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HW 2: Proofs Section

i) Knowing that S follows the form $S^T = -S$, we can conclude that the diagonal entries of S are 0. Thus, S follows the format

$$S = \begin{bmatrix} 0 & s_{12} & \cdots & \cdots & s_{1n} \\ -s_{12} & 0 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & s_{(n-1)n} \\ -s_{1n} & \cdots & \cdots & -(s_{(n-1)n}) & 0 \end{bmatrix}.$$

The eigenvalues of S follow the format $S\vec{x} = \lambda\vec{x}$ s.t. eigenvector $\vec{x} \in \mathbb{C}^n$.

Simplifying

$$S\vec{x} = \lambda\vec{x}$$

$$\Rightarrow \vec{x}^* S\vec{x} = \vec{x}^* \lambda\vec{x}$$

$$\Rightarrow \vec{x}^* S\vec{x} = \lambda\vec{x}^*\vec{x}$$

$$\Rightarrow \vec{x}^* S\vec{x} = \lambda\|\vec{x}\|_2^2$$ (by 2-norm definition).

Aditionally,

$$S\vec{x} = -S^T\vec{x} = \lambda\vec{x}$$

$$\Rightarrow -\vec{x}^* S^T\vec{x} = \lambda\|\vec{x}\|_2^2.$$

Using our definitions,

$$\lambda \|\vec{x}\|_2^2 = \vec{x}^* \lambda \vec{x}$$

$$= \vec{x}^* (-S^* \vec{x})$$ (by equivalence: skew symmetric matrices definitions)

$$= -\vec{x}^* S^* \vec{x}$$

$$= -(S\vec{x})^* \vec{x}$$

$$= -(\lambda \vec{x})^* \vec{x}$$

$$= -\vec{x}^* \lambda^* \vec{x}$$

$$= -\overline{\lambda} \, \vec{x}^* \vec{x}$$

$$= -\overline{\lambda} \|\vec{x}\|_2^2 .$$

Thus,

$$\lambda \|\vec{x}\|_2^2 = -\bar{\lambda} \|\vec{x}\|_2^2$$

$$\Rightarrow \lambda = -\bar{\lambda}$$

$$\Rightarrow a + bi = -\overline{(a+bi)}, \quad \text{s.t. } a \in \mathbb{R} ; b \in \mathbb{R}$$

$$\Rightarrow a + bi = -a + bi.$$

This equivalence only holds when $a = 0$.

$\therefore$ Any non-zero eigenvalue of S must be purely imaginary.

$\blacksquare$

Proof by Contradiction

ii) Assume that S is invertible and hence has a determinant of all n values s.t. $n \in \mathbb{Z}^+$.

We know that by properties of determinants, $|A| = |A^T|$ for an arbitrary $A \in \mathbb{R}^{n \times n}$.

Applying this property to S,

$$|S| = |S^T|.$$

Manipulating the skew-symmetric property,

$$\begin{aligned} |S| &= |-S^T| \\ &= |S^T| \\ &= |-S|. \end{aligned}$$

However, another property of determinants is that $|c \cdot A| = c^n |A|$ for some arbitrary $c \in \mathbb{R}$ & $A \in \mathbb{R}^{n \times n}$.

Back to our case,

$$|-S^T| = |S^T|$$

$$\Rightarrow |-S| = |S|$$

$$\Rightarrow |-1 \cdot S| = |S|$$

$$\Rightarrow (-1)^n |S| = |S|.$$

Observing, in the case that $|S| \neq 0$, which is the invertible condition that we already stated must be true, there exists some contradictions.

when $(-1)^n = 1$, LHS & RHS are equivalent, but not when $(-1)^n = -1$, a contradiction.

$\therefore$ The statement that when S is invertible, then n must be even is true. If n is odd, S cannot be invertible.

$\blacksquare$

iii) Given our definition of S in part i, the matrix $I - S$ follows the format

< Proof by contradiction? >

$$I - S = \begin{bmatrix} 1 & -s_{12} & \cdots & \cdots & -s_{1n} \\ s_{12} & 1 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 1 & -s_{(n-1)n} \\ s_{1n} & \cdots & \cdots & (s_{(n-1)n}) & 1 \end{bmatrix}.$$

Simplifying, $I - S = I + S^T$.

In an attempt to find a nontrivial solution $\vec{x} \in \mathbb{C}^n$ of $(I + S^T)\vec{x} = \vec{0}$ that we guarantee exists,

$$(I + S^T)\vec{x} = 0$$

$$\Rightarrow \vec{x} + S^T\vec{x} = 0$$

$$\Rightarrow \vec{x} = -S^T\vec{x}$$

$$\Rightarrow \vec{x} = S\vec{x}$$

$$\Rightarrow \vec{x}^*\vec{x} = \vec{x}^* S\vec{x}$$

$$\Rightarrow \|\vec{x}\|_2^2 = (S^*\vec{x})^*\vec{x}$$

$$\Rightarrow \|\vec{x}\|_2^2 = (S^T\vec{x})^*\vec{x}$$

$$\Rightarrow \|\vec{x}\|_2^2 = (-S\vec{x})^*\vec{x}$$

$$\Rightarrow \|\vec{x}\|_2^2 = (-\vec{x})^*\vec{x} \quad (\text{by } S\vec{x} = \vec{x})$$

$$\Rightarrow \|\vec{x}\|_2^2 = -\vec{x}^*\vec{x}$$

$$\Rightarrow \|\vec{x}\|_2^2 = -\|\vec{x}\|_2^2$$

$\Rightarrow \|x\|_2^2 = 0$

$\Rightarrow \vec{x} = \vec{0}$, a contradiction.

No nontrivial solution $\vec{x}$ exists in $(I + S^T)\vec{x} = 0$, which implies the same for $(I - S)\vec{x} = 0$.

$\therefore$ $I - S$ is always invertible.

$\blacksquare$

iv.) unitary proof

↳ inverse is equal to conj transpose.

For a real matrix to be unitary, its inverse must equal its conjugate transpose.

Multiplying the Cayley transform by its transpose,

$$C^TC = \left[(I+S)(I-S)^{-1}\right]^T (I+S)(I-S)^{-1}$$

$$= (I-S^T)^{-1}(I+S^T)(I+S)(I-S)^{-1}$$

$$= (I+S)^{-1}(I-S)(I+S)(I-S)^{-1}.$$

$$= \left[(I+S)^{-1}(I-S)\right]C.$$

If $I-S$ is invertible, then $I+S$ is invertible as well, since $I-S = I+S^T$, which maintains the same properties as $I+S$ becase S^T is a skew-symmetric matrix.

Set $A := (I+S)$, $B := (I-S)^{-1}$.

By definition, $C = AB$.

So,

$$C^{-1} = (AB)^{-1}$$

$$= A^{-1}B^{-1} \left(\text{because } A \text{ \& } B \text{ are both invertible} \right)$$

$$= (I+S)^{-1}(I-S)C.$$

Substituting back into our previous problem,

$$C^T C = \left[(I+S)^{-1}(I-S)\right]C$$

$$= C^{-1}C.$$

$$\Rightarrow C^T = C^{-1}.$$

$\therefore$ $C(S)$ is a unitary matrix.

$\blacksquare$

v) $C = (I+S)(I-S)^{-1}$

Eigenvalues of S are denoted as λ_i $\forall_i$, $i=1,2,\dots,n$, and they are all imaginary for all non-zero eigenvectors $\vec{x}$ as seen in part i. Finding the eigenvalues of $I+S$,

$$(I+S)\vec{x} = \vec{x} + S\vec{x}$$
$$= \vec{x} + \lambda_i \vec{x}$$
$$= (1+\lambda_i)\vec{x}.$$

So, the eigenvalues of $I+S$ are equivalent to $1+\lambda_i$ for all i.

Likewise for $I-S$,

$$(I-S)\vec{x} = \vec{x} - S\vec{x}$$
$$= \vec{x} - \lambda_i \vec{x}$$
$$= (1-\lambda_i)\vec{x}.$$

So, the eigenvalues of $I-S$ are equivalent to $1-\lambda_i$ for all i.

Moreover,

$$(I-S)\vec{x} = (1-\lambda_i)\vec{x}$$

$$\Rightarrow (I-S)^{-1}(I-S)\vec{x} = (1-\lambda_i)(I-S)^{-1}\vec{x}$$

$$\Rightarrow \left(\frac{1}{1-\lambda_i}\right)\vec{x} = (I-S)^{-1}\vec{x}.$$

So, the eigenvalues of $(I-S)^{-1}$ are

$$\left(\frac{1}{1-\lambda_i}\right).$$

Now addressing the Cayley transform,

let β_i denote its eigenvalues for all i associated w/ λ_i. That being said,

$$(I+S)(I-S)^{-1}\vec{x}_i$$

$$= \frac{1}{1-\lambda_i}(I+S)\vec{x}$$

$$= \frac{1}{1-\lambda_i}(1+\lambda_i)\vec{x}$$

$$= \left(\frac{1+\lambda_i}{1-\lambda_i}\right)\vec{x}$$

$$= \beta_i \vec{x}.$$

$\therefore$ The eigenvalues of $C(S)$ are

$$\frac{1+\lambda_i}{1-\lambda_i} \quad \text{for all } i = 1, 2, \dots, n.$$