Hard & Very Hard Functions - Save My Exams

Question 1

HARD MARKS: 4

State whether the following mappings are one-to-one, many-to-one, one-to-many or many-to-many.

(i) $f: x \mapsto 2 - x^3$ one to one

(ii) $f: x \mapsto \sin x$ Many to one

(iii) $f: x \mapsto \frac{1}{x^2}$ Many to one

(iv) $f: x \mapsto \ln x$ one to one

Question 2a

HARD MARKS: 1

It is given

$$f(x) = \frac{2}{x}$$

(a) Write down the domain of the function $f(x)$.

$\mathbb{R} - \{0\}$

Question 2b

HARD MARKS: 3

(b) Sketch the graph of $y = f(x)$, stating the coordinates of any intersections with the coordinate axes and the equations of any asymptotes.

$y = \frac{2}{x} \quad x > 0$

$y = \frac{2}{x} \quad x < 0$

Question 2c

HARD MARKS: 1

(c) Write down the range of $f(x)$.

$f(x) \in \mathbb{R} - \{0\}$

Question 3a

HARD MARKS: 1

The function $f(x)$ is defined as

$$f(x) = x(x+3)^2 + 1 \qquad x \geq 0$$

(a) Work out the range of $f(x)$.

$f(0) = 0(0+3)^2 + 1$

$f(0) = 1$

$f(x) \geq 1$

Question 3b

HARD MARKS: 2

(b) If the domain of $f(x)$ is changed to $x \leq 0$, what is the range of $f(x)$?

$f(x) \leq 1$

Question 4a

HARD MARKS: 1

The functions $f(x)$ and $g(x)$ are defined as follows

$$f(x) = 3x^2 + 2 \qquad x \in \mathbb{R}$$
$$g(x) = 1 - 3x \qquad x \in \mathbb{R}$$

(a) Write down the range of $f(x)$.

$f(x) \geq 2$

$3(1-3x)^2+2$, $1-3(3x^2+2)$

Question 4b

HARD MARKS: 4

(b) Find (i) $fg(x)$
(ii) $gf(x)$

1) $3(1-6x+9x^2)+2$
$3-6x+27x^2+2$
$27x^2-6x+5$

2) $1-3(3x^2+2)$
$1-9x^2-6$
$-(9x^2+5)$

Question 4c

HARD MARKS: 2

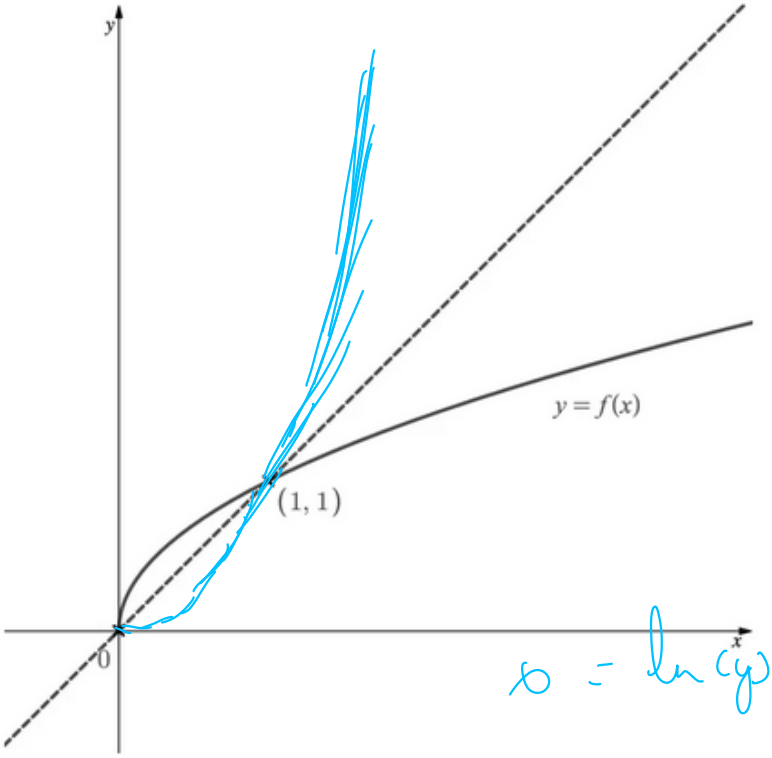
(c) Solve the equation $f(x) = g(x) + 1$

$$3x^2 + 2 = 1 - 3x + 1$$
$$3x^2 = -3x$$
$$3x = -3$$
$$\boxed{x = -1}$$

Question 5a

HARD MARKS: 3

The graph of $y = f(x)$ is shown below.



$x = \ln(y) + 2$

(a) (i) Use the graph to write down the domain and range of $f(x)$.
(ii) Write down the equation of the dotted line on the graph.

$x \geq 0$

$f(x) \geq 0$

$y = b$

Question 5b

HARD MARKS: 2

(b) On the diagram above sketch the graph of $y = f^{-1}(x)$.

Question 6a

HARD MARKS: 3

The functions $f(x)$ and $g(x)$ are defined as follows

$$f(x) = e^{x-2} \qquad x \in \mathbb{R}$$
$$g(x) = 2 + \ln x \qquad x \in \mathbb{R}, x > 0$$

(a) Find (i) $fg(x)$
(ii) $gf(x)$

Confirmed

$f(x) = e^{x-2}$

$g(x) = 2 + \ln(x)$

$fg(x) = e^{\ln(x)}$

$fg(x) = x$ where $x > 0$

$gf(x) = 2 + \ln(e^{x-2})$

$gf(x) = 2 + x - 2$

$= x$

Question 6b

HARD MARKS: 2

(b) Write down $f^{-1}(x)$ and state its domain and range.

$f(x) = e^{x-2}$

$x = e^{y-2}$

~~$\ln x = y - 2 + \ln(e)$~~

$\ln x + 2 = y$

$x > 0$

$f(x) \in \mathbb{R}$

Question 6c

HARD MARKS: 2

(c) The graphs of $f(x)$ and $f^{-1}(x)$ are drawn on the same axes.
Describe the transformation that would map one graph onto the other.

$f(x)$ is reflection of $f^{-1}(x)$ against the line $y = x$

Question 1a

V. HARD MARKS: 2

It is given

$$f(x) = 4x^3 + 4x^2 - 7x + 2$$

(a) Write down the domain and range of the function $f(x)$.

$x \in \mathbb{R}$
$f(x) \in \mathbb{R}$

Question 1b

V. HARD MARKS: 3

(b) Sketch the graph of $y = f(x)$, stating the coordinates of any intersections with the coordinate axes. (You do not need to give the coordinates of any turning points.)

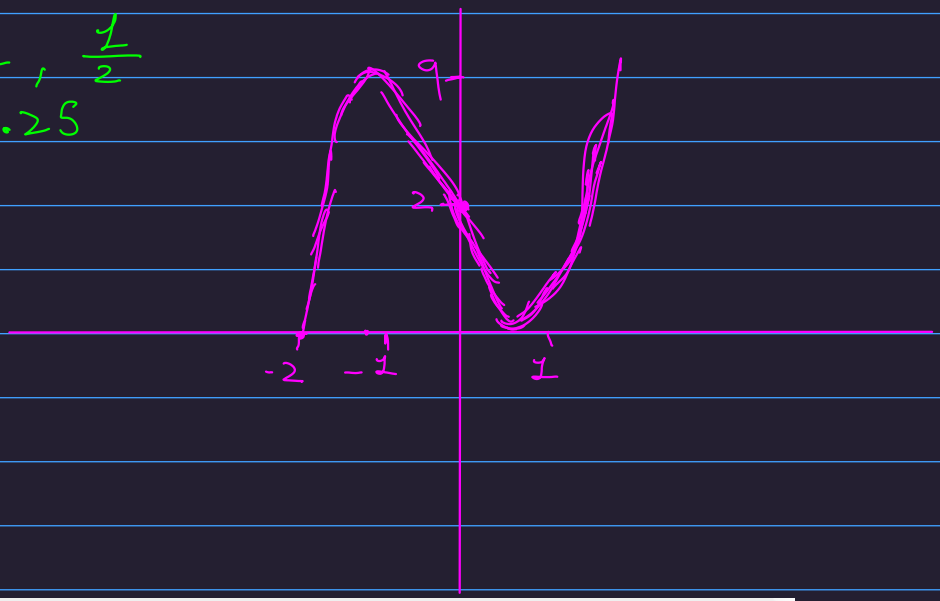
Assess your score

$4x^3 + 4x^2 - 7x + 2$

y intercept = 2

x intercept = $-2, \frac{1}{2}$

mid-point = -1.25



Question 2a

V. HARD MARKS: 2

The function $f(x)$ is defined as

$$f(x) = (x-3)^2(x-4)^2 \qquad 2 \le x \le 5$$

(a) Work out the range of $f(x)$.

$0 \le f(x) \le 4$

Question 2b

V. HARD MARKS: 1

(b) If the domain of $f(x)$ is changed to $x \leq 2$, what is the range of $f(x)$?

$f(x) \geq 4$

Question 2c

V. HARD MARKS: 1

(c) State another domain for $f(x)$ that would have the same effect as that in part (b).

$b \geq 5$

$f(x) \geq 0$

Question 3a

V. HARD MARKS: 1

The functions $f(x)$ and $g(x)$ are defined as follows

$$f(x) = x^2 - 2 \qquad x \in \mathbb{R}$$
$$g(x) = 1 - \frac{2}{x} \qquad x \in \mathbb{R}, x \neq 0$$

(a) Write down the range of $f(x)$.

$f(x) \geq 0$

Question 3b

V. HARD MARKS: 3

(b) Leaving your answers as single fractions, find

(i) $fg(x)$
(ii) $gf(x)$

$$fg(x) = \left(1 - \frac{2}{x}\right)^2 - 2$$
$$= 1 - \frac{4}{x} + \frac{4}{x^2} - 2$$
$$= \frac{4 - 4x - x^2}{x^2}$$

$$gf(x) = 1 - \frac{2}{x^2 - 2}$$
$$gf(x) = \frac{x^2 - 2 - 2}{x^2 - 2} = \frac{x^2 - 4}{x^2 - 2}$$

Question 3c

V. HARD MARKS: 2

(c) Solve the equation $f(x) = g(x)$

$f(b) = b^2 - 2$

$g(b) = 1 - \frac{2}{b}$

$b^2 - 2 = 1 - \frac{2}{b}$

$b(b^2 - 3) = -2$

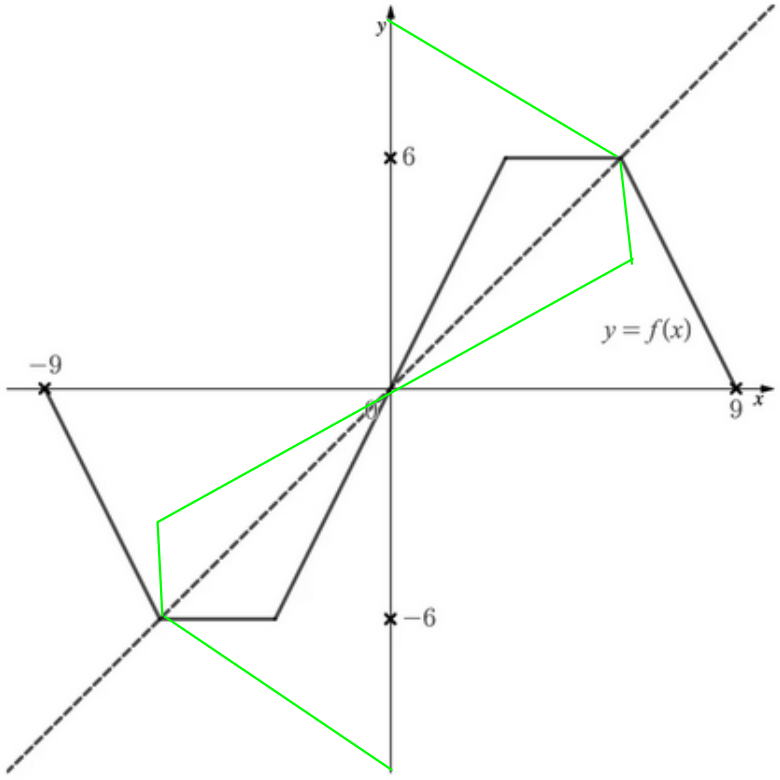
$b^3 - 3b + 2 = 0$

$b = -2, \quad b = 1$

Question 4a

V. HARD MARKS: 3

The graphs of $y = f(x)$ and $y = x$ (dotted line) are shown in the diagram below.



(a) (i) Use the graph to write down the domain and range of $f(x)$.

(ii) Hence, or otherwise, write down the domain and range of $f^{-1}(x)$.

$-9 \leq x \leq 9$

$-6 \leq f(x) \leq 6$

$-9 \leq f(x)^{-1} \leq 9$

$-6 \leq x \leq 6$

Question 4b

V. HARD MARKS: 2

(b) On the diagram above sketch the graph of $y = f^{-1}(x)$.

Is it even possible?

Question 5a

V. HARD MARKS: 1

The function $f(x)$ is defined as

$$f: x \mapsto \sqrt{25 - x^2} \qquad x \in \mathbb{R}, -5 \leq x \leq 5$$

(a) Explain why the inverse of $f(x)$ does not exist.

Only One - One functions has inverse
Many - One does not exist

$f(4) = 3$
$f(-4) = 3$

Many to One

Question 5b

V. HARD MARKS: 2

(b) Suggest an adaption to the domain of $f(x)$ so the following conditions are met:

- the inverse of $f(x)$ exists,
- the graph of $y = f(x)$ lies in the first quadrant only, and,
- the domain of $f(x)$ is as large as possible.

State the range for your adapted $f(x)$.

$\sqrt{25 - b^2}$

$f(b) < 5$

$b > 0$

Question 5c

V. HARD MARKS: 3

(c) The domain of $f(x)$ is changed to $-5 \leq x \leq 0$.

Find an expression for $f^{-1}(x)$ and state its domain and range.

$y = \sqrt{25 - b^2}$

$\sqrt{25 - b^2} = y$

Domain $= 0 \leq b \leq 5$

$= -5 \leq f(b) \leq 0$

Question 6a

V. HARD MARKS: 3

The functions $f(x)$ and $g(x)$ are defined as follows

$$f(x) = (x-1)^2 - 4 \qquad x \in \mathbb{R}, x \geq 1$$
$$g(x) = 1 + \sqrt{x+4} \qquad x \in \mathbb{R}, x \geq -4$$

(a) Find (i) $fg(x)$
(ii) $gf(x)$

Question 6b

V. HARD MARKS: 2

(b) Write down $f^{-1}(x)$ and state its domain and range.

$b = (y-1)^2 - 4$

$\sqrt{b+4} + 1 = y$

$D_{f^{-1}} \in \mathbb{R} \quad b \geq 1$

$b \geq -4$

Question 6c

V. HARD MARKS: 2

(c) The graphs of $f(x)$ and $f^{-1}(x)$ are drawn on the same axes.
Describe the transformation that would map one graph onto the other.

Question 6d

V. HARD MARKS: 2

(d) Find the coordinates of the point where the graphs of $y = f(x)$ and $y = f^{-1}(x)$ meet.

$$x = (x-1)^2 - 4$$
$$(x-1)^2 - 4 - x$$
$$x^2 - 2x + 1 - 4 - x$$
$$x^2 - 3x - 3$$