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Homework 2

MTH 361 - Introduction to ProbabilitySection 1.7 39, 41, 47, 53, and 55.

1.7 #39

Exercise

How many times should a coin be tossed so that the probability of at least one head is $\geq 99\%$?

Solution

Complement

The probability of flipping 1 or more heads is the complement of the probabilty of flipping 0 heads.

$$P(X \geq 1) = 1 - P(X = 0)$$

Applied

The probablity of flipping 0 heads is the probabilty of flipping n tails in a row.

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ 0.99 &\leq 1 - (\frac{1}{2})^n \\ -0.01 &\leq -(\frac{1}{2})^n \\ 0.01 &\leq (\frac{1}{2})^n \\ n &\geq log_{\frac{1}{2}}(0.01) \\ n &\geq 6.64 \\ n &= 7 \end{aligned}$$

And as verification.

$$
\begin{aligned}
P(X \geq 1) &\stackrel{?}{\geq} 1 - P(X = 0) \\
0.99 &\stackrel{?}{\geq} 1 - (\frac{1}{2})^7 \\
&\stackrel{?}{\geq} 1 - 0.0078 \\
&\stackrel{\checkmark}{\geq} 0.9922
\end{aligned}
$$

Thus 7 trials gurantee at least a 99% chance of flipping at least one heads.

1.7 #41

Exercise

A bet is said to carry 3 to 1 odds if you win \$3 for every \$1 you bet. What must the probability of winning be for this to be a fair bet?

Solution

Fair Bet

A fair bet is taken to mean an expectation of zero.

Odds

Setup the expectation equation.

$$
\begin{aligned}
E[X] &= \$3P(\alpha) - \$1P(\alpha^c) \\
\$0 &= \$3P(\alpha) - \$1P(\alpha^c)
\end{aligned}
$$

Account for total probability of 1.

$$1 = P(\alpha) + P(\alpha^c)$$

Solve for $P(\alpha^c)$

$$P(\alpha^c) = 1 - P(\alpha)$$

Insert into the first equation.

$$
\begin{aligned}
\$0 &= \$3P(\alpha) - \$1(1 - P(\alpha)) \\
0 &= 3P(\alpha) - 1 + 1P(\alpha) \\
1 &= 4P(\alpha) \\
P(\alpha) &= \frac{1}{4}
\end{aligned}
$$

P(Winning)

Thus the probabilty of winning must be $\frac{1}{4}$.

Check

$$\begin{aligned}\$0 &\stackrel{?}{=} \$3(\frac{1}{4}) - \$1(\frac{3}{4}) \\ &\stackrel{\checkmark}{=} \$0.75 - \$0.75\end{aligned}$$

1.7 #47

Exercise

Five people play a game of 'odd man out" to determine who will pay for the pizza they ordered. Each flips a coin. If only one person gets heads (or tails) while the other four get tails (or heads) then he is the odd man and has to pay. Otherwise, they flip again. What is the expected number of tosses needed to determine who will pay?

Solution

P(Game End)

The probability of successfully deciding who pays is the number of ways the game can end divide by the total number of outcomes.

Total Outcomes

Each coin can be either heads or tails.

$$\begin{aligned}|\Omega| &= 2^n \\ &= 2^5 \\ &= 32\end{aligned}$$

Number of Game Ending Events

The nuber of ways the game can end is the number of ways to place either a tails or a heads in a set of their opposites.

$$\begin{aligned}|E| &= 2(\text{Number of Ways to Place in 5}) \\ &= 2 * 5 \\ &= 10\end{aligned}$$

Game Ending Probability

Thus the probability of the game ending on any one round is $\frac{10}{32} = \frac{5}{16}$.

Geometric Distribution

This is related to the geometric distribution wherin X is the number of trials before a success and p is the probabilty of success.

$$P(X = k) = p(1 - p)^{k-1}$$

Expected Value of the Geometric Distribution

$$\begin{aligned}
\mu_x &= \sum_{k=1}^{\infty} (\text{Value k}) P(X = k) \\
&= \sum_{k=1}^{\infty} kp(1 - p)^{k-1} \\
&= p \sum_{k=1}^{\infty} k(1 - p)^{k-1} \\
&= p[1(1 - p)^0 + 2(1 - p)^1 + 3(1 - p)^2 + ...] \\
\mu_x(1 - p) &= p[1(1 - p)^1 + 2(1 - p)^2 + 3(1 - p)^3 + ...]
\end{aligned}$$

Subtract the above two lines...

$$\begin{aligned}
\mu_x - \mu_x(1 - p) &= p[(1 - p)^0 + (1 - p)^1 + (1 - p)^2 + ...] \\
\mu_x(1 - (1 - p)) &= p[(1 - p)^0 + (1 - p)^1 + (1 - p)^2 + ...] \\
\mu_x &= [(1 - p)^0 + (1 - p)^1 + (1 - p)^2 + ...]
\end{aligned}$$

Which is an infinite geometric series with $\sum = \frac{1}{1-r}$

$$\begin{aligned}
\mu_x &= \frac{1}{1 - (1 - p)} \\
&= \frac{1}{p}
\end{aligned}$$

Expected Number of Rounds

The probability of a game ending event is $p = \frac{5}{16}$. Therefore the expected number of rounds before a game ending event is $E_x = \frac{1}{\frac{5}{16}} = \frac{16}{5} = 3.2$ rounds.

1.7 #47

Exercise

The Elm Tree golf course in Corland, NY is a par 70 layout with 3 par fives, 5 par threes, and 10 par fours. Find the mean and variance of par on this course.

Solution

Expectation

The total number of holes is required to find the expectation.

$$\begin{aligned} \sum &= 3 + 5 + 10 \\ &= 18 \end{aligned}$$

The mean is the total par over the total number of holes.

$$\begin{aligned} \mu_x &= \frac{70}{18} \\ &= 3.889 \end{aligned}$$

Variance

$$\begin{aligned} \sigma_x^2 &= \sum (x_i - \mu_x)^2 p_i \\ &= (5 - 3.889)^2 \frac{3}{18} + (3 - 3.889)^2 \frac{5}{18} + (4 - 3.889)^2 \frac{10}{18} \\ &= (1.111)^2 \frac{3}{18} + (-0.889)^2 frac518 + (0.111)^2 \frac{10}{18} \\ &= 0.2056 + 0.220 + 0.007 \\ &= 0.4326 \end{aligned}$$

1.7 #55

Exercise

Can we have a random variable with $E[X] = 3$ and $E[X^2] = 8$?

Solution

Reasoning

An expectation of 3 means that all of the values times their probability summed equals 3.

A second moment of 8 means that those values squared times their respective probabilities summed equals 8.
I don't see why this wouldn't be doable.

Attempt to Fabricate Example

$$\begin{aligned}
\mu_x &= 2(1/3) + 2(1/3) + 2(1/3) \\
E[X^2] &= 3(3/4) + 1(1/4) \\
E[X^2] &= 16(3/4)
\end{aligned}$$

After a few attempts it becomes apparent that I don't actually know an easy way to solve this. It may or may not be possible, but I believe it is as there are so many possible combinations of outcomes and probabilities. I will have to ask the solution after class.