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Fall, 2017

Homework 6 Chapter 9

EE 520 Random Processes

Problems: 9.1, 9.5, & 9.14.

Exercise 9.1

Let $X[n]$ be a real-valued stationary random sequence with mean $E[X[n]] = \mu_x$ and auto-correlation function $E[X[n+m]X[n]] = R_{XX}[m]$. If $X[n]$ is the input to a D/A converter, the continuous-time output can be idealized as the analog random process $X_a(t)$ with

$$X_a(t) \triangleq X[n], \text{ for } n \leq t < n+1, \ \forall n$$

(a) Find the mean $E[X_a(t)] = \mu_a(t)$ as a function of μ_x.
Here, $n \leq t < n$ is equivalent to saying $\lfloor t \rfloor = n$

$$\begin{aligned} E[X_a(t)] &= E[X[\lfloor t \rfloor]] \\ &= \mu_x[\lfloor t \rfloor] \\ &= \mu_x \end{aligned}$$

(b) Find the correlation $E[X_a(t1)X_a(t_2)] = R_{X_aX_a}(t_1t_2)$ in terms of $R_{XX}[m]$

$$\begin{aligned} E[X_a(t1)X_a(t_2)] &= E[X_a(\lfloor t1 \rfloor)X_a(\lfloor t_2 \rfloor)] \\ &= E[X[\lfloor t1 \rfloor]X[\lfloor t_2 \rfloor]] \\ &= R_{XX}[X[\lfloor t1 \rfloor]X[\lfloor t_2 \rfloor]] \\ &= R_{XX}[\lfloor t1 \rfloor \lfloor t_2 \rfloor] \end{aligned}$$

Here, $\lfloor t_2 \rfloor - \lfloor t_1 \rfloor \neq t_2 - t_1$, so it cannot be assumed that R_{XX} is stationary.

Exercise 8.22

Let $N(t)$ be a Poisson random process defined on $0 \leq t < \infty$ with $N(0) = 0$ and mean arrival rate $\lambda > 0$.

(a) Find the joint probability $P[N(t_1) = n_1, N(t_2) = n_2]$ for $t_2 > t_1$.

Because the poisson process can be broken down into a sum of independent increments, this is really, what is the probability that $N(t_1) = n_1$, and that the difference between $N(t_1)$ and $N(t_2)$ is the difference between n_2 and n_1.

Thus for the poisson process pmf

$$P(X = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad n = 0, 1, 2 \ldots, \text{ and } t \geq 0$$

$$\begin{aligned} P(N(t_1) = n_1,\ N(t_2) = n_2) &= P(N(t_1) = n_1,\ N(t_2) - N(t_1) = n_2 - n_1) \\ &= P(N(t_1) = n_1) P(N(t_2) - N(t_1) = n_2 - n_1), \quad \text{by independence} \\ &= \left[\frac{(\lambda t_1)^{n_1}}{n_1!} e^{-\lambda t_1} u(n_1)\right] \left[\frac{(\lambda t_2 - t_1)^{n_2 - n_1}}{n_2 - n_1!} e^{-\lambda(t_2 - t_1)} u(n_2 - n_1)\right] \\ &= \frac{(\lambda t_1)^{n_1} e^{-\lambda t_1} (\lambda t_2 - t_1)^{n_2 - n_1} e^{-\lambda(t_2 - t_1)} u(n_1) u(n_2 - n_1)}{n_1!(n_2 - n_1)!} \\ &= \frac{\lambda^{n2} (t_1)^{n_1} e^{-\lambda t_2} (t_2 - t_1)^{n_2 - n_1} u(n_2 - n_1)}{n_1!(n_2 - n_1)!} \end{aligned}$$

(b) Find an expression for the Kth order joint PMF, $P_N(n_1, \ldots, n_K; t_1, \ldots, t_K)$.

For $t_i = 0$ and $n_0 = 0$

$$P_N(n_1, \ldots, n_k; t_1, \ldots, t_k) = \lambda^{n_k} e^{-\lambda t_k} \Pi_{i=1}^{k} \frac{((t_i - t_{i-1}))^{n_i - n_{i-1}}}{(n_i - n_{i-1})!} u(n_i - n_{i-1})$$

Exercise 9.14

Let $W(t)$ be a *standard* Wiener process, defined over $[0, \infty)$. Find the joint density $f_W(a1, a2; t1, t2)$ for $0 < t_1 < t_2$

A standard Wiener process is normally distributed on independent increments with mean dependent upon the previous increment. As such, for increment 1 $f_X(x) = N(0, \alpha t)$ then for increment 2 it is $f_X(x2) = N(x_1, \alpha(t2 - t1))$

Here, α is simply 1.

$$\begin{aligned} f_W(a_1, a_2; t_1, t_2) &= f_W(a_1; t_1) f_W(a_2 | a_1; t_1, t_2) \\ &= f_{N(0,t_1)}(a_1) f_{N(a_1,t_2)}(a_2) \\ &= \frac{e^{-\frac{a_1}{2t_1}}}{\sqrt{2\pi t_1}} \frac{e^{-\frac{(a_2-a_1)}{2(t_2-t_1)}}}{\sqrt{2\pi(t_2 - t_1)}} \end{aligned}$$