Goals:

- → Direction fields for systems w/ complex e-vals
- → All 2x2 Dir fields
- → General Q's

$x' = Ax$, $A = \begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix}$ $\lambda = -2 \pm i$

$-2+i \rightarrow \begin{pmatrix} 1 \\ i \end{pmatrix}$

$-2-i \rightarrow \begin{pmatrix} 1 \\ -i \end{pmatrix}$

Recall Let $f(t) = e^{(-2+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix}$

$g(t) = e^{(-2-i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

Then

(A) $\bar{f} = g$ & $\bar{g} = f$

(B) General Solution (complex) is $c_1 f(t) + c_2 g(t) = c_1 f(t) + c_2 \bar{f}(t)$

(C) If $\bar{c_1} = c_2$ (same as $c_1 = \bar{c_2}$)

Then is Real

$\{f, g\} \rightarrow \{\underbrace{\tfrac{1}{2}f + \tfrac{1}{2}g}_{a(t)}, \underbrace{-\tfrac{i}{2}f + \tfrac{i}{2}g}_{b(t)}\}$

Both of these are a fundamental set of Solns.

But: Right one will always be Real-valued

Follows from: If $z = a + bi$

Then $\begin{cases} a = \tfrac{1}{2}z + \tfrac{1}{2}\bar{z} \\ b = -\tfrac{i}{2}z + \tfrac{i}{2}\bar{z} \end{cases}$

Note: In notation from last lecture $a(t) = k(t)$, $b(t) = l(t)$

write $f(t) = a(t) + i b(t)$

$\Rightarrow \{a(t), b(t)\}$ is a real fundamental set of Solns.

In general $c_1 e^{(a+bi)t} v_1 + c_2 e^{(a-bi)t} v_2$

$= e^{at}\left[c_1 e^{bit} v_1 + c_2 e^{-bit} v_2\right]$

Recall $e^{bit} = \cos(bt) + i\sin(bt)$

$e^{-bit} = \cos(bt) - i\sin(bt)$

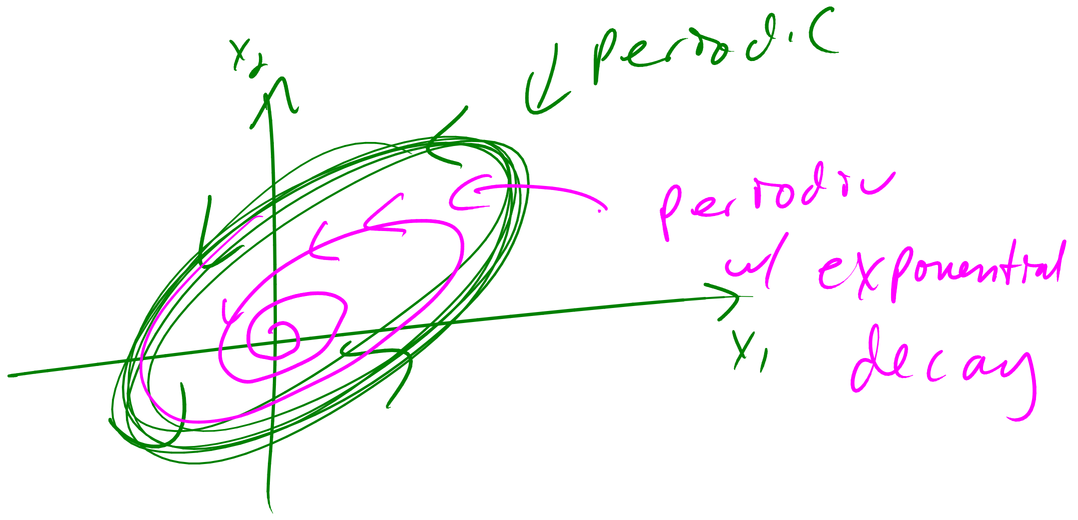
So

$$\underbrace{e^{at}}_{(\text{exponential})}\left[c_1 e^{bit} + c_2 e^{-bit}\right]$$

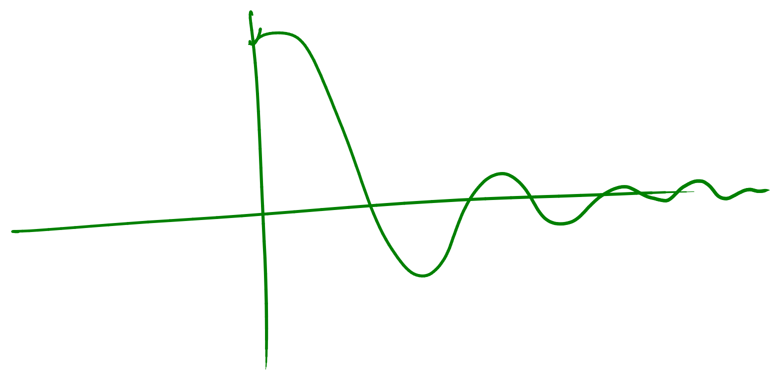
$(\text{exponential}) \cdot (\text{Periodic})$

Consider

$$\underbrace{e^{-2t}}_{\text{exponential decay [i.e. goes to } 0 \text{ as } t \to \infty]}\underbrace{\left[c_1 e^{it}\begin{pmatrix}1\\i\end{pmatrix} + c_2 e^{-it}\begin{pmatrix}1\\-i\end{pmatrix}\right]}_{\text{periodic}}$$

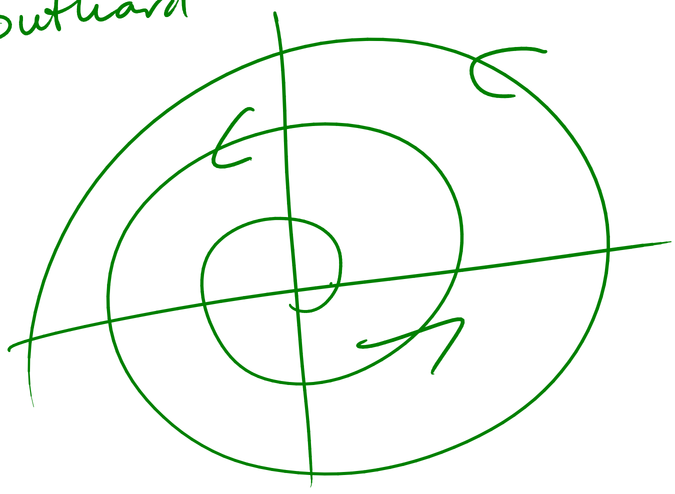


$x_1(t)$:

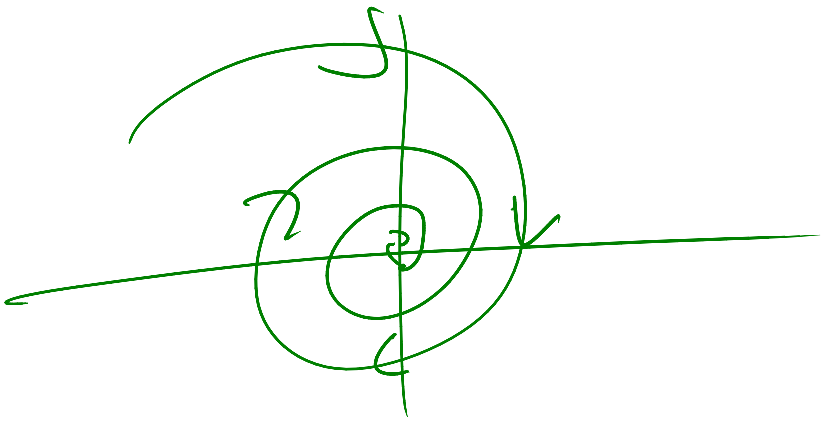


$a \pm bi$ e vals

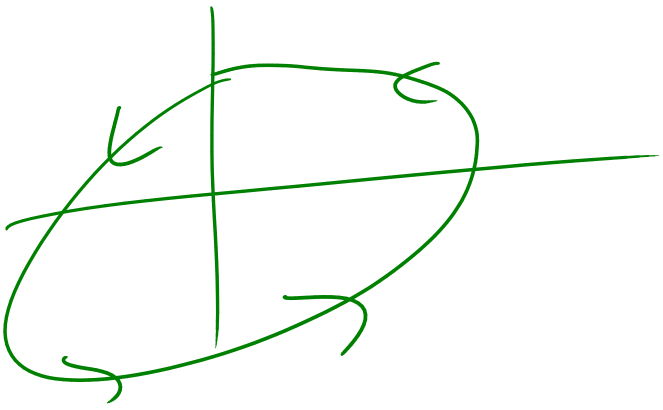
$a > 0 \Rightarrow$ outward



$a < 0 \rightarrow$ inward spiral



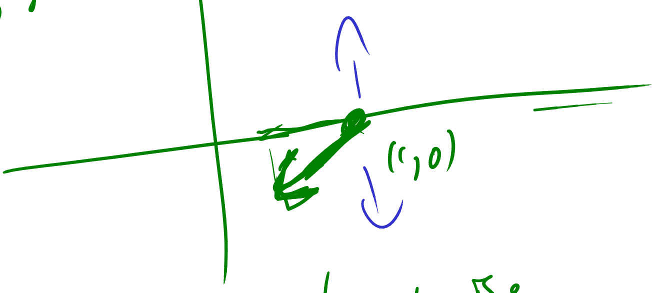
$a = 0$



Clockwise vs. Counterclockwise

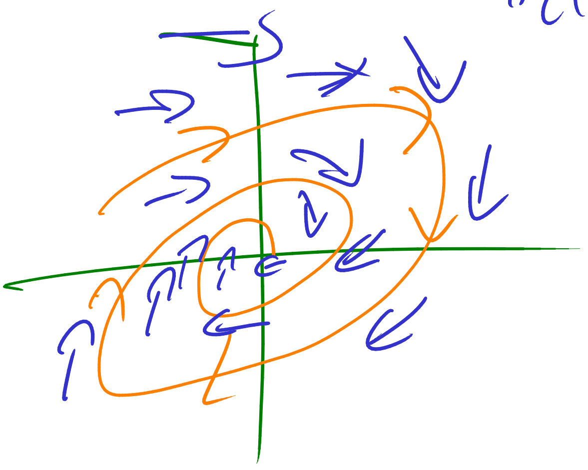
Eg:

$$A = \begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$



So has to be clockwise

Approx



"clockwise inward spiral"

If $A = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix}$

w/ Complex e-vals

then if $\begin{cases} A_{12} > 0 \Rightarrow \text{Counterclockwise} \\ A_{12} < 0 \Rightarrow \text{clockwise} \end{cases}$

If $A_{12} = 0 \Rightarrow A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}A_{11}\\0\end{pmatrix} = A_{11}\begin{pmatrix}1\\0\end{pmatrix}$

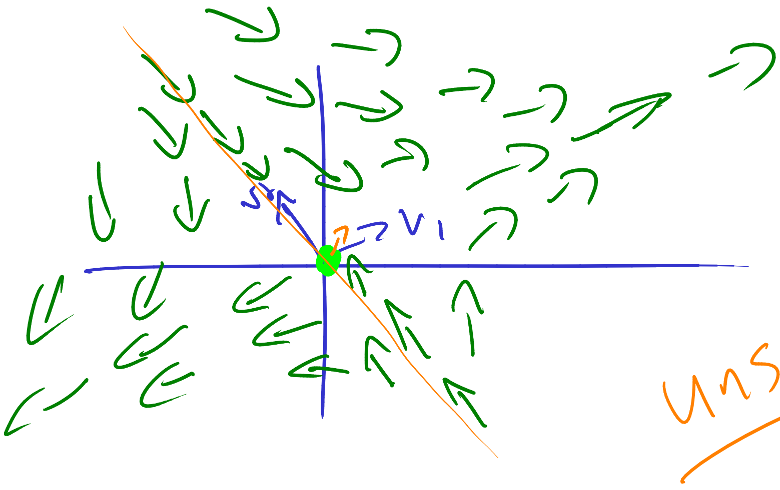
which can't happen w/ Complex e-vals

Stability of Steady state

2x2 ODE
$x' = Ax \Rightarrow \underline{x(t) = \vec{0}}$ is a steadystate

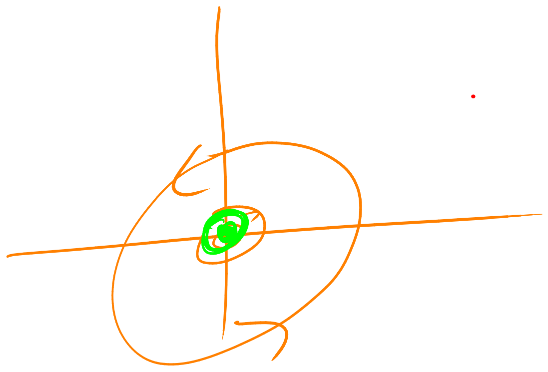
- If Real part of any e-val of A is positive $\Rightarrow$ "unstable"
- If Real part of Both e-vals neg. Then "asymptotically Stable"
- If $\pm bi \Rightarrow$ "Stable"

For ex. If $\lambda_1 = 3$, $\lambda_2 = -1$



unstable

$\lambda = \pm bi$



$\lambda_1 = -1, \quad \lambda_2 = -2$