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Homework 2 Chapter 2

EE 520 Random Processes

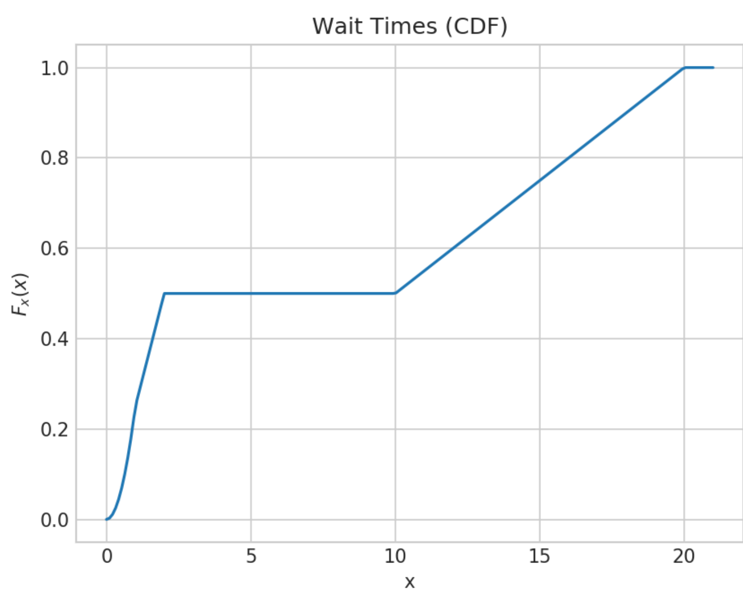
Problems: 2.3, 2.7, 2.9, 2.10, 2.19, 2.22, 2.28, 2.30, 2.31, and 2.38

2.3

In a restaurant known for its unusual service, the time X, in minutes, that a customer has to wait before he captures the attention of a waiter is specified by the following.

$$F_X(x) = \begin{cases} (\frac{x}{2})^2, & \text{for } 0 \leq x \leq 1, \\ \frac{x}{4}, & \text{for } 1 \leq x \leq 2, \\ \frac{1}{2}, & \text{for } 2 \leq x \leq 10, \\ \frac{x}{20}, & \text{for } 10 \leq x \leq 20, \\ 1, & \text{for } x \geq 20, \end{cases}$$

(a) Sketch $F_x(x)$.



(b) Compute and sketch the PDF $f_x(x)$

The PDF is the differential of the CDF, so the derivative of $F_x(X)$ will be the PDF $f_x(x)$.

For this problem, I am assuming that $F_x(x)$ is piecewise differentiable.

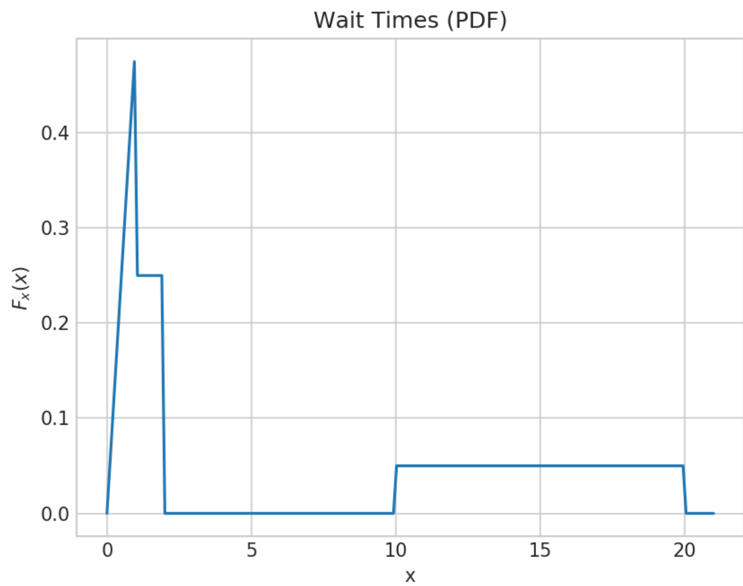
$$\begin{aligned}
\frac{d}{dx}((\frac{x}{2})^2) &= \frac{d}{dx}(\frac{x^2}{4}) \\
&= 2\frac{x}{4} \\
&= \frac{x}{2}
\end{aligned}$$

$$\frac{d}{dx}(\frac{x}{4}) = \frac{1}{4}$$

$$\frac{d}{dx}(\frac{1}{2}) = 0$$

$$\frac{d}{dx}(\frac{x}{20}) = \frac{1}{20}$$

$$F_X(x) = \begin{cases} \frac{x}{2}, & \text{for } 0 \leq x \leq 1, \\ \frac{1}{4}, & \text{for } 1 \leq x \leq 2, \\ 0, & \text{for } 2 \leq x \leq 10, \\ \frac{1}{20}, & \text{for } 10 \leq x \leq 20, \\ 0, & \text{for } x \geq 20, \end{cases}$$



(c) What is the probability that the customer will have to wait (1) at least 10 minutes, (2) less than 5 minutes, (3) between 5 and 10 minutes, (4) exactly 1 minute?

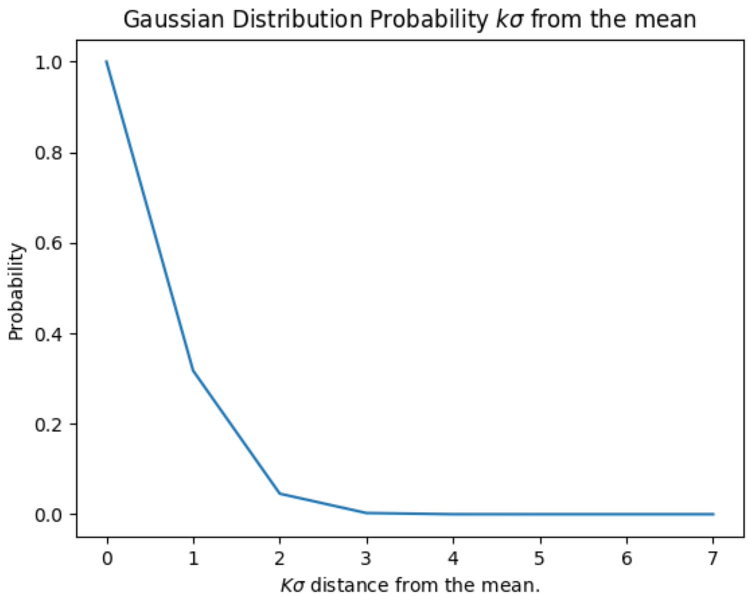
(1) $P(X \geq 10) = P(x \leq \infty) - P(x \leq 10) = F_x(20) - F_x(10) = 1 - 1/2 = 1/2$

(2) $P(X \leq 5) = F_x(5) = 1/2$

(3) $P(5 \leq x \leq 10) = P(x \leq 10) - P(x \leq 5) = 1/2 - 1/2 = 0$

(4) $P(x = 1) = f_x(1) =$ either 1/2, 1/4, or more plausably 0.

Because the likelihood of being served at *exactly* 1 minute is zero.



2.7

A noisy resistor produces a voltage $v_n(t)$. At $t = t_1$, the noise level $X \triangleq v_n(t_1)$ is known to be a Gaussian RV with PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$$

Compute and plot the probability that $|X| > k\sigma$ for $k = 1, 2, \ldots$

2.9

Write PDFs using δ functions for the Bernoulli binomial, and Poisson PMF's.

- Bernoulli
  With probability of success p.

$$\begin{aligned} f_X(x) &= \begin{cases} p^x(1-p)^{1-x}, & \text{for } x = 0, 1 \\ 0, & \text{for all others} \end{cases} \\ &= \delta(x-1)p + \delta(x)(1-p) \end{aligned}$$

- Binomial

$$P[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$$

$$f_X(x) = \sum_{k=0}^{n} \delta(x-k) \binom{n}{k} p^k (1-p)^{n-k}$$

- Poisson

$$P[X = k] = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

$$f_X(x) = \sum_{k=0}^{\infty} \delta(x-k) \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

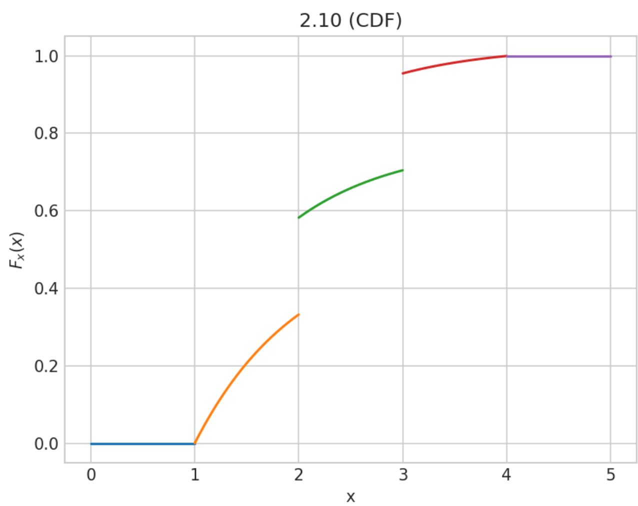
2.10

The PDF of a RV X is shown in figure P2.10. The numbers in the parenthesis indicate area.

(a) Compute the value of A.

$$
\begin{aligned}
1 &= \int_1^4 Ae^{-x}dx + \int_1^4 1/4[\delta(x-2)+\delta(x-3)]dx \\
&= A[(-1)e^{-x}]\Big|_1^4 + 0.5 \\
\frac{1}{2} &= A[-e^{-4}+e] \\
\frac{1}{2} &= A(0.350) \\
A &= \frac{1}{0.70} \\
A &= 1.429
\end{aligned}
$$

(b) Sketch the CDF.



(c) Compute $P[2 \le X < 3]$ From the CDF

$$
\begin{aligned}
P[2 \le X < 3] &= P[X \le 3] - P[X \le 2] \\
&= F_X^-(3) - F_X^-(2) \\
&= 0.372
\end{aligned}
$$

(d) Compute $P[2 < X \le 3]$ The same but with the jump at 3 included instead.
0.372

(e) Compute $F_x(3)$.

Here I used python to calculate the value, but it's simply $\int_1^3 Ae^{-x}dx + 1/2$.

$$F_X(3) = 0.955$$

2.19

It has been found that the number of people Y waiting in a queue in the bank on payday obeys the Poisson law as

$$P[Y = k|X = x] = e^{-x}\frac{x^k}{k!}, k \geq 0, x > 0$$

Given that the normalized serviing time of the teller x is constant. However, the serving time is more accurately modeled as an RV X. For simplicity let X be a uniform RV with

$$f_X(x) = \frac{1}{5}[u(x) - u(x - 5)]$$

Then $P[Y = k|X = x]$ is still Poisson but $P[Y = k]$ is something else. Compute $P[Y = k]$ for $k = 0, 1,$ and 2. The answer for general k may be difficult.

$$\begin{aligned}
P[Y = k] &= \int_{\infty}^{\infty} P[Y = k|X = x]f_X(x)dx \\
&= \int_{\infty}^{\infty} \frac{x^k}{k!}e^{-x}\frac{1}{5}(u(x) - u(x - 5))dx \\
&= \frac{1}{5}\int_0^5 \frac{x^k}{k!}e^{-x}dx
\end{aligned}$$

From here I simply solved numerically in python.

k=1 0.199

k=2 0.192

k=3 0.175

2.22

Consider the joint PDF of X and Y:

$$f_{XY}(x,y) = \frac{1}{3\pi} e^{-\frac{1}{2}[(x/3)^2+(y/2)^2]} u(x)u(y)$$

1. Are X and Y independent RVs?
   Yes, X and Y are independent RVs.

$$f_{XY}(x,y) = \frac{1}{3\pi} e^{-\frac{1}{2}(\frac{X}{3})^2} u(x) e^{-\frac{1}{2}(\frac{y}{2})^2} u(y)$$

   Here, $\frac{1}{3\pi}$ is the equivalent of $\frac{C}{2\pi\sigma_x\sigma_y}$ and $\sigma_x = 3$, $\sigma_y = 2$

$$\begin{aligned}\frac{1}{3\pi} &= \frac{C}{2\pi\sigma_X\sigma_Y} \\ &= \frac{C}{2\pi(3)(2)} \\ &= \frac{C}{\pi 12}\end{aligned}$$

$$\begin{aligned}C_{12} &= \frac{12}{3} \\ &= 4\end{aligned}$$

   Thus C_1 and $C_2 = 2$.

   And finally,

$$f_X(x) = \frac{2}{\sqrt{2\pi}3} e^{-\frac{1}{2}(\frac{x}{3})^2}, \quad \text{and} \quad f_Y(y) = \frac{2}{\sqrt{2\pi}2} e^{-\frac{1}{2}(\frac{y}{2})^2}$$

2. Compute the probability of $\{0 < X \leq 3, 0 < Y \leq 2\}$.

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx$$

$$\begin{aligned}P[\{0 < X \leq 3,\ 0 < Y \leq 2\}] &= \int_0^2 \int_0^3 f_X(x) f_Y(y) dx\ dy \\ &= F_X(3)F_Y(2) \\ &= \frac{1}{3} erf(\frac{x}{3\sqrt{2}}) \frac{1}{2} erf(\frac{y}{2\sqrt{2}}) \\ &= \frac{1}{3} erf(\frac{3}{3\sqrt{2}}) \frac{1}{2} erf(\frac{2}{2\sqrt{2}}) \\ &= erf(\frac{1}{\sqrt{2}}) erf(\frac{1}{\sqrt{2}}) \\ &= 0.466\end{aligned}$$

2.28

Let X be a random variable with a PDF of

$$f_X(x) = \begin{cases} 0, & x < 0, \\ ce^{-2x}, & x \geq 0, \end{cases}$$

for, $c > 0$.

(a) Find c
The pdf must integrate to 1.

$$\begin{aligned} F_X(\infty) = c\int_0^\infty e^{-2x}dx = 1 \\ = c\frac{-1}{2}e^{-2x}\Big|_0^\infty \\ = c\frac{1}{2} \\ c = 2 \end{aligned}$$

(b) Let $a > 0$, $x > 0$, find $P[X \geq x + a]$

$$\begin{aligned} P[X \geq x + a] &= \int_{x+a}^\infty 2e^{-2x}dx \\ &= -e^{-2x}\Big|_{x+a}^\infty \\ &= e^{-2(x+a)} \end{aligned}$$

(c) Let $a > 0$, $x > 0$, find $P[X \geq x + a|X \geq a]$

$$\begin{aligned} P[X \geq x + a|X > a] &= \frac{P[(X \geq x + a) \cap (X \geq a)]}{P[X \geq a]} \\ &= \frac{e^{-2(x+a)}}{e^{-2a}} \\ &= e^{-2x} \end{aligned}$$

2.30

A U.S. defense radar scans the skies for unidentified flying objects (UFOs). Let M be the event that a UFO is present and M^c the event that the UFO is absent. Let $f_{X|M}(x|M) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-r)^2}$ be the conditional pdf of the radar return signal X when a UFO is actually there, and let $f_{X|M^c} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x)^2}$ be the conditional pdf of the radar return signal X when there is no UFO. To be specific, let $r = 1$ and let the alert level be $x_A = 1/2$. Let A denote the event of an alert, that is $X > x_a$.
Here,

$$P[A] = P[X > 1/2] = \int_{1/2}^{\infty} f_X(x)dx$$

$$f_X(x) = f_{X|M^c}(x)P[M^c] + f_{X|M}(x)P[M]$$

Compute

1. $P[A|M] =$
   Here the $P[M] = 1$

$$\begin{aligned} P[A|M] &= \int_{1/2}^{\infty} f_{X|M}(x)dx \\ &= \int_{1/2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-r)^2} dx \\ &= \int_{1/2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-1)^2} dx \\ &= 0.691 \end{aligned}$$

2. $P[A^c|M]$

$$\begin{aligned} P[A^c|M] &= \int_{0}^{1/2} f_{X|M}(x)dx \\ &= \int_{0}^{1/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-r)^2} dx \\ &= 0.309 \end{aligned}$$

3. $P[A|M^c]$

$$\begin{aligned} P[A|M^c] &= \int_{1/2}^{\infty} f_{X|M}(x)dx \\ &= \int_{1/2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x)^2} dx \\ &= 0.309 \end{aligned}$$

4. $P[A^c|M^c]$

$$\begin{aligned} P[A^c|M^c] &= 1 - P[A|M^c] \\ &= 0.691 \end{aligned}$$

2.31

In the previous problem assume that $P[M] = 10^-3$. Compute

$$P[M|A], P[M|A^c], P[M^c|A], P[M^c|A^c]. \text{ Repeat for } P[M] = 10^-6$$

$$P[A] = P[A|M]P[M] + P[A|M^c]P[M^c] \quad = 0.691(10^-3) + 0.309(1 - 10^-3) = 0.3089$$

1. $P[M|A]$

$$\begin{aligned} P[M|A] &= \frac{P[M \cap A]}{P[A]} \\ &= \frac{0.691(10^-3)}{0.3089} \\ &= 0.00224 \end{aligned}$$

2. $P[M^c|A]$

$$\begin{aligned} P[M^c|A] &= 1 - P[M|A] \\ &= 0.998 \end{aligned}$$

3. $P[M|A^c]$

$$\begin{aligned} P[M^c|A] &= \frac{P[M^c \cap A]}{1 - P[A]} \\ &= \frac{0.3085(10^-3)}{0.6911} \\ &= 0.000446 \end{aligned}$$

4. $P[M^c|A^c]$

$$\begin{aligned} P[M^c|A] &= 1 - P[M|A^c] \\ &= 0.9996 \end{aligned}$$

And again with $P[M] = 10^-6$.

$$P[A] = P[A|M]P[M] + P[A|M^c]P[M^c] \quad = 0.691(10^-6) + 0.309(1 - 10^-6) = 0.3085$$

1. $P[M|A]$

$$\begin{aligned} P[M|A] &= \frac{P[M \cap A]}{P[A]} \\ &= \frac{0.691(10^-6)}{0.3089} \\ &= 0.000224 \end{aligned}$$

2. $P[M^c|A]$

$$\begin{aligned} P[M^c|A] &= 1 - P[M|A] \\ &= 0.999998 \end{aligned}$$

3. $P[M|A^c]$

$$\begin{aligned} P[M^c|A] &= \frac{P[M^c \cap A]}{1 - P[A]} \\ &= \frac{0.3085(10^-6)}{0.6911} \\ &= 0.0000446 \end{aligned}$$

4. $P[M^c|A^c]$

$$\begin{aligned} P[M^c|A] &= 1 - P[M|A^c] \\ &= 0.9999996 \end{aligned}$$

2.38

A laser used to scan the bar code on supermarket items is assumed to have a constant conditional failure rate of $\lambda (> 0)$. What is the maximum value of λ that will yield a probability of a first breakdown in 100 hours of operation less than or equal to 0.05?

Because λ is constant, it can be assumed that $F_X(x) = 1 - e^{-\lambda t} u(t)$

$$
\begin{aligned}
P[X \leq 100] &\leq 0.005 \\
F_X(100) &= 1 - e^{-\lambda 100} \\
0.95 &\leq e^{-\lambda 100} \\
\ln(0.95) &\leq \ln(e^{-\lambda 100}) \\
-0.0513 &\leq -\lambda 100 \\
\lambda 100 &\leq 0.0513 \\
\lambda &\leq \frac{0.0513}{100} \\
\lambda &\leq 0.000513
\end{aligned}
$$

Here the solution says it should be negative, but I believe it actually should not.

The solutions at one point write $\frac{-\ln(0.95)}{100} = -0.000513$, but $\ln(0.95) = -0.05129$.