Chapter 13

PARTIAL DERIVATIVES

(Follows from [1] Anton, …)

13.1 FUNCTIONS OF TWO OR MORE VARIABLES

NOTATION AND TERMINOLOGY

$$z = f(x, y)$$

$$u = f(x, y, z)$$

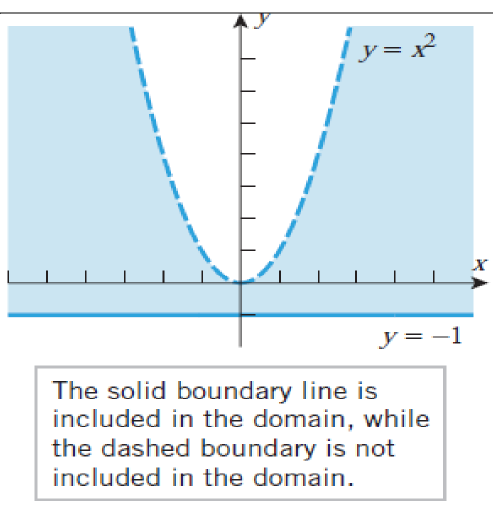
$$u = f(x_1, x_2, \cdots, x_n)$$

13.1.1 Definition A ***function f of two variables***, x and y, is a rule that assigns a unique real number $f(x, y)$ to each point (x, y) in some set D in the xy-plane.

13.1.2 Definition A ***function f of three variables***, x, y, and z, is a rule that assigns a unique real number $f(x, y, z)$ to each point (x, y, z) in some set D in three dimensional space.

Example 1 Sketch the natural domain of

$$f(x, y) = \sqrt{y + 1} + \ln(x^2 - y).$$

	$y + 1 \geq 0$ & $x^2 - y > 0$ => $y \geq -1$ & $y < x^2$

Example 2 Find the natural domain of $f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$.

$$1 - x^2 - y^2 - z^2 \geq 0 \quad => \quad x^2 + y^2 + z^2 \leq 1$$

Therefore, the natural domain of f consists of all points on or within the sphere

$$x^2 + y^2 + z^2 = 1.$$

Graphs of Functions of Two Variables

Recall that for a function f of one variable, the graph of $f(x)$ in the xy-plane was defined to be the graph of the equation $y = f(x)$. Similarly, if f is a function of two variables, we define the ***graph*** of $f(x, y)$ in xyz-space to be the graph of the equation $z = f(x, y)$. In general, such a graph will be a surface in 3-space.

Example 3 In each part, describe the graph of the function in an xyz-coordinate system.

(a) $f(x, y) = 1 - x - \frac{1}{2}y$

(b) $f(x, y) = \sqrt{1 - x^2 - y^2}$

(c) $f(x, y) = -\sqrt{x^2 + y^2}$

Solution (a) By definition, the graph of the given function is the graph of the equation $z = 1 - x - \frac{1}{2}y$ which is a plane. A triangular portion of the plane can be sketched by plotting the intersections with the coordinate axes and joining them with line segments (Figure 13.1.2*a*).

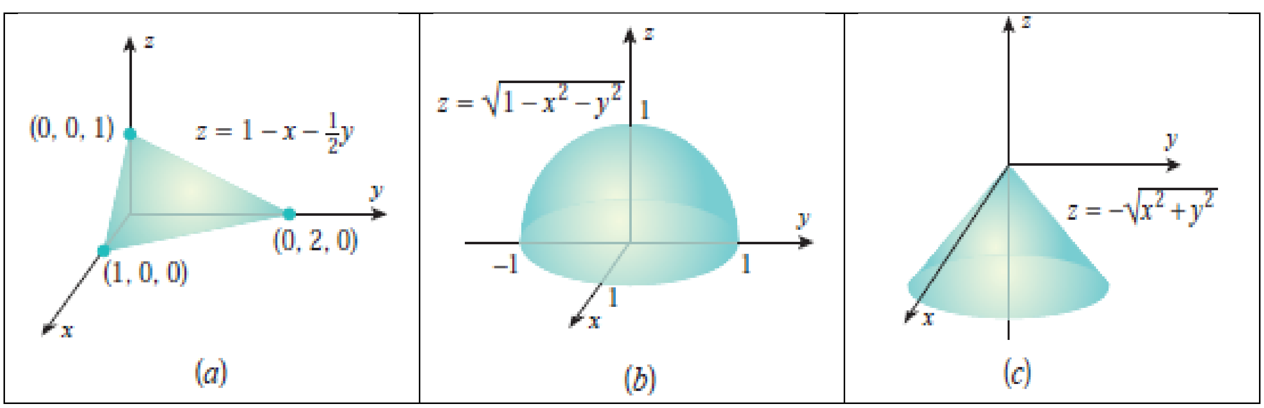


Figure 13.1.2

Solution (b) After squaring both sides, this can be rewritten as

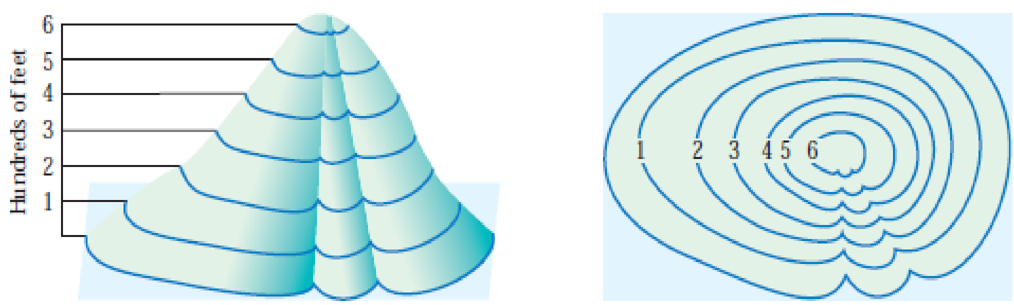
$$x^2 + y^2 + z^2 = 1$$

which represents a sphere of radius 1, centered at the origin. Since (b) imposes the added condition that $z \geq 0$, the graph is just the upper hemisphere (Figure 13.1.2b).

Solution (c)
After squaring, we obtain $z^2 = x^2 + y^2$ which is the equation of a circular cone (see Table 11.7.1). Since (c) imposes the condition that $z \leq 0$, the graph is just the lower nappe of the cone (Figure 13.1.2c).

LEVEL CURVES

We are all familiar with the topographic (or contour) maps in which a three-dimensional landscape, such as a mountain range, is represented by two-dimensional contour lines or curves of constant elevation. Consider, for example, the model hill and its contour map shown in Figure 13.1.3. The contour map is constructed by passing planes of constant elevation through the hill, projecting the resulting contours onto a flat surface, and labeling the contours with their elevations. In Figure 13.1.3, note how the two gullies appear as indentations in the contour lines and how the curves are close together on the contour map where the hill has a steep slope and become more widely spaced where the slope is gradual.



A perspective view of a model hill with two gullies

A contour map of the model hill

Figure 13.1.3

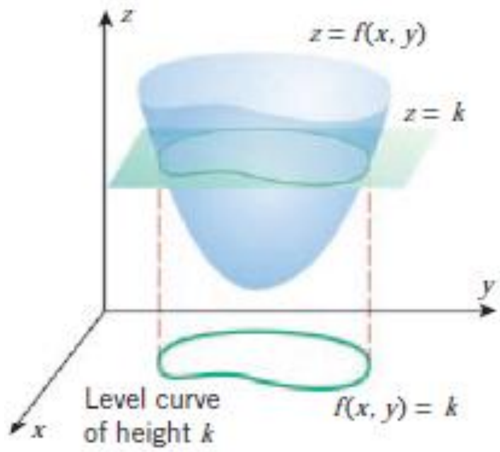
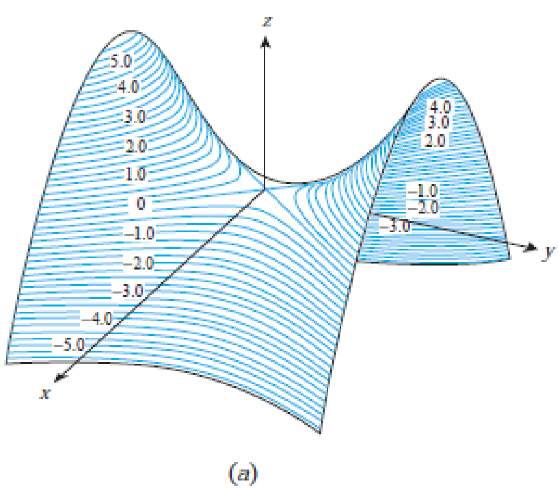


Figure 13.1.4

Contour maps are also useful for studying functions of two variables. If the surface $z = f(x, y)$ is cut by the horizontal plane $z = k$, then at all points on the intersection we have $f(x, y) = k$. The projection of this intersection onto the *xy*-plane is called the ***level curve of height k*** or the ***level curve with constant k*** (Figure 13.1.4). A set of level curves for $z = f(x, y)$ is called a ***contour plot*** or ***contour map*** of *f*.

Example 4 The graph of the function $f(x, y) = y^2 - x^2$ in *xyz*-space is the hyperbolic paraboloid (saddle surface) shown in Figure 13.1.5*a*. The level curves have equations of the form $y^2 - x^2 = k$. For $k > 0$ these curves are hyperbolas opening along lines parallel to the *y*-axis; for $k < 0$ they are hyperbolas opening along lines parallel to the *x*-axis; and for $k = 0$ the level curve consists of the intersecting lines $y + x = 0$ and $y - x = 0$ (Figure 13.1.5*b*).



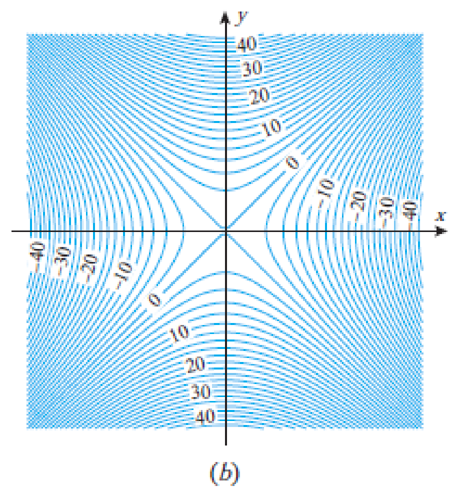


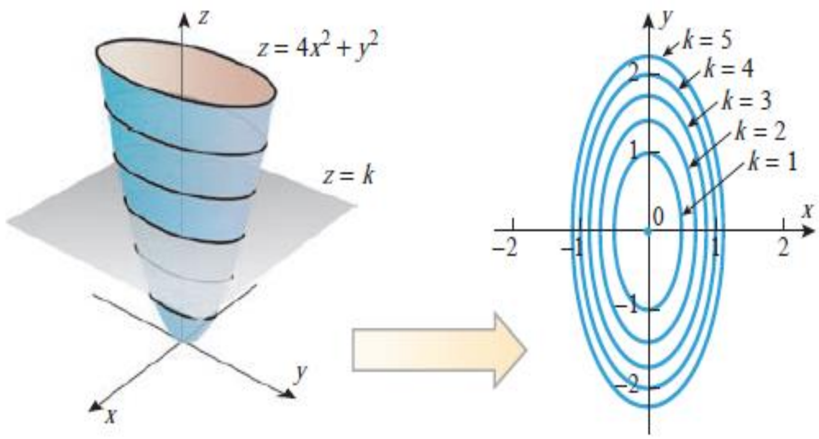
Figure 13.1.5

Example 5 Sketch the contour plot of $f(x, y) = 4x^2 + y^2$ using level curves of height $k = 0, 1, 2, 3, 4, 5$.

Solution. The graph of the surface $z = 4x^2 + y^2$ is the paraboloid shown in the left part of Figure 13.1.6, so we can reasonably expect the contour plot to be a family of ellipses centered at the origin. The level curve of height k has the equation $4x^2 + y^2 = k$. If $k = 0$, then the graph is the single point $(0, 0)$. For $k > 0$ we can rewrite the equation as

$$\frac{x^2}{k/4} + \frac{y^2}{k} = 1$$

which represents a family of ellipses with x-intercepts $\pm\sqrt{k}/2$ and y-intercepts $\pm\sqrt{k}$. The contour plot for the specified values of k is shown in the right part of Figure 13.1.6.



Exercise 13.1

23–26 Sketch the domain of f. Use solid lines for portions of the boundary included in the domain and dashed lines for portions not included.

23. $f(x, y) = \ln(1 - x^2 - y^2)$ **24.** $f(x, y) = \sqrt{x^2 + y^2 - 4}$

25. $f(x, y) = \dfrac{1}{x - y^2}$ **26.** $f(x, y) = \ln xy$

35–42 Sketch the graph of f.

33. $f(x, y) = 3$

34. $f(x, y) = \sqrt{9 - x^2 - y^2}$

35. $f(x, y) = \sqrt{x^2 + y^2}$

36. $f(x, y) = x^2 + y^2$

37. $f(x, y) = x^2 - y^2$

38. $f(x, y) = 4 - x^2 - y^2$

39. $f(x, y) = \sqrt{x^2 + y^2 + 1}$

40. $f(x, y) = \sqrt{x^2 + y^2 - 1}$

41. $f(x, y) = y + 1$

42. $f(x, y) = x^2$

43–44 In each part, select the term that best describes the level curves of the function f. Choose from the terms lines, circles, noncircular ellipses, parabolas, or hyperbolas.

43. (a) $f(x, y) = 5x^2 - 5y^2$ (b) $f(x, y) = y - 4x^2$
(c) $f(x, y) = x^2 + 3y^2$ (d) $f(x, y) = 3x^2$

44. (a) $f(x, y) = x^2 - 2xy + y^2$ (b) $f(x, y) = 2x^2 + 2y^2$
(c) $f(x, y) = x^2 - 2x - y^2$ (d) $f(x, y) = 2y^2 - x$

13.2 Limits and Continuity

Limits along Curves

There are infinitely many different curves along which one point can approach another (Figure 13.2.1). Our first objective in this section is to define the limit of $f(x, y)$ as (x, y) approaches a point (x_0, y_0) along a curve C (and similarly for functions of three variables).

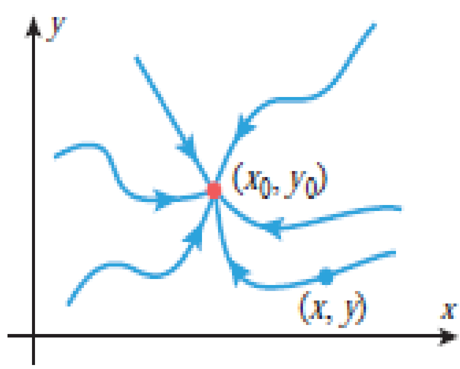


Figure 13.2.1

If C is a smooth parametric curve in 2-space or 3-space that is represented by the equations

$$x = x(t), \quad y = y(t) \qquad \text{or} \qquad x = x(t), \quad y = y(t), \quad z = z(t)$$

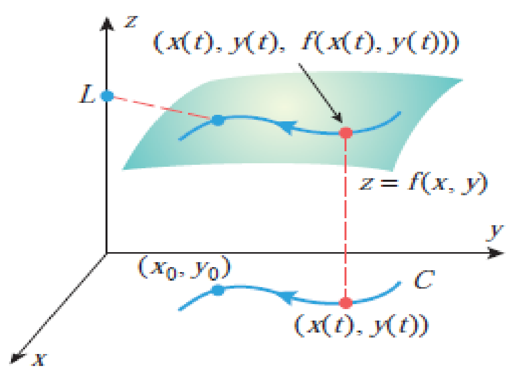
and if $x_0 = x(t_0)$, $y_0 = y(t_0)$, and $z_0 = z(t_0)$, then the limits

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\ (\text{along } C)}} f(x,y) \quad \text{and} \quad \lim_{\substack{(x,y,z)\to(x_0,y_0,z_0)\\ (\text{along } C)}} f(x,y,z)$$

are defined by

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\ (\text{along } C)}} f(x,y) = \lim_{t\to t_0} f(x(t), y(t)) \tag{1}$$

$$\lim_{\substack{(x,y,z)\to(x_0,y_0,z_0)\\ (\text{along } C)}} f(x,y,z) = \lim_{t\to t_0} f(x(t), y(t), z(t)) \tag{2}$$



$$\lim_{\substack{(x,y)\to(x_0,y_0)\\ (\text{along } C)}} f(x,y) = L$$

In these formulas the limit of the function of t must be treated as a one-sided limit if (x_0, y_0) or (x_0, y_0, z_0) is an endpoint of C.

Example 1 Given that

$$f(x,y) = -\frac{xy}{x^2+y^2}.$$

Find the limit along

(a) the x-axis (b) the y-axis (c) the line $y = x$

(d) the line $y = -x$ (e) the parabola $y = x^2$

Solution (a). The x-axis has parametric equations, $x = t, y = 0$, with $(0,0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=0)}} f(x,y) = \lim_{t\to 0} f(t,0) = \lim_{t\to 0}\left(-\frac{0}{t^2}\right) = \lim_{t\to 0} 0 = 0$$

Solution (b). The y-axis has parametric equations x = 0, y = t , with $(0,0)$ corresponding to t = 0, so

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } x=0)}} f(x,y) = \lim_{t\to 0} f(0,t) = \lim_{t\to 0}\left(-\frac{0}{t^2}\right) = \lim_{t\to 0} 0 = 0$$

Solution (c) The line $y = x$ has parametric equations $x = t, y = t$, with $(0,0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=x)}} f(x,y) = \lim_{t\to 0} f(t,t) = \lim_{t\to 0}\left(-\frac{t^2}{2t^2}\right) = \lim_{t\to 0}\left(-\frac{1}{2}\right) = -\frac{1}{2}$$

Solution (d). The line $y = -x$ has parametric equations $x = t$, $y = -t$, with $(0,0)$corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=-x)}} f(x,y) = \lim_{t\to 0} f(t,-t) = \lim_{t\to 0}\frac{t^2}{2t^2} = \lim_{t\to 0}\frac{1}{2} = \frac{1}{2}$$

Solution (e). The parabola $y = x^2$ has parametric equations $x = t$, $y = t$, with $(0,0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=x^2)}} f(x,y) = \lim_{t\to 0} f(t,t^2) = \lim_{t\to 0}\left(-\frac{t^3}{t^2+t^4}\right) = \lim_{t\to 0}\left(-\frac{t}{1+t^2}\right) = 0$$

GENERAL LIMITS OF FUNCTIONS OF TWO VARIABLES

The statement

is intended to convey the idea that the value of $f(x, y)$ can be made as close as we like to the number L by restricting the point (x, y) to be sufficiently close to (but different from) the point (x_0, y_0).

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

13.2.1 DEFINITION Let f be a function of two variables, and assume that f is defined at all points of some open disk centered at (x_0, y_0), except possibly at (x_0, y_0). We will write

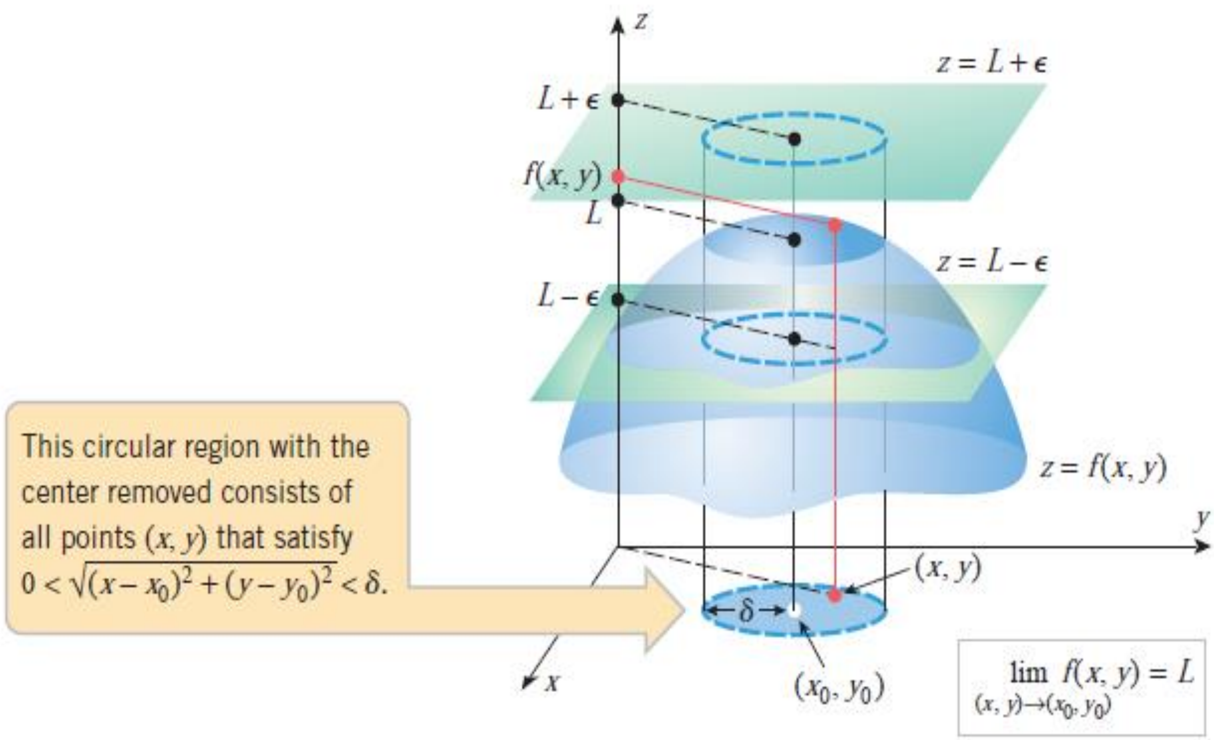
$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L \tag{3}$$

if given any number $\epsilon > 0$, we can find a number $\delta > 0$ such that $f(x, y)$ satisfies

$$|f(x, y) - L| < \epsilon$$

whenever the distance between (x, y) and (x_0, y_0) satisfies

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$



Example 2

$$\lim_{(x,y)\to(1,4)} [5x^3y^2 - 9] = \lim_{(x,y)\to(1,4)} [5x^3y^2] - \lim_{(x,y)\to(1,4)} 9$$
$$= 5\left[\lim_{(x,y)\to(1,4)} x\right]^3\left[\lim_{(x,y)\to(1,4)} y\right]^2 - 9$$
$$= 5(1)^3(4)^2 - 9 = 71 \blacktriangleleft$$

Relationships between General Limits and Limits along smooth Curves

13.2.2 THEOREM

(*a*) *If* $f(x, y) \to L$ *as* $(x, y) \to (x_0, y_0)$, *then* $f(x, y) \to L$ *as* $(x, y) \to (x_0, y_0)$ *along any smooth curve.*

(*b*) *If the limit of* $f(x, y)$ *fails to exist as* $(x, y) \to (x_0, y_0)$ *along some smooth curve, or if* $f(x, y)$ *has different limits as* $(x, y) \to (x_0, y_0)$ *along two different smooth curves, then the limit of* $f(x, y)$ *does not exist as* $(x, y) \to (x_0, y_0)$.

Example 3

The limit

$$\lim_{(x,y)\to(0,0)} -\frac{xy}{x^2 + y^2}$$

does not exist because in Example 1 we found two different smooth curves along which this limit had different values. Specifically,

$$\lim_{\substack{(x,y)\to(0,0) \\ (\text{along } x=0)}} -\frac{xy}{x^2 + y^2} = 0 \quad \text{and} \quad \lim_{\substack{(x,y)\to(0,0) \\ (\text{along } y=x)}} -\frac{xy}{x^2 + y^2} = -\frac{1}{2} \blacktriangleleft$$

CONTINUITY

13.2.3 DEFINITION A function $f(x, y)$ is said to be ***continuous at*** (x_0, y_0) if $f(x_0, y_0)$ is defined and if

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$$

In addition, if f is continuous at every point in an open set D, then we say that f is ***continuous on*** D, and if f is continuous at every point in the xy-plane, then we say that f is ***continuous everywhere***.

13.2.3 DEFINITION A function $f(x, y)$ is said to be ***continuous at*** (x_0, y_0) if $f(x_0, y_0)$ is defined and if

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$$

In addition, if f is continuous at every point in an open set D, then we say that f is ***continuous on*** D, and if f is continuous at every point in the xy-plane, then we say that f is ***continuous everywhere***.

13.2.4 THEOREM

(*a*) *If $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0, then $f(x, y) = g(x)h(y)$ is continuous at (x_0, y_0).*

(*b*) *If $h(x, y)$ is continuous at (x_0, y_0) and $g(u)$ is continuous at $u = h(x_0, y_0)$, then the composition $f(x, y) = g(h(x, y))$ is continuous at (x_0, y_0).*

(*c*) *If $f(x, y)$ is continuous at (x_0, y_0), and if $x(t)$ and $y(t)$ are continuous at t_0 with $x(t_0) = x_0$ and $y(t_0) = y_0$, then the composition $f(x(t), y(t))$ is continuous at t_0.*

Example 4

Use Theorem 13.2.4 to show that the functions $f(x, y) = 3x^2y^5$ and $f(x, y) = \sin(3x^2y^5)$ are continuous everywhere.

Solution. The polynomials $g(x) = 3x^2$ and $h(y) = y^5$ are continuous at every real number, and therefore by part (*a*) of Theorem 13.2.4, the function $f(x, y) = 3x^2y^5$ is continuous at every point (x, y) in the xy-plane. Since $3x^2y^5$ is continuous at every point in the xy-plane and $\sin u$ is continuous at every real number u, it follows from part (*b*) of Theorem 13.2.4 that the composition $f(x, y) = \sin(3x^2y^5)$ is continuous everywhere. ◀

Theorem 13.2.4 is one of a whole class of theorems about continuity of functions in two or more variables. The content of these theorems can be summarized informally with three basic principles:

Recognizing Continuous Functions

- A composition of continuous functions is continuous.
- A sum, difference, or product of continuous functions is continuous.
- A quotient of continuous functions is continuous, except where the denominator is zero.

By using these principles and Theorem 13.2.4, you should be able to confirm that the following functions are all continuous everywhere:

$$xe^{xy} + y^{2/3}, \quad \cosh(xy^3) - |xy|, \quad \frac{xy}{1 + x^2 + y^2}$$

Example 5

Evaluate $\displaystyle\lim_{(x,y)\to(-1,2)} \frac{xy}{x^2 + y^2}$.

Solution

Since $f(x, y) = xy/(x^2 + y^2)$ is continuous at $(-1, 2)$ (why?), it follows from the definition of continuity for functions of two variables that

$$\lim_{(x,y)\to(-1,2)} \frac{xy}{x^2 + y^2} = \frac{(-1)(2)}{(-1)^2 + (2)^2} = -\frac{2}{5}$$

Example 6

Since the function

$$f(x, y) = \frac{x^3y^2}{1 - xy}$$

is a quotient of continuous functions, it is continuous except where $1 - xy = 0$. Thus, $f(x, y)$ is continuous everywhere except on the hyperbola $xy = 1$. ◀

LIMITS AT DISCONTINUITIES

Sometimes it is easy to recognize when a limit does not exist. For example, it is evident that

$$\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2} = +\infty$$

which implies that the values of the function approach $+\infty$ as $(x, y)\to(0, 0)$ along any smooth curve (Figure 13.2.9). However, it is not evident whether the limit

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$$

exists because it is an indeterminate form of type $0\cdot\infty$. Although L'Hôpital's rule cannot be applied directly, the following example illustrates a method for finding this limit by converting to polar coordinates.

Example 7

Find $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$.

Solution

Let (r, θ) be polar coordinates of the point (x, y) with $r \geq 0$. Then we have

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2+y^2$$

Moreover, since $r \geq 0$ we have $r = \sqrt{x^2+y^2}$, so that $r\to 0^+$ if and only if $(x, y)\to(0, 0)$. Thus, we can rewrite the given limit as

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) &= \lim_{r\to 0^+} r^2\ln r^2 \\
&= \lim_{r\to 0^+} \frac{2\ln r}{1/r^2} && \boxed{\text{This converts the limit to an indeterminate form of type } \infty/\infty.} \\
&= \lim_{r\to 0^+} \frac{2/r}{-2/r^3} && \boxed{\text{L'Hôpital's rule}} \\
&= \lim_{r\to 0^+} (-r^2) = 0 \blacktriangleleft
\end{aligned}$$

Exercise 13.2

1–6 Use limit laws and continuity properties to evaluate the limit.

1. $\displaystyle\lim_{(x,y)\to(1,3)} (4xy^2 - x)$

2. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4x - y}{\sin y - 1}$

3. $\displaystyle\lim_{(x,y)\to(-1,2)} \frac{xy^3}{x + y}$

4. $\displaystyle\lim_{(x,y)\to(1,-3)} e^{2x-y^2}$

5. $\displaystyle\lim_{(x,y)\to(0,0)} \ln(1 + x^2y^3)$

6. $\displaystyle\lim_{(x,y)\to(4,-2)} x\sqrt[3]{y^3 + 2x}$

7–8 Show that the limit does not exist by considering the limits as $(x, y) \to (0, 0)$ along the coordinate axes.

7. (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3}{x^2 + 2y^2}$ (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x + y}{2x^2 + y^2}$

8. (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x - y}{x^2 + y^2}$ (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\cos xy}{x^2 + y^2}$

9–12 Evaluate the limit using the substitution $z = x^2 + y^2$ and observing that $z \to 0^+$ if and only if $(x, y) \to (0, 0)$.

9. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2}$

10. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1 - \cos(x^2 + y^2)}{x^2 + y^2}$

11. $\displaystyle\lim_{(x,y)\to(0,0)} e^{-1/(x^2+y^2)}$

12. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{-1/\sqrt{x^2+y^2}}}{\sqrt{x^2 + y^2}}$

13–22 Determine whether the limit exists. If so, find its value.

13. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2}$

14. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 - 16y^4}{x^2 + 4y^2}$

15. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{3x^2 + 2y^2}$

16. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1 - x^2 - y^2}{x^2 + y^2}$

23–26 Evaluate the limits by converting to polar coordinates, as in Example 7. ■

23. $\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}\ln(x^2+y^2)$

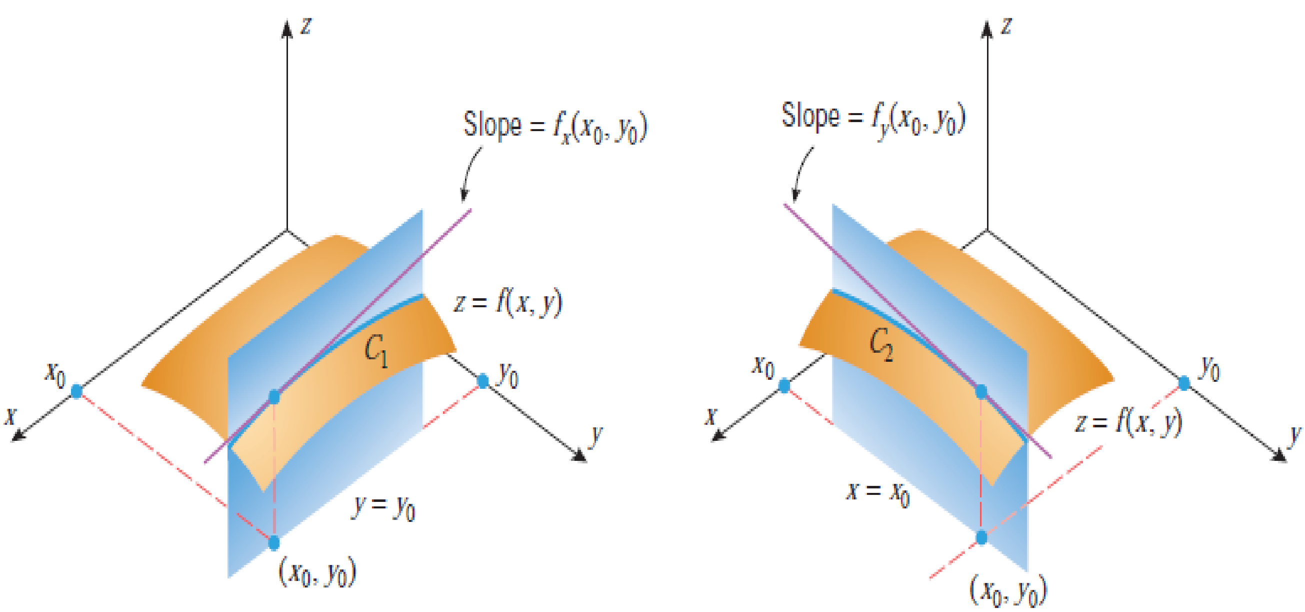
24. $\lim_{(x,y)\to(0,0)} y\ln(x^2+y^2)$ **25.** $\lim_{(x,y)\to(0,0)} \dfrac{x^2y^2}{\sqrt{x^2+y^2}}$

26. $\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+2y^2}}$

13.3 PARTIAL DERIVATIVES

Partial Derivatives of Functions of Two Variables

Partial Derivatives Viewed As Rates of Change and Slopes



13.3.1 Definition If $z = f(x, y)$ and (x_0, y_0) is a point in the domain of f, then the ***partial derivative of f with respect to x*** at (x_0, y_0) [also called the ***partial derivative of z with respect to x*** at (x_0, y_0)] is the derivative at x_0 of the function that results when $y = y_0$ is held fixed and x is allowed to vary. This partial derivative is denoted by $f_x\,(x_0, y_0)$ and is given by

$$f_x(x_0, y_0) = \frac{d}{dx}[f(x, y_0)]\bigg|_{x=x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \qquad (1)$$

Similarly, the ***partial derivative of f with respect to y*** at (x_0, y_0) [also called the ***partial derivative of z with respect to y*** at (x_0, y_0)] is the derivative at y_0 of the function that results when $x = x_0$ is held fixed and y is allowed to vary. This partial derivative is denoted by $f_y\,(x_0, y_0)$ and is given by

$$f_y(x_0, y_0) = \frac{d}{dy}[f(x_0, y)]\bigg|_{y=y_0} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} \qquad (2)$$

Example 1

Find $f_x(1, 3)$ and $f_y(1, 3)$ for the function $f(x, y) = 2x^3y^2 + 2y + 4x$.

For

$$f_x(x, 3) = \frac{d}{dx}[f(x, 3)] = \frac{d}{dx}[18x^3 + 4x + 6] = 54x^2 + 4$$

We obtain

$$f_x(1, 3) = 54 + 4 = 58$$

Also since,

$$f_y(1, y) = \frac{d}{dy}[f(1, y)] = \frac{d}{dy}[2y^2 + 2y + 4] = 4y + 2$$

We obtain,

$$f_y(1, 3) = 4(3) + 2 = 14.$$

The Partial Derivative Functions

Formulas (1) and (2) define the partial derivatives of a function at a specific point (x_0, y_0). However, often it will be desirable to omit the subscripts and think of the partial derivatives as functions of the variables x and y. These functions are

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

Example 2

Find $f_x(x, y)$ and $f_y(x, y)$ for $f(x, y) = 2x^3y^2 + 2y + 4x$,
and hence compute

$$f_x(1, 3) \text{ and } f_y(1, 3).$$

Solution

Keeping y fixed and differentiating with respect to x yields

$$f_x(x, y) = \frac{d}{dx}[2x^3y^2 + 2y + 4x] = 6x^2y^2 + 4$$

and keeping x fixed and differentiating with respect to y yields

$$f_y(x, y) = \frac{d}{dy}[2x^3y^2 + 2y + 4x] = 4x^3y + 2$$

Thus,

$$f_x(1, 3) = 6(1^2)(3^2) + 4 = 58 \quad \text{and} \quad f_y(1, 3) = 4(1^3)3 + 2 = 14$$

Partial Derivative Notation

If $z = f(x, y)$, then the partial derivatives f_x and f_y are also denoted by the symbols

$$\frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}, \quad \frac{\partial z}{\partial y}$$

Some typical notations for the partial derivatives of $z = f(x, y)$ at a point (x_0, y_0) are

$$\left.\frac{\partial f}{\partial x}\right|_{x=x_0, y=y_0}, \quad \left.\frac{\partial z}{\partial x}\right|_{(x_0, y_0)}, \quad \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)}, \quad \frac{\partial f}{\partial x}(x_0, y_0), \quad \frac{\partial z}{\partial x}(x_0, y_0)$$

Example 3

Find $\partial z/\partial x$ and $\partial z/\partial y$ if $z = x^4 \sin(xy^3)$.

Solution

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[x^4 \sin(xy^3)] = x^4 \frac{\partial}{\partial x}[\sin(xy^3)] + \sin(xy^3) \cdot \frac{\partial}{\partial x}(x^4)$$

$$= x^4 \cos(xy^3) \cdot y^3 + \sin(xy^3) \cdot 4x^3 = x^4 y^3 \cos(xy^3) + 4x^3 \sin(xy^3)$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[x^4 \sin(xy^3)] = x^4 \frac{\partial}{\partial y}[\sin(xy^3)] + \sin(xy^3) \cdot \frac{\partial}{\partial y}(x^4)$$

$$= x^4 \cos(xy^3) \cdot 3xy^2 + \sin(xy^3) \cdot 0 = 3x^5 y^2 \cos(xy^3)$$

Example 5

Let $f(x, y) = x^2 y + 5y^3$.

(a) Find the slope of the surface $z = f(x, y)$ in the x-direction at the point $(1, -2)$.

(b) Find the slope of the surface $z = f(x, y)$ in the y-direction at the point $(1, -2)$.

Solution.

(a)

Differentiating f with respect to x with y held fixed yields

$$f_x(x, y) = 2xy$$

Thus, the slope in the x-direction is $f_x(1, -2) = -4$; that is, z is decreasing at the rate of 4 units per unit increase in x.

(b)

Differentiating f with respect to y with x held fixed yields

$$f_y(x, y) = x^2 + 15y^2$$

Thus, the slope in the y-direction is $f_y(1, -2) = 61$; that is, z is increasing at the rate of 61 units per unit increase in y. ◀

Home work

Example 7, Example 8.

Partial Derivatives and Continuity

In contrast to the case of functions of a single variable, the existence of partial derivatives for a multivariable function does not guarantee the continuity of the function. This fact is
shown in the following example.

Example 9 Let

$$f(x, y) = \begin{cases} -\dfrac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases} \qquad (3)$$

(a) Show that $f_x(x, y)$ and $f_y(x, y)$ exist at all points (x, y).

(b) Explain why f is not continuous at $(0, 0)$.

Except at (0, 0), the partial derivatives of *f* are

$$f_x(x, y) = -\frac{(x^2 + y^2)y - xy(2x)}{(x^2 + y^2)^2} = \frac{x^2 y - y^3}{(x^2 + y^2)^2}$$

$$f_y(x, y) = -\frac{(x^2 + y^2)x - xy(2y)}{(x^2 + y^2)^2} = \frac{xy^2 - x^3}{(x^2 + y^2)^2} \qquad (4-5)$$

It is not evident whether *f* has partial derivatives at (0, 0), and if so, what the values of those derivatives are. To answer that question we will have to use the definitions of the partial derivatives (Definition 13.3.1). Applying Formulas (1) and (2) to (3) we obtain

$$f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0$$

$$f_y(0, 0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} = 0$$

This shows that f has partial derivatives at (0, 0) and the values of both partial derivatives are 0 at that point.

Solution (b) We saw in Example 3 of Section 13.2 that

$$\lim_{(x,y)\to(0,0)} -\frac{xy}{x^2+y^2}$$

does not exist. Thus, f is not continuous at (0, 0).

Partial Derivatives of Functions With More Than Two Variables

For a function $f(x, y, z)$ of three variables, there are three ***partial derivatives***:

$$f_x(x,y,z),\ \ f_y(x,y,z),\ \ f_z(x,y,\mathrm{z})$$

The partial derivative f_x is calculated by holding y and z constant and differentiating with respect to x. For f_y the variables x and z are held constant, and for f_z the variables x and y are held constant. If a dependent variable

$$w\ =\ f(x,y,z)$$

is used, then the three partial derivatives of f can be denoted by

$$\frac{\partial w}{\partial x},\qquad \frac{\partial w}{\partial y},\qquad \frac{\partial w}{\partial z}.$$

Home work

Example 10, Example 11.

HIGHER-ORDER PARTIAL DERIVATIVES

Suppose that f is a function of two variables x and y. Since the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are also functions of x and y, these functions may themselves have partial derivatives. This gives rise to four possible ***second-order*** partial derivatives of f, which are defined by

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

Differentiate twice with respect to x.

Differentiate twice with respect to y.

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy} \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

Differentiate first with respect to x and then with respect to y.

Differentiate first with respect to y and then with respect to x.

The last two cases are called the ***mixed second-order partial derivatives*** or the ***mixed second partials***. Also, the derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are often called the ***first-order partial derivatives*** when it is necessary to distinguish them from higher-order partial derivatives. Similar conventions apply to the second-order partial derivatives of a function of three variables.

Example 12 Find the second-order partial derivatives of

Solution. We have

$$f(x, y) = x^2y^3 + x^4y.$$

$$\frac{\partial f}{\partial x} = 2xy^3 + 4x^3y \quad \text{and} \quad \frac{\partial f}{\partial y} = 3x^2y^2 + x^4$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2xy^3 + 4x^3y) = 2y^3 + 12x^2y$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3x^2y^2 + x^4) = 6x^2y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(3x^2y^2 + x^4) = 6xy^2 + 4x^3$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2xy^3 + 4x^3y) = 6xy^2 + 4x^3$$

13.3.2 THEOREM *Let f be a function of two variables. If f_{xy} and f_{yx} are continuous on some open disk, then $f_{xy} = f_{yx}$ on that disk.*

EXERCISE SET 13.3

3–10 Evaluate the indicated partial derivatives

3. $z = 9x^2y - 3x^5y$; $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$

4. $f(x, y) = 10x^2y^4 - 6xy^2 + 10x^2$; $f_x(x, y), f_y(x, y)$

5. $z = (x^2 + 5x - 2y)^8$; $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$

6. $f(x, y) = \dfrac{1}{xy^2 - x^2y}$; $f_x(x, y), f_y(x, y)$

7. $\dfrac{\partial}{\partial p}(e^{-7p/q}), \dfrac{\partial}{\partial q}(e^{-7p/q})$

8. $\dfrac{\partial}{\partial x}(xe^{\sqrt{15xy}}), \dfrac{\partial}{\partial y}(xe^{\sqrt{15xy}})$

9. $z = \sin(5x^3y + 7xy^2)$; $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$

10. $f(x, y) = \cos(2xy^2 - 3x^2y^2)$; $f_x(x, y), f_y(x, y)$

11. Let $f(x, y) = \sqrt{3x + 2y}$.

(a) Find the slope of the surface $z = f(x, y)$ in the x-direction at the point $(4, 2)$.

(b) Find the slope of the surface $z = f(x, y)$ in the y-direction at the point $(4, 2)$.

12. Let $f(x, y) = xe^{-y} + 5y$.

(a) Find the slope of the surface $z = f(x, y)$ in the x-direction at the point $(3, 0)$.

(b) Find the slope of the surface $z = f(x, y)$ in the y-direction at the point $(3, 0)$.

13. Let $z = \sin(y^2 - 4x)$.

(a) Find the rate of change of z with respect to x at the point $(2, 1)$ with y held fixed.

(b) Find the rate of change of z with respect to y at the point $(2, 1)$ with x held fixed.

25–30 Find $\partial z/\partial x$ and $\partial z/\partial y$.

25. $z = 4e^{x^2y^3}$

26. $z = \cos(x^5y^4)$

27. $z = x^3 \ln(1 + xy^{-3/5})$

28. $z = e^{xy} \sin 4y^2$

29. $z = \dfrac{xy}{x^2 + y^2}$

30. $z = \dfrac{x^2y^3}{\sqrt{x + y}}$

31–36 Find $f_x(x, y)$ and $f_y(x, y)$.

31. $f(x, y) = \sqrt{3x^5y - 7x^3y}$

32. $f(x, y) = \dfrac{x + y}{x - y}$

33. $f(x, y) = y^{-3/2} \tan^{-1}(x/y)$

34. $f(x, y) = x^3e^{-y} + y^3 \sec \sqrt{x}$

35. $f(x, y) = (y^2 \tan x)^{-4/3}$

36. $f(x, y) = \cosh(\sqrt{x}) \sinh^2(xy^2)$

43–46 Find f_x, f_y, and f_z.

43. $f(x, y, z) = z \ln(x^2y \cos z)$

44. $f(x, y, z) = y^{-3/2} \sec\left(\dfrac{xz}{y}\right)$

45. $f(x, y, z) = \tan^{-1}\left(\dfrac{1}{xy^2z^3}\right)$

46. $f(x, y, z) = \cosh(\sqrt{z}) \sinh^2(x^2yz)$

47–50 Find $\partial w/\partial x$, $\partial w/\partial y$, and $\partial w/\partial z$.

47. $w = ye^z \sin xz$

48. $w = \dfrac{x^2 - y^2}{y^2 + z^2}$

49. $w = \sqrt{x^2 + y^2 + z^2}$

50. $w = y^3e^{2x+3z}$

55. A point moves along the intersection of the elliptic paraboloid $z = x^2 + 3y^2$ and the plane $y = 1$. At what rate is z changing with respect to x when the point is at $(2, 1, 7)$?

56. A point moves along the intersection of the elliptic paraboloid $z = x^2 + 3y^2$ and the plane $x = 2$. At what rate is z changing with respect to y when the point is at $(2, 1, 7)$?

57. A point moves along the intersection of the plane $y = 3$ and the surface $z = \sqrt{29 - x^2 - y^2}$. At what rate is z changing with respect to x when the point is at $(4, 3, 2)$?

58. Find the slope of the tangent line at $(-1, 1, 5)$ to the curve of intersection of the surface $z = x^2 + 4y^2$ and

(a) the plane $x = -1$ (b) the plane $y = 1$.

15.5 THE CHAIN RULE

13.5.1 THEOREM (*Chain Rules for Derivatives*) *If $x = x(t)$ and $y = y(t)$ are differentiable at t, and if $z = f(x, y)$ is differentiable at the point $(x, y) = (x(t), y(t))$, then $z = f(x(t), y(t))$ is differentiable at t and*

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \tag{5}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y).

If each of the functions $x = x(t)$, $y = y(t)$, and $z = z(t)$ is differentiable at t, and if $w = f(x, y, z)$ is differentiable at the point $(x, y, z) = (x(t), y(t), z(t))$, then the function $w = f(x(t), y(t), z(t))$ is differentiable at t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \tag{6}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y, z).

Example 1 Suppose that

$$z = x^2y, x = t^2, y = t^3$$

Use the chain rule to find *dz/dt*, and check the result by expressing *z* as a function of *t* and differentiating directly.

Solution. By the chain rule [Formula (5)],

By the chain rule [Formula (5)],

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2xy)(2t) + (x^2)(3t^2)$$

$$= (2t^5)(2t) + (t^4)(3t^2) = 7t^6$$

we can express z directly as a function of t,

$$z = x^2y = (t^2)^2(t^3) = t^7 \text{ to obtain } dz/dt = 7t^6.$$

Example 2 H.W.

13.5.2 THEOREM (*Chain Rules for Partial Derivatives*) *If $x = x(u, v)$ and $y = y(u, v)$ have first-order partial derivatives at the point (u, v), and if $z = f(x, y)$ is differentiable at the point $(x, y) = (x(u, v), y(u, v))$, then $z = f(x(u, v), y(u, v))$ has first-order partial derivatives at the point (u, v) given by*

$$\boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}} \quad \text{and} \quad \boxed{\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}} \tag{7–8}$$

If each function $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$ has first-order partial derivatives at the point (u, v), and if the function $w = f(x, y, z)$ is differentiable at the point $(x, y, z) = (x(u, v), y(u, v), z(u, v))$, then $w = f(x(u, v), y(u, v), z(u, v))$ has first-order partial derivatives at the point (u, v) given by

$$\boxed{\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}} \quad \text{and} \quad \boxed{\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}} \tag{9–10}$$

Example 3 Given that

$$z = e^{xy}, \quad x = 2u + v, \quad y = u/v.$$

Find

$\partial z/\partial u$ and $\partial z/\partial v$ using the chain rule.

Solution.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (ye^{xy})(2) + (xe^{xy})\left(\frac{1}{v}\right) = \left[2y + \frac{x}{v}\right]e^{xy}$$

$$= \left[\frac{2u}{v} + \frac{2u+v}{v}\right]e^{(2u+v)(u/v)} = \left[\frac{4u}{v} + 1\right]e^{(2u+v)(u/v)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (ye^{xy})(1) + (xe^{xy})\left(-\frac{u}{v^2}\right)$$
$$= \left[y - x\left(\frac{u}{v^2}\right)\right]e^{xy} = \left[\frac{u}{v} - (2u+v)\left(\frac{u}{v^2}\right)\right]e^{(2u+v)(u/v)}$$
$$= -\frac{2u^2}{v^2}e^{(2u+v)(u/v)}$$

Example 4

Suppose that

$$w = e^{xyz}, \quad x = 3u + v, \quad y = 3u - v, \quad z = u^2 v$$

Use appropriate forms of the chain rule to find $\partial w/\partial u$ and $\partial w/\partial v$.

Solution.

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

$$\frac{\partial w}{\partial u} = yze^{xyz}(3) + xze^{xyz}(3) + xye^{xyz}(2uv) = e^{xyz}(3yz + 3xz + 2xyuv)$$

$$\frac{\partial w}{\partial v} = yze^{xyz}(1) + xze^{xyz}(-1) + xye^{xyz}(u^2) = e^{xyz}(yz - xz + xyu^2)$$

IMPLICIT DIFFERENTIATION

Consider the special case where $z = f(x, y)$ is a function of x and y and y is a differentiable function of x. Equation (5) then becomes

$$\frac{dz}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \tag{12}$$

This result can be used to find derivatives of functions that are defined implicitly. For example, suppose that the equation

$$f(x, y) = c \tag{13}$$

defines y implicitly as a differentiable function of x and we are interested in finding dy/dx. Differentiating both sides of (13) with respect to x and applying (12) yields

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

Thus, if $\partial f/\partial y \neq 0$, we obtain

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} \tag{14}$$

Example 7

Given that

$$x^3 + y^2x - 3 = 0.$$

Find dy/dx using (14), and check the result using implicit differentiation.

Solution.

By (14) with $f(x, y) = x^3 + y^2x - 3$,

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{3x^2 + y^2}{2yx}$$

Alternatively, differentiating implicitly yields

$$3x^2 + y^2 + x\left(2y\frac{dy}{dx}\right) - 0 = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{3x^2 + y^2}{2yx}$$

The chain rule also applies to implicit partial differentiation. Consider the case where $w = f(x, y, z)$ is a function of x, y, and z and z is a differentiable function of x and y. It follows from Theorem 13.5.2 that

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} \tag{15}$$

If the equation

$$f(x, y, z) = c \tag{16}$$

defines z implicitly as a differentiable function of x and y, then taking the partial derivative of each side of (16) with respect to x and applying (15) gives

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0$$

If $\partial f/\partial z \neq 0$, then

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z}$$

A similar result holds for $\partial z/\partial y$.

13.5.4 THEOREM *If the equation $f(x, y, z) = c$ defines z implicitly as a differentiable function of x and y, and if $\partial f/\partial z \neq 0$, then*

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}$$

Example 8

Consider the sphere $x^2 + y^2 + z^2 = 1$. Find $\partial z/\partial x$ and $\partial z/\partial y$ at the point (2/3, 1/3, 2/3)

Solution. By Theorem 13.5.4 with $f(x, y, z) = x^2 + y^2 + z^2$,

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} = -\frac{2x}{2z} = -\frac{x}{z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z} = -\frac{2y}{2z} = -\frac{y}{z}$$

At the point $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$, evaluating these derivatives gives $\partial z/\partial x = -1$ and $\partial z/\partial y = -\frac{1}{2}$.

EXERCISE SET 13.5

1–5 Use an appropriate form of the chain rule to find dz/dt.

1. $z = 3x^2y^3;\ x = t^4,\ y = t^2$

2. $z = \ln(2x^2 + y);\ x = \sqrt{t},\ y = t^{2/3}$

3. $z = 3\cos x - \sin xy;\ x = 1/t,\ y = 3t$

4. $z = \sqrt{1 + x - 2xy^4};\ x = \ln t,\ y = t$

5. $z = e^{1-xy};\ x = t^{1/3},\ y = t^3$

7–10 Use an appropriate form of the chain rule to find dw/dt.

7. $w = 5x^2y^3z^4;\ x = t^2,\ y = t^3,\ z = t^5$

8. $w = \ln(3x^2 - 2y + 4z^3);\ x = t^{1/2},\ y = t^{2/3},\ z = t^{-2}$

9. $w = 5\cos xy - \sin xz;\ x = 1/t,\ y = t,\ z = t^3$

10. $w = \sqrt{1 + x - 2yz^4x};\ x = \ln t,\ y = t,\ z = 4t$

13. Suppose that $z = f(x, y)$ is differentiable at the point $(4, 8)$ with $f_x(4, 8) = 3$ and $f_y(4, 8) = -1$. If $x = t^2$ and $y = t^3$, find dz/dt when $t = 2$.

14. Suppose that $w = f(x, y, z)$ is differentiable at the point $(1, 0, 2)$ with $f_x(1, 0, 2) = 1$, $f_y(1, 0, 2) = 2$, and $f_z(1, 0, 2) = 3$. If $x = t$, $y = \sin(\pi t)$, and $z = t^2 + 1$, find dw/dt when $t = 1$.

17–22 Use appropriate forms of the chain rule to find $\partial z/\partial u$ and $\partial z/\partial v$.

17. $z = 8x^2y - 2x + 3y;\ x = uv,\ y = u - v$

18. $z = x^2 - y\tan x;\ x = u/v,\ y = u^2v^2$

19. $z = x/y;\ x = 2\cos u,\ y = 3\sin v$

20. $z = 3x - 2y;\ x = u + v\ln u,\ y = u^2 - v\ln v$

21. $z = e^{x^2y};\ x = \sqrt{uv},\ y = 1/v$

22. $z = \cos x \sin y;\ x = u - v,\ y = u^2 + v^2$

23. Let $T = x^2y - xy^3 + 2$; $x = r\cos\theta$, $y = r\sin\theta$. Find $\partial T/\partial r$ and $\partial T/\partial\theta$.

24. Let $R = e^{2s-t^2}$; $s = 3\phi$, $t = \phi^{1/2}$. Find $dR/d\phi$.

25. Let $t = u/v$; $u = x^2 - y^2$, $v = 4xy^3$. Find $\partial t/\partial x$ and $\partial t/\partial y$.

26. Let $w = rs/(r^2 + s^2)$; $r = uv$, $s = u - 2v$. Find $\partial w/\partial u$ and $\partial w/\partial v$.

27. Let $z = \ln(x^2 + 1)$, where $x = r\cos\theta$. Find $\partial z/\partial r$ and $\partial z/\partial\theta$.

28. Let $u = rs^2 \ln t$, $r = x^2$, $s = 4y + 1$, $t = xy^3$. Find $\partial u/\partial x$ and $\partial u/\partial y$.

29. Let $w = 4x^2 + 4y^2 + z^2$, $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$. Find $\partial w/\partial\rho$, $\partial w/\partial\phi$, and $\partial w/\partial\theta$.

30. Let $w = 3xy^2z^3$, $y = 3x^2 + 2$, $z = \sqrt{x - 1}$. Find dw/dx.

35. Let a and b denote two sides of a triangle and let θ denote the included angle. Suppose that a, b, and θ vary with time in such a way that the area of the triangle remains constant. At a certain instant $a = 5$ cm, $b = 4$ cm, and $\theta = \pi/6$ radians, and at that instant both a and b are increasing at a rate of 3 cm/s. Estimate the rate at which θ is changing at that instant.

36. The voltage, V (in volts), across a circuit is given by Ohm's law: $V = IR$, where I is the current (in amperes) flowing through the circuit and R is the resistance (in ohms). If two circuits with resistances R_1 and R_2 are connected in parallel, then their combined resistance, R, is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

Suppose that the current is 3 amperes and is increasing at 10^{-2} ampere/s, R_1 is 2 ohms and is increasing at 0.4 ohm/s, and R_2 is 5 ohms and is decreasing at 0.7 ohm/s. Estimate the rate at which the voltage is changing.

41–44 Use Theorem 13.5.3 to find dy/dx and check your result using implicit differentiation.

41. $x^2y^3 + \cos y = 0$

42. $x^3 - 3xy^2 + y^3 = 5$

43. $e^{xy} + ye^y = 1$

44. $x - \sqrt{xy} + 3y = 4$

45–48 Find $\partial z/\partial x$ and $\partial z/\partial y$ by implicit differentiation, and confirm that the results obtained agree with those predicted by the formulas in Theorem 13.5.4.

45. $x^2 - 3yz^2 + xyz - 2 = 0$

46. $\ln(1 + z) + xy^2 + z = 1$

47. $ye^x - 5\sin 3z = 3z$

48. $e^{xy}\cos yz - e^{yz}\sin xz + 2 = 0$

50. (a) Let $z = f(x^2 - y^2)$. Use the result in Exercise 49(a) to show that

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$$

(b) Let $z = f(xy)$. Use the result in Exercise 49(a) to show that

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 0$$

(c) Confirm the result in part (a) in the case where $z = \sin(x^2 - y^2)$.

(d) Confirm the result in part (b) in the case where $z = e^{xy}$.

51. Let f be a differentiable function of one variable, and let $z = f(x + 2y)$. Show that

$$2\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$$

52. Let f be a differentiable function of one variable, and let $z = f(x^2 + y^2)$. Show that

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$$

57. Suppose that the equation $z = f(x, y)$ is expressed in the polar form $z = g(r, \theta)$ by making the substitution $x = r\cos\theta$ and $y = r\sin\theta$.

(a) View r and θ as functions of x and y and use implicit differentiation to show that

$$\frac{\partial r}{\partial x} = \cos\theta \quad \text{and} \quad \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}$$

(b) View r and θ as functions of x and y and use implicit differentiation to show that

$$\frac{\partial r}{\partial y} = \sin\theta \quad \text{and} \quad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

(c) Use the results in parts (a) and (b) to show that

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial z}{\partial \theta}\sin\theta$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial z}{\partial \theta}\cos\theta$$

(d) Use the result in part (c) to show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

(e) Use the result in part (c) to show that if $z = f(x, y)$ satisfies Laplace's equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

then $z = g(r, \theta)$ satisfies the equation

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r} = 0$$

and conversely. The latter equation is called the ***polar form of Laplace's equation***.

13.6 DIRECTIONAL DERIVATIVES AND GRADIENTS

The partial derivatives $f_x(x, y)$ and $f_y(x, y)$ represent the rates of change of $f(x, y)$ in directions parallel to the x- and y-axes. In this section we will investigate rates of change of $f(x, y)$ in other directions.

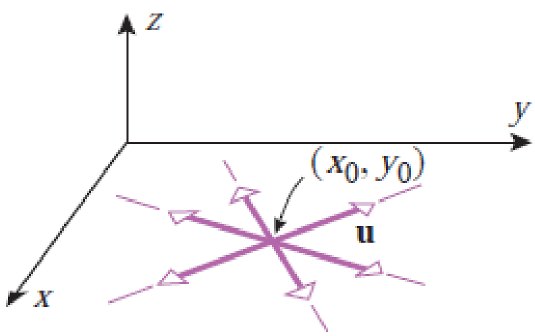


Figure 13.6.1

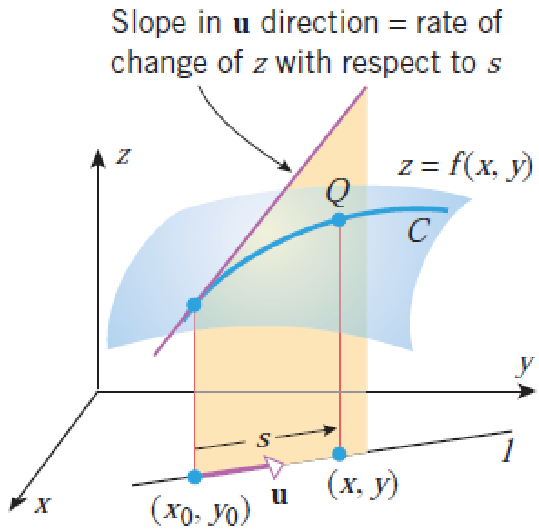


Figure 13.6.2

13.6.1 DEFINITION If $f(x, y)$ is a function of x and y, and if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector, then the ***directional derivative of f in the direction of*** **u** at (x_0, y_0) is denoted by $D_\mathbf{u} f(x_0, y_0)$ and is defined by

$$D_\mathbf{u} f(x_0, y_0) = \frac{d}{ds}\left[f(x_0 + su_1, y_0 + su_2)\right]_{s=0} \tag{2}$$

provided this derivative exists.

Geometrically, $D_u f(x_0, y_0)$ can be interpreted as the ***slope of the surface z = f (x, y) in the direction of*** **u** at the point $(x_0, y_0, f(x_0, y_0))$ (Figure 13.6.2). Usually the value of $D_u f(x_0, y_0)$ will depend on both the point (x_0, y_0) and the direction **u**. Thus, at a fixed point the slope of the surface may vary with the direction (Figure 13.6.3). Analytically, the directional derivative represents the ***instantaneous rate of change of f (x, y) with respect to distance in the direction of*** **u** at the point (x_0, y_0).

Example 1

Let $f(x, y) = xy$. Find and interpret $D_{\mathbf{u}} f(1, 2)$ for the unit vector

$$\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$$

Solution Using (2),

$$D_{\mathbf{u}} f(1, 2) = \frac{d}{ds}\left[f\left(1 + \frac{\sqrt{3}s}{2}, 2 + \frac{s}{2}\right)\right]_{s=0}$$

Since

$$f\left(1 + \frac{\sqrt{3}s}{2}, 2 + \frac{s}{2}\right) = \left(1 + \frac{\sqrt{3}s}{2}\right)\left(2 + \frac{s}{2}\right) = \frac{\sqrt{3}}{4}s^2 + \left(\frac{1}{2} + \sqrt{3}\right)s + 2$$

we have

$$\begin{aligned} D_{\mathbf{u}} f(1, 2) &= \frac{d}{ds}\left[\frac{\sqrt{3}}{4}s^2 + \left(\frac{1}{2} + \sqrt{3}\right)s + 2\right]_{s=0} \\ &= \left[\frac{\sqrt{3}}{2}s + \frac{1}{2} + \sqrt{3}\right]_{s=0} = \frac{1}{2} + \sqrt{3} \end{aligned}$$

Since $\frac{1}{2} + \sqrt{3} \approx 2.23$, we conclude that if we move a small distance from the point (1, 2) in the direction of $\mathbf{u}$, the function $f(x, y) = xy$ will increase by about 2.23 times the distance moved.

13.6.2 DEFINITION If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ is a unit vector, and if $f(x, y, z)$ is a function of x, y, and z, then the ***directional derivative of f in the direction of*** $\mathbf{u}$ at (x_0, y_0, z_0) is denoted by $D_{\mathbf{u}} f(x_0, y_0, z_0)$ and is defined by

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \frac{d}{ds}\left[f(x_0 + su_1, y_0 + su_2, z_0 + su_3)\right]_{s=0} \tag{3}$$

provided this derivative exists.

13.6.3 THEOREM

(a) *If $f(x, y)$ is differentiable at (x_0, y_0), and if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector, then the directional derivative $D_\mathbf{u} f(x_0, y_0)$ exists and is given by*

$$D_\mathbf{u} f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2 \tag{4}$$

(b) *If $f(x, y, z)$ is differentiable at (x_0, y_0, z_0), and if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ is a unit vector, then the directional derivative $D_\mathbf{u} f(x_0, y_0, z_0)$ exists and is given by*

$$D_\mathbf{u} f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0)u_1 + f_y(x_0, y_0, z_0)u_2 + f_z(x_0, y_0, z_0)u_3 \tag{5}$$

Example 2

Find the directional derivative of $f(x, y) = e^{xy}$ at $(-2, 0)$ in the direction of the unit vector that makes an angle of $\pi/3$ with the positive x-axis.

Solution. The partial derivatives of f are

$$f_x(x, y) = ye^{xy}, \quad f_y(x, y) = xe^{xy}$$

$$f_x(-2, 0) = 0, \quad f_y(-2, 0) = -2$$

The unit vector $\mathbf{u}$ that makes an angle of $\pi/3$ with the positive x-axis is

$$\mathbf{u} = \cos(\pi/3)\mathbf{i} + \sin(\pi/3)\mathbf{j} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

Thus, from (7)

$$\begin{aligned} D_\mathbf{u} f(-2, 0) &= f_x(-2, 0)\cos(\pi/3) + f_y(-2, 0)\sin(\pi/3) \\ &= 0(1/2) + (-2)(\sqrt{3}/2) = -\sqrt{3} \end{aligned}$$

Example 3

Find the directional derivative of $f(x, y, z) = x^2y - yz^3 + z$ at the point $(1, -2, 0)$ in the direction of the vector $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

Solution.

$$f_x(x, y, z) = 2xy, \quad f_y(x, y, z) = x^2 - z^3, \quad f_z(x, y, z) = -3yz^2 + 1$$
$$f_x(1, -2, 0) = -4, \quad f_y(1, -2, 0) = 1, \quad f_z(1, -2, 0) = 1$$

Since $\mathbf{a}$ is not a unit vector, we normalize it, getting

$$\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{1}{\sqrt{9}}(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

Formula (5) then yields

$$D_{\mathbf{u}} f(1, -2, 0) = (-4)\left(\frac{2}{3}\right) + \frac{1}{3} - \frac{2}{3} = -3 \blacktriangleleft$$

THE GRADIENT

Formula (4) can be expressed in the form of a dot product as

$$\begin{aligned} D_{\mathbf{u}} f(x_0, y_0) &= (f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}) \cdot (u_1\mathbf{i} + u_2\mathbf{j}) \\ &= (f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}) \cdot \mathbf{u} \end{aligned}$$

Similarly, Formula (5) can be expressed as

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = (f_x(x_0, y_0, z_0)\mathbf{i} + f_y(x_0, y_0, z_0)\mathbf{j} + f_z(x_0, y_0, z_0)\mathbf{k}) \cdot \mathbf{u}$$

In both cases the directional derivative is obtained by dotting the direction vector $\mathbf{u}$ with a new vector constructed from the first-order partial derivatives of f.

13.6.4 DEFINITION

(a) If f is a function of x and y, then the ***gradient of f*** is defined by

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} \tag{8}$$

(b) If f is a function of x, y, and z, then the ***gradient of f*** is defined by

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k} \tag{9}$$

The symbol ∇ (read "del") is an inverted delta. (It is sometimes called a "nabla" because of its similarity in form to an ancient Hebrew ten-stringed harp of that name.)

Formulas (4) and (5) can now be written as

$$D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u} \tag{10}$$

and

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \mathbf{u} \tag{11}$$

respectively. For example, using Formula (11) our solution to Example 3 would take the form

$$\begin{aligned} D_{\mathbf{u}} f(1, -2, 0) = \nabla f(1, -2, 0) \cdot \mathbf{u} &= (-4\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot \left(\tfrac{2}{3}\mathbf{i} + \tfrac{1}{3}\mathbf{j} - \tfrac{2}{3}\mathbf{k}\right) \\ &= (-4)\left(\tfrac{2}{3}\right) + \tfrac{1}{3} - \tfrac{2}{3} = -3 \end{aligned}$$

Formula (10) can be interpreted to mean that the slope of the surface $z = f(x, y)$ at the point (x_0, y_0) in the direction of $\mathbf{u}$ is the dot product of the gradient with $\mathbf{u}$ (Figure 13.6.4).

PROPERTIES OF THE GRADIENT

The gradient is not merely a notational device to simplify the formula for the directional derivative; we will see that the length and direction of the gradient ∇f provide important information about the function f and the surface $z = f(x, y)$. For example, suppose that $\nabla f(x, y) \neq \mathbf{0}$, and let us use Formula (4) of Section 11.3 to rewrite (10) as

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u} = \|\nabla f(x, y)\| \|\mathbf{u}\| \cos\theta = \|\nabla f(x, y)\| \cos\theta \tag{12}$$

where θ is the angle between $\nabla f(x, y)$ and $\mathbf{u}$. Equation (12) tells us that the maximum value of $D_{\mathbf{u}} f$ at the point (x, y) is $\|\nabla f(x, y)\|$, and this maximum occurs when $\theta = 0$, that is, when $\mathbf{u}$ is in the direction of $\nabla f(x, y)$. Geometrically, this means:

At (x, y), the surface $z = f(x, y)$ has its maximum slope in the direction of the gradient, and the maximum slope is $\|\nabla f(x, y)\|$.

Similarly, (12) tells us that the minimum value of $D_{\mathbf{u}} f$ at the point (x, y) is $-\|\nabla f(x, y)\|$, and this minimum occurs when $\theta = \pi$, that is, when $\mathbf{u}$ is oppositely directed to $\nabla f(x, y)$. Geometrically, this means:

At (x, y), the surface $z = f(x, y)$ has its minimum slope in the direction that is opposite to the gradient, and the minimum slope is $-\|\nabla f(x, y)\|$.

13.6.5 THEOREM *Let f be a function of either two variables or three variables, and let P denote the point $P(x_0, y_0)$ or $P(x_0, y_0, z_0)$, respectively. Assume that f is differentiable at P.*

(a) *If $\nabla f = \mathbf{0}$ at P, then all directional derivatives of f at P are zero.*

(b) *If $\nabla f \neq \mathbf{0}$ at P, then among all possible directional derivatives of f at P, the derivative in the direction of ∇f at P has the largest value. The value of this largest directional derivative is $\|\nabla f\|$ at P.*

(c) *If $\nabla f \neq \mathbf{0}$ at P, then among all possible directional derivatives of f at P, the derivative in the direction opposite to that of ∇f at P has the smallest value. The value of this smallest directional derivative is $-\|\nabla f\|$ at P.*

Example 4

Let $(x, y) = x^2 e^y$. Find the maximum value of a directional derivative at $(-2, 0)$, and find the unit vector in the direction in which the maximum value occurs.

Solution.

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = 2xe^y\mathbf{i} + x^2e^y\mathbf{j}$$

the gradient of f at $(-2, 0)$ is

$$\nabla f(-2, 0) = -4\mathbf{i} + 4\mathbf{j}$$

By Theorem 13.6.5, the maximum value of the directional derivative is

$$\|\nabla f(-2, 0)\| = \sqrt{(-4)^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$$

This maximum occurs in the direction of $\nabla f(-2, 0)$. The unit vector in this direction is

$$\mathbf{u} = \frac{\nabla f(-2, 0)}{\|\nabla f(-2, 0)\|} = \frac{1}{4\sqrt{2}}(-4\mathbf{i} + 4\mathbf{j}) = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \blacktriangleleft$$

1–8 Find $D_{\mathbf{u}} f$ at P.

1. $f(x, y) = (1 + xy)^{3/2}$; $P(3, 1)$; $\mathbf{u} = \dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j}$

2. $f(x, y) = \sin(5x - 3y)$; $P(3, 5)$; $\mathbf{u} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$

3. $f(x, y) = \ln(1 + x^2 + y)$; $P(0, 0)$;
$\mathbf{u} = -\dfrac{1}{\sqrt{10}}\mathbf{i} - \dfrac{3}{\sqrt{10}}\mathbf{j}$

4. $f(x, y) = \dfrac{cx + dy}{x - y}$; $P(3, 4)$; $\mathbf{u} = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$

5. $f(x, y, z) = 4x^5y^2z^3$; $P(2, -1, 1)$; $\mathbf{u} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$

6. $f(x, y, z) = ye^{xz} + z^2$; $P(0, 2, 3)$; $\mathbf{u} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$

7. $f(x, y, z) = \ln(x^2 + 2y^2 + 3z^2)$; $P(-1, 2, 4)$;
$\mathbf{u} = -\frac{3}{13}\mathbf{i} - \frac{4}{13}\mathbf{j} - \frac{12}{13}\mathbf{k}$

8. $f(x, y, z) = \sin xyz$; $P\left(\frac{1}{2}, \frac{1}{3}, \pi\right)$;
$\mathbf{u} = \dfrac{1}{\sqrt{3}}\mathbf{i} - \dfrac{1}{\sqrt{3}}\mathbf{j} + \dfrac{1}{\sqrt{3}}\mathbf{k}$

9–18 Find the directional derivative of f at P in the direction of $\mathbf{a}$. ■

9. $f(x, y) = 4x^3y^2$; $P(2, 1)$; $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j}$

10. $f(x, y) = 9x^3 - 2y^3$; $P(1, 0)$; $\mathbf{a} = \mathbf{i} - \mathbf{j}$

11. $f(x, y) = y^2 \ln x$; $P(1, 4)$; $\mathbf{a} = -3\mathbf{i} + 3\mathbf{j}$

12. $f(x, y) = e^x \cos y$; $P(0, \pi/4)$; $\mathbf{a} = 5\mathbf{i} - 2\mathbf{j}$

13. $f(x, y) = \tan^{-1}(y/x)$; $P(-2, 2)$; $\mathbf{a} = -\mathbf{i} - \mathbf{j}$

14. $f(x, y) = xe^y - ye^x$; $P(0, 0)$; $\mathbf{a} = 5\mathbf{i} - 2\mathbf{j}$

15. $f(x, y, z) = xy + z^2$; $P(-3, 0, 4)$; $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

16. $f(x, y, z) = y - \sqrt{x^2 + z^2}$; $P(-3, 1, 4)$; $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$

17. $f(x, y, z) = \dfrac{z - x}{z + y}$; $P(1, 0, -3)$; $\mathbf{a} = -6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

18. $f(x, y, z) = e^{x+y+3z}$; $P(-2, 2, -1)$; $\mathbf{a} = 20\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$

19–22 Find the directional derivative of f at P in the direction of a vector making the counterclockwise angle θ with the positive x-axis. ■

19. $f(x, y) = \sqrt{xy}$; $P(1, 4)$; $\theta = \pi/3$

20. $f(x, y) = \dfrac{x - y}{x + y}$; $P(-1, -2)$; $\theta = \pi/2$

21. $f(x, y) = \tan(2x + y)$; $P(\pi/6, \pi/3)$; $\theta = 7\pi/4$

22. $f(x, y) = \sinh x \cosh y$; $P(0, 0)$; $\theta = \pi$

23. Find the directional derivative of

$$f(x, y) = \frac{x}{x + y}$$

at $P(1, 0)$ in the direction of $Q(-1, -1)$.

24. Find the directional derivative of $f(x, y) = e^{-x} \sec y$ at $P(0, \pi/4)$ in the direction of the origin.

25. Find the directional derivative of $f(x, y) = \sqrt{xy}e^{y}$ at $P(1, 1)$ in the direction of the negative y-axis.

26. Let

$$f(x, y) = \frac{y}{x + y}$$

Find a unit vector $\mathbf{u}$ for which $D_{\mathbf{u}} f(2, 3) = 0$.

27. Find the directional derivative of

$$f(x, y, z) = \frac{y}{x + z}$$

at $P(2, 1, -1)$ in the direction from P to $Q(-1, 2, 0)$.

28. Find the directional derivative of the function

$$f(x, y, z) = x^3 y^2 z^5 - 2xz + yz + 3x$$

at $P(-1, -2, 1)$ in the direction of the negative z-axis.

53–60 Find a unit vector in the direction in which f increases most rapidly at P, and find the rate of change of f at P in that direction. ■

53. $f(x, y) = 4x^3y^2$; $P(-1, 1)$

54. $f(x, y) = 3x - \ln y$; $P(2, 4)$

55. $f(x, y) = \sqrt{x^2 + y^2}$; $P(4, -3)$

56. $f(x, y) = \dfrac{x}{x + y}$; $P(0, 2)$

57. $f(x, y, z) = x^3z^2 + y^3z + z - 1$; $P(1, 1, -1)$

58. $f(x, y, z) = \sqrt{x - 3y + 4z}$; $P(0, -3, 0)$

59. $f(x, y, z) = \dfrac{x}{z} + \dfrac{z}{y^2}$; $P(1, 2, -2)$

60. $f(x, y, z) = \tan^{-1}\left(\dfrac{x}{y + z}\right)$; $P(4, 2, 2)$

61–66 Find a unit vector in the direction in which f decreases most rapidly at P, and find the rate of change of f at P in that direction. ■

61. $f(x, y) = 20 - x^2 - y^2$; $P(-1, -3)$

62. $f(x, y) = e^{xy}$; $P(2, 3)$

63. $f(x, y) = \cos(3x - y)$; $P(\pi/6, \pi/4)$

64. $f(x, y) = \sqrt{\dfrac{x - y}{x + y}}$; $P(3, 1)$

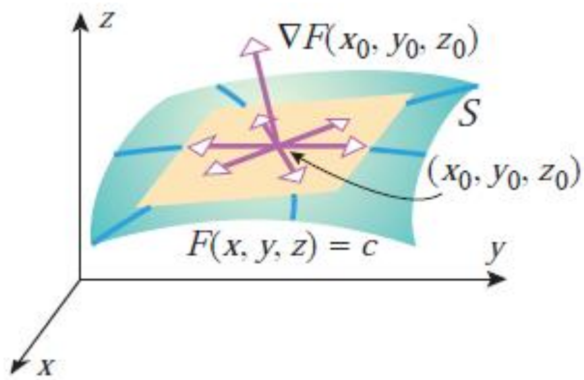
65. $f(x, y, z) = \dfrac{x + z}{z - y}$; $P(5, 7, 6)$

66. $f(x, y, z) = 4e^{xy} \cos z$; $P(0, 1, \pi/4)$

13.7 TANGENT PLANES AND NORMAL VECTORS

13.7.1 DEFINITION Assume that $F(x, y, z)$ has continuous first-order partial derivatives and that $P_0(x_0, y_0, z_0)$ is a point on the level surface $S: F(x, y, z) = c$. If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, then $\mathbf{n} = \nabla F(x_0, y_0, z_0)$ is a ***normal vector*** to S at P_0 and the ***tangent plane*** to S at P_0 is the plane with equation

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 \quad (3)$$



The line through the point P_0 parallel to the normal vector $\mathbf{n}$ is perpendicular to the tangent plane (3). We will call this the ***normal line***, or sometimes more simply the ***normal*** to the surface $F(x, y, z) = c$ at P_0. It follows that this line can be expressed parametrically as

$$x = x_0 + F_x(x_0, y_0, z_0)t, \quad y = y_0 + F_y(x_0, y_0, z_0)t, \quad z = z_0 + F_z(x_0, y_0, z_0)t \quad (4)$$

Example 1 Consider the ellipsoid $x^2 + 4y^2 + z^2 = 18$.

(a) Find an equation of the tangent plane to the ellipsoid at the point $(1, 2, 1)$.
(b) Find parametric equations of the line that is normal to the ellipsoid at the point $(1, 2, 1)$.
(c) Find the acute angle that the tangent plane at the point $(1, 2, 1)$ makes with the xy-plane.

Solution (a). We apply Definition 13.7.1 with $F(x, y, z) = x^2 + 4y^2 + z^2$ and $(x_0, y_0, z_0) = (1, 2, 1)$. Since

$$\nabla F(x, y, z) = \langle F_x(x, y, z), F_y(x, y, z), F_z(x, y, z)\rangle = \langle 2x, 8y, 2z\rangle$$

we have

$$\mathbf{n} = \nabla F(1, 2, 1) = \langle 2, 16, 2\rangle$$

Hence, from (3) the equation of the tangent plane is

$$2(x-1) + 16(y-2) + 2(z-1) = 0 \quad \text{or} \quad x + 8y + z = 18$$

Solution (b). Since **n** = $< 2, 16, 2 >$ at the point $(1, 2, 1)$, it follows from (4) that parametric equations for the normal line to the ellipsoid at the point (1, 2, 1) are $x = 1 + 2t$, $y = 2 + 16t$, $z = 1 + 2t$

Solution (c). To find the acute angle θ between the tangent plane and the *xy*-plane, we
will apply Formula (9) of Section 11.6 with $\mathbf{n}_1 = \mathbf{n} = \langle 2, 16, 2\rangle$ and $\mathbf{n}_2 = \langle 0, 0, 1\rangle$. This yields

$$\cos\theta = \frac{|\langle 2, 16, 2\rangle \cdot \langle 0, 0, 1\rangle|}{\|\langle 2, 16, 2\rangle\| \, \|\langle 0, 0, 1\rangle\|} = \frac{2}{(2\sqrt{66})(1)} = \frac{1}{\sqrt{66}}$$

Thus,

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{66}}\right) \approx 83^\circ$$

TANGENT PLANES TO SURFACES OF THE FORM $z = f(x, y)$

To find a tangent plane to a surface of the form $z = f(x, y)$, we can use Equation (3) with the function $F(x, y, z) = z - f(x, y)$.

Example 2 Find an equation for the tangent plane and parametric equations for the normal line to the surface $z = x^2 y$ at the point (2, 1, 4).

Solution. Let $F(x, y, z) = z - x^2 y$. Then $F(x, y, z) = 0$ on the surface, so we can find the find the gradient of F at the point (2, 1, 4):

$$\nabla F(x, y, z) = -2xy\mathbf{i} - x^2\mathbf{j} + \mathbf{k}$$
$$\nabla F(2, 1, 4) = -4\mathbf{i} - 4\mathbf{j} + \mathbf{k}$$

From (3) the tangent plane has equation

$$-4(x-2) - 4(y-1) + 1(z-4) = 0 \quad \text{or} \quad -4x - 4y + z = -8$$

and the normal line has equations

$$x = 2 - 4t, \qquad y = 1 - 4t, \qquad z = 4 + t \blacktriangleleft$$

13.7.2 THEOREM *If $f(x, y)$ is differentiable at the point (x_0, y_0), then the tangent plane to the surface $z = f(x, y)$ at the point $P_0(x_0, y_0, f(x_0, y_0))$ [or (x_0, y_0)] is the plane*

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{5}$$

EXERCISE SET 13.7

1. Consider the ellipsoid $x^2 + y^2 + 4z^2 = 12$.
   (a) Find an equation of the tangent plane to the ellipsoid at the point $(2, 2, 1)$.
   (b) Find parametric equations of the line that is normal to the ellipsoid at the point $(2, 2, 1)$.
   (c) Find the acute angle that the tangent plane at the point $(2, 2, 1)$ makes with the xy-plane.
2. Consider the surface $xz - yz^3 + yz^2 = 2$.
   (a) Find an equation of the tangent plane to the surface at the point $(2, -1, 1)$.
   (b) Find parametric equations of the line that is normal to the surface at the point $(2, -1, 1)$.
   (c) Find the acute angle that the tangent plane at the point $(2, -1, 1)$ makes with the xy-plane.

3–12 Find an equation for the tangent plane and parametric equations for the normal line to the surface at the point P.

3. $x^2 + y^2 + z^2 = 25$; $P(-3, 0, 4$

4. $x^2y - 4z^2 = -7$; $P(-3, 1, -2$

5. $x^2 - xyz = 56$; $P(-4, 5, 2)$

6. $z = x^2 + y^2$; $P(2, -3, 13)$

7. $z = 4x^3y^2 + 2y$; $P(1, -2, 12)$

8. $z = \frac{1}{2}x^7y^{-2}$; $P(2, 4, 4)$

9. $z = xe^{-y}$; $P(1, 0, 1)$

10. $z = \ln\sqrt{x^2 + y^2}$; $P(-1, 0, 0)$

11. $z = e^{3y}\sin 3x$; $P(\pi/6, 0, 1)$

12. $z = x^{1/2} + y^{1/2}$; $P(4, 9, 5)$

13.8 MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

EXTREMA

If we imagine the graph of a function f of two variables to be a mountain range (Figure 13.8.1), then the mountaintops, which are the high points in their immediate vicinity,

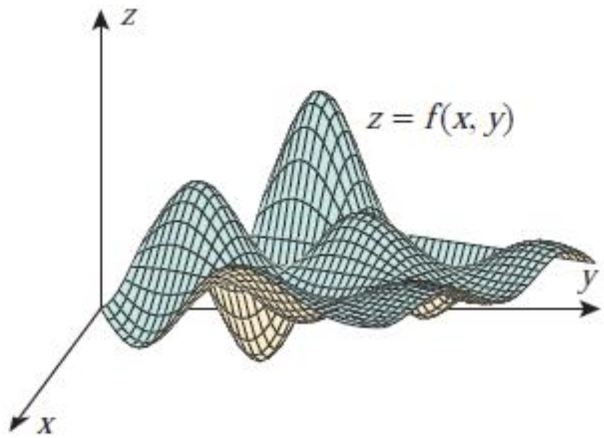


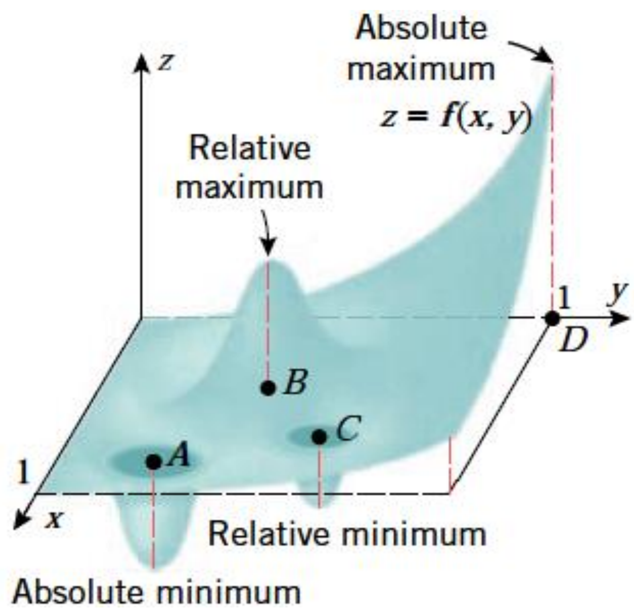
Figure 13.8.1

are called *relative maxima* of f , and the valley bottoms, which are the low points in their immediate vicinity, are called *relative minima* of f.

Just as a geologist might be interested in finding the highest mountain and deepest valley in an entire mountain range, so a mathematician might be interested in finding the largest and smallest values of $f(x, y)$ over the *entire* domain of f. These are called the *absolute maximum* and *absolute minimum values* of f. The following definitions make these informal ideas precise.

13.8.1 DEFINITION A function f of two variables is said to have a ***relative maximum*** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an ***absolute maximum*** at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) in the domain of f.

13.8.2 DEFINITION A function f of two variables is said to have a ***relative minimum*** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an ***absolute minimum*** at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) in the domain of f.



13.8.3 THEOREM (*Extreme-Value Theorem*) *If $f(x, y)$ is continuous on a closed and bounded set R, then f has both an absolute maximum and an absolute minimum on R.*

FINDING RELATIVE EXTREMA

Recall that if a function g of one variable has a relative extremum at a point x_0 where g is differentiable, then $g(x_0) = 0$. To obtain the analog of this result for functions of two variables, suppose that $f(x, y)$ has a relative maximum at a point (x_0, y_0) and that the partial derivatives of f exist at (x_0, y_0). It seems plausible geometrically that the traces of the surface $z = f(x, y)$ on the planes $x = x_0$ and $y = y_0$ have horizontal tangent lines at (x_0, y_0) (Figure 13.8.4), so $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.

The same conclusion holds if f has a relative minimum at (x_0, y_0), all of which suggests the following result, which we state without formal proof.

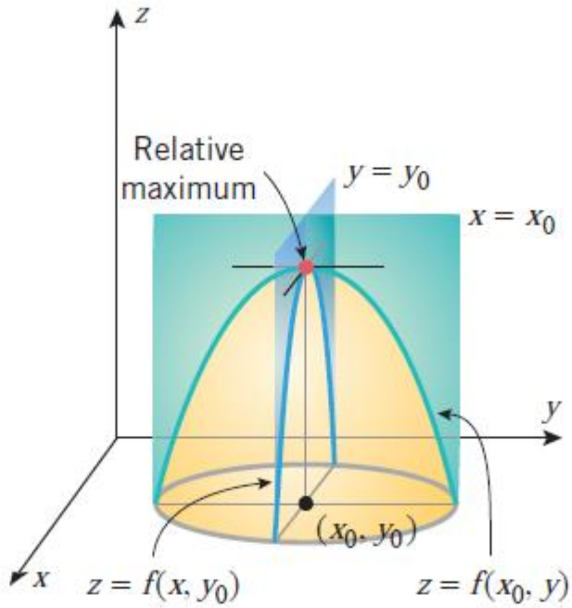


Figure 13.8.4

13.8.4 Theorem If f has a relative extremum at a point (x_0, y_0), and if the first order partial derivatives of f exist at this point, then

$$f_x(x_0, y_0) = 0 \text{ and } f_y(x_0, y_0) = 0.$$

13.8.5 Definition A point (x_0, y_0) in the domain of a function $f(x, y)$ is called a ***critical point*** of the function if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ or if one or both partial derivatives do not exist at (x_0, y_0).

13.8.6 THEOREM (*The Second Partials Test*) *Let f be a function of two variables with continuous second-order partial derivatives in some disk centered at a critical point (x_0, y_0), and let*

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$$

(*a*) *If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a relative minimum at (x_0, y_0).*

(*b*) *If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a relative maximum at (x_0, y_0).*

(*c*) *If $D < 0$, then f has a saddle point at (x_0, y_0).*

(*d*) *If $D = 0$, then no conclusion can be drawn.*

Example 3

Locate all relative extrema and saddle points of

$$f(x, y) = 3x^2 - 2xy + y^2 - 8y$$

Solution. Since $f_x(x, y) = 6x - 2y$ and $f_y(x, y) = -2x + 2y - 8$, the critical points of f satisfy the equations

$$6x - 2y = 0$$
$$-2x + 2y - 8 = 0$$

Solving these for x and y yields $x = 2$, $y = 6$ (verify), so $(2, 6)$ is the only critical point. To apply Theorem 13.8.6 we need the second-order partial derivatives

$$f_{xx}(x, y) = 6, \quad f_{yy}(x, y) = 2, \quad f_{xy}(x, y) = -2$$

At the point $(2, 6)$ we have

$$D = f_{xx}(2, 6) f_{yy}(2, 6) - f_{xy}^2(2, 6) = (6)(2) - (-2)^2 = 8 > 0$$

and

$$f_{xx}(2, 6) = 6 > 0$$

so f has a relative minimum at $(2, 6)$ by part (a) of the second partials test.

Example 4 Locate all relative extrema and saddle points of

$$f(x,y) = 4xy - x^4 - y^4$$

Solution. Since

$$\begin{aligned} f_x(x, y) &= 4y - 4x^3 \\ f_y(x, y) &= 4x - 4y^3 \end{aligned} \tag{1}$$

the critical points of f have coordinates satisfying the equations

$$\begin{aligned} 4y - 4x^3 &= 0 \\ 4x - 4y^3 &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} y &= x^3 \\ x &= y^3 \end{aligned} \tag{2}$$

Substituting the top equation in the bottom yields $x = (x^3)^3$ or, equivalently, $x^9 - x = 0$ or $x(x^8 - 1) = 0$, which has solutions $x = 0$, $x = 1$, $x = -1$. Substituting these values in the top equation of (2), we obtain the corresponding y-values $y = 0$, $y = 1$, $y = -1$. Thus,

$$f_x(x, y) = 4y - 4x^3$$
$$f_y(x, y) = 4x - 4y^3 \tag{1}$$

the critical points of f have coordinates satisfying the equations

$$\begin{aligned} 4y - 4x^3 &= 0 \\ 4x - 4y^3 &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} y &= x^3 \\ x &= y^3 \end{aligned} \tag{2}$$

Substituting the top equation in the bottom yields $x = (x^3)^3$ or, equivalently, $x^9 - x = 0$ or $x(x^8 - 1) = 0$, which has solutions $x = 0, x = 1, x = -1$. Substituting these values in the top equation of (2), we obtain the corresponding y-values $y = 0, y = 1, y = -1$. Thus, the critical points of f are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

From (1),

$$f_{xx}(x, y) = -12x^2, \quad f_{yy}(x, y) = -12y^2, \quad f_{xy}(x, y) = 4$$

which yields the following table:

CRITICAL POINT (x_0, y_0)	$f_{xx}(x_0, y_0)$	$f_{yy}(x_0, y_0)$	$f_{xy}(x_0, y_0)$	$D = f_{xx} f_{yy} - f_{xy}^2$
(0, 0)	0	0	4	–16
(1, 1)	–12	–12	4	128
(–1, –1)	–12	–12	4	128

At the points (1, 1) and $(-1, -1)$, we have $D > 0$ and $f_{xx} < 0$, so relative maxima occur at these critical points. At (0, 0) there is a saddle point since $D < 0$.

9–20 Locate all relative maxima, relative minima, and saddle points, if any.

9. $f(x, y) = y^2 + xy + 3y + 2x + 3$

10. $f(x, y) = x^2 + xy - 2y - 2x + 1$

11. $f(x, y) = x^2 + xy + y^2 - 3x$

12. $f(x, y) = xy - x^3 - y^2$

13. $f(x, y) = x^2 + y^2 + \dfrac{2}{xy}$

14. $f(x, y) = xe^y$

15. $f(x, y) = x^2 + y - e^y$

16. $f(x, y) = xy + \dfrac{2}{x} + \dfrac{4}{y}$

17. $f(x, y) = e^x \sin y$

18. $f(x, y) = y \sin x$

19. $f(x, y) = e^{-(x^2+y^2+2x)}$

20. $f(x, y) = xy + \dfrac{a^3}{x} + \dfrac{b^3}{y} \quad (a \neq 0, b \neq 0)$

42. An international airline has a regulation that each passenger can carry a suitcase having the sum of its width, length, and height less than or equal to 129 cm. Find the dimensions of the suitcase of maximum volume that a passenger can carry under this regulation.

43. A closed rectangular box with a volume of 16 ft^3 is made from two kinds of materials. The top and bottom are made of material costing 10¢ per square foot and the sides from material costing 5¢ per square foot. Find the dimensions of the box so that the cost of materials is minimized.

44. A manufacturer makes two models of an item, standard and deluxe. It costs \$40 to manufacture the standard model and \$60 for the deluxe. A market research firm estimates that if the standard model is priced at x dollars and the deluxe at y dollars, then the manufacturer will sell $500(y - x)$ of the standard items and $45{,}000 + 500(x - 2y)$ of the deluxe each year. How should the items be priced to maximize the profit?