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Homework 5
MTH 361 - Introduction to Probability

Exercise 1

Starting with \$2, you play a game where there is an equal chance of winning or losing. If you win, you gain \$1, and if you lose, you lose \$1. You continue playing until you win \$4, or lose all your money.
This is a version of "Gambler's Ruin".

(a) Write down a Markov chain describing this situation.

The state space is $\{\$0, \$1, \$2, \$3, \$4\}$

The initial distribution is $\boldsymbol{i} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ as the gambler starts with \$2.

The transition matrix is $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

With sink states in rows 0 and 4 representing the fact that once our gambler reaches either \$0 or \$4, he stops gambling.

$$f_x(x|n=2) = \boldsymbol{i}\boldsymbol{T}^2$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0.25 & 0 & 0.25 & 0 \\ 0.25 & 0 & 0.5 & 0 & 0.25 \\ 0 & 0.25 & 0 & 0.25 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.25 & 0 & 0.5 & 0 & 0.25 \end{bmatrix}$$

(b) The chance of going broke after two rounds is $1/4$.

(c) The chance that the gambler is still playing is $1/2$ in which case he still has \$2.

Excercise 2

There are two habitats that a certain kind of migratory bird likes to occupy. When you start studying these birds, 30% are in habitat A and 70% are in habitat B. Each year, the birds migrate according to a Markov chain with transition matrix:

$$\boldsymbol{P} = \begin{bmatrix} 7/8 & 1/8 \\ 2/8 & 6/8 \end{bmatrix}$$

(a) What fraction of the population is in each habitat after 1 year?

Here the initial distribution $\boldsymbol{i} = \begin{bmatrix} 3/10 & 7/10 \end{bmatrix}$.

$$f_x(x|n=1) = \boldsymbol{i}\boldsymbol{P}^1$$
$$= \begin{bmatrix} 3/10 & 7/10 \end{bmatrix} \begin{bmatrix} 7/8 & 1/8 \\ 2/8 & 6/8 \end{bmatrix}$$
$$= \begin{bmatrix} 0.4375 & 0.5625 \end{bmatrix}$$

As such, 43.75% are in habitat A and 56.25% are in habitat B.

(b) Which of the following describes the equilibrium distribution for this Markov chain?

$$\pi_1 = \begin{bmatrix} 3/4 & 1/4 \end{bmatrix} \qquad \text{or} \qquad \pi_2 = \begin{bmatrix} 2/3 & 1/3 \end{bmatrix}$$

Here π_2 describes the equlibrium distribution for this Markov chain. The reason is when multiplied by $\boldsymbol{P}$ the output remains π_2.

(c) As $n_{years} \to \infty$ the distribution of birds between the two habitats eventually reach the equilibrium distribution of π_2.

Exercise 3

At a certain college, 63% of premed students switch to liberal arts before graduation and 18% of liberal arts majors switch to premed. Given that 60% of incoming students are premed when they matriculate and 40% are liberal arts majors, what percent of the graduating class will be premed?

Here, the initial distribution $i = \begin{bmatrix} 6/10 & 4/10 \end{bmatrix}$.

And the transition matrix $P = \begin{bmatrix} 0.37 & 0.63 \\ 0.18 & 0.82 \end{bmatrix}$

The transition is simply one time step or $n = 1$.

$$f_X(x|n = 1) = iP^1$$
$$= \begin{bmatrix} 6/10 & 4/10 \end{bmatrix} \begin{bmatrix} 0.37 & 0.63 \\ 0.18 & 0.82 \end{bmatrix}$$
$$= \begin{bmatrix} 0.294 & 0.706 \end{bmatrix}$$

Here the percent that will graduate premed is 29.4%.

Exercise 4

Classify the following Markov chains as irreducible, aperiodic, both, or neither.

(a)
$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

This is both irreducible and periodic.

It is irreduccible as both states communicate.

The state always return to itself with a period of $n = 2$

(b)
$$P = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix}$$

This isn't irreducible as it contains two states in which one is a sink. This means the two states cannot be viewed as one.

This is aperiodic as it can stay in the same state over one transition.

(c)
$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

This is irreducible as all states communicate.

This is aperiodic as it can return to state B in any number of steps greater than 1 and contains states that can return to themselves.

Exercise 5

A given stock behaves according to the following Markov chain. At the end of any day, the stock price may increase, stay the same, or decrease. Call these states I, S, and D respectively.

$$P = \begin{bmatrix} 2/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 2/3 \end{bmatrix}$$

Given that a certain stock started at had increased on the previous day, what is the probability that the stock will not decrease 3 days from now?

The input distribution $\boldsymbol{i} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

This is simply $P(x \neq D | n = 3) = 1 - P(x = D | n = 3)$.

where

$$f_X(x | n = 3) = \boldsymbol{i} \boldsymbol{p}^3$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.48 & 0.33 & 0.19 \\ 0.33 & 0.33 & 0.33 \\ 0.19 & 0.33 & 0.48 \end{bmatrix}$$

$$= \begin{bmatrix} 0.48 & 0.33 & 0.19 \end{bmatrix}$$

The above is rounded, so the actual value of $P(x = D | n = 3) = 0.\overline{185}$

Finally $P(x \neq D | n = 3) = 1 - 0.\overline{185} = 0.\overline{814}$