

# Satellite Constellation Scheduling to MILP/MIQP Problem

## 1. Given : Contact Matrix without SAT Operational Constraint

- From Satellite constellation simulation, we could get unconstrained SAT–GS contact chart.
- Index for every contact constant variable

SAT	GS	$t_m$	
$a_{11}$	$a_{12}$	$a_{13}$	Total number of contacts: $N$
$a_{21}$	$a_{22}$	$a_{23}$	Number of SATs: $p \rightarrow a_{i1} \in \{1, 2, \dots, p\}$
$a_{31}$	$a_{32}$	$a_{33}$	Number of GS: $q \rightarrow a_{i2} \in \{1, 2, \dots, q\}$
$\vdots$	$\vdots$	$\vdots$	Aceding $t_m$ : $a_{14} \leq a_{24} \leq \dots \leq a_{N4}$
$a_{N1}$	$a_{N2}$	$a_{N3}$	

## 2. Key parameters for the input of MILP/MIQP problem

- Unconstrained Satellite contact matrix:  $A$
- Time Vector:  $t = A(:, 3)$
- Initial and Final time  $t_{\text{start}}, t_{\text{end}}$
- Satellite Cadence Constraint:  $\tau$
- Number of SATs:  $p$ , Number of GSs:  $q$
- Total number of contact:  $N$
- Number of contact for each SAT:  $|S_i|$
- Number of contact for each GS:  $|G_j|$
- Optimization variable:  $x \in \{0, 1\}^N$

## 3. Selection matrix generation from given constant parameters

### 3.1. $E_{S_i}^1$ : Selection matrix from A-matrix to each satellite's contact sequence

$E_{S_i}^1 : |S_i| \times N$  matrix, Satellite index  $i = 1, \dots, p$

1. Initialize:  $\text{zeros}(|S_i|, N)$
2. List row index:  $A(:, 1) == i \Rightarrow a_{i1}, a_{i2}, \dots, a_{i|S_i|}$
3.  $E_{S_i}^1(1, a_{i1}) = 1, E_{S_i}^1(2, a_{i1}) = 1 \dots E_{S_i}^1(|S_i|, a_{i1}) = 1$   
 $\Rightarrow$  We have  $E_{S_i}^1: E_{S_1}^1, E_{S_2}^1, \dots, E_{S_p}^1$  which configures  $x_{Si} = E_{S_i}^1 x, t_{Si} = E_{S_i}^1 t$

**3.2.  $E_{S_i,x}^2, E_{S_i,t}^2$ : Selection matrix for  $\Delta t$  of each satellite from  $|S_i|$**

$$E_{S_i,x}^2 = \frac{|S_i|(|S_i| - 1)}{2} \times |S_i| \text{ matrix, SAT index } i = 1, \dots, p$$

for  $\alpha = 1 \dots |S_i| - 1$

$$E_{S_i,x,\alpha}^2 = \text{zeros}(|S_i| - \alpha, |S_i|)$$

$$E_{S_i,x,\alpha}^2(:, \alpha) = 1$$

for  $\beta = \alpha + 1 \dots |S_i|$

$$E_{S_i,x,\alpha}^2(\beta - \alpha, \beta) = 1$$

end

$$E_{S_i,x}^2 = \begin{bmatrix} E_{S_i,x}^2 \\ E_{S_i,x,\alpha}^2 \end{bmatrix}$$

end

$\Rightarrow$  We have  $E_{S_i,x}^2 : E_{S_1,x}^2, E_{S_2,x}^2, \dots, E_{S_p,x}^2$  from  $|S_i| : |S_1|, |S_2|, \dots, |S_p|$

$$E_{S_i,t}^2 = \frac{|S_i|(|S_i| - 1)}{2} \times |S_i| \text{ matrix, SAT index } i = 1, \dots, p$$

for  $\alpha = 1 \dots |S_i| - 1$

$$E_{S_i,t,\alpha}^2 = \text{zeros}(|S_i| - \alpha, |S_i|)$$

$$E_{S_i,t,\alpha}^2(:, \alpha) = -1$$

for  $\beta = \alpha + 1 \dots |S_i|$

$$E_{S_i,t,\alpha}^2(\beta - \alpha, \beta) = 1$$

end

$$E_{S_i,t}^2 = \begin{bmatrix} E_{S_i,t}^2 \\ E_{S_i,t,\alpha}^2 \end{bmatrix}$$

end

$\Rightarrow$  We have  $E_{S_i,t}^2 : E_{S_1,t}^2, E_{S_2,t}^2, \dots, E_{S_p,t}^2$  from  $|S_i| : |S_1|, |S_2|, \dots, |S_p|$

### 3.3. $E_{G_j}^1$ : Selection matrix from A-matrix to each Ground Point's revisit sequence

$E_{G_j}^1 : (|G_j| + 2) \times (N + 2)$  matrix, GS index  $j = 1, \dots, q$

1. Initialize: zeros( $|G_j| + 2, N + 2$ )
2. List row index:  $A(:, 2) == j \Rightarrow b_{j1}, b_{j2}, \dots, b_{j|G_j|}$
3.  $E_{G_j}^1(1, 1) = 1$
4. for  $\alpha = 1 \dots |G_j|$

$$E_{G_j}^1(\alpha + 1, b_{j\alpha} + 1) = 1$$

end

$$5. E_{G_j}^1(|G_j| + 2, N + 2) = 1$$

$\Rightarrow$  We have  $E_{G_j}^1 : E_{G_1}^1, E_{G_2}^1, \dots, E_{G_q}^1$  from  $\mathbf{x}, \mathbf{t}, t_{start}, t_{end}$  to  $\mathbf{x}_{G_j}, \mathbf{t}_{G_j}$

$$\mathbf{x}_{G_j} = E_{G_j}^1 \begin{bmatrix} 1 \\ \mathbf{x} \\ 1 \end{bmatrix} = E_{G_j}^1 \left\{ \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ \mathbf{0}_{N \times 1} \\ 1 \end{bmatrix} \right\}$$

$$\mathbf{x}_{G_j} = \underbrace{E_{G_j}^1 \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix}}_{\text{Constant}} \mathbf{x} + \underbrace{E_{G_j}^1 \begin{bmatrix} 1 \\ \mathbf{0}_{N \times 1} \\ 1 \end{bmatrix}}_{\text{Constant}} \Rightarrow \mathbf{x}_{G_j} \text{ is affine w.r.t } \mathbf{x}$$

$$\mathbf{t}_{G_j} = E_{G_j}^1 \begin{bmatrix} t_{start} \\ \mathbf{t} \\ t_{end} \end{bmatrix} = E_{G_j}^1 \left\{ \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix} \mathbf{t} + \begin{bmatrix} t_{start} \\ \mathbf{0}_{N \times 1} \\ t_{end} \end{bmatrix} \right\}$$

$$\mathbf{t}_{G_j} = \underbrace{E_{G_j}^1 \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix}}_{\text{Constant}} \mathbf{t} + \underbrace{E_{G_j}^1 \begin{bmatrix} t_{start} \\ \mathbf{0}_{N \times 1} \\ t_{end} \end{bmatrix}}_{\text{Constant}} \Rightarrow \mathbf{t}_{G_j} \text{ is affine w.r.t } \mathbf{t}$$

**3.4.  $E_{G_j,x}^2, E_{G_j,t}^2$ : Selection matrix for  $\Delta t$  of each GS from  $|G_j|$**

$$E_{G_j,x}^2 = \frac{(|G_j| + 2)(|G_j| + 1)}{2} \times (|G_j| + 2) \text{ matrix, GS index } j = 1, \dots, q$$

for  $\alpha = 1 \dots |G_j| + 1$

$$E_{G_j,x,\alpha}^2 = \text{zeros}(|G_j| + 2 - \alpha, |G_j| + 2)$$

$$E_{G_j,x,\alpha}^2(:, \alpha) = 1$$

for  $\beta = \alpha + 1 \dots |G_j| + 2$

if  $\alpha + 1 \leq \beta - 1$

$$E_{G_j,x,\alpha}^2(\beta - \alpha, \alpha + 1 \dots \beta - 1) = -1$$

end

$$E_{G_j,x,\alpha}^2(\beta - \alpha, \beta) = 1$$

end

$$E_{G_j,x}^2 = \begin{bmatrix} E_{G_j,x}^2 \\ E_{G_j,x,\alpha}^2 \end{bmatrix}$$

end

$\Rightarrow$  We have  $E_{G_j,x}^2 : E_{G_1,x}^2, E_{G_2,x}^2, \dots, E_{G_q,x}^2$  from  $|G_j| : |G_1|, |G_2|, \dots, |S_q|$

$$E_{G_j,t}^2 = \frac{(|G_j| + 2)(|G_j| + 1)}{2} \times (|G_j| + 2) \text{ matrix, GS index } j = 1, \dots, q$$

for  $\alpha = 1 \dots |G_j| + 1$

$$E_{G_j,t,\alpha}^2 = \text{zeros}(|G_j| + 2 - \alpha, |G_j| + 2)$$

$$E_{G_j,t,\alpha}^2(:, \alpha) = -1$$

for  $\beta = \alpha + 1 \dots |G_j| + 2$

$$E_{G_j,t,\alpha}^2(\beta - \alpha, \beta) = 1$$

end

$$E_{G_j,t}^2 = \begin{bmatrix} E_{G_j,t}^2 \\ E_{G_j,t,\alpha}^2 \end{bmatrix}$$

end

$\Rightarrow$  We have  $E_{G_j,t}^2 : E_{G_1,t}^2, E_{G_2,t}^2, \dots, E_{G_q,t}^2$  from  $|G_j| : |G_1|, |G_2|, \dots, |S_q|$

## 4. Derivation of $L_1$ , $L_2$ , $L_\infty$ revisit time problem to MILP/MIQP

### 4.1. Satellite operation cadence constraint to linear inequality

From the big- $M$  method, for each satellite  $S_i$  we have the cadence constraint  $\tau$ :

$$\mathbf{t}_{S_i,\ell} - \mathbf{t}_{S_i,k} \geq \tau - \tau(2 - \mathbf{x}_{S_i,\ell} - \mathbf{x}_{S_i,k}), \quad \forall \ell > k, i = 1, \dots, p$$

We can interpret the above depending on the values of the binary variables:

- if  $\mathbf{x}_{S_i,\ell} = 1$ ,  $\mathbf{x}_{S_i,k} = 1 \Rightarrow \mathbf{t}_{S_i,\ell} - \mathbf{t}_{S_i,k} \geq \tau \Rightarrow$  Active constraint,
- if  $\mathbf{x}_{S_i,\ell} = 1$ ,  $\mathbf{x}_{S_i,k} = 0 \Rightarrow \mathbf{t}_{S_i,\ell} + \mathbf{t}_{S_i,k} \geq 0 \Rightarrow$  Non-active constraint,
- if  $\mathbf{x}_{S_i,\ell} = 0$ ,  $\mathbf{x}_{S_i,k} = 0 \Rightarrow \mathbf{t}_{S_i,\ell} - \mathbf{t}_{S_i,k} \geq -\tau \Rightarrow$  Non-active constraint

From the selection matrices  $E_{S_i}^1$  we have

$$\mathbf{x}_{S_i} = E_{S_i}^1 \mathbf{x}, \quad \mathbf{t}_{S_i} = E_{S_i}^1 \mathbf{t},$$

From the difference-selection matrices  $E_{S_i,x}^2$  and  $E_{S_i,t}^2$  we obtain

$$E_{S_i,x}^2 \mathbf{x}_{S_i} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \\ \vdots & & & & \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_{S_i,1} \\ \mathbf{x}_{S_i,2} \\ \vdots \\ \mathbf{x}_{S_i,|S_i|} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{S_i,1} + \mathbf{x}_{S_i,2} \\ \mathbf{x}_{S_i,2} + \mathbf{x}_{S_i,3} \\ \vdots \\ \mathbf{x}_{S_i,|S_i|-1} + \mathbf{x}_{S_i,|S_i|} \end{bmatrix},$$

$$E_{S_i,t}^2 \mathbf{t}_{S_i} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \\ \vdots & & & & \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{t}_{S_i,1} \\ \mathbf{t}_{S_i,2} \\ \vdots \\ \mathbf{t}_{S_i,|S_i|} \end{bmatrix} = \begin{bmatrix} -\mathbf{t}_{S_i,1} + \mathbf{t}_{S_i,2} \\ -\mathbf{t}_{S_i,2} + \mathbf{t}_{S_i,3} \\ \vdots \\ -\mathbf{t}_{S_i,|S_i|-1} + \mathbf{t}_{S_i,|S_i|} \end{bmatrix}$$

Therefore, the cadence constraint can be written compactly as

$$E_{S_i,t}^2 \mathbf{t}_{S_i} \geq \tau \mathbf{1} - \tau(2\mathbf{1} - E_{S_i,x}^2 \mathbf{x}_{S_i}) \iff E_{S_i,x}^2 \mathbf{x}_{S_i} \leq \mathbf{1} + \frac{1}{\tau} E_{S_i,t}^2 \mathbf{t}_{S_i} \quad i = 1, \dots, p$$

Stacking these inequalities for all satellites  $S_i$  ( $i = 1, \dots, p$ ) gives

$$\text{diag} \left( \begin{bmatrix} E_{S_1,x}^2 \\ \vdots \\ E_{S_p,x}^2 \end{bmatrix} \right) \begin{bmatrix} \mathbf{x}_{S_1} \\ \vdots \\ \mathbf{x}_{S_p} \end{bmatrix} \leq \mathbf{1} + \frac{1}{\tau} \text{diag} \left( \begin{bmatrix} E_{S_1,t}^2 \\ \vdots \\ E_{S_p,t}^2 \end{bmatrix} \right) \begin{bmatrix} \mathbf{t}_{S_1} \\ \vdots \\ \mathbf{t}_{S_p} \end{bmatrix}.$$

Using  $\mathbf{x}_{S_i} = E_{S_i}^1 \mathbf{x}$  and  $\mathbf{t}_{S_i} = E_{S_i}^1 \mathbf{t}$ ,

$$\text{diag} \left( \begin{bmatrix} E_{S_1,x}^2 \\ \vdots \\ E_{S_p,x}^2 \end{bmatrix} \right) \begin{bmatrix} E_{S_1}^1 \mathbf{x} \\ \vdots \\ E_{S_p}^1 \mathbf{x} \end{bmatrix} \leq \mathbf{1} + \frac{1}{\tau} \text{diag} \left( \begin{bmatrix} E_{S_1,t}^2 \\ \vdots \\ E_{S_p,t}^2 \end{bmatrix} \right) \begin{bmatrix} E_{S_1}^1 \mathbf{t} \\ \vdots \\ E_{S_p}^1 \mathbf{t} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} E_{S_1,x}^2 E_{S_1}^1 \\ \vdots \\ E_{S_p,x}^2 E_{S_p}^1 \end{bmatrix}}_{\text{Constant } A} \mathbf{x} \leq \underbrace{\mathbf{1} + \frac{1}{\tau} \begin{bmatrix} E_{S_1,t}^2 E_{S_1}^1 \\ \vdots \\ E_{S_p,t}^2 E_{S_p}^1 \end{bmatrix}}_{\text{Constant } b} \mathbf{t}$$

Therefore, satellite operation cadence constraint

$$\mathbf{t}_{S_i,\ell} + \mathbf{t}_{S_i,k} \geq \mathbf{1} - \tau(2 - x_{S_i,\ell} - x_{S_i,k}), \quad \forall \ell > k, i = 1, \dots, p,$$

can be transformed into the standard linear inequality

$$A\mathbf{x} \leq \mathbf{b}, \quad A = \begin{bmatrix} E_{S_1,x}^2 E_{S_1}^1 \\ E_{S_2,x}^2 E_{S_2}^1 \\ \vdots \\ E_{S_p,x}^2 E_{S_p}^1 \end{bmatrix}, \quad \mathbf{b} = \mathbf{1} + \frac{1}{\tau} \begin{bmatrix} E_{S_1,t}^2 E_{S_1}^1 \\ E_{S_2,t}^2 E_{S_2}^1 \\ \vdots \\ E_{S_p,t}^2 E_{S_p}^1 \end{bmatrix} \mathbf{t}$$

#### 4.2. $L_1$ optimization problem by maximizing number of activated contacts

$$\begin{aligned} & \max \mathbf{1}^\top \mathbf{x} \\ \text{s.t. } & A\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \in \{0, 1\}^N \\ & A = \begin{bmatrix} E_{S_1,x}^2 E_{S_1}^1 \\ E_{S_2,x}^2 E_{S_2}^1 \\ \vdots \\ E_{S_p,x}^2 E_{S_p}^1 \end{bmatrix}, \quad \mathbf{b} = \mathbf{1} + \frac{1}{\tau} \begin{bmatrix} E_{S_1,t}^2 E_{S_1}^1 \\ E_{S_2,t}^2 E_{S_2}^1 \\ \vdots \\ E_{S_p,t}^2 E_{S_p}^1 \end{bmatrix} \mathbf{t} \end{aligned}$$

### 4.3. $L_\infty$ optimization problem by minimizing maximum revisit time of GS

We can configure minimize-the-maximum-revisit-time problem as follows:

$$\begin{aligned} & \min R \\ \text{s.t. } & A\mathbf{x} \leq \mathbf{b}, \\ & R \geq \mathbf{t}_{G_j,\ell} - \mathbf{t}_{G_j,k} - M_G(2 - x_{G_j,\ell} - \mathbf{x}_{G_j,k} + \mathbf{x}_{G_j,m}), \quad \forall \ell > m > k, j = 1, \dots, q \end{aligned}$$

take  $M_G = \mathbf{t}_{G_j,\ell} - \mathbf{t}_{G_j,k}$ , then

$$R \geq (\mathbf{t}_{G_j,\ell} - \mathbf{t}_{G_j,k})(\mathbf{x}_{G_j,\ell} + \mathbf{x}_{G_j,k} - \mathbf{x}_{G_j,m} - 1).$$

Since  $\mathbf{x} \in \{0,1\}^N$ , the term

$$\mathbf{x}_{G_j,\ell} + \mathbf{x}_{G_j,k} - \mathbf{x}_{G_j,m} - 1 \leq 1$$

and this becomes 1 (activates inequality) if and only if

$$\mathbf{x}_{G_j,\ell} = 1, \quad \mathbf{x}_{G_j,k} = 1, \quad \mathbf{x}_{G_j,m} = 0,$$

otherwise,

$$\mathbf{x}_{G_j,\ell} + \mathbf{x}_{G_j,k} - \mathbf{x}_{G_j,m} - 1 \leq 0$$

which deactivates the inequality.

We can combine this condition as follows:

$$\begin{aligned} R &\geq (\mathbf{t}_{G_j,k} - \mathbf{t}_{G_j,k})(\mathbf{x}_{G_j,k} - \mathbf{x}_{G_j,k+1} - \mathbf{x}_{G_j,k+2} - \dots - \mathbf{x}_{G_j,\ell-1} + \mathbf{x}_{G_j,\ell}) \quad \forall l > k+1 \\ &\mathbf{x}_{G_j,k} - \mathbf{x}_{G_j,k+1} - \mathbf{x}_{G_j,k+2} - \dots - \mathbf{x}_{G_j,\ell-1} + \mathbf{x}_{G_j,\ell} \leq 1 \quad \text{and this becomes 1 if and only if} \\ &\mathbf{x}_{G_j,\ell} = \mathbf{x}_{G_j,k} = 1 \quad \text{and} \quad \mathbf{x}_{G_j,k+1} = \mathbf{x}_{G_j,k+2} = \dots = \mathbf{x}_{G_j,\ell-1} = 0 \end{aligned}$$

if  $\ell = k+1$

$$R \geq (\mathbf{t}_{G_j,\ell} - \mathbf{t}_{G_j,k})(\mathbf{x}_{G_j,\ell} + \mathbf{x}_{G_j,k}).$$

Define the stacked decision vector and corresponding contact times for GS  $G_j$  as

$$\mathbf{x}_{G_j} = \begin{bmatrix} \mathbf{x}_{G_j,1} \\ \mathbf{x}_{G_j,2} \\ \vdots \\ \mathbf{x}_{G_j,|G_j|} \end{bmatrix} \quad \mathbf{t}_{G_j} = \begin{bmatrix} \mathbf{t}_{G_j,1} \\ \mathbf{t}_{G_j,2} \\ \vdots \\ \mathbf{t}_{G_j,|G_j|} \end{bmatrix}$$

Then we introduce a difference-selection matrix  $E_{G_j, \mathbf{x}}^2$  and  $E_{G_j, \mathbf{t}}^2$  such that

$$E_{G_j, \mathbf{x}}^2 \mathbf{x}_{G_j} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -1 & -1 & \cdots & 1 \\ \vdots & & & & \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_{G_j,1} \\ \mathbf{x}_{G_j,2} \\ \vdots \\ \mathbf{x}_{G_j,|G_j|} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{G_j,1} + \mathbf{x}_{G_j,2} \\ \mathbf{x}_{G_j,1} - \mathbf{x}_{G_j,2} + \mathbf{x}_{G_j,3} \\ \vdots \\ \mathbf{x}_{G_j,1} - \mathbf{x}_{G_j,2} \dots - \mathbf{x}_{G_j,(|G_j|-1)} + \mathbf{x}_{G_j,|G_j|} \\ \mathbf{x}_{G_j,2} + \mathbf{x}_{G_j,3} \\ \mathbf{x}_{G_j,2} - \mathbf{x}_{G_j,3} + \mathbf{x}_{G_j,4} \\ \vdots \\ \mathbf{x}_{G_j,2} - \mathbf{x}_{G_j,3} \dots - \mathbf{x}_{G_j,(|G_j|-1)} + \mathbf{x}_{G_j,|G_j|} \\ \vdots \\ \mathbf{x}_{G_j,(|G_j|-2)} - \mathbf{x}_{G_j,(|G_j|-1)} + \mathbf{x}_{G_j,|G_j|} \\ \mathbf{x}_{G_j,(|G_j|-1)} + \mathbf{x}_{G_j,|G_j|} \end{bmatrix}$$

$$E_{G_j, \mathbf{t}}^2 \mathbf{t}_{G_j} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \\ \vdots & & & & \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{t}_{G_j,1} \\ \mathbf{t}_{G_j,2} \\ \vdots \\ \mathbf{t}_{G_j,|G_j|} \end{bmatrix} = \begin{bmatrix} -\mathbf{t}_{G_j,1} + \mathbf{t}_{G_j,2} \\ -\mathbf{t}_{G_j,1} + \mathbf{t}_{G_j,3} \\ \vdots \\ -\mathbf{t}_{G_j,1} + \mathbf{t}_{G_j,|G_j|} \\ \vdots \\ -\mathbf{t}_{G_j,(|G_j|-1)} + \mathbf{t}_{G_j,|G_j|} \end{bmatrix}$$

Therefore, the given inequality is equivalent to

$$R \mathbf{1} \geq \text{diag}(E_{G_j, t}^2 \mathbf{t}_{G_j}) (E_{G_j, x}^2 \mathbf{x}_{G_j} - \mathbf{1})$$

Since

$$\mathbf{x}_{G_j} = E_{G_j}^1 \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix} = E_{G_j}^1 \left\{ \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ \mathbf{0}_{N \times 1} \\ 1 \end{bmatrix} \right\},$$

we substitute this into the previous inequality and obtain

$$\begin{aligned} R \mathbf{1} &\geq \text{diag}(E_{G_j, t}^2 \mathbf{t}_{G_j}) \left[ E_{G_j, x}^2 E_{G_j}^1 \left\{ \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ \mathbf{0}_{N \times 1} \\ 1 \end{bmatrix} \right\} - \mathbf{1} \right] \\ &= \underbrace{\text{diag}(E_{G_j, t}^2 \mathbf{t}_{G_j}) E_{G_j, x}^2 E_{G_j}^1}_{\text{constant } C_j} \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix} \mathbf{x} + \underbrace{\text{diag}(E_{G_j, t}^2 \mathbf{t}_{G_j}) \left\{ E_{G_j, x}^2 E_{G_j}^1 \begin{bmatrix} 1 \\ \mathbf{0}_{N \times 1} \\ 1 \end{bmatrix} - \mathbf{1} \right\}}_{\text{constant } d_j} \end{aligned}$$

We can define

$$C_j := \text{diag}(E_{G_j, t}^2 \mathbf{t}_{G_j}) E_{G_j, x}^2 E_{G_j}^1 \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix} \Rightarrow C_j : \text{constant},$$

$$d_j := \text{diag}(E_{G_j, t}^2 \mathbf{t}_{G_j}) \left\{ E_{G_j, x}^2 E_{G_j}^1 \begin{bmatrix} 1 \\ \mathbf{0}_{N \times 1} \\ 1 \end{bmatrix} - \mathbf{1} \right\} \Rightarrow d_j : \text{constant}.$$

Since

$$\mathbf{t}_{G_j} = E_{G_j}^1 \begin{bmatrix} t_{\text{start}} \\ \mathbf{t} \\ t_{\text{end}} \end{bmatrix} = E_{G_j}^1 \left\{ \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix} \mathbf{t} + \begin{bmatrix} t_{\text{start}} \\ \mathbf{0}_{N \times 1} \\ t_{\text{end}} \end{bmatrix} \right\},$$

We can get:

$$C_j := \text{diag}(E_{G_j,t}^2 E_{G_j}^1 \begin{bmatrix} t_{\text{start}} \\ \mathbf{t} \\ t_{\text{end}} \end{bmatrix}) E_{G_j,x}^2 E_{G_j}^1 \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix} \Rightarrow C_j : \text{constant},$$

$$\mathbf{d}_j := \text{diag}(E_{G_j,t}^2 E_{G_j}^1 \begin{bmatrix} t_{\text{start}} \\ \mathbf{t} \\ t_{\text{end}} \end{bmatrix}) \left\{ E_{G_j,x}^2 E_{G_j}^1 \begin{bmatrix} 1 \\ \mathbf{0}_{N \times 1} \\ 1 \end{bmatrix} - \mathbf{1} \right\} \Rightarrow \mathbf{d}_j : \text{constant}$$

Therefore, the inequality becomes

$$R\mathbf{1} \geq C_j \mathbf{x} + \mathbf{d}_j$$

Stacking  $j = 1 \dots q$

$$R\mathbf{1} \geq \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_q \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \vdots \\ \mathbf{d}_q \end{bmatrix} = C\mathbf{x} + \mathbf{d}$$

#### 4.4. $L_\infty$ optimization problem

$$\min R$$

$$\text{s.t. } A\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \in \{0, 1\}^N$$

$$R\mathbf{1} \geq C\mathbf{x} + \mathbf{d}$$

$$A = \begin{bmatrix} E_{S_1,x}^2 E_{S_1}^1 \\ E_{S_2,x}^2 E_{S_2}^1 \\ \vdots \\ E_{S_p,x}^2 E_{S_p}^1 \end{bmatrix}, \quad \mathbf{b} = \mathbf{1} + \frac{1}{\tau} \begin{bmatrix} E_{S_1,t}^2 E_{S_1}^1 \\ E_{S_2,t}^2 E_{S_2}^1 \\ \vdots \\ E_{S_p,t}^2 E_{S_p}^1 \end{bmatrix} \mathbf{t}, \quad C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_q \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \vdots \\ \mathbf{d}_q \end{bmatrix}$$

$$C_j := \text{diag}(E_{G_j,t}^2 E_{G_j}^1 \begin{bmatrix} t_{\text{start}} \\ \mathbf{t} \\ t_{\text{end}} \end{bmatrix}) E_{G_j,x}^2 E_{G_j}^1 \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix} \Rightarrow C_j : \text{constant},$$

$$\mathbf{d}_j := \text{diag}(E_{G_j,t}^2 E_{G_j}^1 \begin{bmatrix} t_{\text{start}} \\ \mathbf{t} \\ t_{\text{end}} \end{bmatrix}) \left\{ E_{G_j,x}^2 E_{G_j}^1 \begin{bmatrix} 1 \\ \mathbf{0}_{N \times 1} \\ 1 \end{bmatrix} - \mathbf{1} \right\} \Rightarrow \mathbf{d}_j : \text{constant}$$

#### 4.5. $L_2$ optimization problem by minimizing square sum of revisit time for each GS

From previous section, we could get:

$$E_{G_j, \mathbf{x}}^2 \mathbf{x}_{G_j} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -1 & -1 & \cdots & 1 \\ \vdots & & & & \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_{G_j,1} \\ \mathbf{x}_{G_j,2} \\ \vdots \\ \mathbf{x}_{G_j,|G_j|} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{G_j,1} + \mathbf{x}_{G_j,2} \\ \mathbf{x}_{G_j,1} - \mathbf{x}_{G_j,2} + \mathbf{x}_{G_j,3} \\ \vdots \\ \mathbf{x}_{G_j,1} - \mathbf{x}_{G_j,2} \dots - \mathbf{x}_{G_j,(|G_j|-1)} + \mathbf{x}_{G_j,|G_j|} \\ \mathbf{x}_{G_j,2} + \mathbf{x}_{G_j,3} \\ \mathbf{x}_{G_j,2} - \mathbf{x}_{G_j,3} + \mathbf{x}_{G_j,4} \\ \vdots \\ \mathbf{x}_{G_j,2} - \mathbf{x}_{G_j,3} \dots - \mathbf{x}_{G_j,(|G_j|-1)} + \mathbf{x}_{G_j,|G_j|} \\ \vdots \\ \mathbf{x}_{G_j,(|G_j|-2)} - \mathbf{x}_{G_j,(|G_j|-1)} + \mathbf{x}_{G_j,|G_j|} \\ \mathbf{x}_{G_j,(|G_j|-1)} + \mathbf{x}_{G_j,|G_j|} \end{bmatrix}$$

$$E_{G_j, \mathbf{x}}^2 \mathbf{t}_{G_j} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \\ \vdots & & & & \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{t}_{G_j,1} \\ \mathbf{t}_{G_j,2} \\ \vdots \\ \mathbf{t}_{G_j,|G_j|} \end{bmatrix} = \begin{bmatrix} -\mathbf{t}_{G_j,1} + \mathbf{t}_{G_j,2} \\ -\mathbf{t}_{G_j,1} + \mathbf{t}_{G_j,3} \\ \vdots \\ -\mathbf{t}_{G_j,1} + \mathbf{t}_{G_j,|G_j|} \\ \vdots \\ -\mathbf{t}_{G_j,(|G_j|-1)} + \mathbf{t}_{G_j,|G_j|} \end{bmatrix}$$

Let's define  $\mathbf{y} \in R^M$ ,  $M = \sum_{j=1}^q \frac{(|G_j|+2)(|G_j|+1)}{2}$  and  $0 \leq \mathbf{y} \leq \mathbf{1}$ , then we can add follow constraints:

$$\mathbf{y} \geq \begin{bmatrix} E_{G_1, \mathbf{x}}^2 \mathbf{x}_{G_1} \\ E_{G_2, \mathbf{x}}^2 \mathbf{x}_{G_2} \\ \vdots \\ E_{G_q, \mathbf{x}}^2 \mathbf{x}_{G_q} \end{bmatrix} = \begin{bmatrix} E_{G_1, \mathbf{x}}^2 E_{G_1}^1 \\ E_{G_2, \mathbf{x}}^2 E_{G_2}^1 \\ \vdots \\ E_{G_q, \mathbf{x}}^2 E_{G_q}^1 \end{bmatrix} \left\{ \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ \mathbf{0}_{N \times 1} \\ 1 \end{bmatrix} \right\}$$

Therefore, the inequality becomes

$$\mathbf{y} \geq \underbrace{\begin{bmatrix} E_{G_1, \mathbf{x}}^2 E_{G_1}^1 \\ E_{G_2, \mathbf{x}}^2 E_{G_2}^1 \\ \vdots \\ E_{G_q, \mathbf{x}}^2 E_{G_q}^1 \end{bmatrix}}_E \underbrace{\begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ \mathbf{0}_{N \times 1} \\ 1 \end{bmatrix}}_f$$

Also, we can define square-sum of revisit time by  $\mathbf{y}$  and  $E_{G_j, t}^2 \mathbf{t}_{G_j}$

$$\left\| \begin{bmatrix} E_{G_1, t}^2 \mathbf{I}_{G_1} \\ E_{G_2, t}^2 \mathbf{I}_{G_2} \\ \vdots \\ E_{G_q, t}^2 \mathbf{I}_{G_q} \end{bmatrix} \mathbf{y} \right\|_2^2 = \mathbf{y}^\top \begin{bmatrix} E_{G_1, t}^2 \mathbf{t}_{G_1} \\ E_{G_2, t}^2 \mathbf{t}_{G_2} \\ \vdots \\ E_{G_q, t}^2 \mathbf{t}_{G_q} \end{bmatrix} \begin{bmatrix} E_{G_1, t}^2 \mathbf{t}_{G_1} \\ E_{G_2, t}^2 \mathbf{t}_{G_2} \\ \vdots \\ E_{G_q, t}^2 \mathbf{t}_{G_q} \end{bmatrix}^\top \mathbf{y}$$

Since we know

$$\begin{bmatrix} E_{G_1,t}^2 \mathbf{t}_{G_1} \\ E_{G_2,t}^2 \mathbf{t}_{G_2} \\ \vdots \\ E_{G_q,t}^2 \mathbf{t}_{G_q} \end{bmatrix} = \begin{bmatrix} E_{G_1,t}^2 E_{G_1}^1 \\ E_{G_2,t}^2 E_{G_2}^1 \\ \vdots \\ E_{G_q,t}^2 E_{G_q}^1 \end{bmatrix} \begin{bmatrix} t_{\text{start}} \\ \mathbf{t} \\ t_{\text{end}} \end{bmatrix}$$

The cost function becomes

$$\mathbf{y}^\top \underbrace{\begin{bmatrix} E_{G_1,t}^2 E_{G_1}^1 \\ E_{G_2,t}^2 E_{G_2}^1 \\ \vdots \\ E_{G_q,t}^2 E_{G_q}^1 \end{bmatrix} \begin{bmatrix} t_{\text{start}} \\ \mathbf{t} \\ t_{\text{end}} \end{bmatrix} \begin{bmatrix} t_{\text{start}} \\ \mathbf{t} \\ t_{\text{end}} \end{bmatrix}^\top \begin{bmatrix} E_{G_1,t}^2 E_{G_1}^1 \\ E_{G_2,t}^2 E_{G_2}^1 \\ \vdots \\ E_{G_q,t}^2 E_{G_q}^1 \end{bmatrix}}_{\text{Constant } G} \mathbf{y}$$

#### 4.6. $L_2$ optimization problem

$$\min \mathbf{y}^\top G \mathbf{y}$$

s.t.  $A\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \in \{0, 1\}^N$

$$\mathbf{y} \geq E\mathbf{x} + \mathbf{f}, \quad \mathbf{0} \leq \mathbf{y} \leq \mathbf{1}$$

$$\mathbf{y} \in R^M, M = \sum_{j=1}^q \frac{(|G_j| + 2)(|G_j| + 1)}{2}$$

$$A = \begin{bmatrix} E_{S_1,x}^2 E_{S_1}^1 \\ E_{S_2,x}^2 E_{S_2}^1 \\ \vdots \\ E_{S_p,x}^2 E_{S_p}^1 \end{bmatrix}, \quad \mathbf{b} = \mathbf{1} + \frac{1}{\tau} \begin{bmatrix} E_{S_1,t}^2 E_{S_1}^1 \\ E_{S_2,t}^2 E_{S_2}^1 \\ \vdots \\ E_{S_p,t}^2 E_{S_p}^1 \end{bmatrix} \mathbf{t}$$

$$E = \begin{bmatrix} E_{G_1,x}^2 E_{G_1}^1 \\ E_{G_2,x}^2 E_{G_2}^1 \\ \vdots \\ E_{G_q,x}^2 E_{G_q}^1 \end{bmatrix} \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} E_{G_1,x}^2 E_{G_1}^1 \\ E_{G_2,x}^2 E_{G_2}^1 \\ \vdots \\ E_{G_q,x}^2 E_{G_q}^1 \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{0}_{N \times 1} \\ 1 \end{bmatrix}$$

$$G = \begin{bmatrix} E_{G_1,t}^2 E_{G_1}^1 \\ E_{G_2,t}^2 E_{G_2}^1 \\ \vdots \\ E_{G_q,t}^2 E_{G_q}^1 \end{bmatrix} \begin{bmatrix} t_{\text{start}} \\ \mathbf{t} \\ t_{\text{end}} \end{bmatrix} \begin{bmatrix} t_{\text{start}} \\ \mathbf{t} \\ t_{\text{end}} \end{bmatrix}^\top \begin{bmatrix} E_{G_1,t}^2 E_{G_1}^1 \\ E_{G_2,t}^2 E_{G_2}^1 \\ \vdots \\ E_{G_q,t}^2 E_{G_q}^1 \end{bmatrix}^\top$$