Chapter 7. Relative Motion and Rendezvous

WorkerInSpace

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Chapter Outline

7.1 Introduction

In a rendezvous maneuver, two orbiting vehicles observe one another from each of their own free-falling, rotating, clearly non-inertial frames of reference.

- To base impulsive maneuvers on observations made from a moving platform requires transforming relative velocity and acceleration measurements into an inertial frame.
- Otherwise, the true thrusting forces cannot be sorted out from the fictitious 'inertial forces' that appear in Newton's law when it is written incorrectly ad $\boldsymbol{F}_{\mathrm{net}} = m\boldsymbol{a}_{\mathrm{rel}}$.
- The purpose of this chapter is to use relative motion analysis to gain some familiarity with the problem of maneuvering one spacecraft relative to another, especially when they are in close proximity.

7.2 Relative Motion in Orbit

A rendezvous maneuver usually involves a target vehicle, which is passive and non-maneuvering, and a chase vehicle which is active and performs the maneuvers required to bring itself alongside the target.

The **position vector of the target** in the geocentric equatorial frame is $\boldsymbol{r}_0$.

- This outward radial is sometimes called 'r-bar'.
- The moving frame of reference has its origin at the target as illustrated in Figure 7.1
- The x axis is directed along $\boldsymbol{r}_0$, the outward radial to the target.
- The y axis is perpendicular to $\boldsymbol{r}_0$ and points in the direction of the target satellite's local horizon.
- The x and y axes therefore lie in the target's orbital plane, and the z axis is normal to that plane.

The angular velocity of the xyz axes attached to the target is just the angular velocity of the position vector $\boldsymbol{r}_0$, and it is obtained from the fact that:

$$\boldsymbol{h} = \boldsymbol{r}_0 \times \boldsymbol{v}_0 = \left(r_0 v_{0_\perp}\right)\widehat{\boldsymbol{k}} = \left(r_0^2 \Omega\right)\widehat{\boldsymbol{k}} = r_0^2 \Omega$$

Which means that angular velocity is:

$$\Omega = \frac{\boldsymbol{r}_0 \times \boldsymbol{v}_0}{\boldsymbol{r}_0^2}$$

To fine the angular acceleration $\dot{\Omega}$, we take the derivative of Ω to obtain

$$\dot{\Omega} = \frac{1}{r_0^2}\left(\dot{\boldsymbol{r}}_0 \times \boldsymbol{v}_0 + \boldsymbol{r}_0 \times \dot{\boldsymbol{v}}_0\right) - \frac{2}{r_0^3}\dot{r}_0(\boldsymbol{r}_0 \times \boldsymbol{v}_0)$$

Here,

$$\dot{\boldsymbol{r}}_0 \times \boldsymbol{v}_0 = \boldsymbol{v}_0 \times \boldsymbol{v}_0 = 0$$

$$\dot{\boldsymbol{v}}_0 = -\frac{\mu}{r_0^3}\boldsymbol{r}_0 \rightarrow \boldsymbol{r}_0 \times \dot{\boldsymbol{v}}_0 = - -\frac{\mu}{r_0^3}\boldsymbol{r}_0 \times \boldsymbol{r}_0 = 0$$

$$\therefore \dot{\Omega} = -\frac{2}{r_0^3}\dot{r}_0(\boldsymbol{r}_0 \times \boldsymbol{v}_0) = -\frac{2}{r_0}\dot{\boldsymbol{r}}_0\Omega$$

Also,

$$\frac{d}{\mathrm{dt}}\left(r_0^2\right) = 2r_0\dot{r}_0 = 2\boldsymbol{r}_0 \cdot \boldsymbol{v}_0 = \frac{d}{\mathrm{dt}}(\boldsymbol{r}_0)^2$$

$$\Rightarrow \dot{r}_0 = \frac{\boldsymbol{r}_0 \cdot \boldsymbol{v}_0}{r_0}$$

Therefore,

$$\begin{cases} \Omega = \dfrac{\boldsymbol{r}_0 \times \boldsymbol{v}_0}{\boldsymbol{r}_0^2} \\ \dot{\Omega} = -\dfrac{2(\boldsymbol{r}_0 \cdot \boldsymbol{v}_0)}{r_0^2}\Omega \end{cases}$$

These equations are the means of determining the angular velocity and acceleration of the co-moving frame for use in the relative velocity and acceleration formulas

$$\begin{cases} \boldsymbol{v}_B = \boldsymbol{v}_A + \Omega_A \times \boldsymbol{r}_{\mathrm{rel}} + \boldsymbol{v}_{\mathrm{rel}} \\ \boldsymbol{a}_B = \boldsymbol{a}_A + \dot{\Omega}_A \times \boldsymbol{r}_{\mathrm{rel}} + \Omega_A \times (\Omega_A \times \boldsymbol{r}_{\mathrm{rel}}) + 2\Omega_A \times \boldsymbol{v}_{\mathrm{rel}} + \boldsymbol{a}_{\mathrm{rel}} \end{cases} , \boldsymbol{r}_{\mathrm{rel}} = \boldsymbol{r}_B - \boldsymbol{r}_A$$

7.2.1 Note: Relative coordinate from inertial coordinate

Starting from Spacecraft A's inertial position and velocity information

$$\boldsymbol{r}_A = -266.74\hat{\mathbf{I}} + 3865.4\hat{\mathbf{J}} + 5425.7\hat{\mathbf{K}}$$
$$\boldsymbol{v}_A = -6.4842\hat{\mathbf{I}} - 3.6201\hat{\mathbf{J}} + 2.4159\hat{\mathbf{K}}$$

The unit vector $\hat{\mathbf{i}}$ along the x axis of spacecraft A's rigid, co-moving frame of reference is

$$\hat{\mathbf{i}} = \frac{\boldsymbol{r}_A}{r_A} = -0.040008\hat{\mathbf{I}} + 0.57977\hat{\mathbf{J}} + 0.81380\hat{\mathbf{K}}$$

Since the z axis in the direction of $\boldsymbol{h}_A$,

$$\boldsymbol{h}_A = \boldsymbol{r}_A \times \boldsymbol{v}_A = 28980\hat{\mathbf{I}} - 34537\hat{\mathbf{J}} + 26030\hat{\mathbf{K}}$$
$$\rightarrow \hat{\mathbf{k}} = \frac{\boldsymbol{h}_A}{h_A} = 0.55667\hat{\mathbf{I}} - 0.66341\hat{\mathbf{J}} + 0.5000\hat{\mathbf{K}}$$

Finally, $\hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{I}}$, so that

$$\hat{\mathbf{j}} = -0.82977\hat{\mathbf{I}} - 0.47302\hat{\mathbf{J}} + 0.29260\hat{\mathbf{K}}$$

Therefore, the rotational matrix from inertial coordinate to relative coordinate

$$\begin{bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{bmatrix} = \begin{bmatrix} -0.040008 & 0.57977 & 0.81380 \\ -0.82977 & -0.47302 & 0.29260 \\ 0.55667 & -0.66341 & 0.5000 \end{bmatrix}$$

7.3 Linearization of the equation of relative motion in orbit

Note :

$\boldsymbol{r}$: Inertial position vector of chase vehicle

$\boldsymbol{r}_0$: Inertial position vector of target vehicle

$\delta\boldsymbol{r}$: Relative position vector of chase vehicle from with repect to target vehicle

The **inertial position vector** of the target vehicle A is $\boldsymbol{r}_0$, and that of the chase vehicle B is $\boldsymbol{r}$.

- The position vector of the chase vehicle to the target is $\delta\boldsymbol{r}$, so that $\boldsymbol{r} = \boldsymbol{r}_0 + \delta\boldsymbol{r}$
- The symbol δ is used to represent the fact that the relative position vector has a magnitude which is very small compared to the magnitude of $\boldsymbol{r}_0$ (and $\boldsymbol{r}$) $\frac{\delta r}{r_0} \ll 1$.
- This is true if the two vehicles are in close proximity to each other, as is the case in a rendezvous maneuver.
- Our purpose in this section is **to seek the equations of motion of the chase vehicle relative to the target**.

The equation of motion of the chase vehicle B is:

$$\ddot{\boldsymbol{r}} = -\mu\frac{\boldsymbol{r}}{r^3}, \text{ where } r = |\boldsymbol{r}|$$

From $\boldsymbol{r} = \boldsymbol{r}_0 + \delta\boldsymbol{r}$, the equation of motion of the chaser relative to the target is:

$$\delta\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{r}} - \ddot{\boldsymbol{r}}_0 = -\ddot{\boldsymbol{r}}_0 - \mu\frac{\boldsymbol{r}}{r^3} = -\ddot{\boldsymbol{r}}_0 - \mu\frac{\boldsymbol{r}_0 + \delta\boldsymbol{r}}{r^3}$$

$$\rightarrow \delta\ddot{\boldsymbol{r}} = -\ddot{\boldsymbol{r}}_0 - \mu\frac{\boldsymbol{r}_0 + \delta\boldsymbol{r}}{r^3}$$

We will simplify this equation by making use of the fact that $|\delta\boldsymbol{r}|$ is very small. First note that:

$$r^2 = \boldsymbol{r}\cdot\boldsymbol{r} = (\boldsymbol{r}_0 + \delta\boldsymbol{r})\cdot(\boldsymbol{r}_0 + \delta\boldsymbol{r}) = \boldsymbol{r}_0\cdot\boldsymbol{r}_0 + 2\boldsymbol{r}_0\cdot\delta\boldsymbol{r} + \delta\boldsymbol{r}\cdot\delta\boldsymbol{r}$$

Since $\boldsymbol{r}_0\cdot\boldsymbol{r}_0 = r_0^2$ and $\delta\boldsymbol{r}\cdot\delta\boldsymbol{r} = \delta r^2$, we can factor out r_0^2 on the right to obtain

$$r^2 = r_0^2 + 2\boldsymbol{r}_0\cdot\delta\boldsymbol{r} + \delta r^2$$

$$r^2 = r_0^2\left[1 + \frac{2\boldsymbol{r}_0\cdot\delta\boldsymbol{r}}{r_0^2} + \left(\frac{\delta r}{r_0}\right)^2\right]$$

$$\text{since } \frac{\delta r}{r_0} \ll 1$$

$$r^2 = r_0^2\left[1 + \frac{2\boldsymbol{r}_0\cdot\delta\boldsymbol{r}}{r_0^2}\right]$$

In fact, we will neglect all powers of $\delta r/r_0$ greater than unity, wherever they appear.

$$(r^2)^{-\frac{3}{2}} = \left(r_0^2\left[1 + \frac{2\boldsymbol{r}_0 \cdot \delta\boldsymbol{r}}{r_0^2}\right]\right)^{-\frac{3}{2}}$$

$$r^{-3} = r_0^{-3}\left[1 + \frac{2\boldsymbol{r}_0 \cdot \delta\boldsymbol{r}}{r_0^2}\right]^{-\frac{3}{2}}$$

$$\left[1 + \frac{2\boldsymbol{r}_0 \cdot \delta\boldsymbol{r}}{r_0^2}\right]^{-\frac{3}{2}} \approx 1 - \frac{3}{2}\frac{2\boldsymbol{r}_0 \cdot \delta\boldsymbol{r}}{r_0^2} = 1 - \frac{3}{r_0^2}\boldsymbol{r}_0 \cdot \delta\boldsymbol{r}$$

$$\therefore r^{-3} = r_0^{-3}\left(1 - \frac{3}{r_0^2}\boldsymbol{r}_0 \cdot \delta\boldsymbol{r}\right)$$

$$\Leftrightarrow \frac{1}{r^3} = \frac{1}{r_0^3} - \frac{3}{r_0^5}\boldsymbol{r}_0 \cdot \delta\boldsymbol{r}$$

From $\delta\ddot{\boldsymbol{r}} = -\ddot{\boldsymbol{r}}_0 - \mu\frac{\boldsymbol{r}_0 + \delta\boldsymbol{r}}{r^3}$ and substituting $r^{-3} = \frac{1}{r_0^3} - \frac{3}{r_0^5}\boldsymbol{r}_0 \cdot \delta\boldsymbol{r}$

$$\begin{aligned}
\delta\ddot{\boldsymbol{r}} &= -\ddot{\boldsymbol{r}}_0 - \mu\left(\frac{1}{r_0^3} - \frac{3}{r_0^5}\boldsymbol{r}_0 \cdot \delta\boldsymbol{r}\right)(\boldsymbol{r}_0 + \delta\boldsymbol{r}) \\
&= -\ddot{\boldsymbol{r}}_0 - \mu\left[\frac{\boldsymbol{r}_0 + \delta\boldsymbol{r}}{r_0^3} - \frac{3}{r_0^5}(\boldsymbol{r}_0 \cdot \delta\boldsymbol{r})(\boldsymbol{r}_0 + \delta\boldsymbol{r})\right] \\
&= -\ddot{\boldsymbol{r}}_0 - \mu\left[\frac{\boldsymbol{r}_0}{r_0^3} + \frac{\delta\boldsymbol{r}}{r_0^3} - \frac{3}{r_0^5}(\boldsymbol{r}_0 \cdot \delta\boldsymbol{r})\boldsymbol{r}_0 - \frac{3}{r_0^5}(\boldsymbol{r}_0 \cdot \delta\boldsymbol{r})\delta\boldsymbol{r}\right] \\
&\approx -\ddot{\boldsymbol{r}}_0 - \mu\left[\frac{\boldsymbol{r}_0}{r_0^3} + \frac{\delta\boldsymbol{r}}{r_0^3} - \frac{3}{r_0^5}(\boldsymbol{r}_0 \cdot \delta\boldsymbol{r})\boldsymbol{r}_0\right] \\
\therefore \delta\ddot{\boldsymbol{r}} &= -\ddot{\boldsymbol{r}}_0 - \mu\frac{\boldsymbol{r}_0}{r_0^3} - \frac{\mu}{r_0^3}\left[\delta\boldsymbol{r} - \frac{3}{r_0^2}(\boldsymbol{r}_0 \cdot \delta\boldsymbol{r})\boldsymbol{r}_0\right]
\end{aligned}$$

Since the equation of motion of the target vehicle is $\ddot{\boldsymbol{r}}_0 = -\mu\frac{\boldsymbol{r}_0}{r_0^3}$,

$$\delta\ddot{\boldsymbol{r}} = -\frac{\mu}{r_0^3}\left[\delta\boldsymbol{r} - \frac{3}{r_0^2}(\boldsymbol{r}_0 \cdot \delta\boldsymbol{r})\boldsymbol{r}_0\right]$$

This is the linearized version of $\delta\ddot{\boldsymbol{r}} = -\ddot{\boldsymbol{r}}_0 - \mu\frac{\boldsymbol{r}_0 + \delta\boldsymbol{r}}{r^3}$, the equation which governs the motion of the chaser with respect to the target.

- The expression is linear because $\delta\boldsymbol{r}$ appears only in the numerator and only to the first power throughout.
- We achieved this by dropping a lot of terms that are insignificant when Equation $\frac{\delta r}{r_0} \ll 1$ is valid.

7.4 Clohessy-Wiltshire equations

Note :

$\boldsymbol{r}$: Inertial position vector of chase vehicle

$\boldsymbol{r}_0$: Inertial position vector of target vehicle

$\delta\boldsymbol{r}$: Relative position vector of chase vehicle from with repect to target vehicle

Let us attach a **moving frame of reference** xyz **to the target vehicle** A.

- The origin of the moving system is at A.
- The x axis lies long $\boldsymbol{r}_0$, so that $\hat{\mathbf{i}} = \dfrac{\boldsymbol{r}_0}{r_0}$
- The y axis is in the direction of the local horizon, and the z axis is normal to the orbital plane A, such that $\hat{\mathbf{k}} = \hat{\mathbf{i}} \times \hat{\mathbf{j}}$.
- The inertial angular velocity of the moving frame of reference is Ω, and the inertial angular acceleration is $\dot{\Omega}$.

According to the relative acceleration formula, we have

$$
\begin{aligned}
&\boldsymbol{a}_B = \boldsymbol{a}_A + \dot{\Omega}_A \times \boldsymbol{r}_{\text{rel}} + \Omega_A \times (\Omega_A \times \boldsymbol{r}_{\text{rel}}) + 2\Omega_A \times \boldsymbol{v}_{\text{rel}} + \boldsymbol{a}_{\text{rel}} \\
&\rightarrow\ \ddot{\boldsymbol{r}} = \ddot{\boldsymbol{r}}_0 + \dot{\Omega} \times \delta\boldsymbol{r} + \Omega \times (\Omega \times \delta\boldsymbol{r}) + 2\Omega \times \delta\boldsymbol{v}_{\text{rel}} + \delta\boldsymbol{a}_{\text{rel}}
\end{aligned}
$$

where, in terms of their components in the moving frame, the relative position, velocity and acceleration are given by

$$
\begin{aligned}
\delta\boldsymbol{r} &= \delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} + \delta z\hat{\mathbf{k}} \\
\delta\boldsymbol{v}_{\text{rel}} &= \delta\dot{x}\hat{\mathbf{i}} + \delta\dot{y}\hat{\mathbf{j}} + \delta\dot{z}\hat{\mathbf{k}} \\
\delta\boldsymbol{a}_{\text{rel}} &= \delta\ddot{x}\hat{\mathbf{i}} + \delta\ddot{y}\hat{\mathbf{j}} + \delta\ddot{z}\hat{\mathbf{k}}
\end{aligned}
$$

For simplicity, we assume at this point that the orbit of the target vehicle A is circular. Then, $\dot{\Omega} = 0$.

$$
\begin{aligned}
&\delta\ddot{\boldsymbol{r}} = \Omega \times (\Omega \times \delta\boldsymbol{r}) + 2\Omega \times \delta\boldsymbol{v}_{\text{rel}} + \delta\boldsymbol{a}_{\text{rel}} \\
&\text{applying bac} - \text{cab rule,} \\
&\delta\ddot{\boldsymbol{r}} = \Omega(\Omega \cdot \delta\boldsymbol{r}) - \Omega^2\delta\boldsymbol{r} + 2\Omega \times \delta\boldsymbol{v}_{\text{rel}} + \delta\boldsymbol{a}_{\text{rel}}
\end{aligned}
$$

Since the orbit of A is circular, we may write the angular velocity as $\Omega = n\hat{\mathbf{k}}$, where n, the mean motion, is constant.

$$
\begin{aligned}
&\Omega \cdot \delta\boldsymbol{r} = n\hat{\mathbf{k}} \cdot \left(\delta x\hat{\mathbf{i}} + \delta y\hat{\mathbf{j}} + \delta z\hat{\mathbf{k}}\right) = n\delta z \\
&\Omega \times \delta\boldsymbol{v}_{\text{rel}} = n\hat{\mathbf{k}} \times \left(\delta\dot{x}\hat{\mathbf{i}} + \delta\dot{y}\hat{\mathbf{j}} + \delta\dot{z}\hat{\mathbf{k}}\right) = -n\delta\dot{y}\hat{\mathbf{i}} + n\delta\dot{x}\hat{\mathbf{j}}
\end{aligned}
$$

From $\delta\ddot{\boldsymbol{r}} = \Omega(\Omega \cdot \delta\boldsymbol{r}) - \Omega^2\delta\boldsymbol{r} + 2\Omega \times \delta\boldsymbol{v}_{\text{rel}} + \delta\boldsymbol{a}_{\text{rel}}$

$$\delta\ddot{\boldsymbol{r}} = n\widehat{\mathbf{k}}(n\delta z) - n^2\left(\delta x\widehat{\mathbf{i}} + \delta y\widehat{\mathbf{j}} + \delta z\widehat{\mathbf{k}}\right) + 2\left(-n\delta\dot{y}\widehat{\mathbf{i}} + n\delta\dot{x}\widehat{\mathbf{j}}\right) + \delta\ddot{x}\widehat{\mathbf{i}} + \delta\ddot{y}\widehat{\mathbf{j}} + \delta\ddot{z}\widehat{\mathbf{k}}$$
$$\therefore\ \delta\ddot{\boldsymbol{r}} = (-n^2\delta x - 2n\delta\dot{y} + \delta\ddot{x})\widehat{\mathbf{i}} + (-n^2\delta y + 2n\delta\dot{x} + \delta\ddot{y})\widehat{\mathbf{j}} + \delta\ddot{z}\widehat{\mathbf{k}}$$

This expression gives the components of the chaser's absolute relative acceleration vector in terms of quantities measured in the moving reference.

Since the orbit of A is circular, the mean motion is found as

$$n = \frac{v}{r_0} = \frac{1}{r_0}\sqrt{\frac{\mu}{r_0}} = \sqrt{\frac{\mu}{r_0^3}} \quad \rightarrow \quad \frac{\mu}{r_0^3} = n^2$$

We also note that

$$\boldsymbol{r}_0 \cdot \delta\boldsymbol{r} = \left(\boldsymbol{r}_0\widehat{\mathbf{i}}\right) \cdot \left(\delta x\widehat{\mathbf{i}} + \delta y\widehat{\mathbf{j}} + \delta z\widehat{\mathbf{k}}\right) = r_0\delta x$$

From linearized equation of motion, $\delta\ddot{\boldsymbol{r}} = -\dfrac{\mu}{r_0^3}\left[\delta\boldsymbol{r} - \dfrac{3}{r_0^2}(\boldsymbol{r}_0 \cdot \delta\boldsymbol{r})\boldsymbol{r}_0\right]$

$$\delta\ddot{\boldsymbol{r}} = -n^2\left[\delta x\widehat{\mathbf{i}} + \delta y\widehat{\mathbf{j}} + \delta z\widehat{\mathbf{k}} - \frac{3}{r_0^2}(r_0\delta x)r_0\widehat{\mathbf{i}}\right]$$
$$\therefore\ \delta\ddot{\boldsymbol{r}} = 2n^2\delta x\widehat{\mathbf{i}} - n^2\delta y\widehat{\mathbf{j}} - n^2\delta z\widehat{\mathbf{k}}$$

Combining a kinematic relationship and the equation of motion, we can obtain the follows:

$$\text{Kinematic relationship}:\ \delta\ddot{\boldsymbol{r}} = (-n^2\delta x - 2n\delta\dot{y} + \delta\ddot{x})\widehat{\mathbf{i}} + (-n^2\delta y + 2n\delta\dot{x} + \delta\ddot{y})\widehat{\mathbf{j}} + \delta\ddot{z}\widehat{\mathbf{k}}$$
$$\text{Equation of motion}:\ \delta\ddot{\boldsymbol{r}} = 2n^2\delta x\widehat{\mathbf{i}} - n^2\delta y\widehat{\mathbf{j}} - n^2\delta z\widehat{\mathbf{k}}$$
$$\Rightarrow (-n^2\delta x - 2n\delta\dot{y} + \delta\ddot{x})\widehat{\mathbf{i}} + (-n^2\delta y + 2n\delta\dot{x} + \delta\ddot{y})\widehat{\mathbf{j}} + \delta\ddot{z}\widehat{\mathbf{k}} = 2n^2\delta x\widehat{\mathbf{i}} - n^2\delta y\widehat{\mathbf{j}} - n^2\delta z\widehat{\mathbf{k}}$$

Upon collecting terms to the left-hand side, we get

$$(\delta\ddot{x} - 3n^2\delta x - 2n\delta\dot{y})\widehat{\mathbf{i}} + (\delta\ddot{y} + 2n\delta\dot{x})\widehat{\mathbf{j}} + (\delta\ddot{z} + n^2\delta z) = 0$$

That is

$$\delta\ddot{x} - 3n^2\delta x - 2n\delta\dot{y} = 0$$
$$\delta\ddot{y} + 2n\delta\dot{x} = 0$$
$$\delta\ddot{z} + n^2\delta z = 0$$

These are the Clohessy-Wiltshire (CW) equations. When using these equations, we will refer to the moving frame of reference in which they were derived as the Clohessy-Wiltshire frame (or CW frame).

- These equations are a set of coupled, second order differential equations with constant coefficients.

- The initial conditions are:

$$\text{At } t=0 \;\; \delta x = \delta x_0 \quad \delta y = \delta y_0 \quad \delta z = \delta z_0$$
$$\delta \dot{x} = \delta \dot{x}_0 \quad \delta \dot{y} = \delta \dot{y}_0 \quad \delta \dot{z} = \delta \dot{z}_0$$

From equation $\delta \ddot{y} + 2n\delta \dot{x} = 0$ we can induce

$$\begin{aligned}
&\frac{d}{dt}(\delta \dot{y} + 2n\delta x) = 0 \\
&\rightarrow \delta \dot{y} + 2n\delta x = \text{constant} \\
&\therefore \delta \dot{y} + 2n\delta x = \delta \dot{y}_0 + 2n\delta x_0 \\
&\text{so that} \\
&\delta \dot{y} = \delta \dot{y}_0 + 2n(\delta x_0 - \delta x)
\end{aligned}$$

Substituting this result into $\delta \ddot{x} - 3n^2\delta x - 2n\delta \dot{y} = 0$ yields:

$$\begin{aligned}
&\delta \ddot{x} - 3n^2\delta x - 2n\left[\delta \dot{y}_0 + 2n(\delta x_0 - \delta x)\right] = 0 \\
&\delta \ddot{x} + n^2\delta x = 2n\delta \dot{y}_0 + 4n^2\delta x_0
\end{aligned}$$

The solution of this differential equation is

$$\begin{aligned}
&\delta x = A \sin nt + B \cos nt + \frac{1}{n^2}\left(2n\delta \dot{y}_0 + 4n^2\delta x_0\right) \\
&\delta x = A \sin nt + B \cos nt + \frac{2}{n}\delta \dot{y}_0 + 4\delta x_0 \\
&\text{Differentiating this } w.r.t \text{ time, we obtain} \\
&\delta \dot{x} = nA \cos nt - nB \sin nt
\end{aligned}$$

Plug $t = 0$ at δx and $\delta \dot{x}$

$$\begin{aligned}
&\delta x_0 = B + \frac{2}{n}\delta \dot{y}_0 + 4\delta x_0 \rightarrow B = -3\delta x_0 - 2\frac{\delta \dot{y}_0}{n} \\
&\delta \dot{x}_0 = nA \rightarrow A = \frac{\delta \dot{x}_0}{n}
\end{aligned}$$

Substituting these initial values into δx solution leads to

$$\begin{aligned}
&\delta x = \frac{\delta \dot{x}_0}{n} \sin nt + \left(-3\delta x_0 - 2\frac{\delta \dot{y}_0}{n}\right)\cos nt + \frac{2}{n}\delta \dot{y}_0 + 4\delta x_0 \\
&\delta x = (4 - 3\cos nt)\delta x_0 + \frac{\sin nt}{n}\delta \dot{x}_0 + \frac{2}{n}(1 - \cos nt)\delta \dot{y}_0 \\
&\therefore \delta \dot{x} = 3n \sin nt \, \delta x_0 + \cos nt \, \delta \dot{x}_0 + 2 \sin nt \delta \dot{y}_0
\end{aligned}$$

From $\delta \dot{y} = \delta \dot{y}_0 + 2n(\delta x_0 - \delta x)$ and $\delta x = (4 - 3\cos nt)\delta x_0 + \frac{\sin nt}{n}\delta \dot{x}_0 + \frac{2}{n}(1 - \cos nt)\delta \dot{y}_0$

$$\delta \dot{y} = \delta \dot{y}_0 + 2n\left[\delta x_0 - (4 - 3\cos nt)\delta x_0 - \frac{\sin nt}{n}\delta \dot{x}_0 - \frac{2}{n}(1 - \cos nt)\delta \dot{y}_0\right]$$

Which simplie to become

$$\delta \dot{y} = 6n(\cos nt - 1)\delta x_0 - 2\sin nt\, \delta \dot{x}_0 + (4\cos nt - 3)\delta \dot{y}_0$$

Integrating this expression with respect to time, we find

$$\delta y = 6n\left(\frac{1}{n}\sin nt - t\right)\delta x_0 + \frac{2}{n}\cos nt\delta \dot{x}_0 + \left(\frac{4}{n}\sin nt - 3t\right)\delta \dot{y}_0 + C$$

Evaluating δy at $t = 0$ yields

$$\delta y_0 = \frac{2}{n}\delta \dot{x}_0 + C \;\Rightarrow C = \delta y_0 - \frac{2}{n}\delta \dot{x}_0$$

$$\therefore \delta y = 6n\left(\frac{1}{n}\sin nt - t\right)\delta x_0 + \frac{2}{n}\cos nt\delta \dot{x}_0 + \left(\frac{4}{n}\sin nt - 3t\right)\delta \dot{y}_0 + \delta y_0 - \frac{2}{n}\delta \dot{x}_0$$

$$\delta y = 6(\sin nt - nt)\delta x_0 + \delta y_0 + \frac{2}{n}(\cos nt - 1)\delta \dot{x}_0 + \left(\frac{4}{n}\sin nt - 3t\right)\delta \dot{y}_0$$

Finally, the solution of $\delta \ddot{z} + n^2\delta z = 0$ is

$$\delta z = D\cos nt + E\sin nt$$

$$\delta \dot{z} = -nD\sin nt + nE\cos nt$$

Evaluating δz and $\delta \dot{z}$ at $t = 0$ yields

$$\delta z_0 = D$$

$$\delta \dot{z}_0 = nE \rightarrow E = \frac{\delta \dot{z}_0}{n}$$

Therefore:

$$\delta z = \cos nt\, \delta z_0 + \frac{1}{n}\sin nt\, \delta \dot{z}_0$$

$$\delta \dot{z} = -n\sin nt\, \delta z_0 + \cos nt\, \delta \dot{z}_0$$

Now that we have finished solving the Clohessy-Wiltshire equations, let us change our notation a bit and denote the x, y, and z components of relative velocity in the moving frame as δu, δv and δw, respectively That is:

$$\delta u = \delta \dot{x} \quad \delta v = \delta \dot{y} \quad \delta w = \delta \dot{z}$$

The initial conditions on the relative velocity components are then written

$$\delta u = \delta \dot{x}_0 \quad \delta v = \delta \dot{y}_0 \quad \delta w_0 = \delta \dot{z}_0$$

So far, we have derived equations of $\delta x, \delta \dot{x}, \delta y, \delta \dot{y}, \delta z, \delta \dot{z}$

$$\delta x = (4 - 3\cos nt)\delta x_0 + \frac{\sin nt}{n}\delta \dot{x}_0 + \frac{2}{n}(1 - \cos nt)\delta \dot{y}_0$$

$$\delta \dot{x} = 3n \sin nt\, \delta x_0 + \cos nt\, \delta \dot{x}_0 + 2 \sin nt \delta \dot{y}_0$$

$$\delta y = 6(\sin nt - nt)\delta x_0 + \delta y_0 + \frac{2}{n}(\cos nt - 1)\delta \dot{x}_0 + \left(\frac{4}{n}\sin nt - 3t\right)\delta \dot{y}_0$$

$$\delta \dot{y} = 6n(\cos nt - 1)\delta x_0 - 2 \sin nt\, \delta \dot{x}_0 + (4\cos nt - 3)\delta \dot{y}_0$$

$$\delta z = \cos nt\, \delta z_0 + \frac{1}{n}\sin nt\, \delta \dot{z}_0$$

$$\delta \dot{z} = -n \sin nt\, \delta z_0 + \cos nt\, \delta \dot{z}_0$$

We can rewrite this into following format

$$\delta x = (4 - 3\cos nt)\delta x_0 + \frac{\sin nt}{n}\delta u_0 + \frac{2}{n}(1 - \cos nt)\delta v_0$$

$$\delta y = 6(\sin nt - nt)\delta x_0 + \delta y_0 + \frac{2}{n}(\cos nt - 1)\delta u_0 + \left(\frac{4}{n}\sin nt - 3t\right)\delta v_0$$

$$\delta z = \cos nt\, \delta z_0 + \frac{1}{n}\sin nt\, \delta w_0$$

$$\delta u = 3n \sin nt\, \delta x_0 + \cos nt\, \delta u_0 + 2 \sin nt \delta v_0$$

$$\delta v = 6n(\cos nt - 1)\delta x_0 - 2 \sin nt\, \delta u_0 + (4\cos nt - 3)\delta v_0$$

$$\delta w = -n \sin nt\, \delta z_0 + \cos nt\, \delta w_0$$

Let us introduce matrix notation to define the relative position and velocity vectors and their initial values

$$[\delta \boldsymbol{r}(t)] = \begin{bmatrix} \delta x(t) \\ \delta y(t) \\ \delta z(t) \end{bmatrix} \quad [\delta \boldsymbol{v}(t)] = \begin{bmatrix} \delta u(t) \\ \delta v(t) \\ \delta w(t) \end{bmatrix}$$

$$[\delta \boldsymbol{r}_0] = \begin{bmatrix} \delta x_0 \\ \delta y_0 \\ \delta z_0 \end{bmatrix} \quad [\delta \boldsymbol{v}_0] = \begin{bmatrix} \delta u_0 \\ \delta v_0 \\ \delta w_0 \end{bmatrix}$$

Observe that we have dropped the subscript introduced in $\delta \boldsymbol{r} = \delta x \hat{\mathbf{i}} + \delta y \hat{\mathbf{j}} + \delta z \hat{\mathbf{k}}$ and $\delta \boldsymbol{v}_{\text{rel}} = \delta \dot{x} \hat{\mathbf{i}} + \delta \dot{y} \hat{\mathbf{j}} + \delta \dot{z} \hat{\mathbf{k}}$ because it is superfluous in rendezvous analysis, where all kinematic equations are relative to the Clohessy-Wiltshire frame.

Also, we introduce Clohessy-Wiltshire matrices:

$$[\delta \boldsymbol{r}(t)] = [\Phi_{\text{rr}}(t)][\delta \boldsymbol{r}_0] + [\Phi_{\text{rv}}(t)][\delta \boldsymbol{v}_0]$$
$$[\delta \boldsymbol{v}(t)] = [\Phi_{\text{vr}}(t)][\delta \boldsymbol{r}_0] + [\Phi_{\text{vv}}(t)][\delta \boldsymbol{v}_0]$$

In which

$$[\Phi_{\mathrm{rr}}(t)] = \begin{bmatrix} 4 - 3\cos nt & 0 & 0 \\ 6(\sin nt - nt) & 1 & 0 \\ 0 & 0 & \cos nt\, \delta z_0 \end{bmatrix}$$

$$[\Phi_{\mathrm{rv}}(t)] = \begin{bmatrix} \dfrac{\sin nt}{n} & \dfrac{2}{n}(1 - \cos nt) & 0 \\ \dfrac{2}{n}(\cos nt - 1) & \dfrac{4}{n}\sin nt - 3t & 0 \\ 0 & 0 & \dfrac{1}{n}\sin nt \end{bmatrix}$$

$$[\Phi_{\mathrm{vr}}(t)] = \begin{bmatrix} 3n \sin nt & 0 & 0 \\ 6n(\cos nt - 1) & 0 & 0 \\ 0 & 0 & -n \sin nt \end{bmatrix}$$

$$[\Phi_{\mathrm{vv}}(t)] = \begin{bmatrix} \cos nt & 2 \sin nt & 0 \\ -2 \sin nt & 4\cos nt - 3 & 0 \\ 0 & 0 & \cos nt \end{bmatrix}$$

7.5 Two-Impulse Rendezvous Maneuvers

Note :

$\boldsymbol{r}$: Inertial position vector of chase vehicle

$\boldsymbol{r}_0$: Inertial position vector of target vehicle

$\delta\boldsymbol{r}$: Relative position vector of chase vehicle from with repect to target vehicle

Figure 7.6 illustrates the rendezvous problem.

- At time $t = 0^-$(the instant preceding $t = 0$), the position $\delta\boldsymbol{r}_0$ and velocity $\delta\boldsymbol{v}_0^-$ of the chase vehicle B relative to the target A are known.
- At $t = 0$ and impulsive maneuver instantaneously changes the relative velocity to $\delta\boldsymbol{v}_0^+$ at $t = 0^+$(the instant after t=0).
- The components of $\delta\boldsymbol{v}_0^+$ are shown in figure 7.6.
- We must determine the values of δu_0^+, δv_0^+, δw_0^+, at the beginning of the rendezvous trajectory, so that B will arrive at the target in a specified time t_f.

The delta-v required to place B on the rendezvous trajectory is

$$[\Delta v_0] = \left[\delta\boldsymbol{v}_0^+\right] - \left[\delta\boldsymbol{v}_0^-\right] = \begin{bmatrix} \delta u_0^+ \\ \delta v_0^+ \\ \delta w_0^+ \end{bmatrix} - \begin{bmatrix} \delta u_0^- \\ \delta v_0^- \\ \delta w_0^- \end{bmatrix}$$

At time t_f, B arrives at A, at the origin of the co-moving frame, which means

$$[\delta\boldsymbol{r}_f] = [\delta\boldsymbol{r}(t_f)] = \left[\vec{0}\right]$$

Evaluating the equation $[\delta\boldsymbol{r}(t)] = [\Phi_{\mathrm{rr}}(t)][\delta\boldsymbol{r}_0] + [\Phi_{\mathrm{rv}}(t)][\delta\boldsymbol{v}_0]$ at t_f, we find follows:

$$\left[\vec{0}\right] = [\Phi_{\mathrm{rr}}(t_f)][\delta\boldsymbol{r}_0] + [\Phi_{\mathrm{rv}}(t_f)]\left[\delta\boldsymbol{v}_0^+\right]$$

Solving for initial chasing vehicle velocity $\left[\delta\boldsymbol{v}_0^+\right]$ yields

$$\left[\delta\boldsymbol{v}_0^+\right] = -[\Phi_{\mathrm{rv}}(t_f)]^{-1}[\Phi_{\mathrm{rr}}(t_f)][\delta\boldsymbol{r}_0]$$

$[\Phi_{\mathrm{rv}}(t_f)]^{-1}$ is the matrix inverse of $[\Phi_{\mathrm{rv}}(t_f)]$. We know the velocity $\delta\boldsymbol{v}_0^+$ at the beginning of the rendezvous path substituting $\left[\delta\boldsymbol{v}_0^+\right]$ value into $[\delta\boldsymbol{v}(t)] = [\Phi_{\mathrm{vr}}(t)][\delta\boldsymbol{r}_0] + [\Phi_{\mathrm{vv}}(t)][\delta\boldsymbol{v}_0]$ we obtain the velocity $\delta\boldsymbol{v}_f^-$ at which B arrives at the target A, when $t = t_f^-$:

$$
\begin{aligned}
\left[\delta \boldsymbol{v}_f^-\right] &= [\Phi_{\mathrm{vr}}(t_f)][\delta \boldsymbol{r}_0] + [\Phi_{\mathrm{vv}}(t_f)]\left[\delta \boldsymbol{v}_0^+\right] \\
&= [\Phi_{\mathrm{vr}}(t_f)][\delta \boldsymbol{r}_0] + [\Phi_{\mathrm{vv}}(t_f)]\left(-[\Phi_{\mathrm{rv}}(t_f)]^{-1}[\Phi_{\mathrm{rr}}(t_f)][\delta \boldsymbol{r}_0]\right) \\
&= \left([\Phi_{\mathrm{vr}}(t_f)] - [\Phi_{\mathrm{vv}}(t_f)][\Phi_{\mathrm{rv}}(t_f)]^{-1}[\Phi_{\mathrm{rr}}(t_f)]\right)[\delta \boldsymbol{r}_0]
\end{aligned}
$$

Obviously, an impulsive delta-v maneuver is required at $t = t_f$ to bring vehicle B to rest relative to A $\left(\delta \boldsymbol{v}_f^+ = 0\right)$:

$$
[\Delta \boldsymbol{v}_f] = \left[\delta \boldsymbol{v}_f^+\right] - \left[\delta \boldsymbol{v}_f^-\right] = \left[\vec{0}\right] - \left[\delta \boldsymbol{v}_f^-\right] = -\left[\delta \boldsymbol{v}_f^-\right]
$$

Note that in Equations $[\Delta v_0] = \left[\delta \boldsymbol{v}_0^+\right] - \left[\delta \boldsymbol{v}_0^-\right]$ and $[\Delta \boldsymbol{v}_f] = \left[\delta \boldsymbol{v}_f^+\right] - \left[\delta \boldsymbol{v}_f^-\right]$, we are using the difference between relative velocities to calculate delta-v, which is the difference in absolute velocities.

To show that this is valid, use relative velocity wrt rotating relative frame

$$
\begin{aligned}
\boldsymbol{v}^- &= \boldsymbol{v}_0^- + \Omega^- \times \boldsymbol{r}_{\mathrm{rel}}^- + \boldsymbol{v}_{\mathrm{rel}}^- \\
\boldsymbol{v}^+ &= \boldsymbol{v}_0^+ + \Omega^+ \times \boldsymbol{r}_{\mathrm{rel}}^+ + \boldsymbol{v}_{\mathrm{rel}}^+
\end{aligned}
$$

Since the target is passive, the impulsive maneuver has no effect on its state of motion, which means $\boldsymbol{v}_0^+ = \boldsymbol{v}_0^-$ and $\Omega^+ = \Omega^-$.

- Furthermore, by definition of an impulsive maneuver, there is no change in the position, i.e., $\boldsymbol{r}_{\mathrm{rel}}^+ = \boldsymbol{r}_{\mathrm{rel}}^-$.

$$
\boldsymbol{v}^+ - \boldsymbol{v}^- = \boldsymbol{v}_{\mathrm{rel}}^+ - \boldsymbol{v}_{\mathrm{rel}}^- \quad \text{or} \quad \Delta \boldsymbol{v} = \Delta \boldsymbol{v}_{\mathrm{rel}}
$$