

Satellite Constellation Scheduling to MILP/MIQP Problem

1. Given : Contact Matrix without SAT Operational Constraint

- From Satellite constellation simulation, we could get unconstrained SAT–GS contact chart.
- Index for every contact constant variable

SAT	GS	t_m	
a_{11}	a_{12}	a_{13}	Total number of contacts: N
a_{21}	a_{22}	a_{23}	Number of SATs: $p \rightarrow a_{i1} \in \{1, 2, \dots, p\}$
a_{31}	a_{32}	a_{33}	Number of GS: $q \rightarrow a_{i2} \in \{1, 2, \dots, q\}$
\vdots	\vdots	\vdots	Aceding t_m : $a_{14} \leq a_{24} \leq \dots \leq a_{N4}$
a_{N1}	a_{N2}	a_{N3}	

2. Key parameters for the input of MILP/MIQP problem

- Unconstrained Satellite contact matrix: A
- Time Vector: $t = A(:, 3)$
- Initial and Final time $t_{\text{start}}, t_{\text{end}}$
- Satellite Cadence Constraint: τ
- Number of SATs: p , Number of GSs: q
- Total number of contact: N
- Number of contact for each SAT: $|S_i|$
- Number of contact for each GS: $|G_j|$
- Optimization variable: $x \in \{0, 1\}^N$

3. Selection matrix generation from given constant parameters

3.1. $E_{S_i}^1$: Selection matrix from A-matrix to each satellite's contact sequence

$E_{S_i}^1 : |S_i| \times N$ matrix, Satellite index $i = 1, \dots, p$

1. Initialize: $\text{zeros}(|S_i|, N)$
2. List row index: $A(:, 1) == i \Rightarrow a_{i1}, a_{i2}, \dots, a_{i|S_i|}$
3. $E_{S_i}^1(1, a_{i1}) = 1, E_{S_i}^1(2, a_{i1}) = 1 \dots E_{S_i}^1(|S_i|, a_{i1}) = 1$
 \Rightarrow We have $E_{S_i}^1: E_{S_1}^1, E_{S_2}^1, \dots, E_{S_p}^1$ which configures $x_{Si} = E_{S_i}^1 x, t_{Si} = E_{S_i}^1 t$

3.2. $E_{S_i,x}^2, E_{S_i,t}^2$: Selection matrix for Δt of each satellite from $|S_i|$

$$E_{S_i,x}^2 = \frac{|S_i|(|S_i| - 1)}{2} \times |S_i| \text{ matrix, SAT index } i = 1, \dots, p$$

for $\alpha = 1 \dots |S_i| - 1$

$$E_{S_i,x,\alpha}^2 = \text{zeros}(|S_i| - \alpha, |S_i|)$$

$$E_{S_i,x,\alpha}^2(:, \alpha) = 1$$

for $\beta = \alpha + 1 \dots |S_i|$

$$E_{S_i,x,\alpha}^2(\beta - \alpha, \beta) = 1$$

end

$$E_{S_i,x}^2 = \begin{bmatrix} E_{S_i,x}^2 \\ E_{S_i,x,\alpha}^2 \end{bmatrix}$$

end

\Rightarrow We have $E_{S_i,x}^2 : E_{S_1,x}^2, E_{S_2,x}^2, \dots, E_{S_p,x}^2$ from $|S_i| : |S_1|, |S_2|, \dots, |S_p|$

$$E_{S_i,t}^2 = \frac{|S_i|(|S_i| - 1)}{2} \times |S_i| \text{ matrix, SAT index } i = 1, \dots, p$$

for $\alpha = 1 \dots |S_i| - 1$

$$E_{S_i,t,\alpha}^2 = \text{zeros}(|S_i| - \alpha, |S_i|)$$

$$E_{S_i,t,\alpha}^2(:, \alpha) = -1$$

for $\beta = \alpha + 1 \dots |S_i|$

$$E_{S_i,t,\alpha}^2(\beta - \alpha, \beta) = 1$$

end

$$E_{S_i,t}^2 = \begin{bmatrix} E_{S_i,t}^2 \\ E_{S_i,t,\alpha}^2 \end{bmatrix}$$

end

\Rightarrow We have $E_{S_i,t}^2 : E_{S_1,t}^2, E_{S_2,t}^2, \dots, E_{S_p,t}^2$ from $|S_i| : |S_1|, |S_2|, \dots, |S_p|$

3.3. $E_{G_j}^1$: Selection matrix from A-matrix to each Ground Point's revisit sequence

$E_{G_j}^1 : (|G_j| + 2) \times (N + 2)$ matrix, GS index $j = 1, \dots, q$

1. Initialize: zeros($|G_j| + 2, N + 2$)
2. List row index: $A(:, 2) == j \Rightarrow b_{j1}, b_{j2}, \dots, b_{j|G_j|}$
3. $E_{G_j}^1(1, 1) = 1$
4. for $\alpha = 1 \dots |G_j|$

$$E_{G_j}^1(\alpha + 1, b_{j\alpha} + 1) = 1$$

end

$$5. E_{G_j}^1(|G_j| + 2, N + 2) = 1$$

\Rightarrow We have $E_{G_j}^1 : E_{G_1}^1, E_{G_2}^1, \dots, E_{G_q}^1$ from $\mathbf{x}, \mathbf{t}, t_{start}, t_{end}$ to $\mathbf{x}_{G_j}, \mathbf{t}_{G_j}$

$$\mathbf{x}_{G_j} = E_{G_j}^1 \begin{bmatrix} 1 \\ \mathbf{x} \\ 1 \end{bmatrix} = E_{G_j}^1 \left\{ \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ \mathbf{0}_{N \times 1} \\ 1 \end{bmatrix} \right\}$$

$$\mathbf{x}_{G_j} = \underbrace{E_{G_j}^1 \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix}}_{\text{Constant}} \mathbf{x} + \underbrace{E_{G_j}^1 \begin{bmatrix} 1 \\ \mathbf{0}_{N \times 1} \\ 1 \end{bmatrix}}_{\text{Constant}} \Rightarrow \mathbf{x}_{G_j} \text{ is affine w.r.t } \mathbf{x}$$

$$\mathbf{t}_{G_j} = E_{G_j}^1 \begin{bmatrix} t_{start} \\ \mathbf{t} \\ t_{end} \end{bmatrix} = E_{G_j}^1 \left\{ \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix} \mathbf{t} + \begin{bmatrix} t_{start} \\ \mathbf{0}_{N \times 1} \\ t_{end} \end{bmatrix} \right\}$$

$$\mathbf{t}_{G_j} = \underbrace{E_{G_j}^1 \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix}}_{\text{Constant}} \mathbf{t} + \underbrace{E_{G_j}^1 \begin{bmatrix} t_{start} \\ \mathbf{0}_{N \times 1} \\ t_{end} \end{bmatrix}}_{\text{Constant}} \Rightarrow \mathbf{t}_{G_j} \text{ is affine w.r.t } \mathbf{t}$$

3.4. $E_{G_j,x}^2, E_{G_j,t}^2$: Selection matrix for Δt of each GS from $|G_j|$

$$E_{G_j,x}^2 = \frac{(|G_j| + 2)(|G_j| + 1)}{2} \times (|G_j| + 2) \text{ matrix, GS index } j = 1, \dots, q$$

for $\alpha = 1 \dots |G_j| + 1$

$$E_{G_j,x,\alpha}^2 = \text{zeros}(|G_j| + 2 - \alpha, |G_j| + 2)$$

$$E_{G_j,x,\alpha}^2(:, \alpha) = 1$$

for $\beta = \alpha + 1 \dots |G_j| + 2$

if $\alpha + 1 \leq \beta - 1$

$$E_{G_j,x,\alpha}^2(\beta - \alpha, \alpha + 1 \dots \beta - 1) = -1$$

end

$$E_{G_j,x,\alpha}^2(\beta - \alpha, \beta) = 1$$

end

$$E_{G_j,x}^2 = \begin{bmatrix} E_{G_j,x}^2 \\ E_{G_j,x,\alpha}^2 \end{bmatrix}$$

end

\Rightarrow We have $E_{G_j,x}^2 : E_{G_1,x}^2, E_{G_2,x}^2, \dots, E_{G_q,x}^2$ from $|G_j| : |G_1|, |G_2|, \dots, |S_q|$

$$E_{G_j,t}^2 = \frac{(|G_j| + 2)(|G_j| + 1)}{2} \times (|G_j| + 2) \text{ matrix, GS index } j = 1, \dots, q$$

for $\alpha = 1 \dots |G_j| + 1$

$$E_{G_j,t,\alpha}^2 = \text{zeros}(|G_j| + 2 - \alpha, |G_j| + 2)$$

$$E_{G_j,t,\alpha}^2(:, \alpha) = -1$$

for $\beta = \alpha + 1 \dots |G_j| + 2$

$$E_{G_j,t,\alpha}^2(\beta - \alpha, \beta) = 1$$

end

$$E_{G_j,t}^2 = \begin{bmatrix} E_{G_j,t}^2 \\ E_{G_j,t,\alpha}^2 \end{bmatrix}$$

end

\Rightarrow We have $E_{G_j,t}^2 : E_{G_1,t}^2, E_{G_2,t}^2, \dots, E_{G_q,t}^2$ from $|G_j| : |G_1|, |G_2|, \dots, |S_q|$

4. Derivation of L_1 , L_2 , L_∞ revisit time problem to MILP/MIQP

4.1. Satellite operation cadence constraint to linear inequality

From the big- M method, for each satellite S_i we have the cadence constraint τ :

$$\mathbf{t}_{S_i,\ell} - \mathbf{t}_{S_i,k} \geq \tau - \tau(2 - \mathbf{x}_{S_i,\ell} - \mathbf{x}_{S_i,k}), \quad \forall \ell > k, i = 1, \dots, p$$

We can interpret the above depending on the values of the binary variables:

- if $\mathbf{x}_{S_i,\ell} = 1$, $\mathbf{x}_{S_i,k} = 1 \Rightarrow \mathbf{t}_{S_i,\ell} - \mathbf{t}_{S_i,k} \geq \tau \Rightarrow$ Active constraint,
- if $\mathbf{x}_{S_i,\ell} = 1$, $\mathbf{x}_{S_i,k} = 0 \Rightarrow \mathbf{t}_{S_i,\ell} + \mathbf{t}_{S_i,k} \geq 0 \Rightarrow$ Non-active constraint,
- if $\mathbf{x}_{S_i,\ell} = 0$, $\mathbf{x}_{S_i,k} = 0 \Rightarrow \mathbf{t}_{S_i,\ell} - \mathbf{t}_{S_i,k} \geq -\tau \Rightarrow$ Non-active constraint

From the selection matrices $E_{S_i}^1$ we have

$$\mathbf{x}_{S_i} = E_{S_i}^1 \mathbf{x}, \quad \mathbf{t}_{S_i} = E_{S_i}^1 \mathbf{t},$$

From the difference-selection matrices $E_{S_i,x}^2$ and $E_{S_i,t}^2$ we obtain

$$E_{S_i,x}^2 \mathbf{x}_{S_i} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \\ \vdots & & & & \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_{S_i,1} \\ \mathbf{x}_{S_i,2} \\ \vdots \\ \mathbf{x}_{S_i,|S_i|} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{S_i,1} + \mathbf{x}_{S_i,2} \\ \mathbf{x}_{S_i,2} + \mathbf{x}_{S_i,3} \\ \vdots \\ \mathbf{x}_{S_i,|S_i|-1} + \mathbf{x}_{S_i,|S_i|} \end{bmatrix},$$

$$E_{S_i,t}^2 \mathbf{t}_{S_i} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \\ \vdots & & & & \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{t}_{S_i,1} \\ \mathbf{t}_{S_i,2} \\ \vdots \\ \mathbf{t}_{S_i,|S_i|} \end{bmatrix} = \begin{bmatrix} -\mathbf{t}_{S_i,1} + \mathbf{t}_{S_i,2} \\ -\mathbf{t}_{S_i,2} + \mathbf{t}_{S_i,3} \\ \vdots \\ -\mathbf{t}_{S_i,|S_i|-1} + \mathbf{t}_{S_i,|S_i|} \end{bmatrix}$$

Therefore, the cadence constraint can be written compactly as

$$E_{S_i,t}^2 \mathbf{t}_{S_i} \geq \tau \mathbf{1} - \tau(2\mathbf{1} - E_{S_i,x}^2 \mathbf{x}_{S_i}) \iff E_{S_i,x}^2 \mathbf{x}_{S_i} \leq \mathbf{1} + \frac{1}{\tau} E_{S_i,t}^2 \mathbf{t}_{S_i} \quad i = 1, \dots, p$$

Stacking these inequalities for all satellites S_i ($i = 1, \dots, p$) gives

$$\text{diag} \left(\begin{bmatrix} E_{S_1,x}^2 \\ \vdots \\ E_{S_p,x}^2 \end{bmatrix} \right) \begin{bmatrix} \mathbf{x}_{S_1} \\ \vdots \\ \mathbf{x}_{S_p} \end{bmatrix} \leq \mathbf{1} + \frac{1}{\tau} \text{diag} \left(\begin{bmatrix} E_{S_1,t}^2 \\ \vdots \\ E_{S_p,t}^2 \end{bmatrix} \right) \begin{bmatrix} \mathbf{t}_{S_1} \\ \vdots \\ \mathbf{t}_{S_p} \end{bmatrix}.$$

Using $\mathbf{x}_{S_i} = E_{S_i}^1 \mathbf{x}$ and $\mathbf{t}_{S_i} = E_{S_i}^1 \mathbf{t}$,

$$\text{diag} \left(\begin{bmatrix} E_{S_1,x}^2 \\ \vdots \\ E_{S_p,x}^2 \end{bmatrix} \right) \begin{bmatrix} E_{S_1}^1 \mathbf{x} \\ \vdots \\ E_{S_p}^1 \mathbf{x} \end{bmatrix} \leq \mathbf{1} + \frac{1}{\tau} \text{diag} \left(\begin{bmatrix} E_{S_1,t}^2 \\ \vdots \\ E_{S_p,t}^2 \end{bmatrix} \right) \begin{bmatrix} E_{S_1}^1 \mathbf{t} \\ \vdots \\ E_{S_p}^1 \mathbf{t} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} E_{S_1,x}^2 E_{S_1}^1 \\ \vdots \\ E_{S_p,x}^2 E_{S_p}^1 \end{bmatrix}}_{\text{Constant } A} \mathbf{x} \leq \underbrace{\mathbf{1} + \frac{1}{\tau} \begin{bmatrix} E_{S_1,t}^2 E_{S_1}^1 \\ \vdots \\ E_{S_p,t}^2 E_{S_p}^1 \end{bmatrix}}_{\text{Constant } b} \mathbf{t}$$

Therefore, satellite operation cadence constraint

$$\mathbf{t}_{S_i,\ell} + \mathbf{t}_{S_i,k} \geq \mathbf{1} - \tau(2 - x_{S_i,\ell} - x_{S_i,k}), \quad \forall \ell > k, i = 1, \dots, p,$$

can be transformed into the standard linear inequality

$$A\mathbf{x} \leq \mathbf{b}, \quad A = \begin{bmatrix} E_{S_1,x}^2 E_{S_1}^1 \\ E_{S_2,x}^2 E_{S_2}^1 \\ \vdots \\ E_{S_p,x}^2 E_{S_p}^1 \end{bmatrix}, \quad \mathbf{b} = \mathbf{1} + \frac{1}{\tau} \begin{bmatrix} E_{S_1,t}^2 E_{S_1}^1 \\ E_{S_2,t}^2 E_{S_2}^1 \\ \vdots \\ E_{S_p,t}^2 E_{S_p}^1 \end{bmatrix} \mathbf{t}$$

4.2. L_1 optimization problem by maximizing number of activated contacts

$$\begin{aligned} & \max \mathbf{1}^\top \mathbf{x} \\ \text{s.t. } & A\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \in \{0, 1\}^N \\ & A = \begin{bmatrix} E_{S_1,x}^2 E_{S_1}^1 \\ E_{S_2,x}^2 E_{S_2}^1 \\ \vdots \\ E_{S_p,x}^2 E_{S_p}^1 \end{bmatrix}, \quad \mathbf{b} = \mathbf{1} + \frac{1}{\tau} \begin{bmatrix} E_{S_1,t}^2 E_{S_1}^1 \\ E_{S_2,t}^2 E_{S_2}^1 \\ \vdots \\ E_{S_p,t}^2 E_{S_p}^1 \end{bmatrix} \mathbf{t} \end{aligned}$$

4.3. L_∞ optimization problem by minimizing maximum revisit time of GS

We can configure minimize-the-maximum-revisit-time problem as follows:

$$\begin{aligned} & \min R \\ \text{s.t. } & A\mathbf{x} \leq \mathbf{b}, \\ & R \geq \mathbf{t}_{G_j,\ell} - \mathbf{t}_{G_j,k} - M_G(2 - x_{G_j,\ell} - \mathbf{x}_{G_j,k} + \mathbf{x}_{G_j,m}), \quad \forall \ell > m > k, j = 1, \dots, q \end{aligned}$$

take $M_G = \mathbf{t}_{G_j,\ell} - \mathbf{t}_{G_j,k}$, then

$$R \geq (\mathbf{t}_{G_j,\ell} - \mathbf{t}_{G_j,k})(\mathbf{x}_{G_j,\ell} + \mathbf{x}_{G_j,k} - \mathbf{x}_{G_j,m} - 1).$$

Since $\mathbf{x} \in \{0,1\}^N$, the term

$$\mathbf{x}_{G_j,\ell} + \mathbf{x}_{G_j,k} - \mathbf{x}_{G_j,m} - 1 \leq 1$$

and this becomes 1 (activates inequality) if and only if

$$\mathbf{x}_{G_j,\ell} = 1, \quad \mathbf{x}_{G_j,k} = 1, \quad \mathbf{x}_{G_j,m} = 0,$$

otherwise,

$$\mathbf{x}_{G_j,\ell} + \mathbf{x}_{G_j,k} - \mathbf{x}_{G_j,m} - 1 \leq 0$$

which deactivates the inequality.

We can combine this condition as follows:

$$\begin{aligned} R &\geq (\mathbf{t}_{G_j,k} - \mathbf{t}_{G_j,k})(\mathbf{x}_{G_j,k} - \mathbf{x}_{G_j,k+1} - \mathbf{x}_{G_j,k+2} - \dots - \mathbf{x}_{G_j,\ell-1} + \mathbf{x}_{G_j,\ell}) \quad \forall l > k+1 \\ &\mathbf{x}_{G_j,k} - \mathbf{x}_{G_j,k+1} - \mathbf{x}_{G_j,k+2} - \dots - \mathbf{x}_{G_j,\ell-1} + \mathbf{x}_{G_j,\ell} \leq 1 \quad \text{and this becomes 1 if and only if} \\ &\mathbf{x}_{G_j,\ell} = \mathbf{x}_{G_j,k} = 1 \quad \text{and} \quad \mathbf{x}_{G_j,k+1} = \mathbf{x}_{G_j,k+2} = \dots = \mathbf{x}_{G_j,\ell-1} = 0 \end{aligned}$$

if $\ell = k+1$

$$R \geq (\mathbf{t}_{G_j,\ell} - \mathbf{t}_{G_j,k})(\mathbf{x}_{G_j,\ell} + \mathbf{x}_{G_j,k}).$$

Define the stacked decision vector and corresponding contact times for GS G_j as

$$\mathbf{x}_{G_j} = \begin{bmatrix} \mathbf{x}_{G_j,1} \\ \mathbf{x}_{G_j,2} \\ \vdots \\ \mathbf{x}_{G_j,|G_j|} \end{bmatrix} \quad \mathbf{t}_{G_j} = \begin{bmatrix} \mathbf{t}_{G_j,1} \\ \mathbf{t}_{G_j,2} \\ \vdots \\ \mathbf{t}_{G_j,|G_j|} \end{bmatrix}$$

Then we introduce a difference-selection matrix $E_{G_j,x}^2$ and $E_{G_j,t}^2$ such that

$$E_{G_j,x}^2 \mathbf{x}_{G_j} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -1 & -1 & \cdots & 1 \\ \vdots & & & & \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_{G_j,1} \\ \mathbf{x}_{G_j,2} \\ \vdots \\ \mathbf{x}_{G_j,|G_j|} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{G_j,1} + \mathbf{x}_{G_j,2} \\ \mathbf{x}_{G_j,1} - \mathbf{x}_{G_j,2} + \mathbf{x}_{G_j,3} \\ \vdots \\ \mathbf{x}_{G_j,1} - \mathbf{x}_{G_j,2} \dots - \mathbf{x}_{G_j,(|G_j|-1)} + \mathbf{x}_{G_j,|G_j|} \\ \mathbf{x}_{G_j,2} + \mathbf{x}_{G_j,3} \\ \mathbf{x}_{G_j,2} - \mathbf{x}_{G_j,3} + \mathbf{x}_{G_j,4} \\ \vdots \\ \mathbf{x}_{G_j,2} - \mathbf{x}_{G_j,3} \dots - \mathbf{x}_{G_j,(|G_j|-1)} + \mathbf{x}_{G_j,|G_j|} \\ \vdots \\ \mathbf{x}_{G_j,(|G_j|-2)} - \mathbf{x}_{G_j,(|G_j|-1)} + \mathbf{x}_{G_j,|G_j|} \\ \mathbf{x}_{G_j,(|G_j|-1)} + \mathbf{x}_{G_j,|G_j|} \end{bmatrix}$$

$$E_{G_j,t}^2 \mathbf{t}_{G_j} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \\ \vdots & & & & \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{t}_{G_j,1} \\ \mathbf{t}_{G_j,2} \\ \vdots \\ \mathbf{t}_{G_j,|G_j|} \end{bmatrix} = \begin{bmatrix} -\mathbf{t}_{G_j,1} + \mathbf{t}_{G_j,2} \\ -\mathbf{t}_{G_j,1} + \mathbf{t}_{G_j,3} \\ \vdots \\ -\mathbf{t}_{G_j,1} + \mathbf{t}_{G_j,|G_j|} \\ \vdots \\ -\mathbf{t}_{G_j,(|G_j|-1)} + \mathbf{t}_{G_j,|G_j|} \end{bmatrix}$$

Therefore, the given inequality is equivalent to

$$R \mathbf{1} \geq \text{diag}(E_{G_j,t}^2 \mathbf{t}_{G_j}) (E_{G_j,x}^2 \mathbf{x}_{G_j} - \mathbf{1})$$

Since

$$\mathbf{x}_{G_j} = E_{G_j}^1 \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix} = E_{G_j}^1 \left\{ \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ \mathbf{0}_{N \times 1} \\ 1 \end{bmatrix} \right\},$$

we substitute this into the previous inequality and obtain

$$\begin{aligned} R \mathbf{1} &\geq \text{diag}(E_{G_j,t}^2 \mathbf{t}_{G_j}) \left[E_{G_j,x}^2 E_{G_j}^1 \left\{ \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ \mathbf{0}_{N \times 1} \\ 1 \end{bmatrix} \right\} - \mathbf{1} \right] \\ &= \underbrace{\text{diag}(E_{G_j,t}^2 \mathbf{t}_{G_j}) E_{G_j,x}^2 E_{G_j}^1}_{\text{constant } C_j} \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix} \mathbf{x} + \underbrace{\text{diag}(E_{G_j,t}^2 \mathbf{t}_{G_j}) \left\{ E_{G_j,x}^2 E_{G_j}^1 \begin{bmatrix} 1 \\ \mathbf{0}_{N \times 1} \\ 1 \end{bmatrix} - \mathbf{1} \right\}}_{\text{constant } \mathbf{d}_j} \end{aligned}$$

We can define

$$C_j := \text{diag}(E_{G_j,t}^2 \mathbf{t}_{G_j}) E_{G_j,x}^2 E_{G_j}^1 \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix} \Rightarrow C_j : \text{constant},$$

$$\mathbf{d}_j := \text{diag}(E_{G_j,t}^2 \mathbf{t}_{G_j}) \left\{ E_{G_j,x}^2 E_{G_j}^1 \begin{bmatrix} 1 \\ \mathbf{0}_{N \times 1} \\ 1 \end{bmatrix} - \mathbf{1} \right\} \Rightarrow \mathbf{d}_j : \text{constant}.$$

Since

$$\mathbf{t}_{G_j} = E_{G_j}^1 \begin{bmatrix} t_{\text{start}} \\ \mathbf{t} \\ t_{\text{end}} \end{bmatrix} = E_{G_j}^1 \left\{ \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix} \mathbf{t} + \begin{bmatrix} t_{\text{start}} \\ \mathbf{0}_{N \times 1} \\ t_{\text{end}} \end{bmatrix} \right\},$$

We can get:

$$C_j := \text{diag} \left(E_{G_j,t}^2 E_{G_j}^1 \begin{bmatrix} t_{\text{start}} \\ \mathbf{t} \\ t_{\text{end}} \end{bmatrix} \right) E_{G_j,x}^2 E_{G_j}^1 \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix} \Rightarrow C_j : \text{constant},$$

$$\mathbf{d}_j := \text{diag} \left(E_{G_j,t}^2 E_{G_j}^1 \begin{bmatrix} t_{\text{start}} \\ \mathbf{t} \\ t_{\text{end}} \end{bmatrix} \right) \left\{ E_{G_j,x}^2 E_{G_j}^1 \begin{bmatrix} 1 \\ \mathbf{0}_{N \times 1} \\ 1 \end{bmatrix} - \mathbf{1} \right\} \Rightarrow \mathbf{d}_j : \text{constant}$$

Therefore, the inequality becomes

$$R\mathbf{1} \geq C_j \mathbf{x} + \mathbf{d}_j$$

Stacking $j = 1 \dots q$

$$R\mathbf{1} \geq \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_q \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \vdots \\ \mathbf{d}_q \end{bmatrix} = C\mathbf{x} + \mathbf{d}$$

4.4. L_∞ optimization problem

$$\min R$$

$$\text{s.t. } A\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \in \{0,1\}^N$$

$$R\mathbf{1} \geq C\mathbf{x} + \mathbf{d}$$

$$A = \begin{bmatrix} E_{S_1,x}^2 E_{S_1}^1 \\ E_{S_2,x}^2 E_{S_2}^1 \\ \vdots \\ E_{S_p,x}^2 E_{S_p}^1 \end{bmatrix}, \quad \mathbf{b} = \mathbf{1} + \frac{1}{\tau} \begin{bmatrix} E_{S_1,t}^2 E_{S_1}^1 \\ E_{S_2,t}^2 E_{S_2}^1 \\ \vdots \\ E_{S_p,t}^2 E_{S_p}^1 \end{bmatrix} \mathbf{t}, \quad C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_q \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \vdots \\ \mathbf{d}_q \end{bmatrix}$$

$$C_j := \text{diag} \left(E_{G_j,t}^2 E_{G_j}^1 \begin{bmatrix} t_{\text{start}} \\ \mathbf{t} \\ t_{\text{end}} \end{bmatrix} \right) E_{G_j,x}^2 E_{G_j}^1 \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix} \Rightarrow C_j : \text{constant},$$

$$\mathbf{d}_j := \text{diag} \left(E_{G_j,t}^2 E_{G_j}^1 \begin{bmatrix} t_{\text{start}} \\ \mathbf{t} \\ t_{\text{end}} \end{bmatrix} \right) \left\{ E_{G_j,x}^2 E_{G_j}^1 \begin{bmatrix} 1 \\ \mathbf{0}_{N \times 1} \\ 1 \end{bmatrix} - \mathbf{1} \right\} \Rightarrow \mathbf{d}_j : \text{constant}$$

4.5. L_2 optimization problem by minimizing square sum of revisit time for each GS

From previous section, we could get:

$$E_{G_j, \mathbf{x}}^2 \mathbf{x}_{G_j} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -1 & -1 & \cdots & 1 \\ \vdots & & & & \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_{G_j,1} \\ \mathbf{x}_{G_j,2} \\ \vdots \\ \mathbf{x}_{G_j,|G_j|} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{G_j,1} + \mathbf{x}_{G_j,2} \\ \mathbf{x}_{G_j,1} - \mathbf{x}_{G_j,2} + \mathbf{x}_{G_j,3} \\ \vdots \\ \mathbf{x}_{G_j,1} - \mathbf{x}_{G_j,2} \dots - \mathbf{x}_{G_j,(|G_j|-1)} + \mathbf{x}_{G_j,|G_j|} \\ \mathbf{x}_{G_j,2} + \mathbf{x}_{G_j,3} \\ \mathbf{x}_{G_j,2} - \mathbf{x}_{G_j,3} + \mathbf{x}_{G_j,4} \\ \vdots \\ \mathbf{x}_{G_j,2} - \mathbf{x}_{G_j,3} \dots - \mathbf{x}_{G_j,(|G_j|-1)} + \mathbf{x}_{G_j,|G_j|} \\ \vdots \\ \mathbf{x}_{G_j,(|G_j|-2)} - \mathbf{x}_{G_j,(|G_j|-1)} + \mathbf{x}_{G_j,|G_j|} \\ \mathbf{x}_{G_j,(|G_j|-1)} + \mathbf{x}_{G_j,|G_j|} \end{bmatrix}$$

$$E_{G_j, \mathbf{t}}^2 \mathbf{t}_{G_j} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \\ \vdots & & & & \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{t}_{G_j,1} \\ \mathbf{t}_{G_j,2} \\ \vdots \\ \mathbf{t}_{G_j,|G_j|} \end{bmatrix} = \begin{bmatrix} -\mathbf{t}_{G_j,1} + \mathbf{t}_{G_j,2} \\ -\mathbf{t}_{G_j,1} + \mathbf{t}_{G_j,3} \\ \vdots \\ -\mathbf{t}_{G_j,1} + \mathbf{t}_{G_j,|G_j|} \\ \vdots \\ -\mathbf{t}_{G_j,(|G_j|-1)} + \mathbf{t}_{G_j,|G_j|} \end{bmatrix}$$

Let's define $\mathbf{y} \in R^M$, $M = \sum_{j=1}^q \frac{(|G_j|+2)(|G_j|+1)}{2}$ and $\mathbf{1} \leq \mathbf{y} \leq 2 \times \mathbf{1}$, then we can add follow constraints:

$$\mathbf{y} \geq \begin{bmatrix} E_{G_1, \mathbf{x}}^2 \mathbf{x}_{G_1} \\ E_{G_2, \mathbf{x}}^2 \mathbf{x}_{G_2} \\ \vdots \\ E_{G_q, \mathbf{x}}^2 \mathbf{x}_{G_q} \end{bmatrix} = \begin{bmatrix} E_{G_1, \mathbf{x}}^2 E_{G_1}^1 \\ E_{G_2, \mathbf{x}}^2 E_{G_2}^1 \\ \vdots \\ E_{G_q, \mathbf{x}}^2 E_{G_q}^1 \end{bmatrix} \left\{ \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ \mathbf{0}_{N \times 1} \\ 1 \end{bmatrix} \right\}$$

Therefore, the inequality becomes

$$\mathbf{y} \geq \underbrace{\begin{bmatrix} E_{G_1, \mathbf{x}}^2 E_{G_1}^1 \\ E_{G_2, \mathbf{x}}^2 E_{G_2}^1 \\ \vdots \\ E_{G_q, \mathbf{x}}^2 E_{G_q}^1 \end{bmatrix}}_E \underbrace{\begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix}}_f \mathbf{x} + \underbrace{\begin{bmatrix} E_{G_1, \mathbf{x}}^2 E_{G_1}^1 \\ E_{G_2, \mathbf{x}}^2 E_{G_2}^1 \\ \vdots \\ E_{G_q, \mathbf{x}}^2 E_{G_q}^1 \end{bmatrix}}_f \begin{bmatrix} 1 \\ \mathbf{0}_{N \times 1} \\ 1 \end{bmatrix}$$

Also, we can define square-sum of revisit time by \mathbf{y} and $E_{G_j, \mathbf{t}}^2 \mathbf{t}_{G_j}$

$$\left\| \begin{bmatrix} E_{G_1, \mathbf{t}}^2 \mathbf{t}_{G_1} \\ E_{G_2, \mathbf{t}}^2 \mathbf{t}_{G_2} \\ \vdots \\ E_{G_q, \mathbf{t}}^2 \mathbf{t}_{G_q} \end{bmatrix}^\top (\mathbf{y} - \mathbf{1}) \right\|_2^2 = (\mathbf{y} - \mathbf{1})^\top \left[\text{diag} \left(\begin{bmatrix} E_{G_1, \mathbf{t}}^2 \mathbf{t}_{G_1} \\ E_{G_2, \mathbf{t}}^2 \mathbf{t}_{G_2} \\ \vdots \\ E_{G_q, \mathbf{t}}^2 \mathbf{t}_{G_q} \end{bmatrix} \right) \right]^2 (\mathbf{y} - \mathbf{1})$$

Since we know

$$\begin{bmatrix} E_{G_1,t}^2 \mathbf{t}_{G_1} \\ E_{G_2,t}^2 \mathbf{t}_{G_2} \\ \vdots \\ E_{G_q,t}^2 \mathbf{t}_{G_q} \end{bmatrix} = \underbrace{\begin{bmatrix} E_{G_1,t}^2 E_{G_1}^1 \\ E_{G_2,t}^2 E_{G_2}^1 \\ \vdots \\ E_{G_q,t}^2 E_{G_q}^1 \end{bmatrix}}_{\text{Constant } G} \begin{bmatrix} t_{\text{start}} \\ \mathbf{t} \\ t_{\text{end}} \end{bmatrix}$$

The cost function becomes

$$(\mathbf{y} - \mathbf{1})^\top \underbrace{\text{diag} \left(\begin{bmatrix} E_{G_1,t}^2 E_{G_1}^1 \\ E_{G_2,t}^2 E_{G_2}^1 \\ \vdots \\ E_{G_q,t}^2 E_{G_q}^1 \end{bmatrix} \begin{bmatrix} t_{\text{start}} \\ \mathbf{t} \\ t_{\text{end}} \end{bmatrix} \right)}_{\text{Constant } G} (\mathbf{y} - \mathbf{1})$$

If we substitute $\mathbf{y} - \mathbf{1} = \mathbf{z}$, then $\mathbf{0} \leq \mathbf{z} \leq \mathbf{1}$ and $\mathbf{z} \geq E\mathbf{x} + \mathbf{f} - \mathbf{1}$. Also, the objective function $(\mathbf{y} - \mathbf{1})^\top G(\mathbf{y} - \mathbf{1})$ is equivalent to $\mathbf{1}^\top G\mathbf{z}$, which is linear constraint.

4.6. L_2 optimization problem

$$\begin{aligned} & \min \mathbf{1}^\top G\mathbf{z} \\ \text{s.t. } & A\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \in \{0, 1\}^N \\ & \mathbf{z} \geq E\mathbf{x} + \mathbf{f} - \mathbf{1}, \quad \mathbf{z} \geq \mathbf{0} \\ & \mathbf{z} \in R^M, M = \sum_{j=1}^q \frac{(|G_j| + 2)(|G_j| + 1)}{2} \\ & A = \begin{bmatrix} E_{S_1,x}^2 E_{S_1}^1 \\ E_{S_2,x}^2 E_{S_2}^1 \\ \vdots \\ E_{S_p,x}^2 E_{S_p}^1 \end{bmatrix}, \quad \mathbf{b} = \mathbf{1} + \frac{1}{\tau} \begin{bmatrix} E_{S_1,t}^2 E_{S_1}^1 \\ E_{S_2,t}^2 E_{S_2}^1 \\ \vdots \\ E_{S_p,t}^2 E_{S_p}^1 \end{bmatrix} \mathbf{t} \\ & E = \begin{bmatrix} E_{G_1,x}^2 E_{G_1}^1 \\ E_{G_2,x}^2 E_{G_2}^1 \\ \vdots \\ E_{G_q,x}^2 E_{G_q}^1 \end{bmatrix} \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} E_{G_1,x}^2 E_{G_1}^1 \\ E_{G_2,x}^2 E_{G_2}^1 \\ \vdots \\ E_{G_q,x}^2 E_{G_q}^1 \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{0}_{N \times 1} \\ 1 \end{bmatrix} \\ & G = \left[\text{diag} \left(\begin{bmatrix} E_{G_1,t}^2 E_{G_1}^1 \\ E_{G_2,t}^2 E_{G_2}^1 \\ \vdots \\ E_{G_q,t}^2 E_{G_q}^1 \end{bmatrix} \begin{bmatrix} t_{\text{start}} \\ \mathbf{t} \\ t_{\text{end}} \end{bmatrix} \right) \right]^2 \end{aligned}$$

4.7. Rewriting L_∞ optimization problem

$$\begin{aligned}
& \min R \\
\text{s.t. } & A\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \in \{0, 1\}^N \\
& R\mathbf{1} \geq H(E\mathbf{x} + \mathbf{f} - \mathbf{1}) \\
A = & \begin{bmatrix} E_{S_1,x}^2 E_{S_1}^1 \\ E_{S_2,x}^2 E_{S_2}^1 \\ \vdots \\ E_{S_p,x}^2 E_{S_p}^1 \end{bmatrix}, \quad \mathbf{b} = \mathbf{1} + \frac{1}{\tau} \begin{bmatrix} E_{S_1,t}^2 E_{S_1}^1 \\ E_{S_2,t}^2 E_{S_2}^1 \\ \vdots \\ E_{S_p,t}^2 E_{S_p}^1 \end{bmatrix} \mathbf{t} \\
E = & \begin{bmatrix} E_{G_1,x}^2 E_{G_1}^1 \\ E_{G_2,x}^2 E_{G_2}^1 \\ \vdots \\ E_{G_q,x}^2 E_{G_q}^1 \end{bmatrix} \begin{bmatrix} \mathbf{0}_{1 \times N} \\ \mathbf{I}_{N \times N} \\ \mathbf{0}_{1 \times N} \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} E_{G_1,x}^2 E_{G_1}^1 \\ E_{G_2,x}^2 E_{G_2}^1 \\ \vdots \\ E_{G_q,x}^2 E_{G_q}^1 \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{0}_{N \times 1} \\ 1 \end{bmatrix} \\
H = & \text{diag} \left(\begin{bmatrix} E_{G_1,t}^2 E_{G_1}^1 \\ E_{G_2,t}^2 E_{G_2}^1 \\ \vdots \\ E_{G_q,t}^2 E_{G_q}^1 \end{bmatrix} \begin{bmatrix} t_{\text{start}} \\ \mathbf{t} \\ t_{\text{end}} \end{bmatrix} \right)
\end{aligned}$$