Probability and Stochastic Process
by Joshua Reed, page 1 of 2

Events and their operations

An event is a collection of outcomes of a random experiment

$S = \{$collection of all outcomes of the experiment$\}$

$\phi = \{$empty set$\}$

If $A \cap B = \phi$,
then A and B are mutually exclusive events

DeMorgan's $\overline{(A \cup B)} = (\overline{A} \cap \overline{B})$

Probability–Axioms and Properties

Axioms

I. $P(A) \geq 0$

II. $P(S) = 1$

III. If $(A \cap B) = \phi$,
then $P(A \cup B) = P(A) + P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(\overline{A}) = 1 - P(A)$

Independence

If $P\{A \cap B\} = P\{A\}P\{B\}$,
then A and B are independent

If $P(A \cap B|C) = P(A|C)P(B|C)$,
A and B are **conditionally** independent given event C

Mutually Exclusivity

If $P\{A \cap B\} = \phi$,
then A and B are M.E. And, in this case
$P(A|B) = P(A)$ and $P(B|A) = P(B)$

Conditional Probability

$P(A|B) = P(A \cap B)/P(B)$ $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$

Bayes' rule

$P(B|A) = P(A|B)P(B)/P(A)$,

Total Probability

If $B_i \cap B_j = \phi$ and
$B_1 \cup B_2 \cup \ldots \cup B_{n-1} \cup B_n = S$,
then: $P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$

Bayes for this situation,
$P(B_k|A) = P(A|B_k)P(B_k)/P(A)$,

Binomial Distribution

General

X = the number of successes in n trials.

Mass Function

$P\{X = k\} = \binom{n}{k} p^k q^{n-k}$, for
$k = 0, 1, 2, \ldots, n$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Mean

$m_x = np$

Variance

$Var(x) = np(1-p)$

Geometric Distribution

General

X is the number of trials before the first success

p is the probability of success

Mass Function

$P\{X = k\} = p(1-p)^{k-1}$, for
$k = 1, 2, 3, \ldots$

Mean

$m_x = 1/p$

Variance

$Var(X) = (1-p)/p^2$

Poisson Distribution

General

X is the number of arrivals in a time interval t

λ is the arrival rate

Mass Function

$P\{X = k\} = \frac{(\lambda t)^k}{k!} e^{\lambda t}$, for
$k = 0, 1, 2, 3, \ldots$

Mean and Variance

$m_x = Var(X) = \lambda t$

Uniform Distribution

General

X has equal likeliness of taking any value in the interval $[a, b]$

Probability Density Function

$f_x(u) = \frac{1}{b-a}$, for $a < u < b$, and is 0 elsewhere

Cumulative Distribution Function

$$F_x(u) = \begin{cases} 0, & u < a \\ \frac{u-a}{b-a}, & a < u < b \\ 1, & b < u \end{cases}$$

Mean

$m_x = (a+b)/2$

Variance

$Var(X) = \frac{(b-a)^2}{12}$

Triangular Distribution

General

Upon adding two uniform distributions, we get the triangular density function.
The function only has value over $[2a, 2b]$

Density

$$f_x(\alpha) = \begin{cases} \frac{\alpha - 2a}{(b-a)^2}, & 2a < \alpha < (a+b) \\ \frac{2b-\alpha}{(b-a)^2}, & (a+b) < \alpha < (2b) \\ 0, & otherwise \end{cases}$$

Cumulative Distribution

This was not listed in the summary, and I need to review to understand why.

Gaussian Distribution

General

The normal distribution

Probability Density Function

With mean m and standard deviation σ

$f_x(u) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(u-m)^2}{2\sigma^2}}$

Unit Gaussian (normal) $\sigma = 1$, $m = 0$

$f_x(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}$

Unit Gaussian Cumulative Distribution

$\phi(u)$

$\phi(u)$ was used to compute the following

$P\{a < X < b\} = \phi(\frac{b-m}{\sigma}) - \phi(\frac{a-m}{\sigma})$

Mean and Variance

m and σ are the mean and standard deviation

σ_x^2 is the variance

Exponential Distribution

General

X is the time to arrival or time to failure, where arrival rate is λ

X can also be viewed as departure time with departure rate μ

Probability Density Function

$f_x(t) = \lambda e^{-\lambda t}$, for $t > 0$, and is 0 elsewhere.

Cumulative Distribution Function

$F_x(t) = 1 - e{-}\lambda t$, for $t \geq 0$, and is 0 elsewhere.

Mean and Variance

$m_x = \sigma_x = 1/\lambda$

Expectations

General

The expectation E of a function g of a random variable x, $E\{g(X)\}$:

$E\{g(X)\} = \int_{-\infty}^{\infty} g(u) f_x(u) du$

A sum can be substituted for the integral in the discrete case unless using impulse functions

Properties

$E\{C\} = C$

$E\{ag(X) + bh(X)\} = aE\{g(x)\} + bE\{h(X)\}$

If $g(X) \geq 0$, then $E\{g(X)\} \geq 0$

Mean and Variance

Mean of X

$m_x = E\{X\} = \int_{-\infty}^{\infty} u f_x(u) du$

Variance of X

$\sigma_x^2 = E\{[X - m_x]^2\} = E\{X^2\} - m_x^2$

Properties

If $Y = aX + b$,
then $m_y = am_x + b$
and $\sigma_y^2 = a^2 \sigma_x^2$

Functions of Random Variables

A Single Random Variable

If $Y = g(X)$, where X is a random variable, then

$f_y(v) = P\{Y \leq v\} = P\{g(X) \leq v\}$

If $g(u)$ is monotonic, then

$f_y(v) = [\frac{f_x(u)}{g'(u)}]_{u = g^{-1}(v)}$

Two Random Variables

Joint distribution function of X and Y

$F_{XY}(u, v) = P\{X \leq u, Y \leq v\}$

Properties

$P\{(a < X \leq b) \; and \; (c < Y \leq d)\} = F_{XY}(b, d) - F_{XY}(a, d) - F_{XY}(b, c) + F_{XY}(a, c) \geq 0$

$F_{XY}(-\infty, v) = 0, F_{XY} = (u, -\infty) = 0$

$F_{XY}(\infty, \infty) = 1$

Marginal Distributions

$F_{XY}(u, \infty) = F_X(u)$

$F_{XY}(\infty, v) = F_y(v)$

Joint Probability Mass Function

$p_{XY}(a_i, b_j) = P\{X = a_i, Y = b_j\}$,
where X and Y take values $\{a_i\}$ and $\{b_i\}$

Joint Probability Density Function

$f_{XY}(u, v) = \frac{\delta^2 F_{XY}(u,v)}{\delta u \delta v}$

Properties

$f_{XY}(u, v) \geq 0$

$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(u, v) du \, dv = 1$

$P\{(a < X \leq b \; and \; (c < Y \leq d)\} = \int_c^d \int_a^b f_{XY}(u, v) du \, dv = 1$

$F_{XY}(b, d) = \int_{-\infty}^{d} \int_{-\infty}^{b} f_{XY}(u, v) du \, dv$

Marginal Densities

$f_X(u) = \int_{-\infty}^{\infty} f_{XY}(u, v) dv$, and

$f_Y(v) = \int_{-\infty}^{\infty} f_{XY}(u, v) du$

Independent Random Variables

$f_{XY}(u, v) = f_X(u) f_Y(u)$

$F_{XY}(u, v) = F_X(u) F_Y(v)$

Conditional Densities

$f_{X|A}(u) = \frac{d}{du} P\{X \leq u|A\} = \frac{d}{du} P\{(X \leq u) \cap A\}/P\{A\}$

Two Cases

$A = a < X \leq b : f_{X|A}(u|A) = f_x(u)/P\{A\}$,
for $a < u \leq b$, and 0 elsewhere

$A = \{Y = v\} : f_{X|Y}(u|v) = f_{XY}(u, v)/f_y(v)$

The second way can be represented in two ways

$f_{XY}(u, v) = f_{X|Y}(v|u) f_y(v) = f_{Y|X}(u|v) f_X(u)$

Total Probability and Bayes' for Random Vars

$f_x(u) = \int_{-\infty}^{\infty} f_{X|Y}(u|v) f_Y(v) dv$

$f_{Y|X}(v|u) = f_{X|Y}(u|v) f_Y(v)/f_X(v)$

In the discrete case the integrals can be replaced by sums, and the densities can be replaced by probabilities

Jointly Gaussian Rondom Variable

Placeholder

Conditional Densities

Placeholder

Functions of Two Random Variables

Expectations

$E\{g(X, Y)\} =$

Discrete Case

$\sum_j \sum_i g(a_i, b_j) P\{X = a_i, Y = b_j\}$

Continuous Case

$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(u, v) f_{XY}(u, v) du \, dv$

Properties

1. $E\{C\} = C$
2. $E\{ag(X, Y)\} = aE\{g(X, Y)\}$
3. $E\{g(X, Y) + h(X, Y)\} = E\{g(X, Y)\} + E\{h(X, Y)\}$
4. If $g(X, Y) \geq 0$, then $E\{g(X, Y)\} \geq 0$
5. IF X and Y are independent, then $E\{g(X)h(Y)\} = E\{g(X)\}E\{h(Y)\}$

Correlation and Covariance

Correlation between X and Y:

$R_{XY} = E\{X, Y\}$

Covariance of X and Y:

$Cov(X, Y) = C_{XY} = E\{(X - m_x)(Y - m_y)\} = R_{XY} - m_x m_y$

Correlations Coefficient:

$\rho_{XY} = C_{XY}/(\sigma_x \sigma_y) - 1 \leq \rho_{XY} \leq 1$

If $\rho_{XY} = \pm 1$,
then X and Y are perfectly correlated

If $\rho_{XY} = \pm 0$,
then X and Y are uncorrelated

Linear Approximation

Extimating X from the values of Y:

$\hat{X} = m_x + (\rho_{XY} \sigma_x / \sigma_y)(Y - m_y)$

Mean-Squared Error:

$E\{[\hat{X} - X]^2\} = \sigma_x^2 (1 - \rho_{XY}^2)$

Gaussian random variables:

If X and Y are Gaussian and uncorrelated,
then they are independent

The linear transformation of Gaussian random variables is also Gaussian

Functions of Two RVs

$Z = g(X, Y)$

$F_Z(w) = P\{g(X, Y) \leq w\}$

Sums of Two RVs

$Z = X + Y$, then

$f_Z(w) = \int_{-\infty}^{\infty} f_{XY}(u, w - u) du$

If X and Y are independent:

$f_Z(w) = \int_{-\infty}^{\infty} f_Y(w - u) f_x(u) du$

Mean and Variance of a Sum

$E\{Z\} = E\{X\} + E\{Y\}$

$Var(Z) = Var(X) + Var(Y) + 2Cov(X, Y)$

For uncorrelated variables, variance of a sum is the sum of the variance.

Subjects not yet added

Rayleigh Density

Estimation

Maximum a-posteriori probability (MAP) estimate of X given Y

Minimum mean-squared-error estimate

Linear Estimate

Random Processes

First- and Second-order distribution of a rp

$F_{X(t)}(x; t) = P\{X(t) \leq x\}$

$F_{X(t)}(x_1, x_2; t_1, t_2) = P\{X(t_1) \leq x_1 \; and \; X(t_2) \leq x_2\}$

Mean, autocorrelation, and autocovariance functions

$m_x(t) = E\{x(t)\}$

$R_X(t_1, t_2) = E\{X(t_1)X(t_2)\}$

$C_X(t_1, t_2) = E\{[X(t_1) - m_x(t_1)][X(t_2) - m_x(t_2)]\} = R_x(t_1, t_2) - m_x(t_1) m_x(t_2)$

Wide-sense stationary process

$m_x = E\{X(t)\}$

$R_X(\tau) = E\{X(t)X(t + \tau)\}$

$C_x(\tau) = E\{[X(t) - m_x][X(t+\tau) - m_x]\} = R_x(\tau) - [m_x]^2$

Cross-correlation function

for the general case,
$R_{XY}(t_1, t_2) = E\{X(t_1)Y(t_2)\}$
for jointly WSS processes,
$R_{XY}(\tau) = E\{X(t)Y(t+\tau)\}$