Homework 1 Solutions

1 Substitution

Solve the following initial value problem by substitution

$$\begin{cases} x_1' &= x_1 + 3x_2 \\ x_2' &= 4x_2 \\ x_1(0) &= 7 \\ x_2(0) &= 8 \end{cases}$$

- $x(t) = 8e^{4t} - e^{3t}$, $y(t) = 8e^{4t}$.

2 Diagonalization

Diagonalize each of the following matrices.

A. $A_1 = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$

- $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1}$

B. $A_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

- $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$

C. $A_3 = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

- $\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}^{-1}$

3 Vector Spaces

Determine whether each of the following sets forms a vector space. For the ones that do not, you must do one of the following:

1. find two elements v, w of the set such that $v + w$ is not in the set.
2. find an element, v, of the set, and a real number α such that αv is not in the set.

For instance, if I wanted to show that

$$\{\text{continuous functions, } f : [0,1] \to \mathbb{R}, \text{ such that } f(1/2) = 1\}$$

is *not* a vector space, I would let $f(x) = 2x$ and $\alpha = 2$. Then I would observe that $(\alpha f)(1/2) = 2 \neq 1$, so (αf) is not in the set, even though f is in the set and α is a real number.

A. The elements of V_1 are standard vectors (x, y, z) in $\mathbb{R}^3$ such that $x + y + z = 1$.

- Let $v = (1, 0, 0)$ and $\alpha = 0$. Then $v \in V_1$ but $(\alpha v) \notin V_1$.

B. The elements of V_2 are differentiable functions $x : \mathbb{R} \to \mathbb{R}^2$ such that $x'(5) = (0, 0)$.

- This is a vector space.

C. The elements of V_3 are continuous functions $f : [0,1] \to \mathbb{R}$ such that $\int_0^{1/2} f(t)dt = \int_{1/2}^{1} f(t)dt$.

- This is a vector space

D. The elements of V_4 are differentiable functions $x : \mathbb{R} \to \mathbb{R}^2$ such that $x(0) = (1, 2)$.

- Let $x(t) = (1, 2)$ (the constant function) and $\alpha = 0$. Then $(\alpha x) \notin V_4$.

E. The elements of V_5 are functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(t + 5) = f(t)$ for all t in $\mathbb{R}$. (i.e. functions that if you shift by 5 units, you get the original function back).

- This is a vector space.

4 Linear Transformations

Determine whether each of the following transformations is linear. If you determine that a transformation $T : V \to W$ is *not* linear, you must do one of the following:

1. find two elements v_1, v_2 of V of the set such that $T(v_1 + v_2) \neq T(v_1) + T(v_2)$
2. find an element v of V and a real number α such that $T(\alpha v) \neq \alpha T(v)$.

For instance, if I wanted to show that $T : C([0,1]) \to \mathbb{R}$ defined by

$$T(f) = \int_0^1 [f(t)]^2 dt$$

is *not* linear, then I might let $f(t) = 2$ (the constant function whose value is 2), and I might let $\alpha = 3$. Then I would compute $\alpha T(f) = 3\int_0^1 2^2 dt = 12$. But $T(\alpha f) = \int_0^1 (3 \cdot 2)^2 dt = 36 \neq 12$.

A. $T_1 : \mathbb{R}^3 \to \mathbb{R}$ defined by $T_1(a, b, c) = a + b + c$.

- This is a linear transformation.

B. $T_2 : C([0,1]) \to \mathbb{R}$ defined by $T_2(f) = f(1) - f(0)$

- This is a linear transformation.

C. $T_3 : \{\text{polynomials on } [0,1] \text{ of order at most } 3\} \to \mathbb{R}^2$ defined by $T_3(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = (a_0, a_2)$. Here you must consider any "missing terms" to just be terms where the coefficient is zero. As an example, $2 + x + 3x^3 = 2 + x + 0x^2 + 3x^3$, so $T_3(2 + x + 3x^3) = (2, 0)$.

- This is a linear transformation.

D. $T_4 : C(\mathbb{R}) \to C(\mathbb{R})$ defined by $T_4(f)(x) = f(f(x))$ (i.e. if $f(x) = \sin(x)$, then $T_4(f)(x) = \sin(\sin(x))$).

- Let $f(x) = x^2$ and $g(x) = 1$. Then $T_4(f + g) = (x^2 + 1)^2 + 1$ But $T_4(f) + T_4(g) = x^4 + 1$, so T is not linear.

E. $T_5 : \mathbb{R}^3 \to \mathbb{R}$ defined by $T_5(a, b, c) = a + b + c + 1$.

- Let $v = (0, 0, 0)$ and $\alpha = 0$. Then $\alpha T(v) = 0$ but $T(\alpha v) = 1$