Empirical Methods HW 2

Group 9: Linqi Huang, Abhesh Kumar, Yu Onohara, Maitrayee Patil, Redmond Xia

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Problem 1

1.

The first order autocorrelation of the ARMA(1,1) model can be written as follow.

$$\rho_1 = \phi_1 - \theta_1 \frac{\sigma_\epsilon^2}{\gamma_0}$$

Hence, we will compute γ_0 first, and then also compute ρ_1.

```
# Assumptions
phi1 = 0.95
theta1 = 0.9
sigma = 0.05

# Caluculating the covariance with lag 0 (= Variance of yt)
gamma0 = sigma^2 * (1 + theta1^2 - 2 * phi1 * theta1) / (1 - phi1^2)
gamma0
```

```
## [1] 0.002564103
```

```
# Using gamma above, compute the first order autocorrelation
rho1 = phi1 - theta1 * (sigma^2 / gamma0)
rho1
```

```
## [1] 0.0725
```

2.

The jth ($j > 1$) order autocorrelation of the ARMA(1,1) model can be written as follow.

$$\rho_j = \phi_1 \rho_(j-1)$$

Therefore, we will compute the second order autocorrelation of the ARMA model as follow.

```
# Using the first order autocorrelation, compute the second order autocorrelation
rho2 = phi1 * rho1
rho2
```

```
## [1] 0.068875
```

```
rho2 / rho1 # = phi1
```

```
## [1] 0.95
```

The ratio of the second-order to the first-order autocorrelation equals to ϕ_1. since this fact implies that the autocorrelation of the this model ($\phi_1 < 1$) converge to 0 ($\lim_{j\to\infty} \rho_j = 0$), this process is stationary.

3.

We will compute the conditional expectation of y_{t+1} and y_{t+2} as follow.

$$\begin{aligned}
E[y_{t+1}|y_t] &= E[\phi_1 y_t - \theta_1 \epsilon_t + \epsilon_{t+1}] \\
&= \phi_1 y_t - \theta_1 \epsilon_t + 0 \\
&= \phi_1 y_t - \theta_1 \epsilon_t
\end{aligned}$$

$$\begin{aligned}
E[y_{t+2}|y_t] &= E[\phi_1 y_{t+1} - \theta_1 \epsilon_{t+1} + \epsilon_{t+2}] \\
&= \phi_1 E[y_{t+1}|Y_t] - \theta_1 E[\epsilon_{t+1}] + E[\epsilon_{t+2}] \\
&= \phi_1(\phi_1 y_t - \theta_1 \epsilon_t) \\
&= \phi_1^2 y_t - \phi_1 \theta_1 \epsilon_t
\end{aligned}$$

```
# Calculatingt the conditional expectations of yt+1 and yt+2
yt = 0.6
epsiront = 0.1
E_yt1 = phi1 * yt - theta1 * epsiront
E_yt2 = phi1^2 * yt - phi1 * theta1 * epsiront
E_yt1
```

```
## [1] 0.48
```

```
E_yt2
```

```
## [1] 0.456
```

4.

Denote the unconditional mean of y_t be μ. From the results and implications of above questions, this process is stationary. Hence, the unconditional mean of y_{t+1} equals to the expectation of y_t.

$$E[\hat{x_t}] = \mu$$

Problem 2

1.

From the assumption $\phi = 0$, we can rewrite the e_t as follow.

$$e_t = e_{t-1} + \epsilon_t$$

Then, using the recursive calculation of e_t, we can obtain e_{t-4} as follow.

$$\begin{aligned}
e_{t-4} &= e_{t-3} - \epsilon_{t-3} \\
e_{t-3} &= e_{t-2} - \epsilon_{t-2} \\
e_{t-2} &= e_{t-1} - \epsilon_{t-1}
\end{aligned}$$

Hence,

$$e_{t-4} = e_{t-1} - \epsilon_{t-1} - \epsilon_{t-2} - \epsilon_{t-3}$$

Therefore,

$$\begin{aligned}
y_t &= e_t - e_{t-4} \\
y_t &= e_{t-1} + \epsilon t - (e_{t-1} - \epsilon_{t-1} - \epsilon_{t-2} - \epsilon_{t-3}) \\
y_t &= \epsilon_t + \epsilon_{t-1} + \epsilon_{t-2} + \epsilon_{t-3}
\end{aligned}$$

2.

To calculate the autocovariances of this process, we will use the following property of the covariance.

$$Cov(aX + bY, cW + dV) = acCov(X, W) + adCov(X, Z) + bcCov(Y, W) + bdCov(Y, V)$$

Since ϵ_t is i.i.d, $Cov(\epsilon_t, \epsilon_{t-j}) = 0$ if any $j > 0$. Therefore, we can calculate the autocovariance of order 0 through 5 as follow.

$$\begin{aligned}
Cov(y_t, y_t) &= Var(\epsilon_t) + Var(\epsilon_{t-1}) + Var(\epsilon_{t-2}) + Var(\epsilon_{t-3}) \\
&= 4\sigma^2 \\
Cov(y_t, y_{t-1}) &= Var(\epsilon_{t-1}) + Var(\epsilon_{t-2}) + Var(\epsilon_{t-3}) \\
&= 3\sigma^2 \\
Cov(y_t, y_{t-2}) &= Var(\epsilon_{t-2}) + Var(\epsilon_{t-3}) \\
&= 2\sigma^2 \\
Cov(y_t, y_{t-3}) &= Var(\epsilon_{t-3}) \\
&= \sigma^2 \\
Cov(y_t, y_{t-4}) &= 0 \\
Cov(y_t, y_{t-5}) &= 0
\end{aligned}$$

3.