Lossy Image Compression

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***Abstract*—**

I. Introduction

Image compression is an important area of study due to the ever-increasing consumption of media and explosion of big data image processing. The goal of lossy image compression is to preserve the perceptually relevant portions of an image so that the image appears unaltered while drastically reducing the number of bits required to store the image.

In this paper, I will discuss several methods of image compression and evaluate their performance on metrics of image quality and compression ratio.

II. Background

A. Perception

Images are often represented as 3 separate channels of the red, green, and blue intensity at each pixel in the image. However, because humans are more sensitive to some colors but not others, and are more sensitive to different hues when they are of different brightnesses or intensities, this is not the best way to encode images for compression. JPEG uses the YC_bC_r image representation, where Y is the luma or brightness component and C_b and C_r are the blue and red difference components (i.e. $C_b = B - Y$, $C_r = R - Y$). The human eye is most sensitive to the information in the Y channel, but less sensitive to changes in red and blue color, so this leaves an opportunity to apply higher compression to the C_b and C_r channels to achieve higher compression ratios while maintaining similar image quality.

Humans perceive the different spatial frequencies in images with varying degrees of sensitivity. State of the art lossy image compression systems (e.g. JPEG and JPEG2000 from the Joint Photographic Experts Group) leverage this to dramatically shrink the number of bits required to encode an image while maintaining a strong resemblance of the original image. In order to do this, the original image is transformed into a frequency representation and the frequency representation is quantized in order to reduce the number of bits required to store it. The quantized frequency-domain representation is then passed through lossless entropy coding to produce a compressed bitstream that can be stored or transmitted and then decoded to reconstruct the original image. The two most popular approaches for this transformation are the discrete cosine transform (DCT) and discrete wavelet transform (DWT).

B. Discrete Transforms in 1D and 2D

A frequency transform in general can be viewed as a series of inner products between the input function and a set of basis vectors. Often, these basis vectors are chosen to be orthogonal (such as in the DCT and many variants of the DWT). A one-dimensional transform of an input vector $\mathbf{x}$ can be represented as:

$$\widetilde{\mathbf{x}} = \mathbf{\Phi}\mathbf{x} \tag{1}$$

$$\mathbf{\Phi} = \begin{pmatrix} — & \boldsymbol{\varphi}_1^T & — \\ & \vdots & \\ — & \boldsymbol{\varphi}_N^T & — \end{pmatrix} \tag{2}$$

where $\boldsymbol{\varphi}_k$ are the basis vectors of the transform. Each of the elements $\widetilde{x}_k$ of $\widetilde{\mathbf{x}}$ is a coefficient which measures the degree to which the input signal $\mathbf{x}$ is orthogonal to the corresponding basis vector $\boldsymbol{\varphi}_k$. For a discrete fourier transform, these basis functions are simply complex exponentials, with different frequencies for different basis vectors:

$$\boldsymbol{\varphi}_k = \left(1, e^{-\frac{2i\pi k}{N}}, e^{-\frac{4i\pi k}{N}}, \dots e^{\frac{2(N-1)\pi k}{N}}\right)^T \tag{3}$$

Intuitively, a discrete transform of 2D data (e.g. a matrix) can be performed by first taking a discrete transform of each individual column:

$$\widetilde{\mathbf{X}}_c = \begin{pmatrix} — & \boldsymbol{\varphi}_1^T & — \\ & \vdots & \\ — & \boldsymbol{\varphi}_N^T & — \end{pmatrix} \begin{pmatrix} | & & | \\ \mathbf{x}_1 & \dots & \mathbf{x}_N \\ | & & | \end{pmatrix} \tag{4}$$

Then taking a discrete transform of each row of the partially transformed matrix:

$$\widetilde{\mathbf{X}}^T = \mathbf{\Phi}\widetilde{\mathbf{X}}_c^T \tag{5}$$

$$\widetilde{\mathbf{X}} = \mathbf{\Phi}\mathbf{X}\mathbf{\Phi}^T \tag{6}$$

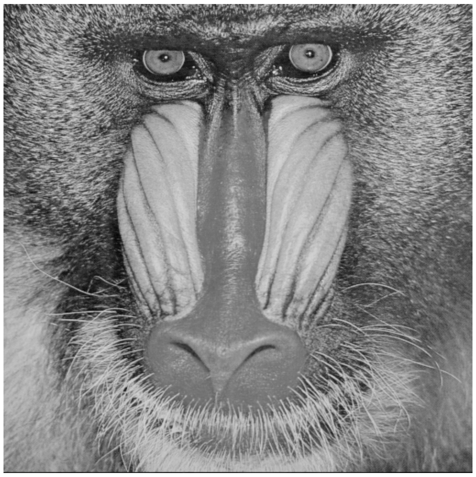
C. Discrete Cosine Transform and Discrete Wavelet Transform

For a discrete cosine transform, the basis vectors are evenly spaced in frequency. They are given by:

$$\boldsymbol{\varphi}_k = \left(\cos\frac{\pi k}{2N}, \cos\frac{3\pi k}{2N}, \dots \cos\frac{(2N-1)\pi k}{2N}\right)^T$$

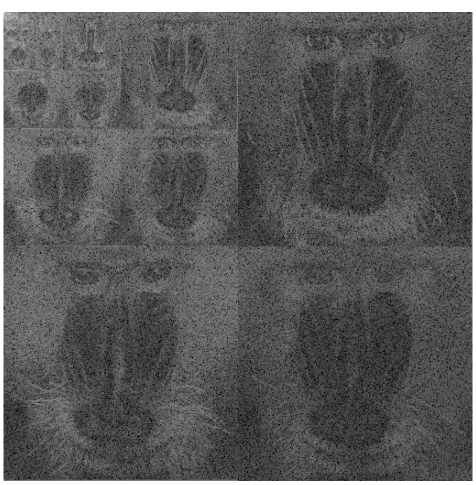
The angular frequency of the kth basis vector is $k\pi/N$ and the phase offset is $k\pi/2N$.

Unlike the discrete cosine transform, which allocates equal bandwidth for each coefficient, discrete wavelet transform basis vectors (or wavelets) trade off frequency and spatial resolution. The wavelets which have low frequency content (which will be maximally orthogonal to signals with low frequency and thus yield a large transform coefficient value) have poor spatial resolution but very fine frequency resolution. Similarly, basis vectors with high frequency content are broad-band in nature due to their temporally-localized structure. The



(a) Grayscale Image of Mandrill

(b) Linear Scale DWT of Image



(c) Log Scale DWT of Image

Fig. 1: Discrete Wavelet Transform of an image. The Cohen-Daubechies-Feauveau 9/7 (cdf97) wavelet used in JPEG2000 was selected for this transform.

discrete wavelet transform is typically implemented as a set of cascaded filters and downsamplers. The input signal is passed through a quadrature mirror filter consisting of a high pass and low pass filter. Both filter outputs are decimated by a factor of 2, and the low pass filter output is recursively cascaded through another quadrature mirror filter pair. This pattern of downsampling and recursively filtering the low pass output can be seen in Figure 1c. Because this is in two dimensions, each portion of the image is filtered twice: once by row and once by column.

D. Singular Value Decomposition

A matrix can be represented as a singular value decomposition. Singular values σ_i of a matrix $\mathbf{A}$ satisfy

$$\mathbf{A}\mathbf{v}_i = \sigma_i \mathbf{u}_i \tag{7}$$

with orthonormal $\mathbf{v}_i$ and $\mathbf{u}_i$ forming an eigenbasis for $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$ (respectively). The singular value decomposition of an $m \times n$ matrix $\mathbf{A}$ of rank r can be written as so

$$\mathbf{A} = \begin{pmatrix} | & & | \\ \mathbf{u}_1 & \cdots & \mathbf{u}_m \\ | & & | \end{pmatrix} \begin{pmatrix} \sigma_1 & & & 0 \\ & \ddots & & \\ & & \sigma_r & \\ 0 & & & 0 \end{pmatrix} \begin{pmatrix} \text{—} & \boldsymbol{v}_1^T & \text{—} \\ & \vdots & \\ \text{—} & \boldsymbol{v}_n^T & \text{—} \end{pmatrix} \tag{8}$$

Due to Eckart and Young, we know that the rank k matrix $\mathbf{A}_k$ which is constructed from the first k singular values of $\mathbf{A}$ minimizes the mean square error $\|\mathbf{A} - \mathbf{A}_k\|$.

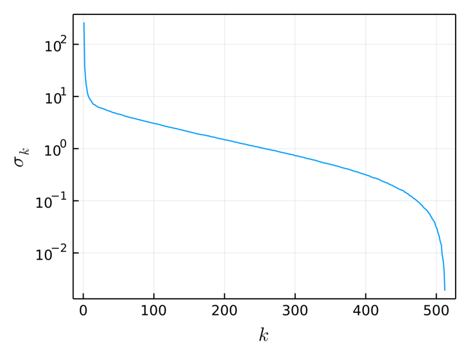


Fig. 2: Singular Values of Mandrill Image

As shown in Figure 2, some singular values are very small compared to the largest singular values. This leaves an opportunity for image compression through rank reduction. The number of nonzero values required to store the SVD of a matrix can be calculated from the number of nonzero values in the 3 matrices of the SVD. In $\mathbf{U}$, each basis vector $\mathbf{u}_i \in \mathbb{R}^m$ has m elements. For a rank k truncation of the SVD, only $\mathbf{u}_1$ to $\mathbf{u}_k$ are needed, meaning $\mathbf{U}$ can be stored with mk nonzero values. Similarly, nk nonzero values are required to store $\mathbf{V}^T$, and k values to store the diagonal matrix of singular values $\mathbf{\Sigma}$. Thus, the SVD can be compactly stored with $k(m+n+1)$ nonzero values, provided $k \ll m, n$. Storing the original $m \times n$

matrix of rank r takes mn nonzero values, so the rank-reduced SVD representation is not useful for compression if $k \approx m, n$.

E. Lossless Compression - Entropy Coding

The focus of this paper is not in entropy coding and lossless compression so there will be minimal discussion of these topics beyond this section. Rank truncation and quantization reduce the number of nonzero values required to represent an image, but in order to actually store the information required to reconstruct the image, this sparsity must be leveraged by an entropy coding scheme. JPEG (the 1992 standard) uses run length encoding and Huffman codes to create a compact representation of the quantized DCT representation of the image. JPEG2000 uses arithmetic coding. Because run length encoding is quite simple to understand and implement, I selected to also perform run length encoding and use the bitlength of a run-length-coded image as a compression ratio metric. However, I will also report the sparsity of the transform-domain representation of the image (relative reduction in nonzero values in the image) as a compression ratio metric.

III. Methods

A. Proposed Image Compression Techniques

I implemented two DCT-based image compression techniques: one using the JPEG tiling and quantization scheme and one using an adaptive thresholding scheme. I also implemented several DWT-based schemes. I chose the Cohen-Daubechies-Feauveau 9/7 (cdf97) wavelet for computing the DWT because it is used in the JPEG2000 compression algorithm. One is quite similar to JPEG2000, and uses scalar quantization. Another uses the same adaptive thresholding scheme as in the DCT-thresholding compression technique. I also implemented a rank-truncation method which uses the SVD of the transform-domain coefficients to compress the image. Finally, I implemented a hybrid rank-truncation and adaptive thresholding method.

B. Quantifying Image Quality

The most common metric to quantify image quality is the mean squared error of the image, which is defined for a single channel as

$$MSE_1 = \frac{1}{mn} \sum_{m,n} \left(\mathbf{X}_{m,n} - \hat{\mathbf{X}}_{m,n} \right)^2 \tag{9}$$

where $\mathbf{X}$ is the original image and $\hat{\mathbf{X}}$ is the compressed image. The mean squared error for the euclidean RGB distance between two colors is calculated as

$$MSE_{RGB} = \frac{1}{3mn} \sum_{\mathbf{X} \in \{\mathbf{R},\mathbf{G},\mathbf{B}\}} \sum_{m,n} \left(\mathbf{X}_{m,n} - \hat{\mathbf{X}}_{m,n} \right)^2 \tag{10}$$

Finally, based on the perceptual model from CIE for ΔE (CIEDE2000), a metric of perceived color similarity, the perceived MSE can be calculated as

$$MSE_{\Delta E} = \frac{1}{mn} \sum_{m,n} \Delta E \left(\mathbf{X}, \hat{\mathbf{X}} \right).^2 \tag{11}$$

where $\Delta E(\mathbf{X}, \hat{\mathbf{X}})$ computes the perceived difference between the 3-channel colors $\mathbf{X}$ and $\hat{\mathbf{X}}$.

Often, these mean squared error quantities are reported as a peak signal-to-noise ratio or PSNR:

$$PSNR = 10 \log_{10} \left(\frac{MAX_I^2}{MSE} \right) \tag{12}$$

where MAX_I is the maximum pixel value of the input $\mathbf{X}$ used to compute the mean squared error.

C. Estimating Compression Ratio

As mentioned in the section on Entropy Coding, the focus of this work is not on lossless compression techniques, so the compression ratio of these methods will be estimated with the relative increase in sparsity of the transform of the image and the number of bits required to store a run length encoding of the transform.

IV. Results

V. Discussion

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	Table column subhead	***Subhead***	***Subhead***
copy	More table copy[a]		

[a]Sample of a Table footnote.

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REFERENCES

[1] G. Eason, B. Noble, and I. N. Sneddon, "On certain integrals of Lipschitz-Hankel type involving products of Bessel functions," Phil. Trans. Roy. Soc. London, vol. A247, pp. 529–551, April 1955.

[2] J. Clerk Maxwell, A Treatise on Electricity and Magnetism, 3rd ed., vol. 2. Oxford: Clarendon, 1892, pp.68–73.

[3] I. S. Jacobs and C. P. Bean, "Fine particles, thin films and exchange anisotropy," in Magnetism, vol. III, G. T. Rado and H. Suhl, Eds. New York: Academic, 1963, pp. 271–350.

[4] K. Elissa, "Title of paper if known," unpublished.

[5] R. Nicole, "Title of paper with only first word capitalized," J. Name Stand. Abbrev., in press.

[6] Y. Yorozu, M. Hirano, K. Oka, and Y. Tagawa, "Electron spectroscopy studies on magneto-optical media and plastic substrate interface," IEEE Transl. J. Magn. Japan, vol. 2, pp. 740–741, August 1987 [Digests 9th Annual Conf. Magnetics Japan, p. 301, 1982].

[7] M. Young, The Technical Writer's Handbook. Mill Valley, CA: University Science, 1989.

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