Empirical Methods HW 2

Group 9: Linqi Huang, Abhesh Kumar, Yu Onohara, Maitrayee Patil, Redmond Xia

January 19, 2020

Problem 1

1.

The first order autocorrelation of the ARMA(1,1) model can be written as follow.

$$\rho_1 = \phi_1 - \theta_1 \frac{\sigma_\epsilon^2}{\gamma_0}$$

Hence, we will compute γ_0 first, and then also compute ρ_1.

```r
# Assumptions
phi1 = 0.95
theta1 = 0.9
sigma = 0.05

# Caluculating the covariance with lag 0 (= Variance of yt)
gamma0 = sigma^2 * (1 + theta1^2 - 2 * phi1 * theta1) / (1 - phi1^2)
gamma0
```

```
## [1] 0.002564103
```

```r
# Using gamma above, compute the first order autocorrelation
rho1 = phi1 - theta1 * (sigma^2 / gamma0)
rho1
```

```
## [1] 0.0725
```

2.

The jth ($j > 1$) order autocorrelation of the ARMA(1,1) model can be written as follow.

$$\rho_j = \phi_1 \rho_(j - 1)$$

Therefore, we will compute the second order autocorrelation of the ARMA model as follow.

```r
# Using the first order autocorrelation, compute the second order autocorrelation
rho2 = phi1 * rho1
rho2
```

```
## [1] 0.068875
```

```r
rho2 / rho1 # = phi1
```

```
## [1] 0.95
```

The ratio of the second-order to the first-order autocorrelation equals to ϕ_1. since this fact implies that the autocorrelation of the this model ($\phi_1 < 1$) converge to 0 ($\lim_{j\to\infty} \rho_j = 0$), this process is stationary.

3.

We will compute the conditional expectation of y_{t+1} and y_{t+2} as follow.

$$
\begin{aligned}
E[y_{t+1}|y_t] &= E[\phi_1 y_t - \theta_1 \epsilon_t + \epsilon_{t+1}] \\
&= \phi_1 y_t - \theta_1 \epsilon_t + 0 \\
&= \phi_1 y_t - \theta_1 \epsilon_t
\end{aligned}
$$

$$
\begin{aligned}
E[y_{t+2}|y_t] &= E[\phi_1 y_{t+1} - \theta_1 \epsilon_{t+1} + \epsilon_{t+2}] \\
&= \phi_1 E[y_{t+1}|Y_t] - \theta_1 E[\epsilon_{t+1}] + E[\epsilon_{t+2}] \\
&= \phi_1(\phi_1 y_t - \theta_1 \epsilon_t) \\
&= \phi_1^2 y_t - \phi_1 \theta_1 \epsilon_t
\end{aligned}
$$

```r
# Calculatingt the conditional expectations of yt+1 and yt+2
yt = 0.6
epsiront = 0.1
E_yt1 = phi1 * yt - theta1 * epsiront
E_yt2 = phi1^2 * yt - phi1 * theta1 * epsiront
E_yt1
```

```
## [1] 0.48
```

```r
E_yt2
```

```
## [1] 0.456
```

4. We will compute the unconditional $E[\hat{x}_t]$, $Var[\hat{x}_t]$ and ρ as follow.

$$
\begin{aligned}
E[\hat{x}_t] &= E[\phi_1 y_t - \theta_1 \epsilon_t] \\
&= \phi_1 E[y_t] - \theta_1 E[\epsilon_t] \\
&= \phi_1 \frac{\phi_0}{1-\phi_1} - \theta_1 * 0 \\
&= \phi_1 \frac{0}{1-\phi_1} \\
&= 0
\end{aligned}
$$

```r
var_yt = gamma0
var_yt1 = phi1^2*var_yt + theta1^2*sigma^2 - 2*phi1*theta1*sigma^2
sd_yt1 = sqrt(var_yt1)
sd_yt1
```

```
## [1] 0.008006408
```

$$
\begin{aligned}
Var[\hat{x_t}] &= Var[\phi_1 y_t - \theta_1 \epsilon_t] \\
&= \phi_1^2 Var[y_t] + \theta_1^2 Var[\epsilon_t] - 2\phi_1\theta_1 Cov[y_t, \epsilon_t] \\
&= \phi_1^2 \sigma^2 \frac{1 + \theta_1^2 - 2\phi_1\theta_1}{1 - \phi_1^2} + \theta_1^2\sigma^2 - 2\phi_1\theta_1\sigma^2 \\
&= 0.008006408
\end{aligned}
$$

```
gamma1 = phi1*gamma0 - theta1*sigma^2
rho_f = phi1^2*gamma1 - phi1*theta1*(phi1-theta1)*sigma^2
rho_f
```

```
## [1] 6.089744e-05
```

$$
\begin{aligned}
Cov[\hat{x_t}, \hat{x_{t-1}}] &= Cov[\phi_1 y_t - \theta_1 \epsilon_t, \phi_1 y_{t-1} - \theta_1 \epsilon_{t-1}] \\
&= \phi_1^2 Cov[y_t, y_{t-1}] - \phi_1\theta_1 Cov[y_t, \epsilon_{t-1}] - \phi_1\theta_1 Cov[y_{t-1}, \epsilon_t] + \theta_1^2 Cov[\epsilon_t, \epsilon_{t-1}] \\
&= \phi_1^2\gamma_1 - \phi_1\theta_1 Cov[\phi_1 y_{t-1} + \epsilon_t - \theta_1\epsilon_{t-1}, \epsilon_{t-1}] - 0 - 0 \\
&= \phi_1^2\gamma_1 - \phi_1\theta_1(\phi_1 Var[\epsilon_{t-1}] - \theta_1 Var[\epsilon_{t-1}]) \\
&= \phi_1^2\gamma_1 - \phi_1\theta_1(\phi_1 - \theta_1)\sigma^2 \\
&= 0.000061
\end{aligned}
$$

Problem 2

1.

From the assumption $\phi = 0$, we can rewrite the e_t as follow.

$$
e_t = e_{t-1} + \epsilon_t
$$

Then, using the recursive calculation of e_t, we can obtain e_{t-4} as follow.

$$
\begin{aligned}
e_{t-4} &= e_{t-3} - \epsilon_{t-3} \\
e_{t-3} &= e_{t-2} - \epsilon_{t-2} \\
e_{t-2} &= e_{t-1} - \epsilon_{t-1}
\end{aligned}
$$

Hence,

$$
e_{t-4} = e_{t-1} - \epsilon_{t-1} - \epsilon_{t-2} - \epsilon_{t-3}
$$

Therefore,

$$
\begin{aligned}
&y_t = e_t - e_{t-4} \\
&y_t = e_{t-1} + \epsilon t - (e_{t-1} - \epsilon_{t-1} - \epsilon_{t-2} - \epsilon_{t-3}) \\
&y_t = \epsilon_t + \epsilon_{t-1} + \epsilon_{t-2} + \epsilon_{t-3}
\end{aligned}
$$

To calculate the autocovariances of this process, we will use the following property of the covariance.

$$Cov(aX + bY, cW + dV) = acCov(X, W) + adCov(X, Z) + bcCov(Y, W) + bdCov(Y, V)$$

Since ϵ_t is i.i.d, $Cov(\epsilon_t, \epsilon_{t-j}) = 0$ if any $j > 0$. Therefore, we can calculate the autocovariance of order 0 through 5 as follow.

$$\begin{aligned}
Cov(y_t, y_t) &= Var(\epsilon_t) + Var(\epsilon_{t-1}) + Var(\epsilon_{t-2}) + Var(\epsilon_{t-3}) \\
&= 4\sigma^2 \\
Cov(y_t, y_{t-1}) &= Var(\epsilon_{t-1}) + Var(\epsilon_{t-2}) + Var(\epsilon_{t-3}) \\
&= 3\sigma^2 \\
Cov(y_t, y_{t-2}) &= Var(\epsilon_{t-2}) + Var(\epsilon_{t-3}) \\
&= 2\sigma^2 \\
Cov(y_t, y_{t-3}) &= Var(\epsilon_{t-3}) \\
&= \sigma^2 \\
Cov(y_t, y_{t-4}) &= 0 \\
Cov(y_t, y_{t-5}) &= 0
\end{aligned}$$

3.

This model has no AR structure because we have no y_{t-j} terms in the model. However, ϵ_t has 3 ϵ terms. Hence, this model is ARMA(0,4) model (MA(4) model). Also, each coefficient of ϵ terms equals to 1.

Problem 3

1.

$$\begin{aligned}
Var(R^e_{t+1}) = \beta^2 Var(x_t) + Var(\epsilon_{t+1}) = 1 * (0.05)^2 + 0.15^2 = 0.025 \\
SD(R^e_{t+1}) = \sqrt{0.025} = .15811
\end{aligned}$$

2.

$$R^2 = \rho^2 = (\frac{Cov(R^e_{t+1}, x_t)}{\sigma_{R^e_{t+1}}\sigma_{x_t}})^2 = (\frac{Cov(\beta x_t + \epsilon_{t-1}, x_t)}{\sigma_{R^e_{t+1}}\sigma_{x_t}})^2 = (\frac{Var(x_t)}{\sigma_{R^e_{t+1}}\sigma_{x_t}})^2 = \frac{Var(x_t)}{Var(R^e_{t+1})} = \frac{0.05^2}{0.025} = 0.1$$

3.

Sharpe Ratio $= \dfrac{E[R^e_{t+1}]}{\sigma_{mkt}} = \dfrac{0.05}{\sqrt{0.025}} = 0.31622$

4.

$$\begin{aligned}
\gamma &= \frac{40}{9} \\
\alpha_t &= \frac{E[x_t]}{\gamma\sigma^2_t[R^e_{t+1}]} = \frac{E[x_t]}{\gamma(\sigma^2_t[x_t] + \sigma^2_t[\epsilon_{t+1}])} \\
So,\ if\ x = 0,\ then\ \alpha_t &= \frac{0}{positive} = 0 \implies Sharpe\ Ratio = 0 \\
if\ x = .1, \alpha_t &= \frac{.1}{\frac{40}{9}(0.15^2)} = 1 \implies Sharpe\ Ratio = \frac{.1}{.15} = 0.67
\end{aligned}$$

5.

a) Below is the Expected value of the return

```
prob <- 0.5
xt_1 <- 0
xt_2 <- 0.1
alphat_1 <- 0
alphat_2 <- 1
sigma_e <- 0.15
E_alpR <- prob*alphat_1*xt_1 + prob*alphat_2*xt_2
E_alpR
```

```
## [1] 0.05
```

b) The output below is the unconditional standard deviation

```
var_alpR <- prob*alphat_2^2*(xt_2^2+sigma_e^2) - (prob*alphat_2*xt_2)^2
sqrt(var_alpR)
```

```
## [1] 0.1172604
```

c) Below is the Sharpe ratio

```
E_alpR / sqrt(var_alpR)
```

```
## [1] 0.4264014
```

d) i.Below is the implied R^2

```
xt_3 <- -0.05
xt_4 <- 0.15
E_xt <- prob*xt_3 + prob*xt_4
Var_xt <- prob*(xt_3-E_xt)^2 + prob*(xt_4-E_xt)^2
R_sq2 <- Var_xt / (Var_xt+sigma_e^2)
R_sq2
```

```
## [1] 0.3076923
```

ii. Below is the higher Sharpe ratio

```
gamma_t <- 40/9
alphat_3 <- xt_3 / (gamma_t*sigma_e^2)
alphat_4 <- xt_4 / (gamma_t*sigma_e^2)
E_alpR2 <- prob*alphat_3*xt_3 + prob*alphat_4*xt_4
var_alpR2 <- prob*alphat_3^2*(xt_3^2+sigma_e^2) + prob*alphat_4^2*(xt_4^2+sigma_e^2) - E_alpR2^2
E_alpR2 / sqrt(var_alpR2)
```

```
## [1] 0.6401844
```