Homework 2 Solution

Due: Monday, April 12, at 11:00 PM Seattle time on Gradescope

1 Inner Products

Indentify which of the following are inner products. For the ones that are *not* inner products, show a property of inner products that they do not satisfy (i.e. `symmetric`, `linear in the first argument`, or `positive definite`). As an example, consider $\langle , \rangle_0$ defined on $C([0,1])$ by

$$\langle f, g \rangle_0 = f(1/2)g(1/2)$$

Then $\langle , \rangle_0$ is symmetric, and linear in the first argument (can you show this?) but it is *not* positive definite. Let $f(x) = x - 1/2$. Then f is not identically zero, but $\langle f, f \rangle_0 = 0$.

A. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Define $\langle , \rangle_1$ on $\mathbb{R}^2$ by

$$\langle v, w \rangle_1 = (Av) \bullet (Aw)$$

- This is an inner product

B. Let $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Define $\langle , \rangle_2$ on $\mathbb{R}^2$ by

$$\langle v, w \rangle_2 = (Bv) \bullet (Bw)$$

- This is not an inner product. It is not positive definite. To see this, let $v = (0, 1)$. Then $\langle v, v \rangle_2 = 0$.

C. Define $\langle , \rangle_3$ on $C^1([0,1])$ by

$$\langle f, g \rangle_3 = \int_0^1 f'(s)g'(s)ds$$

- This is not an inner product. It is not positive definite. To see this, let $f(x) = 1$ Then $\langle f, f \rangle_3 = 0$.

D. Define $\langle , \rangle_4$ on $C([0,1])$ by

$$\langle f, g \rangle_4 = \int_0^1 (1 + s^2) f(s) g(s) ds$$

- This is an inner product

E. Define $\langle , \rangle_5$ on $C([0,1])$ by

$$\langle f, g \rangle_5 = \int_0^1 f(s) g(s) ds + f(1/2)$$

- This is not an inner product. It is not symmetric. To see this, let $f(x) = 1$ and $g(x) = 1$. Then $\langle f, g \rangle_5 = 1$ but $\langle g, f \rangle_5 = 0$. It is also not positive definite, although this is a bit trickier to find a counterexample. If you let $h(x) = (x - 1/2)^2 - 1/4$ then $\langle h, h \rangle_5 = -13/60$.

2 Initial Value Problems

For each of the following initial value problems

- Solve the IVP
- Compute $\lim_{t \to \infty} |x(t)|$, for the solution $|x(t)|$.

You may use a computer to diagonalize matrices and solve for coefficients, once you are confident with your skills. In particular, the coefficients in (D.) and (F.) are tricky to solve for by hand.

A.

$$\begin{cases} x' &= \begin{pmatrix} 11 & -6 \\ 18 & -10 \end{pmatrix} x \\ x(0) &= (4, 7) \end{cases}$$

- $x(t) = 1e^{2t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + 2e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
- The limit is ∞

B.

$$\begin{cases} x' &= \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix} x \\ x(0) &= (3, -2) \end{cases}$$

- $x(t) = -1e^{3t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} - 1e^{2t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$
- The limit is ∞

C.

$$\begin{cases} x' &= \begin{pmatrix} 8 & 10 \\ -5 & -7 \end{pmatrix} x \\ x(0) &= (1, -1) \end{cases}$$

- $x(t) = e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
- The limit is 0

D.

$$\begin{cases} x' &= \begin{pmatrix} 8 & 10 \\ -5 & -7 \end{pmatrix} x \\ x(0) &= (1.01, -1) \end{cases}$$

- $x(t) = 0.01e^{3t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} + 0.99e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
- The limit is ∞

E.

$$\begin{cases} x' &= \begin{pmatrix} -4 & -2 \\ 1 & -1 \end{pmatrix} x \\ x(0) &= (2, -1) \end{cases}$$

- $x(t) = e^{-3t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$
- The limit is 0

F.

$$\begin{cases} x' &= \begin{pmatrix} -4 & -2 \\ 1 & -1 \end{pmatrix} x \\ x(0) &= (2.01, -1) \end{cases}$$

- $x(t) = 1.01e^{-3t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} + 0.01e^{-2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$