Nonlinear Systems and Control
Lecture # 40

Observers

High-Gain Observers
Stabilization

$$\begin{aligned}
\dot{x} &= Ax + B\phi(x, z, u) \\
\dot{z} &= \psi(x, z, u) \\
y &= Cx \\
\zeta &= q(x, z)
\end{aligned}$$

$$u \in R^p,\ y \in R^m,\ \zeta \in R^s,\ x \in R^\rho,\ z \in R^\ell$$

A, B, C are block diagonal matrices

$$A_i = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}_{\rho_i \times \rho_i}, \quad B_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{\rho_i \times 1}$$

$$C_i = \left[\begin{array}{ccccc} 1 & 0 & \cdots & \cdots & 0 \end{array}\right]_{1\times\rho_i}, \quad \rho = \sum_{i=1}^{m} \rho_i$$

- Normal form
- Mechanical and electromechanical systems

Example: Magnetic Suspension

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= g - \frac{k}{m}x_2 - \frac{L_0 a x_3^2}{2m(a+x_1)^2} \\
\dot{x}_3 &= \frac{1}{L(x_1)}\left[-Rx_3 + \frac{L_0 a x_2 x_3}{(a+x_1)^2} + u\right]
\end{aligned}$$

Stabilizing (partial) state feedback controller:

$$u = \gamma(x, \zeta)$$

$$\dot{\vartheta} = \Gamma(\vartheta, x, \zeta), \quad u = \gamma(\vartheta, x, \zeta)$$

Closed-loop system under state feedback:

$$\dot{\mathcal{X}} = f(\mathcal{X}), \quad \mathcal{X} = (x, z, \vartheta)$$

The origin of $\dot{\mathcal{X}} = f(\mathcal{X})$ is asymptotically stable

Observer:

$$\dot{\hat{x}} = A\hat{x} + B\phi_0(\hat{x}, \zeta, u) + H(y - C\hat{x})$$

H is block diagonal

$$H_i = \begin{bmatrix} \alpha_1^i/\varepsilon \\ \alpha_2^i/\varepsilon^2 \\ \vdots \\ \alpha_{\rho_i-1}^i/\varepsilon^{\rho_i-1} \\ \alpha_{\rho_i}^i/\varepsilon^{\rho_i} \end{bmatrix}_{\rho_i\times 1}$$

$$s^{\rho_i} + \alpha_1^i s^{\rho_i-1} + \cdots + \alpha_{\rho_i-1}^i s + \alpha_{\rho_i}^i$$

is Hurwitz and $\varepsilon > 0$ (small)

$\phi_0(x, \zeta, u)$ is a nominal model of $\phi(x, z, u)$, which is globally bounded in x

Theorem 14.6 (Nonlinear Separation Principle:

Suppose the origin of $\dot{\mathcal{X}} = f(\mathcal{X})$ is asymptotically stable and $\mathcal{R}$ is its region of attraction. Let $\mathcal{S}$ be any compact set in the interior of $\mathcal{R}$ and $\mathcal{Q}$ be any compact subset of R^{ρ}. Then,

- $\exists\ \varepsilon_1^* > 0$ such that, for every $0 < \varepsilon \le \varepsilon_1^*$, the solutions $(\mathcal{X}(t), \hat{x}(t))$ of the closed-loop system, starting in $\mathcal{S} \times \mathcal{Q}$, are bounded for all $t \ge 0$

- given any $\mu > 0$, $\exists\ \varepsilon_2^* > 0$ and $T_2 > 0$, dependent on μ, such that, for every $0 < \varepsilon \le \varepsilon_2^*$, the solutions of the closed-loop system, starting in $\mathcal{S} \times \mathcal{Q}$, satisfy

$$\|\mathcal{X}(t)\| \le \mu \quad \text{and} \quad \|\hat{x}(t)\| \le \mu, \quad \forall\ t \ge T_2$$

- given any $\mu > 0$, $\exists\ \varepsilon_3^* > 0$, dependent on μ, such that, for every $0 < \varepsilon \leq \varepsilon_3^*$, the solutions of the closed-loop system, starting in $\mathcal{S} \times \mathcal{Q}$, satisfy

$$\|\mathcal{X}(t) - \mathcal{X}_r(t)\| \leq \mu, \quad \forall\, t \geq 0$$

  where $\mathcal{X}_r$ is the solution of $\dot{\mathcal{X}} = f(\mathcal{X})$, starting at $\mathcal{X}(0)$

- if the origin of $\dot{\mathcal{X}} = f(\mathcal{X})$ is exponentially stable, then $\exists\ \varepsilon_4^* > 0$ such that, for every $0 < \varepsilon \leq \varepsilon_4^*$, the origin of the closed-loop system is exponentially stable and $\mathcal{S} \times \mathcal{Q}$ is a subset of its region of attraction.

Key ideas of the proof:

- Representation of the closed-loop system as a singularly perturbed one with $\mathcal{X}$ as the slow and η (scaled estimation error) as the fast
- Use of a converse Lyapunov theorem to construct positively invariant sets
- Use of global boundedness in $\hat{x}$ to show that η reaches $O(\varepsilon)$ while $\mathcal{X}$ is inside a positively invariant set
- Nonlocal versus local analysis

Novel Feature: Performance recovery

Example 14.19:

$$m\ell^2\ddot{\theta} + mg_0\ell\sin\theta + k_0\ell^2\dot{\theta} = u$$

Stabilization at $(\theta = \pi, \dot{\theta} = 0)$. From Section 14.1

$$u = -k \,\mathrm{sat}\left(\frac{a_1(\theta - \pi) + \dot{\theta}}{\mu}\right)$$

Suppose we only measure θ

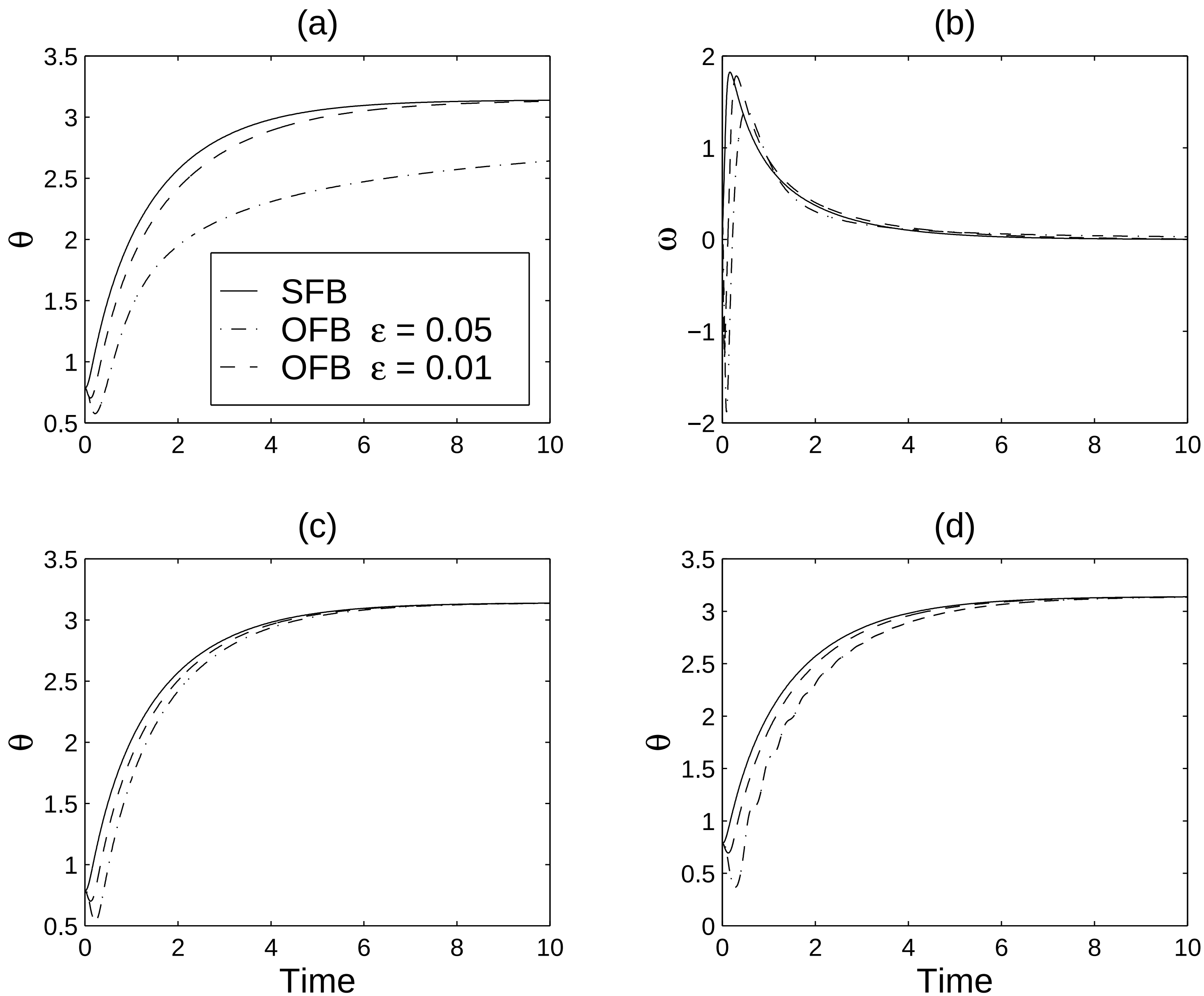
$$\begin{aligned}\dot{\hat{\theta}} &= \hat{\omega} + (2/\varepsilon)(\theta - \hat{\theta}) \\ \dot{\hat{\omega}} &= \phi_0(\hat{\theta}, u) + (1/\varepsilon^2)(\theta - \hat{\theta})\end{aligned}$$

$$\phi_0 = -\hat{a}\sin\hat{\theta} + \hat{c}u$$

$$u = -k \,\mathrm{sat}\left(\frac{a_1(\hat{\theta} - \pi) + \hat{\omega}}{\mu}\right)$$

or

$$u = -k \,\mathrm{sat}\left(\frac{a_1(\theta - \pi) + \hat{\omega}}{\mu}\right)$$


(a)
(b)
(c)
(d)
θ
ω
Time
SFB
OFB ε = 0.05
OFB ε = 0.01