Lecture 8 Notes

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Contents

1 Last time

- Complex Eigenvalues
- Complex "looking" solution

$$x(t) = (3-4i)e^{(-2+i)t}\begin{pmatrix}1\\i\end{pmatrix} + (3+4i)e^{(-2-i)t}\begin{pmatrix}1\\-i\end{pmatrix}$$

Let

$$f(t) = (3-4i)e^{(-2+i)t}\begin{pmatrix}1\\i\end{pmatrix}$$
$$g(t) = (3+4i)e^{(-2-i)t}\begin{pmatrix}1\\-i\end{pmatrix}$$

Then $g(t) = \overline{f(t)}$, so $x(t) = f(t) + \overline{f(t)}$ is *real*

2 Fundamental Set of Solutions

Let $x' = Ax$ be a 2×2 system. Then there will be two linearly independent solutions $f_1(t), f_2(t)$ that span the set of solutions. In such a situation, we say $\{f_1, f_2\}$ is a **fundamental set of solutions** for $x' = Ax$.

- For example, if λ_1, λ_2 are distinct eigenvalues of A with eigenvectors v_1, v_2. Then $\{e^{\lambda_1 t}v_1, e^{\lambda_2 t}v_2\}$ is a fundamental set of solutions.

Claim: if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible, then $\{af_1 + bf_2, cf_1 + df_2\}$ is a fundamental set of solutions if and only if $\{f_1, f_2\}$ is a fundamental set of solutions.

Proof: Try this on your own. Idea: anything I can write as a sum of f_1, f_2, I can also write as a sum of $(af_1 + bf_2)$ and $(cf_1 + df_2)$, and vice versa.

Example: $\{e^{\lambda_1 t}v_1 + e^{\lambda_2 t}v_2, e^{\lambda_1 t}v_1 - e^{\lambda_2 t}v_2\}$ is a fundamental set of solutions because $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is invertible.

3 Goal for Today

Let $h(t) = e^{(-2+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $j(t) = e^{(-2-i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix}$ Take $\{h(t), j(t)\}$ and transform into a *real* fundamental set of solution.

3.1 Recall

If $z = a + bi$, then

$$a = \frac{1}{2}z + \frac{1}{2}\overline{z}$$
$$b = \frac{-i}{2}z + \frac{i}{2}\overline{z}$$

In particular a and b are *real*. It follows that if

$$k(t) = \frac{1}{2}h(t) + \frac{1}{2}j(t)$$
$$l(t) = \frac{-i}{2}h(t) + \frac{i}{2}j(t)$$

then k, l are *real* vector valued functions because $\overline{h(t)} = j(t)$.

Exercise: Check that

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-i}{2} & \frac{i}{2} \end{pmatrix}$$

is invertible.

We conclude that $\{k(t), l(t)\}$ forms a real fundamental set of solutions.

4 How to compute k and l?

Observe that from formulas above with a and b, we know that $k(t) = \mathbb{R}\mathrm{e}(h(t))$ and $l(t) = \mathbb{I}\mathrm{m}(h(t))$. So we compute

$$\begin{aligned}
h(t) &= e^{(-2+i)t}\begin{pmatrix}1\\ i\end{pmatrix} \\
&= e^{-2t}e^{it}\begin{pmatrix}1\\ i\end{pmatrix} \\
&= e^{-2t}(\cos t + i\sin t)\begin{pmatrix}1\\ i\end{pmatrix} \\
&= e^{-2t}\begin{pmatrix}\cos t + i\sin t\\ i\cos t + i^2\sin t\end{pmatrix} \\
&= e^{-2t}\begin{pmatrix}\cos t + i\sin t\\ -\sin t + i\cos t\end{pmatrix} \\
&= e^{-2t}\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix} + ie^{-2t}\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}
\end{aligned}$$

So we conclude that

$$\begin{aligned}
k(t) &= e^{-2t}\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix} \\
l(t) &= e^{-2t}\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}
\end{aligned}$$

Exercise: compute k and l manually, using formulas from the previous section, and verify.

5 Using real fundamental set of solutions to solve IVP

$x' = Ax$ with same A as last lecture. Solve $x' = Ax$, $x(0) = (5, 1)$. Then I know

$$\begin{aligned}
x(t) &= c_1 k(t) + c_2 l(t) \\
x(0) &= c_1 k(0) + c_2 l(0) \\
&= c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 5 \\ 1 \end{pmatrix}
\end{aligned}$$

So, $c_1 = 5$, $c_2 = 1$. So the solution is

$$x(t) = 5e^{-2t} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + 1e^{-2t} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$