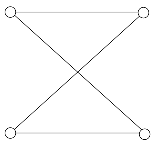
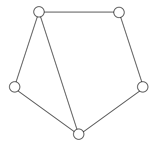
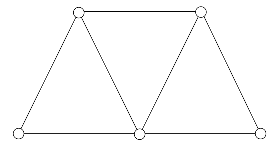
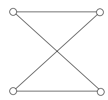
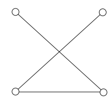
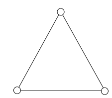
Math 316 – Solutions To Sample Final Exam Problems

1. Find the chromatic polynomials of the three graphs below. Clearly show your steps.

G_1  G_2  G_3 

Solution:

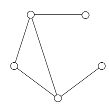
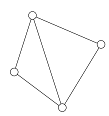
(a)

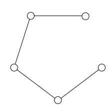
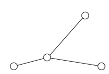
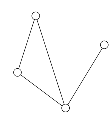
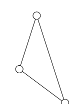
$p\Bigg($  $\Bigg) \quad = \quad p\Bigg($  $\Bigg) \quad - \quad p\Bigg($  $\Bigg)$

$$= \quad k(k-1)^3 \quad - \quad k(k-1)(k-2)$$

After simplifying, we see that $p_{G_1}(k) \;=\; k(k-1)\big((k-1)^2 \;-\; (k-2)\big)$.

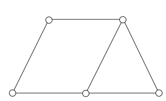
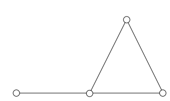
(b)

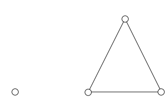
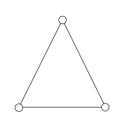
$p(G_2) \quad = \quad p\Bigg($  $\Bigg) \quad - \quad p\Bigg($  $\Bigg)$

$= \quad p\Bigg($  $\Bigg) \quad - \quad p\Bigg($  $\Bigg) \quad - \quad p\Bigg($  $\Bigg) \quad + \quad p\Bigg($  $\Bigg)$

$$\begin{aligned} &= \quad k(k-1)^4 \quad - \quad k(k-1)^3 \quad - \quad k(k-1)(k-2)(k-1) \quad + \quad k(k-1)(k-2) \\ &= \quad k(k-1)^4 \quad - \quad k(k-1)^3 \quad - \quad k(k-1)^2(k-2) \quad + \quad k(k-1)(k-2) \end{aligned}$$

(c) Note that removing one of the "interior edges" of G_3 produces a graph that is isomorphic to G_2 from the previous part of the problem. Therefore, we have

$p(G_3) \quad = \quad p\Bigg($  $\Bigg) \quad - \quad p\Bigg($  $\Bigg)$

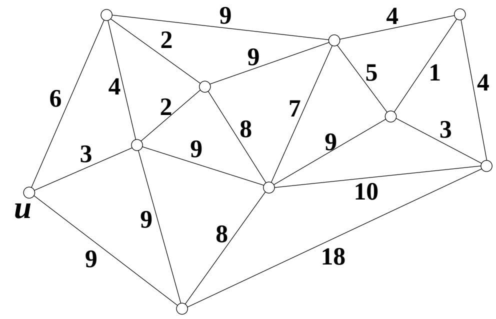
$= \quad p(G_2) \quad - \quad p\Bigg($  $\Bigg) \quad + \quad p\Bigg($  $\Bigg)$

$$= \quad p(G_2) \quad - \quad k \cdot k(k-1)(k-2) \quad + \quad k(k-1)(k-2),$$

which, when combined with our answer from part (b) and simplified a bit, yields

$$p_{G_3}(k) \;=\; k(k-1)^4 \,-\, k(k-1)^3 \,-\, k(k-1)^2(k-2) \,+\, 2k(k-1)(k-2) \,-\, k^2(k-1)(k-2).$$

2. Given to the right is a graph (not drawn to scale) representing a network of dirt roads connecting 10 towns. The edge labels represent distances between towns, in miles.



   (a) Which roads should be paved so that the distance from the town labeled "u" to any other town along paved roads is as small as possible? Also indicate the minimum distance that each town is away from u.

   (b) Which roads should be paved so that it is possible to drive between any two towns on only paved roads, and so that the highway department paves the minimum total distance? What is the minimum total distance that must be paved?

Solution:

Figure 1. Distance tree

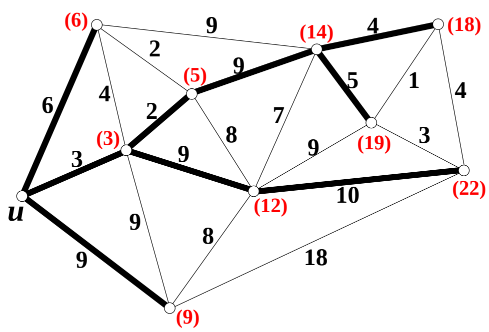
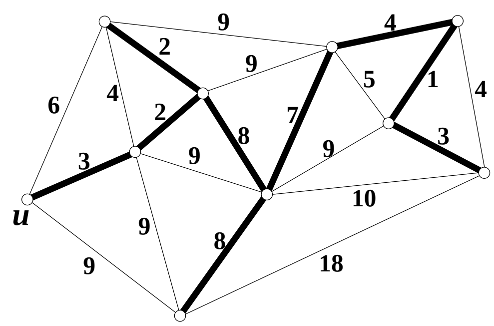


Figure 2. Minimum weight spanning tree



(a) Here, we are being asked to find a minimum distance tree from the vertex u. After applying Dijkstra's algorithm, we end up paving the roads indicated by the shaded edges in Figure 1 above, and the distances from u to each town are indicated by the numbers in parentheses.

(b) In this case, we are being asked to find a minimum weight spanning tree. Applying Prim's algorithm, we end up paving the roads indicated by the shaded edges in Figure 2 above. By adding the edge weights in the spanning tree, we see that the total distance that needs to be paved is

$$3 + 2 + 2 + 8 + 7 + 4 + 1 + 3 + 8 \ = \ 38 \text{ miles.}$$

Note that the edge weights above are listed in the order in which the edges were selected by Prim's algorithm.

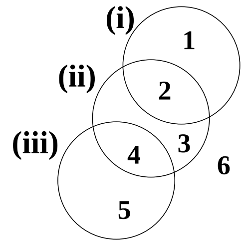
3. For each of the items below, we will restrict ourselves to graphs, not multigraphs or general graphs. In other words, loops and/or multiple edges are not allowed.

   (a) Consider the following properties that a graph G may or may not have: (i) G has a Hamilton cycle, (ii) G is bipartite, (iii) G has a bridge. Draw a Venn diagram to illustrate the relationship between these three properties. Then, find an example of a graph in each of the distinct "zones" created by your Venn diagram. (For example, find a graph which is bipartite and has a bridge, a graph that is not bipartite but has a bridge, etc., until you've exhausted all possible combinations of properties.)

   (b) Repeat part (a) with the following properties: (i) G is planar, (ii) G is bipartite, (iii) G is isomorphic to K_n for some n.

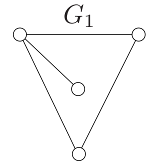
Solution:

(a) The Venn diagram is shown to the right. Note that the circles for properties (i) and (iii) do not intersect because it is not possible for a graph to have both a bridge and a Hamilton cycle. These circles divide the plane into 6 regions, which are numbered in the diagram. We will now find examples of graphs that fit into each of the six zones.

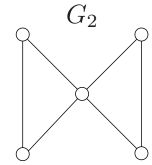


1. C_3 has a Hamilton cycle but is not bipartite and has no bridges.
2. C_4 has a Hamilton cycle and is bipartite but has no bridges.
3. $K_{3,2}$ has no Hamilton cycle, is bipartite, but has no bridges.
4. K_2 has no Hamilton cycle, is bipartite, and has a bridge.
5. The graph G_1 shown to the right has no Hamilton cycle, is not bipartite, but has a bridge.
6. The graph G_2 shown to the far right has no Hamilton cycle, is not bipartite, and has no bridge.

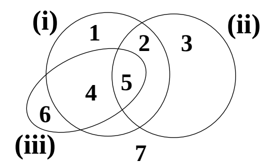
G_1



G_2

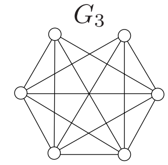


(b) The Venn diagram is shown to the right. Note that there are seven, not eight zones because it is not possible for a graph to be isomorphic to K_n, bipartite, and not planar all at the same time. This is because, for K_n to not be planar, we must have $n \geq 5$, meaning that K_n will have odd cycles (like K_3) as subgraphs, preventing it from being bipartite. We will now find examples of graphs that fit into each of the seven zones.



1. C_5 is planar, not bipartite, and not complete.
2. C_4 is planar, bipartite, but not complete.
3. $K_{3,3}$ is not planar, is bipartite, but is not isomorphic to K_n for any n.
4. K_4 is planar, not bipartite, but is complete.
5. K_2 is planar, bipartite, and complete.
6. K_5 is not planar, not bipartite, but is complete.
7. The graph G_3 shown to the right, which is simply K_6 with one edge removed, is not planar, not bipartite, and not complete. Note that it is not planar by Kuratowski's Theorem (it has K_5 as a subgraph).

G_3



4. Prove that, if a tree has a vertex of degree p, then it has at least p pendent vertices.

Solution: Let T be any tree of order n that has a vertex of degree p. Let k be the number of pendent vertices in T. We will show that $k \geq p$.

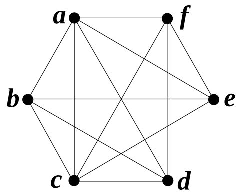
Let $d_1, d_2, \ldots, d_n$ represent the vertex degrees, and suppose that d_1 through d_k give the degrees of the pendent vertices, and that d_n corresponds to the vertex having degree p. Then $d_1 = d_2 = \cdots = d_k = 1$, and $d_n = p$, and $d_i \geq 2$ for all i satisfying $k+1 \leq i < n$. We therefore have

$$\begin{aligned} 2(n-1) &= d_1 + d_2 + \cdots + d_n \\ &= (d_1 + d_2 + \cdots + d_k) + (d_{k+1} + d_{k+2} + \cdots + d_{n-1}) + d_n \\ &\geq k + (n-1-k) \cdot 2 + p \\ &= k + 2(n-1) - 2k + p \end{aligned}$$

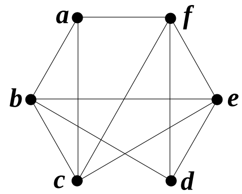
After subtracting $2(n-1)$ from both sides of the above inequality and simplifying, we obtain $0 \geq -k + p$, which is equivalent to $k \geq p$. This completes the proof.

5. Determine which of the following graphs are planar. If they are planar, draw a planar representation. If they are not planar, show that they are not planar using Kuratowski's Theorem or some other method.

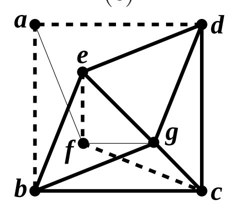
(a)



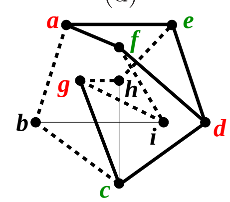
(b)



(c)



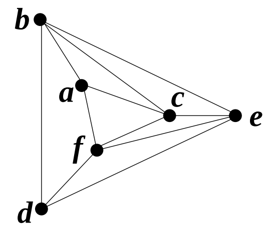
(d)



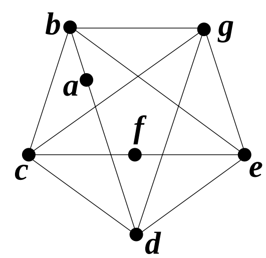
Solution:

(a) This graph is <u>not</u> planar. To see this, note that the graph has $e = 13$ edges and $n = 6$ vertices, so $3n - 6 = 12$, which means that $e > 3n - 6$.

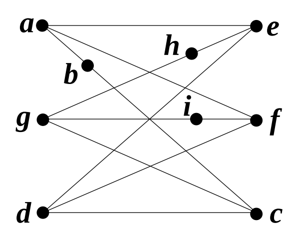
(b) This graph <u>is</u> planar, as illustrated by the planar representation to the right. Note that this planar representation can be obtained from the original graph by simply rearranging the positions of the vertices and allowing the edges to "stretch" or "shrink" accordingly.



(c) This graph is <u>not</u> planar. To see this, we highlight the edges of a K_5 subdivision subgraph in the original graph (see the original graph given above with the statement of the problem); the bold, solid edges are actual edges in the desired subgraph, while the dashed edges are the subdivided edges. Note that this subgraph is isomorphic to the K_5 subdivision shown to the right, which means that the given graph is not planar by Kuratowski's Theorem.



(d) This graph is <u>not</u> planar. To see this, we highlight the edges of a $K_{3,3}$ subdivision subgraph in the original graph (see the original graph given above with the statement of the problem); the bold, solid edges are actual edges in the desired subgraph, while the dashed edges are the subdivided edges. Note that this subgraph is isomorphic to the $K_{3,3}$ subdivision shown to the right, which means that the given graph is not planar by Kuratowski's Theorem.



6. In the game of Yahtzee, five standard six-sided dice are rolled simultaneously. For simplicity, let us assume the dice are different colors so that we can tell them apart. Determine the number of different ways in which each of the following outcomes can occur if the five dice are rolled once.
   (a) a full house (three dice sharing one number, two dice sharing a second number)
   (b) four of a kind (four dice sharing the same number)
   (c) three of a kind (but not four of a kind, five of a kind, or a full house)
   (d) a large straight (five consecutive numbers)
   (e) a small straight (four consecutive numbers, but not five consecutive)
   (f) a Yahtzee (five of a kind)

Solution:

(a) We break the task into several parts:

- Choose a common number for the three dice $\longleftarrow$ 6 ways
  (We can choose three 1's, three 2's, three 3's, ..., or three 6's, giving 6 choices.)
- Choose a common number for the two dice $\longleftarrow$ 5 ways
  (We can choose any number except the one chosen at the previous step.)
- Choose colors for the three dice $\longleftarrow$ $C(5,3)$ ways
- Choose colors for the two dice $\longleftarrow$ $C(2,2)$ ways
  (There are only 2 colors left to choose from after the three colors are chosen in the previous step.)

Therefore, our answer is $\boxed{6 \cdot 5 \cdot \binom{5}{3} \cdot \binom{2}{2}}$.

(b) We organize the task as follows:

- Choose a common number for the four dice $\longleftarrow$ 6 ways
- Choose colors for the four dice $\longleftarrow$ $C(5,4)$ ways
- Choose a number for the fifth die $\longleftarrow$ 5 ways
- Choose a color for the fifth die $\longleftarrow$ only 1 way

Therefore, our answer is $6 \cdot 5 \cdot \binom{5}{4}$.

(c) We organize the task as follows:

- Choose a common number for the three dice $\longleftarrow$ 6 ways
- Choose two different numbers for the remaining two dice $\longleftarrow$ $C(5,2)$ ways
- Choose colors for the three dice having a common number $\longleftarrow$ $C(5,3)$ ways
- Choose colors for the two remaining dice $\longleftarrow$ $2 \cdot 1$ ways
  (After the three colors are chosen for the dice in the previous step, there are 2 color choices for the fourth die, and only 1 choice for the fifth die.)

Therefore, our answer is $6 \cdot \binom{5}{2} \cdot \binom{5}{3} \cdot 2$.

(d) We break into cases as follows:

Case 1. (The large straight involves the numbers $1, 2, 3, 4, 5$.) Since the numbers are determined, the only thing remaining to do is to assign colors. There are 5 color choices for the die with a one on it, 4 remaining color choices for the die with a two on it, etc., down to only one color choice for the die with a five on it. Therefore, there are 5! such large straights.

Case 2. (The large straight involves the numbers $2, 3, 4, 5, 6$.) By the same logic as above, there are 5! of this type of large straight.

Combining Cases 1 and 2, we see that our answer is $2 \cdot 5!$.

(e) We break into cases as follows:

Case 1. (The small straight involves the numbers $1, 2, 3, 4$.) We organize the task as follows:

- Choose colors for the four sequential dice $\longleftarrow$ $P(5,4)$ ways
- Choose a number for the fifth die $\longleftarrow$ 5 ways
  (The fifth die cannot be a five; otherwise, we would have a large straight. Therefore, there are only 5 numbers to choose from.)
- Choose a color for the fifth die $\longleftarrow$ only 1 way

Therefore, there are $5 \cdot P(5,4)$ such small straights.

Case 2. (The small straight involves the numbers $2, 3, 4, 5$.) In this case, the task can be broken into the same three parts as in Case 1 above. The only difference is in the second step, when we choose a number for the fifth die. This time, it cannot be a one and it cannot be a six, since both of these outcomes would give us a large straight. Therefore, there are only 4 choices for the number on the fifth die, which yields an answer of $4 \cdot P(5,4)$ such small straights.

Case 3. (The small straight involves the numbers $3, 4, 5, 6$.) Using analogous reasoning as in Case 1, there are $5 \cdot P(5,4)$ such small straights.

Combining the above cases, our answer is

$$5P(5,4) + 4P(5,4) + 5P(5,4) = 14 \cdot P(5,4).$$

(f) The only ways to get five of a kind are five ones, five twos, five threes, five fours, five fives, or five sixes, so our answer is just 6.

7. Let P be a set of any n integers, where $n \geq 2$.

(a) Prove that if none of the integers in P are multiples of n, then there are at least two integers whose difference is a multiple of n. (**Hint:** Think about remainders when dividing by n.)

(b) Now, suppose that we allow integers in P that are multiples of n. Can we still conclude that there are two integers whose difference is a multiple of n? Justify your answer.

Solution:

(a) Let $P = \{k_1, k_2, \ldots, k_n\}$ be any set of n integers such that no k_i is a multiple of n. For each i, divide the integer k_i by n to obtain a quotient q_i and a remainder r_i, so that

$$k_i = nq_i + r_i.$$

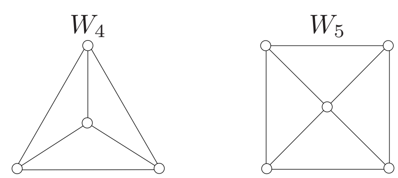
Observe that, since none of the k_i's are multiples of n, none of the remainders r_i can be zero, meaning that $1 \leq r_i \leq n-1$ for all i. Therefore, we have n remainders $(r_1, r_2, \ldots, r_n)$, but only $n-1$ different values that they can have. Thus, by the Pigeonhole Principle, we must have $r_i = r_j$ for some $i \neq j$, so

$$\begin{aligned} k_i - k_j &= n(q_i - q_j) + (r_i - r_j) \\ &= n(q_i - q_j), \qquad \longleftarrow \qquad \text{since } r_i = r_j \end{aligned}$$

and we see that the difference between k_i and k_j is a multiple of n.

(b) No, we cannot draw the same conclusion. For example, if we choose $P = \{1, 2, 3, \ldots, n\}$, then the difference between any two of the integers in P is between 1 and $n-1$, so none of the differences is a multiple of n.

8. For $n \geq 4$, the "wheel graph," W_n, is a graph that consists of the cycle C_{n-1} on the outside, with the nth vertex adjacent to all vertices on the cycle. The graphs W_4 and W_5 are shown to the right. In this problem, you will find the chromatic polynomial of W_n. Assume that there are k colors available.



(a) If we start by coloring the center vertex, how many color choices are there?

(b) Can the same color that was used for the center vertex be used for any of the remaining $n-1$ vertices?

(c) Using the multiplication principle and the result of Problem 13.14 from your textbook, write down the chromatic polynomial for W_n.

Solution:

(a) There are k color choices.

(b) No, it cannot because, by definition of W_n, the center vertex is adjacent to all other vertices in the graph.

(c) By Problem 13.14, we have $p_{C_n}(k) = (k-1)^n + (-1)^n(k-1)$. After coloring the center vertex of W_n, there are only $k-1$ colors left with which to color the remaining $n-1$ vertices of W_n, meaning that the outer C_{n-1} subgraph can be colored in $p_{C_{n-1}}(k-1)$ ways. Therefore, we have

$$\begin{aligned} p_{W_n}(k) &= (\#\text{ ways to color center vertex}) \cdot (\#\text{ ways to color outer } C_{n-1} \text{ subgraph}) \\ &= k \cdot p_{C_{n-1}}(k-1) \\ &= k\big((k-2)^{n-1} + (-1)^{n-1}(k-2)\big). \end{aligned}$$

9. In each of the following items, find the number of 20-digit sequences that only contain 0's, 1's, 2's, and/or 3's and have the specified properties.
   (a) The sequence contains at least 1 zero.
   (b) The sequence contains 5 zeros, 4 ones, 3 twos, and 8 threes.
   (c) The sequence does <u>not</u> contain consecutive zeros. (**Hint:** First, find and solve an appropriate recurrence relation. To make the calculations easier, you may use the fact that $a_0 = 1$.)

Solution:

(a) First, we note that, without any restrictions, there are 4 choices for each digit, giving 4^{20} total sequences. On the other hand, there are 3^{20} total sequences with no zeros, since there are only three choices for each of the 20 digits. Therefore, we have

$$\begin{aligned} \#\text{ sequences with at least 1 zero} &= 4^{20} - (\#\text{ sequences having no zeros}) \\ &= 4^{20} - 3^{20}. \end{aligned}$$

(b) Think of breaking this process into 4 sequential tasks: first, we choose 5 of 20 available positions for the zeros, which can be done in $C(20,5)$ ways. Second, we choose 4 of the 15 remaining positions for the ones, which can be done in $C(15,4)$ ways. Continuing in this fashion, there are $C(11,3)$ ways to place the twos, and $C(8,8) = 1$ way to place the threes. One way to complete this process is shown below:

_	_	_	_	**0**	_	_	_	**0**	_	_	**0**	**0**	_	_	_	**0**	_	_	_
_	_	_	_	**0**	**1**	_	_	**0**	_	**1**	**0**	**0**	_	**1**	_	**0**	_	**1**	_
_	_	**2**	_	**0**	**1**	_	_	**0**	**2**	**1**	**0**	**0**	**2**	**1**	_	**0**	_	**1**	_
3	**3**	**2**	**3**	**0**	**1**	**3**	**3**	**0**	**2**	**1**	**0**	**0**	**2**	**1**	**3**	**0**	**3**	**1**	**3**

This gives a total of $C(20,5) \cdot C(15,4) \cdot C(11,3) \cdot C(8,8)$ total sequences. Another way to look at this same problem is to observe that there are 20! ways to arrange the 20 digits in a row, but 5! 4! 3! 8! of the arrangements will look the same because of repeated digits. Therefore, the total number of permutations can be written as either

$$\binom{20}{5}\binom{15}{4}\binom{11}{3}\binom{8}{8} \qquad \text{or} \qquad \frac{20!}{5!\,4!\,3!\,8!}.$$

(c) Let a_n represent the number of the desired types of digit strings of length n that can be made. First, we determine our initial conditions. We were asked to assume that $a_0 = 1$, and note that $a_1 = 4$ because there are four sequences of length one: 0, 1, 2, or 3.

To determine a recurrence formula, we use the diagram to the right as an aid. If a string of length n starts with any digit other than 0, then the remaining $n-1$ spaces can filled in a_{n-1} ways. On the other hand, if a string of length n starts with 0, then the first two digits must look like $01, 02$, or 03 in order to avoid repeated zeros. Each of these three configurations can be completed in a_{n-2} ways, yielding the following recursion formula:

Length n string	String Count
1 _ _ _ _ _	a_{n-1}
2 _ _ _ _ _	a_{n-1}
3 _ _ _ _ _	a_{n-1}
0 1 _ _ _ _	a_{n-2}
0 2 _ _ _ _	a_{n-2}
0 3 _ _ _ _	a_{n-2}

$$a_n = 3a_{n-1} + 3a_{n-2} \quad \text{for } n \geq 2.$$

The characteristic equation of this recurrence is $x^2 - 3x - 3 = 0$, which, after using the quadratic formula, yields the following roots:

$$x = \frac{3 \pm \sqrt{9 - 4(1)(-3)}}{2} = \frac{3 \pm \sqrt{21}}{2}$$

Therefore, the general solution has the form

$$a_n = c\left(\frac{3+\sqrt{21}}{2}\right)^n + d\left(\frac{3-\sqrt{21}}{2}\right)^n.$$

Using the initial conditions $a_0 = 1$ and $a_1 = 4$, we obtain the following:

$$\begin{array}{rcrcl} c & + & d & = & 1 \\ \left(\frac{3+\sqrt{21}}{2}\right) c & + & \left(\frac{3-\sqrt{21}}{2}\right) d & = & 4 \end{array} \quad \Longrightarrow \quad \begin{array}{rcrcl} c & + & d & = & 1 \\ (3+\sqrt{21})\, c & + & (3-\sqrt{21})\, d & = & 8 \end{array}$$

Solving the first equation for d yields $d = 1 - c$, and substituting into the second equation, we have

$$\begin{aligned} (3+\sqrt{21})c + (3-\sqrt{21})(1-c) \;=\; 8 \quad &\Longrightarrow \quad 3c + c\sqrt{21} + 3 - \sqrt{21} - 3c + c\sqrt{21} \;=\; 8 \\ &\Longrightarrow \quad 2c\sqrt{21} \;=\; \sqrt{21} + 5 \\ &\Longrightarrow \quad c \;=\; \frac{\sqrt{21}+5}{2\sqrt{21}} \end{aligned}$$

Therefore,

$$d \;=\; 1 - \left(\frac{\sqrt{21}+5}{2\sqrt{21}}\right) \;=\; \frac{\sqrt{21}-5}{2\sqrt{21}}.$$

Therefore, the number of 20-digit sequences with no consecutive zeros is given by

$$a_{20} \;=\; \frac{\sqrt{21}+5}{2\sqrt{21}} \left(\frac{3+\sqrt{21}}{2}\right)^{20} + \frac{\sqrt{21}-5}{2\sqrt{21}} \left(\frac{3-\sqrt{21}}{2}\right)^{20}.$$

10. Suppose that we have plentiful supplies of apples, celery sticks, walnuts, and grapes. Assume that the objects in each category are identical.

(a) Use counting techniques to find the number of different selections of 20 of the available food items in which no more than 2 apples are chosen.

(b) Find a generating function for the number of different selections of n food items that contain no more than 2 apples.

(c) Use algebra and your answer to part (b) to confirm your answer to part (a).

Solution:

(a) We break the process into cases according to how many apples are chosen.

Case 1. (0 apples) In this case, we are trying to choose 20 food items from 3 different categories: celery sticks, walnuts, or grapes. We therefore have 20 stars and 2 bars, giving $C(22, 20)$ selections.

Case 2. (1 apple) We break the task into two parts:

- Choose 1 apple $\longleftarrow$ only 1 way
- Choose the other 19 items $\longleftarrow$ $C(21, 19)$ ways
  (In this case, there are 19 remaining food items (stars) to be picked, and since the apples are already picked, there are 3 food categories, giving 2 bars.)

Therefore, there are $C(21, 19)$ such selections.

Case 3. (2 apples) We break the task into two parts:

- Choose the 2 apples $\longleftarrow$ only 1 way
- Choose the other 18 items $\longleftarrow$ $C(20, 18)$ ways
  (Reasoning as in Case 2, there are 18 stars and 2 bars.)

Therefore, there are $C(20, 18)$ such selections.

Adding, we obtain an answer of $\binom{22}{20} + \binom{21}{19} + \binom{20}{18}$.

(b) Our generating function f is given by

$$\begin{aligned} f(x) &= \underbrace{(1+x+x^2)}_{\text{apples}} \cdot \underbrace{(1+x+x^2+x^3+\cdots)}_{\text{celery}} \cdot \underbrace{(1+x+x^2+x^3+\cdots)}_{\text{walnuts}} \cdot \underbrace{(1+x+x^2+x^3+\cdots)}_{\text{grapes}} \\ &= (1+x+x^2)\cdot(1+x+x^2+x^3+\cdots)^3 \\ &= \frac{1+x+x^2}{(1-x)^3}. \end{aligned}$$

(c) Our final answer will be the coefficient of the "x^{20}" term in our formula for $f(x)$ from part (c). Calculating, we have

$$\begin{aligned} f(x) = \frac{1+x+x^2}{(1-x)^3} &= (1+x+x^2)\cdot\frac{1}{(1-x)^3} \\ &= (1+x+x^2)\cdot\sum_{k=0}^{\infty}\binom{3+k-1}{k}x^k \\ &= (1+x+x^2)\cdot\sum_{k=0}^{\infty}\binom{k+2}{k}x^k, \end{aligned}$$

and we note that there are three ways to get an "x^{20}" term in the above expression: by taking 1 from the "$(1+x+x^2)$" portion of the expression times the term with $k=20$ in the infinite summation, by taking x from the "$(1+x+x^2)$" portion of the expression times the term with $k=19$ in the infinite summation, and by taking x^2 from the "$(1+x+x^2)$" portion of the expression times the term with $k=18$ in the infinite summation. In other words, the three terms in the expansion of $f(x)$ that involve x^{20} can be organized as follows:

$$1\cdot\binom{22}{20}x^{20} + x\cdot\binom{21}{19}x^{19} + x^2\cdot\binom{20}{18}x^{18} = \left[\binom{22}{20}+\binom{21}{19}+\binom{20}{18}\right]x^{20}$$

We therefore see that the coefficient on the "x^{20}" term agrees with our answer to part (a).

11. Let m, n, and k be positive integers with $k \le m$ and $k \le n$. Give a combinatorial proof of the following identity:

$$\binom{m+n}{k} = \binom{m}{k}\binom{n}{0} + \binom{m}{k-1}\binom{n}{1} + \binom{m}{k-2}\binom{n}{2} + \cdots + \binom{m}{0}\binom{n}{k}$$

Hint: Think of choosing committees from a group of m men and n women.

Solution: We claim that both sides of the identity count the number of ways to choose a committee of k people from a group of m men and n women available to serve on the committee. We demonstrate below:

Left side: By definition, $C(m+n,\ k)$ counts the number of ways to choose k people from a group of $m+n$ total people, which is exactly what we claimed above.

Right side: This side counts the desired quantity by cases according to how many women are on the committee. To see this, consider any value of i, where $0 \le i \le k$. We break the task of choosing a k-person committee with i women into two parts:

- Choose the women $\longleftarrow$ $C(n,i)$ ways
  (Here, we are choosing i women from the n total women available.)
- Choose the men $\longleftarrow$ $C(m,k-i)$ ways
  (To complete our committee, we need to choose the remaining $k-i$ committee members from the m available men.)

Therefore, there are $\binom{m}{k-i}\binom{n}{i}$ such committees with exactly i women. Since a k-person committee can have anywhere from $i = 0$ to $i = k$ women serving on it, we get the total number of committees by calculating

$$\sum_{i=0}^{k} \binom{m}{k-i}\binom{n}{i}.$$

Since this sum is equal to the right-hand side of our identity, this completes our proof.