Fractal Geometry and Computer-Generated Art

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Date

Contents

1 Abstract

In this project I will delve into fractal geometry, computer generated visual art, and how the two tie together. I will draw from set theory, topology, manifold theory, the theory of dynamics.

2 Preliminaries

I lay out some fundamental concepts that we will build on. Some of this will get advanced so skim over material if it seems too dense.

2.1 Fractals - Some Definitions and Notions

The term 'fractal' is most likely familiar to you though one runs the risk of being imprecise with its use. We briefly state how we use the term.

Definition. A **fractal** is a natural phenomenon or a mathematical set that exhibits a repeating pattern that displays at every scale. If the replication is exactly the same at every scale, it is called a self-similar pattern. [4]

These are very general and somewhat wishy-washy definitions - not really definitions at all. However they will suffice for our purposes as long as we note a couple things. Firstly, in nature recursive patterns must bottom out, so we use the phrase "at all scales" very loosely. A classic example of the fractal geometry being exhibited in nature is looking at a coastline. There is, at any 'reasonable' scale (read *not too close* and *not too far away*), a boundary between a land mass and the water that surrounds it. When we ask what the length of the coastline is we will get differing results, depending on what are base unit of measurement is.

In fact, coast lines exhibit a fractal nature. If you were given just the line of the coast (say a part of the British coast, small enough to not have distinctive features) and asked to guess what scale you were looking at it would be impossible to tell. This is the essence of fractals. Note that there is not 'pattern' featured here, but rather the lack of a pattern. However we may notice that the lack of a pattern is consistent - that is, the 'roughness' of the coastline is approximately the same at all scales. This brings us to another definition of fractals which addresses this nuance.

Figure 1: The length of the coast of Britain with increasing precision - Taken from Wikipedia.com

Definition. A **fractal** is a curve or geometric figure, each part of which has the same statistical character as the whole.[3]

Here we aren't talking about patterns anymore but statistical characteristics of a set or curve. But this still doesn't quite get to the core of the matter. We want to be able to cast aside degenerate examples such as lines or planes. But the real line looks identical at all scales and fulfills both of our definitions so far.

Theorem 1. *An open interval is homeomorphic to the real line.*

This is a topological theorem and we will later describe both what topology is and what a homeomorphism is. For now it suffices to understand that in mathematics a morphism is a structure-preserving map from one object to another. Morphisms are a way of formalizing similarities between different mathematical structures.

Proof. Let $a < b$ be real numbers and so that the interval (a, b) is well defined. We would like a bijective continuous open map $\phi : (a, b) \to \mathbb{R}$. Without loss of generality assume that $a = -\pi/2$ and $b = \pi/2$ (a simple scale and translation allows us to do this and meets the requirements of ϕ). Taking $\phi(x) = \arctan(x)$ we have our desired result. $\square$

This means that the real line is *topologically identical* to an open interval. A similar statement is true for an open disc in $\mathbb{R}^2$ and $\mathbb{R}^2$ itself. This brings us to our third and final definition of a fractal:

Definition. fractal: You'll know it when you see it.

In truth we can supply more rigour by talking about *fractional dimensions*, and we will, though perhaps not at a great enough depth to classify exactly *why* an open interval is not fractal.

2.2 Types of Fractals

There are many types of fractals and to my knowledge their categorizion is not standardized. We enumerate a few of them here.

Deterministic Iterated Funcion Systems (IFS) This is a method of constructing fractals that involves taking the union of several images of an initial figure under *contractive* [1] maps $f_1, f_2, \ldots, f_n$ and repeating the process *ad infinitum.* Often these maps are required to be linear or affine though many IFS fractals are created with nonlinear functions. The classic example is the Sierpinski Gaskett. This mathematical oddity has many interesting properties, some of which we will have a chance to look at. The basic affine transformation is to make three copies of the original image contracted to half their original size and place them in a triangular formation.

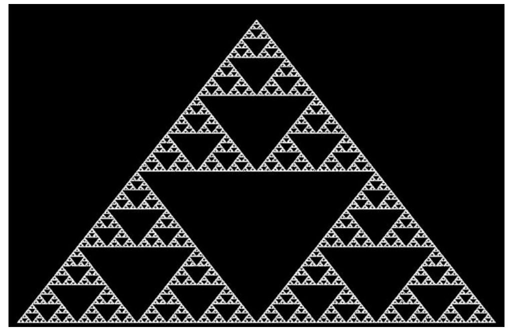


Figure 2: The Sierpinski Gaskett is a classic example of an IFS. Rendered in C++ using OpenGL

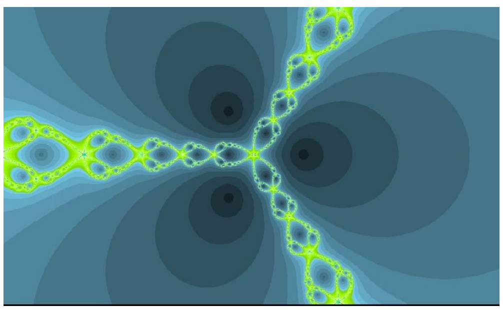
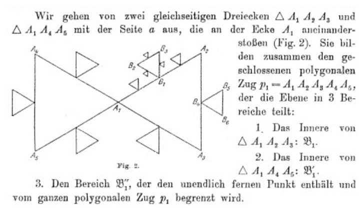


Figure 3: The newton fractal for $f(z) = z^3$. Rendered in C++ using OpenGL.

Complex-Dynamic Fractals (CDF)

i **Newton Fractals:** Newton fractals use Newton's method of approximation in the complex plane to observe how a function tends to its roots. While in the real case it is rather easy to predict where point will end up, in the complex case it is much more difficult.

ii **Julia Sets:** The Julia set, named after mathematician Gaston Julia, is another fractal generated in the complex plane. From a very simple equation and endless iteration we get a glimpse at how dynamic the complex plane really is.



Wir gehen von zwei gleichseitigen Dreiecken $\triangle A_1 A_2 A_3$ und $\triangle A_1 A_4 A_5$ mit der Seite a aus, die an der Ecke A_1 aneinanderstoßen (Fig. 2). Sie bilden zusammen den geschlossenen polygonalen Zug $p_1 = A_1 A_2 A_3 A_4 A_5$, der die Ebene in 3 Bereiche teilt:

1. Das Innere von $\triangle A_1 A_2 A_3$: $\mathfrak{B}_1$.

2. Das Innere von $\triangle A_1 A_4 A_5$: $\mathfrak{B}_1'$.

3. Den Bereich $\mathfrak{B}_1''$, der den unendlich fernen Punkt enthält und vom ganzen polygonalen Zug p_1 begrenzt wird.

Figure 4: The Julia Set before computer graphics, drawn in 1924 by Hubert Cremer [5]

Physical Simulations We have recently begun to notice the order inherant in chaos - that behind much of the worlds seeming randomness there is actually a very fine order; a beauty behind the clutter. Perhaps one of the most striking visualizations of this is the *(Magnetic) Chaos Pendulum.* Here is the basic setup: We have a table on which lie three magnets of equal charge and polarity, all equadistant from some center point p and equadistant frome one another; that is they lie on a circle with center p and describe an equilateral triangle.

Now associate with each magnet a color: Red, Yellow and Blue. We conduct an experiment where we hang a magnetic pendulum above the table so that it hangs down above the point p. Assume that the pendulum is attracted to each of the magnets, and assume that the attraction is powerful enough that the pendulum can be 'affixed' at an angle so that it is held steady by one of the magnets.

We take a birds eye view and let the pendulum be moved to occupy a point above the table. We then release it and see, after a while, which magnet the pendulum eventually ends up at. We color the point over which the pendulum initially hung the color associated with the magnet. The image on your left is the result.

[1]A contractive map $f : U \to V$ is a map with the property that for any two points in its domain x and y, $|f(x) - f(y)| < |x - y|$.

2.3 Fractals in Nature

2.4 Complex Numbers

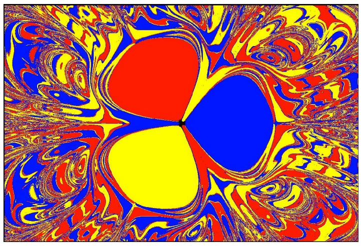


Figure 5: The magnetic chaos pendulum, rendered in C++ using OpenGL.

We assume some familiarity with complex numbers. We briefly state some facts that will be useful. First we talk about the ***field*** *of complex numbers* and denote it $\mathbb{C}$. We will also refer to complex numbers as existing in the **complex plane**. This gives us a geometric handle on how these objects act.

We denote complex numbers in a few ways. First we write them as ordered pairs $z = (x, y)$ where x is called the *real part* and y is called the *imaginary part*. This notation lets us work with $\mathbb{C}$ as the real plane $\mathbb{R}^2$ with some extra structure (the multiplication operation). We may also write $z = (x, y) = x + iy$ where i represents the square root of negative one: $i = \sqrt{-1}$. Note that there are actually two square roots of negative one: $i^2 = (-i)^2 = -1$. Finally it is often convenient to write elements of $\mathbb{C}^* = \mathbb{C} \setminus (0, 0)$ in polar form: $z = x + iy = r(\cos\theta + i \sin\theta)$, where $r = \sqrt{x^2 + y^2}$ and θ is the angle of elevation from the positive real line (up to an additive term of $2\pi i$). By Euler's formula we can write this as $re^{i\theta}$. This is called the exponential form. Note that exponential and polar form are not bijective.

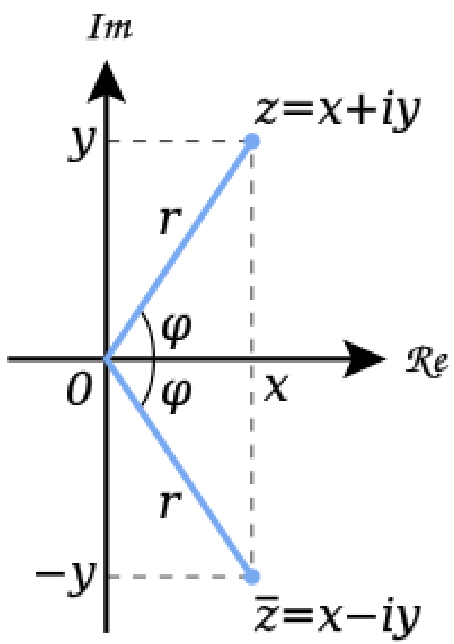


Figure 6: Complex conjugate of $z = x + iy$ with exponential form. Taken from Wikipedia.com

We define addition in the same manner as addition in $\mathbb{R}^2$: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$. Let $w = x_1 + iy_1$ and $z = x_2 + iy_2$. We define multiplication $w \cdot z$ as

$$w \cdot z = (x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2,\ x_1y_2 + x_2y_1).$$

Letting $re^{i\theta} = z,\ se^{i\phi} = w$ be the exponential forms of z, w we can write $zw = rse^{i(\phi+\theta)}$ and we see that complex multiplication is a rotation composed with a dilation.

As stated, $\mathbb{C}$ is a field under addition and operation, which means

1. $\mathbb{C}$ is an abelian group under addition;

2. $\mathbb{C} \setminus \{0\}$ is an abelian group under multiplication.

Both of these properties are easy to check. Further we note that $\mathbb{C}$ is the **extension field** of $\mathbb{R}$ - that is, an n-degree polynomial $P[X]$ over $\mathbb{R}$ has exactly n solutions counting multiplicity in $\mathbb{C}$, and $\mathbb{C}$ has no proper subset with this property (Fundamental Theorem of Algebra).

It will at times be useful to view $\mathbb{C}$ as a subset of $\mathbb{S}$, the Riemann Sphere. We can imagine attaching a point at infinity, written $\mathbb{S} = \mathbb{C} \cup \{\infty\}$. This has the very nice property that $\mathbb{S}$ contains all of its limit points. We say it is **limit-point compact** or simply **compact**. Compactness is a topological notion and in general compactness is stronger than limit-point compactness. However in the Euclidean plane this the two are equivalent and we lose no rigor switching between the terms.

Since $\mathbb{S}$ is compact we call $\mathbb{S}$ the **one-point compactification of** $\mathbb{C}$. Compact topological spaces have many nice features. Our main focus for considering $\mathbb{S}$ instead of $\mathbb{C}$ is that we can consider functions such as $\frac{1}{z}$ as a map from $\mathbb{S}$ to $\mathbb{S}$ if we define $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$. Additionally we will be discussing fractal basins, attractors in $\mathbb{C}$ for iterated functions. By dealing with $\mathbb{S}$ instead of $\mathbb{C}$ this will unify our theory and simplify concepts.

2.5 Topology and Manifolds

We will use some notions from topology and manifold theory. First we define an open disc $D(z_0, r)$ to be the set of points $\{z \in \mathbb{C} \mid |z - z_0| < r\}$. This is the familiar definition. Then an **open set** in the complex plane is a set that can be formed as

$$U = \bigcup_{\alpha \in J} D_\alpha,$$

where J is an arbitrary index set and D_α is an open disc. The collection $\{D(z_0, r) \mid z_0 \in \mathbb{C},\ r \in \mathbb{R}^+\}$ is called the **basis that generates** $\mathbb{C}$.

An n-manifold is a topological space that is locally euclidean, Hausdorff and second-countable. We inspect these ideas briefly.

Definition. Let X, Y be topological spaces. We say that $f : X \to Y$ is **continuous** if for every $V \subset Y$ the pre-image $U = f^{-1}(V) \subset X$ is open. We say that f is an **open map** if for every open set $U \subset X$ the image $f(U)$ is open in Y. We say that the map f is **bijective** if it is onto and into. Finally we say that f is a **homeomorphism** if it is a continuous open bijection. If $f : X \to Y$ is a homeomorphic then we say that X is homeomorphic to Y and write $X \approxeq Y$.

We leave the following statement unproved (it is a simple exercise if the reader is interested).

Theorem 2. *Homeomorphism is an equivalence relation.*

We define three terms needed to define a manifold.

Definition. A space X is **locally Euclidean** if for every point $x \in X$ we may find an open set $U \subset X$ with a homeomorphism $\phi : U \to \tilde{U} \subset \mathbb{R}^n$.

Definition. A space X is **Hausdörff** if for every pair of distinct points $x, y \in X$ we may find open sets $U, V \subset X$ so that $x \in U, y \in V,\ U \cap V = \varnothing$.

Definition. A space X is **Second Countable** if it has a countable basis.

The most

3 Fractals: A Closer Look

Here we will flesh out some ideas mentioned above. First, we will explicitly construct some of our fractals and look at some of the truly bizarre mathematical properties they have. We begin with a family of fractals named after Wacław Sierpiński, though the ideas that he studied were not coined by him, but rather have been along for a long time.

3.1 Sierpinski's Gasket

3.1.1 Wacław Sierpiński

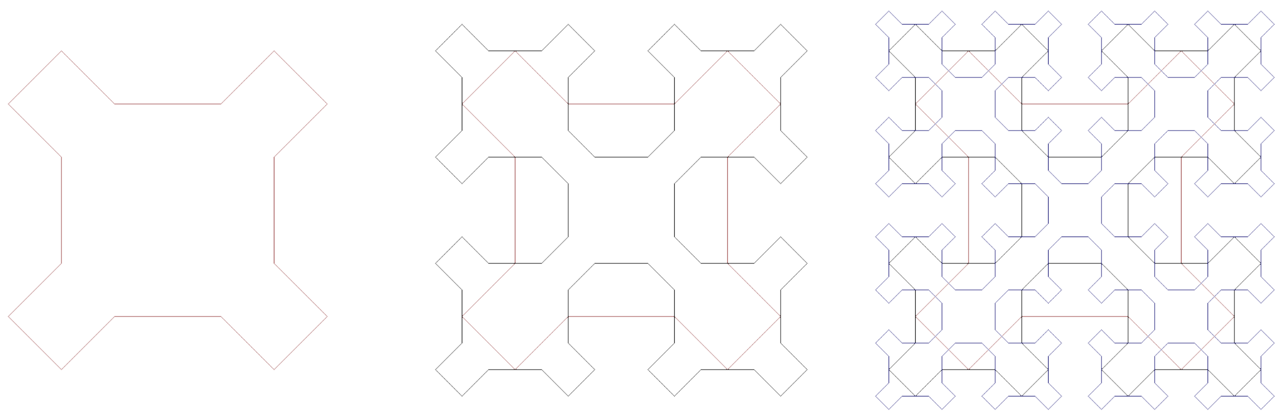
Wacław Sierpiński, a Polish mathematician, worked on set theory, number theory and topology. One of his first major accomplishments was to improve upon an inequality set by Gauss [6] pertaining to a famous problem on lattice points, often called the 'Gauss Circle Problem'. A basic statement of the problem, and a summary of Sierpiński's contributions, are described in the MacTutor History of Mathematics archive.

Let $R(r)$ denote the number of points (m, n), with $m, n \in \mathbb{Z}$, contained in a circle of radius r. Then there exists a constant C and a number k such that

$$|R(r) - \pi r^2| < Cr^k.$$

Let d be the minimal value of k. Gauss had proved in 1837 that $d \leq 1$. Sierpinski's major contribution was to show that it was possible to improve the inequality to $d < 2/3$.[6]

Another interesting result given by Sierpinski was the first ever example of a *regular number* - a number whose digits occur with equal frequency in whichever base it is written.[6] In addition he studied the *Sierpinski Curve* (Figure 7), which is a a closed path which contains every interior point of a square. As a result it is a space filling curve and has a *Hausdorff dimension* of 2 (the same as a region of the plane).



(a) C_1, the first step in creating the Sierpinski Curve

(b) C_2, the second step in creating the Sierpinski Curve

(c) C_3, the third step in creating the Sierpinski Curve

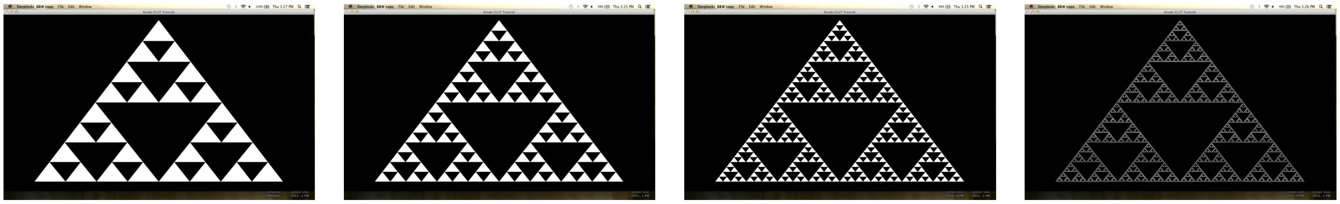
Figure 7: Successive depths of Sierpinski's Curve

Of note, the Sierpinksi Curve is recursively defined. The curve itself is the *limit* of a process. We define C_1 (Figure 7a) and then C_{n+1} in terms of C_n for all $n \geq 1$. Thus at no *finite* point do we have Sierpinski's Curve.

Sierpinski's Curve has infinite length, which should not be surprising. However what *is* surprising is that the curve encloses a non-zero area, specifically 5/12 of the area of the original square (CITATION NEEDED).

3.1.2 Generating Sierpinski's Gasket

Sierpinski is perhaps most famous for the triangle that takes his namesake; the *Sierpinski Triangle* (or *Sierpinski Gaskett*). Like the Sierpinski Curve, this is (usually) created with a recursive algorithm. We begin with a triangle, call it S_0. Now for $k > 0$ we work as follows: to form S_{k+1} from S_k inspect each of the triangles fomring S_k. For each triangle we mark the midpoint of each of its sides. Then we connect these marked points with line segments, thus cutting each of S_k's sub-triangles into four new triangles congruent to S_k but with dimensions halved. Removing the middle triangles (specifically the *interior* of the triangle, leaving the lines just drawn untouched), we obtain S_{k+1}.



(a) S_3: the third iteration of the construction of Sierpinski's Triangle.

(b) S_4: the fourth iteration of the construction of Sierpinski's Triangle.

(c) S_5: the fifth iteration of the construction of Sierpinski's Triangle.

(d) S_8: the eighth iteration of the construction of Sierpinski's Triangle.

Figure 8: Construction of Sierpinski's Triangle. Generated using C++ and OpenGL

3.2 Some Properties

One can ask about some of the basic properties of Sierpinski's Gasket. Firstly, what is its area? In each iteration from S_k to S_{k+1} the middle of each unremoved triangle (colored white in 8) is removed. Thus we compute the area of S_{k+1} as

$$\mathcal{A}(S_{k+1}) = \frac{3\mathcal{A}(S_k)}{4},$$

Assuming our original triangle S_0 has unit area then we have the closed form equation for the area: $\mathcal{A}(S_k) = \left(\frac{3}{4}\right)^k \mathcal{A}(S_0) = \left(\frac{3}{4}\right)^k$. It is easy to see that the limit of the area goes to zero,

$$\lim_{k\to\infty} \mathcal{A}(S_k) = 0.$$

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