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Homework 5

EE 520 - Random Processes

Problems: 5.1, 5.4, 5.20, 5.29, and 5.30

5.1

Exercise

Let $f_{\vec{x}}(\vec{x})$ be given as.

$$f_{\vec{x}}(\vec{x}) = Ke^{-\vec{x}^T\vec{\Lambda}}u(\vec{x}),$$

where $\vec{\Lambda} = (\lambda_1, \ldots, \lambda_n)^T$ with $\lambda_i > 0$ for all i, $\vec{x} = (x_i, \ldots, x_n)^T$, $u(\vec{x}) = 1$ if $x_i \geq 0$, $i = 1, \ldots, n$, and zero otherwise, and K is a constant to be determined. What value of K will enable $f_{\vec{x}}(\vec{x})$ to be a pdf?

For $f_{\vec{x}}(\vec{x})$ to be a pdf it must equal 1 under indefinite integration.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} f_{\vec{x}}(\vec{x}) d\vec{x} \\
&= \int_{-\infty}^{\infty} Ke^{-\vec{x}^T\vec{\Lambda}}u(\vec{x}) d\vec{x} \\
&= K\int_{0}^{\infty} e^{-\vec{x}^T\vec{\Lambda}} d\vec{x}
\end{aligned}$$

and here, $\vec{x}^T\vec{\Lambda}$ is a scalar product of all $x_i\lambda_i$.

$$\vec{x}^T\vec{\Lambda} = \sum_{1}^{n} x_i\lambda_i$$

$$
\begin{aligned}
1 &= K \int_0^\infty e^{-\vec{x}^T\vec{\Lambda}} d\vec{x} \\
&= K \int_0^\infty e^{-\sum_1^n x_i\lambda_i} d\vec{x} \\
&= K \int_0^\infty \prod_1^n e^{-x_i\lambda_i} d\vec{x} \\
&= K \prod_1^n \int_0^\infty e^{-x_i\lambda_i} dx_i \\
&= K \prod_1^n \left.\frac{-e^{-x_i\lambda_i}}{\lambda_i}\right|_0^\infty \\
&= K \prod_1^n \frac{1}{\lambda_i}
\end{aligned}
$$

and finally...

$$
K = \prod_1^n \lambda_i
$$

5.4

Exercise

Let X_1, X_2, X_3, be three standard Normal RV's. For $i = 1, 2, 3$ let $Y_i \in X_1, X_2, X_3$ such that $Y_1 < Y_2 < Y_3$ i.e. the ordered—by—signed magnitude of the X_i. Compute the joint pdf $f_{Y_1Y_2Y_3}(y_1, y_2, y_3)$.

$$
\begin{aligned}
f_{Y_1Y_2Y_3}(y_1, y_2, y_3,) &= \begin{cases} n! \prod_1^n f_x(y_i), & \text{for } y_1 < y_2 < y_3 \\ 0, & \forall \; other \end{cases} \\
&= \begin{cases} 6! \prod_1^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{y_i^2}{2}}, & \text{for } y_1 < y_2 < y_3 \\ 0, & \forall \; other \end{cases}
\end{aligned}
$$

5.20

Exercise

Let $\vec{X}_i, i = 1, \ldots, n$, be n mutually orthogonal random vectors. Show that

$$E\left[\left\|\sum_{i=1}^{n} \vec{X}_i\right\|^2\right] = \sum_{i=1}^{n} E\left[\left\|\vec{X}_i\right\|^2\right]$$

I solved this problem without either of the two below hints, but I believe my solution to still be correct. I have attempted to justify my solution below.

- (*Hint*: Use the definition $\left\|\vec{X}\right\|^2 \triangleq \vec{X}^{\,T}\vec{X}$)
- Note: $\vec{X}_i\vec{X}_j$ for $j \neq i$, is zero because they are orthoganol. Therefore: $\sum_i^n \sum_j^n \vec{x}_i\vec{x}_j = \sum_i^n \vec{x}_i^{\,2}$

From the embedded python script and accompanying output, it can be seen that the magnitude of a sum of orthogonal vectors is equal to the square root of the sum of the squared magnitudes of the individual vectors.

$$\left\|\sum_{i}^{n} \vec{X}\right\| = \sqrt{\sum_{i}^{n}\left\|\vec{X}_i\right\|^2}$$

Which will be used below.

$$
\begin{aligned}
E\left[\left\|\sum_{i=1}^{n}\vec{X}_i\right\|^2\right] &= \sum_{i=1}^{n} E\left[\left\|\vec{X}_i\right\|^2\right] \\
E\left[\left\|\vec{X}_1+\cdots+\vec{X}_n\right\|^2\right] &= \sum_{i=1}^{n} E\left[\left\|\vec{X}_i\right\|^2\right] \\
E\left[\sqrt{\left\|\vec{X}_1\right\|^2+\cdots+\left\|\vec{X}_n\right\|^2}^{\,2}\right] &= \sum_{i=1}^{n} E\left[\left\|\vec{X}_i\right\|^2\right] \\
E\left[\left\|\vec{X}_1\right\|^2+\cdots+\left\|\vec{X}_n\right\|^2\right] &= \sum_{i=1}^{n} E\left[\left\|\vec{X}_i\right\|^2\right] \\
E\left[\left\|\vec{X}_1\right\|^2\right]+\cdots+E\left[\left\|\vec{X}_n\right\|^2\right] &= \sum_{i=1}^{n} E\left[\left\|\vec{X}_i\right\|^2\right] \\
\sum_{i=1}^{n} E\left[\left\|\vec{X}_i\right\|^2\right] &\overset{\checkmark}{=} \sum_{i=1}^{n} E\left[\left\|\vec{X}_i\right\|^2\right]
\end{aligned}
$$

```
#!/usr/bin/env python3
import numpy as np
import numpy.linalg as LA


V1 = np.array([[1],[0],[0]])
V2 = np.array([[0],[1],[1]])

print("V1 orthogonal V2")
print()
print("V1: \n", V1)
print("V1 norm: \n", LA.norm(V1))
print()

print("V2:\n", V2)
print("V2 norm:\n", LA.norm(V2))
print()

print("V2 + V1:\n", V2+V1)
print("norm(V2 + V1) :\n", LA.norm(V2+V1))
print("norm(V2) + norm(V1) :\n", LA.norm(V2)+LA.norm(V1))

print("sqrt(norm(V2)^2 + norm(V1)^2):\n", (LA.norm(V2)**2+LA.norm(V1)**2)**(0
```

```
V1 orthogonal V2

V1:
 [[1]
 [0]
 [0]]
V1 norm:
 1.0

V2:
 [[0]
 [1]
 [1]]
V2 norm:
 1.41421356237

V2 + V1:
 [[1]
 [1]
 [1]]

norm(V2 + V1) :
 1.73205080757

norm(V2) + norm(V1) :
 2.41421356237

sqrt(norm(V2)^2 + norm(V1)^2):
 1.73205080757
```

5.29

Excercise

Let $\vec{X} = (X_1, X_2, X_3)^T$ be a random vector with $\vec{\mu} \triangleq E[\vec{X}]$ given by $\vec{\mu} = (5, -5, 6)^T$.

And covariance given by

$$\vec{K} = \begin{bmatrix} 5 & 2 & -1 \\ 5 & 5 & 0 \\ -1 & 0 & 4 \end{bmatrix}$$

Calculate the mean and variance of

$$Y = \vec{A}^T\vec{X} + B$$

Where

$$\vec{A} = (2, -1, 2)^T \text{and } B = 5$$

$$\begin{aligned}
E[Y] &= E[\vec{A}^T\vec{X} + B] \\
&= E[\vec{A}^T]E[\vec{X}] + E[B] \\
&= E\left[\begin{bmatrix} 2 & -1 & 2 \end{bmatrix}\right] E\left[\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}\right] + E[5] \\
&= \begin{bmatrix} 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ -5 \\ 6 \end{bmatrix} + 5 \\
&= 2(5) + -1(-5) + 2(6) + 5 \\
&= 10 + 5 + 12 + 5 \\
&= 32
\end{aligned}$$

First...

$$\begin{aligned}
cov(\vec{X}) &= E[(\vec{X} - \vec{\mu_X})(\vec{X} - \vec{\mu_X})^T] \\
&= E[(\vec{X} - \vec{\mu_X})(\vec{X}^T - \vec{\mu_X}^T)] \\
&= E[\vec{X}\vec{X}^T - \vec{X}\vec{\mu_X}^T - \vec{\mu_X}\vec{X}^T - \vec{\mu_X}\vec{\mu_X}^T] \\
&= E[\vec{X}\vec{X}^T] - E[\vec{X}\vec{\mu_X}^T] - E[\vec{\mu_X}\vec{X}^T] - \vec{\mu_X}\vec{\mu_X}^T \\
&= E[\vec{X}\vec{X}^T] - \vec{\mu_X}\vec{\mu_X}^T - \vec{\mu_X}\vec{\mu_X}^T - \vec{\mu_X}\vec{\mu_X}^T \\
&= E[\vec{X}\vec{X}^T] - \vec{\mu_X}\vec{\mu_X}^T
\end{aligned}$$

Now...

$$
\begin{aligned}
\sigma_Y^2 &= E[(Y - E[Y])^2] \\
&= E[(\vec{A}^T\vec{X} + 5 - 32)^2] \\
&= E[(\vec{A}^T\vec{X} - 27)^2] \\
&= E[(\vec{A}^T\vec{X})^2 - 2(27)\vec{A}^T\vec{X} + 27^2] \\
&= E[(\vec{A}^T\vec{X})^2] - 2(27)E[\vec{A}^T\vec{X}] + E[27^2] \\
&= E[(\vec{A}^T\vec{X})^2] - 2(27)^2 + 27^2 \\
&= E[(\vec{A}^T\vec{X})^2] - 27^2 \\
&= \vec{A}^T E[\vec{X}\vec{X}^T]\vec{A} - 729 \\
&= \vec{A}^T(cov(\vec{X}) + \vec{\mu}\vec{\mu}^T)\vec{A} - 1429 \\
&= 754 - 729 \\
&= 25
\end{aligned}
$$

5.30

Exercise

Two jointly normal variables X_1, and X_2 have joint pdf $f_{X_1X_2}$ given by:

$$
f_{X_1X_2}(X_1, X_2) = \frac{2}{\pi\sqrt{7}}^{-\frac{8}{7}(X_1^2 + \frac{3}{2}X_1X_2 + X_2^2)}
$$

With

$$
\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \vec{A}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}
$$

Find a non-trivial transformation A such that Y_1 and Y_2 are independent.

I'm still working on completely understanding this one. I may turn in a complete solution in the future, but I didn't want to just copy from the solutions.