Chapter 7. Relative Motion and Rendezvous

WorkerInSpace

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Example 7.2

Problem Statement

A space station and spacecraft are in orbits with the following parameters

$$\begin{bmatrix} & \text{Space station} & \text{Spacecraft} \\ \text{Perigee} \times \text{apogee(altitude)} & \text{300km circular} & 318.5 \times 515.51 \text{ km} \\ \text{Period (computed using above data)} & 1.508 \text{ hr} & 1.548 \text{ hr} \\ \text{True anomaly, } \theta & 60° & 349.65° \\ \text{Inclination, } i & 40° & 40.130° \\ \text{RA, } \Omega & 20° & 19.819° \\ \text{Argument of perigee, } \omega & 0° \text{ (arbitrary)} & 70.662° \end{bmatrix}$$

Compute the total delta-v required for an eight-hour, two-impulse rendezvous trajectory.

Solution

We use the given data in Algorithm 4.1 to obtain the state vectors of the two spacecraft in the geocentric equatorial frame.

Space station:

$$r_0 = 1622.39\hat{\mathbf{I}} + 5305.10\hat{\mathbf{J}} + 3717.44\hat{\mathbf{K}} \text{ (km)}$$
$$v_0 = -7.29977\hat{\mathbf{I}} + 0.492357\hat{\mathbf{J}} + 2.48318\hat{\mathbf{K}} \text{ (km/}s\text{)}$$

Spacecraft:

$$r = 1612.75\hat{\mathbf{I}} + 5310.19\hat{\mathbf{J}} + 3750.33\hat{\mathbf{K}} \text{ (km)}$$
$$v = -7.35521\hat{\mathbf{I}} + 0.463856\hat{\mathbf{J}} + 2.46920\hat{\mathbf{K}} \text{ (km/}s\text{)}$$

The space station reference frame unit vectors (at this instant) are, by definition

$$\hat{\mathbf{i}} = \frac{r_0}{|r_0|} = 0.242945\hat{\mathbf{I}} + 0.794415\hat{\mathbf{J}} + 0.556670\hat{\mathbf{K}}$$

$$\hat{\mathbf{j}} = \frac{v_0}{|v_0|} = -0.944799\hat{\mathbf{I}} + 0.063725\hat{\mathbf{J}} + 0.321392\hat{\mathbf{K}}$$

$$\hat{\mathbf{k}} = \hat{\mathbf{i}} \times \hat{\mathbf{j}} = 0.219846\hat{\mathbf{I}} - 0.604023\hat{\mathbf{J}} + 0.766044\hat{\mathbf{K}}$$

Therefore, the transformation matrix from the geocentric equatorial frame into space station frame is (at this instant)

$$[\boldsymbol{Q}]_{\mathrm{Xx}} = \begin{bmatrix} 0.242945 & 0.794415 & 0.556670 \\ -0.944799 & 0.063725 & 0.321394 \\ 0.219846 & -0.604023 & 0.766044 \end{bmatrix}$$

The position vector of the spacecraft relative to the space station (in the geocentric equatorial frame) is

$$\delta\boldsymbol{r} = \boldsymbol{r} - \boldsymbol{r}_0 = -9.63980\hat{\mathbf{I}} + 5.08240\hat{\mathbf{J}} + 32.8821\hat{\mathbf{K}}\ (\mathrm{km})$$

The relative velocity is given by the formula

$$\begin{aligned} &\boldsymbol{v} = \boldsymbol{v}_0 + \Omega_{\text{space station}} \times \boldsymbol{r}_{\text{rel}} + \boldsymbol{v}_{\text{rel}} \\ &\boldsymbol{r}_{\text{rel}} = \delta\boldsymbol{r}, \boldsymbol{v}_{\text{rel}} = \delta\boldsymbol{v} \\ &\delta\boldsymbol{v} = \boldsymbol{v} - \boldsymbol{v}_0 - \Omega_{\text{space station}} \times \delta\boldsymbol{r} \end{aligned}$$

where $\Omega_{\text{space station}} = n\hat{\mathbf{k}}$ and n, the mean motion of the space station, is

$$n = \frac{v_0}{r_0} = \frac{7.72627}{6678} = 0.00115697\ \mathrm{rad}/s$$

Thus,

$$\begin{aligned} \delta\boldsymbol{v} &= \boldsymbol{v} - \boldsymbol{v}_0 - \Omega_{\text{space station}} \times \delta\boldsymbol{r} = \boldsymbol{v} - \boldsymbol{v}_0 - n\hat{\mathbf{k}} \times \delta\boldsymbol{r} \\ &= -7.35521\hat{\mathbf{I}} + 0.463856\hat{\mathbf{J}} + 2.46920\hat{\mathbf{K}} - \left(-7.29977\hat{\mathbf{I}} + 0.492357\hat{\mathbf{J}} + 2.48318\hat{\mathbf{K}}\right) \\ &\quad - 0.00115697 \cdot \left(0.219846\hat{\mathbf{I}} - 0.604023\hat{\mathbf{J}} + 0.766044\hat{\mathbf{K}}\right) \times \left(9.63980\hat{\mathbf{I}} + 5.08240\hat{\mathbf{J}} + 32.8821\hat{\mathbf{K}}\right) \\ &= -0.024854\hat{\mathbf{I}} - 0.01159370\hat{\mathbf{J}} - 0.00853577\hat{\mathbf{K}}\ (\mathrm{km}/s) \end{aligned}$$

In space station coordinates, the relative position vector $\delta\boldsymbol{r}_0$ at the beginning of the rendezvous maneuver is

$$\begin{aligned} [\delta\boldsymbol{r}_0] = [\boldsymbol{Q}]_{\mathrm{Xx}}\delta\boldsymbol{r} &= \begin{bmatrix} 0.242945 & 0.794415 & 0.556670 \\ -0.944799 & 0.063725 & 0.321394 \\ 0.219846 & -0.604023 & 0.766044 \end{bmatrix} \begin{bmatrix} -9.63980 \\ 5.08240 \\ 32.8821 \end{bmatrix} \\ &= \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix} (\mathrm{km}) \end{aligned}$$

Likewise, the relative velocity $\delta\boldsymbol{v}_0^-$ just before launch into the rendezvous trajectory is

$$\left[\delta\boldsymbol{v}_0^-\right] = [\boldsymbol{Q}]_{\mathrm{Xx}}[\delta\boldsymbol{v}] = \begin{bmatrix} 0.242945 & 0.794415 & 0.556670 \\ -0.944799 & 0.063725 & 0.321394 \\ 0.219846 & -0.604023 & 0.766044 \end{bmatrix} \begin{bmatrix} -0.024854 \\ -0.0115937 \\ -0.00853577 \end{bmatrix}$$
$$= \begin{bmatrix} -0.02000 \\ 0.02000 \\ -0.005000 \end{bmatrix} (\mathrm{km}/s)$$

The Clohessy-Wiltshire matrices, for $t = t_f = 8\mathrm{hr} = 28800s$ and $n = 0.00115697\mathrm{rad}/s$, are

$$[\delta\boldsymbol{r}(t)] = [\Phi_{\mathrm{rr}}(t)][\delta\boldsymbol{r}_0] + [\Phi_{\mathrm{rv}}(t)][\delta\boldsymbol{v}_0]$$
$$[\delta\boldsymbol{v}(t)] = [\Phi_{\mathrm{vr}}(t)][\delta\boldsymbol{r}_0] + [\Phi_{\mathrm{vv}}(t)][\delta\boldsymbol{v}_0]$$

$$[\Phi_{\mathrm{rr}}(t)] = \begin{bmatrix} 4 - 3\cos nt & 0 & 0 \\ 6(\sin nt - nt) & 1 & 0 \\ 0 & 0 & \cos nt \end{bmatrix} = \begin{bmatrix} 4.98383 & 0 & 0 \\ -194.257 & 1 & 0 \\ 0 & 0 & -0.327942 \end{bmatrix}$$

$$[\Phi_{\mathrm{rv}}(t)] = \begin{bmatrix} \dfrac{\sin nt}{n} & \dfrac{2}{n}(1 - \cos nt) & 0 \\ \dfrac{2}{n}(\cos nt - 1) & \dfrac{4}{n}\sin nt - 3t & 0 \\ 0 & 0 & \dfrac{1}{n}\sin nt \end{bmatrix} = \begin{bmatrix} 816.525 & 2295.54 & 0 \\ -2205.54 & -83133.9 & 0 \\ 0 & 0 & 816.525 \end{bmatrix}$$

$$[\Phi_{\mathrm{vr}}(t)] = \begin{bmatrix} 3n\sin nt & 0 & 0 \\ 6n(\cos nt - 1) & 0 & 0 \\ 0 & 0 & -n\sin nt \end{bmatrix} = \begin{bmatrix} 0.00327897 & 0 & 0 \\ -0.00921837 & 0 & 0 \\ 0 & 0 & -0.00109299 \end{bmatrix}$$

$$[\Phi_{\mathrm{vv}}(t)] = \begin{bmatrix} \cos nt & 2\sin nt & 0 \\ -2\sin nt & 4\cos nt - 3 & 0 \\ 0 & 0 & \cos nt \end{bmatrix} = \begin{bmatrix} -0.327942 & 1.88940 & 0 \\ -1.88940 & -4.31177 & 0 \\ 0 & 0 & -0.327942 \end{bmatrix}$$

From equation $\left[\delta\boldsymbol{v}_0^+\right] = -[\Phi_{\mathrm{rv}}(t_f)]^{-1}[\Phi_{\mathrm{rr}}(t_f)][\delta\boldsymbol{r}_0]$, we find $\delta\boldsymbol{v}_0^+$

$$\begin{bmatrix} \delta u_0^+ \\ \delta v_0^+ \\ \delta w_0^+ \end{bmatrix} = -\begin{bmatrix} 816.525 & 2295.54 & 0 \\ -2205.54 & -83133.9 & 0 \\ 0 & 0 & 816.525 \end{bmatrix}^{-1} \begin{bmatrix} 4.98383 & 0 & 0 \\ -194.257 & 1 & 0 \\ 0 & 0 & -0.327942 \end{bmatrix} \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix}$$
$$= -\begin{bmatrix} 816.525 & 2295.54 & 0 \\ -2205.54 & -83133.9 & 0 \\ 0 & 0 & 816.525 \end{bmatrix}^{-1} \begin{bmatrix} 99.6765 \\ -3865.14 \\ -6.55884 \end{bmatrix}$$
$$= \begin{bmatrix} 0.00936084 \\ -0.0467514 \\ 0.00803263 \end{bmatrix} (\mathrm{km}/s)$$

From equation $\left[\delta \boldsymbol{v}_f^-\right] = [\Phi_{\mathrm{vr}}(t_f)][\delta \boldsymbol{r}_0] + [\Phi_{\mathrm{vv}}(t_f)]\left[\delta \boldsymbol{v}_0^+\right]$

$$\begin{bmatrix} \delta u_f^- \\ \delta v_f^- \\ \delta w_f^- \end{bmatrix} = \begin{bmatrix} 0.00327897 & 0 & 0 \\ -0.00921837 & 0 & 0 \\ 0 & 0 & -0.00109299 \end{bmatrix}\begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix} + \begin{bmatrix} -0.327942 & 1.88940 & 0 \\ -1.88940 & -4.31177 & 0 \\ 0 & 0 & -0.327942 \end{bmatrix}\begin{bmatrix} 0.00936084 \\ -0.0467514 \\ 0.00803263 \end{bmatrix}$$
$$= \begin{bmatrix} -0.2528223 \\ -0.000472444 \\ -0.0222449 \end{bmatrix}(\mathrm{km}/s)$$

Delta-v at the beginning of the rendezvous maneuver is found as

$$[\Delta \boldsymbol{v}_0] = \left[\delta \boldsymbol{v}_0^+\right] - \left[\delta \boldsymbol{v}_0^-\right] = \begin{bmatrix} 0.00936084 \\ -0.0467514 \\ 0.00803263 \end{bmatrix} - \begin{bmatrix} -0.02000 \\ 0.02000 \\ -0.005000 \end{bmatrix} = \begin{bmatrix} 0.0293608 \\ -0.0667514 \\ 0.0130326 \end{bmatrix}$$

Delta-v at the conclusion of the maneuver is

$$[\Delta \boldsymbol{v}_f] = \left[\delta \boldsymbol{v}_f^+\right] - \left[\delta \boldsymbol{v}_f^-\right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -0.2528223 \\ -0.000472444 \\ -0.0222449 \end{bmatrix} = \begin{bmatrix} 0.2528223 \\ 0.000472444 \\ 0.0222449 \end{bmatrix}(\mathrm{km}/s)$$

The total delta-v requirement is

$$\Delta v_{\mathrm{total}} = |\Delta \boldsymbol{v}_0| + |\Delta \boldsymbol{v}_f| = 0.0740787 + 0.03559465 = 0.109673\mathrm{km}/s = 109.7 m/s$$

From equation $[\delta \boldsymbol{r}(t)] = [\Phi_{\mathrm{rr}}(t)][\delta \boldsymbol{r}_0] + [\Phi_{\mathrm{rv}}(t)][\delta \boldsymbol{v}_0]$, we have, for $0 < t < t_f$,

$$\begin{bmatrix} \delta x(t) \\ \delta y(t) \\ \delta z(t) \end{bmatrix} = \begin{bmatrix} 4 - 3\cos nt & 0 & 0 \\ 6(\sin nt - nt) & 1 & 0 \\ 0 & 0 & \cos nt \end{bmatrix}\begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix} + \begin{bmatrix} \dfrac{\sin nt}{n} & \dfrac{2}{n}(1 - \cos nt) & 0 \\ \dfrac{2}{n}(\cos nt - 1) & \dfrac{4}{n}\sin nt - 3t & 0 \\ 0 & 0 & \dfrac{1}{n}\sin nt \end{bmatrix}\begin{bmatrix} 0.00936084 \\ -0.0467514 \\ 0.00803263 \end{bmatrix}$$

Substituting $n = 0.00115697\mathrm{rad}/s$, we obtain the relative position vector as a function of time.

Example 7.3

Problem statement

A target and a chase vehicle are in the same 300km circular earth orbit. The chaser is 2km behind the target when the chaser initiates a two-impulse rendezvous maneuver so as to rendezvous with the target in 1.49 hours. Find the total delta-v requirement.

Solution:

For the circular orbit

$$v = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{398600}{6378 + 300}} = 7.726 \text{ km/}s$$

So that the mean motion is

$$n = \frac{v}{r} = \frac{7.726}{6678} = 0.0011569 \text{ rad/}s$$

For this mean motion and the rendezvous trajectory time $t = 1.49 \text{ hr} = 5364s$, the Clohessy-Wiltshire matrices are

$$[\Phi_{rr}(t)] = \begin{bmatrix} 4 - 3\cos nt & 0 & 0 \\ 6(\sin nt - nt) & 1 & 0 \\ 0 & 0 & \cos nt \end{bmatrix} = \begin{bmatrix} 1.0090 & 0 & 0 \\ -37.699 & 1 & 0 \\ 0 & 0 & 0.99700 \end{bmatrix}$$

$$[\Phi_{rv}(t)] = \begin{bmatrix} \dfrac{\sin nt}{n} & \dfrac{2}{n}(1 - \cos nt) & 0 \\ \dfrac{2}{n}(\cos nt - 1) & \dfrac{4}{n}\sin nt - 3t & 0 \\ 0 & 0 & \dfrac{1}{n}\sin nt \end{bmatrix} = \begin{bmatrix} -66.946 & 5.1928 & 0 \\ -5.1928 & -16360 & 0 \\ 0 & 0 & -66.946 \end{bmatrix}$$

$$[\Phi_{vr}(t)] = \begin{bmatrix} 3n\sin nt & 0 & 0 \\ 6n(\cos nt - 1) & 0 & 0 \\ 0 & 0 & -n\sin nt \end{bmatrix} = \begin{bmatrix} -2.6881 \times 10^{-4} & 0 & 0 \\ -2.0851 \times 10^{-5} & 0 & 0 \\ 0 & 0 & 8.9603 \times 10^{-5} \end{bmatrix}$$

$$[\Phi_{vv}(t)] = \begin{bmatrix} \cos nt & 2\sin nt & 0 \\ -2\sin nt & 4\cos nt - 3 & 0 \\ 0 & 0 & \cos nt \end{bmatrix} = \begin{bmatrix} 0.99700 & -0.15490 & 0 \\ 0.15490 & 0.98798 & 0 \\ 0 & 0 & 0.99700 \end{bmatrix}$$

The initial and final positions of the chaser in the CW frame are

$$[\delta \boldsymbol{r}_0] = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} (\text{km}) \quad [\delta \boldsymbol{r}_f] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, solving the first CW equation, $[\delta \boldsymbol{r}_f] = [\Phi_{\text{rr}}(t_f)][\delta \boldsymbol{r}_0] + [\Phi_{\text{rv}}(t_f)]\left[\delta \boldsymbol{v}_0^+\right]$, for $\left[\delta \boldsymbol{v}_0^+\right]$, we get

$$\begin{aligned}\left[\delta \boldsymbol{v}_0^+\right] &= -\begin{bmatrix} -66.946 & 5.1928 & 0 \\ -5.1928 & -16360 & 0 \\ 0 & 0 & -66.946 \end{bmatrix}^{-1} \begin{bmatrix} 1.0090 & 0 & 0 \\ -37.699 & 1 & 0 \\ 0 & 0 & 0.99700 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -9.4824 \times 10^{-6} \\ -1.2225 \times 10^{-4} \\ 0 \end{bmatrix} (\text{km}/s)\end{aligned}$$

Therefore, the second CW equation, $\left[\delta \boldsymbol{v}_f^-\right] = [\Phi_{\text{vr}}(t)][\delta \boldsymbol{r}_0] + [\Phi_{\text{vv}}(t)]\left[\delta \boldsymbol{v}_0^+\right]$

$$\begin{aligned}\left[\delta \boldsymbol{v}_f^-\right] &= \begin{bmatrix} -2.6881 \times 10^{-4} & 0 & 0 \\ -2.0851 \times 10^{-5} & 0 & 0 \\ 0 & 0 & 8.9603 \times 10^{-5} \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.99700 & -0.15490 & 0 \\ 0.15490 & 0.98798 & 0 \\ 0 & 0 & 0.99700 \end{bmatrix} \begin{bmatrix} -9.4824 \times 10^{-6} \\ -1.2225 \times 10^{-4} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -2.839 \times 10^{-5} \\ 1.1193 \times 10^{-4} \\ 0 \end{bmatrix} (\text{km}/s)\end{aligned}$$

Since the chaser is in the same circular orbit as the tareget, its relative velocity is initially zero, i.e., $\left[\delta \boldsymbol{v}_0^-\right] = [0]$. Thus,

$$[\Delta \boldsymbol{v}_0] = \left[\delta \boldsymbol{v}_0^+\right] - \left[\delta \boldsymbol{v}_0^-\right] = \begin{bmatrix} -9.4824 \times 10^{-6} \\ -1.2225 \times 10^{-4} \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -9.4824 \times 10^{-6} \\ -1.2225 \times 10^{-4} \\ 0 \end{bmatrix} (\text{km}/s)$$

Which implies

$$|\Delta \boldsymbol{v}_0| = 0.1226\, m/s$$

At the end of the rendezvous maneuver, $\left[\delta \boldsymbol{v}_f^+\right] = [0]$, so that

$$[\Delta \boldsymbol{v}_f] = \left[\delta \boldsymbol{v}_f^+\right] - \left[\delta \boldsymbol{v}_f^-\right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -2.839 \times 10^{-5} \\ 1.1193 \times 10^{-4} \\ 0 \end{bmatrix} = \begin{bmatrix} 2.839 \times 10^{-5} \\ -1.1193 \times 10^{-4} \\ 0 \end{bmatrix} (\text{km}/s)$$

Therefore,

$$|\Delta \boldsymbol{v}_f| = 0.1226 m/s$$

The total delta-v required is

$$\Delta \boldsymbol{v}_{\text{total}} = |\Delta \boldsymbol{v}_0| + |\Delta \boldsymbol{v}_f| = 0.2452\ m/s$$

Observe that in this case the motion takes place entirely in the plane of the target orbit. There is no motion normal to the plane (in the z direction). The coplanar rendezvous trajectory relative to the CW frame is sketched.