$$

$$|s| \le c \ \Rightarrow\ \dot{V} \le -\alpha_3(\|\eta\|), \ \text{ for } \|\eta\| \ge \gamma(c)$$

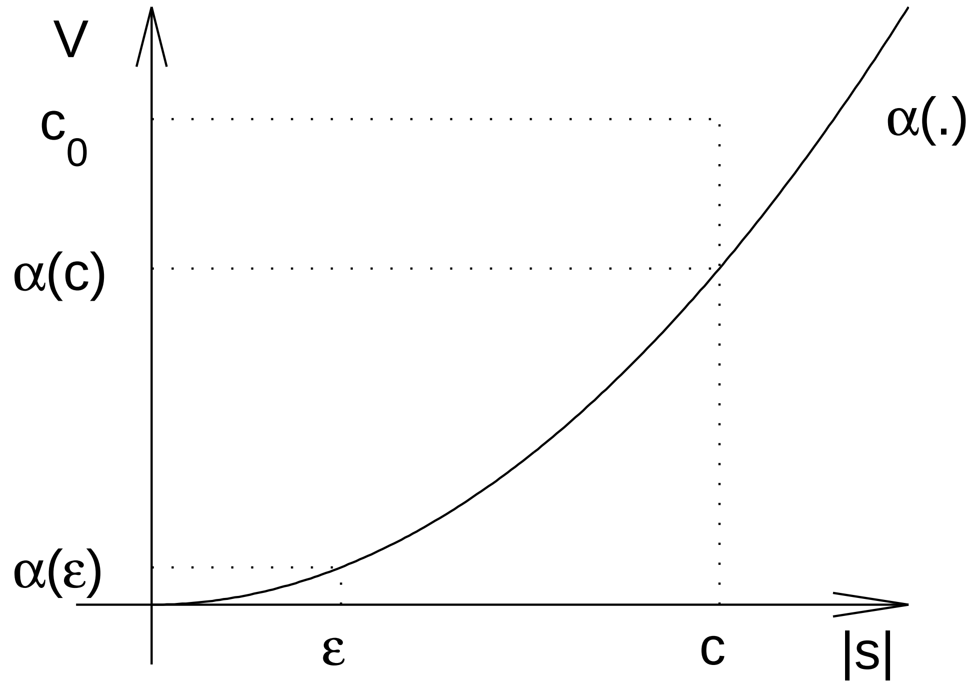
$$\alpha(r) = \alpha_2(\gamma(r))$$

$$V(\eta) \ge \alpha(c) \Leftrightarrow V(\eta) \ge \alpha_2(\gamma(c)) \ \Rightarrow \alpha_2(\|\eta\|) \ge \alpha_2(\gamma(c))$$

$$\Rightarrow \|\eta\| \ge \gamma(c) \ \Rightarrow\ \dot{V} \le -\alpha_3(\|\eta\|) \le -\alpha_3(\gamma(c))$$

The set $\{V(\eta) \le c_0\}$ with $c_0 \ge \alpha(c)$ is positively invariant

$$\Omega = \{V(\eta) \le c_0\} \times \{|s| \le c\}, \ \text{with } c_0 \ge \alpha(c)$$



$$\boldsymbol{\Omega = \{V(\eta) \leq c_0\} \times \{|s| \leq c\},\ \text{with}\ c_0 \geq \alpha(c)}$$

is positively invariant and all trajectories starting in $\boldsymbol{\Omega}$ reach $\boldsymbol{\Omega_\varepsilon = \{V(\eta) \leq \alpha(\varepsilon)\} \times \{|s| \leq \varepsilon\}}$ in finite time

Theorem 14.1: Suppose all the assumptions hold over Ω. Then, for all $(\eta(0), \xi(0)) \in \Omega$, the trajectory $(\eta(t), \xi(t))$ is bounded for all $t \geq 0$ and reaches the positively invariant set Ω_ε in finite time. If the assumptions hold globally and $V(\eta)$ is radially unbounded, the foregoing conclusion holds for any initial state

Theorem 14.2: Suppose all the assumptions hold over Ω

- $\varrho(0) = 0$, $\kappa_0 = 0$
- The origin of $\dot{\eta} = f_a(\eta, \phi(\eta))$ is exponentially stale

Then there exits $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, the origin of the closed-loop system is exponentially stable and Ω is a subset of its region of attraction. If the assumptions hold globally, the origin will be globally uniformly asymptotically stable

Example

$$\dot{x}_1 = x_2 + \theta_1 x_1 \sin x_2, \qquad \dot{x}_2 = \theta_2 x_2^2 + x_1 + u$$

$$|\theta_1| \le a, \quad |\theta_2| \le b$$

$$x_2 = -kx_1 \;\Rightarrow\; \dot{x}_1 = -kx_1 + \theta_1 x_1 \sin x_2$$

$$V_1 = \tfrac{1}{2}x_1^2 \;\Rightarrow\; x_1\dot{x}_1 \le -kx_1^2 + ax_1^2$$

$$s = x_2 + kx_1, \quad k > a$$

$$\dot{s} = \theta_2 x_2^2 + x_1 + u + k(x_2 + \theta_1 x_1 \sin x_2)$$

$$u = -x_1 - kx_2 + v \;\Rightarrow\; \dot{s} = v + \Delta(x)$$

$$\Delta(x) = \theta_2 x_2^2 + k\theta_1 x_1 \sin x_2$$

$$\Delta(x) = \theta_2 x_2^2 + k\theta_1 x_1 \sin x_2$$

$$|\Delta(x)| \le ak|x_1| + bx_2^2$$

$$\beta(x) = ak|x_1| + bx_2^2 + \beta_0, \quad \beta_0 > 0$$

$$u = -x_1 - kx_2 - \beta(x)\,\mathrm{sgn}(s)$$

Will

$$u = -x_1 - kx_2 - \beta(x)\,\mathrm{sat}\left(\frac{s}{\varepsilon}\right)$$

stabilize the origin?

Example: Normal Form

$$
\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{\xi}_i &= \xi_{i+1}, \quad 1 \le i \le \rho - 1 \\
\dot{\xi}_\rho &= L_f^\rho h(x) + L_g L_f^{\rho-1} h(x)\, u \\
y &= \xi_1
\end{aligned}
$$

View ξ_ρ as input to the system

$$
\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi_1, \cdots, \xi_{\rho-1}, \xi_\rho) \\
\dot{\xi}_i &= \xi_{i+1}, \quad 1 \le i \le \rho - 2 \\
\dot{\xi}_{\rho-1} &= \xi_\rho
\end{aligned}
$$

Design $\xi_\rho = \phi(\eta, \xi_1, \cdots, \xi_{\rho-1})$ to stabilize the origin

$$s = \xi_\rho - \phi(\eta, \xi_1, \cdots, \xi_{\rho-1})$$

Minimum Phase Systems: The origin of $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable

$$s = \xi_\rho + k_1\xi_1 + \cdots + k_{\rho-1}\xi_{\rho-1}$$

$$\dot{\eta} = f_0(\eta, \xi_1, \cdots, \xi_{\rho-1}, -k_1\xi_1 - \cdots - k_{\rho-1}\xi_{\rho-1})$$

$$\begin{bmatrix} \dot{\xi}_1 \\ \vdots \\ \\ \dot{\xi}_{\rho-1} \end{bmatrix} = \begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ -k_1 & & & -k_{\rho-1} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \\ \xi_{\rho-1} \end{bmatrix}$$

Multi-Input Systems

$$\begin{aligned}\dot{\eta} &= f_a(\eta, \xi)\\ \dot{\xi} &= f_b(\eta, \xi) + G(\eta, \xi)E(\eta, \xi)u + \delta(t, \eta, \xi, u)\end{aligned}$$

$$\eta \in R^{n-p},\ \xi \in R^p,\ u \in R^p$$

$$f_a(0,0) = 0,\ f_b(0,0) = 0,\ \det(G) \neq 0,\ \det(E) \neq 0$$

$$G = \mathrm{diag}[g_1,\ g_2,\ \cdots, g_m],\ \ g_i(\eta, \xi) \geq g_0 > 0$$

Design ϕ s.t. the origin of $\dot{\eta} = f_a(\eta, \phi(\eta))$ is asymp. stable

$$s = \xi - \phi(\eta)$$

$$\dot{s} = f_b(\eta, \xi) - \frac{\partial \phi}{\partial \eta} f_a(\eta, \xi) + G(\eta, \xi)E(\eta, \xi)u + \delta(t, \eta, \xi, u)$$

$$\dot{s} = f_b(\eta,\xi) - \frac{\partial \phi}{\partial \eta} f_a(\eta,\xi) + G(\eta,\xi)E(\eta,\xi)u + \delta(t,\eta,\xi,u)$$

$$u = E^{-1}\left\{-L\left[\hat{f}_b - \frac{\partial \phi}{\partial \eta}\hat{f}_a\right] + v\right\}, \quad L = \hat{G}^{-1} \text{ or } L = 0$$

$$\dot{s}_i = g_i(\eta,\xi)v_i + \Delta_i(t,\eta,\xi,v), \quad 1 \le i \le p$$

$$\left|\frac{\Delta_i(t,\eta,\xi,v)}{g_i(\eta,\xi)}\right| \le \varrho(\eta,\xi) + \kappa_0 \max_{1\le i\le p} |v_i|, \quad \forall\, 1 \le i \le p$$

$$\varrho(\eta,\xi) \ge 0, \quad 0 \le \kappa_0 < 1 \quad (Known)$$

$$\beta(x) \ge \frac{\varrho(x)}{1-\kappa_0} + \beta_0, \quad \beta_0 > 0$$

$$s_i\dot{s}_i = s_i g_i v_i + s_i \Delta_i \le g_i\{s_i v_i + |s_i|[\varrho + \kappa_0 \max_{1\le i\le p} |v_i|]\}$$

$$v_i = -\beta\,\mathrm{sgn}(s_i), \quad 1 \le i \le p$$

$$\begin{aligned} s_i\dot{s}_i &\le g_i[-\beta + \varrho + \kappa_0\beta]|s_i| \\ &= g_i[-(1-\kappa_0)\beta + \varrho]|s_i| \\ &\le g_i[-\varrho - (1-\kappa_0)\beta_0 + \varrho]|s_i| \\ &\le -g_0\beta_0(1-\kappa_0)|s_i| \end{aligned}$$

Now use

$$v_i = -\beta\,\mathrm{sat}\left(\frac{s_i}{\varepsilon}\right), \quad 1 \le i \le p$$

Read Theorem 14.1 and 14.2 in the textbook