Nonlinear high-gain observers

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Slides tratte da: *"Nonlinear observers and separation principle"*, Handout III, A. Isidori
PhD school of Bertinoro, SIDRA 2008,.

The linear case: reminiscences...

When the system $P(A,B,C,D)$ is linear and time-invariant (LTI, in continuous or discrete time) a Luenberger-type observer can be designed to a obtain global exponential converging estimate $\hat{x}$ of the plant state x.
The problem of estimating the state at time t is equivalent to estimate the initial condition $x(0)$ ($t_0 = 0$) given that $x(t) = e^{At}x(0) + \int_0^t e^{At-\tau}Bu(\tau)d\tau$.
The classical Luenberger observer can be designed implementing a LTI which consists into a copy of the plant P and a *linear* correction term as

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}), \\ \hat{y} = C\hat{x} + Du. \end{cases} \tag{1}$$

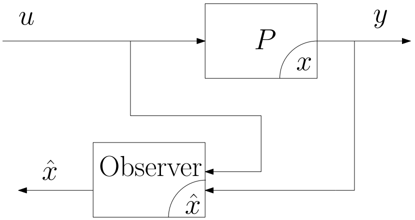

Figure: The observer scheme.

The linear case: reminiscences...

Theorem (Observer for LTI systems)

Let (A, B, C, D) and $u(t)$ be known for all $t \geq 0$ and assume that the system is **detectable**, *i.e.*

$$\text{(detectability)} \quad \text{rank}\left(\begin{bmatrix} C \\ A - \lambda_i I \end{bmatrix}\right) = n, \ \forall \lambda_i \in \sigma\{A\} \cap \mathbb{C}_{bad}, \tag{2}$$

where $n = \dim(x)$, $\mathbb{C}_{bad} \triangleq \{s \in \mathbb{C} : \ \mathrm{Re}(s) \geq 0\}$ if the $t \in \mathbb{R}$ or $\mathbb{C}_{bad} \triangleq \{s \in \mathbb{C} : |s| \geq 1\}$ if $t \in \mathbb{Z}$.

Then there exists a correction matrix L such that $A - LC$ can be rendered **Hurwitz** *if $t \in \mathbb{R}$ or* **Shurr** *if $t \in \mathbb{Z}$, yielding*

$$\Downarrow$$

$$\lim_{t\to\infty} ||x(t) - \hat{x}(t)|| \leq \lim_{t\to\infty} c_0||x(0) - \hat{x}(0)||e^{-\gamma t} = 0, \tag{3}$$

for some $c_0 > 0$ and $\gamma > 0$, i.e. the origin $e = x - \hat{x} = 0$ of the estimation error system is globally exponentially stable.

The linear case: reminiscences...

LTI systems: proof .

The continuous/discrete time (we use in this case the operator Δ) dynamics of the estimation error $e(t)$ are described by

$$\begin{aligned} \Delta e = \Delta x - \Delta\hat{x} &= A\hat{x} + Bu - (A\hat{x} + Bu + L(y - \hat{y})) \\ &= A(x - \hat{x}) + LC(x - \hat{x}) = (A - LC)e. \end{aligned} \quad (4)$$

Then, it is a linear autonomous system whose time evolution can be evaluated analytically in continuous and discrete time (within the square brackets) as

$$e(t) = e^{(A-LC)t}x_0, \quad [e(t) = (A - LC)^t x_0].$$

Assume without lack of generality that the matrix $A - LC$ has all distinct eigenvalues and reduce to the continuous time case. To clearly retrieve the bound in (3), perform the change of co-ordinates $z(t) = Te(t)$ with the invertible matrix T that has as rows the left eigenvectors of the matrix $A - LC$. Then $\dot{z}(t) = \Lambda z(t)$ with $\Lambda =$diag$\{\gamma_i\} = T(A - LC)T^{-1}$ and $\gamma_i \in \sigma\{A - LC\}$ for $i = 1 \ldots n$ and

$$\begin{aligned} ||e(t)|| &= ||T^{-1}z(t)|| = ||T^{-1}e^{\Lambda t}z_0|| = ||T^{-1}e^{\Lambda t}Te_0|| \\ &\le ||T^{-1}||||T||||e^{\Lambda t}||||e_0|| \le \sqrt{\frac{\lambda_{max}(T'T)}{\lambda_{min}(T'T)}}e^{-\gamma_{min}t}||e_0||. \end{aligned} \quad (5)$$

The linear case: reminiscences...

Remark (Observability)

Observability implies **detectability**, where

$$\text{(observability)} \quad \text{rank}\left(\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = n \Leftrightarrow \text{rank}\left(\begin{bmatrix} C \\ A - \lambda_i I \end{bmatrix}\right) = n,\ \forall \lambda_i \in \sigma\{A\} \tag{6}$$

Furthermore, if the system P is **observable**, the parameter λ in (3) can be arbitrarily chosen picking wisely L and even more, all the eigenvalues of the estimation error dynamic matrix $A - LC$ can be freely assigned ($\exists!\, L$).

Remark (Discrete time: Finite time estimation error convergence)

If $t \in \mathbb{Z}$ and the system is **observable** or even less it is **finite-time detectable**, i.e.

$$\text{(finite-time detectability)} \quad \text{rank}\left(\begin{bmatrix} C \\ A - \lambda_i I \end{bmatrix}\right) = n,\ \forall \lambda_i \in \sigma\{A\} : \lambda_i \neq 0, \tag{7}$$

then there exists L such that $\sigma\{A - LC\} = \{0\}$ yielding $e(t) = 0$ for all $t \geq n$ and any e_0.

Observability Canonical Form

Consider the system

$$\begin{cases} \dot{x} & = f(x, u), \\ y & = h(x, u), \end{cases} \tag{8}$$

with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ and $y \in \mathbb{R}$, with $f(\cdot)$ and $h(\cdot)$ sufficiently smooth. *High-gain observers* investigated by **Gauthier and Kupca** are strongly related to a specific global diffeomorfism $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $z = \Phi(x)$ such that the system (8) is rewritten into the **Gauthier-Kupca's Observability Canonical Form**

$$\begin{cases} \dot{z}_1 & = \tilde{f}_1(z_1, z_2, u), \\ \dot{z}_2 & = \tilde{f}_2(z_1, z_2, z_3, u), \\ & \vdots \\ \dot{z}_{n-1} & = \tilde{f}_{n-1}(z_1, z_2, \ldots, z_{n-1}, z_n, u), \\ \dot{z}_n & = \tilde{f}_n(z_1, z_2, \ldots, z_{n-1}, z_n, u), \\ y & = \tilde{h}(z_1, u), \end{cases} \tag{9}$$

(note the *Brunovsky-like* + *lower-triangular* structure) where

$$\frac{\partial \tilde{h}}{\partial z_1} \neq 0, \quad \frac{\partial \tilde{f}_i}{\partial z_{i+1}} \neq 0, \quad \forall i \in \{1, 2, \ldots, n-1\}, z \in \mathbb{R}^n, u \in \mathbb{R}^m. \tag{10}$$

Observability Canonical Form: necessary conditions

Let $f(0,0) = 0$, $h(0,0) = 0$ and define recursively

$$\varphi_1(x,u) := h(x,u), \quad \varphi_i(x,u) := \frac{\partial \varphi_{i-1}}{\partial x} f(x,u), \tag{11}$$

for $i = 1, 2, \ldots, n$ and the incremental i-vector-valued functions

$$\Phi_i(x,u) := \begin{bmatrix} \varphi_1(x,u) \\ \vdots \\ \varphi_i(x,u) \end{bmatrix}. \tag{12}$$

Let $K_i(x,u)$ be the null subspace of $\partial \Phi_i / \partial x$ evaluated at (x,u), i.e.

$$K_i(x,u) = \ker \left[\frac{\partial \Phi_i}{\partial x} \right]_{(x,u)}, \tag{13}$$

and note that the map $D_i(u) : x \to K_i(x,u)$ is a distribution on $\mathbb{R}^n$, the collection of which is called the **canonical flag** (u is the “wind” that changes the manifold)

Canonical flag: uniformity

The **canonical flag** $D_i(u)$ is said to be **uniform** if for all $i = 1, 2, \ldots, n$ and all $x \in \mathbb{R}^n$ it holds

(regularity): $\dim(K_i(x,u)) = n - i$ for all $u \in \mathbb{R}^m$;

(u-independency): $K_i(x,u)$ is independent of u (same subspace).

Proposition

System (8) is globally diffeomorfic to a system in **Gauthier-Kupca's observability canonical form** *only if its* **canonical flag** *is* **uniform***.*

Canonical flag: uniformity

Proposition

System (8) is globally diffeomorfic to a system in **Gauthier-Kupca's observability canonical form** *only if its* **canonical flag** *is* **uniform**.

Sketch of the proof: If a system is already in the observability canonical form then

$$y = \tilde{h}(z_1, u)(= h(x, u) = \varphi_1(x, u))$$

yielding (as φ was evaluated with respect to (8) with (h, f), now $\tilde{\varphi}$ is evaluated with the system in the observability canonical form, i.e. with respect to $(\tilde{h}, \tilde{f})$)

$$\tilde{\varphi}_1(z_1, u) \rightarrow \tilde{\varphi}_i(z_1, \ldots, z_i, u),$$

and by assumption (obs. canonical form)

$$\frac{\partial \tilde{\varphi}_i}{\partial z_i} \neq 0, \ \forall z_1, \ z_2, \ \ldots, z_i, \ u.$$

$$\Downarrow$$

$$K_i(z, u) = \text{span} \begin{bmatrix} 0 \\ I_{n-i} \end{bmatrix}, \forall i = 1, \ldots, n,$$

then the canonical flag (of systems in canonical observability form) is **uniform**. This property is left unchanged by a diffeomorfism.□

Observability Canonical Form

Remark

The **uniform** (necessary) condition is also *sufficient* for the existence of a **local** diffeomorfism to transform (8) into (9)

Observability Canonical Form: sufficient conditions

Proposition

Consider (8) and the map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, $z = \Phi(x)$ *as*

$$\Phi(x) := \begin{bmatrix} \varphi_1(x,0) \\ \varphi_2(x,0) \\ \vdots \\ \varphi_n(x,0) \end{bmatrix},$$

and suppose that

i) *the canonical flag of (8) is* **uniform***;*

ii) $\Phi(\cdot)$ *is a global diffeomorfism.*

Then, system (8) is globally diffeomorfic, via $\Phi(x)$*, to a system in Gauthier-Kupca's observability canonical form.*

Proof: By Assumption

$$\dim\left(\ker\left[\frac{\partial \Phi_i}{\partial x}\right]_{(x,u)}\right) = n - i,\ \forall u \in \mathbb{R}^m.$$

Let $T : \mathbb{R}^n \to \mathbb{R}^n$, $T(z) = x$, i.e. $T(\Phi(x)) = x$.

Observability Canonical Form: sufficient conditions - proof

Since by definition $\Phi(x) := \Phi_n(x,u)$, then $\Phi_n(x,0) = z = \Phi_n(T(z),0)$ whose partial derivatives with respect to z yield

$$\left(\left[\frac{\partial \Phi_n}{\partial x} \right]_{\substack{x = T(z) \\ u = 0}} \right) \frac{\partial T}{\partial z} = I,\ \forall z \in \mathbb{R}^n.$$

This implies that the left-side matrix in the above equality is such that for all $j > i$ it holds

$$\left(\left[\frac{\partial \Phi_i}{\partial x} \right]_{\substack{x = T(z) \\ u = 0}} \right) \frac{\partial T}{\partial z_j} = 0,\ \forall z \in \mathbb{R}^n, \tag{14}$$

$$\Updownarrow \tag{15}$$

$$\frac{\partial T}{\partial z_j} \in \ker \left(\left[\frac{\partial \Phi_i}{\partial x} \right]_{\substack{x = T(z) \\ u = 0}} \right)$$

$$\Downarrow (\text{u-independency of } K_i)$$

$$\frac{\partial T}{\partial z_j} \in \ker \left(\left[\frac{\partial \Phi_i}{\partial x} \right]_{\substack{x = T(z) \\ u = \mathbf{u}}} \right),\ \forall j > i,\ z, u. \tag{16}$$

Observability Canonical Form: sufficient conditions - proof

Exploit the change of co-ordinates $z = \Phi(x)$ on (8), then

$$\begin{cases} \dot{z} & = \tilde{f}(z,u), \\ y & = \tilde{h}(z,u), \end{cases} , \tag{17}$$

where

$$\tilde{h}(z,u) = h(T(z),u)(\text{prove that} = \tilde{h}(z_1,u) = \tilde{\varphi}_1(z_1,u)),$$

$$\tilde{f}(z,u) = \frac{\partial \Phi(x)}{\partial x} f(x,u) = \left(\left[\frac{\partial \Phi_n}{\partial x} \right]_{\substack{x = T(z) \\ u = 0}} \right) f(T(z),u) = \begin{cases} \tilde{f}_1(z_1,z_2,u), \\ \tilde{f}_2(z_1,z_2,z_3,u), \\ \vdots \\ \tilde{f}_{n-1}(z_1,\dots,z_n,u), \\ \tilde{f}_n(z_1,\dots,z_n,u), \end{cases}$$

Define now

$$\tilde{\varphi}_1(z,u) = \tilde{h}(z,u), \quad \tilde{\varphi}_i(z,u) := \frac{\partial \tilde{\varphi_{i-1}}}{\partial z} \tilde{f}(z,u), \, \forall i = 1,\dots,n,$$

and note that by selection of $\varphi_i(\cdot)$ it holds

$$\tilde{\varphi}_1(z,u) = \varphi_1(T(z),u), \quad \tilde{\varphi}_i(z,u) = \varphi_i(T(z),u),$$

yielding...

Observability Canonical Form: sufficient conditions - proof

...yielding

$$\tilde{\Phi}_i(x,u) := \begin{bmatrix} \tilde{\varphi}_1(z,u) \\ \tilde{\varphi}_2(z,u) \\ \vdots \\ \tilde{\varphi}_n(z,u) \end{bmatrix} = \Phi_i(T(z),u).$$

Equation (15) with $i=1$ yields

$$\left(\left[\frac{\partial \Phi_1}{\partial x} \right]_{\substack{x = T(z) \\ u = 0}} \right) \frac{\partial T}{\partial z_j} = 0$$

for all $j > 1$, i.e. $\tilde{\varphi}_1(z,u)$ depends only by z_1 then

$$\frac{\partial \tilde{\varphi}_1}{\partial z} = \left[\frac{\partial \tilde{\varphi}_1}{\partial z_1}, 0, \ldots, 0 \right],$$

and by uniformity we have also

$$\frac{\partial \tilde{h}}{\partial z_1} = \frac{\partial \tilde{\varphi}_1}{\partial z_1} \neq 0.$$

Observability Canonical Form: sufficient conditions - proof

By the definition of $\tilde{\varphi}_i$ and what has been highlighted before, then

$$\tilde{\varphi}_2(z,u) = \frac{\partial \tilde{h}}{\partial z_1}\tilde{f}_1(z,u).$$

Reiterating (15) for $i=2$ and $j>2$ then

$$\left(\left[\frac{\partial \Phi_{2}}{\partial x}\right]_{\substack{x=T(z)\\ u=0}} \right) \frac{\partial T}{\partial z_{j}} = 0,$$

hence $\tilde{\varphi}_2(z,x)$ depends only on z_1 and z_2, which implies that

$$\frac{\partial \tilde{f}_1}{\partial z_j} = 0,\ \forall j>2,$$

and also $\tilde{f}_1(z,u)$ depends only on z_1 and z_2. Furthermore

$$\frac{\partial \tilde{\varphi}_2(z,u)}{\partial z} = \left(\star,\ \frac{\partial}{\partial z_2}\left(\frac{\partial \tilde{h}(z,u)}{\partial z_1}\tilde{f}_1(z,u)\right),0,\dots,0\right) = \left(\star,\ \frac{\partial \tilde{h}(z,u)}{\partial z_1}\frac{\partial \tilde{f}_1(z,u)}{\partial z_2},0,\dots,0\right)$$

Observability Canonical Form: sufficient conditions - proof

...and by uniformity assumption

$$\frac{\partial \tilde{f}_1(z,u)}{\partial z_2} \neq 0, \; \forall z_1, z_2 \text{ and } u.$$

Iterating the same procedure the result follows, i.e. the change of co-ordinates $z = \Phi(x)$ (equivalently $x = T(z)$...but T could be very difficult to find!) transforms (8) into

$$\begin{cases} \dot{z}_1 & = \tilde{f}_1(z_1, z_2, u), \\ \dot{z}_2 & = \tilde{f}_2(z_1, z_2, z_3, u), \\ & \vdots \\ \dot{z}_{n-1} & = \tilde{f}_{n-1}(z_1, z_2, \dots, z_{n-1}, z_n, u), \\ \dot{z}_n & = \tilde{f}_n(z_1, z_2, \dots, z_{n-1}, z_n, u), \\ y & = \tilde{h}(z_1, u). \end{cases} \tag{18}$$

$\square$

The change of co-ordinate in case of input-affine systems

Consider the input-affine system described by

$$\begin{cases} \dot{x} & = f(x) + g(x)u, \\ y & = h(x), \end{cases} \tag{19}$$

where the functions $\varphi_i(x,u)$ are given by (the output y and its derivatives up to the $n-1$ order with $u(t) \equiv 0$)

$$\begin{aligned}
\varphi_1(x,u) &= h(x), \\
\varphi_2(x,u) &= \frac{\partial h}{\partial x}(f(x) + g(x)u) = L_f h(x) + L_g h(x) u, \\
\varphi_2(x,u) &= L_f^2 h(x) + (L_g L_f h(x) + L_f L_g h(x))\, u + L_g^2 h(x)\, u^2, \\
&\ \ \vdots
\end{aligned}$$

that by $u - independency$ can be rewritten picking $u = 0$ as

$$\Phi_n(x,0) = \begin{bmatrix} h(x) \\ L_f h(x) \\ L_f^2 h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix} := \Phi(x). \tag{20}$$

The change of co-ordinate in case of input-affine systems

If the canonical flag of (19) is **uniform** and $\Phi(x)$ is a **global diffeomorfism**, the system is transformable via $\Phi(x)$ into a *uniform observability canonical form*

$$\begin{cases} \dot{z} & = \tilde{f}(z) + \tilde{g}(z)u, \\ y & = \tilde{h}(z), \end{cases} \tag{21}$$

where

$$\tilde{f}(z) = \left(\frac{\partial \Phi(x)}{\partial x} f(x) \right)_{x=\Phi^{-1}(z):=T(z)}, \tag{22}$$

$$\tilde{g}(z) = \left(\frac{\partial \Phi(x)}{\partial x} g(x) \right)_{x=\Phi^{-1}(z)}, \tag{23}$$

$$\tilde{h}(z) = h(\Phi^{-1}(x)). \tag{24}$$

Since $\varphi_1(x) = h(x)$ and $z = \Phi(x)$, then $z_1 = h(x) = y$ yielding $\tilde{h}(z) = z_1$.
Moreover,

$$z_2 = \varphi_2(x) = \left(\frac{d\,y(t)}{dt} \right)_{\substack{x = \Phi^{-1}(z), \\ u = 0.}} = \left(\frac{\partial \varphi_1(x,u)}{\partial x} (f(x) + g(x)u) \right)_{\substack{x = \Phi^{-1}(z), \\ u = 0.}}$$

$$= \dot{z}_1 \text{.....then....}$$

The change of co-ordinate in case of input-affine systems

If the canonical flag of (19) is **uniform** and $\Phi(x)$ is a **global diffeomorfism**, the system is transformable via $\Phi(x)$ into a *uniform observability canonical form*

$$\begin{cases} \dot{z} & = \tilde{f}(z) + \tilde{g}(z)u, \\ y & = \tilde{h}(z), \end{cases} \tag{25}$$

where

$$\tilde{f}(z) = \begin{bmatrix} z_2 \\ z_3 \\ \vdots \\ z_n \\ \tilde{f}_n(z_1, z_2, \dots, z_n) \end{bmatrix} \tag{26}$$

is a chain of $n-1$ integrators and knowing the fact that $\tilde{f}_i(z) + \tilde{g}_i(z)u$ depends only on $(z_1, z_2, \dots, z_i, z_{i+1})$ then

$$\tilde{g}(z) = \begin{bmatrix} \tilde{g}_1(z_1, z_2) \\ \tilde{g}_2(z_1, z_2, z_3) \\ \vdots \\ \tilde{g}_{n-1}(z_1, z_2, \dots, z_n) \\ \tilde{g}_n(z_1, z_2, \dots, z_n) \end{bmatrix} \tag{27}$$

The change of co-ordinate in case of input-affine systems

It is possible to show that g_i does not depend on z_{i+1}. In fact,

$$\tilde{\varphi}_1(z,u) = z_1, \tag{28}$$

$$\tilde{\varphi}_2(z,u) = z_2 + \tilde{g}_1(z_1,z_2)u, \tag{29}$$

whose Jacobian is

$$\begin{bmatrix} \frac{\partial \tilde{\varphi}_1(z,u)}{\partial z} \\ \frac{\partial \tilde{\varphi}_2(z,u)}{\partial z} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \frac{\partial \tilde{g}_1(z,u)}{\partial z_1}u & \left(1 + \frac{\partial \tilde{g}_1(z1,z_2)}{\partial z_2}u\right) & 0 & \dots & 0 \end{bmatrix}$$

yielding, by **uniformity** (that is no changed by the diffeomorfism Φ) condition for all (z_1, z_2), that $\frac{\partial \tilde{g}_1}{\partial z_2} = 0$. This holds also for $\tilde{g}_i$ with $i > 4$.

The change of co-ordinate in case of input-affine systems

The **uniform observability canonical form** of **input-affine** systems is

$$\left\{\begin{array}{ll} \dot{x} & = f(x) + g(x)u, \\ y & = h(x), \end{array}\right. \quad \overset{z = \Phi(x)}{\Longrightarrow} \quad \left\{\begin{array}{ll} \dot{z}_1 & = z_2 + \tilde{g}_1(z_1)u, \\ \dot{z}_2 & = z_3 + \tilde{g}_2(z_1, z_2)u, \\ & \vdots \\ \dot{z}_{n-1} & = z_n + \tilde{g}_{n-1}(z_1, \ldots, z_n)u, \\ \dot{z}_n & = \tilde{f}_n(z_1, \ldots, z_n, u) + \tilde{g}_n(z_1, \ldots, z_n)u, \\ y & = z_1. \end{array}\right. \tag{30}$$

High-gain global asymptotic observer: framework

Let

$$\mathbf{z_i} = [z_1,\ z_2,\ \ldots,\ z_i]',$$

and rewrite the observer canonical form concisely (with some abuse of notation, they should be $\tilde{f}(\cdot)$ and $\tilde{g}(\cdot)$) as

$$\begin{cases} \dot{z}_1 & = f_1(\mathbf{z_1}, z_2, u), \\ \dot{z}_2 & = f_2(\mathbf{z_2}, z_3, u), \\ & \vdots \\ \dot{z}_{n-1} & = f_{n-1}(\mathbf{z_{n-1}}, z_n, u), \\ \dot{z}_n & = f_n(\mathbf{z_n}, u), \\ y & = h(z_1, u). \end{cases} \tag{31}$$

High-gain global asymptotic observer: framework

Let

$$\mathbf{z_i} = [z_1,\, z_2,\, \ldots, z_i]',$$

and rewrite the observer canonical form concisely (with some abuse of notation, they should be $\tilde{f}(\cdot)$ and $\tilde{g}(\cdot)$) as

$$\left\{\begin{array}{ll} \dot{z}_1 & = f_1(\mathbf{z_1}, z_2, u), \\ \dot{z}_2 & = f_2(\mathbf{z_2}, z_3, u), \\ & \vdots \\ \dot{z}_{n-1} & = f_{n-1}(\mathbf{z_{n-1}}, z_n, u), \\ \dot{z}_n & = f_n(\mathbf{z_n}, u), \\ y & = h(z_1, u). \end{array}\right. \tag{31}$$

Assumption (Technical)

i) $\|f_i(\mathbf{z_i}, z_{i+1}, u) - f_i(\overline{\mathbf{z}}_\mathbf{i}, z_{i+1}, u)\| \leq L\|z_i - \overline{z}_i\|$ *for some* $L > 0$*, all* z_i *and* $\overline{z}_i$ *belonging to* $\mathbb{R}^n$ *and uniformly in* z_{i+1}*,* u *and* $i = 1, \ldots, n$ *(Globally Lipschitz);*

ii) *There exists real numbers* $\beta > \alpha > 0$ *such that*

$$\beta \geq \left\| \frac{\partial h}{\partial z_1} \right\| \geq \alpha, \beta \geq \left\| \frac{\partial f_i}{\partial z_{i+1}} \right\| \geq \alpha,\ \forall z \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ i = 1, \ldots, \mathbf{n\text{-}1}.$$

High-gain global asymptotic observer: techincal assumptions

Assumption (Technical)

i) $\|f_i(\mathbf{z_i}, z_{i+1}, u) - f_i(\overline{\mathbf{z}}_\mathbf{i}, z_{i+1}, u)\| \leq L\|z_i - \overline{z}_i\|$ *for some* $L > 0$*, all* z_i *and* $\overline{z}_i$ *belonging to* $\mathbb{R}^n$ *and uniformly in* z_{i+1}, u *and* $i = 1, \ldots, n$ *(Globally Lipschitz)*;

ii) *There exists real numbers* $\beta > \alpha > 0$ *such that*

$$\beta \geq \left|\frac{\partial h}{\partial z_1}\right| \geq \alpha, \beta \geq \left|\frac{\partial f_i}{\partial z_{i+1}}\right| \geq \alpha, \forall z \in \mathbb{R}^n, u \in \mathbb{R}^m, i = 1, \ldots, \mathbf{n\text{-}1}.$$

Remark

Assumption $i)$ is automatically satisfied if it is known - a priori - that $z(t) \subset \mathcal{Z}$ for all $t \geq 0$ and $\mathcal{Z}$ is a compact subset of $\mathbb{R}^n$.

The high-gain observer

The high-gain observer consists in a copy of the system (31) plus a correction term proportional to the output estimation error $y(t) - \hat{y}(t)$, namely

$$\left\{\begin{array}{lll} \dot{\hat{z}}_1 & = f_1(\hat{\mathbf{z}}_1, \hat{z}_2, u) & + k\, c_{n-1}\, (y - \hat{y}), \\ \dot{\hat{z}}_2 & = f_2(\hat{\mathbf{z}}_2, \hat{z}_3, u) & + k^2\, c_{n-2}\, (y - \hat{y}), \\ & \quad \vdots & \quad \vdots \\ \dot{\hat{z}}_{n-1} & = f_{n-1}(\hat{\mathbf{z}}_{n-1}, \hat{z}_n, u) & + k^{n-1}\, c_1\, (y - \hat{y}), \\ \dot{\hat{z}}_n & = f_n(\hat{\mathbf{z}}_n, u) & + k^n\, c_0\, (y - \hat{y}), \\ \hat{y} & = h(\hat{z}_1, u). & \end{array}\right. \tag{32}$$

where the observer gain $k > 0$ and parameters $(c_{n-1}, \dots, c_0)$ have to be selected.

Theorem (Global asymptotic high-gain observer)

Consider the system (31) and the observer (32). Let the technical assumption hold and select the vector parameters $(c_{n-1}, \dots, c_0)$ such that the polynomial $p(s) = s^n + c_{n-1}s^{n-1} + \cdots + c_0$ is Hurwitz.
Then, there exists a "sufficiently high" value of k (high-gain) such that the origin of the estimation error $e = \hat{z} - z$ system is uniformly ***globally*** *asymptotically (exponentially) stable.*

The high-gain observer dynamics (proof of the main result)

Proof: the mean value theorem yields

$$f_i(\hat{\mathbf{z}}_i, \hat{z}_{i+1}, u) - f_i(\mathbf{z}_i, z_{i+1}, u) + f_i(\mathbf{z}_i, \hat{z}_{i+1}, u) - f_i(\mathbf{z}_i, \hat{z}_{i+1}, u) =$$
$$f_i(\hat{\mathbf{z}}_i, \hat{z}_{i+1}, u) - f_i(\mathbf{z}_i, \hat{z}_{i+1}, u) + f_i(\mathbf{z}_i, \hat{z}_{i+1}, u) - f_i(\mathbf{z}_i, z_{i+1}, u) =$$
$$F_i(\mathbf{z}_i(t), \hat{\mathbf{z}}_i(t), \hat{z}_{i+1}(t), u(t)) + \frac{\partial f_i}{\partial z_{i+1}}(\mathbf{z_i}(t), \delta_i(t), u(t))e_{i+1},$$

where $\delta_i(t) \in [\hat{z}_{i+1}(t), z_{i+1}(t)]$ (swap their order if necessary), similarly

$$y(t) - \hat{y}(t) = y(t) - h(\hat{z}_1(t), u(t)) = \frac{\partial h}{\partial z_1}(\delta_0(t), u(t))e_1,$$

with $\delta_0(t) \in [\hat{z}_1(t), z_1(t)]$ (swap their order if necessary). Define

$$g_{i+1}(t) := \frac{\partial f_i}{\partial z_{i+1}}(\mathbf{z_i}(t), \delta_i(t), u(t)),\ i \geq 0, \quad \text{and } g_1(t) := \frac{\partial h}{\partial z_1}(\delta_0(t), u(t)),$$

then, since $e_i = \hat{z}_i - z_i$, it holds

$$\dot{e}_i(t) = g_{i+1}(t)\, e_{i+1} - k^i c_{n-i} g_1(t)\, e_1 + F_i(\mathbf{z}_i(t), \hat{\mathbf{z}}_i(t), \hat{z}_{i+1}(t), u(t))$$

The high-gain observer dynamics (proof of the main result...)

The estimation error system is

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \vdots \\ \dot{e}_{n-1} \\ \dot{e}_n \end{bmatrix} = \begin{bmatrix} -k\, c_{n-1} g_1(t) & g_2(t) & 0 & \dots & 0 & 0 \\ -k^2\, c_{n-2} g_1(t) & 0 & g_3(t) & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \dots & \ddots & \vdots \\ -k^{n-1}\, c_1 g_1(t) & 0 & 0 & \dots & 0 & g_n(t) \\ -k^n\, c_0 g_1(t) & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{n-1} \\ e_n \end{bmatrix} + \begin{bmatrix} F_1(\cdot) \\ F_2(\cdot) \\ \vdots \\ F_{n-1}(\cdot) \\ F_n(\cdot) \end{bmatrix} \tag{33}$$

Consider now the **scaled** estimation error defined as

$$\tilde{e}_i = \frac{e_i}{k^i}, \quad i = 1, 2, \dots, n\,, \tag{34}$$

yielding

$$\begin{aligned} \dot{\tilde{e}}_i = k^{-i}\dot{e}_i &= k^{-i}\left(g_{i+1}(t)e_{i+1} + c_{n-i}g_1(t)k^i e_1 + F_i(\cdot)\right) \\ &= g_{i+1}(t)k^{-i}e_{i+1} + c_{n-i}g_1(t)e_1 + k^{-i}F_i(\cdot) \\ &= g_{i+1}(t)k\tilde{e}_{i+1} + c_{n-i}g_1(t)k\tilde{e}_1 + k^{-i}F_i(\cdot). \end{aligned}$$

The high-gain observer dynamics (proof of the main result...)

The scaled estimation error system is

$$\begin{bmatrix} \dot{\tilde{e}}_1 \\ \dot{\tilde{e}}_2 \\ \vdots \\ \dot{\tilde{e}}_{n-1} \\ \dot{\tilde{e}}_n \end{bmatrix} = k \begin{bmatrix} -c_{n-1}g_1(t) & g_2(t) & 0 & \dots & 0 & 0 \\ -c_{n-2}g_1(t) & 0 & g_3(t) & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \dots & \ddots & \vdots \\ -c_1 g_1(t) & 0 & 0 & \dots & 0 & g_n(t) \\ -c_0 g_1(t) & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \\ \vdots \\ \tilde{e}_{n-1} \\ \tilde{e}_n \end{bmatrix} + \begin{bmatrix} k^{-1}F_1(\cdot) \\ k^{-2}F_2(\cdot) \\ \vdots \\ k^{-n+1}F_{n-1}(\cdot) \\ k^{-n}F_n(\cdot) \end{bmatrix}$$

$$= \underbrace{k\, A(t)\tilde{e}}_{\text{linear in } e \text{ (time-varying)}} + \underbrace{\tilde{F}_k(z(t), \hat{z}(t), u(t))}_{\text{nonlinear in } e = \hat{z} - z \text{ (time-varying)}} . \qquad (35)$$

The globally Lipschitz Assumption $i)$ on f_i is such that $||F_i(\cdot)|| \leq L||\hat{\mathbf{z}}_i(t) - \mathbf{z}_i(t)||$, then

$$||F_i(\cdot)|| \leq L||\mathbf{e}_i|| = L\sqrt{e_1^2 + e_2^2 + \dots + e_i^2} = L\sqrt{k^2\tilde{e}_1^2 + k^4\tilde{e}_2^2 + \dots + k^{2i}\tilde{e}_i^2}.$$

Furthermore, if $k \geq 1$, the following holds

$$||k^{-i}F_i(\cdot)|| = L\sqrt{\frac{k^2\tilde{e}_1^2 + k^4\tilde{e}_2^2 + \dots + k^{2i}\tilde{e}_i^2}{k^{2i}}} \underbrace{\leq}_{k\geq 1} L||\tilde{\mathbf{e}}_i|| \underbrace{\leq}_{i\leq n} L||\tilde{e}||.$$

The proof continues relying on the next lemma...

The high-gain observer dynamics (proof of the main result...)

Lemma

Consider a time-varying matrix of the form

$$A(t) = \begin{bmatrix} 0 & g_2(t) & 0 & \dots & 0 & 0 \\ 0 & 0 & g_3(t) & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \dots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & g_n(t) \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} - \begin{bmatrix} c_{n-1} \\ c_{n-2} \\ \vdots \\ c_1 \\ c_0 \end{bmatrix} [g_1(t),\, 0,\, 0, \dots,\, 0,\, 0]\,, \tag{36}$$

and suppose there exists real numbers [a] $\beta > \alpha > 0$ *such that*

$$\beta \geq g_i(t) \geq \alpha, \quad \forall t \geq 0 \text{ and } i = 1,\, 2,\, \dots, n. \tag{37}$$

Then, there exist a set of real numbers $c_0,\, c_1,\, \dots,\, c_{n-1},\, \lambda > 0$ *and a symmetric positive definite* **constant** *matrix* $S \in \mathbb{R}^{n\times n}$*, with* λ *and* S *only depending on* α*,* β*, and the parameters* c_i *such that*

$$A'(t)S + SA(t) \leq -\lambda I. \tag{38}$$

---

[a] In our proof, they are exactly the ones in the Technical Assumption whereas $A(t)$ is the one defined in (35).

The high-gain observer dynamics (proof of the main result...)

Consider the candidate Lyapunov function $V(\tilde{e}) = \tilde{e}' S \tilde{e}$, then

$$\begin{aligned}\dot{V}(\tilde{e}(t)) &= k\tilde{e}'(t)\left(A'(t)S + SA(t)\right)\tilde{e}(t) + 2\tilde{e}'(t)S\tilde{F}_k(t) \\ &\leq -k\,\lambda\,\|\tilde{e}(t)\|^2 + 2\|S\|\,L\underbrace{\sqrt{n}}_{\text{n components of } \tilde{F}_k}\|\tilde{e}(t)\|^2 = -\left(k\lambda - 2\|S\|L\sqrt{n}\right)\|\tilde{e}(t)\|^2,\end{aligned} \tag{39}$$

and defining

$$k^{\star} := \frac{2\|S\|\,L\sqrt{n}}{\lambda},$$

if k is sufficiently **high**, i.e. $k > k^{\star}$ (and $k > 1$), then[1]

$$\dot{V}(\tilde{e}(t)) \leq -\frac{c_k}{\overline{\lambda}_S} V(\tilde{e}(t)) \tag{40}$$

where $\overline{\lambda}_S$ is the largest eigenvalue of S, $c_k := (k\lambda - 2\|S\|L\sqrt{n}) > 0$ yielding

$$\lim_{t\to\infty} \tilde{e}(t) = 0 \Rightarrow \lim_{t\to\infty} e(t) = 0$$

□

[1] S and L do not depend on k.

The high-gain observer dynamics (proof of the main result...)

Remark (Peaking phenomenon)

The differential inequality (40) together with the selected Lyapunov function, $V(\tilde{e}) = \tilde{e}' S \tilde{e}$, yield

$$V(\tilde{e}(t)) \leq V(\tilde{e}(0)) e^{-\frac{c_k}{\overline{\lambda}_S} t}.$$

Since for any $x \in \mathbb{R}^n$ it holds $\underline{\lambda}_S ||x||^2 \leq x' S x = V(x) \leq \overline{\lambda}_S ||x||^2$ where $(\underline{\lambda}_S, \overline{\lambda}_S)$ are the smaller and the largest eigenvalues of S, respectively, then

$$||\tilde{e}(t)||^2 \leq \frac{1}{\underline{\lambda}_S} V(\tilde{e}(t)) \leq \frac{\overline{\lambda}_S}{\underline{\lambda}_S} ||\tilde{e}(0)||^2 e^{-\frac{c_k}{\overline{\lambda}_S} t},$$

which is an appealing bound for the **scaled** estimation error $\tilde{e}(t)$... nevertheless

$$||e(t)|| = \left\| \begin{bmatrix} k & 0 & \dots & 0 \\ 0 & k^2 & \dots & 0 \\ \vdots & \dots & \ddots & 0 \\ 0 & 0 & \dots & k^n \end{bmatrix} \tilde{e}(t) \right\| \underbrace{\leq}_{k>1} k^n ||\tilde{e}(t)|| \leq k^n ||e(0)|| \sqrt{\frac{\overline{\lambda}_S}{\underline{\lambda}_S}} \, e^{-\frac{c_k}{2\overline{\lambda}_S} t}. \tag{41}$$

The higher the k is, the faster $||e(t)||$ converges to zero...but the transient amplitude of $e(t)$ increases polynomially in k as well! **[Peaking phenomenon]**

The high-gain observer dynamics (proof of the lemma...)

We now prove Lemma 1 used in the proof of Theorem 1. As first, note that

$$g_i(t) = \alpha\delta_i(t) + (1-\delta_i(t))\beta = \beta - (\beta-\alpha)\delta_i(t),$$

for some functions $\delta_i(t) \in [0,1]$. The matrix $A(t)$ in (36) can be rewritten as

$$A(t) = \begin{bmatrix} -c_{n-1}g_1(t) & g_2(t) & 0 & \dots & 0 & 0 \\ -c_{n-2}g_1(t) & 0 & g_3(t) & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \dots & \ddots & \vdots \\ -c_1 g_1(t) & 0 & 0 & \dots & 0 & g_n(t) \\ -c_0 g_1(t) & 0 & 0 & \dots & 0 & 0 \end{bmatrix} =$$

$$\beta\left(\begin{bmatrix} -c_{n-1} & 1 & 0 & \dots & 0 & 0 \\ -c_{n-2} & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \dots & \ddots & \vdots \\ -c_1 & 0 & 0 & \dots & 0 & 1 \\ -c_0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} - \gamma \begin{bmatrix} -c_{n-1}\delta_1(t) & \delta_2(t) & 0 & \dots & 0 & 0 \\ -c_{n-2}\delta_1(t) & 0 & \delta_3(t) & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \dots & \ddots & \vdots \\ -c_1\delta_1(t) & 0 & 0 & \dots & 0 & \delta_n(t) \\ -c_0\delta_1(t) & 0 & 0 & \dots & 0 & 0 \end{bmatrix}\right)$$

Then, defining A_c and $\Delta_i(t)$ accordingly to the above equation, $A(t)$ is rewritten as

$$A(t) = \beta\left(A_c - \gamma\Delta_c(t)\right), \quad \text{where } \gamma := \frac{\beta-\alpha}{\beta}, \quad 1 > \gamma > 0 \ \ (\text{since } \beta > \alpha > 0).$$

The high-gain observer dynamics (proof of the lemma...)

Given the special (companion form - observer) structure of A_c, then it is possible to select the parameters c_i $(i = 1, 2, \ldots, n)$ such that the characteristic polynomial of A_c, $p_{A_c}(s) = s^n + c_{n-1}s^{n-1} + \cdots + c_1 s + c_0$, is Hurwitz.

Then, there exists a matrix $S = S' \succ 0$ such that

$$A_c' S + S A_c = -hI,$$

for a desired $h > 0$ $(c_i(h))$. This choice yields, for any $x \in \mathbb{R}^n$,

$$\begin{aligned}
x'\left(A'(t)S + SA(t)\right)x &= \beta x'\left(A_c'S + SA_c' - \gamma\left(\Delta_c'(t)S + S\Delta_c(t)\right)\right)x, \\
&\leq \beta\left(-h\|x\|^2 + \gamma|x'\left(\Delta_c'(t)S + S\Delta_c(t)\right)x|\right) \\
&\leq \beta\left(-h\|x\|^2 + \gamma \max_{i=1,\ldots,n}\{\delta_i(t)\}|\,x'\left(A_c'S + SA_c\right)x\,|\right) \\
&\leq \beta\, h\left(-1 + \gamma \max_{i=1,\ldots,n}\{\delta_i(t)\}\right)||x||^2 \\
&\leq -\beta h(1-\gamma)||x||^2 \underbrace{=}_{\lambda := \beta h(1-\gamma) > 0} -\lambda\,||x||^2,
\end{aligned}$$

that complete the proof of the Lemma. $\square$

Example: the Van der Pol oscillator

The Van der Pol oscillator is described by

$$\ddot{y}(t) = \mu(1 - y^2(t))\dot{y}(t) - y(t),$$

with $\mu > 0$ represents a nonlinear damping coefficient. The system is already written in the observer canonical form[2] ($z = I\,x$), in fact, let $x_1 = y$ and $x_2 = \dot{y} = \dot{x}_1$, then

$$\begin{cases} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 + \mu(1 - x_1^2)x_2, \\ y &= x_1, \end{cases} \tag{42}$$

Can we implement the high-gain observer just introduced?

[2] This is always the case when the system differential equations are written with respect to the output.

Example: the Van der Pol oscillator

The Van der Pol oscillator is described by

$$\ddot{y}(t) = \mu(1 - y^2(t))\dot{y}(t) - y(t),$$

with $\mu > 0$ represents a nonlinear damping coefficient. The system is already written in the observer canonical form[3] ($z = I\,x$), in fact, let $z_1 = y$ and $z_2 = \dot{y} = \dot{z}_1$, then

$$\begin{cases} \dot{z}_1 & = z_2, \\ \dot{z}_2 & = -z_1 - \mu(1 - z_1^2)z_2, \\ y & = z_1, \end{cases} \tag{43}$$

☹: The Technical assumption $i)$ is not satisfied with respect to $f_2(\cdot)$, in fact $f_2(z)$ is not **globally** Lipschitz (whereas $f_1(z_1) = z_1$ is Lipschitz, and $ii)$ is satisfied for any $0 < \alpha < 1$ and $\beta \geq 1$).

☺: the state of a Van der Pol oscillator remains bounded.

[3] This is always the case when the system differential equations are written with respect to the output.

Example: the Van der Pol oscillator

The high-gain observer for the oscillator is

$$\left\{ \begin{array}{lll} \dot{\hat{z}}_1 & = \hat{z}_2 & + k\, c_1\, (y - \hat{y}), \\ \dot{\hat{z}}_2 & = -\hat{z}_1 + \mu(1 - \hat{z}_1^2)\hat{z}_2 & + k^2\, c_0\, (y - \hat{y}), \\ \hat{y} & = \hat{z}_1 . & \end{array} \right. \tag{44}$$

The polynomial $p_c(s) = s^2 + c_1 s + c_0$ is Hurwitz iff (Cartesio's law) $c_1 > 0$ and $c_0 > 0$. We assume now to know the set $\mathcal{D}$ where the attractive limit cycle belongs to. Then, we can evaluate L such that, for all $z \in \mathcal{D}$ it holds

$$\left\| \hat{z}_1 + \mu(1 - \hat{z}_1^2)\hat{z}_2 \; - \; \left(-z_1 + \mu(1 - z_1^2)z_2\right) \right\| \leq L \, \|\hat{z} - z\| = L||e||.$$

We pick $c_0 = 2$, $c_1 = 1$ and different $k \in \{1, 10, 100\}$ (the $k^\star$ has not been evaluated numerically....laziness...).

The initial condition of the observer is set equal to $\hat{z}_0 = (0, 0)$ whereas $z_0 = (1, 1)$.

Example: the Van der Pol oscillator

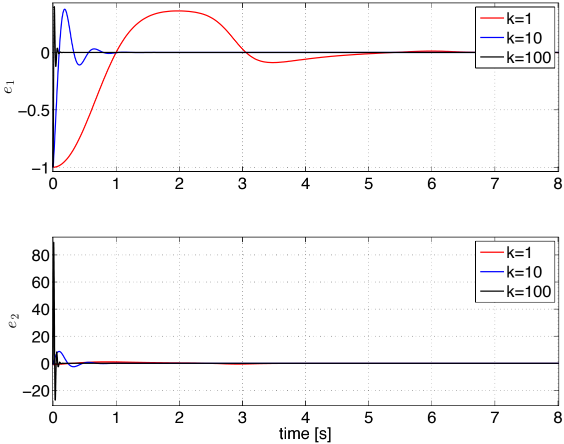

Figure: High gain observer for the Van der Pol oscillator: estimation errors for different k.

Example: the Van der Pol oscillator

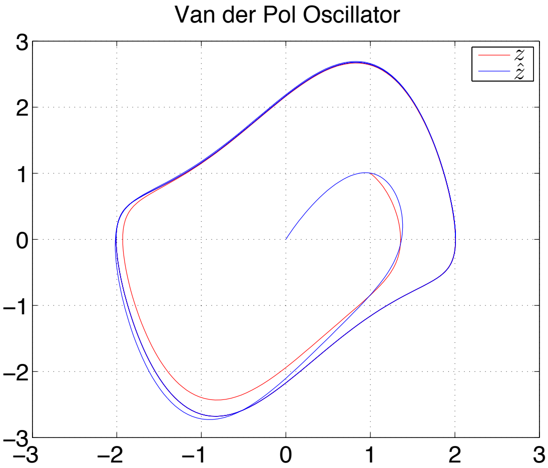

Figure: High gain observer for the Van der Pol oscillator: phase plot with $k = 1$.

Example: the Van der Pol oscillator

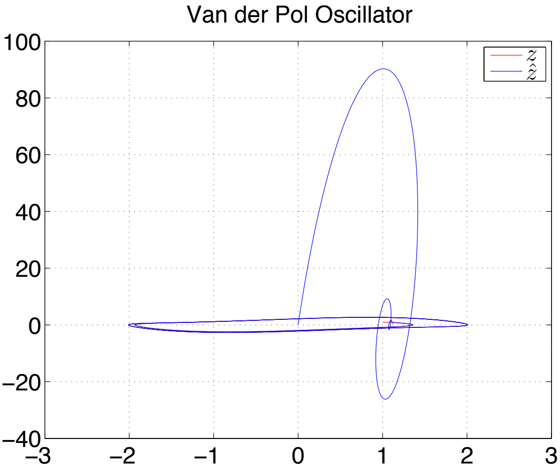

Figure: High gain observer for the Van der Pol oscillator: phase plot with $k = 100$.

Example: the Van der Pol oscillator

What does it happen if the parameter μ is not known exactly?

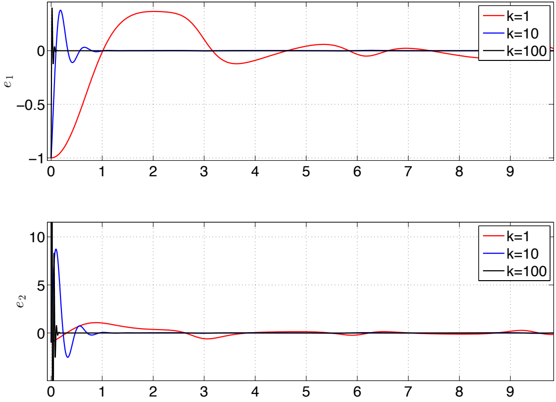

Figure: High gain observer for the Van der Pol oscillator with $\bar{\mu} = 0.8\mu$ and different values of k (note the peaking phenomenon visible for $k = 100$).

Example: the Van der Pol oscillator

A constant estimate $\overline{\mu}$ is considered and the observer is

$$\left\{\begin{array}{ll} \dot{\hat{z}}_1 & = \hat{z}_2 \qquad\qquad\qquad\qquad + k\,c_1\,(y-\hat{y}), \\ \dot{\hat{z}}_2 & = -\hat{z}_1 + \overline{\mu}(1-\hat{z}_1^2)\hat{z}_2 \quad + k^2\,c_0\,(y-\hat{y}), \\ \hat{y} & = \hat{z}_1. \end{array}\right. \tag{45}$$

The scaled estimation error dynamics are described by

$$\begin{bmatrix} \dot{\tilde{e}}_1 \\ \dot{\tilde{e}}_2 \end{bmatrix} = k \begin{bmatrix} -c_1 g_1(t) & g_2(t) \\ -c_0 g_1(t) & 0 \end{bmatrix} \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{bmatrix} + \begin{bmatrix} k^{-1} F_1(\cdot) \\ k^{-2} F_2(\cdot) \end{bmatrix}, \tag{46}$$

where $g_1(t) = g_2(t) = 1$, $F_1(\cdot) = 0$ and

$$\begin{aligned} F_2(\cdot) &= -e_1 + \bar{\mu}(1-\hat{z}_1^2)\hat{z}_2 - \mu(1-\hat{z}_1^2)\hat{z}_2 = -e_1 + \mu\left(e_2 - \hat{z}_1^2\hat{z}_2 + z_1^2 z_2\right), \\ &= \underbrace{-e_1 + \mu(e_2 - \hat{z}_1^2\hat{z}_2 + z_1^2 z_2)}_{=0 \text{ if } e=0} + \underbrace{(\overline{\mu}-\mu)(1-\hat{z}_1^2)\hat{z}_2}_{\text{can be } \neq 0 \text{ when } e=0} \quad . \end{aligned} \tag{47}$$

There is a *non-vanishing* perturbation term that ruins asymptotic convergence. Nevertheless, the gain k gives certain degree of robustness to un-modeled dynamics reducing the perturbation term.

Hi: the Van der Pol oscillator

...take it (very) easy.....

$$\left\{\begin{array}{ll} \dot{\hat{z}}_1 & = \hat{z}_2 \quad + k\, c_1\, (y - \hat{y}), \\ \dot{\hat{z}}_2 & = \quad\quad + k^2\, c_0\, (y - \hat{y}), \\ \hat{y} & = \hat{z}_1. \end{array}\right. \tag{48}$$

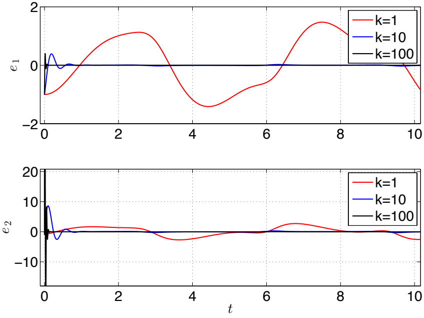

Figure: The High gain observer for the Van der Pol oscillator in (48) for different values of k.

Example: the Van der Pol oscillator

...wait, not too easy...

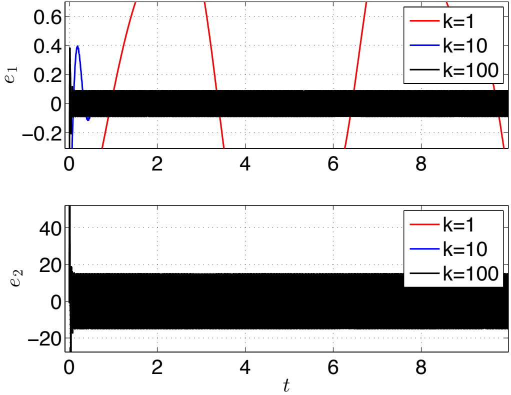

Figure: The High gain observer for the Van der Pol oscillator in (48) for different values of k in case of **measurement noise** $d = 0.2\sin(2\pi\,50\,t)$.

Example: the Van der Pol oscillator

Highlighting the main features of the high-gain observer:

☺: very easy to implement when the model is written with differential equation involving the measurable output y

☺: robust to un-modeled dynamics

☺: easy to tune.

☹: finding the nonlinear map $\Phi(\cdot)$ often requires complicated calculation

☹: it suffers of **peaking phenomenon** and it is sensitive to measurement noise (related to "high" gains).

☹: the globally Lipschitz assumption $i)$ is very restrictive (as well as $ii)$).

Discretization: the discretized version (Euler...) can be implemented but in this case the sampling and quantization introduce un-modeled dynamics (good...) and noise (ouch!).

A nonlinear separation principle: introduction

Consider a system in the observability canonical form rewritten as

$$\begin{cases} \dot{z} & = f(z,u), \\ y & = h(z,u), \end{cases} \tag{49}$$

with $f(0,0) = h(0,0) = 0$ and assume there exists a **feedback law** $u = \alpha(z)$ such that the equilibrium $z = 0$ ($\Rightarrow \alpha(0) = 0$) of the closed-loop system

$$\dot{z} = f(z, \alpha(z))$$

is **GAS** (Globally Asymptotically Stable). Outlines of the proof:

- A saturated control $u = \sigma_\gamma(\alpha(z))$ with the estimate $\hat{z}$ is considered in place of $u = \alpha(z)$ (only semi-global stability is achieved).
- It is shown that, independently by the selection of the observer gain k, the state of the closed-loop system remains bounded for a bounded time T no matter the estimation error is.
- The observer gain k is selected such that within a time T the estimation error is smaller than a desired value ε.
- It is shown that $|z(t)|$ goes to zero since the estimation error goes to zero.

Preliminaries

By the inverse Lyapunov theorem, if $u = \alpha(z)$ renders the origin of (49) GAS, then there exist a smooth function $W(z)$, class $\mathcal{K}_\infty$ functions $\overline{\alpha}(\cdot)$, $\underline{\alpha}(\cdot)$ and $w(\cdot)$ such that

$$\underline{\alpha}(|z|) \leq W(z) \leq \overline{\alpha}(|z|), \quad \frac{\partial W}{\partial z} f(z, \alpha(z)) \leq -w(|z|), \quad \forall\, z \in \mathbb{R}^n.$$

Let the Technical Assumption $i)$ and $ii)$ hold (they will be removed) and the high-gain observer rewritten in a more general form as

$$\dot{\hat{z}} = \underbrace{f(\hat{z}, u)}_{\text{copy of the plant}} + \underbrace{G(y - h(\hat{z}, u))}_{\text{output-injection}}. \tag{50}$$

Remark (From state feedback $u = \alpha(z)$ to output feedback $u = \alpha(\hat{z})$)

In the linear case the **separation principle** holds, then it is possible to design the observer and the control law independently and let $u = \alpha(\hat{z})$.
In the nonlinear case the stability analysis of the closed loop system with $u = \alpha(\hat{z})$ have to be performed taking into account the estimation error dynamics.
In the latter case, in fact, exponential convergent estimates could even lead to finite escape time.

Saturation and peaking phenomenon

Note that the estimation error e is related to $\tilde{e}$ by

$$e = \hat{z} - z = \mathrm{diag}\{k, k^2, \ldots, k^n\}\tilde{e} = D_k \tilde{e},$$

yielding

$$\dot{z} = f(z, \alpha(\hat{z})) = f(z, \alpha(z + D_k \tilde{e})). \tag{51}$$

For large k, the estimation error $e = D_k \tilde{e}$ exhibits transients with large amplitude values. To cope with the issues induced by the peaking phenomenon, the input u is "saturated" as

$$u = \sigma_\gamma(\alpha(\hat{z})),$$

where $\sigma_\gamma(s) = s$ for all $s : |s| \leq \gamma$ and is bounded elsewhere such that $\sigma_\gamma(s) \leq 2\gamma$. Then, **global** stability is lost (in general) and only **semi-global** stability is guaranteed.

On the other way around, it will be proven that the input saturation guarantees certain degree of decoupling between the observer design (selection of the parameter k) and the output feedback law.

The closed-loop analysis

Consider the system in closed loop with the observer, i.e.

$$\dot{z} = f(z, \sigma_\gamma(\alpha(\hat{z}))) = f(z, \sigma_\gamma(\alpha(z + D_k\tilde{e}))) = F(z) + H(z, \tilde{e}), \quad (52)$$

$$\dot{\hat{z}} = f(\hat{z}, \sigma_\gamma(\alpha(\hat{z}))) + G(h(z, \sigma_\gamma(\alpha(\hat{z}))) - h(\hat{z}, \sigma_\gamma(\alpha(\hat{z})))), \quad (53)$$

where $F(z) = f(z, \alpha(z))$, $\quad H(z, \tilde{e}) = f(z, \sigma_\gamma(\alpha(z + D_k\tilde{e}))) - f(z, \alpha(z))$.

Select $R > 0$ such that $(z(0), \tilde{e}(0)) \in \mathcal{A} := \mathcal{B}_R \times \mathcal{B}_R$ (then $\hat{z}(0) = 0$), where $\mathcal{B}_R := \{x \in \mathbb{R}^n : |x| \leq R\}$, and

$$\Omega_c := \{z \in \mathbb{R}^n : W(z) \leq c\} \supset \mathcal{B}_R, \quad \Omega_{c+1} \supset \Omega_c \supset \mathcal{B}_R.$$

To leave unchanged the original control law within the set Ω_{c+1}, pick

$$\gamma = \max_{z \in \Omega_{c+1}} \{\alpha(z)\} + 1.$$

Then there exist positive numbers M_0, M_1, δ and M_2 depending on R (the set Ω_c and Ω_{c+1} depend on R) such that

$$\underbrace{\|H(z, \tilde{e})\| \leq M_0,}_{\forall z \in \Omega_{c+1},\, \tilde{e} \in \mathbb{R}^n} \quad \underbrace{\|H(z, \tilde{e})\| \leq M_1 \|D_k \tilde{e}\|,}_{\forall z \in \Omega_{c+1},} \quad \underbrace{\left\|\frac{\partial W}{\partial z}\right\| \leq M_2}_{\forall z \in \Omega_{c+1}}.$$

The closed-loop analysis: picture of the extended state space

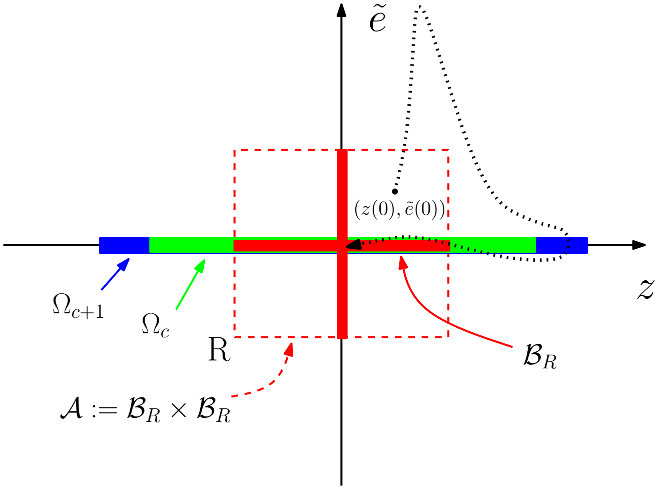

Figure: The extended state space when $\dim(z) = \dim(\tilde{e}) = 1$. The set $\mathcal{B}_R$ in this case is simply a segment.

The closed-loop analysis: the saturation benefits

Due to the saturated input, regardless the values of $\tilde{e}(t)$, as long as $z(t) \in \Omega_{c+1}$, it holds

$$\dot{W}(z(t)) = \frac{\partial W}{\partial z}\left(F(z) + H(z,\tilde{e})\right) \leq -w(|z|) + \frac{\partial W}{\partial z} M_0 \leq -w(|z|) + M_2\, M_0,$$

Denoting with $M = M_2 M_0$, the above inequality yields

$$\dot{W}(z(t)) \leq M \Rightarrow W(z(t)) \leq W(z(0)) + M\, t \Rightarrow z(t) \in \Omega_{c+1}\, \forall\, t \in [0, 1/M), \tag{54}$$

that is obtained by imposing $W(T) = c + 1 = W(0) + MT = c + MT$. Note that the time $T = 1/M$ at which $z(t)$ might leave (a worst case analysis has been performed) the set Ω_{c+1} does not depend on k but on R.

$$\Downarrow$$

The $z(t)$, thanks to the function $\sigma_\gamma(\cdot)$ selected, remains bounded (within the set Ω_{c+1}) at least a time T.

The closed-loop analysis: the estimation error convergence

We recall that

$$V(\tilde{e}) = \tilde{e}' S \tilde{e}, \quad \dot{V}(\tilde{e}(t)) \leq -c_k\, V(\tilde{e}(t)), \text{ with } c_k := k\,\lambda - 2||S||\sqrt{n}.$$

yielding

$$||\tilde{e}(t)||^2 \leq \frac{1}{\underline{\lambda}_S} V(\tilde{e}(t)) \leq \frac{\overline{\lambda}_S}{\underline{\lambda}_S} ||\tilde{e}(0)||^2 e^{-c_k\, t},$$

where $\underline{\lambda}_S ||\tilde{e}||^2 \leq \tilde{e}' S \tilde{e} = V(x) \leq \overline{\lambda}_S ||\tilde{e}||^2$, and $(\underline{\lambda}_S, \overline{\lambda}_S)$ are the smaller and the largest eigenvalues of S, respectively. Then

$$||\tilde{e}(t)|| \leq \sqrt{\frac{\overline{\lambda}_S}{\underline{\lambda}_S}} ||\tilde{e}(0)|| e^{-\frac{c_k}{2}\, t},$$

and with the suggested selection $\hat{z}(0) = 0$ and $k \geq 1$ yields

$$\|D_k \tilde{e}(t)\| \leq k^n \sqrt{\frac{\overline{\lambda}_S}{\underline{\lambda}_S}}\, R\, e^{-\frac{c_k}{2}\, t}. \tag{55}$$

It is now clear that, due to the convergence properties of the exponential which depends on k, it is possible to select a sufficiently high value $k^\star$ such that if $k \geq k^\star$ then $\|D_k \tilde{e}(t)\| \leq \varepsilon$ for any desired $\varepsilon > 0$ before $z(t)$ leaves the set Ω_{c+1}, i.e. for $t < T = 1/M$.

The closed-loop analysis: the estimation error convergence

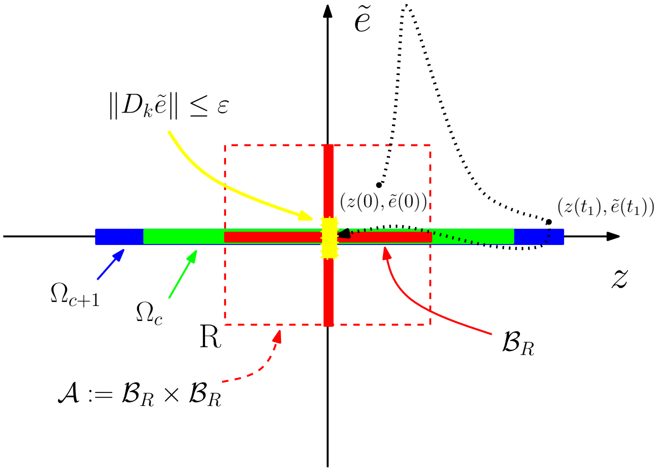

Figure: The set in yellow is such that $\|D_k\tilde{e}\| \leq \varepsilon$ and there exists a sufficiently high value $k^\star$ such that if $k \geq k^\star$ then $\|D_k\tilde{e}(t)\| \leq \varepsilon$ for all $t \geq t_1$ with $t_1 < T$.

The closed-loop analysis: finite time convergence

Let now $\varepsilon \leq \delta$, then the previous inequality

$$\dot{W}(z(t)) \leq -w(|z(t)|) + M_2\, M_1\, \delta$$

certainly holds true if $t \leq T$. Consider the "annular" compact set

$$\Omega^d_{c+1} := \{z \in \mathbb{R}^n : d \leq W(z) \leq c+1\},$$

with $d \ll c$ and define its "inner radius" r as

$$r := \min_{z \in \Omega^d_{c+1}} \{\|z\|\}.$$

Since $w(\cdot) \in \mathcal{K}_\infty$, then $w(\|z\|) \geq w(r)$ for all $z \in \Omega^d_{c+1}$. This yield the existence of a **sufficiently small values of** δ such that

$$M_2 M_1 \delta \leq \frac{1}{2} w(r) \;\Rightarrow\; \dot{W}(z(t)) \leq -\frac{1}{2} w(r).$$

Hence $z(t)$ not only **does not leave** the set Ω_{c+1}, but it **enters in finite time** the set Ω_d. We have proved so far that if $(z(0), \tilde{e}(0)) \in \mathcal{A}$, saturating opportunely the input and selecting a sufficiently high value of k, then $\tilde{e}(t)$ decays to zero and $z(t)$ remains bounded.

The closed-loop analysis: $\omega-$limit set

To conclude, the previous analysis highlight the existence of an $\omega-$limit set and since $\tilde{e}(t)$ converges to zero and the restriction of the closed-loop system to $\tilde{e} = 0$ is

$$\dot{z} = f(z, \sigma_\gamma(\alpha(z))) = f(z, \alpha(z)),$$

in which $z = 0$ is GAS, then the $\omega-$limit set is constituted just by the origin of the extended state space $(z, \tilde{e}) = (0, 0)$.

A separation principle

A separation principle

Consider the system (49) and assume that there exists a feedback law $u = \alpha(z)$ that renders the origin $z = 0$ GAS. Furthermore, let the Technical Assumptions hold so that the high-gain observer of the form (32) can be considered. Then, there exists a saturation function $\sigma_\gamma(\cdot)$ and a sufficiently large value of k such that the origin of the extended system

$$\dot{z} = f(z, \sigma_\gamma(\alpha(\hat{z}))), \tag{56}$$

$$\dot{\hat{z}} = f(\hat{z}, \sigma_\gamma(\alpha(\hat{z}))) + G(h(z, \sigma_\gamma(\alpha(\hat{z}))) - h(\hat{z}, \sigma_\gamma(\alpha(\hat{z})))), \tag{57}$$

is semi-globally (with respect to $z(0)$) asymptotically stable.

Remark

The requirement on the Technical Assumptions can be relaxed. In fact, it has been proven that, independently of $\tilde{e}(t)$, the saturated control $u = \sigma_\gamma(\alpha(z))$ yield $z(t) \in \Omega_{c+1}$ for at least $t \leq T = 1/M$. Then, evaluating α and β of the Technical Assumption over the set Ω_{c+1}, allows to completely define the high-gain observer.

Observers: A different point of view...

Consider the continuous time nonlinear system described by

$$\dot{x} = f(x, u)\,, \tag{58}$$

$$y = h(x, u)\,, \tag{59}$$

where $x(t) \in \mathbb{R}^n$ is the state variable, $u(t) \in \mathbb{R}^m$ denotes the input and $y(t) \in \mathbb{R}^p$ is the output of the system.

The input u is **piece-wise constant**, as produced by a sample-and-hold device, with sampling time T.

Observers: A different point of view...

Define as y_k the sampled value of the output such as $y_i := y(t_i)$, with $t_{i+1} - t_i = T$, for all $i \geq 0$. Similarly define the sampled input $u_i := u(t_i)$.

Let $Y_i^q \in \mathbb{R}^{qp}$ and $U_i^q \in \mathbb{R}^{qm}$ be defined as

$$Y_i^q = \begin{bmatrix} y_{i-N+1} \\ \vdots \\ y_{i-N+q} \end{bmatrix}, \quad U_i^q = \begin{bmatrix} u_{i-N+1} \\ \vdots \\ u_{i-N+q} \end{bmatrix}.$$

Moreover, define $H(x_{i-N+1}, U_i^N) : \mathbb{R}^{n \times Nm} \mapsto \mathbb{R}^{Np}$ as

$$H(x_{i-N+1}, U_i^N) \triangleq \begin{bmatrix} h(x_{i-N+1}, u_{i-N+1}) \\ h(F_T(x_{i-N+1}, U_i^1), u_{i-N+2}) \\ \vdots \\ h(F_{(N-1)T}(x_{i-N+1}, U_i^{N-1}), u_i) \end{bmatrix}, \tag{60}$$

with $Y_i^N = H(x_{i-N+1}, U_i^N)$, where F_{qT} is such that $x_{i-N+1+q} = F_{qT}(x_{i-N+1}, U_i^q)$.

Note that in general the analytical expression of F is unknown.

Observers: A different point of view...minimization!

The estimation problem can be reformulated as follows.

Estimation problem: Find the value $\hat{x}_{i-N+1}$ such that

$$Y_i^N - H(\hat{x}_{i-N+1}, U_i^N) = 0, \tag{61}$$

holds, and select a suitable N (assuming that it exists[4]) such that if (61) holds, than $\hat{x}_k = x_k$ for all $k \geq i - N + 1$. $\square$

The problem of estimation can be recast into a multi-parametric minimization problem for $V_i(Y_i^N - H(\hat{x}_{i-N+1}, U_i^N))$, with a positive definite $V : \mathbb{R}^N \to \mathbb{R}_{\geq} 0$ such that V_i is zero iff $\hat{x}_{i-N+1} = x_{i-N+1}$.

The map $F_{qT}(\cdot)$ is not known in general, but it can be numerically evaluated integrating the system vector field between sampling times.

Typically $N \geq 2n$ works fine, but a larger N should be considered when the measurements are affected by noise.

[4] Then the system is defined **N-observable**

Considerations

- There are available a great number of techniques for multi-parametric minimization problems (line search, Gradient-like, Newton-like, Monte Carlo methods ecc...)
- The computational cost of the selected method should be carefully considered since it has to run on-line (and generically has to converge, or at least give a "better approximation" with respect to the previous one within a time T...)
- Gradient and Newton algorithms can not be used in general since the map $F_{qT}(\cdot)$ is not known (so its explicit dependence by x_{i-N+1} is unknown) and approximation methods have to be considered.
- Generally the "course of dimensionality" affects this (brutal force) approach.

...nevertheless, it is quite effective in a number of applications.

This observer could be exploited to define a "good" initial condition for your more sophisticated observer (a local, semi-global,...).

...

The goal is:

$$\text{find } \hat{x} \text{ such that } \hat{x} = \operatorname{argmin} V_i(x).$$

Gradient algorithm:

$$\dot{\hat{x}}(t) = -\gamma \frac{\partial V_i(\hat{x}(t))}{\partial \hat{x}(t)}, \quad \hat{x}(k+1) = x(k) - \gamma \frac{\partial V_i(\hat{x}(k))}{\partial \hat{x}(k)},$$

If V_i is **strictly quasi-convex** the convergence toward the **unique minimum** is **global** (it may be slow especially around the minimum and in discrete time the chattering phenomenon is likely to happen).

Newton algorithm:

$$\hat{x}(k+1) = \hat{x}(k) - \left(\frac{\partial^2 V_i(\hat{x}(k))}{\partial^2 \hat{x}(k)} \right)^{-1} \frac{\partial V(\hat{x}(k))}{\partial \hat{x}(k)},$$

The global convergence is not guaranteed in general. However, under some technical assumptions (see for example the Kantorovich's Theorem), (locally) the algorithm **converges quadratically**. When the Jacobian of V_i is not square, the Penn-Rose pseudo-inverse has to be considered.