# Nonlinear Systems and Control
# Lecture # 23

# Controller Form

Definition: A nonlinear system is in the controller form if

$$\dot{x} = Ax + B\gamma(x)[u - \alpha(x)]$$

where $(A, B)$ is controllable and $\gamma(x)$ is a nonsingular

$$u = \alpha(x) + \gamma^{-1}(x)v \quad \Rightarrow \quad \dot{x} = Ax + Bv$$

The $n$-dimensional single-input (SI) system

$$\dot{x} = f(x) + g(x)u$$

can be transformed into the controller form if $\exists\ h(x)$ s.t.

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

has relative degree $n$. Why?

Transform the system into the normal form

$$\dot{z} = A_c z + B_c \gamma(z)[u - \alpha(z)], \qquad y = C_c z$$

On the other hand, if there is a change of variables $\zeta = S(x)$ that transforms the SI system

$$\dot{x} = f(x) + g(x)u$$

into the controller form

$$\dot{\zeta} = A\zeta + B\gamma(\zeta)[u - \alpha(\zeta)]$$

then there is a function $h(x)$ such that the system

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

has relative degree $n$. Why?

For any controllable pair $(A, B)$, we can find a nonsingular matrix $M$ that transforms $(A, B)$ into a controllable canonical form:

$$MAM^{-1} = A_c + B_c\lambda^T, \qquad MB = B_c$$

$$z = M\zeta = MS(x) \stackrel{\text{def}}{=} T(x)$$

$$\dot{z} = A_c z + B_c\gamma(\cdot)[u - \alpha(\cdot)]$$

$$h(x) = T_1(x)$$

In summary, the $n$-dimensional SI system

$$\dot{x} = f(x) + g(x)u$$

is transformable into the controller form if and only if $\exists\ h(x)$ such that

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

has relative degree $n$
Search for a smooth function $h(x)$ such that

$$L_g L_f^{i-1} h(x) = 0,\ \ i = 1, 2, \ldots, n-1, \ \text{ and } \ L_g L_f^{n-1} h(x) \neq 0$$

$$T(x) = \left[ \begin{array}{cccc} h(x), & L_f h(x), & \cdots & L_f^{n-1} h(x) \end{array} \right]$$

**The Lie Bracket:** For two vector fields $f$ and $g$, the *Lie bracket* $[f,g]$ is a third vector field defined by

$$[f,g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x)$$

Notation:

$$ad_f^0 g(x) = g(x), \qquad ad_f g(x) = [f,g](x)$$

$$ad_f^k g(x) = [f, ad_f^{k-1} g](x), \;\; k \geq 1$$

Properties:

- $[f,g] = -[g,f]$
- For constant vector fields $f$ and $g$, $[f,g] = 0$

# Example

$$\boldsymbol{f} = \begin{bmatrix} \boldsymbol{x_2} \\ -\sin \boldsymbol{x_1} - \boldsymbol{x_2} \end{bmatrix}, \quad \boldsymbol{g} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{x_1} \end{bmatrix}$$

$$[\boldsymbol{f}, \boldsymbol{g}] = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x_2} \\ -\sin \boldsymbol{x_1} - \boldsymbol{x_2} \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\cos \boldsymbol{x_1} & -\mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \boldsymbol{x_1} \end{bmatrix}$$

$$\boldsymbol{ad_f g} = [\boldsymbol{f}, \boldsymbol{g}] = \begin{bmatrix} -\boldsymbol{x_1} \\ \boldsymbol{x_1} + \boldsymbol{x_2} \end{bmatrix}$$

$$f = \begin{bmatrix} x_2 \\ -\sin x_1 - x_2 \end{bmatrix}, \quad ad_f g = \begin{bmatrix} -x_1 \\ x_1 + x_2 \end{bmatrix}$$

$$\begin{aligned} ad_f^2 g = [f, ad_f g] = & \\ & \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ -\sin x_1 - x_2 \end{bmatrix} \\ & - \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -1 \end{bmatrix} \begin{bmatrix} -x_1 \\ x_1 + x_2 \end{bmatrix} \\ = & \begin{bmatrix} -x_1 - 2x_2 \\ x_1 + x_2 - \sin x_1 - x_1 \cos x_1 \end{bmatrix} \end{aligned}$$

**Distribution:** For vector fields $f_1, f_2, \ldots, f_k$ on $D \subset R^n$, let

$$\Delta(x) = \mathrm{span}\{f_1(x), f_2(x), \ldots, f_k(x)\}$$

The collection of all vector spaces $\Delta(x)$ for $x \in D$ is called a *distribution* and referred to by

$$\Delta = \mathrm{span}\{f_1, f_2, \ldots, f_k\}$$

If $\dim(\Delta(x)) = k$ for all $x \in D$, we say that $\Delta$ is a nonsingular distribution on $D$, generated by $f_1, \ldots, f_k$
A distribution $\Delta$ is *involutive* if

$$g_1 \in \Delta \ \text{ and } \ g_2 \in \Delta \ \Rightarrow \ [g_1, g_2] \in \Delta$$

Lemma: If $\Delta$ is a nonsingular distribution, generated by $f_1, \ldots, f_k$, then it is involutive if and only if

$$[f_i, f_j] \in \Delta, \quad \forall\, 1 \leq i, j \leq k$$

Example: $D = R^3$; $\Delta = \text{span}\{f_1, f_2\}$

$$f_1 = \begin{bmatrix} 2x_2 \\ 1 \\ 0 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 1 \\ 0 \\ x_2 \end{bmatrix}, \quad \dim(\Delta(x)) = 2, \quad \forall\, x \in D$$

$$[f_1, f_2] = \frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathrm{rank}\,[f_1(x), f_2(x), [f_1, f_2](x)] =$$

$$\mathrm{rank}\begin{bmatrix} 2x_2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & x_2 & 1 \end{bmatrix} = 3, \quad \forall\, x \in D$$

$\Delta$ is not involutive

Example: $D = \{x \in R^3 \mid x_1^2 + x_3^2 \neq 0\}$; $\Delta = \text{span}\{f_1, f_2\}$

$$f_1 = \begin{bmatrix} 2x_3 \\ -1 \\ 0 \end{bmatrix}, \; f_2 = \begin{bmatrix} -x_1 \\ -2x_2 \\ x_3 \end{bmatrix}, \; \dim(\Delta(x)) = 2, \; \forall\, x \in D$$

$$[f_1, f_2] = \frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2 = \begin{bmatrix} -4x_3 \\ 2 \\ 0 \end{bmatrix}$$

$$\text{rank} \begin{bmatrix} 2x_3 & -x_1 & -4x_3 \\ -1 & -2x_2 & 2 \\ 0 & x_3 & 0 \end{bmatrix} = 2, \;\; \forall\, x \in D$$

$\Delta$ is involutive

Theorem: The $n$-dimensional SI system

$$\dot{x} = f(x) + g(x)u$$

is transformable into the controller form if and only if there is a domain $D_0$ such that

$$\text{rank}[g(x), ad_f g(x), \ldots, ad_f^{n-1} g(x)] = n, \quad \forall\, x \in D_0$$

and

$$\text{span}\ \{g, ad_f g, \ldots, ad_f^{n-2} g\}\ \text{ is involutive in } D_0$$

# Example

$$\dot{x} = \begin{bmatrix} a \sin x_2 \\ -x_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$ad_f g = [f, g] = -\frac{\partial f}{\partial x} g = \begin{bmatrix} -a \cos x_2 \\ 0 \end{bmatrix}$$

$$[g(x), ad_f g(x)] = \begin{bmatrix} 0 & -a \cos x_2 \\ 1 & 0 \end{bmatrix}$$

$$\text{rank}[g(x), ad_f g(x)] = 2, \ \forall\, x \text{ such that } \cos x_2 \neq 0$$

$$\text{span}\{g\} \text{ is involutive}$$

Find $h$ such that $L_g h(x) = 0$, and $L_g L_f h(x) \neq 0$

$$\frac{\partial h}{\partial x} g = \frac{\partial h}{\partial x_2} = 0 \quad \Rightarrow \quad h \text{ is independent of } x_2$$

$$L_f h(x) = \frac{\partial h}{\partial x_1} a \sin x_2$$

$$L_g L_f h(x) = \frac{\partial (L_f h)}{\partial x} g = \frac{\partial (L_f h)}{\partial x_2} = \frac{\partial h}{\partial x_1} a \cos x_2$$

$$L_g L_f h(x) \neq 0 \text{ in } D_0 = \{x \in R^2 | \cos x_2 \neq 0\} \text{ if } \frac{\partial h}{\partial x_1} \neq 0$$

$$\text{Take } h(x) = x_1 \quad \Rightarrow \quad T(x) = \begin{bmatrix} h \\ L_f h \end{bmatrix} = \begin{bmatrix} x_1 \\ a \sin x_2 \end{bmatrix}$$

## Example (Field-Controlled DC Motor)

$$\dot{x} = \begin{bmatrix} -ax_1 \\ -bx_2 + k - cx_1x_3 \\ \theta x_1 x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$ad_f g = \begin{bmatrix} a \\ cx_3 \\ -\theta x_2 \end{bmatrix}; \quad ad_f^2 g = \begin{bmatrix} a^2 \\ (a+b)cx_3 \\ (b-a)\theta x_2 - \theta k \end{bmatrix}$$

$$[g(x), ad_f g(x), ad_f^2 g(x)] = \begin{bmatrix} 1 & a & a^2 \\ 0 & cx_3 & (a+b)cx_3 \\ 0 & -\theta x_2 & (b-a)\theta x_2 - \theta k \end{bmatrix}$$

$$\det[\cdot] = c\theta(-k + 2bx_2)x_3$$

rank $[\cdot] = 3$ for $x_2 \neq k/2b$ and $x_3 \neq 0$

$\text{span}\{g, ad_f g\}$ is involutive if $[g, ad_f g] \in \text{span}\{g, ad_f g\}$

$$[g, ad_f g] = \frac{\partial(ad_f g)}{\partial x} g = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -\theta & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad \text{span}\{g, ad_f g\}$ is involutive

$$D_0 = \{x \in R^3 \mid x_2 > \frac{k}{2b} \text{ and } x_3 > 0\}$$

Find $h$ such that $L_g h(x) = L_g L_f h(x) = 0; \;\; L_g L_f^2 h(x) \neq 0$

$$x^* = [0, k/b, \omega_0]^T, \quad h(x^*) = 0$$

$$\frac{\partial h}{\partial x} g = \frac{\partial h}{\partial x_1} = 0 \;\; \Rightarrow \;\; h \text{ is independent of } x_1$$

$$L_f h(x) = \frac{\partial h}{\partial x_2}[-bx_2 + k - cx_1 x_3] + \frac{\partial h}{\partial x_3}\theta x_1 x_2$$

$$[\partial(L_f h)/\partial x] g = 0 \;\; \Rightarrow \;\; c x_3 \frac{\partial h}{\partial x_2} = \theta x_2 \frac{\partial h}{\partial x_3}$$

$$h = c_1[\theta x_2^2 + c x_3^2] + c_2, \;\; L_g L_f^2 h(x) = -2c_1 c\theta(k - 2bx_2)x_3$$

$$h(x^*) = c_1[\theta(k/b)^2 + c\omega_0^2] + c_2$$

$$c_1 = 1, \;\; c_2 = -\theta(k/b)^2 - c\omega_0^2$$