EE520 - Random Processes

HW2

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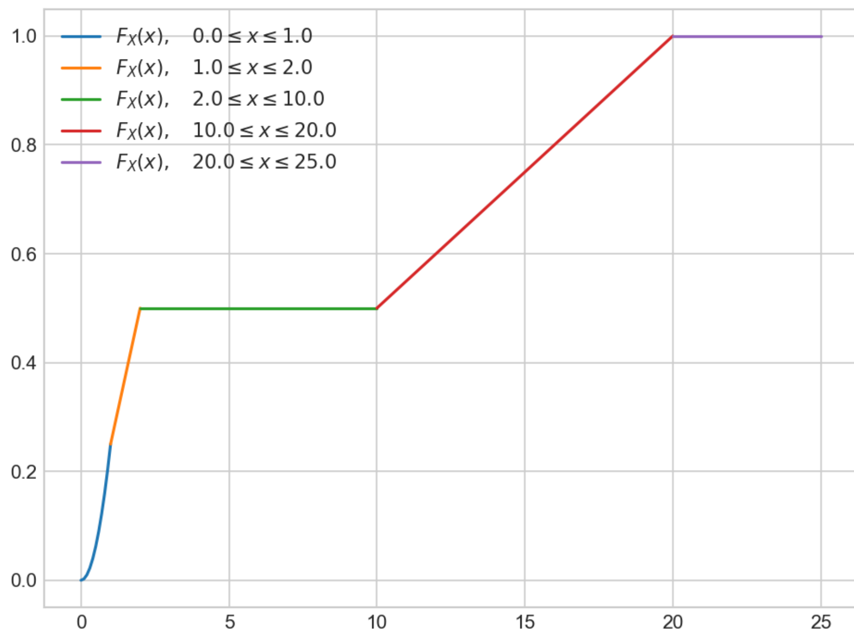
2.3

We are told that the wait time's CDF is:

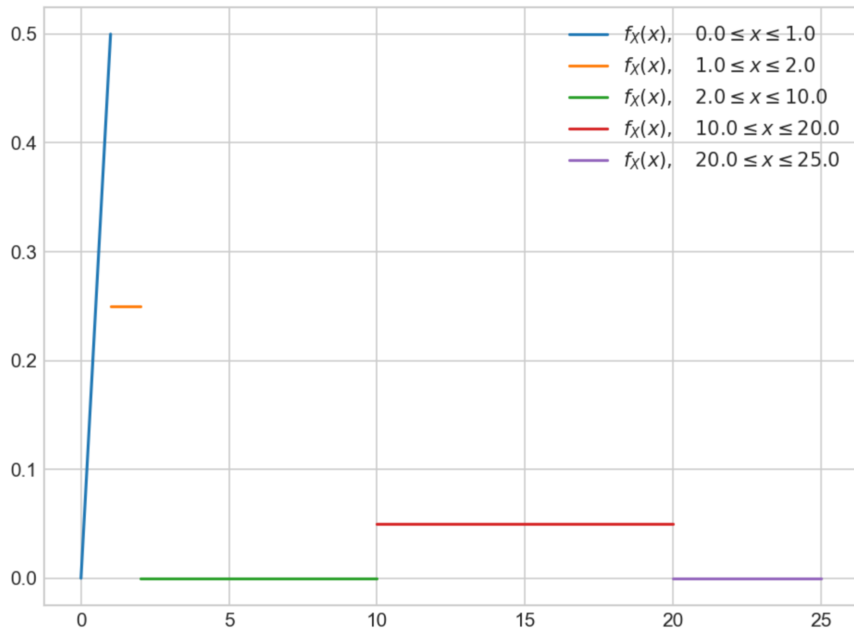
$$F_X(x) = \begin{cases} \left(\frac{x}{2}\right)^2, & 0 \le x \le 1, \\ \frac{x}{4}, & 1 \le x \le 2, \\ \frac{1}{2}, & 2 \le x \le 10, \\ \frac{x}{20}, & 10 \le x \le 20, \\ 1, & 20 \le x. \end{cases}$$

a. Can be sketched as a piecewise function:



b. PDF via derivative of each piece:

$$f_X(x) = \begin{cases} \frac{x}{2}, & 0 \le x \le 1, \\ \frac{1}{4}, & 1 \le x \le 2, \\ 0, & 2 \le x \le 10, \\ \frac{1}{20}, & 10 \le x \le 20, \\ 0, & 20 \le x. \end{cases}$$



c.

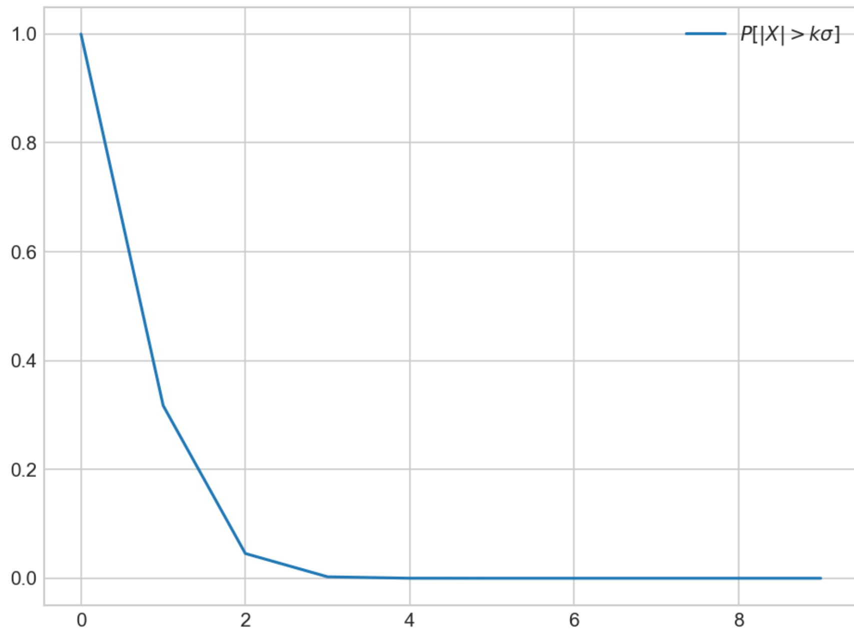
$$
\begin{aligned}
P[X > 10] &= 1 - F_X(10) = 0.5, \\
P[X \leq 5] &= F_X(5) = 0.5, \\
P[(X > 5) \cap (X \leq 10)] &= P[X \leq 10] - P[X \leq 5], \\
&= F_X(10) - F_X(5) = 0, \\
P[X = 1] &= \int_{1^-}^{1^+} f_X(x)dx, \\
&= 0 \text{ (no delta functions)}.
\end{aligned}
$$

2.7

Want $P[|X| > k\sigma] = 1 - P[-k\sigma < X < k\sigma]$. Assuming $F_X(x)$ is continuous, this is the same as $1 - (F_X(k\sigma) - F_X(-k\sigma))$ which is the same as $1 + F_X(-k\sigma) - F_X(k\sigma)$.

$$
\begin{aligned}
F_X(x) &= \frac{1}{2}\left[1 + erf\left(\frac{x}{\sigma\sqrt{2}}\right)\right], \\
1 + F_X(-k\sigma) - F_X(k\sigma) &= \frac{1}{2}\left[2 + erf\left(\frac{-k}{\sqrt{2}}\right) - erf\left(\frac{k}{\sqrt{2}}\right)\right].
\end{aligned}
$$

$$
\begin{aligned}
k = 0 &\implies 1.00, \\
k = 1 &\implies 0.317, \\
k = 2 &\implies 0.0455, \\
k = 3 &\implies 0.00270.
\end{aligned}
$$



2.9

These are accomplished by turning summations into integrations across a sum of delta functions.

- Bernoulli:

$$P[X=0]=1-p,$$
$$P[X=1]=p,$$
$$f_X(x)=(1-p)\delta(x)+p\delta(x-1).$$

- Binomial:

$$P[X=k]=\binom{n}{k}p^k(1-p)^{n-k},$$
$$f_X(x)=\sum_{k=0}^{n}\delta(x-k)\binom{n}{k}p^k(1-p)^{n-k}.$$

- Poisson:

$$P[X=k]=\frac{\mu^k}{k!}e^{-\mu},$$
$$f_X(x)=\sum_{k=0}^{\infty}\delta(x-k)\frac{\mu^k}{k!}e^{-\mu}.$$

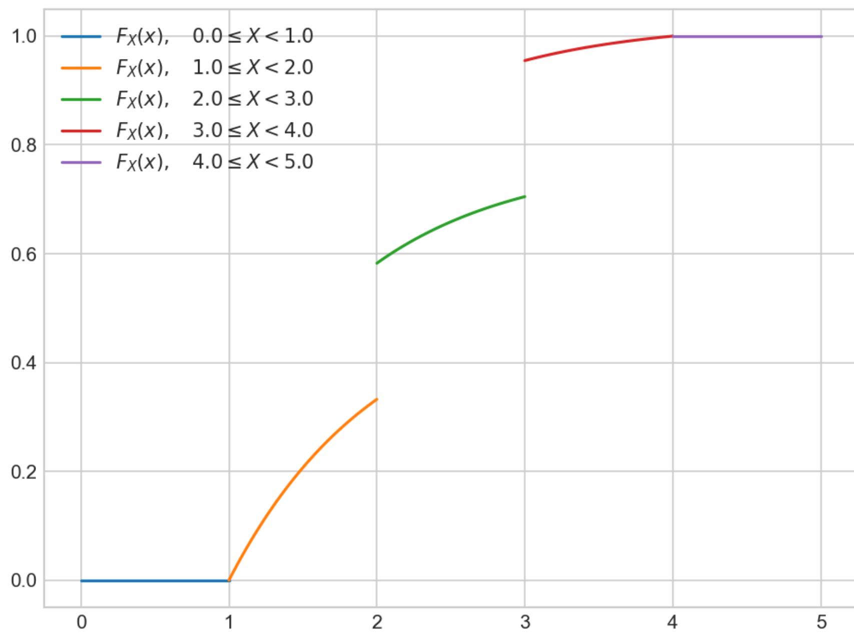
2.10

a. We know that PDFs must integrate to 1, and that delta functions (indicated by the arrows) integrate to their magnitude (the number above each arrow). Therefore:

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty} f_X(x)dx, \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{4}\delta(x-2) + \frac{1}{4}\delta(x-3) + Ae^{-x}(u(x-1) - u(x-4))\right] dx, \\
&= 0.5 + \left(-Ae^{-x}\right|_{x=1}^{4}, \\
&= 0.5 + 0.350A, \\
A &\approx 1.430.
\end{aligned}
$$

b. The CDF will raise as an exponential, with discontinuous bumps at the deltas:

$$
F_X(x) = \begin{cases} 0, & x \leq 1, \\ \int_1^x Ae^{-y}dy, & 1 \leq x < 2 \\ \int_1^x Ae^{-y}dy + \frac{1}{4}, & 2 \leq x < 3 \\ \int_1^x Ae^{-y}dy + \frac{1}{2}, & 3 \leq x < 4 \\ 1. & 4 \leq x \end{cases}
$$



c. The best way to approach this sort of problem with a discontinuous CDF is:

- Convert all operations to $<$ or $\leq$.
- For each $<$, use the left-limit of the CDF.
- For each $\leq$, use the right-limit of the CDF.

Below, we'll use $F_X^+(x)$ to mean the CDF from the right-limit, and $F_X^-(x)$ to mean the left-limit of the CDF.

$$
\begin{aligned}
P[2 \leq X < 3] &= P[X < 3] - P[X < 2], \\
&= F_X^-(3) - F_X^-(2), \\
&= \int_1^3 Ae^{-y}dy + \frac{1}{4} - (\int_1^2 Ae^{-y}dy), \\
&= 0.705 - 0.333, \\
&= 0.372.
\end{aligned}
$$

d.

$$
\begin{aligned}
P[2 < X \leq 3] &= P[X \leq 3] - P[X \leq 2], \\
&= F_X^+(3) - F_X^+(2), \\
&= \int_1^3 Ae^{-y}dy + \frac{1}{2} - (\int_1^2 Ae^{-y}dy + \frac{1}{4}), \\
&= 0.955 - 0.583, \\
&= 0.372.
\end{aligned}
$$

e.

$$
\begin{aligned}
F_X(3) &= P[X \leq 3], \\
&= F_X^+(3), \\
&= \int_1^3 Ae^{-y}dy + \frac{1}{2}, \\
&= 0.955.
\end{aligned}
$$

2.19

$$
\begin{aligned}
P[Y=k|X=x] &= \frac{x^k}{k!}e^{-x}, \quad k \geq 0, x > 0,\\
f_X(x) &= \frac{1}{5}[u(x)-u(x-5)],\\
P[Y=k] &= \int_{-\infty}^{\infty} P[Y=k|X=x]f_X(x)dx,\\
&= \frac{1}{5}\int_0^5 \frac{x^k}{k!}e^{-x}dx,\\
&= \frac{1}{5}\int_0^5 \frac{x^k}{k!}\left[\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}x^i\right]dx,\\
&= \frac{1}{5k!}\int_0^5 \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}x^{i+k}dx,\\
&= \frac{1}{5k!}\sum_{i=0}^{\infty}\frac{(-1)^i}{(i+k+1)i!}x^{i+k+1}\Big|_{x=0}^{5},\\
&= \frac{1}{5k!}\sum_{i=0}^{\infty}\frac{(-1)^i(5)^{i+k+1}}{(i+k+1)i!}.
\end{aligned}
$$

I'm sure there is a further reduction of this expression, but as there are no more integrals, and the summation rapidly decreases as it is divided by $i!$, this function is easy to compute. Numerically, this function generates the table:

$$
\begin{aligned}
0 &\implies 0.199\\
1 &\implies 0.192\\
2 &\implies 0.175\\
3 &\implies 0.147\\
4 &\implies 0.112\\
5 &\implies 0.077\\
6 &\implies 0.048\\
7 &\implies 0.027\\
8 &\implies 0.014\\
9 &\implies 0.006
\end{aligned}
$$

2.22

$$
f_{XY}(x,y) = \frac{1}{3\pi}e^{-\frac{1}{2}[(x/3)^2+(y/2)^2]}u(x)u(y).
$$

- Are they independent? Yes:

$$f_X(x) = C_X e^{-\frac{1}{2}[(x/3)^2]} u(x),$$
$$f_Y(y) = C_Y e^{-\frac{1}{2}[(y/2)^2]} u(y),$$
$$f_{XY}(x,y) = f_X(x) f_Y(y).$$

Note that C_X, C_Y are normalizing constants for each marginal PDF. That is, each PDF should individually integrate to 1. Since they are normalizing Gaussian distributions, we know that σ must be the divisor of each's variable, and that the normalizing constant should be $1/\sqrt{2\pi\sigma^2}$ (compare to Gaussian PDF equation to see this). Our distributions are only half Gaussian, so the normalizing constants need to be twice what they normally would be. Thus:

$$C_X = \frac{2}{\sqrt{18\pi}},$$
$$C_Y = \frac{2}{\sqrt{8\pi}},$$
$$C_X C_Y = \frac{1}{3\pi}.$$

- What is $P[X \leq 3, Y \leq 2]$?

Since independent, can split into product of two integrals:

$$P[X \leq 3, Y \leq 2] = \iint f_{XY}(x,y)dxdy,$$
$$= \int f_X(x)dx \int f_Y(y)dy.$$

So, just need to multiply our CDFs, which have a PDF of exactly double a traditional Gaussian PDF. Thus, take the CDF of a Gaussian between $[0,3]$ for X and $[0,2]$ for Y, and double each result:

$$P[X \leq 3, Y \leq 2] = \frac{4}{4} erf(\frac{3}{3\sqrt{2}}) erf(\frac{2}{2\sqrt{2}}),$$
$$= erf\left(\frac{1}{\sqrt{2}}\right)^2,$$
$$= 0.466.$$

2.28

a. We know that a PDF integrates to 1, so:

$$
\begin{aligned}
1 &= \int_0^\infty ce^{-2x}dx, \\
&= \frac{c}{-2}e^{-2x}\bigg|_{x=0}^{\infty}, \\
&= \frac{c}{2}, \\
c &= 2.
\end{aligned}
$$

b. Note that $a > 0, x > 0$. Therefore, we know that we are in the region of $f_X(x)$ that is non-zero. To avoid confusion, we'll use $f_X(y)$ rather than $f_X(x)$, since x is used in our bound.

$$
\begin{aligned}
P[X \geq x+a] &= \int_{x+a}^\infty 2e^{-2y}dy, \\
&= -e^{-2y}\big|_{y=x+a}^{\infty}, \\
&= e^{-2x-2a}.
\end{aligned}
$$

c. To find $P[X \geq x+a | X \geq a]$, we must expand that expression to its definition:

$$
P[X \geq x+a | X \geq a] = \frac{P[(X \geq x+a) \cap (X \geq a)]}{P[X \geq a]}.
$$

Since $x > 0, a > 0$, we know that the intersection of these two events is the case where $X \geq x + a$. Therefore, we get the result of part (b) divided by $P[X \geq a]$:

$$
\begin{aligned}
P[X \geq x+a | X \geq a] &= \frac{e^{-x-a}}{\int_a^\infty 2e^{-2x}dx}, \\
&= \frac{e^{-2x-2a}}{e^{-2a}}, \\
&= e^{-2x}.
\end{aligned}
$$

2.30

$$
\begin{aligned}
A &= \{X > x_A\}, \\
P[A] &= \int_{x_A}^\infty f_X(x)dx, \\
f_{X|M}(x) &= \frac{1}{\sqrt{2\pi}}\exp\left(-0.5[x-r]^2\right), \\
f_{X|M^c}(x) &= \frac{1}{\sqrt{2\pi}}\exp\left(-0.5x^2\right), \\
f_X(x) &= f_{X|M}(x)P[M] + f_{X|M^c}(x)P[M^c].
\end{aligned}
$$

When given M, $P[M] = 1$. Then:

$$
\begin{aligned}
P[A|M] &= \int_{x_A}^{\infty} f_{X|M}(x)dx, \\
&= 1 - F_{X|M}(x_A), \\
&= 1 - \frac{1}{2}\left[1 + erf(\frac{x_A - r}{\sqrt{2}})\right], \\
&= 0.691. \\
P[A^c|M] &= 1 - P[A|M], \\
&= 0.309. \\
P[A|M^c] &= \int_{x_A}^{\infty} f_{X|M^c}(x)dx, \\
&= 1 - F_{X|M^c}(x_A), \\
&= 1 - \frac{1}{2}\left[1 + erf(\frac{x_A}{\sqrt{2}})\right], \\
&= 0.309. \\
P[A^c|M^c] &= 1 - P[A|M^c], \\
&= 0.691.
\end{aligned}
$$

2.31

We want $P[M|A] = P[A|M]P[M]/P[A]$. The only piece of information we are missing is $P[A]$. Through partitioning, we know that $P[A] = P[A|M]P[M] + P[A|M^c]P[M^c]$, as the UFO is either present or not, and never both.

For $P[M] = 10^{-3}$:

$$
\begin{aligned}
P[A] &= (0.691)(10^{-3}) + (0.3085)(1 - 10^{-3}), \\
&= 0.3089, \\
P[M|A] &= (0.691)(10^{-3})/0.3089, \\
&= 0.00224. \\
P[M^c|A] &= 1 - P[M|A] = 0.998. \\
P[M|A^c] &= (0.3085)(10^{-3})/(1 - P[A]), \\
&= 0.000446. \\
P[M^c|A^c] &= 1 - P[M|A^c] = 0.9996.
\end{aligned}
$$

For $P[M] = 10^{-6}$:

$$
\begin{aligned}
P[A] &= (0.691)(10^{-6}) + (0.3085)(1-10^{-6}), \\
&= 0.3085, \\
P[M|A] &= P[A|M]P[M]/P[A], \\
&= 2.24 \times 10^{-6}, \\
P[M^c|A] &= 0.999998, \\
P[M|A^c] &= P[A^c|M]P[M]/P[A^c], \\
&= 4.46 \times 10^{-7}, \\
P[M^c|A^c] &= 0.9999996.
\end{aligned}
$$

The primary information to take away here is that if you have an extremely rare event - the presence of a UFO, in this instance - then this rarity magnifies any inaccuracy in your measurement device. That is, an alarm system must be more discerning the more rare the event it is detecting. This phenomenon is identical to the one seen in medical testing: it is more difficult to design useful tests for rare diseases.

2.38

Method One

For conditional failure rates, we proved in class that if using an exponential function, then λ will be constant. Since this problem specifies that λ is a constant, it is safe to assume that $F_X(x) = 1 - e^{-\lambda t}u(t)$. Therefore, deriving λ is algebraic:

$$
\begin{aligned}
P[X \leq 100] &\leq 0.05, \\
F_X(100) &= 1 - e^{-\lambda 100}u(100) \leq 0.05, \\
0.95 &\leq e^{-\lambda 100}, \\
ln 0.95 &\leq -\lambda 100, \\
\lambda &\leq \frac{-ln 0.95}{100} = -0.000513.
\end{aligned}
$$

Method Two

Any conditional failure rate may be converted to a PDF:

$$
\begin{aligned}
F_X(x) &= 1 - \exp\left[-\int_0^x \alpha(t)dt\right], \\
&= 1 - \exp\left[-\int_0^x \lambda dt\right], \\
&= 1 - \exp\left[-\lambda x\right].
\end{aligned}
$$

The result of the solution is the same as the above.