Fractals - Math 370

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Chapter 1

Introduction

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1.1 What are fractals?

In what follows we will be investigating fractals and some of their interesting properties. First we define a fractal.

Definition. A **fractal** is a curve or geometric figure, each part of which has the same statistical character as the whole.

What this means in layterms is that a fractal is a shape or set that has similar properties at all magnifications. We have a special name for this property: *self similarity*. A classic example is the *Koch Curve* (Fig 1.1) which looks identical at all levels of magnification. We should note that there are certain degenerate cases which satisfy our definition such as a straight line. The distinction between a true fractal and a degenerate case can be made with the notion of *fractal dimension* which we will touch upon in Chapter 4 which deals with measure theory.

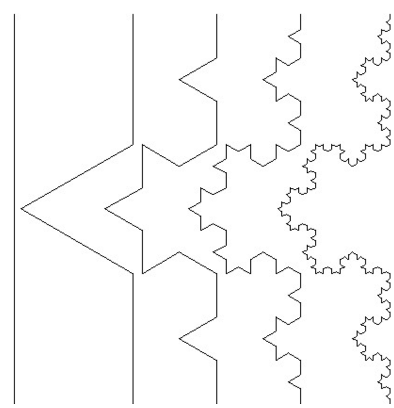

Figure 1.1: The first few iterations in generating the Koch Curve

1.2 Fractal Generation

It is often useful to think of a fractal as a set of points. However it is often difficult to explicitly state which points are in our fractal. A standard technique for fractal generation is to describe the *generation* of a fractal *iteratively* or *corecursively*. Typically we start with a base case, usually a set A_0, and define some sort of operation φ on a set and define $A_{i+1} = \varphi(A_i)$ for $i \in \mathbb{Z}^+$. In Fig 1.1 we see that our base case is a straight line and our operation φ replaces the middle third of each line segment in A_i with two line segments forming the top part of a triangle.

But at what point is our shape a fractal? A_0 obviously is not fractal - it is just a line segment and holds no interest for us. Likewise A_1 is not a fractal since it is just four line segments. What about A_{10} or A_{100}? $A_{1,000,000}$? None of these are in fact fractals. They *appear* to be fractal since we can only see detail larger than a certain scale. In fact, any fractal you have ever seen has only been an approximation of a fractal; true fractals are self similar and look the same at all scales.

So how is our construction useful? Well at no finite stage have we created a fractal but we can steal the analytic notion of the limit. First we defined the distance between a point $\boldsymbol{x} = (x_1, x_2)$ and a set $S \subset \mathbb{R}^2$ as

$$d(\boldsymbol{x}, S) = \inf_{\boldsymbol{y} \in S} \{|x - y|\}.$$

Formally we define the Koch curve as the collection of points $\boldsymbol{x} \in \mathbb{R}^2$ so that for any given $\varepsilon > 0$ there exists an $N > 0$ with $d(\boldsymbol{x}, A_n) < \varepsilon$ for all $n > N$.

We will see in Chapter 5 that there are other ways to generate and define fractals. We should also note that the term fractal is used in a number of ways. We often refer to shapes in nature as fractal even though they often bottom out after a finite, often rather low, number of iterations.

Chapter 2

Fractals in Nature

Chapter 3

Sierpinski's Gasket

Chapter 4

Measure Theory and Fractal Dimension

Chapter 5

Complex Dynamics

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5.1 The Complex Plane

One of the most beautiful areas of mathematics is complex analysis. From linear algebra we know that we may treat $\mathbb{R}^2$ as a vector space with vector addition and scalar multiplication. It would be desirable to define a multiplicative operation $* : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$

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