Satellite Constellation Scheduling to MILP/MIQP Problem

1. Given : Contact Matrix without SAT Operational Constraint

- From Satellite constellation simulation, we could get unconstrained SAT–GS contact chart.
- Index for every contact constant variable

$$A = \begin{array}{c|c|c} \text{SAT} & \text{GS} & t_m \\ \hline a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & a_{N3} \end{array}$$

Total number of contacts: N

Number of SATs: $p \quad \rightarrow \quad a_{i1} \in \{1, 2, \ldots, p\}$

Number of GS: $q \quad \rightarrow \quad a_{i2} \in \{1, 2, \ldots, q\}$

Aceding t_m: $a_{14} \le a_{24} \le \ldots \le a_{N4}$

2. Key parameters for the input of MILP/MIQP problem

- Unconstrained Satellite contact matrix: A
- Time Vector: $\boldsymbol{t} = A(:, 3)$
- Initial and Final time $t_{\text{start}}, t_{\text{end}}$
- Satellite Cadence Constraint: τ
- Number of SATs: p,   Number of GSs: q
- Total number of contact: N
- Number of contact for each SAT: $|S_i|$
- Number of contact for each GS: $|G_j|$
- Optimization variable: $\boldsymbol{x} \in \{0, 1\}^N$

3. Selection matrix generation from given constant parameters

3.1. $E^1_{S_i}$: Selection matrix from A-matrix to each satellite's contact sequence

$E^1_{S_i} : |S_i| \times N$ matrix,   Satellite index $\quad i = 1, \ldots, p$

1. Initialize: $\quad \text{zeros}(|S_i|, N)$
2. List row index: $\quad A(:, 1)$ `==` $i \quad \Rightarrow a_{i1}, a_{i2}, \ldots, a_{i|S_i|}$
3. $E^1_{Si}(1, a_{i1}) = 1, \quad E^1_{Si}(2, a_{i1}) = 1 \quad \ldots \quad E^1_{Si}(|S_i|, a_{i1}) = 1$

$\Rightarrow$ We have E^1_{Si}: $E^1_{S1}, E^1_{S2}, \ldots, E^1_{Sp}$ which configures $\quad \boldsymbol{x}_{Si} = E^1_{Si}\boldsymbol{x}, \quad \boldsymbol{t}_{Si} = E^1_{Si}\boldsymbol{t}$

3.2. $E^2_{S_i,x}, E^2_{S_i,t}$: Selection matrix for Δt of each satellite from $|S_i|$

$$E^2_{S_i,x} = \frac{|S_i|\,(|S_i|-1)}{2} \times |S_i| \text{ matrix}, \quad \text{SAT index} \quad i = 1, \ldots, p$$

for $\alpha = 1 \ldots |S_i| - 1$

$E^2_{S_i,x,\alpha} = \text{zeros}\,(|S_i| - \alpha,\, |S_i|)$

$E^2_{S_i,x,\alpha}(:,\, \alpha) = 1$

for $\beta = \alpha + 1 \ldots |S_i|$

$E^2_{S_i,x,\alpha}(\beta - \alpha,\, \beta) = 1$

end

$$E^2_{S_i,x} = \begin{bmatrix} E^2_{S_i,x} \\ E^2_{S_i,x,\alpha} \end{bmatrix}$$

end

$\Rightarrow$ We have $E^2_{S_i,x} : E^2_{S_1,x},\, E^2_{S_2,x},\, \ldots,\, E^2_{S_p,x}$ from $|S_i| : |S_1|, |S_2|, \ldots, |S_p|$

$$E^2_{S_i,t} = \frac{|S_i|\,(|S_i|-1)}{2} \times |S_i| \text{ matrix}, \quad \text{SAT index} \quad i = 1, \ldots, p$$

for $\alpha = 1 \ldots |S_i| - 1$

$E^2_{S_i,t,\alpha} = \text{zeros}\,(|S_i| - \alpha,\, |S_i|)$

$E^2_{S_i,t,\alpha}(:,\, \alpha) = -1$

for $\beta = \alpha + 1 \ldots |S_i|$

$E^2_{S_i,t,\alpha}(\beta - \alpha,\, \beta) = 1$

end

$$E^2_{S_i,t} = \begin{bmatrix} E^2_{S_i,t} \\ E^2_{S_i,t,\alpha} \end{bmatrix}$$

end

$\Rightarrow$ We have $E^2_{S_i,t} : E^2_{S_1,t},\, E^2_{S_2,t},\, \ldots,\, E^2_{S_p,t}$ from $|S_i| : |S_1|, |S_2|, \ldots, |S_p|$

3.3. $E^1_{G_j}$: Selection matrix from A-matrix to each Ground Point's revisit sequence

$E^1_{G_j}$: $(|G_j|+2)\times(N+2)$ matrix, GS index $j = 1,\dots,q$

1. Initialize: $\text{zeros}(|G_j|+2, N+2)$
2. List row index: $A(:,2)$ `==` j $\Rightarrow b_{j1},\, b_{j2},\, \dots,\, b_{j|G_j|}$
3. $E^1_{G_j}(1,1) = 1$
4. for $\alpha = 1\dots|G_j|$

   $E^1_{G_j}(\alpha+1,\, b_{j\alpha}+1) = 1$

   end
5. $E^1_{G_j}(|G_j|+2,\, N+2) = 1$

$\Rightarrow$ We have $E^1_{G_j}$: $E^1_{G_1},\, E^1_{G_2},\, \dots,\, E^1_{G_q}$ from $\boldsymbol{x}, \boldsymbol{t}, t_{start}, t_{end}$ to $\boldsymbol{x}_{G_j}, \boldsymbol{t}_{G_j}$

$$\boldsymbol{x}_{G_j} = E^1_{G_j}\begin{bmatrix} 1 \\ \boldsymbol{x} \\ 1 \end{bmatrix} = E^1_{G_j}\left\{\begin{bmatrix} \mathbf{0}_{1\times N} \\ \mathbf{I}_{N\times N} \\ \mathbf{0}_{1\times N} \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 1 \\ \mathbf{0}_{N\times 1} \\ 1 \end{bmatrix}\right\}$$

$$\boldsymbol{x}_{G_j} = \underbrace{E^1_{G_j}\begin{bmatrix} \mathbf{0}_{1\times N} \\ \mathbf{I}_{N\times N} \\ \mathbf{0}_{1\times N} \end{bmatrix}}_{\text{Constant}}\boldsymbol{x} \;+\; \underbrace{E^1_{G_j}\begin{bmatrix} 1 \\ \mathbf{0}_{N\times 1} \\ 1 \end{bmatrix}}_{\text{Constant}} \quad \Rightarrow \quad \boldsymbol{x}_{G_j} \text{ is affine w.r.t } \boldsymbol{x}$$

$$\boldsymbol{t}_{G_j} = E^1_{G_j}\begin{bmatrix} t_{start} \\ \boldsymbol{t} \\ t_{end} \end{bmatrix} = E^1_{G_j}\left\{\begin{bmatrix} \mathbf{0}_{1\times N} \\ \mathbf{I}_{N\times N} \\ \mathbf{0}_{1\times N} \end{bmatrix}\boldsymbol{t} + \begin{bmatrix} t_{start} \\ \mathbf{0}_{N\times 1} \\ t_{end} \end{bmatrix}\right\}$$

$$\boldsymbol{t}_{G_j} = \underbrace{E^1_{G_j}\begin{bmatrix} \mathbf{0}_{1\times N} \\ \mathbf{I}_{N\times N} \\ \mathbf{0}_{1\times N} \end{bmatrix}}_{\text{Constant}}\boldsymbol{t} \;+\; \underbrace{E^1_{G_j}\begin{bmatrix} t_{start} \\ \mathbf{0}_{N\times 1} \\ t_{end} \end{bmatrix}}_{\text{Constant}} \quad \Rightarrow \quad \boldsymbol{t}_{G_j} \text{ is affine w.r.t } \boldsymbol{t}$$

3.4. $E^2_{G_j,x}, E^2_{G_j,t}$: Selection matrix for Δt of each GS from $|G_j|$

$$E^2_{G_j,x} = \frac{(|G_j|+2)\,(|G_j|+1)}{2} \times (|G_j|+2) \text{ matrix}, \quad \text{GS index} \quad j = 1, \ldots, q$$

for $\alpha = 1 \ldots |G_j|+1$

$E^2_{G_j,x,\alpha} = \text{zeros}\,(|G_j|+2-\alpha,\ |G_j|+2)$

$E^2_{G_j,x,\alpha}(:,\ \alpha) = 1$

for $\beta = \alpha+1 \ldots |G_j|+2$

if $\alpha+1 \le \beta-1$

$E^2_{G_j,x,\alpha}(\beta-\alpha, \alpha+1 \ldots \beta-1) = -1$

end

$E^2_{G_j,x,\alpha}(\beta-\alpha,\ \beta) = 1$

end

$$E^2_{G_j,x} = \begin{bmatrix} E^2_{G_j,x} \\ E^2_{G_j,x,\alpha} \end{bmatrix}$$

end

$\Rightarrow$ We have $E^2_{G_j,x} : E^2_{G_1,x},\ E^2_{G_2,x},\ \ldots,\ E^2_{G_q,x}$ from $|G_j| : |G_1|, |G_2|, \ldots, |S_q|$

$$E^2_{G_j,t} = \frac{(|G_j|+2)\,(|G_j|+1)}{2} \times (|G_j|+2) \text{ matrix}, \quad \text{GS index} \quad j = 1, \ldots, q$$

for $\alpha = 1 \ldots |G_j|+1$

$E^2_{G_j,t,\alpha} = \text{zeros}\,(|G_j|+2-\alpha,\ |G_j|+2)$

$E^2_{G_j,t,\alpha}(:,\ \alpha) = -1$

for $\beta = \alpha+1 \ldots |G_j|+2$

$E^2_{G_j,t,\alpha}(\beta-\alpha,\ \beta) = 1$

end

$$E^2_{G_j,t} = \begin{bmatrix} E^2_{G_j,t} \\ E^2_{G_j,t,\alpha} \end{bmatrix}$$

end

$\Rightarrow$ We have $E^2_{G_j,t} : E^2_{G_1,t},\ E^2_{G_2,t},\ \ldots,\ E^2_{G_q,t}$ from $|G_j| : |G_1|, |G_2|, \ldots, |S_q|$

4. Derivation of L_1, L_2, L_∞ revisit time problem to MILP/MIQP

4.1. Satellite operation cadence constraint to linear inequality

From the big-M method, for each satellite S_i we have the cadence constraint τ:

$$\boldsymbol{t}_{S_i,\ell} - \boldsymbol{t}_{S_i,k} \;\geq\; \tau - \tau\big(2 - \boldsymbol{x}_{S_i,\ell} - \boldsymbol{x}_{S_i,k}\big), \qquad \forall\, \ell > k,\; i = 1,\dots,p$$

We can interpret the above depending on the values of the binary variables:

$$\begin{aligned}
&\text{if } \boldsymbol{x}_{S_i,\ell} = 1,\; \boldsymbol{x}_{S_i,k} = 1 \quad \Rightarrow \quad \boldsymbol{t}_{S_i,\ell} - \boldsymbol{t}_{S_i,k} \;\geq\; \tau \quad \Rightarrow \quad \text{Active constraint,}\\
&\text{if } \boldsymbol{x}_{S_i,\ell} = 1,\; \boldsymbol{x}_{S_i,k} = 0 \quad \Rightarrow \quad \boldsymbol{t}_{S_i,\ell} + \boldsymbol{t}_{S_i,k} \;\geq\; 0 \quad \Rightarrow \quad \text{Non–active constraint,}\\
&\text{if } \boldsymbol{x}_{S_i,\ell} = 0,\; \boldsymbol{x}_{S_i,k} = 0 \quad \Rightarrow \quad \boldsymbol{t}_{S_i,\ell} - \boldsymbol{t}_{S_i,k} \;\geq\; -\tau \;\Rightarrow \quad \text{Non–active constraint}
\end{aligned}$$

From the selection matrices $E^1_{S_i}$ we have

$$\boldsymbol{x}_{S_i} = E^1_{S_i}\boldsymbol{x}, \qquad \boldsymbol{t}_{S_i} = E^1_{S_i}\boldsymbol{t},$$

From the difference–selection matrices $E^2_{S_i,x}$ and $E^2_{S_i,t}$ we obtain

$$E^2_{S_i,x}\,\boldsymbol{x}_{S_i} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0\\ 1 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 0 & 0 & \cdots & 1\\ & & \vdots & & \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_{S_i,1}\\ \boldsymbol{x}_{S_i,2}\\ \vdots\\ \boldsymbol{x}_{S_i,|S_i|} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_{S_i,1} + \boldsymbol{x}_{S_i,2}\\ \boldsymbol{x}_{S_i,2} + \boldsymbol{x}_{S_i,3}\\ \vdots\\ \boldsymbol{x}_{S_i,|S_i|-1} + \boldsymbol{x}_{S_i,|S_i|} \end{bmatrix},$$

$$E^2_{S_i,t}\,\boldsymbol{t}_{S_i} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0\\ -1 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & & \ddots & \vdots\\ -1 & 0 & 0 & \cdots & 1\\ & & \vdots & & \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{t}_{S_i,1}\\ \boldsymbol{t}_{S_i,2}\\ \vdots\\ \boldsymbol{t}_{S_i,|S_i|} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{t}_{S_i,1} + \boldsymbol{t}_{S_i,2}\\ -\boldsymbol{t}_{S_i,2} + \boldsymbol{t}_{S_i,3}\\ \vdots\\ -\boldsymbol{t}_{S_i,|S_i|-1} + \boldsymbol{t}_{S_i,|S_i|} \end{bmatrix}$$

Therefore, the cadence constraint can be written compactly as

$$E^2_{S_i,t}\,\boldsymbol{t}_{S_i} \;\geq\; \tau\mathbf{1} - \tau\big(2\mathbf{1} - E^2_{S_i,x}\,\boldsymbol{x}_{S_i}\big) \qquad \Longleftrightarrow \qquad E^2_{S_i,x}\,\boldsymbol{x}_{S_i} \;\leq\; \mathbf{1} + \frac{1}{\tau}\,E^2_{S_i,t}\,\boldsymbol{t}_{S_i} \qquad i = 1,\dots,p$$

Stacking these inequalities for all satellites S_i $(i = 1,\dots,p)$ gives

$$\operatorname{diag}\left(\begin{bmatrix} E^2_{S_1,x}\\ \vdots\\ E^2_{S_p,x} \end{bmatrix}\right) \begin{bmatrix} \boldsymbol{x}_{S_1}\\ \vdots\\ \boldsymbol{x}_{S_p} \end{bmatrix} \;\leq\; \mathbf{1} + \frac{1}{\tau}\operatorname{diag}\left(\begin{bmatrix} E^2_{S_1,t}\\ \vdots\\ E^2_{S_p,t} \end{bmatrix}\right) \begin{bmatrix} \boldsymbol{t}_{S_1}\\ \vdots\\ \boldsymbol{t}_{S_p} \end{bmatrix}.$$

Using $\boldsymbol{x}_{S_i} = E^1_{S_i}\boldsymbol{x}$ and $\boldsymbol{t}_{S_i} = E^1_{S_i}\boldsymbol{t}$,

$$\operatorname{diag}\left(\begin{bmatrix} E^2_{S_1,x}\\ \vdots\\ E^2_{S_p,x} \end{bmatrix}\right) \begin{bmatrix} E^1_{S_1}\boldsymbol{x}\\ \vdots\\ E^1_{S_p}\boldsymbol{x} \end{bmatrix} \;\leq\; \mathbf{1} + \frac{1}{\tau}\operatorname{diag}\left(\begin{bmatrix} E^2_{S_1,t}\\ \vdots\\ E^2_{S_p,t} \end{bmatrix}\right) \begin{bmatrix} E^1_{S_1}\boldsymbol{t}\\ \vdots\\ E^1_{S_p}\boldsymbol{t} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} E^2_{S_1,x}E^1_{S_1} \\ \vdots \\ E^2_{S_p,x}E^1_{S_p} \end{bmatrix}}_{\text{Constant A}} \boldsymbol{x} \ \le \ \underbrace{\mathbf{1} + \frac{1}{\tau} \begin{bmatrix} E^2_{S_1,t}E^1_{S_1} \\ \vdots \\ E^2_{S_p,t}E^1_{S_p} \end{bmatrix} \boldsymbol{t}}_{\text{Constant } \boldsymbol{b}}$$

Therefore, satellite operation cadence constraint

$$\boldsymbol{t}_{S_i,\ell} + \boldsymbol{t}_{S_i,k} \ \ge \ \tau - \tau\big(2 - x_{S_i,\ell} - x_{S_i,k}\big), \qquad \forall\, \ell > k,\ i = 1, \ldots, p,$$

can be transformed into the standard linear inequality

$$A\boldsymbol{x} \le \boldsymbol{b}, \qquad A = \begin{bmatrix} E^2_{S_{1,x}}E^1_{S_1} \\ E^2_{S_{2,x}}E^1_{S_2} \\ \vdots \\ E^2_{S_{p,x}}E^1_{S_p} \end{bmatrix}, \qquad \boldsymbol{b} = \mathbf{1} + \frac{1}{\tau} \begin{bmatrix} E^2_{S_{1,t}}E^1_{S_1} \\ E^2_{S_{2,t}}E^1_{S_2} \\ \vdots \\ E^2_{S_{p,t}}E^1_{S_p} \end{bmatrix} \boldsymbol{t}$$

4.2. L_1 optimization problem by maximizing number of activated contacts

$$\begin{aligned} &\max\ \mathbf{1}^{\mathsf{T}}\boldsymbol{x} \\ &\ \text{s.t. } A\boldsymbol{x} \le \boldsymbol{b}, \qquad \boldsymbol{x} \in \{0,1\}^N \\ &\quad A = \begin{bmatrix} E^2_{S_{1,x}}E^1_{S_1} \\ E^2_{S_{2,x}}E^1_{S_2} \\ \vdots \\ E^2_{S_{p,x}}E^1_{S_p} \end{bmatrix}, \qquad \boldsymbol{b} = \mathbf{1} + \frac{1}{\tau} \begin{bmatrix} E^2_{S_{1,t}}E^1_{S_1} \\ E^2_{S_{2,t}}E^1_{S_2} \\ \vdots \\ E^2_{S_{p,t}}E^1_{S_p} \end{bmatrix} \boldsymbol{t} \end{aligned}$$

4.3. L_∞ optimization problem by minimizing maximum revisit time of GS

We can configure minimize-the-maximum-revisit-time problem as follows:

$$\boxed{\begin{aligned}&\min R\\ &\text{s.t. } A\boldsymbol{x} \le \boldsymbol{b},\\ &\quad R \ \ge\ \boldsymbol{t}_{G_j,\ell} - \boldsymbol{t}_{G_j,k} - M_G\big(2 - x_{G_j,\ell} - \boldsymbol{x}_{G_j,k} + \boldsymbol{x}_{G_j,m}\big), \qquad \forall\, \ell > m > k,\ j = 1,\dots,q\end{aligned}}$$

take $M_G = \boldsymbol{t}_{G_j,\ell} - \boldsymbol{t}_{G_j,k}$, then

$$R \ \ge\ (\boldsymbol{t}_{G_j,\ell} - \boldsymbol{t}_{G_j,k})\big(\boldsymbol{x}_{G_j,\ell} + \boldsymbol{x}_{G_j,k} - \boldsymbol{x}_{G_j,m} - 1\big).$$

Since $\boldsymbol{x} \in \{0,1\}^N$, the term

$$\boldsymbol{x}_{G_j,\ell} + \boldsymbol{x}_{G_j,k} - \boldsymbol{x}_{G_j,m} - 1 \le 1$$

and this becomes 1 (activates inequality) if and only if

$$\boldsymbol{x}_{G_j,\ell} = 1, \quad \boldsymbol{x}_{G_j,k} = 1, \quad \boldsymbol{x}_{G_j,m} = 0,$$

otherwise,

$$\boldsymbol{x}_{G_j,\ell} + \boldsymbol{x}_{G_j,k} - \boldsymbol{x}_{G_j,m} - 1 \le 0$$

which deactivates the inequality.

We can combine this condition as follows:

$$R \ge \big(\boldsymbol{t}_{G_j,k} - \boldsymbol{t}_{G_j,k}\big)\big(\boldsymbol{x}_{G_j,k} - \boldsymbol{x}_{G_j,k+1} - \boldsymbol{x}_{G_j,k+2} - \ldots - + \boldsymbol{x}_{G_j,\ell-1} + \boldsymbol{x}_{G_j,\ell}\big) \quad \forall l > k+1$$

$\boldsymbol{x}_{G_j,k} - \boldsymbol{x}_{G_j,k+1} - \boldsymbol{x}_{G_j,k+2} - \ldots - + \boldsymbol{x}_{G_j,\ell-1} + \boldsymbol{x}_{G_j,\ell} \le 1$ and this becomes 1 if and only if

$\boldsymbol{x}_{G_j,\ell} = \boldsymbol{x}_{G_j,\ell} = 1$ and $\boldsymbol{x}_{G_j,k+1} = \boldsymbol{x}_{G_j,k+2} = \ldots = \boldsymbol{x}_{G_j,\ell-1} = 0$

if $\ell = k+1$

$$R \ge \big(\boldsymbol{t}_{G_j,\ell} - \boldsymbol{t}_{G_j,k}\big)\big(\boldsymbol{x}_{G_j,\ell} + \boldsymbol{x}_{G_j,k}\big).$$

Define the stacked decision vector and corresponding contact times for GS G_j as

$$\boldsymbol{x}_{G_j} = \begin{bmatrix} \boldsymbol{x}_{G_j,1} \\ \boldsymbol{x}_{G_j,2} \\ \vdots \\ \boldsymbol{x}_{G_j,|G_j|} \end{bmatrix} \quad \boldsymbol{t}_{G_j} = \begin{bmatrix} \boldsymbol{t}_{G_j,1} \\ \boldsymbol{t}_{G_j,2} \\ \vdots \\ \boldsymbol{t}_{G_j,|G_j|} \end{bmatrix}$$

Then we introduce a difference–selection matrix $E^2_{G_j,\boldsymbol{x}}$ and $E^2_{G_j,\boldsymbol{t}}$ such that

$$E^2_{G_j,\boldsymbol{x}}\boldsymbol{x}_{G_j} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -1 & -1 & \cdots & 1 \\ & & \vdots & & \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_{G_j,1} \\ \boldsymbol{x}_{G_j,2} \\ \vdots \\ \boldsymbol{x}_{G_j,|G_j|} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_{G_j,1} + \boldsymbol{x}_{G_j,2} \\ \boldsymbol{x}_{G_j,1} - \boldsymbol{x}_{G_j,2} + \boldsymbol{x}_{G_j,3} \\ \vdots \\ \boldsymbol{x}_{G_j,1} - \boldsymbol{x}_{G_j,2} \ldots - \boldsymbol{x}_{G_j,(|G_j|-1)} + \boldsymbol{x}_{G_j,|G_j|} \\ \boldsymbol{x}_{G_j,2} + \boldsymbol{x}_{G_j,3} \\ \boldsymbol{x}_{G_j,2} - \boldsymbol{x}_{G_j,3} + \boldsymbol{x}_{G_j,4} \\ \vdots \\ \boldsymbol{x}_{G_j,2} - \boldsymbol{x}_{G_j,3} \ldots - \boldsymbol{x}_{G_j,(|G_j|-1)} + \boldsymbol{x}_{G_j,|G_j|} \\ \vdots \\ \boldsymbol{x}_{G_j,(|G_j|-2)} - \boldsymbol{x}_{G_j,(|G_j|-1)} + \boldsymbol{x}_{G_j,|G_j|} \\ \boldsymbol{x}_{G_j,(|G_j|-1)} + \boldsymbol{x}_{G_j,|G_j|} \end{bmatrix}$$

$$E^2_{G_j,\boldsymbol{t}}\boldsymbol{t}_{G_j} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \\ & & \vdots & & \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{t}_{G_j,1} \\ \boldsymbol{t}_{G_j,2} \\ \vdots \\ \boldsymbol{t}_{G_j,|G_j|} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{t}_{G_j,1} + \boldsymbol{t}_{G_j,2} \\ -\boldsymbol{t}_{G_j,1} + \boldsymbol{t}_{G_j,3} \\ \vdots \\ -\boldsymbol{t}_{G_j,1} + \boldsymbol{t}_{G_j,|G_j|} \\ \vdots \\ -\boldsymbol{t}_{G_j,(|G_j|-1)} + \boldsymbol{t}_{G_j,|G_j|} \end{bmatrix}$$

Therefore, the given inequality is equivalent to

$$R\,\mathbf{1} \;\geq\; \mathrm{diag}\big(E^2_{G_j,t}\,\boldsymbol{t}_{G_j}\big)\big(E^2_{G_j,x}\,\boldsymbol{x}_{G_j} - \mathbf{1}\big)$$

Since

$$\boldsymbol{x}_{G_j} = E^1_{G_j}\begin{bmatrix} 1 \\ \boldsymbol{x} \\ 1 \end{bmatrix} = E^1_{G_j}\left\{ \begin{bmatrix} \mathbf{0}_{1\times N} \\ \mathbf{I}_{N\times N} \\ \mathbf{0}_{1\times N} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ \mathbf{0}_{N\times 1} \\ 1 \end{bmatrix} \right\},$$

we substitute this into the previous inequality and obtain

$$\begin{aligned} R\,\mathbf{1} \;\geq\; & \mathrm{diag}\big(E^2_{G_j,t}\,\boldsymbol{t}_{G_j}\big)\left[E^2_{G_j,x}E^1_{G_j}\left\{ \begin{bmatrix} \mathbf{0}_{1\times N} \\ \mathbf{I}_{N\times N} \\ \mathbf{0}_{1\times N} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ \mathbf{0}_{N\times 1} \\ 1 \end{bmatrix} \right\} - \mathbf{1}\right] \\ = & \underbrace{\mathrm{diag}\big(E^2_{G_j,t}\,\boldsymbol{t}_{G_j}\big)E^2_{G_j,x}E^1_{G_j}\begin{bmatrix} \mathbf{0}_{1\times N} \\ \mathbf{I}_{N\times N} \\ \mathbf{0}_{1\times N} \end{bmatrix}}_{\text{constant } C_j} \boldsymbol{x} + \underbrace{\mathrm{diag}\big(E^2_{G_j,t}\,\boldsymbol{t}_{G_j}\big)\left\{E^2_{G_j,x}E^1_{G_j}\begin{bmatrix} 1 \\ \mathbf{0}_{N\times 1} \\ 1 \end{bmatrix} - \mathbf{1}\right\}}_{\text{constant } \boldsymbol{d}_j} \end{aligned}$$

We can define

$$C_j \;:=\; \mathrm{diag}\big(E^2_{G_j,t}\,\boldsymbol{t}_{G_j}\big)\,E^2_{G_j,x}E^1_{G_j}\begin{bmatrix} \mathbf{0}_{1\times N} \\ \mathbf{I}_{N\times N} \\ \mathbf{0}_{1\times N} \end{bmatrix} \qquad \Rightarrow \qquad C_j : \text{ constant},$$

$$\boldsymbol{d}_j \;:=\; \mathrm{diag}\big(E^2_{G_j,t}\,\boldsymbol{t}_{G_j}\big)\left\{E^2_{G_j,x}E^1_{G_j}\begin{bmatrix} 1 \\ \mathbf{0}_{N\times 1} \\ 1 \end{bmatrix} - \mathbf{1}\right\} \qquad \Rightarrow \qquad \boldsymbol{d}_j : \text{ constant}.$$

Since

$$\boldsymbol{t}_{G_j} = E^1_{G_j} \begin{bmatrix} t_{\text{start}} \\ \boldsymbol{t} \\ t_{\text{end}} \end{bmatrix} = E^1_{G_j} \left\{ \begin{bmatrix} \mathbf{0}_{1\times N} \\ \mathbf{I}_{N\times N} \\ \mathbf{0}_{1\times N} \end{bmatrix} \boldsymbol{t} + \begin{bmatrix} t_{\text{start}} \\ \mathbf{0}_{N\times 1} \\ t_{\text{end}} \end{bmatrix} \right\},$$

We can get:

$$C_j \;:=\; \operatorname{diag}\left(E^2_{G_j,t} E^1_{G_j} \begin{bmatrix} t_{\text{start}} \\ \boldsymbol{t} \\ t_{\text{end}} \end{bmatrix} \right) E^2_{G_j,x} E^1_{G_j} \begin{bmatrix} \mathbf{0}_{1\times N} \\ \mathbf{I}_{N\times N} \\ \mathbf{0}_{1\times N} \end{bmatrix} \qquad \Rightarrow \qquad C_j : \text{ constant},$$

$$\boldsymbol{d}_j \;:=\; \operatorname{diag}\left(E^2_{G_j,t} E^1_{G_j} \begin{bmatrix} t_{\text{start}} \\ \boldsymbol{t} \\ t_{\text{end}} \end{bmatrix} \right) \left\{ E^2_{G_j,x} E^1_{G_j} \begin{bmatrix} 1 \\ \mathbf{0}_{N\times 1} \\ 1 \end{bmatrix} - \mathbf{1} \right\} \qquad \Rightarrow \qquad \boldsymbol{d}_j : \text{ constant}$$

Therefore, the inequality becomes

$$R\mathbf{1} \geq C_j \boldsymbol{x} + \boldsymbol{d}_j$$

Stacking $j = 1 \dots q$

$$R\,\mathbf{1} \;\geq\; \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_q \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \\ \vdots \\ \boldsymbol{d}_q \end{bmatrix} = C\boldsymbol{x} + \boldsymbol{d}$$

4.4. L_∞ optimization problem

$$\begin{aligned}
&\min\ R \\
&\text{s.t. } A\boldsymbol{x} \leq \boldsymbol{b}, \qquad \boldsymbol{x} \in \{0,1\}^N \\
&\qquad R\mathbf{1} \geq C\boldsymbol{x} + \boldsymbol{d} \\
&\qquad A = \begin{bmatrix} E^2_{S_1,x} E^1_{S_1} \\ E^2_{S_2,x} E^1_{S_2} \\ \vdots \\ E^2_{S_p,x} E^1_{S_p} \end{bmatrix}, \qquad \boldsymbol{b} = \mathbf{1} + \frac{1}{\tau} \begin{bmatrix} E^2_{S_1,t} E^1_{S_1} \\ E^2_{S_2,t} E^1_{S_2} \\ \vdots \\ E^2_{S_p,t} E^1_{S_p} \end{bmatrix} \boldsymbol{t}, \quad C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_q \end{bmatrix}, \quad \boldsymbol{d} = \begin{bmatrix} \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \\ \vdots \\ \boldsymbol{d}_q \end{bmatrix} \\
&C_j \;:=\; \operatorname{diag}\left(E^2_{G_j,t} E^1_{G_j} \begin{bmatrix} t_{\text{start}} \\ \boldsymbol{t} \\ t_{\text{end}} \end{bmatrix} \right) E^2_{G_j,x} E^1_{G_j} \begin{bmatrix} \mathbf{0}_{1\times N} \\ \mathbf{I}_{N\times N} \\ \mathbf{0}_{1\times N} \end{bmatrix} \qquad \Rightarrow \qquad C_j : \text{ constant}, \\
&\boldsymbol{d}_j \;:=\; \operatorname{diag}\left(E^2_{G_j,t} E^1_{G_j} \begin{bmatrix} t_{\text{start}} \\ \boldsymbol{t} \\ t_{\text{end}} \end{bmatrix} \right) \left\{ E^2_{G_j,x} E^1_{G_j} \begin{bmatrix} 1 \\ \mathbf{0}_{N\times 1} \\ 1 \end{bmatrix} - \mathbf{1} \right\} \qquad \Rightarrow \qquad \boldsymbol{d}_j : \text{ constant}
\end{aligned}$$

4.5. L_2 optimization problem by minimizing square sum of revisit time for each GS

From previous section, we could get:

$$E^2_{G_j,\boldsymbol{x}}\boldsymbol{x}_{G_j} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -1 & -1 & \cdots & 1 \\ & & \vdots & & \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_{G_j,1} \\ \boldsymbol{x}_{G_j,2} \\ \vdots \\ \boldsymbol{x}_{G_j,|G_j|} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_{G_j,1} + \boldsymbol{x}_{G_j,2} \\ \boldsymbol{x}_{G_j,1} - \boldsymbol{x}_{G_j,2} + \boldsymbol{x}_{G_j,3} \\ \vdots \\ \boldsymbol{x}_{G_j,1} - \boldsymbol{x}_{G_j,2} \ldots - \boldsymbol{x}_{G_j,(|G_j|-1)} + \boldsymbol{x}_{G_j,|G_j|} \\ \boldsymbol{x}_{G_j,2} + \boldsymbol{x}_{G_j,3} \\ \boldsymbol{x}_{G_j,2} - \boldsymbol{x}_{G_j,3} + \boldsymbol{x}_{G_j,4} \\ \vdots \\ \boldsymbol{x}_{G_j,2} - \boldsymbol{x}_{G_j,3} \ldots - \boldsymbol{x}_{G_j,(|G_j|-1)} + \boldsymbol{x}_{G_j,|G_j|} \\ \vdots \\ \boldsymbol{x}_{G_j,(|G_j|-2)} - \boldsymbol{x}_{G_j,(|G_j|-1)} + \boldsymbol{x}_{G_j,|G_j|} \\ \boldsymbol{x}_{G_j,(|G_j|-1)} + \boldsymbol{x}_{G_j,|G_j|} \end{bmatrix}$$

$$E^2_{G_j,\boldsymbol{x}}\boldsymbol{t}_{G_j} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \\ & & \vdots & & \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{t}_{G_j,1} \\ \boldsymbol{t}_{G_j,2} \\ \vdots \\ \boldsymbol{t}_{G_j,|G_j|} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{t}_{G_j,1} + \boldsymbol{t}_{G_j,2} \\ -\boldsymbol{t}_{G_j,1} + \boldsymbol{t}_{G_j,3} \\ \vdots \\ -\boldsymbol{t}_{G_j,1} + \boldsymbol{t}_{G_j,|G_j|} \\ \vdots \\ -\boldsymbol{t}_{G_j,(|G_j|-1)} + \boldsymbol{t}_{G_j,|G_j|} \end{bmatrix}$$

Let's define $\boldsymbol{y} \in R^M$, $M = \sum_{j=1}^{q} \frac{(|G_j|+2)(|G_j|+1)}{2}$ and $\mathbf{1} \le \boldsymbol{y} \le 2\times\mathbf{1}$, then we can add follow constraints:

$$\boldsymbol{y} \ \ge \begin{bmatrix} E^2_{G_1,x}\boldsymbol{x}_{G_1} \\ E^2_{G_2,x}\boldsymbol{x}_{G_2} \\ \vdots \\ E^2_{G_q,x}\boldsymbol{x}_{G_q} \end{bmatrix} = \begin{bmatrix} E^2_{G_1,x}E^1_{G_1} \\ E^2_{G_2,x}E^1_{G_2} \\ \vdots \\ E^2_{G_q,x}E^1_{G_q} \end{bmatrix} \left\{ \begin{bmatrix} \mathbf{0}_{1\times N} \\ \mathbf{I}_{N\times N} \\ \mathbf{0}_{1\times N} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ \mathbf{0}_{N\times 1} \\ 1 \end{bmatrix} \right\}$$

Therefore, the inequality becomes

$$\boldsymbol{y} \ \ge \underbrace{\begin{bmatrix} E^2_{G_1,x}E^1_{G_1} \\ E^2_{G_2,x}E^1_{G_2} \\ \vdots \\ E^2_{G_q,x}E^1_{G_q} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{1\times N} \\ \mathbf{I}_{N\times N} \\ \mathbf{0}_{1\times N} \end{bmatrix}}_{E} \boldsymbol{x} + \underbrace{\begin{bmatrix} E^2_{G_1,x}E^1_{G_1} \\ E^2_{G_2,x}E^1_{G_2} \\ \vdots \\ E^2_{G_q,x}E^1_{G_q} \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{0}_{N\times 1} \\ 1 \end{bmatrix}}_{\boldsymbol{f}}$$

Also, we can define square-sum of revisit time by $\boldsymbol{y}$ and $E^2_{G_j,t}\boldsymbol{t}_{G_j}$

$$\left\| \begin{bmatrix} E^2_{G_1,t}\,\boldsymbol{t}_{G_1} \\ E^2_{G_2,t}\,\boldsymbol{t}_{G_2} \\ \vdots \\ E^2_{G_q,t}\,\boldsymbol{t}_{G_q} \end{bmatrix}^{\mathsf{T}} (\boldsymbol{y} - \mathbf{1}) \right\|_2^2 = (\boldsymbol{y} - \mathbf{1})^{\mathsf{T}} \left[\operatorname{diag}\left(\begin{bmatrix} E^2_{G_1,t}\,\boldsymbol{t}_{G_1} \\ E^2_{G_2,t}\,\boldsymbol{t}_{G_2} \\ \vdots \\ E^2_{G_q,t}\,\boldsymbol{t}_{G_q} \end{bmatrix} \right) \right]^2 (\boldsymbol{y} - \mathbf{1})$$

Since we know

$$\begin{bmatrix} E^2_{G_1,t}\boldsymbol{t}_{G_1} \\ E^2_{G_2,t}\boldsymbol{t}_{G_2} \\ \vdots \\ E^2_{G_q,t}\boldsymbol{t}_{G_q} \end{bmatrix} = \begin{bmatrix} E^2_{G_1,t}E^1_{G_1} \\ E^2_{G_2,t}E^1_{G_2} \\ \vdots \\ E^2_{G_q,t}E^1_{G_q} \end{bmatrix} \begin{bmatrix} t_{\text{start}} \\ \boldsymbol{t} \\ t_{\text{end}} \end{bmatrix}$$

The cost function becomes

$$(\boldsymbol{y}-\mathbf{1})^{\mathsf{T}} \left[\underbrace{\operatorname{diag}\left(\begin{bmatrix} E^2_{G_1,t}E^1_{G_1} \\ E^2_{G_2,t}E^1_{G_2} \\ \vdots \\ E^2_{G_q,t}E^1_{G_q} \end{bmatrix} \begin{bmatrix} t_{\text{start}} \\ \boldsymbol{t} \\ t_{\text{end}} \end{bmatrix} \right)}_{\text{Constant } G} \right]^2 (\boldsymbol{y}-\mathbf{1})$$

If we substitute $\boldsymbol{y}-\mathbf{1}=\boldsymbol{z}$, then $\mathbf{0}\le\boldsymbol{z}\le\mathbf{1}$ and $\boldsymbol{z}\ge E\boldsymbol{x}+\boldsymbol{f}-\mathbf{1}$. Also, the objective function $(\boldsymbol{y}-\mathbf{1})^{\mathsf{T}}G^2(\boldsymbol{y}-\mathbf{1})$ is equivalent to $\mathbf{1}^{\mathsf{T}}G^2\boldsymbol{z}$, which is linear objective function.

4.6. L_2 optimization problem

$$\begin{aligned}
&\min\ \mathbf{1}^{\mathsf{T}}G^2\boldsymbol{z} \\
&\text{s.t. } A\boldsymbol{x}\le\boldsymbol{b}, \qquad \boldsymbol{x}\in\{0,1\}^N \\
&\quad\ \boldsymbol{z}\ge E\boldsymbol{x}+\boldsymbol{f}-\mathbf{1}, \qquad \boldsymbol{z}\ge\mathbf{0} \\
&\quad\ \boldsymbol{z}\in R^M, M=\sum_{j=1}^{q}\frac{(|G_j|+2)(|G_j|+1)}{2} \\
&\quad\ A=\begin{bmatrix} E^2_{S_1,x}E^1_{S_1} \\ E^2_{S_2,x}E^1_{S_2} \\ \vdots \\ E^2_{S_p,x}E^1_{S_p} \end{bmatrix}, \qquad \boldsymbol{b}=\mathbf{1}+\frac{1}{\tau}\begin{bmatrix} E^2_{S_1,t}E^1_{S_1} \\ E^2_{S_2,t}E^1_{S_2} \\ \vdots \\ E^2_{S_p,t}E^1_{S_p} \end{bmatrix}\boldsymbol{t} \\
&\quad\ E=\begin{bmatrix} E^2_{G_1,x}E^1_{G_1} \\ E^2_{G_2,x}E^1_{G_2} \\ \vdots \\ E^2_{G_q,x}E^1_{G_q} \end{bmatrix}\begin{bmatrix} \mathbf{0}_{1\times N} \\ \mathbf{I}_{N\times N} \\ \mathbf{0}_{1\times N} \end{bmatrix}, \quad \boldsymbol{f}=\begin{bmatrix} E^2_{G_1,x}E^1_{G_1} \\ E^2_{G_2,x}E^1_{G_2} \\ \vdots \\ E^2_{G_q,x}E^1_{G_q} \end{bmatrix}\begin{bmatrix} 1 \\ \mathbf{0}_{N\times 1} \\ 1 \end{bmatrix} \\
&\quad\ G=\operatorname{diag}\left(\begin{bmatrix} E^2_{G_1,t}E^1_{G_1} \\ E^2_{G_2,t}E^1_{G_2} \\ \vdots \\ E^2_{G_q,t}E^1_{G_q} \end{bmatrix} \begin{bmatrix} t_{\text{start}} \\ \boldsymbol{t} \\ t_{\text{end}} \end{bmatrix} \right)
\end{aligned}$$

4.7. Rewriting L_∞ optimization problem

$$
\begin{aligned}
&\min\ R \\
&\text{s.t.}\ A\boldsymbol{x} \le \boldsymbol{b}, \qquad \boldsymbol{x} \in \{0,1\}^N \\
&\quad R\mathbf{1} \ge G(E\boldsymbol{x} + \boldsymbol{f} - \mathbf{1}) \\
&\quad A = \begin{bmatrix} E^2_{S_1,x} E^1_{S_1} \\ E^2_{S_2,x} E^1_{S_2} \\ \vdots \\ E^2_{S_p,x} E^1_{S_p} \end{bmatrix}, \qquad \boldsymbol{b} = \mathbf{1} + \frac{1}{\tau} \begin{bmatrix} E^2_{S_1,t} E^1_{S_1} \\ E^2_{S_2,t} E^1_{S_2} \\ \vdots \\ E^2_{S_p,t} E^1_{S_p} \end{bmatrix} \boldsymbol{t} \\
&\quad E = \begin{bmatrix} E^2_{G_1,x} E^1_{G_1} \\ E^2_{G_2,x} E^1_{G_2} \\ \vdots \\ E^2_{G_q,x} E^1_{G_q} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{1\times N} \\ \mathbf{I}_{N\times N} \\ \mathbf{0}_{1\times N} \end{bmatrix}, \quad \boldsymbol{f} = \begin{bmatrix} E^2_{G_1,x} E^1_{G_1} \\ E^2_{G_2,x} E^1_{G_2} \\ \vdots \\ E^2_{G_q,x} E^1_{G_q} \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{0}_{N\times 1} \\ 1 \end{bmatrix} \\
&\quad G = \operatorname{diag}\left(\begin{bmatrix} E^2_{G_1,t} E^1_{G_1} \\ E^2_{G_2,t} E^1_{G_2} \\ \vdots \\ E^2_{G_q,t} E^1_{G_q} \end{bmatrix} \begin{bmatrix} t_{\text{start}} \\ \boldsymbol{t} \\ t_{\text{end}} \end{bmatrix} \right)
\end{aligned}
$$