# Probabilty

## Events

An event is a collection of outcomes of a random experiment
$S = \{$collection of all outcomes of the experiment$\}$
$\phi = \{$empty set$\}$
If $A \cap B = \phi$,
then $A$ and $B$ are mutually exclusive events
DeMorgan's $(\bar{A \cup B}) = (\bar{A} \cap \bar{B})$

## Axioms and Properties

### Axioms

I. $P(A) \geq 0$
II. $P(S) = 1$
III. If $(A \cap B) = \phi$,
then $P(A \cup B) = P(A) + P(B)$
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ $P(\bar{A}) = 1 - P(A)$

### Independence

If $P\{A \cap B\} = P\{A\}P\{B\}$,
then $A$ and $B$ are independent
If $P(A \cap B|C) = P(A|C)P(B|C)$,
$A$ and $B$ are **conditionally** independent given event $C$

### Mutually Exclusivity

If $P\{A \cap B\} = \phi$,
then $A$ and $B$ are M.E. And, in this case
$P(A|B) = P(A)$ and $P(B|A) = P(B)$

### Conditional Probability

$P(A|B) = P(A \cap B)/P(B)$
$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$

### Bayes' rule

$P(B|A) = P(A|B)P(B)/P(A)$,

## PDF and CDF

### PDF

The **P**robability **D**ensity **F**unction is a function that accepts an outcome and returns the probability of that outcome occuring. Written as:
$p(x)$ and $f_x(x)$

## PMF and CMF

Are the discrete time versions of the PDF and CDF

### CDF

The **C**umulative **D**istribution **F**unction. Commonly written as:
$P(x)$ and $F_x(x)$
Is the integral of the PDF. $F_x(x) = \int f_x(x)dx$

# Distributions

## Binomial

### General

$X$ = the number of successes in n trials. This is n trials of a Bernoulli random variable.

### Probability Mass Function

$P\{X = k\} = \binom{n}{k} p^k q^{n-k}$, for $k = 0, 1, 2, ..., n$, where
$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

### Mean

$m_x = np$

### Variance

$Var(x) = np(1-p)$

## Uniform

### General

$X$ has equal likeliness of taking any value in the interval $[a, b]$

### Probability Density Function

$f_x(u) = \frac{1}{b-a}$, for $a < u < b$, and is 0 elsewhere

### Cumulative Distribution Function

$$F_x(u) = \begin{cases} 0, & u < a \\ \frac{u-a}{b-a}, & a < u < b \\ 1, & b < u \end{cases}$$

### Mean

$m_x = (a+b)/2$

### Variance

$Var(X) = \frac{(b-a)^2}{12}$

## Triangular

### General

Upon adding two uniform distributions, we get the triangular density function. The function only has value over $[2a, 2b]$

### Density

$$f_x(\alpha) = \begin{cases} \frac{\alpha - 2a}{(b-a)^2}, & 2a < \alpha < (a+b) \\ \frac{2b-\alpha}{(b-a)^2}, & (a+b) < \alpha < (2b) \\ 0, & otherwise \end{cases}$$

### Cumulative

This was not listed in the summary, and I need to review to understand why.

## Exponential

### General

$X$ is the time to arrival or time to failure, where arrival rate is $\lambda$
$X$ can also be viewed as departure time with departure rate $\mu$

### Probability Density Function

$f_x(t) = \lambda e^{-\lambda t}$, for $t > 0$, and is 0 elsewhere.

### Cumulative Distribution Function

$F_x(t) = 1 - e{-}\lambda t$, for $t \geq 0$, and is 0 elsewhere.

### Mean and Variance

$m_x = \sigma_x = 1/\lambda$

## Gaussian

### General

The normal distribution

### Probability Density Function

With mean $m$ and standard deviation $\sigma$
$f_x(u) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(u-m)^2}{2\sigma^2}}$
Unit Gaussian (normal) $\sigma = 1$, $m = 0$
$f_x(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}$

### Unit Gaussian Cumulative Distribution $\phi(u)$

$\phi(u)$ was used to compute the following
$P\{a < X < b\} = \phi(\frac{b-m}{\sigma}) - \phi(\frac{a-m}{\sigma})$

### Mean and Variance

$m$ and $\sigma$ are the mean and standard deviation
$\sigma_x^2$ is the variance

## Geometric

### General

$X$ is the number of trials before the first success
$p$ is the probability of success

### Mass Function

$P\{X = k\} = p(1-p)^{k-1}$, for $k = 1, 2, 3, ...$

### Mean

$m_x = 1/p$

### Variance

$Var(X) = (1-p)/p^2$

## Poisson

### General

$X$ is the number of arrivals in a time interval $t$
$\lambda$ is the arrival rate

### Mass Function

$P\{X = k\} = \frac{(\lambda t)^k}{k!} e^{\lambda t}$, for $k = 0, 1, 2, 3, ...$

### Mean and Variance

$m_x = Var(X) = \lambda t$

## Moments

## First

### General

The first moment is the mean of the distribution.
Sometimes refered to as the center of mass.

### Formula

Where $p(x)$ is the probabilty of the outcome $x$ occuring.
$\mu_x = E\{X\} = \int p(x)xdx$
And applies via a sum for the discrete case.

## N$^{\text{th}}$ Moment

$E\{X^n\} = \int p(x)x^n dx$

### Properties

If $Y = aX + b$,
then $m_y = am_x + b$
and $\sigma_y^2 = a^2\sigma_x^2$

## Expectation

### General

The expectation $E$ of a function $g$ of a random variable $x$, $E\{g(X)\}$:
$E\{g(X)\} = \int_{-\infty}^{\infty} g(u)f_x(u)du$
A sum can be substituted for the integral in the discrete case unless using impulse functions

### Properties

$E\{C\} = C$
$E\{ag(X) + bh(X)\} = aE\{g(x)\} + bE\{h(X)\}$
If $g(X) \geq 0$, then $E\{g(X)\} \geq 0$

## Variance

$Var(X) = E\left[(X - \mu_x)^2\right] = E[X^2] - \mu_x^2$
$Var(X) = \sigma^2 = \int (X - \mu_x)^2 f_x(x)dx$
$Var(X) = \sigma^2 = \sum_{i=1}^{n} p_i(x_i - \mu_x)^2$

## Covariance

$Var(X) = Cov(X, X)$
$Cov(X, Y) = \sigma_{XY}$ $\sigma_{XY} = E[(X - \mu_x)(Y - \mu_y)]$
$\sigma_{XY} = E[XY] - E[X]E[Y]$ $cov(X, a) = 0$
$cov(X, Y) = cov(Y, X)$ $cov(X + a, Y + b) = abcov(X, y)$
$cov(X + a, Y + b) = cov(X, Y)$

## Covariance for RV Vector

$\sigma_{\vec{X}\vec{Y}} = E[(\vec{X} - E[\vec{X}])(\vec{Y} - E[\vec{Y}])]$
$\sigma_{\vec{X}\vec{Y}} = E[\vec{X}\vec{Y}^T] - E[\vec{X}]E[\vec{Y}]^T$

## Correlation

$corr(X, Y) = \rho_{XY}$
$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$

# Functions of Random Variables

## A Single Random Variable

If $Y = g(X)$, where $X$ is a random variable, then
$f_y(v) = P\{Y \leq v\} = P\{g(X) \leq v\}$
If $g(u)$ is monotonic, then
$f_y(v) = [\frac{f_x(u)}{g'(u)}]_{u=g^{-1}(v)}$

## Two Random Variables

### Joint distribution function of $X$ and $Y$

$F_{XY}(u, v) = P\{X \leq u, Y \leq v\}$

### Properties

$P\{(a < X \leq b)\ and\ (c < Y \leq d)\} =$
$F_{XY}(b, d) - F_{XY}(a, d) - F_{XY}(b, c) + F_{XY}(a, c) \geq 0$
$F_{XY}(-\infty, v) = 0, F_{XY} = (u, -\infty) = 0$
$F_{XY}(\infty, \infty) = 1$

### Marginal Distributions

$F_{XY}(u, \infty) = F_X(u)$
$F_{XY}(\infty, v) = F_y(v)$

### Joint Probability Mass Function

$p_{XY}(a_i, b_j) = P\{X = a_i, Y = b_j\}$, where $X$ and $Y$ take values $\{a_i\}$ and $\{b_i\}$

### Joint Probability Density Function

$f_{XY}(u, v) = \frac{\delta^2 F_{XY}(u,v)}{\delta u \delta v}$

### Properties

$f_{XY}(u, v) \geq 0$
$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(u, v)du\, dv = 1$
$P\{(a < X \leq b\ and\ (c < Y \leq d)\} =$
$\int_c^d \int_a^b f_{XY}(u, v)du\, dv = 1$
$F_{XY}(b, d) = \int_{-\infty}^{d} \int_{-\infty}^{b} f_{XY}(u, v)du\, dv$

### Marginal Densities

$f_X(u) = \int_{-\infty}^{\infty} f_{XY}(u, v)dv$, and
$f_Y(v) = \int_{-\infty}^{\infty} f_{XY}(u, v)du$

### Independent Random Variables

$f_{XY}(u, v) = f_X(u)f_Y(u)$
$F_{XY}(u, v) = F_X(u)F_Y(v)$

### Conditional Densities

$f_{X|A}(u) = \frac{d}{du} P\{X \leq u|A\} = \frac{d}{du} P\{(X \leq u) \cap A\}/P\{A\}$

### Two Cases

$A = a < X \leq b : f_{X|A}(u|A) = f_x(u)/P\{A\}$,
for $a < u \leq b$, and 0 elsewhere
$A = \{Y = v\} : f_{X|Y}(u|v) = f_{XY}(u, v)/f_y(v)$
The second way can be represented in two ways
$f_{XY}(u, v) = f_{X|Y}(v|u)f_y(v) = f_{Y|X}(u|v)f_X(u)$

### Total Probability and Bayes' for Random Vars

$f_x(u) = \int_{-\infty}^{\infty} f_{X|Y}(u|v)f_Y(v)dv$
$f_{Y|X}(v|u) = f_{X|Y}(u|v)f_Y(v)/f_X(v)$
In the discrete case the integrals can be replaced by sums, and the densities can be replaced by probabilities

### Jointly Gaussian Rondom Variable

Placeholder

### Conditional Densities

Placeholder

## Functions of Two Random Variables

### Expectations

$E\{g(X, Y)\} =$
Discrete Case
$\sum_j \sum_i g(a_i, b_j)P\{X = a_i, Y = b_j\}$
Continuous Case
$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(u, v)f_{XY}(u, v)du\, dv$

### Properties

1. $E\{C\} = C$
2. $E\{ag(X, Y)\} = aE\{g(X, Y)\}$
3. $E\{g(X, Y) + h(X, Y)\} = E\{g(X, Y)\} + E\{h(X, Y)\}$
4. If $g(X, Y) \geq 0$, then $E\{g(X, Y)\} \geq 0$
5. IF $X$ and $Y$ are independent, then $E\{g(X)h(Y)\} = E\{g(X)\}E\{h(Y)\}$

### Correlation and Covariance

Correlation between $X$ and $Y$: $R_{XY} = E\{X, Y\}$
Covariance of $X$ and $Y$: $Cov(X, Y) = C_{XY} =$
$E\{(X - m_x)(Y - m_y)\} = R_{XY} - m_x m_y$
Correlations Coefficient:
$\rho_{XY} = C_{XY}/(\sigma_x \sigma_y) - 1 \leq \rho_{XY} \leq 1$
If $\rho_{XY} = \pm 1$,
then $X$ and $Y$ are perfectly correlated
If $\rho_{XY} = \pm 0$,
then $X$ and $Y$ are uncorrelated

### Linear Approximation

Extimating $X$ from the values of $Y$:
$\hat{X} = m_x + (\rho_{XY}\sigma_x/\sigma_y)(Y - m_y)$ Mean-Squared Error:
$E\{[\hat{X} - X]^2\} = \sigma_x^2(1 - \rho_{XY}^2)$
Gaussian random variables:
If $X$ and $Y$ are Gaussian and uncorrelated,
then they are independent
The linear transformation of Gaussian random variables is also Gaussian

### Functions of Two RVs

$Z = g(X, Y)$
$F_Z(w) = P\{g(X,Y) \leq w\}$

### Sums of Two RVs

$Z = X + Y$, then $f_Z(w) = \int_{-\infty}^{\infty} f_{XY}(u, w-u)du$
If $X$ and $Y$ are independent:
$f_Z(w) = \int_{-\infty}^{\infty} f_Y(w-u)f_x(u)du$

### Mean and Variance of a Sum

$E\{Z\} = E\{X\} + E\{Y\}$
$Var(Z) = Var(X) + Var(Y) + 2Cov(X, Y)$
For uncorrelated variables, variance of a sum is the sum of the variance.

### Subjects not yet added

Rayleigh Density
Estimation
Maximum a-posteriori probability (MAP) estimate of $X$ given $Y$
Minimum mean-squared-error estimate
Linear Estimate

# Random Processes

## General

### 1st and 2nd order Distributions of a RP

$F_{X(t)}(x;t) = P\{X(t) \leq x\}$
$F_{X(t)}(x_1, x_2; t_1, t_2) =$
$P\{X(t_1) \leq x_1 \; and \; X(t_2) \leq x_2\}$

### Mean, autocorrelation, and autocovariance functions

$m_x(t) = E[x(t)]$
$R_X(t_1, t_2) = E[X(t_1)X(t_2)]$
$C_X(t_1, t_2) = E[[X(t_1) - m_x(t_1)][X(t_2) - m_x(t_2)]] =$
$R_x(t_1, t_2) - m_x(t_1)m_x(t_2)$

### Wide-sense stationary process

$m_x = E\{X(t)\}$
$R_X(\tau) = E\{X(t)X(t+\tau)\}$
$C_x(\tau) = E\{[X(t) - m_x][X(t+\tau) - m_x]\} = R_x(\tau) - [m_x]^2$

### Cross-correlation function

for the general case,
$R_{XY}(t_1, t_2) = E\{X(t_1)Y(t_2)\}$
for jointly WSS processes,
$R_{XY}(\tau) = E\{X(t)Y(t+\tau)\}$

## Specific Processes

### Poisson Process

Interpreted as a Counting Process
For: $N(0) = 0$; iid; Independend Increments.
$P(N(t) = n) = \frac{(\lambda t)^n}{n!} e^{\lambda t}$
$E[N(t)] = \lambda t$, $\forall \tau = \Delta t$
Interarrival Time Approximated Exponential
With Expected Interarrivel of $\frac{1}{\lambda}$

# Basic Maths

## Series and Sequences

### Geometric Sequence

A series with a constant ration between successive terms.
Ex. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$
Often defined as using $ar$
Ex. $a + ar + ar^2 + ar^3 + \dots$
For $r \neq 1$, the sum of the first $n$ terms is:
$\sum_{k=0}^{n-1} ar^k = a(\frac{1-r^n}{1-r})$
And for infinite sequences:
$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$, for $|r| < 1$

### Arithmetic Series

A series with a constant difference between successive terms.
Ex. $2 + 5 + 8 + 11 + \dots$
Sum of an arithmetic series with $n$ terms starting with $a_1$ and ending with $a_2$:
$\sum = \frac{n(a_1 + a_2)}{2}$

### Power Series

A series of the form:
$\sum_{n=0}^{\infty} = a_n(x-c)^n$
Where often $c = 0$
$\sum_{n=0}^{\infty} = a_n(x)^n$
The power series allows generalization of multiplication, division, subtraction, and addition between like series. It is also possible to integrate or differentiate a power series.

### Taylor Series

The Taylor series of $f(x)$ (a function that is infinetely differentiable at a number a) is the power series:
$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a) + \dots$
$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

## Logarithms

$\log_b c = k$
$b^k = c$
$\ln(xy) = \ln(x) + \ln(y)$
$\ln(x/y) = \ln(x) - \ln(y)$
$\ln(x^y) = yln(x)$
$\ln(e) = 1$
$\ln(1/x) = -\ln(x)$

## Integrals

$\int x^n dx = \frac{1}{n+1}x^{n+1}, n \neq -1$
$\int \frac{1}{x} dx = \ln|x|$
$\int u dv = uv - \int v du$
$\int e^{ax} dx = \frac{1}{a}e^{ax}$ $\int xe^{ax} dx = (\frac{x}{a} - \frac{1}{a^2})e^{ax}$

## Derivatives

## Combinatorics

### Permutations

Number of ways to order $n$ distinct elements:
$n!$

### $k$-Permutations of $n$

Ordered arrangements of a k-element subset of an n-set.
$P(n,k) = \frac{n!}{(n-k)!}$

### Permutations With Repitition

For a set S of size k, the number of n-tuples over S is. $k^n$

### Combination

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

### Binomial Theorem