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Homework for Chapter 8

EE 520 - Random Processes
Problems: 8.1, 8.22, 8.40, 8.44, and 8.51

8.21

Exercise

Given a random sequence $X[n]$ for $n \geq 0$ with conditional pdfs

$$f_X(x_n|x_{n-1}) = \alpha e^{-\alpha(x_n - x_{n-1})} u(x_n - x_{n-1}), \text{ for } n \geq 1,$$

with $u(x)$ the unit-step function and the intitial pdf $f_X(x_0) = \delta(x_0)$. Take $\alpha > 0$.

(a) Find the first-order pdf $f_X(x_n)$ for $n = 2$.

The integration of dx_0 below comes from.

$$\begin{aligned} f(x_0) &= \delta(x_0)\big|_{x_0=0} \\ &= 1 \end{aligned}$$

Building up from the known first-order pdf of x_0.

$$f(x_0, x_1, x_2) = fx(x_0) f_X(x_1|x_0) f_X(x_2|x_1)$$

And then removing the reliance upon x_0 and x_1 via integration.

$$
\begin{aligned}
f(x_2) &= \int\int fx(x_0)f_X(x_1|x_0)f_X(x_2|x_1)\ dx_0\ dx_1 \\
&= \int\int \delta(x_0)f_X(x_1|x_0)f_X(x_2|x_1)\ dx_0\ dx_1 \\
&= \int f_X(x_1|x_0=0)f_X(x_2|x_1)\ dx_1, \qquad \text{here } \int \delta(x)f(x)dx = f(0) \\
&= \int \alpha e^{-\alpha(x_1-0)}u(x_1)\alpha e^{-\alpha(x_2-x_1)}u(x_2-x_1)\ dx_1 \\
&= \alpha^2 \int e^{-\alpha(x_1+x_2-x_1)}u(x_1)u(x_2-x_1)\ dx_1 \\
&= \alpha^2 \int e^{-\alpha x_2}u(x_1)u(x_2-x_1)\ dx_1 \\
&= \alpha^2 \int_{-\infty}^{0} e^{-\alpha x_2}u(x_2-x_1)\ dx_1 \\
&= \alpha^2 \left[e^{-\alpha x_2}ramp(x_2-x_1)\ \Big|_{-\infty}^{0}\right], \qquad \text{where } ramp(x) = xu(x) \\
&= \alpha^2 \left[e^{-\alpha x_2}ramp(x_2) - e^{-\alpha x_2}(0)\right] \\
&= \alpha^2 e^{-\alpha x_2}x_2 u(x_2)
\end{aligned}
$$

(b) Find the first-order pdf $f_X(x_n)$ for arbitrary $n > 1$ using mathematical induction.

From the above it is easy to see what the first three steps of $f_X(x_n)$ are.

$$
\begin{aligned}
f_X(x_0) &= \delta(x_0) \\
f_X(x_1) &= \alpha e^{-\alpha x_1}u(x_1) \\
f_X(x_2) &= \alpha^2 x_2 e^{-\alpha x_2}u(x_2)
\end{aligned}
$$

From this progression it would appear that.

$$f_X(x_n) = \alpha^n x_n^{n-1} e^{-\alpha x_n}u(x_n)$$

Thus assuming the above is correct.

$$\begin{aligned}
f_X(x_n) &= \int f(x_{n-1})f(x_n|x_{n-1})dx_{n-1} \\
&= \int \alpha^{n-1}x_{n-1}^{n-1-1}e^{-\alpha x_{n-1}}u(x_{n-1})\alpha e^{-\alpha(x_n-x_{n-1})}u(x_n-x_{n-1})dx_{n-1} \\
&= \alpha^n \int_0^\infty x_{n-1}^{n-2}e^{-\alpha(x_{n-1}+x_n-x_{n-1})}u(x_n-x_{n-1})dx_{n-1} \\
&= \alpha^n \int_0^\infty x_{n-1}^{n-2}e^{-\alpha x_n}u(x_n-x_{n-1})dx_{n-1} \\
&= \alpha^n e^{-\alpha x_n}u(x_n)\int_0^{x^n} x_{n-1}^{n-2}dx_{n-1} \\
&= \alpha^n e^{-\alpha x_n}u(x_n)\left[\frac{x_{n-1}^{n-2}}{n-1}\Big|_0^{x_n}\right] \\
&= \alpha^n e^{-\alpha x_n}u(x_n)\left[\frac{x_n^{n-1}}{n-1}\right]
\end{aligned}$$

Which at this point doesn't quite match the original assumption as it is missing a factor of $\frac{1}{n-1}$.

What could then be added to the original assumption to make up for this ommission?

Clearly it must be 1 at $n = 1, 2$

Thereafter it needs to be something that when multiplied with $\frac{1}{(n-1)}$ becomes $\frac{1}{(n-1)}$.

Honestly at this point I would never had made this jump. I realize that $\frac{1}{(n-1)!}$ works here as $n-1$ is inserted for n, but I still don't see how I would ever have found that other than guess and check. Sort of a needle in a haystack. Maybe there's a certain pattern I'm missing.

Anyway, modifying the above to become.

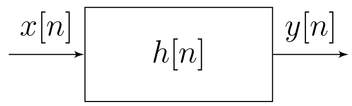
$$f_X(x_n) = \frac{1}{(n-1)!}\alpha^n x_n^{n-1}e^{-\alpha x_n}u(x_n)$$

And plugging through eventually proves this as $\frac{1}{(n-1)!}\Big|_{n=n-1} = \frac{1}{(n-2)}$ and this factor multiplied by $\frac{1}{(n-1)}$ becomes $\frac{1}{(n-1)!}$ The missing component of the above equation.

8.22

Exercise

Let x[n] be a deterministic input to the LSI discrete-time system H shown in the figure below.



(a) Use linearity and shift-invariance properties to show that

$$y[n] = x[n] * h[n] \triangleq \sum_{k=-\infty}^{\infty} x[k]h[n-k] = h[n] * x[n].$$

This is show that

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} x[k]h[n-k] &\overset{?}{=} \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
\sum_{k=-\infty}^{\infty} x[k]h[n-k] &= \sum_{k=-\infty}^{\infty} h[n-m]x[m] \qquad \text{for } m = n-k, \text{ and } k = n-m \\
&= \sum_{n-m=-\infty}^{\infty} h[m-n]x[m] \qquad \text{and shifting such that m is an upper bound} \\
&= \sum_{-\infty}^{m=\infty} h[m-n]x[m] \\
&\overset{\checkmark}{=} \sum_{m=-\infty}^{\infty} h[m-n]x[m] \qquad \text{which is equivalent under summation}
\end{aligned}$$

(b) Define the Fourier transform of a sequence a[n] as

$$A(\omega) \triangleq \sum_{n=-\infty}^{\infty} a[n]e^{-j\omega n}, \; -\pi \leq \omega \leq \pi$$

and show that the inverse Fourier transform is

$$a[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} A(\omega)e^{+jwn}d\omega, \; -\infty < n < \infty$$

I will simply insert the definition of $A(\omega)$ into the inverse FT and solve for $a[n]$

$$
\begin{aligned}
a[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{m=-\infty}^{\infty} a[m]e^{-j\omega m}e^{+jwn}d\omega \\
&= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} a[m]\int_{-\pi}^{\pi} e^{-j\omega m}e^{+jwn}d\omega \\
&= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} a[m]\int_{-\pi}^{\pi} e^{-j\omega(m+n)}d\omega \\
&= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} a[m]\left[\frac{e^{j\omega(n-m)}}{j(n-m)}\Big|_{-\pi}^{\pi}\right] \\
&= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} a[m]\left[\frac{e^{j\pi(n-m)}-e^{-j\pi(n-m)}}{j(n-m)}\right]
\end{aligned}
$$

$$\text{Euler's } sin(t) = \frac{e^{jt}-e^{-jt}}{2j}$$

$$= \frac{1}{\pi}\sum_{m=-\infty}^{\infty} a[m]\frac{sin(\pi(n-m))}{(n-m)}$$

Here, $sin(\pi(n-m))$ becomes zero for $n \neq m$ as $sin(\pi n) = 0$, for $n \in$ integers, Finally for $n = m$, $\lim_{n\to 0}\frac{sin(n)}{n} = \pi$

$$
\begin{aligned}
a[m] &= \frac{1}{\pi}a[m]\frac{sin(\pi(0))}{(0)} \\
&= \frac{1}{\pi}a[m]\pi \\
&\stackrel{\checkmark}{=} a[m]
\end{aligned}
$$

(c) Using the results in (a) and (b), show that

$$Y(\omega) = H(\omega)X(\omega), \ -\pi \leq \omega \leq \pi$$

for an LSI discrete time system.

$$
\begin{aligned}
Y(\omega) &= H(\omega)X(\omega) \\
FT\{Y\}(\omega) &= FT\{H\}(\omega)FT\{X\}(\omega) \\
&= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m} \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h[n]x[m]e^{-j\omega(n+m)}
\end{aligned}
$$

$$
\begin{aligned}
y[v] &= FT^{-1}\left\{FT\{Y\}(\omega)\right\} \\
&= FT^{-1}\left\{\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h[n]x[m]e^{-j\omega(n+m)}\right\} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h[n]x[m]e^{-j\omega(n+m)}\right\} e^{jwv}d\omega \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h[n]x[m]\int_{-\pi}^{\pi} e^{-j\omega(n+m)}e^{jwv}d\omega
\end{aligned}
$$

And as this is nearly the same integral as before, the only time this isn't zero is at $m + n = v$. At this point, the integral equals 2π.

$$
\begin{aligned}
&= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h[n]x[m]\pi \\
&= \sum_{n=-\infty}^{\infty} h[n]x[m]\Big|_{m=v-n} \\
&\stackrel{\checkmark}{=} \sum_{n=-\infty}^{\infty} h[n]x[v-n]
\end{aligned}
$$

8.40

Exercise

Consider using a first-order Markov sequence to model a random sequence $X[n]$ as

$$X[n] = rX[n-1] + Z[n],$$

where $Z[n]$ is white noise of variance σ_Z^2. Thus, we can look at X[n] as the output of passign $Z[n]$ through a linear system. Take $|r| < 1$ and assume the system has been running for a long time, that is, $-\infty < n < \infty$

(a) Find the psd of $X[n]$, that is, $-\infty < n < \infty$.

White noise implies $\mu_z = 0$, $\sigma_Z^2 = c$; c constant $\forall n$.

White noise by definition is uncorrelated. Therefore $E[Z[n]Z[n+l]] = 0$ for $l \neq 0$,

From here, the output PSD S_{XX}.

$$\begin{aligned} S_{XX}(\omega) &= |H(\omega)|^2 Z(\omega), \qquad \text{where } Z(\omega) = \sigma_Z^2 \\ &= |H(\omega)|^2 \sigma_Z^2 \end{aligned}$$

Start by finding the impulse response.

$$\begin{aligned} X[n] &= rX[n-1] + \delta[n] \\ X[n] - rX[n-1] &= \delta[n] \\ h[n] - rh[n-1] &= \delta[n] \\ Z\{h[n] - rh[n-1]\} &= Z\{\delta[n]\} \\ H(Z) - rZ^{-1}H(Z) &= 1 \\ H(Z) &= \frac{1}{1 - rZ^{-1}} \end{aligned}$$

The $FT\{h[n]\}$ is then simply the $Z\{h[n]\}\big|_{Z=e^{-jwn}}$

$H(\omega) = \frac{1}{1-re^{-jwn}}$

$$\begin{aligned} |H(w)|^2 &= \frac{1}{(1-re^{-jwn})(1-re^{-jwn})} \\ &= \frac{1}{1 + (-re^{-jwn})(-re^{-jwn}) + 2re^{-jwn}} \\ &= \frac{1}{-re^{-2jwn} + 2re^{-jwn}} \\ &= \frac{1}{1 + r^2 - 2rcos(\omega)} \qquad \text{which I'm still slightly unsure on.} \end{aligned}$$

Finally $S_{XX} = \frac{\sigma_Z^2}{1+r^2-2rcos(\omega)}$

(b) Find the correlation function $R_{XX}[m]$.

8.44

Exercise

Given a Markov chain X[n] on $n \geq 1$, with the transistion probabilities given as $P\left[x[n]|x[n-1]\right]$, find an expression for the two-step transition probabilities $P\left[x[n]|x[n-2]\right]$. Also show that

$$P\left[x[n+1]x[n-1], x[n-2], \ldots, x[1]\right] = P\left[x[n+1]x[n-1]\right], \text{ for } n \geq 1.$$

8.51

Exercise

Let $X[n]$ be a real-valued random sequence on $n \geq 0$, made up from a stationary and *independent increments*, that is, $X[n] - X[n-1] = W[n]$, "the increment" with $W[n]$ being a stationary and independent random sequence. The random sequence always starts with $X[0] = 0$. We also know that at time $n = 1$, $E[X[1]] = \nu$ and $Var[X[1]] = \sigma^2$.

(a) Fin $\mu_X[n]$ and $\sigma_X^2[n]$, the mean and variance functions of the random sequence X at time n for any time $n > 1$.

(b) Prove that $\frac{X[n]}{n}$ converges in probability to ν as the time n approaches infinity.