Redstone Control Performance

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I. Theoretical Review: PID Controller

1. **Angular Momentum Equation**

$\boldsymbol{H}_{\text{SAT}} = \boldsymbol{H}_{\text{body}} + \boldsymbol{H}_{\text{RWA}} = J\omega_B + J\omega_{\text{RWA}}$

2. **Euler Equations of Rigid Body**

$\frac{d\boldsymbol{H}_{\text{SAT}}}{dt} = J\dot{\omega}_B + I\dot{\omega}_{\text{RWA}} + \omega_B \times (J\omega_B + J\omega_{\text{RWA}}) = \tau_d$

let $\tau_d - \omega_B \times (J\omega_B + J\omega_{\text{RWA}}) = \tau_{\text{dw}}$, and assume $\tau_{\text{dw}}(t)$ is step input

Disturbance term includes gyroscopic torque, bearing friction

$\Rightarrow \tau_{\text{dw}} = J\dot{\omega}_B + I\dot{\omega}_{\text{RWA}}$

3. **Definition of Attitude Error**

Reference input : R^*_{BI} , State input : $R_{\text{BI}} \Rightarrow$ DCM based control

$\Rightarrow R^*_{\text{BI}} = R_E R_{\text{BI}} \iff R_E = R^*_{\text{BI}} R^T_{\text{BI}}$

$\dot{R}^*_{\text{BI}} = 0 = \dot{R}_E R_{\text{BI}} + R_E \dot{R}_{\text{BI}} = \dot{R}_E R_{\text{BI}} + R_E[-\omega_B \times]\, R_{\text{BI}}$

$\Rightarrow \dot{R}_E R_{\text{BI}} = R_E[\omega_B \times]\, R_{\text{BI}} \;\therefore\; \dot{R}_E = R_E[\omega_B \times]$

$$2\boldsymbol{e} = \begin{bmatrix} R_E(2,3) - R_E(3,2) \\ R_E(3,1) - R_E(1,3) \\ R_E(1,2) - R_E(2,1) \end{bmatrix}, \quad 2\dot{\boldsymbol{e}} = \begin{bmatrix} \dot{R}_E(2,3) - \dot{R}_E(3,2) \\ \dot{R}_E(3,1) - \dot{R}_E(1,3) \\ \dot{R}_E(1,2) - \dot{R}_E(2,1) \end{bmatrix}$$

$$2\dot{\boldsymbol{e}} = \begin{bmatrix} R_E(2,1)\,\omega_{\text{By}} - R_E(3,3)\,\omega_{\text{Bx}} - R_E(2,2)\,\omega_{\text{Bx}} + R_E(3,1)\,\omega_{\text{Bz}} \\ R_E(2,2)\,\omega_{\text{Bx}} - R_E(1,1)\,\omega_{\text{By}} - R_E(3,3)\,\omega_{\text{By}} + R_E(3,2)\,\omega_{\text{Bz}} \\ R_E(1,3)\,\omega_{\text{Bx}} - R_E(2,1)\,\omega_{\text{By}} - R_E(1,1)\,\omega_{\text{Bz}} - R_E(2,2)\,\omega_{\text{Bz}} \end{bmatrix}$$

$$\approx \begin{bmatrix} -R_E(3,3)\,\omega_{\text{Bx}} - R_E(2,2)\,\omega_{\text{Bx}} \\ -R_E(1,1)\,\omega_{\text{By}} - R_E(3,3)\,\omega_{\text{By}} \\ -R_E(1,1)\,\omega_{\text{Bz}} - R_E(2,2)\,\omega_{\text{Bz}} \end{bmatrix} \approx -2\omega_B$$

(We Assume non $-$ diagonal terms of R_E are small)

4. **Approximation and Connecting to Euler Angle**

We can approximate $\boldsymbol{e}$ as error euler angle : $\boldsymbol{e} \approx \begin{bmatrix} \delta\phi \\ \delta\theta \\ \delta\phi \end{bmatrix} = \begin{bmatrix} \phi^* - \phi \\ \theta^* - \theta \\ \psi^* - \psi \end{bmatrix}$

We are using $3 - 1 - 2$ errer euler angles

$$\dot{\boldsymbol{e}} \approx -\frac{1}{\cos\delta\phi}\begin{bmatrix} \cos\delta\phi\cos\delta\theta & 0 & \cos\delta\phi\sin\delta\theta \\ \sin\delta\phi\sin\delta\theta & \cos\delta\phi & -\cos\delta\theta\sin\delta\phi \\ -\sin\delta\theta & 0 & \cos\delta\theta \end{bmatrix}\omega_B$$

We are assuming $\delta\phi, \delta\theta, \delta\psi \approx 0$

$$\therefore \dot{\boldsymbol{e}} \approx \begin{bmatrix} \dot{\phi}^* - \dot{\phi} \\ \dot{\theta}^* - \dot{\theta} \\ \dot{\psi}^* - \dot{\psi} \end{bmatrix} \approx -\omega_B \Rightarrow \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \approx \omega_B$$

let $\begin{bmatrix} \phi^* - \phi \\ \theta^* - \theta \\ \psi^* - \psi \end{bmatrix} = \boldsymbol{\phi}^* - \boldsymbol{\phi}$, then we can get following relationships :

$$\boldsymbol{e} \approx \boldsymbol{\phi}^* - \boldsymbol{\phi},\ \ \dot{\boldsymbol{e}} \approx -\dot{\boldsymbol{\phi}} \approx -\omega_B,\ \ \ddot{\boldsymbol{e}} \approx -\ddot{\boldsymbol{\phi}} \approx -\dot{\omega}_B,$$

$$\int \boldsymbol{e}\,\mathrm{dt} \approx \int (\boldsymbol{\phi}^* - \boldsymbol{\phi})\mathrm{dt}$$

5. **Derivation of Transfer Function**

We can Model Reaction Wheel Torque by PID Control input :

$$\tau_{\mathrm{RWA}} = -I_{\mathrm{RWA}}\dot{\omega}_{\mathrm{RWA}} = K_P\,\boldsymbol{e} + K_I\int \boldsymbol{e}\,\mathrm{dt} + K_d\dot{\boldsymbol{e}}$$

$$= K_P\,(\phi^* - \phi) + K_I\int (\phi^* - \phi)\,\mathrm{dt} - K_d\dot{\phi}$$

From angular momentum equation :

$$\tau_{\mathrm{dw}} = J\dot{\omega}_B + I\dot{\omega}_{\mathrm{RWA}} = J\ddot{\phi} + K_P\,(\phi - \phi^*) + K_I\int (\phi - \phi^*)\,\mathrm{dt} + K_d\dot{\phi}$$

$\Rightarrow$ Laplace Transform

$$\tau_{\mathrm{dw}}(s) = \mathrm{Js}^2\phi(s) + K_P(\phi(s) - \phi^*(s)) + \frac{K_I}{s}(\phi(s) - \phi^*(s)) + K_D s\phi(s)$$

$$\left[\mathrm{Js}^2 + K_p + \frac{K_I}{s} + K_D s\right]\phi(s) - \left[K_P + \frac{K_I}{s}\right]\phi^*(s) = \tau_{\mathrm{dw}}(s)$$

$$\therefore\ \phi(s) = \frac{s\,\tau_{\mathrm{dw}}(s) + (K_I + K_P s)\phi^*(s)}{\mathrm{Js}^3 + K_D s + K_P s + K_I}$$

We can assume $\tau_{\mathrm{dw}}(s)$ and $\phi^*(s)$ are step input :

$$\tau_{\mathrm{dw}}(s) = \frac{\tau_{\mathrm{dw}}}{s},\ \ \phi^*(s) = \frac{\phi^*}{s}$$

Applying Final Value Theorem :

$$\lim_{t\to\infty}\phi(t) = \lim_{s\to 0}s\phi(s) = \lim_{s\to 0}\frac{s\,\tau_{\mathrm{dw}} + (K_I + K_P s)\phi^*}{\mathrm{Js}^3 + K_D s + K_P s + K_I} = \phi^*$$

We can conclude that in given system, steady $-$ state error is 0.

Let's assume $|\tau_{\mathrm{dw}}| \ll |\phi^*|$ (Or we can quantify $\tau_{\mathrm{dw}} = k\phi^*$)

then tranfer function is $\dfrac{\phi(s)}{\phi^*(s)} = \dfrac{K_P s + K_I}{\mathrm{Js}^3 + K_D s + K_P s + K_I}$

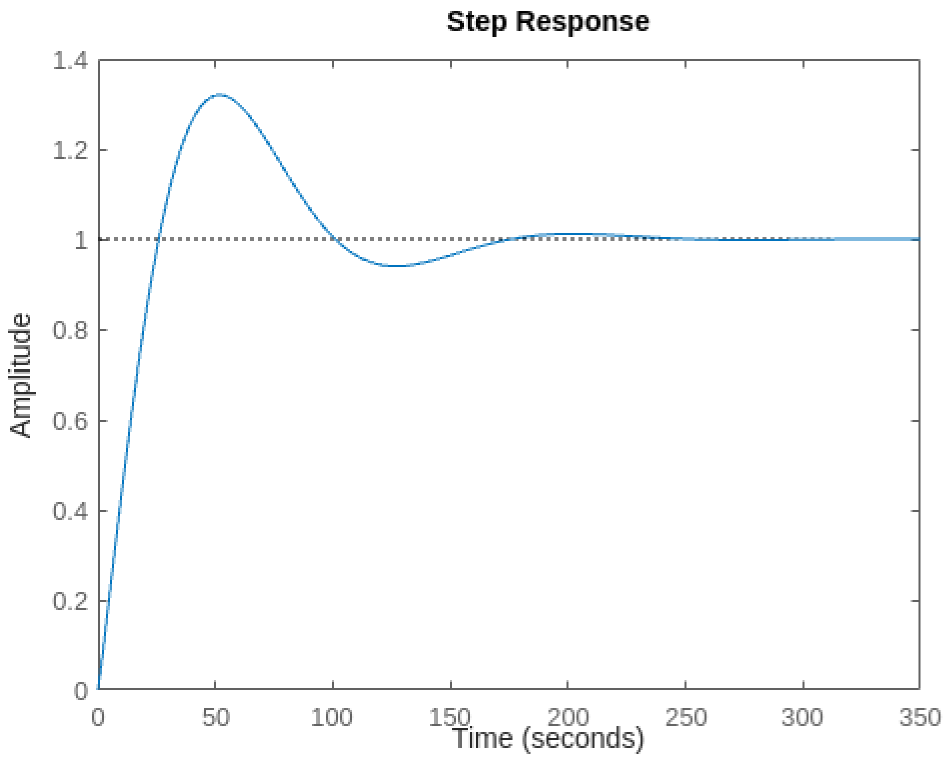
And Closed loop characteristic equation is $\mathrm{Js}^3 + K_D s + K_P s + K_I$

II. Call the Parameters

```
addpath("Desktop/Redstone_Project/RS_LL/RS_LL_2_SAT_Attitude_Controller/")
SATParamsScript;


J = diag([200;100;150]);

K_P = 20;
K_I = 1;
K_D = 450;

Transfer_function_1 = tf([K_P,K_I],[J(2,2),K_D,K_P,K_I])
```

```
Transfer_function_1 =

            20 s + 1
  ----------------------------
  100 s^3 + 450 s^2 + 20 s + 1

Continuous-time transfer function.
Model Properties
```

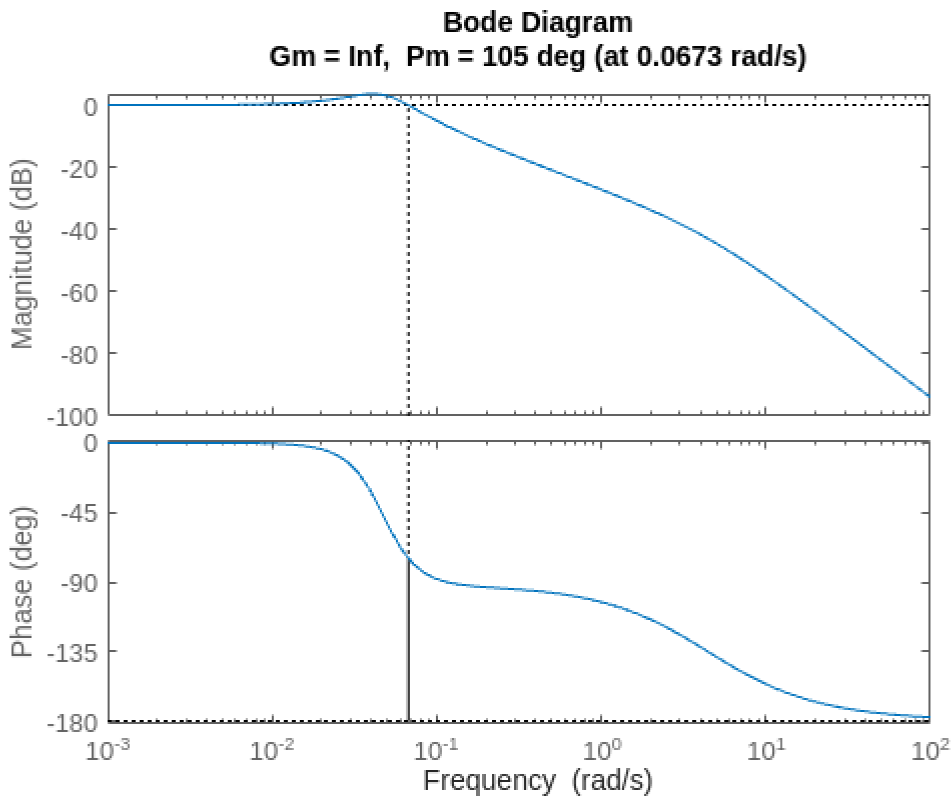
```
step(Transfer_function_1)
```




```
stepinfo(Transfer_function_1)
```

```
ans = struct with fields:
         RiseTime: 20.0024
    TransientTime: 160.2480
     SettlingTime: 160.2480
      SettlingMin: 0.9121
      SettlingMax: 1.3202
        Overshoot: 32.0199
       Undershoot: 0
             Peak: 1.3202
         PeakTime: 51.8794
```

```
margin(Transfer_function_1)
```




```
allmargin(Transfer_function_1)
```

```
ans = struct with fields:
      GainMargin: Inf
     GMFrequency: Inf
     PhaseMargin: [-180 105.0889]
     PMFrequency: [0 0.0673]
     DelayMargin: [Inf 27.2429]
     DMFrequency: [0 0.0673]
          Stable: 1
```

```
bandwidth(Transfer_function_1)
```

```
ans = 0.0850
```

closed loop control bandwidth

III. Theoretical Review: Motor Dynamics

Motor Dynamics Equation :

$$\frac{\omega(s)}{e_a(s)} = \frac{c_m}{\tau_m s + 1}$$

c_m : voltage (V) to Angular Rate (Rad/s)

τ_m : Phase Lag time constant

Basic Equation :

$$\boldsymbol{e}_a = \frac{1}{c_m}\left[-I_{\text{RWA}}^{-1}\tau_{\text{RWA}}\tau_m + \omega_{\text{RWA}}\right]$$

Environment Parameter : $\tau_m, c_m, I_{\text{RWA}}$

Input Parameter : $\tau_{\text{RWA}}, \omega_{\text{RWA}}$

Output Parameter : $\boldsymbol{e}_a$

```
% 1 Axis Simulation
c_m = 1;
tau_m = 1/20;
I_RWA = 1;

omega_RWA = 2;
tau_RWA = linspace(-30,30);

e_a = 1/c_m * (-I_RWA^-1 * tau_m * tau_RWA + omega_RWA);
plot(tau_RWA,e_a)
grid on
```